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23ETEC004002

Digital Signal Processing

Home Work - 1

Q.1 Sampling of $x(t) = \cos(2\pi F t)$ at $F_s = 10$ samples

a) Find F from your name

Using A = 1, B = 2 --- Z = 26

$\hookrightarrow$ S = 19, H = 8, R = 18, E = 5, Y = 25, A = 1, S = 19

Sum: 19 + 8 + 18 + 5 + 25 + 1 + 19 = 95

9 + 5 = 14 ⇒ 1 + 4 = 5. So F = 5 Hz

b) Expression for the discrete-time samples $x[n]$

Continuous time signal:

$x(t) = \cos(2\pi F t) = \cos(2\pi \cdot 5t) = \cos(10\pi t)$

Sampling frequency: $F_s = 10$ samples/sec

⇒ Sampling Period ⇒ $T_s = 1/F_s = 0.1$ sec

Samples: $x[n] = x(nT_s) = \cos(10\pi n T_s)$

$= \cos(10\pi n \cdot 0.1)$

$= \cos(\pi n)$

Using $\cos(\pi n) = (-1)^n$

$x[n] = (-1)^n$

for $n = 0$; $x[0] = 1$

$n = 1$; $x[1] = -1$

$n = 2$; $x[2] = 1$

$n = 3$; $x[3] = -1$

Sequence = 1, -1, 1, -1, ----

c> Reconstructed Analog Signal

$F_s = 10 Hz$ Condition $F_s > 2F$ - oversampled

$F = 5 Hz$ $F_s < 2F$ - undersampled.

$2F = 10 Hz$

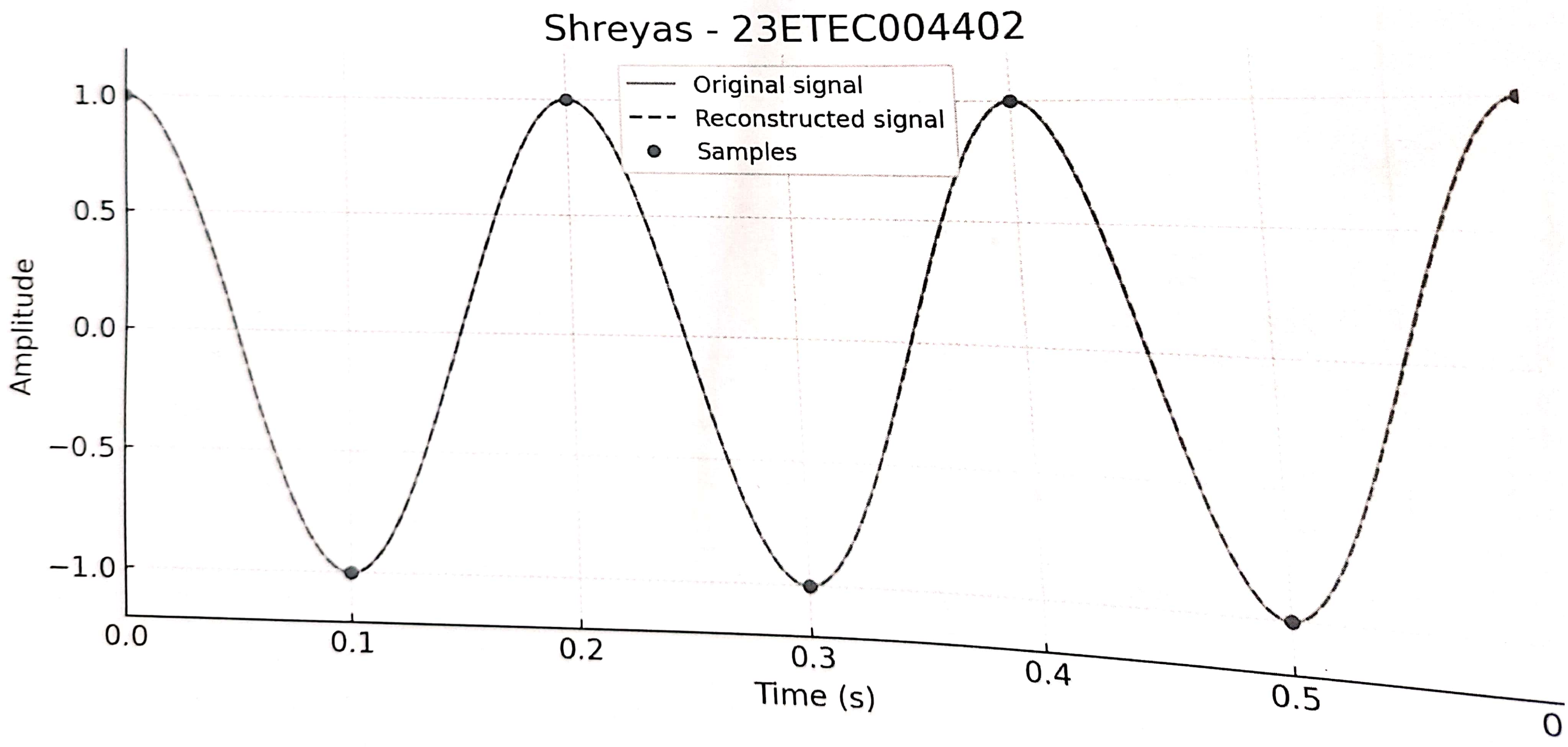
$F_s = F \Rightarrow 10 = 10$ Thus

$x(t)$ is (the) critically sampled (sampled at Nyquist rate).

d> So the signal is critically sampled - neither over nor undersampled.

$\boxed{F_s = 2F} \rightarrow$ Nyquist rate.

$10 Hz = 10 Hz$


Shreyas - 23ETEC004402
Original signal
Reconstructed signal
Samples
1.0
0.5
0.0
−0.5
−1.0
Amplitude
0.0
0.1
0.2
0.3
0.4
0.5
0
Time (s)

Question 2.

a) N-point DFT by direct equation (N=6)

$$X[k] = \sum_{n=0}^{5} x[n] e^{-j\frac{2\pi}{6}kn}, \quad k = 0, 1, \dots 5.$$

Let $x(n) = \{0, 1, -3, 2, 0, 0\}$ $n = 0, 1, 2, 3, 4, 5$

$W_6 = e^{-j2\pi/6} = e^{-j\pi/3}$, $\boxed{W_k^n = e^{-j\frac{2\pi}{N}kn}}$

Let's note the powers,

$W_6^0 = 1, W_6^1 = 1/2 - j\sqrt{3}/2, W_6^2 = -1/2 - j\sqrt{3}/2, W_6^3 = -1$

$W_6^4 = -1/2 + j\sqrt{3}/2 \quad W_6^5 = 1/2 + j\sqrt{3}/2$

Because $x[0] = x[4] = x[5] = 0$, the sum is only over $n = 1, 2, 3$

$$X[k] = x[1]W_6^k + x[2]W_6^{2k} + x[3]W_6^{3k} = 1.W_6^k - 3W_6^{2k} + 2W_6^{3k}$$

$k = 0$, $X[0] = 1.W_6^0 + (-3W_6^0) + (2W_6^0)$

$= +1 - 3 + 2 = 0$

$X[0] = 0$

$k = 1$ $X[1] = W_6^1 - 3W_6^2 + 2W_6^3$

$= (1/2 - j\sqrt{3}/2) - 3(\frac{1}{2} - j\sqrt{3}/2) + 2(-1)$

$= 0 + j\sqrt{3}$

$k = 2$ $X[2] = W_6^2 - 3W_6^4 + 2W_6^6$

$= (-\frac{1}{2} - j\sqrt{3}/2) - 3(-1/2 + j\sqrt{3}/2) + 2.1$

$= 3 - 2j\sqrt{3}$

$k=3$

$X[3] = W_6^3 - 3W_6^6 + 2W_6^0$

$= (-1) - 3(1) + 2(-1)$

$= -1 - 3 - 2 = -6$

$X[3] = -6$

$k=4$

$X[4] = \overline{X[2]}$

$X(4) = (W_6^4 - 3W_6^8 + 2W_6^{12})$ 8 $[W_6^4 - 3W_6^2 + 2.1]$

$= \left(-\frac{1}{2} + j\frac{\sqrt{3}}{2}\right) - 3\left(-\frac{1}{2} - j\frac{\sqrt{3}}{2}\right) + 2$

$X(4) = 3 + j2\sqrt{3}$

$k=5$

$X[5] = \overline{X[1]} = \cancel{j\sqrt{3}}\ -j\sqrt{3}$

$X[5] = W_6^5 - 3W_6^{10} + 2W_6^{15}$

$= W_6^5 - 3W_6^4 + 2W_6^3$

$= -j\sqrt{3}$

$X[0] = 0$

$X[1] = j\sqrt{3}$

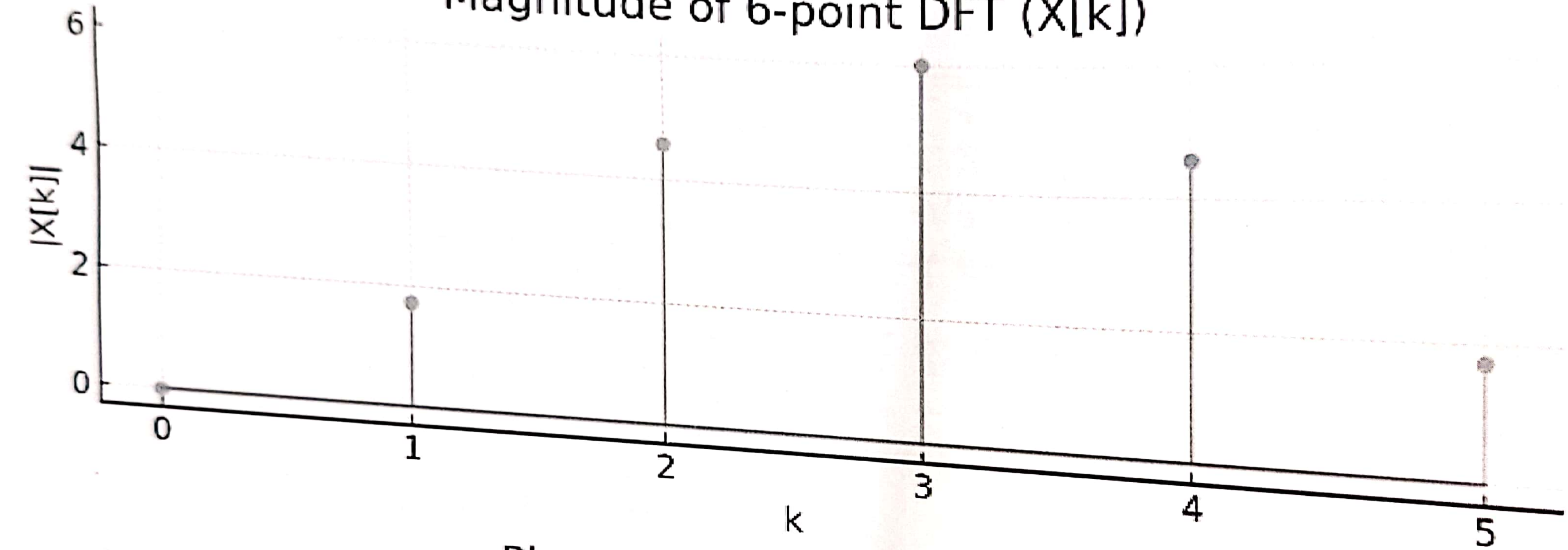
$X[2] = 3 - j2\sqrt{3}$

$X[3] = -6$

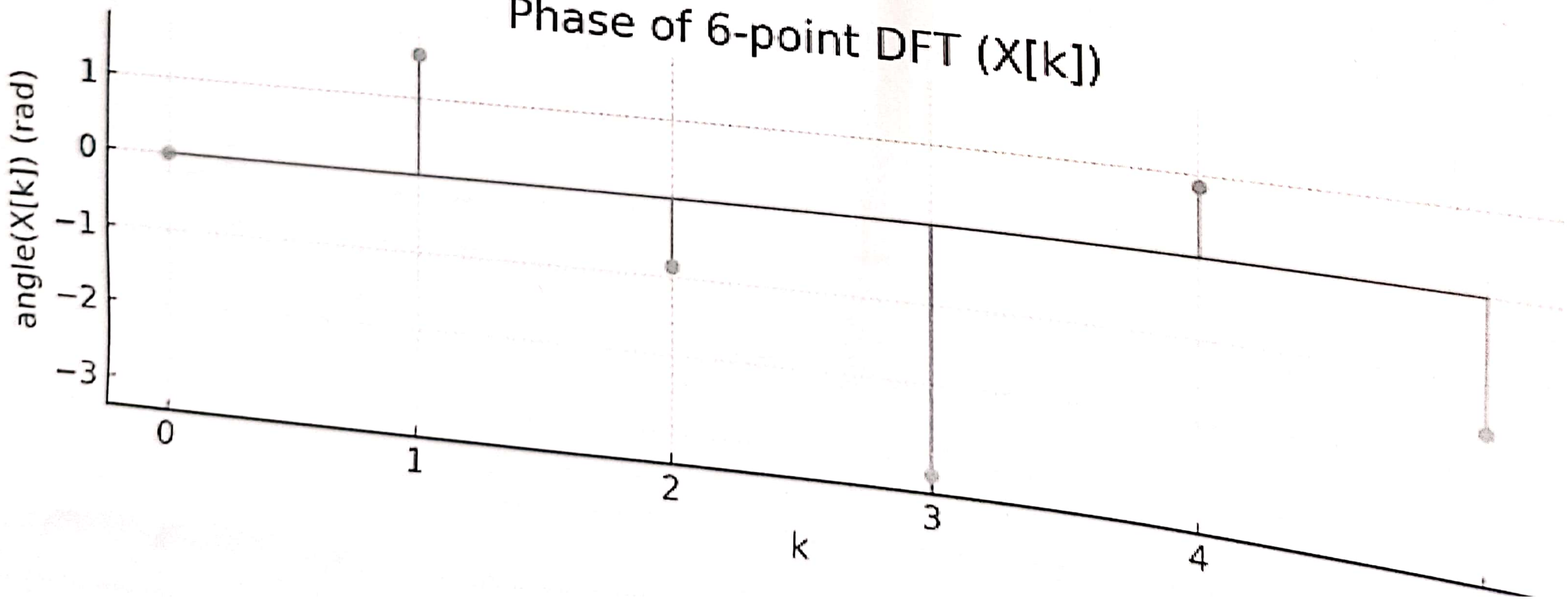
$X[4] = 3 + j2\sqrt{3}$

$X[5] = -j\sqrt{3}$

Magnitude of 6-point DFT (X[k])



Phase of 6-point DFT (X[k])



Q2(b) - Matrix Method.

$$x = \begin{bmatrix} 0 \\ 1 \\ -3 \\ 2 \\ 0 \\ 0 \end{bmatrix}_{6\times 1}, \Rightarrow X = \begin{bmatrix} X[0] \\ X[1] \\ X[2] \\ X[3] \\ X[4] \\ X[5] \end{bmatrix}$$

$$W_6 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & W_6 & W_6^2 & W_6^3 & W_6^4 & W_6^5 \\ 1 & W_6^2 & W_6^4 & 1 & W_6^2 & W_6^4 \\ 1 & W_6^3 & 1 & W_6^3 & 1 & W_6^3 \\ 1 & W_6^4 & W_6^2 & 1 & W_6^4 & W_6^2 \\ 1 & W_6^5 & W_6^4 & W_6^3 & W_6^2 & W_6^1 \end{bmatrix}_{6\times 6}$$

$$\begin{bmatrix} X[0] \\ X[1] \\ X[2] \\ X[3] \\ X[4] \\ X[5] \end{bmatrix} = [0\ 1\ -3\ 2\ 0\ 0] \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & W_6^1 & W_6^2 & W_3^3 & W_4^6 & W_6^5 \\ 1 & W_6^2 & W_6^4 & 1 & W_6^2 & W_6^4 \\ 1 & W_6^3 & 1 & W_6^3 & 1 & W_6^3 \\ 1 & W_5^4 & W_6^2 & 1 & W_6^4 & W_6^2 \\ 1 & W_6^5 & W_6^4 & W_6^3 & W_6^2 & W_6^1 \end{bmatrix}$$

$$= \left[0, j\sqrt{3}, 3-j2\sqrt{3}, -6, 3+j2\sqrt{3}, -j\sqrt{3}\right]$$

Q.2 (c)

$$x[n] = 1/N \sum_{k=0}^{N-1} X[k] e^{j\frac{2\pi}{N}kn}$$

$$= 1/6 \sum_{k=0}^{5} X[k] e^{j\frac{2\pi}{6}kn}$$

let $= e^{j2\pi/6} = e^{j\pi/3} = W_6^{-1}$

$$W_k = e^{j2\pi/6\, kn}$$

For $n=0$

$$x[0] = 1/6 \sum_{k=0}^{5} X[k] = 1/6 \left(0 + j\sqrt{3} + (3 - j2\sqrt{3}) + (-6) + (3 + j2\sqrt{3} - j\sqrt{3})\right)$$

$$= 0$$

For

$n=1$,

$$x[1] = 1/6 \sum_{k=0}^{5} X[k] = 1/6 \{0.1 + j\sqrt{3}\, W_6^{-1} + (3 - j2\sqrt{3})(W_6^{-1})^2 - 6(W_6^{-1})^3 + (3 + j2\sqrt{3})(W_6^{-1})^4 - j\sqrt{3}(W_6^{-1})^5\}.$$

Thus $x[1] = 1$

$x[2] = -3$

$x[3] = 2$

$x[4] = 0$

$x[5] = 0$

Thus the sequence is

$$x(n) = \{0, 1, -3, 2, 0, 0\}$$

Q.2 (d) Twiddle Factor Matrix for $N = 6$

The twiddle factor is:

$$W_6 = e^{-j2\pi/6} = e^{-j\pi/3}$$

The point [6-point] DFT twiddle matrix W_6 is:

$$[W_6]_{k,n} = W_6^{kn}, \quad k,n = 0, \ldots 5.$$

$$W_6 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & W_6 & W_6^2 & W_6^3 & W_6^4 & W_6^5 \\ 1 & W_6^2 & W_6^4 & 1 & W_6^2 & W_6^4 \\ 1 & W_6^3 & 1 & W_6^3 & 1 & W_6^3 \\ 1 & W_6^4 & W_6^2 & 1 & W_6^4 & W_6^2 \\ 1 & W_6^5 & W_6^4 & W_6^3 & W_6^2 & W_6^1 \end{bmatrix}$$

$$W_6 = 1/2 - j\sqrt{3}/2$$

Q.2 (e)

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x = [0 1 -3 2 0 0]
X = fft(x, 6) % should give the same X[k]
x_idft = ifft(X) % should return [0 1 -3 2 0 0]
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Data for Q3, Q4, Q5

$h(n) = \{0, 1, -3, 2\}$

$x(n) = \{N, h(-n)\}$

$h(n) = \{0, 1, -3, 2\} \Rightarrow h(-n) = \{2, -3, 1, 0\}$

$x(n) = [6, 2, -3, 1, 0]$

For convolution:

↪ $x(n)$ has length $L_x = 5$

↪ $h(n)$ has length $L_h = 4$

↪ Zero Padding

$\Rightarrow x(n) = \{6, 2, -3, 1, 0\}$

$h(n) = \{0, 1, -3, 2, 0\}$

Q3. Circular Convolution of $x(n)$ & $h(n)$

3(a). Time-Domain formula

For N-point circular convolution (here $N=5$):

$$y_c[n] = \sum_{k=0}^{N-1} x(k)\, h[(n-k) \bmod N], \quad n = 0 \text{ to } 4.$$

$n = 0$,

$$y_c[0] = x[0]h[0] + x[1]h[4] + x[2]h[3] + x[3]h[2] + x[4]h[1]$$

$$= 6.0 + 2.0 + (-3)(2) + 1(-3) + 0.1$$

$$= -9$$

$n=1,$

$$y_c[1] = x[0]h[1] + x[1]h[0] + x[2]h[4] + x[3]h[3] + x[4]h[2]$$

$$= 6.1 + 2.0 + (-3).0 + 1.2 + 0.(-3)$$

$$= 8$$

$n=2$

$$y_c[2] = x[0]h[2] + x[1]h[1] + x[2]h[0] + x[3]h[4] + x[4]h[3]$$

$$= 6(-3) + 2.1 + (-3).0 + 1.0 + 0.2$$

$$= -16$$

$n=3$

$$y_c[3] = x[0]h[3] + x[1]h[2] + x[2]h[1] + x[3]h[0] + x[4]h[4]$$

$$= 6.2 + 2(-3) + (-3)(1) + 1.0 + 0.0$$

$$= 3$$

$n=4$

$$y_c[4] = x[0]h[4] + x[1]h[3] + x[2]h[2] + x[3]h[1] + x[4]h[0]$$

$$= 6.0 + 2.2 + (-3).(-3) + 1.1 + 0.0$$

$$= 14$$

So the 5-point circular convolution result is:

$$y_c(n) = \{-9, 8, -16, 3, 14\}$$

Q3b) Verification using DFT and IDFT

let $N=5$, and

$$x(n)=\{6, 2, -3, 1, 0\} \qquad \Bigg\{ \text{Zero padding} \Bigg\}$$
$$h(n)=\{0, 1, -3, 2, 0\}$$

compute 5-point DFT's

$$X[k]=\sum_{n=0}^{4} x[n]e^{-j2\pi/5\,kn}, \quad H[k]=\sum_{n=0}^{4} h[n]e^{-j2\pi/5\,kn}$$

for $k=0, 1, 2, 3, 4$.

Multiply in frequency domain:

$$Y[k]=X[k].H[k]$$

$$X[K]=(6, 8.236+j0.45, 3.49-j5.07, 3.49+j5.07, \cancel{8}\ 8.236-j.045)$$

$$H[K]=(0, j\sqrt{3}, 3-j2\sqrt{3}, -6, 3+2j\sqrt{3}, -j\sqrt{3})$$

$$y_c(n)=\{-9, 8, -16, 3, 14\}$$

$\Downarrow$

This Answer is after taking IDFT

$$\text{IDFT} = y(n) = 1/5\left\{\sum_{k=0}^{4} Y[k]e^{j2\pi/5\,kn}\right.$$

So

$$y(n)=\{-9, 8, -16, 3, 14\}$$

Question 4

a) Linear Convolution using DFT and IDFT.

1) $N_L \geq 8$, Taking $N_L = 8$ (length $= 5+4-1 = 8$)

$x_L = \{6, 2, -3, 1, 0, 0, 0, 0\}$ [Zero-padding]
$h_L = \{0, 1, -3, 2, 0, 0, 0, 0\}$

2) Compute 8-point DFTs:

$$X_L[k] = \sum_{n=0}^{7} x_L[n] e^{-j2\pi/8\, kn}, \quad \sum_{n=0}^{7} h_L[n] e^{-j2\pi/8\, kn}$$

$X_L[k] = \{6, 4.646 - j11.914, 12 - 4j, 8.914 - j0.646, 12, 8.914 + j0.646, 12 + 4j, 4.646 + j11.914\}$

$H_L[k] = \{0, -4 - j3.414, -2 - j3, -4 - j0.586, 0, -4 + j0.586, -2 + 3j, -4 + 3.414j\}$

3) $Y_L[k] = X_L[k] \cdot H_L[k]$

$= \{0, 20.084 + j34.792, -36 - j28, -35.32 + j1.732, 0, -35.32 - j1.732, -36 + j28, 20.084 - j34.792\}$

$\Rightarrow$ Inverse IDFT:

$$y_L[n] = 1/8 \sum_{k=0}^{7} Y_L[k] e^{j2\pi/8\, kn}, \quad n = 0, \dots 7.$$

$y_L[n] = \{0, 6, -16, 3, 14, -9, 2, 0\}$

Question-4 Using time-domain / tabular method.

b) $y_L[n] = \sum_{k=0}^{4} x[k]\, h[n-k]$

$n=0$,
$y_L[0] = 6.0 = 0$

$n=1$
$y_L[1] = 6.1 + 2.0 = 6$

$n=2$
$y_L[2] = 6.(-3) + 2(1) + (-3).0 = -18 + 2 = -16$

$n=3$
$y_L[3] = 6.2 + 2(-3) + (-3).1 + 1.0 = 12 - 6 - 3 = 3$

$n=4$
$y_L[4] = 2.2 + (-3).(-3) + 1.1 + 0.0 = 4 + 9 + 1 = 14$

$n=5$
$y_L[5] = (-3).2 + 1(-3) + 0.1 = -6 - 3 = -9$

$n=6$
$y_L[6] = 1.2 + 0(-3) = 2$

$n=7$
$y_L[7] = 0.2 = 0$

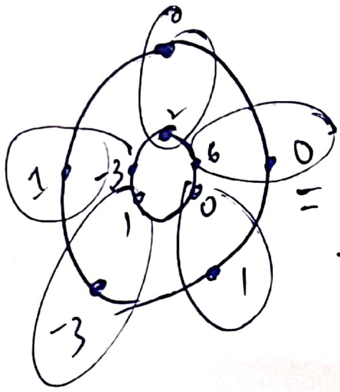
So the linear Convolution

$y_L(n) = \{0, 6, -16, 3, 14, -9, 2, 0\}$

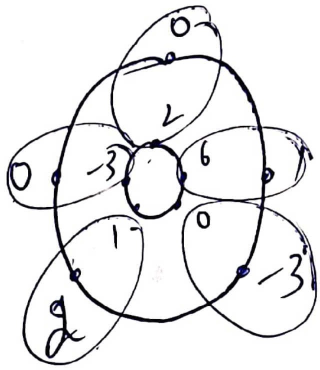
Q.5 Circular Convolution using Concentric Circles Method.

$x(n) = \{6, 2, -3, 1, 0\}$

$h(n) = \{0, 1, -3, 2, 0\}$

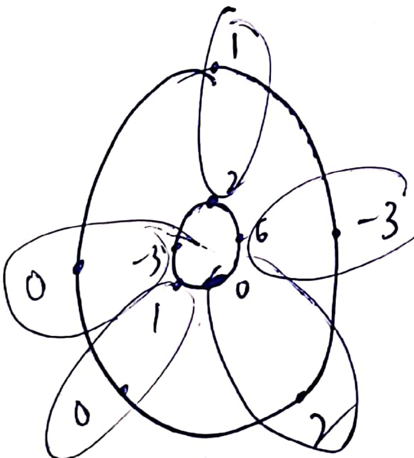


$= -3\times1 + -3\times2 + 0 + 0 + 0 = -9$

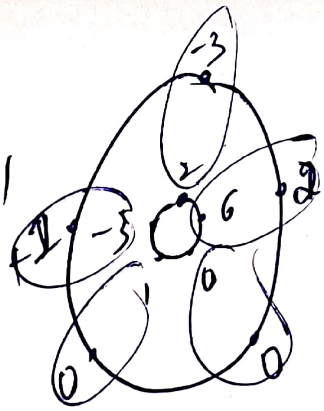


$= 6\times1 + 2\times1$

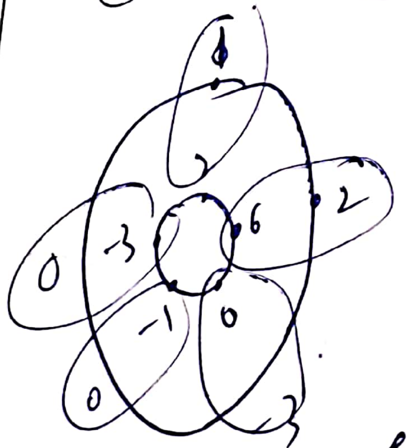
8



$6\times-3 + 2\times1 + 0 + 0 + 0 = -16$



$12 - 6 - 3 + 0 + 0 = 3$



$6\times2 + 1\times2 + 0 + 0 + 0 = 14$

$\{-9, 8, -16, 3, 14\}$