NP-Completeness

Reference: Computers and Intractability: A Guide to the Theory of NP-Completeness
by Garey and Johnson,
W.H. Freeman and Company, 1979.

General Problems, Input Size and Time Complexity

- Time complexity of algorithms :
  polynomial time algorithm ("efficient algorithm") v.s.
  exponential time algorithm ("inefficient algorithm")

f(n) \ n	**10**	**30**	**50**
n	0.00001 sec	0.00003 sec	0.00005 sec
n^5	0.1 sec	24.3 sec	5.2 mins
2^n	0.001 sec	17.9 mins	**35.7 yrs**

“Hard” and “easy’ Problems

- Sometimes the dividing line between “easy” and “hard” problems is a fine one. For example
  - Find the shortest path in a graph from X to Y. (easy)
  - Find the longest path in a graph from X to Y. (with no cycles) (hard)
- View another way – as “yes/no” problems
  - Is there a simple path from X to Y with weight <= M? (easy)
  - Is there a simple path from X to Y with weight >= M? (hard)
- First problem can be solved in polynomial time.
- All known algorithms for the second problem (could) take exponential time .

- Decision problem: The solution to the problem is **"yes"** or **"no".** Most optimization problems can be phrased as decision problems (still have the same time complexity).

  Example : Assume we have a decision algorithm X for 0/1 Knapsack problem with capacity M, i.e. Algorithm X returns "Yes" or "No" to the question "is there a solution with profit ≥ P subject to knapsack capacity ≤ M?"

We can repeatedly run algorithm X for various profits(P values) to find an optimal solution. Example : Use binary search to get the optimal profit,

maximum of lg $\sum p_i$ runs.

(where M is the capacity of the knapsack optimization problem)

Min Bound	Optimal Profit	Max Bound
0	Search for the optimal solution	$\sum p_i$

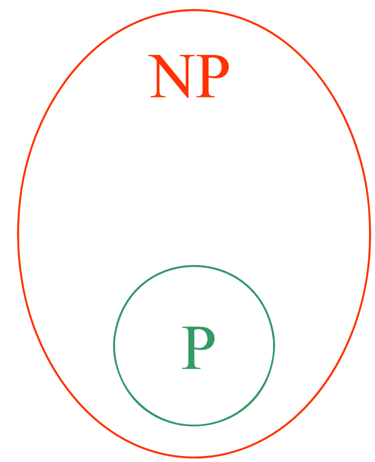
|______________________________|___________________________|

The Classes of P and NP

- The class P and Deterministic Turing Machine
  - Given a decision problem X, if there is a polynomial time ***Deterministic*** Turing Machine program that solves X, then X is belong to **P**
  - Informally, there is a polynomial time algorithm to solve the problem

- The class NP and Non-deterministic Turing Machine
  - Given a decision problem X, if there is a polynomial time ***Non-deterministic*** Turing machine program that solves X, then X belongs to **NP**
  - Given a decision problem X. For every instance I of X, (a) guess solution S for I, and (b) check "is S a solution to I?". If (a) and (b) can be done in polynomial time, then X belongs to NP.

- Obvious : P $\subseteq$ NP, i.e. A problem in P does not need "guess solution". The correct solution can be computed in polynomial time.



- Some problems which are in NP, but may not in P :
  - 0/1 Knapsack Problem
  - PARTITION Problem : Given a finite set of positive integers Z.

    Question : Is there a subset Z' of Z such that

    Sum of all numbers in Z' = Sum of all numbers in Z-Z' ?

    $$\text{i.e. } \sum Z' = \sum (Z-Z')$$

- One of the most important open problem in theoretical compute science : **Is P=NP ?** Most likely "No". Currently, there are many known problems in NP, and there is no solution to show anyone of them in P.

NP-Complete Problems

- Stephen Cook introduced the notion of NP-Complete Problems. This makes the problem "P = NP ?" much more interesting to study.

- The following are several important things presented by Cook :

1. Polynomial Transformation (" $\propto$ ")

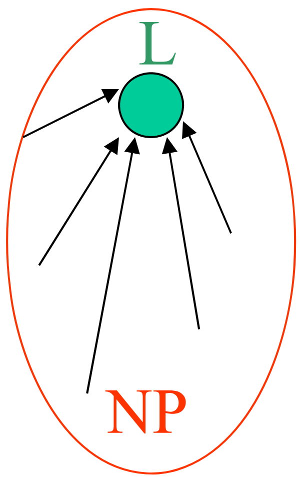
- L1 $\propto$ L2 : There is a polynomial time transformation that transforms arbitrary instance of L1 to some instance of L2.

- If L1 $\propto$ L2 then L2 is in P implies L1 is in P (or L1 is not in P implies L2 is not in P)

- If L1 $\propto$ L2 and L2 $\propto$ L3 then L1 $\propto$ L3

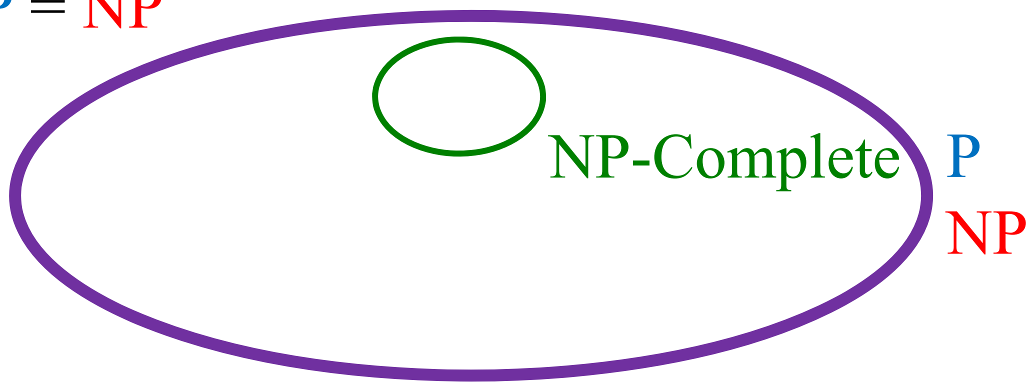
2. Focus on the class of NP – decision problems only. Many intractable problems, when phrased as decision problems, belong to this class.

3. L is NP-Complete if L $\in$NP and for all other L' $\in$ NP, L' $\propto$ L
   - If a problem in NP-complete can be solved in polynomial time then all problems in NP can be solved in polynomial time.
   - If a problem in NP cannot be solved in polynomial time then all problems in NP-complete cannot be solved in polynomial time.

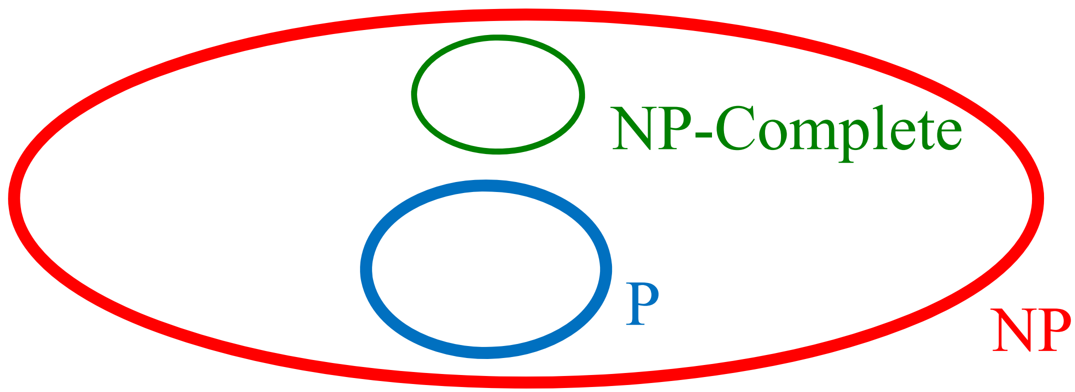


<u>Note that an NP-complete problem is one of those hardest problems in NP.</u>

- So, if an NP-complete problem is in P
  then P = NP



- if P != NP
  then all **NP-complete** problems are in **NP**-**P**



Question : how can we obtain the first NP-complete problem L?

4. Cook Theorem : SATISFIABILITY is NP-Complete. (The first NP-Complete problem)

   Instance : Given a set of variables, U, and a collection of clauses, C, over U.

   Question : Is there a truth assignment for U that satisfies all clauses in C?

Example :

$U = \{x_1, x_2\}$

$C_1 = \{(x_1, \neg x_2), (\neg x_1, x_2)\}$

$= (x_1 \text{ OR } \neg x_2) \text{ AND } (\neg x_1 \text{ OR } x_2)$

if $x_1 = x_2 = \text{True} \rightarrow C_1 = \text{True}$

$C_2 = (x_1, x_2)\,(x_1, \neg x_2)\,(\neg x_1) \rightarrow$ not satisfiable

"$\neg x_i$" = "not x_i"   "OR" = "logical or"   "AND" = "logical and"
This problem is also called "CNF-Satisfiability" since the expression is in CNF – Conjunctive Normal Form (the product of sums).

- With the Cook Theorem, we have the following property :

  Lemma : If L1 and L2 belong to NP, L1 is NP-complete, and L1 $\propto$ L2 then L2 is NP-complete.

  i.e. L1, L2 $\in$NP and for all other L' $\in$ NP, L' $\propto$ L1 and L1 $\propto$ L2 $\rightarrow$ L' $\propto$ L2

- So now, to prove a problem L to be NP-complete problem , we need to

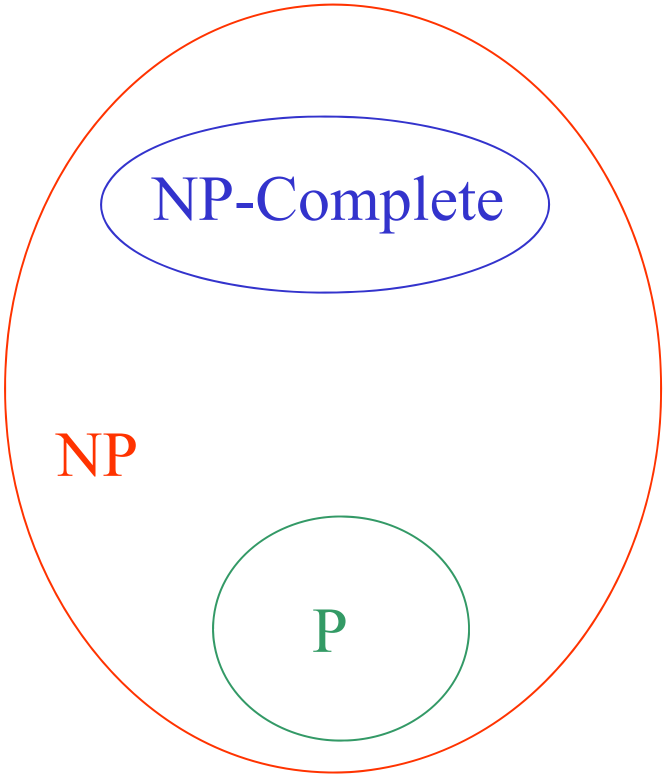
  (#1) show L is in NP

  (#2) select a known NP-complete problem L'

  - construct a polynomial time transformation f from L' to L
  - prove the correctness of f (i.e. L' has a solution if and only if L has a solution) and that f is a polynomial transformation

- P: Decision problems solvable by ***deterministic*** algorithms in polynomial time
- NP: Decision problems solved by ***non-deterministic*** algorithms in polynomial time
- A group of problems, including all of the ones we have discussed (Satisfiability, 0/1 Knapsack, Longest Path, Partition) have an additional important property:

  If any of them can be solved in polynomial time, then they all can!



- These problems are called NP-complete problems.

- Some NP-complete problems :
  - SATISFIABILITY
  - 0/1 Knapsack Decision problem
  - PARTITION
  - Two-Processor Non-Preemptive Schedule Length Decision Problem
  - CLIQUE : An undirected graph G=(V, E) and a positive integer $J \leq |V|$

    Question : Does G contain a clique (complete subgraph) of size J or more?