The Greedy Approach

General idea:

Given a problem with *n* inputs, we are required to obtain a subset that maximizes or minimizes a given objective function subject to some constraints.

Feasible solution — any subset that satisfies some constraints

Optimal solution — a feasible solution that maximizes or minimizes the objective function

procedure Greedy ***(A, n)***

```
begin
    solution ← Ø;
    for i ← 1 to n do
        x ← Select (A);   // based on the objective
                          // function
        if Feasible (solution, x),
            then solution ← Union (solution, x);
end;
```

Select: A greedy procedure, based on a given objective function, which selects input from *A*, removes it and assigns its value to *x*.

Feasible: A boolean function to decide if *x* can be included into solution vector (without violating any given constraint).

About Greedy method

The n inputs are ordered by some selection procedure which is based on some optimization measures.

It works in stages, considering one input at a time. At each stage, a decision is made regarding whether or not a particular input is in an optimal solution.

1. **Minimum Spanning Tree** (For Undirected Graph)

The problem:

1) Tree

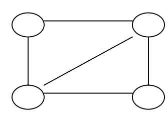
A *Tree* is connected graph with no cycles.

2) Spanning Tree

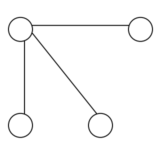
A *Spanning Tree* of *G* is a tree which contains all vertices in *G*.

Example:

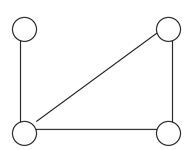
G:



b) Is *G* a Spanning Tree?



Key: Yes



Key: No

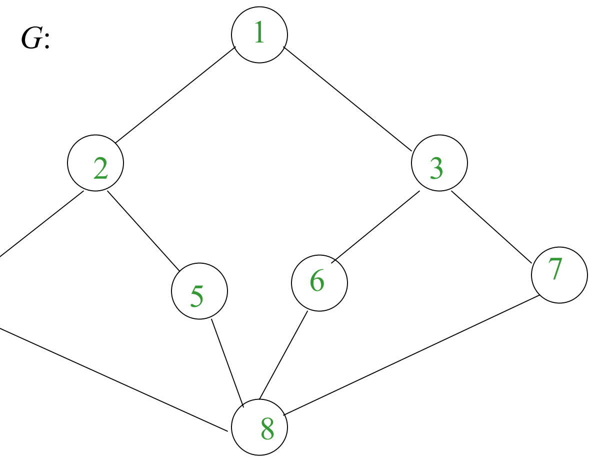
Note: Connected graph with n vertices and exactly $n-1$ edges is Spanning Tree.

3) Minimum Spanning Tree

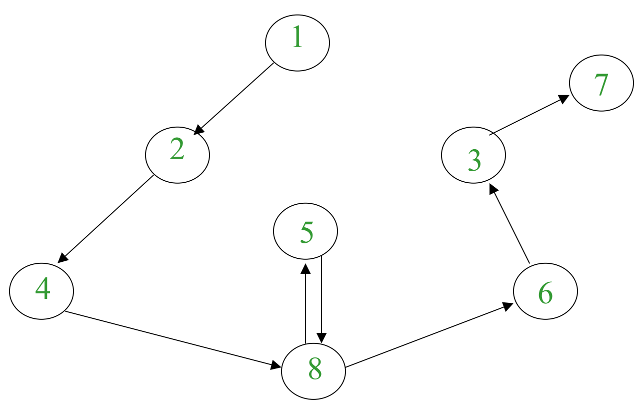
Assign weight to each edge of *G*, then *Minimum Spanning Tree* is the Spanning Tree with minimum total weight.

Example:

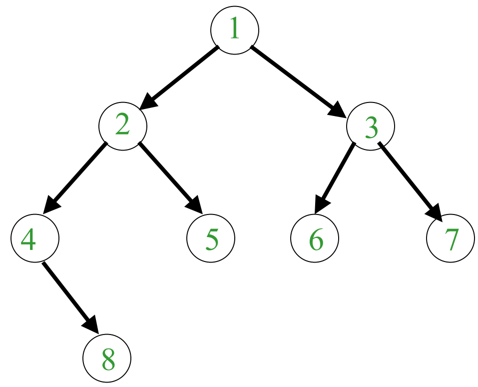
a) Edges have the same weight



- DFS (Depth First Search)

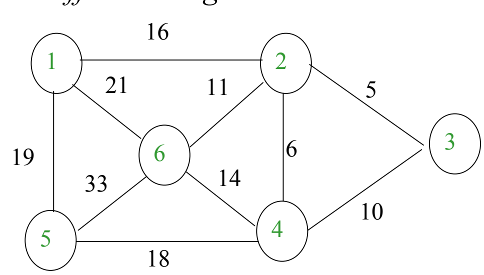


- BFS (Breadth First Search)

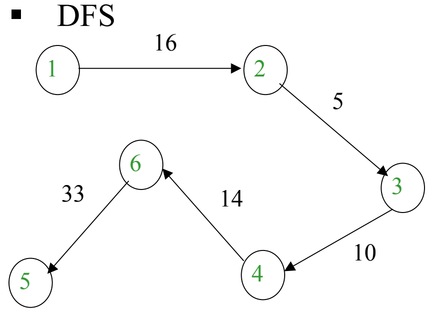


b) Edges have different weights

<u>*G:*</u>

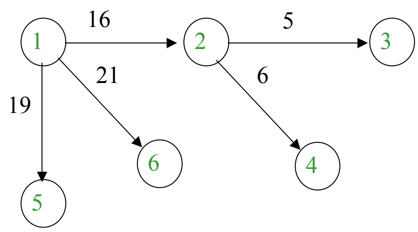


- DFS



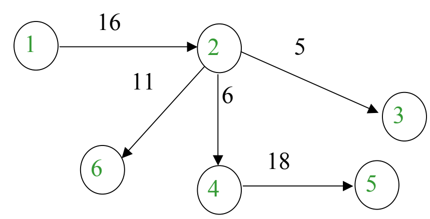
$$Cost = 16 + 5 + 10 + 14 + 33$$
$$= 78$$

- BFS



$$Cost = 16 + 19 + 21 + 5 + 6$$
$$= 67$$

- Minimum Spanning Tree (with the least total weight)



$$Cost = 16 + 5 + 6 + 11 + 18$$
$$= 56$$

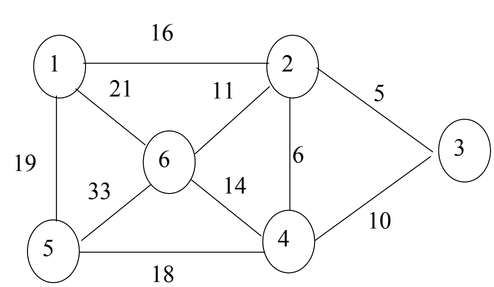
Algorithms:

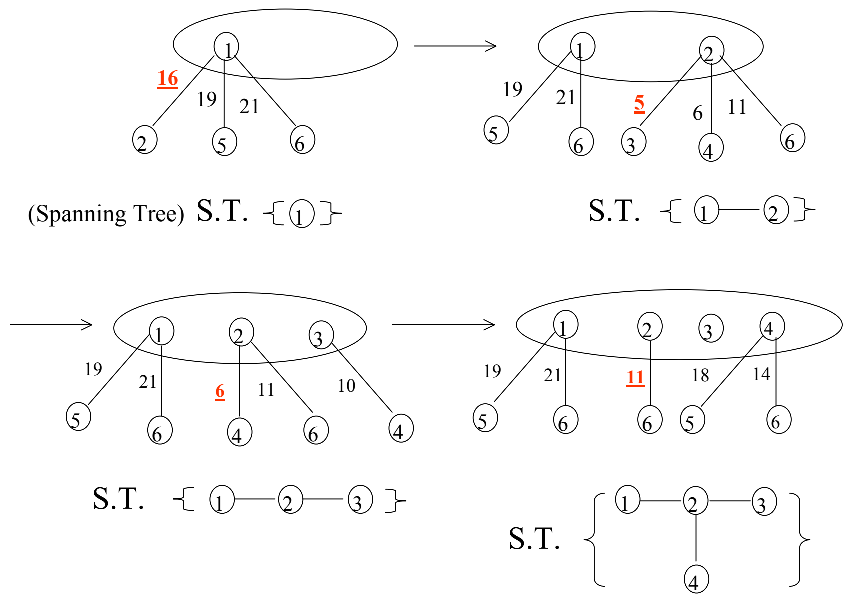
1) Prim's Algorithm (Minimum Spanning Tree)

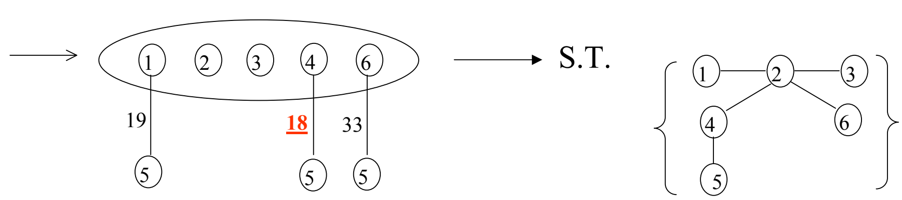
Basic idea:

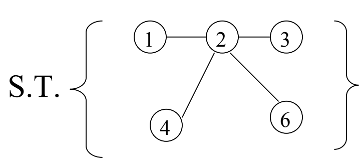
Start from vertex 1 and let $T \leftarrow \emptyset$ (T will contain all edges in the S.T.); the next edge to be included in T is the minimum cost edge(u, v), s.t. u is in the tree and v is not.

Example*:* *G*









Cost = 16+5+6+11+18=56
Minimum Spanning Tree

(*n* – # of vertices, *e* – *# of edges)*
It takes *O(n)* steps. Each step takes *O(e) and* $e \le n(n\text{-}1)/2 \Rightarrow O(n^2)$.
Therefore, it takes $O(n^3)$ *time.*

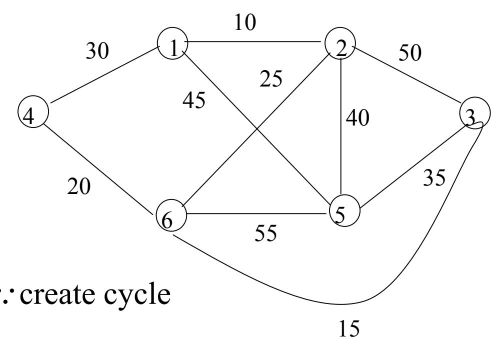
With clever data structure, it can be implemented in $O(n^2)$.

2) Kruskal's Algorithm

Basic idea:

Don't care if T is a tree or not in the intermediate stage, as long as the including of a new edge <u>will not create a cycle</u>, we include the minimum cost edge

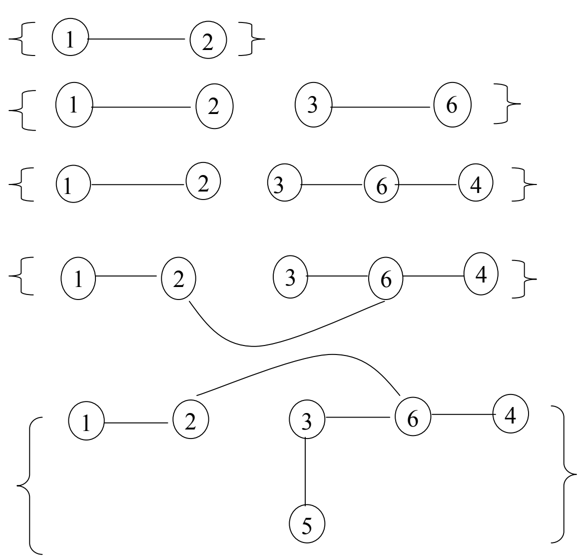
Example:



Step 1: Sort all of edges

(1,2)	10	√	
(3,6)	15	√	
(4,6)	20	√	
(2,6)	25	√	
(1,4)	30	×	reject ∵ create cycle
(3,5)	35	√	

Step 2: T



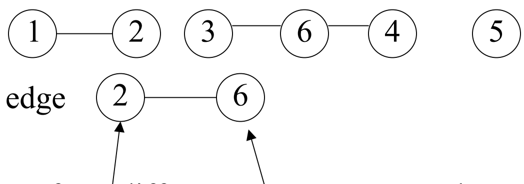
How to check:

adding an edge will create a cycle or not?

If Maintain a set for each group
(initially each node represents a set)

Ex: set1 set2 set3

∴ new edge



from different groups ⇒ no cycle created

Data structure to store sets so that:

i. The group number can be easily found, and

ii. Two sets can be easily merged

Kruskal's algorithm

While (T contains fewer than n-1 edges) and ($E \neq \emptyset$) do
Begin
  Choose an edge (v,w) from E of lowest cost;
  Delete (v,w) from E;
  If (v,w) does not create a cycle in T
  then add (v,w) to T
  else discard (v,w);
End;

With clever data structure, it can be implemented in *O(e log e)*.

So, complexity of Kruskal is $O(eLoge)$

$$\because e \le \frac{n(n-1)}{2} \Rightarrow Loge \le Logn^2 = 2Logn$$

$$\Rightarrow O(eLoge) = O(eLogn)$$

3) Comparing Prim's Algorithm with Kruskal's Algorithm

i. Prim's complexity is $O(n^2)$

ii. Kruskal's complexity is $O(eLogn)$

if G is a complete (dense) graph,
Kruskal's complexity is $O(n^2 Logn)$
if G is a sparse graph,
Kruskal's complexity is $O(nLogn)$.

2. Dijkstra's Algorithm for Single-Source Shortest Paths

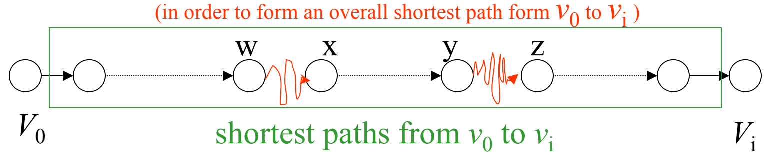
The problem: Given directed graph $G = (V, E)$,
a weight for each edge in G,
a source node v_0,

Goal: determine the (length of) shortest paths from v_0 to all the remaining vertices in G

Def: Length of the path: Sum of the weight of the edges

Observation:

May have more than 1 paths between w and x (y and z)
But each individual path must be minimal length
(in order to form an overall shortest path form v_0 to v_i)



shortest paths from v_0 to v_i

Notation

cost adjacency matrix $Cost$, $\forall 1 \le a,b \le |V|$

$$Cost\,(a,\,b) = \begin{cases} \text{cost from vertex i to vertex j} & \text{if there is a edge} \\ 0 & \text{if } a = b \\ \infty & \text{otherwise} \end{cases}$$

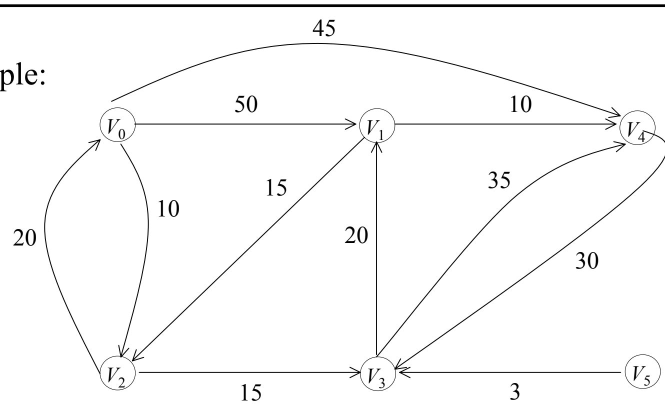
$$s(w) = \begin{cases} 1 & \text{if shortest path } (v,\,w) \text{ is defined} \\ 0 & \text{otherwise} \end{cases}$$

$Dist(j)$ $\quad \forall j$ in the vertex set V

= the length of the shortest path from v to j

$From(j) = i$ $\quad$ if i is the predecessor of j along the shortest path from v to j

Example:

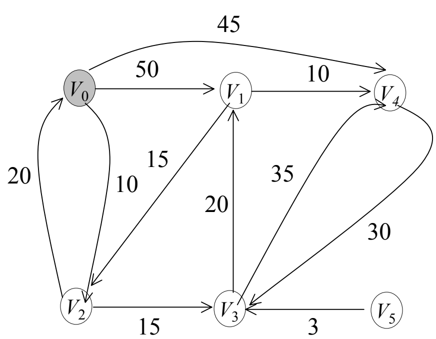


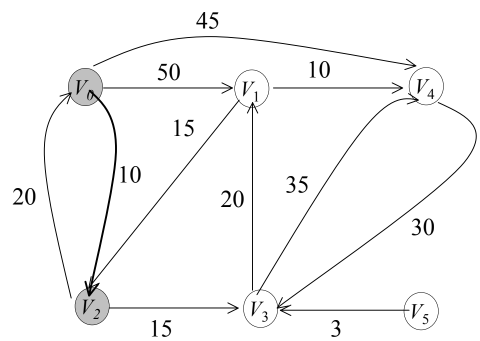
a) Cost adjacent matrix

$$\begin{array}{c c} & \begin{array}{cccccc} v_0 & v_1 & v_2 & v_3 & v_4 & v_5 \end{array} \\ \begin{array}{c} v_0 \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} & \begin{bmatrix} 0 & 50 & 10 & \infty & 45 & \infty \\ \infty & 0 & 15 & \infty & 10 & \infty \\ 20 & \infty & 0 & 15 & \infty & \infty \\ \infty & 20 & \infty & 0 & 35 & \infty \\ \infty & \infty & \infty & 30 & 0 & \infty \\ \infty & \infty & \infty & 3 & \infty & 0 \end{bmatrix} \end{array}$$

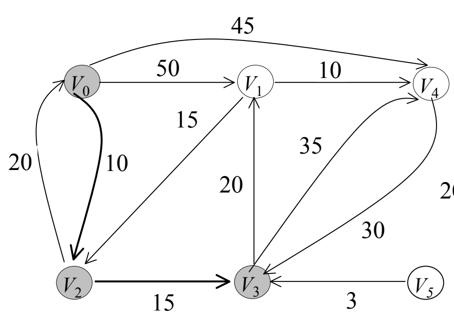
b) Steps in Dijkstra's Algorithm

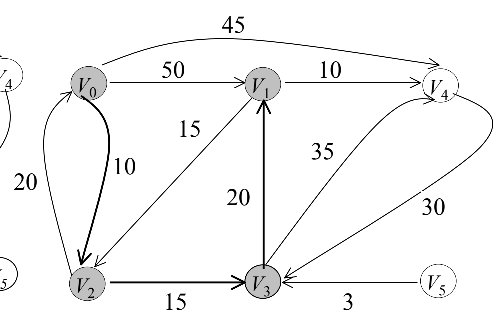
1. *Dist* $(v_0) = 0$, *From* $(v_0) = v_0$ 2. *Dist* $(v_2) = 10$, *From* $(v_2) = v_0$



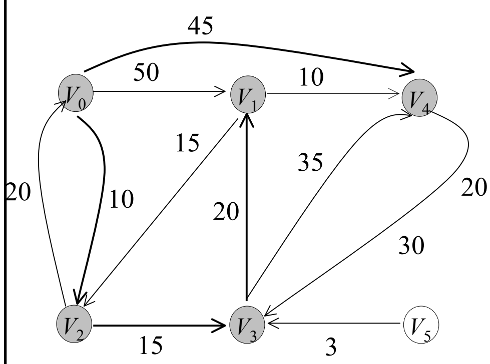


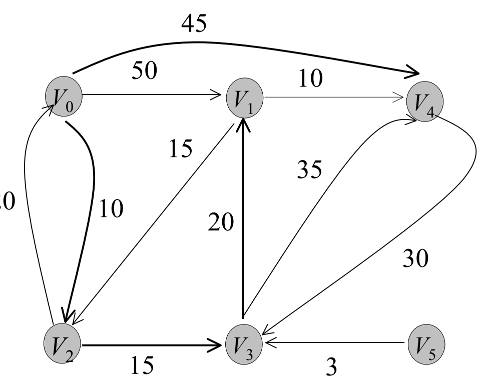
3. *Dist* $(v_3) = 25$, *From* $(v_3) = v_2$ 4. *Dist* $(v_1) = 45$, *From* $(v_1) = v_3$





5. *Dist* $(v_4) = 45$, *From* $(v_4) = v_0$ 6. *Dist* $(5) = \infty$





c) Shortest paths from source v_0

$v_0 \rightarrow v_2 \rightarrow v_3 \rightarrow v_1$	45
$v_0 \rightarrow v_2$	10
$v_0 \rightarrow v_2 \rightarrow v_3$	25
$v_0 \rightarrow v_4$	45
$v_0 \rightarrow v_5$	∞

Dijkstra's algorithm:

procedure Dijkstra (*Cost*, *n*, *v*, *Dist*, *From*)
// *Cost*, *n*, *v* are input, *Dist*, *From* are output
begin
for $i \leftarrow 1$ to n do
$\begin{cases} s(i) \leftarrow 0; \\ Dist(i) \leftarrow Cost(v,i); \\ From(i) \leftarrow v; \end{cases}$
$s(v) \leftarrow 1;$
for $num \leftarrow 1$ to $(n-1)$ do
choose u s.t. $s(u) = 0$ and $Dist(u)$ is minimum;
$s(u) \leftarrow 1;$
for all w with $s(w) = 0$ do
if $(Dist(u) + Cost(u,w) < Dist(w))$
$\begin{cases} Dist(w) \leftarrow Dist(u) + Cost(u,w); \\ From(w) \leftarrow u; \end{cases}$

end;

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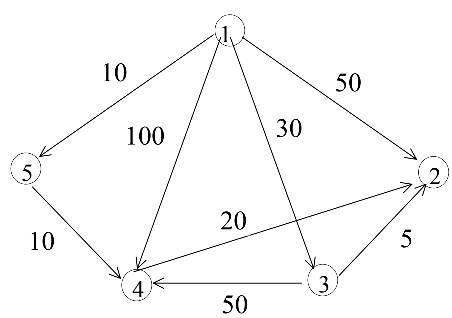
Ex:

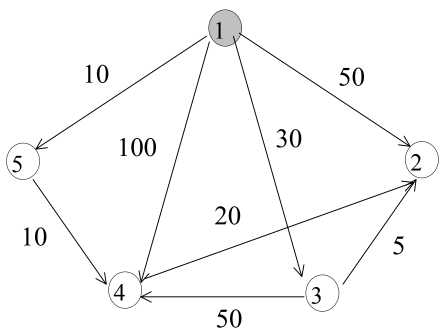


a) Cost adjacent matrix

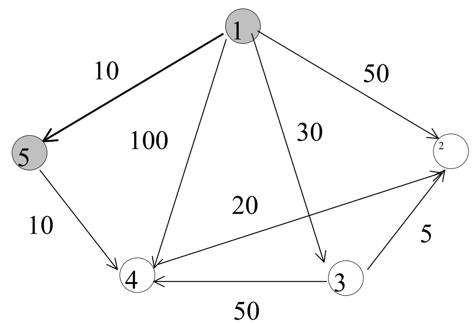
$$\begin{array}{c c} & \begin{array}{ccccc} 01 & 02 & 03 & 04 & 05 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} & \begin{bmatrix} 0 & 50 & 30 & 100 & 10 \\ \infty & 0 & \infty & \infty & \infty \\ \infty & 5 & 0 & 50 & \infty \\ \infty & 20 & \infty & 0 & \infty \\ \infty & \infty & \infty & 10 & 0 \end{bmatrix} \end{array}$$

b) Steps in Dijkstra's algorithm

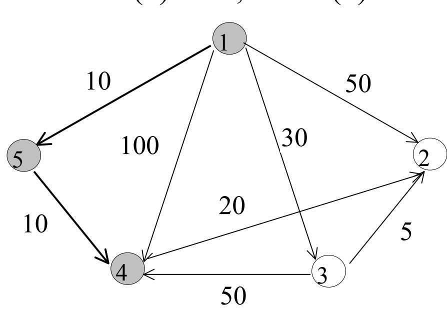
1. *Dist* (1) = 0, *From* (1) = 1



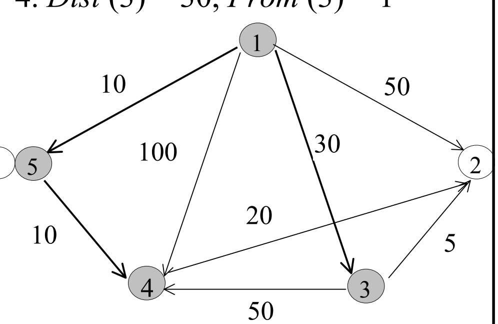
2. *Dist* (5) = 10, *From* (5) = 1



3. *Dist* (4) = 20, *From* (4) = 5



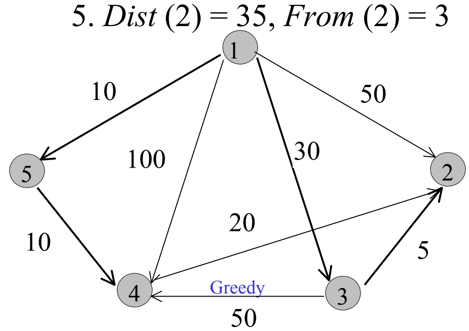
4. *Dist* (3) = 30, *From* (3) = 1



Shortest paths from source *1*

1 → 3 → 2	35
1 → 3	30
1 → 5 → 4	20
1 → 5	10

5. *Dist* (2) = 35, *From* (2) = 3



3. Optimal Storage on Tapes

The problem:

Given n programs to be stored on tape, the lengths of these n programs are $l_1, l_2, \ldots, l_n$ respectively. Suppose the programs are stored in the order of $i_1, i_2, \ldots, i_n$

Let t_j be the time to retrieve program i_j.

Assume that the tape is initially positioned at the beginning.

t_j is proportional to the sum of all lengths of programs stored in front of the program i_j.

The goal is to minimize MRT (Mean Retrieval Time), $\frac{1}{n}\sum_{j=1}^{n} t_j$

i.e. want to minimize $\sum_{j=1}^{n}\sum_{k=1}^{j} l_{i_k}$

Ex: $n = 3, \quad (l_1, l_2, l_3) = (5,10,3)$

There are n! = 6 possible orderings for storing them.

	order	total retrieval time	MRT
1	1 2 3	5+(5+10)+(5+10+3)=38	38/3
2	1 3 2	5+(5+3)+(5+3+10)=31	31/3
3	2 1 3	10+(10+5)+(10+5+3)=43	43/3
4	2 3 1	10+(10+3)+(10+3+5)=41	41/3
5	3 1 2	3+(3+5)+(3+5+10)=29	29/3 ← Smallest
6	3 2 1	3+(3+10)+(3+10+5)=34	34/3

Note: The problem can be solved using greedy strategy, just always let the shortest program goes first.

(Can simply get the right order by using any sorting algorithm)

***Analysis*:**

Try all combination: O(n!)

Shortest-length-First Greedy method: O ($n$log$n$)

Shortest-length-First Greedy method:

Sort the programs s.t. $l_1 \le l_2 \le \ldots \le l_n$
and call this ordering L.

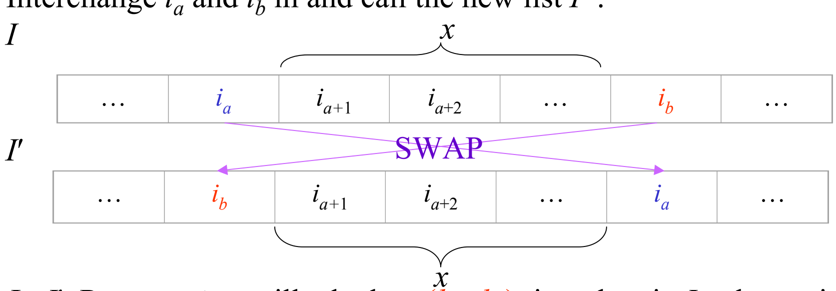
Next is to show that the ordering L is the best

Proof by contradiction:

Suppose Greedy ordering L is not optimal, then there exists some other permutation I that is optimal.

$I = (i_1, i_2, \ldots\ i_n)$ $\quad \exists$ a < b, s.t. $l_{i_a} > l_{i_b}$ (otherwise $I = L$)

Interchange i_a and i_b in and call the new list I' :



In I', Program i_{a+1} will take less $(l_{i_a} - l_{i_b})$ time than in I to be retrieved
In fact, each program i_{a+1} , …, i_{b-1} will take less $(l_{i_a} - l_{i_b})$ time
For i_b, the retrieval time decreases $x + l_{i_a}$
For i_a, the retrieval time increases $x + l_{i_b}$

$$totalRT(I) - totalRT(I') = (b-a-1)(l_{i_a} - l_{i_b}) + (x + l_{i_a}) - (x + l_{i_b})$$
$$= (b-a)(l_{i_a} - l_{i_b}) > 0 \quad (\rightarrow\leftarrow)$$

Contradiction!!

Therefore, greedy ordering L is optimal

4. Knapsack Problem

The problem:

Given a knapsack with a certain capacity M,
n objects, are to be put into the knapsack,
each has a weight $w_1, w_2, \cdots, w_n$ and
a profit if put in the knapsack $p_1, p_2, \cdots, p_n$.

The goal is find $(x_1, x_2, \cdots, x_n)$ where $0 \le x_i \le 1$

$$\text{s.t. } \sum_{i=1}^{n} p_i x_i \text{ is maximized and } \sum_{i=1}^{n} w_i x_i \le M$$

Note: All objects can break into small pieces or x_i can be any fraction between 0 and 1.

Example:

$n = 3$

$M = 20$

$(w_1, w_2, w_3) = (18,15,10)$

$(p_1, p_2, p_3) = (25,24,15)$

Greedy Strategy#1: Profits are ordered in nonincreasing order (1,2,3)

$$(x_1, x_2, x_3) = (1, \frac{2}{15}, 0)$$

$$\sum_{i=1}^{3} p_i x_i = 25 \times 1 + 24 \times \frac{2}{15} + 15 \times 0 = 28.2$$

Greedy Strategy#2: Weights are ordered in nondecreasing order (3,2,1)

$$(x_1, x_2, x_3) = (0, \frac{2}{3}, 1)$$

$$\sum_{i=1}^{3} p_i x_i = 25 \times 0 + 24 \times \frac{2}{3} + 15 \times 1 = 31$$

Greedy Strategy#3: *p/w* are ordered in nonincreasing order (2,3,1)

$$\left.\begin{aligned} \frac{p_1}{w_1} &= \frac{25}{18} = 1.4 \\ \frac{p_2}{w_2} &= \frac{24}{15} = 1.6 \\ \frac{p_3}{w_3} &= \frac{15}{10} = 1.5 \end{aligned}\right\} \Rightarrow (2,3,1)$$

Optimal solution

$$(x_1, x_2, x_3) = (0, 1, \frac{1}{2})$$

$$\sum_{i=1}^{3} p_i x_i = 25 \times 0 + 24 \times 1 + 15 \times \frac{1}{2} = \underline{31.5}$$

Analysis:

Sort the *p/w*, such that $\frac{p_1}{w_1} \geq \frac{p_2}{w_2} \geq \cdots \geq \frac{p_n}{w_n}$

Show that the ordering is the best.

<u>Proof by contradiction:</u>

Given some knapsack instance

Suppose the objects are ordered s.t. $\frac{p_1}{w_1} \geq \frac{p_2}{w_2} \geq \cdots \geq \frac{p_n}{w_n}$

let the greedy solution be $X = (x_1, x_2 \cdots, x_n)$

Show that this ordering is optimal

Case1: $X = (1, 1, \cdots, 1)$ it's optimal

Case2: $X = (1, 1, \cdots, x_j, 0, \cdots, 0)$ s.t. $\sum_{i=1}^{n} w_i x_i = M$

where $0 \leq x_j \leq 1$

Assume X is not optimal, and then there exists $Y=(y_1,y_2,\cdots,y_n)$

s.t. $\sum_{i=1}^{n} p_i y_i > \sum_{i=1}^{n} p_i x_i$ and Y is optimal

examine X and Y, let y_k be the 1st one in Y that $y_k \neq x_k$.

$$X=(\underbrace{x_1,x_2,\cdots\cdots,x_{k-1}}_{\text{same}},x_k,\cdots\cdots,x_n)$$

$$Y=(\overbrace{y_1,y_2,\cdots\cdots,y_{k-1}}^{\text{same}},y_k,\cdots\cdots,y_n)$$

$$y_k \neq x_k \Rightarrow y_k < x_k$$

Now we increase y_k to x_k and decrease as many of $(y_{k+1},\cdots,y_n)$ as necessary, so that the capacity is still M.

Let this new solution be $Z=(z_1,z_2,\cdots,z_n)$

where $z_i = x_i \quad \forall 1 \le i \le k$

$$\text{and } (z_k-y_k)\cdot w_k = \sum_{i=k+1}^{n}(y_i-z_i)\cdot w_i$$

$$\sum_{i=1}^{n} p_i z_i = \sum_{i=1}^{n} p_i y_i + (z_k-y_k)\cdot p_k - \sum_{i=k+1}^{n}(y_i-z_i)\cdot p_i$$

$$\because \frac{p_k}{w_k} \ge \frac{p_i}{w_i} \qquad \forall i \ge k+1$$

$$\therefore p_i \le \left(\frac{p_k}{w_k}\right)\cdot w_i$$

$$\sum_{i=1}^{n} p_i z_i \ge \sum_{i=1}^{n} p_i y_i + \overbrace{\cancel{\left[(z_k-y_k)\cdot w_k - \sum_{i=k+1}^{n}(y_i-z_i)\cdot w_i\right]}}^{0}\cdot\frac{p_k}{w_k} = \sum_{i=1}^{n} p_i y_i$$

So,

if $profit(z) > profit(y)$ $(\rightarrow\leftarrow)$

else $profit(z) = profit(y)$

(Repeat the same process.
At the end, Y can be transformed into X.
$\Rightarrow X$ is also optimal.
Contradiction! $\rightarrow\leftarrow$)