Infinite Impulse Response(IIR) Filters

Lecture 11
IIR Filters

Lecture delivered by:

Viswanath K

Objectives

At the end of this lecture, student will be able to:

- Explain the filter concepts and types
- Describe the standard steps in IIR filter design

Topics

- Comparison of Digital and Analog Filter
- Advantages and Disadvantages of Digital Filter
- Types of Digital Filter
- Classification of Filters Based on Frequency Response
- Performance Constraints
- IIR Filter
- IIR filter Design

Comparison of Digital and Analog Filter

Digital Filter	Analog Filter
1. Operates on digital samples (or sampled version) of the signal.	1. Operates on analog signals (or actual signals).
2. It is governed (or defined) by linear difference equation.	2. It is governed (or defined) by linear differential equation.
3. It consists of adders, multipliers and delays implemented in digital logic (either in hardware or software or both).	3. It consists of electrical components like resistors, capacitors and inductors.
4. In digital filters the filter coefficients are designed to satisfy the desired frequency response.	4. In analog filters the approximation problem is solved to satisfy the desired frequency response.

Types of Analog filters

- Butterworth filters- Maximally flat passband
- Chebyshev filters- Either equiripple in passband or stopband
- Elliptical filters- Equiripple in passband and stopband

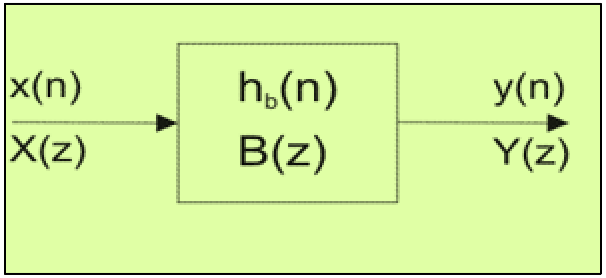
Advantages and Disadvantages of Digital Filter

- Advantages
  - High thermal stability(absence of R,L and C)
  - Programmable
  - Accuracy, dynamic range, stability can be enhanced(increasing length of register)

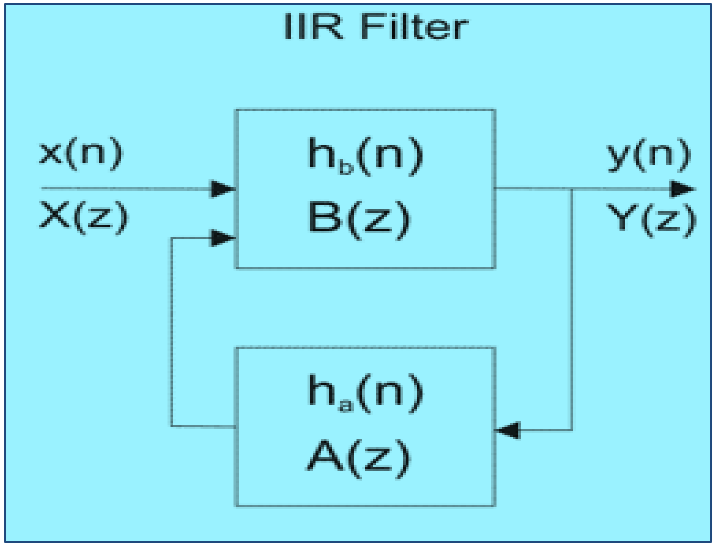
- Disadvantages
  - Bandwidth is limited (by sampling frequency)
  - Performance depends on hardware(bit length of register)

Types of Digital Filter

- FIR(Finite Impulse Response) filter



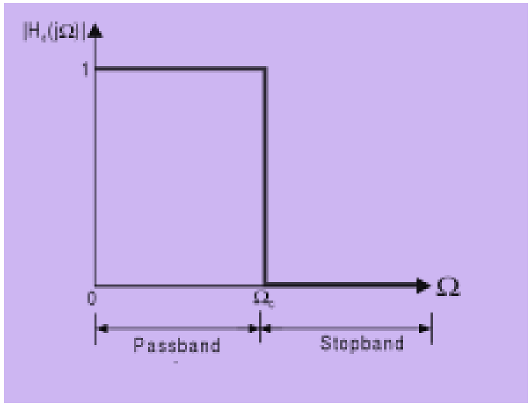
- IIR(Infinite Impulse Response) filter



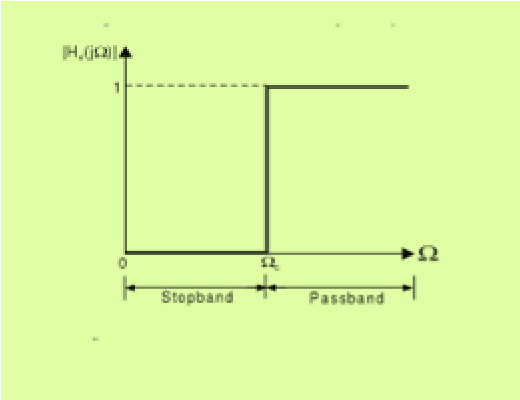
Classification of Filters based on Frequency Response

- Frequency selective devices - pass the spectral content of the input signal in a specified band of frequencies

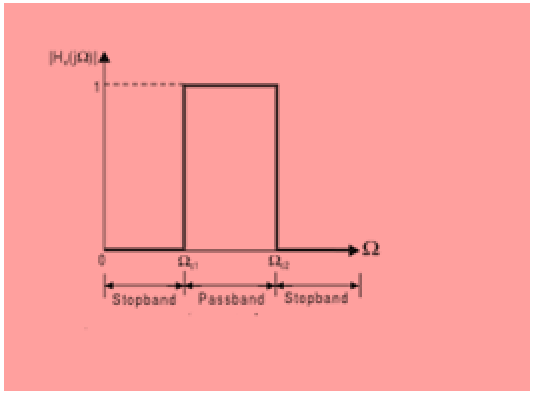
Lowpass



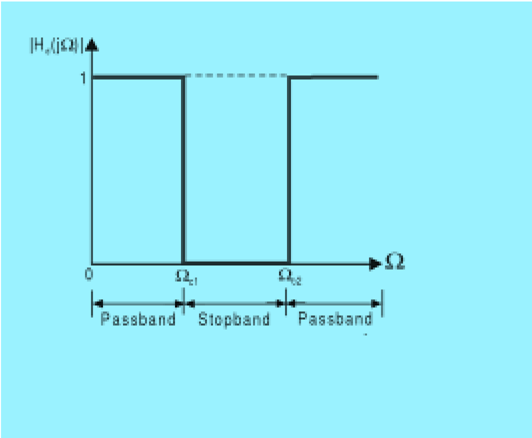
Highpass



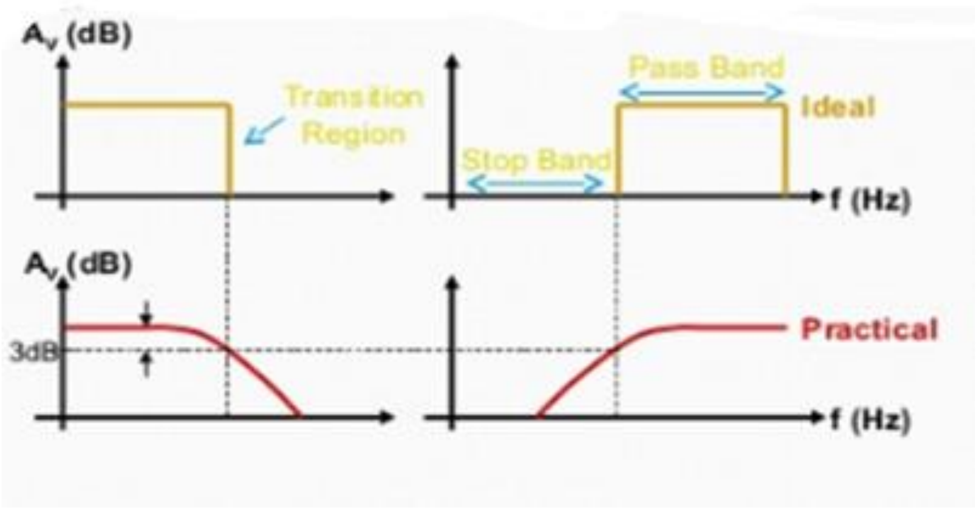
Bandpass

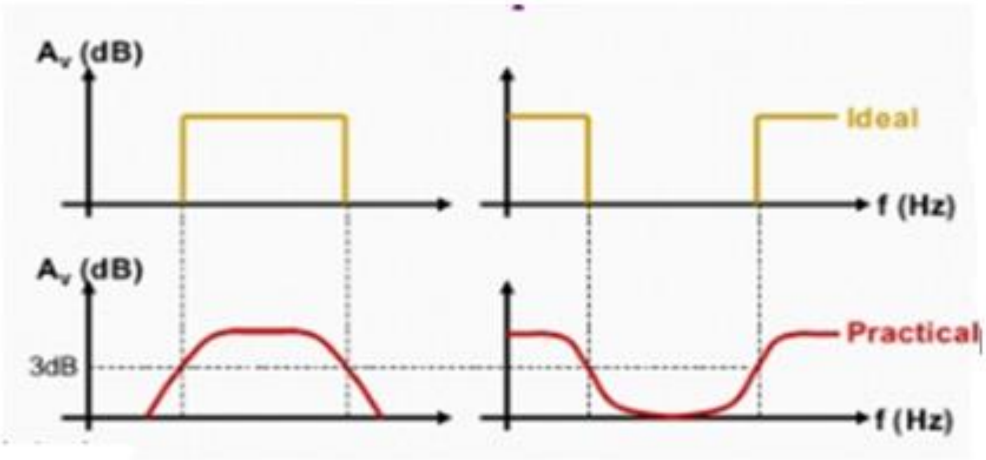


Bandstop



Cont'd





Passband : The range of frequencies seen in the output. It is the bandwidth(BW) of the filter.

Stopband : The range of frequencies blocked by the filter. These frequencies are not seen in the output.

Transition region : The range of frequencies between passband and stopband.

Cutoff frequency : The highest or lowest frequency that is allowed to pass or determines the passband. The cutoff frequency of real time filter is -3dB frequency of that filter.

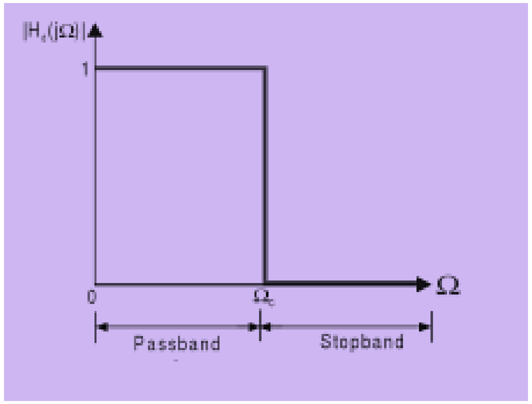
Order of the filter :Highest power in the polynomial. (e.g.) IIR filter(N):Highest power in denominator polynomial

Order of The Filter

- When you give an input to a filter, output is calculated using present inputs, past inputs and past outputs. Past inputs and Past outputs are nothing but delayed inputs and delayed outputs.
- The maximum amount of delay used in the calculation of any output is called the order of filter. Order of the filter is directly proportional to no. of calculations involved or no. of components needed to realize the filter.
- If order increases, simply the number of delay blocks increases making the filter more complex in design, but making it more and more efficient filter.
- The more the past values of signals are accumulated, more experience on about the signal and it's characteristics, and it becomes easier for filter to predict or estimate or make some relevant output by taking large number of sampled input value delay versions

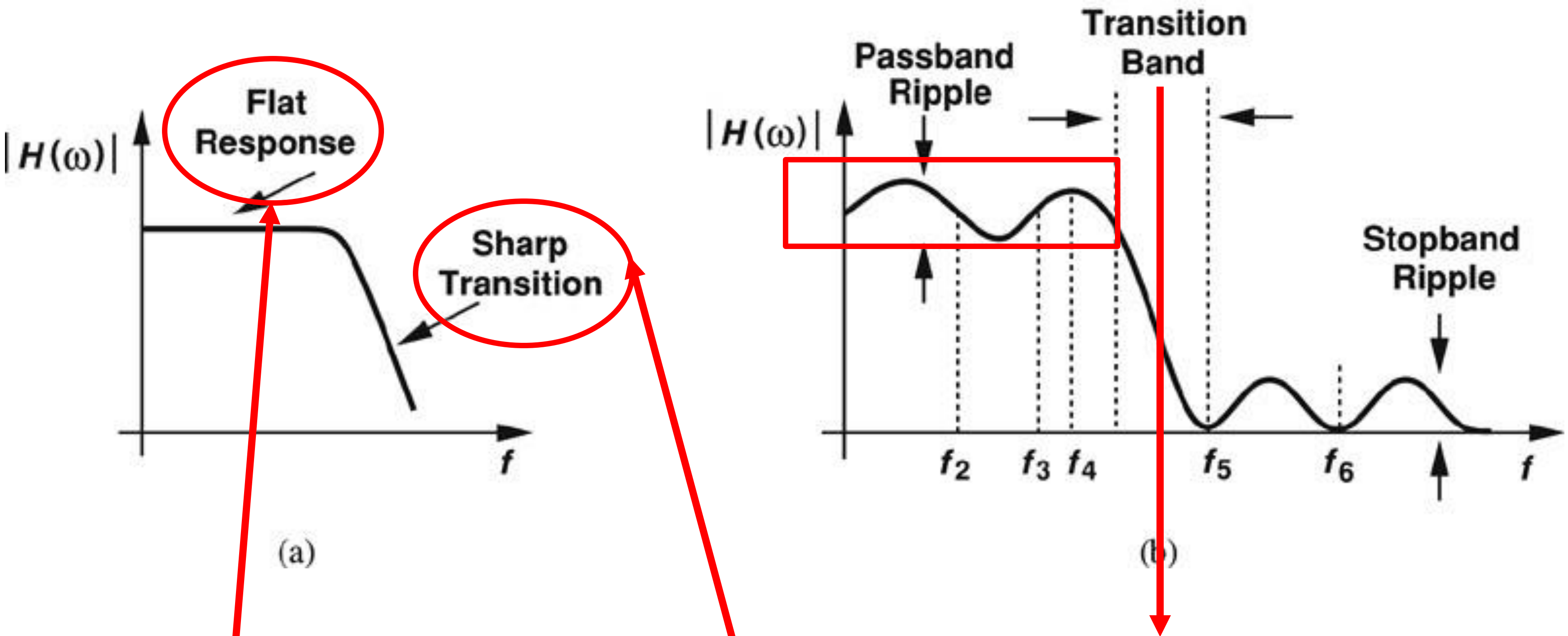
Ideal Filter

- Transmit signal under the passband without attenuation and completely suppress the signal in stopband
- Characteristic
  - Constant gain in passband and zero gain in stopband
  - It has linear phase response
  - It must be causal



Note: A system is said to be causal, if the output of a system y(n)at any time(n) depends only on present and past inputs(i.e x(n-1),x(n)).But doesn't depend on future inputs.

Practical Filter Characteristics



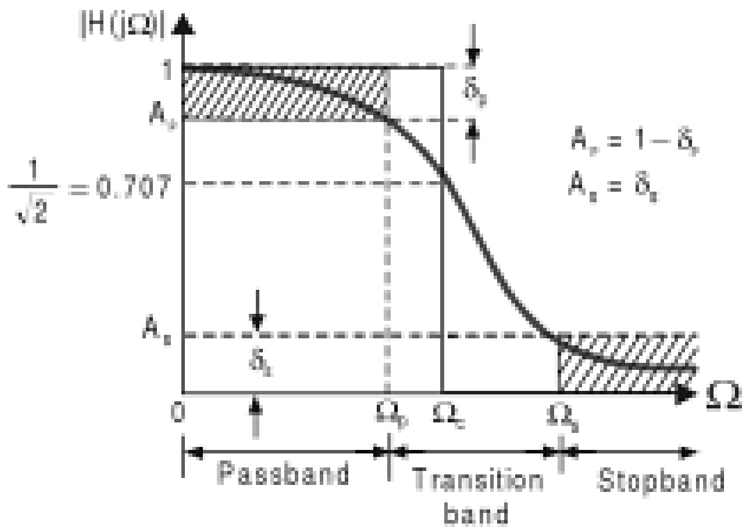
Must not alter the desired signal

Sharp Transition in order to attenuate the interference

Affect selectivity

Frequency Response of Practical Filters

Lowpass Analog filter



Ω_p= Passband edge frequency(rad/second)

Ω_s= Stopband edge frequency(rad/second)

A_p= Gain at passband edge frequency

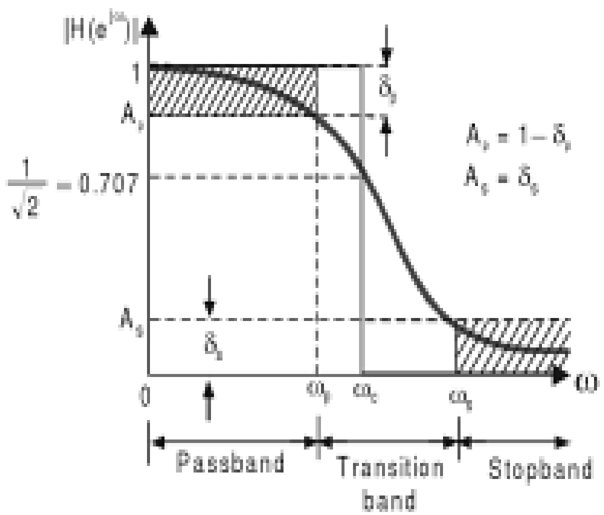
A_s= Gain at stopband edge frequency

-3dB point

- The half of the maximum power
- Defines the bandwidth
- Till this point, gain is resasonably constant,beyond that attenuation is large

Frequency Response of Practical Filters

Lowpass Digital filter



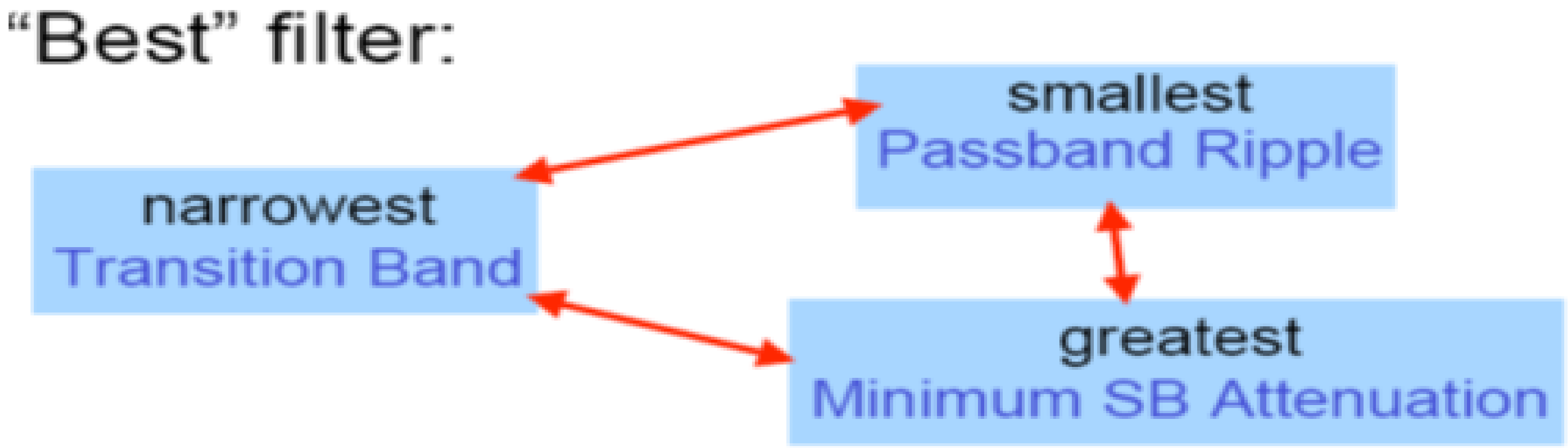
ω_p= Passband edge frequency(rad/sample)

ω_s= Stopband edge frequency(rad/sample)

A_p= Gain at passband edge frequency

A_s= Gain at stopband edge frequency

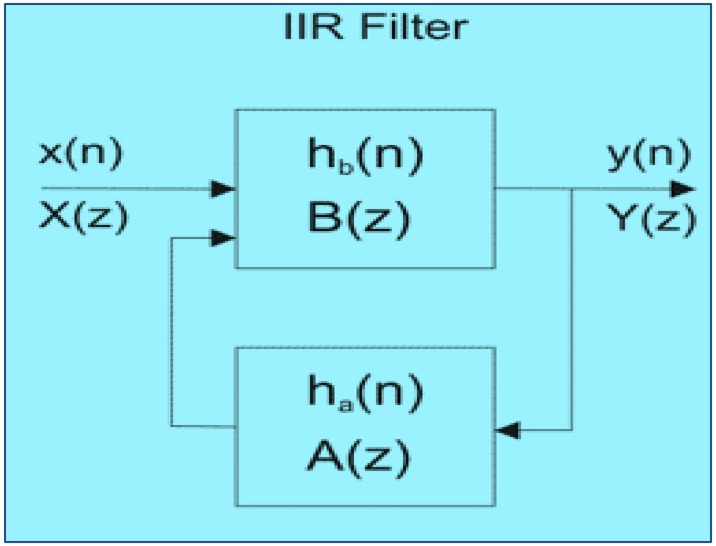
Performance Constraints



- Improving one worsens other
- But increasing order of the filter improves all three measures

IIR Filter

- Designed by the infinite samples of impulse response
- Direct design of IIR is not possible.(processing of infinite samples is not possible in digital domain)
- So IIR filter is designed through analog filter
- Non linear phase filter
- Filter with feedback



IIR Filter Design

- Standard approach
  - Specification : Get digital filter specification and convert to analogue filter specification
  - Approximation :Design an analog prototype low pass filter specifications $H_a(s)$–Butterworth and Chebyshev
  - Transformation: Transform $H_a(s)$ by replacing the complex variable to the digital transfer function $H_d(z)$- bilinear and impulse invariance

IIR Filter Design -Approximation

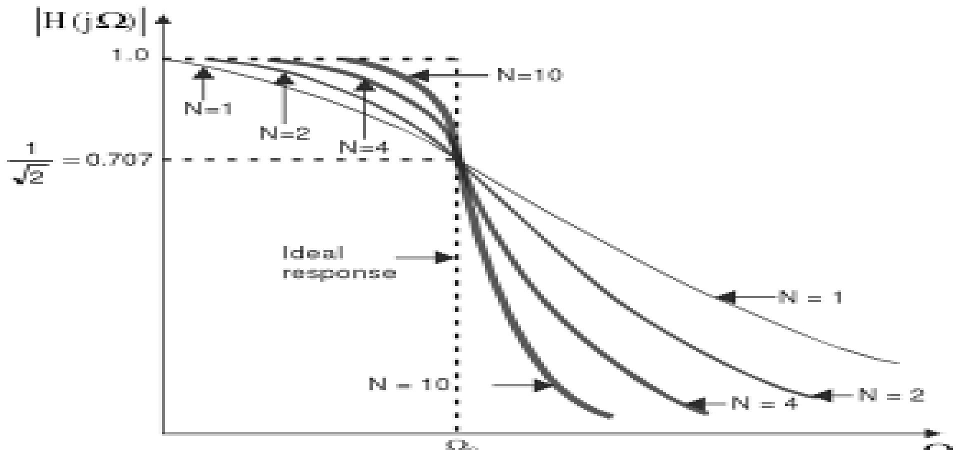
- Analog filter is designed by approximating the ideal frequency response using error function
- Approximation problem is solved to meet a specified tolerance in the passband and stopband
- In passband the magnitude is approximated to unity within an error of δ_p
- In stopband the magnitude is approximated to zero within error of δ_s
- δ_p and δ_s are limits of the tolerance in the passband and stopband (ripples)

IIR Filter Design –Approximation cont'd

- Commonly used types
  - Butterworth,
  - Chebyshev (Type 1,Type 2)

IIR filter design –Approximation cont'd

- Butterworth Approximation
  - The error function is selected such that magnitude is maximally flat in the origin (Ω =0) and monotonically decreasing with increasing Ω

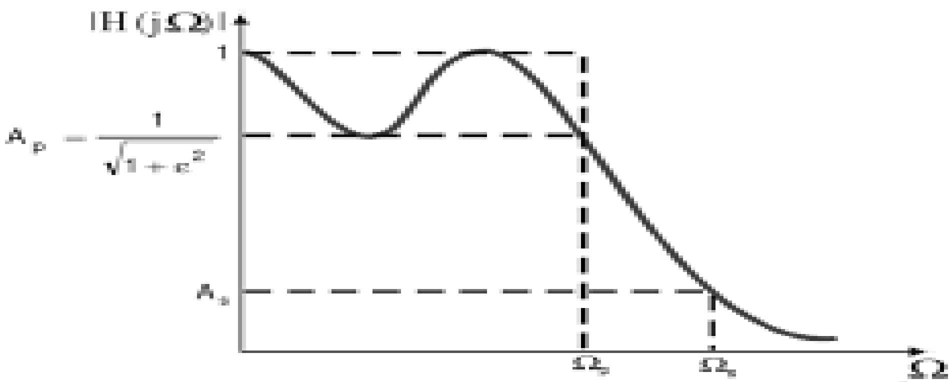


Magnitude response LPF for different values of N

IIR Filter Design –Approximation cont’d

Chebyshev type1

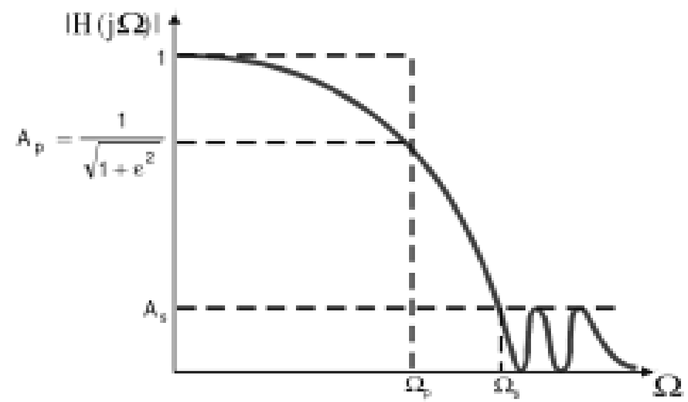
- Equiripple in passband and monotonic in stopband



chebyshev type-1

Chebyshev type 2

- Monotonic in passband and



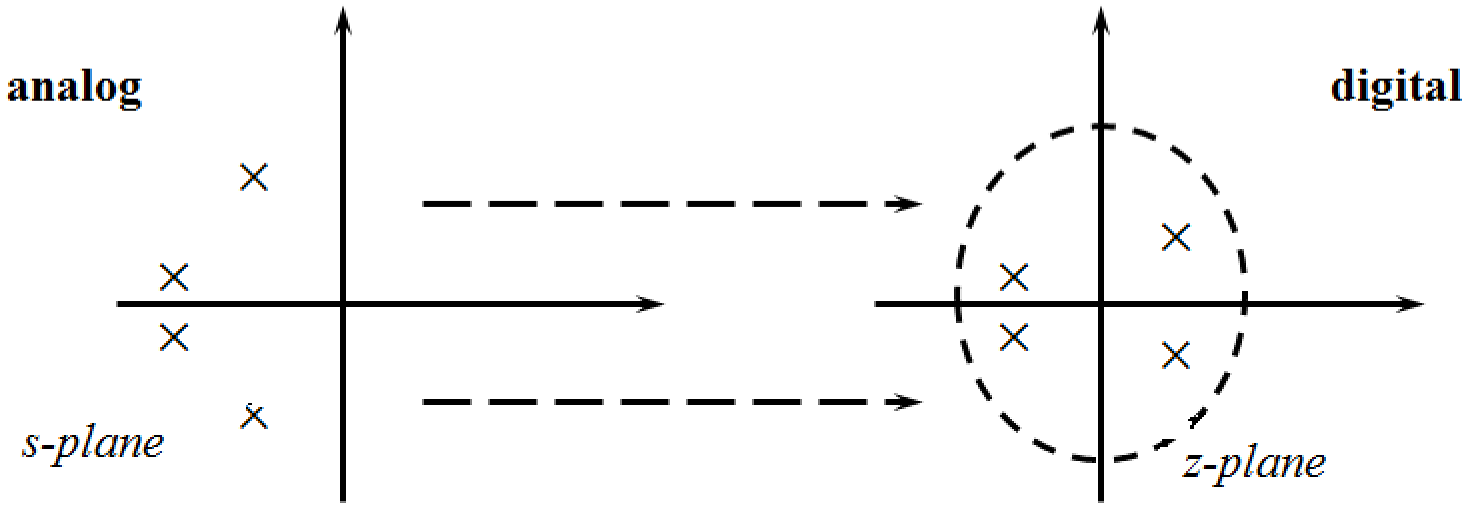
chebyshev type-2

IIR Filter Design - Transformation

- For stability and causality of filter , transfer function should satisfy the following
- Analog filter
  - The H(s) should be a rational function of s and the coefficients of s should be real
  - The poles should lie on the left half of s plane
  - The number of zeros $\leq$ number of poles
- Digital filter
  - The H(z) should be a rational function of z and the coefficients of z should be real
  - The poles should lie inside the unit circle in z plane
  - The number of zeros $\leq$ number of poles

IIR Filter Design - Transformation cont'd

- $H_a(s) \Longrightarrow H_d(z)$



Stability in z-plane

s-plane

1. A system is stable if all poles lie in the left half of s-plane.

2. A system is marginally stable if non repeated poles are on the imaginary axis and all remaining poles are in the left half of s-plane.

3. A system is unstable if any of its pole is in the right half of s-plane or repeated poles are on the imaginary axis.

↑

Stability conditions for C.T. systems

z-plane

1. A system is stable if all poles lie inside the unit circle in z-plane

2. A system is marginally stable if non repeated poles are on the circumference of unit circle in z-plane and all remaining poles are inside the unit circle in z-plane

3. A system is unstable if any of its pole is outside the unit circle in z-plane or repeated poles are on the circumference of unit circle in z-plane

↑

stability conditions for D.T./Digital/sampled data systems

Mapping between s-plane and z-plane

Mapping between s & z plane can be done by

1. Impulse invariance technique
2. Bilinear transformation technique

Impulse invariance technique

$$z = e^{sT}$$

we have,

$$s = \sigma \pm j\omega$$

$$z = e^{(\sigma \pm j\omega)T}$$

$$= e^{\sigma T} e^{\pm j\omega T}$$

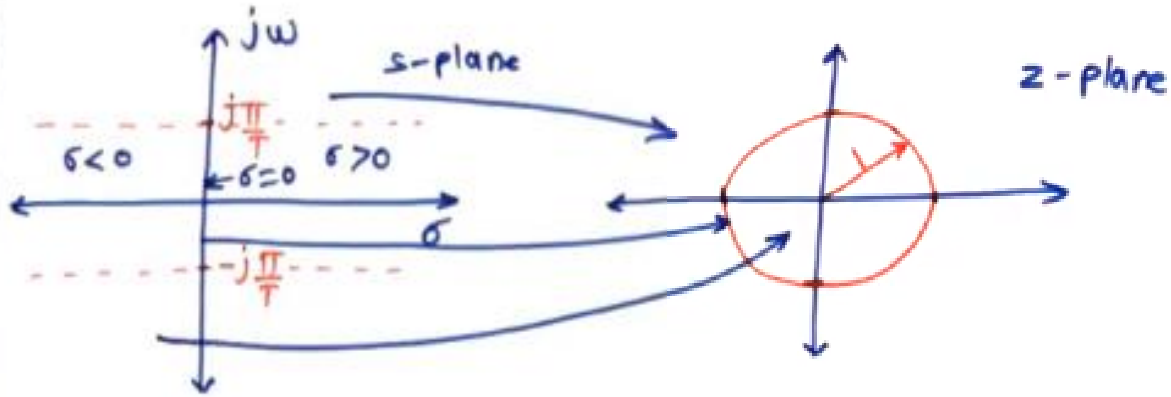
$$\therefore |z| = e^{\sigma T}$$

$$\angle z = \pm \omega T$$

As per sampling theorem, $|\omega T| \leq \pi$

$\therefore \angle z$ will vary from $-\pi$ to π (0 to 2π)

Thus the mapping will be a circle with radius $e^{\sigma T}$ and phase varies from 0 to 2π



For the left half of s-plane, $\sigma<0$

$\therefore |z| = e^{\sigma T} < 1$

For the imaginary axis in s-plane $\sigma=0$

$\therefore |z| = 1$

For the right half of s-plane, $\sigma>0$

$|z| = e^{\sigma T} > 1$

1. Left half of s-plane is mapped into inner part of unit circle in z-plane
2. Imaginary axis in s-plane is mapped into circumference of unit circle in z-plane
3. Right half of s-plane is mapped into outer part of unit circle in z-plane

Bilinear Transformation:

S plane to Z plane Mapping (One to One)
Unique.

$$s = \frac{2}{T}\left[\frac{1-z^{-1}}{1+z^{-1}}\right] \quad \text{or} \quad s = \frac{2}{T}\,\frac{z-1}{z+1} \rightarrow (1)$$

$$\therefore \quad \frac{sT}{2} = \frac{z-1}{z+1}$$

$$\Rightarrow \frac{sT}{2}\cdot[z+1] = z-1$$

$$\Rightarrow \frac{sT}{2}\cdot z + \frac{sT}{2} = z-1$$

$$z\left[\frac{sT}{2}-1\right] = -1-\frac{sT}{2}$$

$$z = \frac{-\left[1+\frac{sT}{2}\right]}{\frac{sT}{2}-1}$$

$$\boxed{z = \frac{1+\frac{sT}{2}}{1-\frac{sT}{2}}} \rightarrow (2)$$

WKT: $\left.\begin{aligned} s &= \sigma + j\Omega \\ z &= r e^{j\omega} \end{aligned}\right\}$ substitute in Eq (2) and equate real and Imaginary parts

We get

$$\sigma = \frac{2}{T}\left[\frac{r^2-1}{r^2+1+2r\cos\omega}\right] \rightarrow (3)$$

$$\Omega = \frac{2}{T}\left[\frac{2r\sin\omega}{r^2+1+2r\cos\omega}\right] \rightarrow (4)$$

Mapping

Case (a) : $\sigma < 0$ then $r < 1$

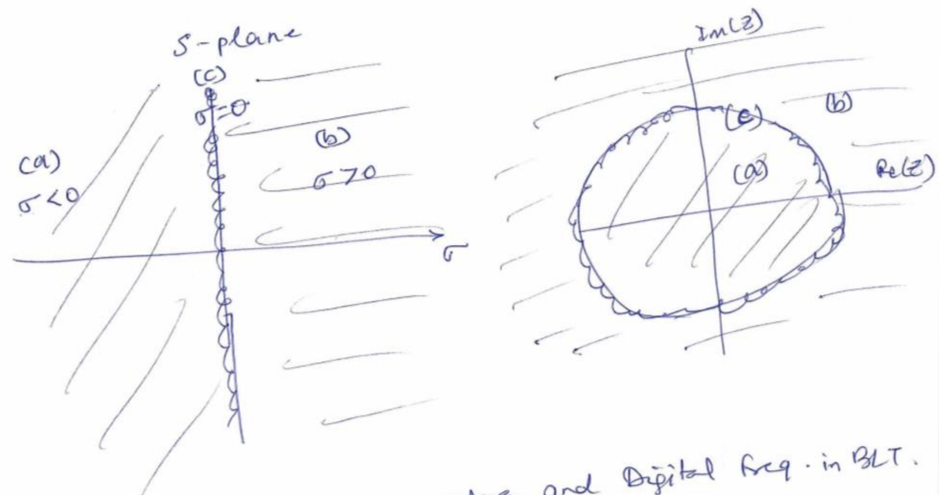
LHS of S plane mapped inside unit circle of Z plane

Case (b) $\sigma > 0$ then $r > 1$

RHS of S plane mapped outside the Unit circle in Z plane

Case (c) $\sigma = 0$ then $r = 1$

$j\Omega$ axis in S plane mapped to Unit circle in Z plane



Relation between analog and Digital freq. in BLT.

Analog freq $\Omega = \frac{2}{T} \tan\left(\frac{W}{2}\right)$

Digital freq $W = 2 \tan^{-1}\left(\frac{\Omega T}{2}\right)$

Summary

- Digital filters are classified as IIR,FIR filters
- The filters designed by considering all the infinite samples of impulse response are called IIR filter
- The IIR filters are designed via analogue filters
- The analogue filter is designed by approximating the ideal frequency response using an error function
- The popular approximation method used for analog filter design are Butterworth and Chebyshev
- For stability of analog filter, the poles should lie on the left half of S plane
- For stability of digital filter, the poles should lie inside the unit circle in z plane

Summary cont'd

- The frequency response of a practical filter will have a passband , transition band and stop band
- For casuality of analog and digital IIR filter the number of zeros should be less than the number of poles
- In Butterworth filter design , the error function is selected such that the magnitude is maximally flat in the passband and monotonically decreasing in stopband
- In type-1 Chebyshev approximation the magnitude response is equiripple in passband and monotonic in stopband
- In type-2 Chebyshev approximation the magnitude response is equiripple in stopband and monotonic in passband

Bilinear Transformation Design Method :

IIR

Digital Filter Specifications

↓ 1. Transformation with frq warping.

Analog filter Specifications

↓ 2. Transformation by lowpass prototype filter.

Analog filter Transfer function

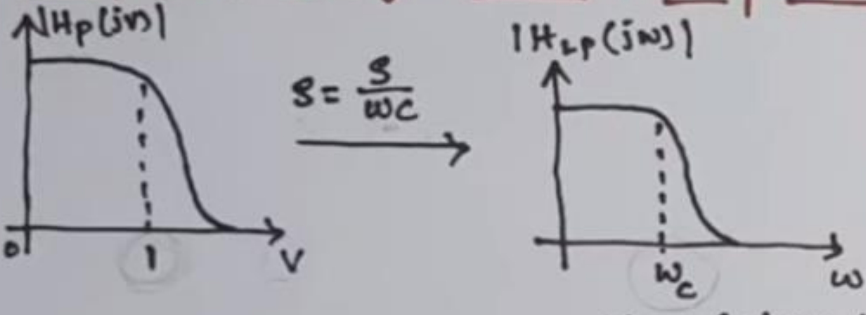
↓ 3. Bilinear Transformation.

Digital filter transfer function & Frequency response Verification

General procedure for IIR Filter design using bilinear transformation.

Analog Filters using Lowpass Prototype Transformation:

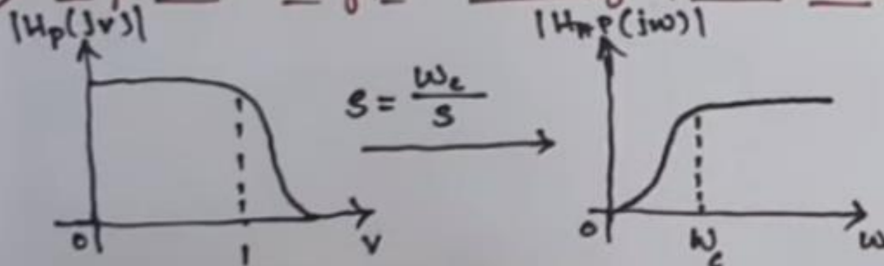
① Lowpass prototype into a lowpass Filter:



$\omega_c \rightarrow$ Cutoff freq

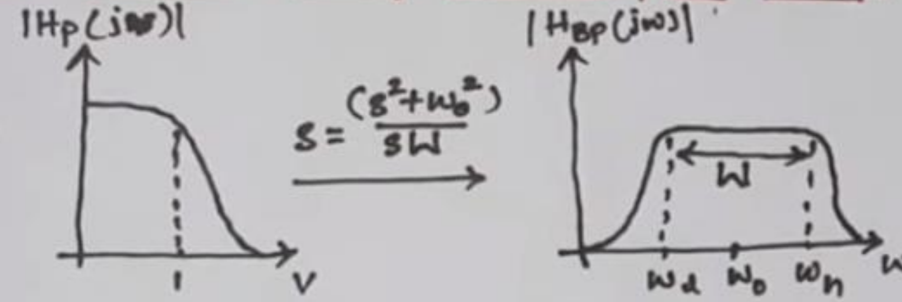
$$\boxed{H_{LP}(s) = H_p(s)\Big|_{s=\frac{s}{\omega_c}}}$$

② Lowpass Prototype to High pass Filter:



$$\boxed{H_{HP}(s) = H_p(s)\Big|_{s=\frac{\omega_c}{s}}}$$

③ Lowpass Prototype to bandpass Filter:



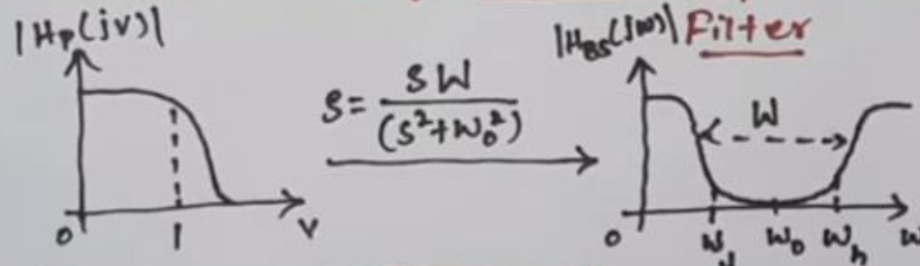
$$\boxed{H_{BP}(s) = H_p(s)\Big|_{s=\frac{(s^2+\omega_0^2)}{sW}}}$$

$\omega_0 = \sqrt{\omega_L \cdot \omega_h}$

$\hookrightarrow$ geometric center freq

$W = \omega_h - \omega_L \rightarrow$ Passband bandwidth.

④ Lowpass Prototype to bandstop [band reject] Filter



$$\boxed{H_{BS}(s) = H_p(s)\Big|_{s=\frac{sW}{(s^2+\omega_0^2)}}}$$

Given, a lowpass prototype $H_p(s) = \frac{1}{s+1}$.
Determine each of the following analog filters & plot their magnitude response from 0 to 200 rad/sec.

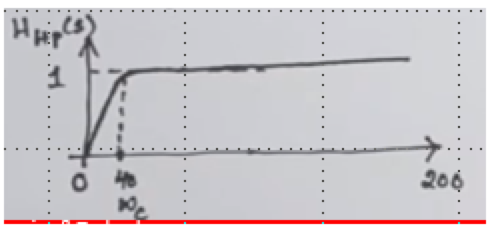
(i) A highpass filter with a cutoff freq of 40 rad/sec.

(ii) A bandpass filter with a cutoff freq of 100 rad/sec. & Bandwidth of 20 rad/sec.

(i) High-pass Filters

$$H_p(s) = \frac{1}{s+1}\Big|_{s = \frac{\omega_c}{s}} \Rightarrow s = \frac{40}{s}$$

$$H_{HP}(s) = \frac{1}{\frac{40}{s}+1} = \frac{s}{40+s}$$



(ii) band pass Filter

$$\omega_0 = \sqrt{\omega_L \omega_h} = 100$$

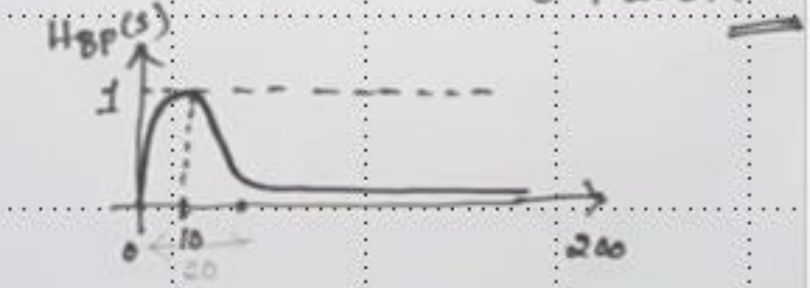
$$\omega_0^2 = 100 \Rightarrow \omega_0 = 10 \text{ rad/sec}$$

$$W = \omega_h - \omega_L = 20 \text{ rad/sec}.$$

$$H_p(s) = \frac{1}{s+1}\Big|_{s = \frac{s^2+\omega_0^2}{sW}}$$

$$s = \frac{s^2+100}{s \cdot 20}$$

$$H_{BP}(s) = \frac{1}{\frac{s^2+100}{20s}+1} = \frac{20s}{s^2+20s+100}$$



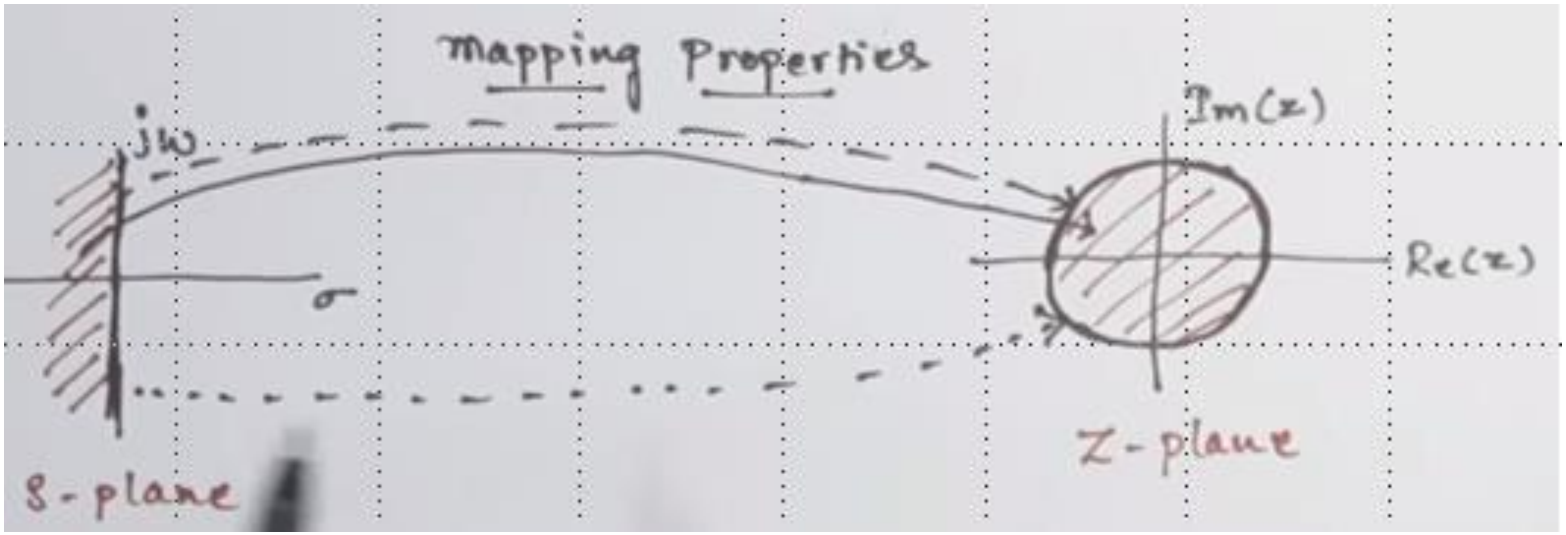
Bilinear Transformation & Frequency Warping

Analog filter transfer fun → Digital filter transfer fun:

$H(s) \rightarrow H(z)$

$$\boxed{H(z) = H(s)\Big|_{s = \frac{2}{T}\cdot\frac{(z-1)}{(z+1)}}}$$

T → Sampling period.



Given an analog filter whose transfer fun$^{=}$ is $H(s) = \frac{10}{s+10}$. Convert it to the digital filter transfer fun$^{=}$ & difference eqn. when the sampling period is given as $T = 0.01$ sec.

Bilinear Transformation,

$$H(z) = H(s)\Big|_{s = \frac{2}{T}\cdot\frac{(z-1)}{(z+1)}} = \frac{10}{s+10}\Bigg|_{s = \frac{2}{0.01}\cdot\frac{(z-1)}{(z+1)}}$$

$$H(z) = \frac{10}{\frac{2}{0.01}\frac{(z-1)}{(z+1)} + 10} = \frac{10}{\frac{200(z-1)}{z+1} + 10} = \frac{0.05}{\frac{(z-1)}{(z+1)} + 0.05}$$

$$= \frac{0.05(z+1)}{(z-1) + 0.05(z+1)} = \frac{0.05z + 0.05}{1.05z - 0.95} = \frac{(0.05z + 0.05)/1.05z}{(1.05z - 0.95)/1.05z}$$

$$\frac{Y(z)}{X(z)} = H(z) = \frac{0.0476 + 0.0476z^{-1}}{1 - 0.9048z^{-1}}$$

$$Y(z) - 0.9048z^{-1}Y(z) = 0.0476X(z) + 0.0476z^{-1}X(z)$$

$$\therefore Y(z) = 0.0476X(z) + 0.0476z^{-1}X(z) + 0.9048z^{-1}Y(z)$$

IZT

$$y(n) = 0.0476x(n) + 0.0476x(n-1) + 0.9048y(n-1)$$

Bilinear Transformation design Procedure :

1. Digital filter freq specifications → Given
   ↳ prewarp to the Analog freq Specification

Lowpass & Highpass filter:

$$W_a = \frac{2}{T}\tan\left(\frac{W_d T}{2}\right) \longrightarrow (1)$$

T → Sampling Period.

bandpass & band stop filter:

$$W_{al} = \frac{2}{T}\tan\left(\frac{W_l T}{2}\right);\ W_{ah} = \frac{2}{T}\tan\left(\frac{W_h T}{2}\right) \longrightarrow (2)$$

& $W_0 = \sqrt{W_{al} \cdot W_{ah}}$ & $W = W_{ah} - W_{al}$

W_l → lower cutoff freq W_0 → Center freq

W_h → higher cutoff freq W → Bandwidth

2. Perform the Prototype transformation using lowpass prototype $H_p(s)$

from lowpass to lowpass:

$$H(s) = H_p(s)\Big|_{s = \frac{s}{W_a}} \longrightarrow (3)$$

from lowpass to highpass:

$$H(s) = H_p(s)\Big|_{s = \frac{W_a}{s}} \longrightarrow (4)$$

from lowpass to bandpass:

$$H(s) = H_p(s)\Big|_{s = \frac{s^2 + W_0^2}{sW}} \longrightarrow (5)$$

from lowpass to bandstop:

$$H(s) = H_p(s)\Big|_{s = \frac{sW}{s^2 + W_0^2}} \longrightarrow (6)$$

3. Obtain digital Filter: $H(z) = H(s)\Big|_{s =}$

The normalized lowpass filter with a cutoff freq of 1 rad|sec is given as $H_p(s) = \frac{1}{s+1}$

use the given $H_p(s)$ & the BLT to design a corresponding digital IIR lowpass filter with cutoff freq of 15 Hz & Sampling rate of 90 Hz.

$$H(z) = \frac{0.5773(z+1)}{\left[\frac{z-1}{z+1} + 0.5773\right](z+1)} = \frac{0.5773z + 0.5773}{(z-1) + 0.5773(z+1)}$$

$$\therefore H(z) = \frac{0.5773z + 0.5773}{\underbrace{1.5773z}_{1} - 0.4227}$$

$\div$ both nr & dr by $1.5773z$

$$\therefore H(z) = \frac{(0.5773z + 0.5773)/1.5773z}{(1.5773z - 0.4227)/1.5773z}$$

$$\therefore H(z) = \frac{0.3660 + 0.3660z^{-1}}{1 - 0.2679z^{-1}}$$

Design a lowpass Butterworth filter with the follow specifications:

1] 3 dB attenuation at passband frq of 1.5 KHz

2] 10 dB Stopband attenuation at frq of 3 KHz

3] Sampling frq of 8,000 Hz.

$$H(s) = \frac{W_{ap}}{s + W_{ap}} = \frac{1.0691 \times 10^4}{s + 1.0691 \times 10^4}$$

3] Apply BLT, $H(z) = H(s)\Big|_{s = \frac{2}{T}\frac{(z-1)}{(z+1)}}$

$$H(z) = \frac{1.0691 \times 10^4}{s + 1.0691 \times 10^4}\Bigg|_{s = 16000\frac{(z-1)}{(z+1)}}$$

$$H(z) = \frac{1.0691 \times 10^4}{\left(16{,}000\,\frac{z-1}{z+1}\right) + 1.0691 \times 10^4}$$

÷ by 16,000

$$H(z) = \frac{0.6682}{\left(\frac{z-1}{z+1}\right) + 0.6682}$$

× (z+1) on both nr & dr

$$H(z) = \frac{0.6682(z+1)}{(z-1) + 0.6682(z+1)} = \frac{0.6682z + 0.6682}{\underbrace{1.6682z - 0.3318}_{1}}$$

÷ Nr & Dr by 1.6682z

$$\therefore \boxed{H(z) = \frac{0.4006 + 0.4006z^{-1}}{1 - 0.1989z^{-1}}}$$

Design a second order digital bandpass Butterworth filt[er] with the following specifications:

i) upper cutoff freq of 2.6 KHz

ii) lower cutoff freq of 2.4 KHz

iii) Sampling freq of 8000 Hz.

Find digital freqies (rad|sec)

$$W_h = 2\pi f_h = 2\pi(2600) = 5200\pi \text{ rad|sec}$$

$$W_L = 2\pi f_L = 2\pi(2400) = 4800\pi \text{ rad|sec}$$

$$T = 1/f_s = 1/8000 \text{ sec.}$$

1] warping eqns

$$W_{ah} = \frac{2}{T}\tan\left(\frac{W_h T}{2}\right) = 16000 \times \tan\left(\frac{5200\pi/8000}{2}\right)$$

$$\boxed{W_{ah} = 2.6110 \times 10^4 \text{ rad|sec}}$$

$$W_{al} = \frac{2}{T}\tan\left(\frac{W_l T}{2}\right) = 16000 \times \tan\left(\frac{4800\pi/8000}{2}\right)$$

$$\boxed{W_{al} = 2.2022 \times 10^4 \text{ rad|sec}}$$

$$W = W_{ah} - W_{al} = 4{,}088 \text{ rad|sec}$$

$$W_0^2 = W_{ah} \times W_{al} = 5.7499 \times 10^8 \text{ rad|sec}$$

2] Prototype transformation

lowpass ⟶ bandpass

$$H_p(s) = \frac{1}{s+1}$$

$$\therefore H(s) = H_p(s)\Big|_{s = \frac{s^2 + W_0^2}{sW}}$$

$$= \frac{Ws}{s^2 + Ws + W_0^2} = \frac{4{,}088\,s}{s^2 + 4088s + 5.7499 \times 10^8}$$

3] Apply BLT.

$$H(z) = \frac{4088s}{s^2 + 4088s + 5.7499 \times 10^8}\Bigg|_{s = 16000\frac{z-1}{z+1}}$$

$$H(z) = \frac{0.0730 - 0.0730z^{-2}}{1 + 0.7117z^{-1} + 0.8541z^{-2}}$$