Unit No. 6: Fuzzy Arithmetic

1.Fuzzy Number:

A fuzzy number is a fuzzy set 'A' define on set of real numbers (R) must satisfies the following three properties.

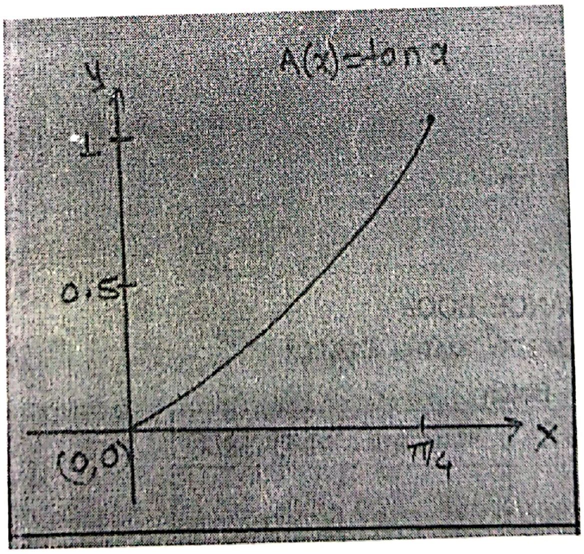
1. Fuzzy set (A) must be a normal fuzzy set. i. e. h (A) = 1.
2. α cut of fuzzy set A is a closed interval for all $\alpha \in [0,1]$.
3. The support of fuzzy set A must be bounded.

Examples

Example 1: Determine following fuzzy sets is a fuzzy numbers or not.

$$A(x) = \begin{cases} \tan x & ,if\ 0 \le x \le \frac{\pi}{4} \\ 0 & ,otherwise \end{cases}$$

Solution: We draw the diagram of given fuzzy set A(x) as,



1. We define height of fuzzy set A as, $h(A) = \sup_{x \in X} A(x)$

$$here,\ A(x) = \tan x \ \Rightarrow A\left(\frac{\pi}{4}\right) = 1 \text{ as, } \frac{\pi}{4} \in \left[0, \frac{\pi}{4}\right]$$

Here h (A) = 1 so A(x) is normal fuzzy set.

2. We define α-cut of fuzzy set A as, ${}^{\alpha}A = \{x \in X / A(x) \geq \alpha\}$

$$\tan x \geq \alpha \quad \Rightarrow x \geq \tan^{-1}\alpha$$

$${}^{\alpha}A = \left[\tan^{-1}\alpha, \frac{\pi}{4}\right] \text{ for every } \alpha \in [0,1]$$

α cut of fuzzy set A is a closed interval for all α ∈[0,1].

3. We define support of fuzzy set A as, $Supp\, A = {}^{0+}A = \{x \in X / A(x) > 0\}$

$$Supp\, A = {}^{0+}A = \left(0, \frac{\pi}{4}\right]$$

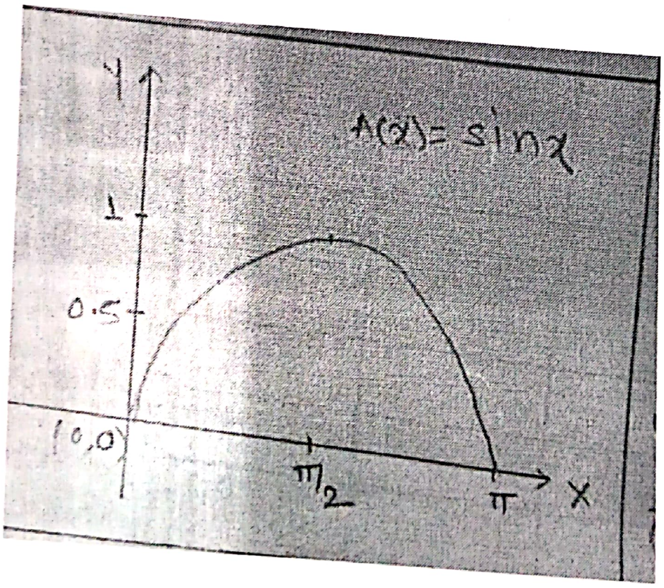
${}^{0+}A$ *is bounded.*

Here, A(x) has satisfied all three properties therefore A(x) is a fuzzy number.

Example 2: Determine following fuzzy sets is a fuzzy numbers or not.

$$A(x) = \begin{cases} \sin x & , if\ 0 \leq x \leq \pi \\ 0 & , otherwise \end{cases}$$

Solution: We draw the diagram of given fuzzy set A(x) as,



1. We define height of fuzzy set A as, $h(A) = \sup_{x \in X} A(x)$

here, $A(x) = \sin x \Rightarrow A\left(\frac{\pi}{2}\right) = 1$ as, $\frac{\pi}{2} \in [0, \pi]$

Here h (A) = 1 so A(x) is normal fuzzy set.

2. We define α-cut of fuzzy set A as, ${}^{\alpha}A = \{x \in X / A(x) \geq \alpha\}$

$$\sin x \geq \alpha \quad \Rightarrow x \geq \sin^{-1}\alpha$$

$${}^{\alpha}A = [\sin^{-1}\alpha, \pi - \sin^{-1}\alpha] \; for\ every\ \alpha \in [0,1]$$

α cut of fuzzy set A is a closed interval for all α ∈[0,1].

3. We define support of fuzzy set A as, $Supp\,A = {}^{0+}A = \{x \in X / A(x) > 0\}$

$$Supp\,A = {}^{0+}A = (0, \pi)$$

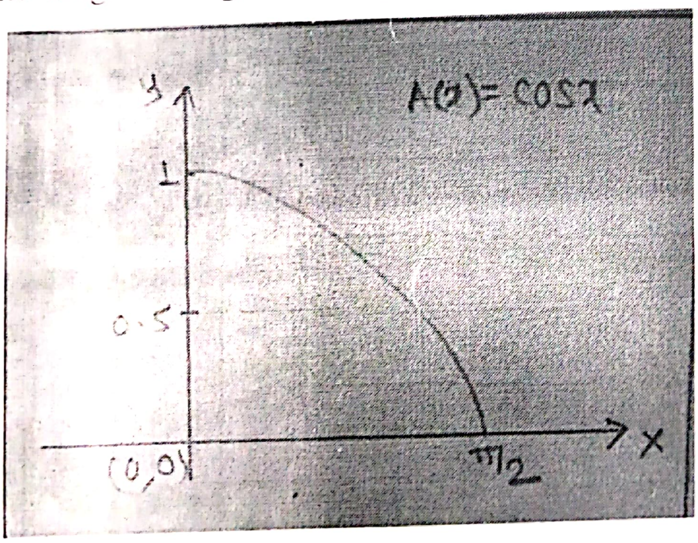
$${}^{0+}A\ is\ bounded.$$

Here, A(x) has satisfied all three properties therefore A(x) is a fuzzy number.

Example 3: Determine following fuzzy sets is a fuzzy numbers or not.

$$A(x) = \begin{cases} \cos x & , if\ 0 \leq x \leq \frac{\pi}{2} \\ 0 & , otherwise \end{cases}$$

Solution: We draw the diagram of given fuzzy set A(x) as,



1. We define height of fuzzy set A as, $h(A) = \sup_{x \in X} A(x)$

$$here,\ A(x) = \cos x \quad \Rightarrow A(0) = 1 \text{ as, } 0 \in \left[0, \frac{\pi}{2}\right]$$

Here h (A) = 1 so A(x) is normal fuzzy set.

2. We define α-cut of fuzzy set A as, ${}^{\alpha}A = \{x \in X / A(x) \geq \alpha\}$

$$\cos x \geq \alpha \quad \Rightarrow x \geq \cos^{-1}\alpha$$

$${}^{\alpha}A = [0, \cos^{-1}\alpha] \quad \forall \; \alpha \in [0,1]$$

α cut of fuzzy set A is a closed interval for all $\alpha \in [0,1]$.

3. We define support of fuzzy set A as, $Supp\,A = {}^{0+}A = \{x \in X \,/\, A(x) > 0\}$

$$Supp\,A = {}^{0+}A = \left[0, \frac{\pi}{2}\right)$$

$${}^{0+}A \text{ is bounded.}$$

Here, A(x) has satisfied all three properties therefore A(x) is a fuzzy number.

Examples for Practice

Example 1: What are the criteria's of a fuzzy set to be a fuzzy number. Determine whether the following fuzzy set is a fuzzy number.

$$A(x) = \begin{cases} \sin 2x & ,if\ 0 \le x \le \frac{\pi}{4} \\ 0 & ,otherwise \end{cases}$$

Example 2: Determine whether the following fuzzy set is a fuzzy number.

$$A(x) = \begin{cases} 1 + x & ,if\ -1 \le x \le 0 \\ 0 & ,otherwise \end{cases}$$

Example 3: Determine whether the following fuzzy set is a fuzzy number.

$$A(x) = \begin{cases} 1 - |x| & ,if\ -1 \le x \le 1 \\ 0 & ,otherwise \end{cases}$$

Example 4: Determine whether the following fuzzy set is a fuzzy number.

$$A(x) = \begin{cases} \frac{x}{10} & ,if\ 0 \le x \le 10 \\ 0 & ,otherwise \end{cases}$$

2.Fuzzy Cardinality:

Fuzzy cardinality is defined as a fuzzy number rather than as a real number, as is the case for a scalar cardinality. When a fuzzy set A has a finite support its fuzzy cardinality is denoted by $|\tilde{A}|$ is a fuzzy set defined on N whose membership function is defined by,

$$|\tilde{A}| = \frac{\alpha}{|{}^{\alpha}A|} \qquad A: N \rightarrow [0, 1]$$

Where, $|{}^{\alpha}A|$ is the cardinality of α - cut of A.

Example 1: Find Fuzzy Cardinality for the fuzzy set, $A(x) = 3^{-x}$ for $x \in \{0,1,..,5\}$

Solution: Given fuzzy set, $A(x) = 3^{-x}$ for universal set X = [0, 1, 2…5]

Fuzzy set A(x) can be represented as,

$$A(x) = \left\{\frac{1}{0} + \frac{0.3333}{1} + \frac{0.1111}{2} + \frac{0.037}{3} + \frac{0.0123}{4} + \frac{0.0041}{5}\right\}$$

We define Level set of fuzzy set A as,

$$\Lambda = \{\alpha / A(x) = \alpha \ for\ some\ x \in X\}$$

$$\Lambda = \{0.0041, 0.0123, 0.037, 0.1111, 0.3333, 1\}$$

We define α-cut of fuzzy set A as, ${}^{\alpha}A = \{x \in X / A(x) \geq \alpha\}$

${}^{0.0041}A = \{x \in X / A(x) \geq 0.0041\}$ ${}^{0.0041}A = \{0, 1, 2, 3, 4,5\}$ $\therefore |{}^{0.0041}A| = 6$

${}^{0.0123}A = \{x \in X / A(x) \geq 0.0123\}$ ${}^{0.0123}A = \{0, 1, 2, 3, 4\}$ $\therefore |{}^{0.0123}A| = 5$

${}^{0.037}A = \{x \in X / A(x) \geq 0.037\}$ ${}^{0.037}A = \{0, 1, 2, 3\}$ $\therefore |{}^{0.037}A| = 4$

${}^{0.1111}A = \{x \in X / A(x) \geq 0.1111\}$ ${}^{0.1111}A = \{0, 1, 2\}$ $\therefore |{}^{0.1111}A| = 3$

${}^{0.3333}A = \{x \in X / A(x) \geq 0.3333\}$ ${}^{0.3333}A = \{0, 1\}$ $\therefore |{}^{0.3333}A| = 2$

${}^{1}A = \{x \in X / A(x) \geq 1\}$ ${}^{1}A = \{0\}$ $\therefore |{}^{1}A| = 1$

Fuzzy cardinality of A (x) is, $|\tilde{A}| = \frac{\alpha}{|{}^{\alpha}A|}$

$$|\tilde{A}| = \left\{\frac{1}{1} + \frac{0.3333}{2} + \frac{0.1111}{3} + \frac{0.037}{4} + \frac{0.0123}{5} + \frac{0.0041}{6}\right\}$$

Example 2: Let A and B be fuzzy sets defined on the universal set X=Z (set of integers) whose membership functions are given by,

$$A(x)=\left\{\frac{0}{1}+\frac{0.2}{1.5}+\frac{0.35}{2}+\frac{0.15}{2.5}+\frac{0.5}{3}+\frac{0.25}{3.5}+\frac{0.4}{4}\right\}$$

$$B(x)=\left\{\frac{1}{1}+\frac{0.15}{1.5}+\frac{0.2}{2}+\frac{0.35}{2.5}+\frac{0.4}{3}+\frac{0.125}{3.5}+\frac{0.4}{4}\right\}$$

Find Fuzzy Cardinality of $\overline{A\cup B}$.

Solution: Given the fuzzy set A and B as,

$$A(x)=\left\{\frac{0}{1}+\frac{0.2}{1.5}+\frac{0.35}{2}+\frac{0.15}{2.5}+\frac{0.5}{3}+\frac{0.25}{3.5}+\frac{0.4}{4}\right\}$$

$$B(x)=\left\{\frac{1}{1}+\frac{0.15}{1.5}+\frac{0.2}{2}+\frac{0.35}{2.5}+\frac{0.4}{3}+\frac{0.125}{3.5}+\frac{0.4}{4}\right\}$$

Union of fuzzy set A and B defined as,

$$A\cup B(x)=Max\{A(x),B(x)\}$$

$$A\cup B(x)=\left\{\frac{1}{1}+\frac{0.2}{1.5}+\frac{0.35}{2}+\frac{0.35}{2.5}+\frac{0.5}{3}+\frac{0.25}{3.5}+\frac{0.4}{4}\right\}$$

Complement of fuzzy set A ∪ B defined as,

$$\overline{A\cup B}(x)=1-A\cup B(x)$$

$$\overline{A\cup B}(x)=\left\{\frac{0}{1}+\frac{0.8}{1.5}+\frac{0.65}{2}+\frac{0.65}{2.5}+\frac{0.5}{3}+\frac{0.75}{3.5}+\frac{0.6}{4}\right\}$$

We define Level set of fuzzy set $\overline{A\cup B}$ as,

$$\Lambda\overline{A\cup B}(x)=\{\alpha/\overline{A\cup B}=\alpha\ for\ some\ x\in X\}$$

$$\Lambda\overline{A\cup B}(x)=\{0,0.5,0.6,0.65,0.75,0.8\}$$

We define α-cut of fuzzy set A as, ${}^{\alpha}A=\{x\in X/A(x)\geq\alpha\}$

$${}^{0}\overline{A\cup B}(x)=\{x\in X/\overline{A\cup B}(x)\geq 0\}$$

$${}^{0}\overline{A\cup B}(x)=\{1,1.5,2,2.5,3,3.5,4\}\qquad \therefore|{}^{0}\overline{A\cup B}|=7$$

$${}^{0.5}\overline{A\cup B}(x)=\{x\in X/\overline{A\cup B}(x)\geq 0.5\}$$

$${}^{0.5}\overline{A\cup B}(x)=\{1.5,2,2.5,3,3.5,4\}\qquad \therefore|{}^{0.5}\overline{A\cup B}|=6$$

$${}^{0.6}\overline{A\cup B}(x)=\{x\in X/\overline{A\cup B}(x)\geq 0.6\}$$

Example 2: Let A and B be fuzzy sets defined on the universal set X=Z (set of integers) whose membership functions are given by,

$$A(x)=\left\{\frac{0}{1}+\frac{0.2}{1.5}+\frac{0.35}{2}+\frac{0.15}{2.5}+\frac{0.5}{3}+\frac{0.25}{3.5}+\frac{0.4}{4}\right\}$$

$$B(x)=\left\{\frac{1}{1}+\frac{0.15}{1.5}+\frac{0.2}{2}+\frac{0.35}{2.5}+\frac{0.4}{3}+\frac{0.125}{3.5}+\frac{0.4}{4}\right\}$$

Find Fuzzy Cardinality of $\overline{A\cup B}$.

Solution: Given the fuzzy set A and B as,

$$A(x)=\left\{\frac{0}{1}+\frac{0.2}{1.5}+\frac{0.35}{2}+\frac{0.15}{2.5}+\frac{0.5}{3}+\frac{0.25}{3.5}+\frac{0.4}{4}\right\}$$

$$B(x)=\left\{\frac{1}{1}+\frac{0.15}{1.5}+\frac{0.2}{2}+\frac{0.35}{2.5}+\frac{0.4}{3}+\frac{0.125}{3.5}+\frac{0.4}{4}\right\}$$

Union of fuzzy set A and B defined as,

$$A\cup B(x)=Max\{\,A(x),B(x)\,\}$$

$$A\cup B(x)=\left\{\frac{1}{1}+\frac{0.2}{1.5}+\frac{0.35}{2}+\frac{0.35}{2.5}+\frac{0.5}{3}+\frac{0.25}{3.5}+\frac{0.4}{4}\right\}$$

Complement of fuzzy set A ∪ B defined as,

$$\overline{A\cup B}(x)=1-A\cup B(x)$$

$$\overline{A\cup B}(x)=\left\{\frac{0}{1}+\frac{0.8}{1.5}+\frac{0.65}{2}+\frac{0.65}{2.5}+\frac{0.5}{3}+\frac{0.75}{3.5}+\frac{0.6}{4}\right\}$$

We define Level set of fuzzy set $\overline{A\cup B}$ as,

$$\Lambda\overline{A\cup B}(x)=\{\alpha/\overline{A\cup B}=\alpha\ for\ some\ x\in X\}$$

$$\Lambda\overline{A\cup B}(x)=\{0,0.5,0.6,0.65,0.75,0.8\}$$

We define α-cut of fuzzy set A as, ${}^{\alpha}A=\{x\in X/A(x)\geq\alpha\}$

$${}^{0}\overline{A\cup B}(x)=\{x\in X/\overline{A\cup B}(x)\geq 0\}$$

$${}^{0}\overline{A\cup B}(x)=\{1,1.5,2,2.5,3,3.5,4\} \qquad \therefore |{}^{0}\overline{A\cup B}|=7$$

$${}^{0.5}\overline{A\cup B}(x)=\{x\in X/\overline{A\cup B}(x)\geq 0.5\}$$

$${}^{0.5}\overline{A\cup B}(x)=\{1.5,2,2.5,3,3.5,4\} \qquad \therefore |{}^{0.5}\overline{A\cup B}|=6$$

$${}^{0.6}\overline{A\cup B}(x)=\{x\in X/\overline{A\cup B}(x)\geq 0.6\}$$

$$^{0.6}\overline{A \cup B}(x) = \{1.5, 2, 2.5, 3.5, 4\} \qquad \therefore |^{0.6}\overline{A \cup B}| = 5$$

$$^{0.65}\overline{A \cup B}(x) = \{x \in X / \overline{A \cup B}(x) \geq 0.65\}$$

$$^{0.65}\overline{A \cup B}(x) = \{1.5, 2, 2.5, 3.5\} \qquad \therefore |^{0.65}\overline{A \cup B}| = 4$$

$$^{0.75}\overline{A \cup B}(x) = \{x \in X / \overline{A \cup B}(x) \geq 0.75\}$$

$$^{0.75}\overline{A \cup B}(x) = \{1.5, 3.5\} \qquad \therefore |^{0.75}\overline{A \cup B}| = 2$$

$$^{0.8}\overline{A \cup B}(x) = \{x \in X / \overline{A \cup B}(x) \geq 0.8\}$$

$$^{0.8}\overline{A \cup B}(x) = \{1.5\} \qquad \therefore |^{0.8}\overline{A \cup B}| = 1$$

Fuzzy cardinality of A (x) is, $|\tilde{A}| = \dfrac{\alpha}{|^{\alpha}A|}$

$$|\tilde{A}| = \left\{\frac{0}{7} + \frac{0.5}{6} + \frac{0.6}{5} + \frac{0.65}{4} + \frac{0.75}{2} + \frac{0.8}{1}\right\}$$

xample 3: Find the degree of subset hood $S(|\tilde{A}|, |\tilde{B}|)$ for the fuzzy sets,

$$A(x) = \frac{x}{x+3}, \quad B(x) = \frac{x}{x+2}, \quad x \in \{0,1,2\}$$

olution: Given fuzzy set, $A(x) = \dfrac{x}{x+3}$ for universal set X = [0, 1, 2]

Fuzzy set A(x) can be represented as,

$$A(x) = \left\{\frac{0}{0} + \frac{0.25}{1} + \frac{0.4}{2}\right\}$$

We define Level set of fuzzy set A as,

$$\Lambda A = \{\alpha / A(x) = \alpha \text{ for some } x \in X\}$$

$$\Lambda A = \{0, 0.25, 0.4\}$$

We define α-cut of fuzzy set A as, $^{\alpha}A = \{x \in X / A(x) \geq \alpha\}$

$$^{0}A = \{x \in X / A(x) \geq 0\} \qquad ^{0}A = \{0, 1, 2\} \qquad \therefore |^{0}A| = 3$$

$$^{0.25}A = \{x \in X / A(x) \geq 0.25\} \qquad ^{0.25}A = \{1, 2\} \qquad \therefore |^{0.25}A| = 2$$

$$^{0.4}A = \{x \in X / A(x) \geq 0.4\} \qquad ^{0.4}A = \{2\} \qquad \therefore |^{0.4}A| = 1$$

Fuzzy cardinality of A (x) is, $|\tilde{A}| = \dfrac{\alpha}{|^{\alpha}A|}$

$$|\tilde{A}| = \left\{ \frac{0}{3} + \frac{0.25}{2} + \frac{0.4}{1} \right\}$$

Scalar cardinality of fuzzy set A as, $\left| |\tilde{A}| \right| = \sum_{x \in X} |\tilde{A}|(x) = 0.65$

Given fuzzy set, $B(x) = \frac{x}{x+2}$ for universal set X = [0, 1, 2]

Fuzzy set A(x) can be represented as,

$$B(x) = \left\{ \frac{0}{0} + \frac{0.3333}{1} + \frac{0.5}{2} \right\}$$

We define Level set of fuzzy set B as,

$$\Lambda B = \{ \alpha / B(x) = \alpha \text{ for some } x \in X \}$$

$$\Lambda B = \{ 0, 0.3333, 0.5 \}$$

We define α-cut of fuzzy set A as, ${}^{\alpha}B = \{ x \in X / B(x) \geq \alpha \}$

${}^{0}B = \{ x \in X / B(x) \geq 0 \}$ ${}^{0}B = \{0, 1, 2\}$ $\therefore |{}^{0}B| = 3$

${}^{0.3333}B = \{ x \in X / B(x) \geq 0.3333 \}$ ${}^{0.3333}B = \{1, 2\}$ $\therefore |{}^{0.3333}B| = 2$

${}^{0.5}B = \{ x \in X / B(x) \geq 0.5 \}$ ${}^{0.5}B = \{2\}$ $\therefore |{}^{0.4}B| = 1$

Fuzzy cardinality of A (x) is, $|\tilde{B}| = \frac{\alpha}{|{}^{\alpha}B|}$

$$|\tilde{B}| = \left\{ \frac{0}{3} + \frac{0.3333}{2} + \frac{0.5}{1} \right\}$$

Scalar cardinality of fuzzy set B as, $\left| |\tilde{B}| \right| = \sum_{x \in X} |\tilde{B}|(x) = 0.8333$

By definition, $|\tilde{A}| \cap |\tilde{B}|(x) = Min\{ |\tilde{A}|(x), |\tilde{B}|(x) \}$

$$|\tilde{A}| \cap |\tilde{B}| = \left\{ \frac{0}{3} + \frac{0.25}{2} + \frac{0.4}{1} \right\}$$

Scalar cardinality of fuzzy set $|\tilde{A}| \cap |\tilde{B}|(x)$ as, $\left| |\tilde{A}| \cap |\tilde{B}| \right| = \sum_{x \in X} |\tilde{A}| \cap |\tilde{B}| = 0.65$

Degree of Subset hood, $S(|\tilde{A}|, |\tilde{B}|) = \frac{\left| |\tilde{A}| \cap |\tilde{B}| \right|}{|\tilde{A}|} = \frac{0.65}{0.65} = 1$

Examples for Practice

Example 1: Find Fuzzy Cardinality for the fuzzy set, $A(x)=3^{-x}$ for $x \in \{0,1,..,5\}$

Example 2: Find Fuzzy Cardinality for the fuzzy set, $A(x)=\dfrac{x+2}{x+5}$ for $x \in \{0,1,..,10\}$

Example 3: Find Fuzzy Cardinality for the fuzzy set, $A(x)=\dfrac{x}{x+2}$ for $x \in \{6,7,..,10\}$

Example 4: Find Fuzzy Cardinality for the fuzzy set, $B(x)=\dfrac{1}{1+10(x-1)^2}$ for $x \in \{0,1,..,5\}$

Example 5: Find Fuzzy Cardinality for the fuzzy set, $A(x)=\dfrac{2x+5}{3x+7}$ for $x \in \{0,1,..,10\}$

Example 6: Find Fuzzy Cardinality for the fuzzy set, $A(x)=\dfrac{x+5}{4x+9}$ for $x \in \{0,1,..,10\}$

Example 7: Find the degree of subset hood $S(|\tilde{A}|,|\tilde{B}|)$ for the fuzzy sets,

$$A(x)=\frac{x}{x+4}\ ,\ B(x)=3^{-x}\ ,\ x \in \{0,1,2\ldots 5\}$$

Example 8: Find the degree of subset hood $S(|\tilde{A}|,|\tilde{B}|)$ for the fuzzy sets,

$$A(x)=\frac{2x+1}{5x+2}\ ,\ B(x)=\frac{x}{x+1}\ ,\ x \in \{0,1,2\ldots 10\}$$

Example 9: Let A and B be fuzzy sets defined on the universal set X=Z (set of integers) whose membership functions are given by,

$$A(x)=\left\{\frac{0}{1}+\frac{0.2}{1.5}+\frac{0.35}{2}+\frac{0.15}{2.5}+\frac{0.5}{3}+\frac{0.25}{3.5}+\frac{0.4}{4}\right\}$$

$$B(x)=\left\{\frac{1}{1}+\frac{0.15}{1.5}+\frac{0.2}{2}+\frac{0.35}{2.5}+\frac{0.4}{3}+\frac{0.125}{3.5}+\frac{0.4}{4}\right\}$$

Find Fuzzy Cardinality of $\overline{A \cap B}$

Example 10: Let fuzzy numbers A and B each defined on its own inverse be

$$A(x)=\left\{\frac{0.5}{1}+\frac{0.6}{2}+\frac{0.9}{3}+\frac{1}{4}+\frac{0.2}{5}\right\}\ ,\quad B(x)=\left\{\frac{0.3}{3}+\frac{0.4}{4}+\frac{0.7}{5}+\frac{1}{6}+\frac{0.3}{7}\right\}$$

Find Fuzzy Cardinality of $A \cup \overline{B}$.

4. Method of developing Fuzzy Arithmetic is based on α – cut sets:

Let A and B are fuzzy numbers, if * be binary operation defined on set of numbers (R) then, A* B by defining it's α – cut.

$$^{\alpha}A * B = {}^{\alpha}A * {}^{\alpha}B$$

The four arithmetic operations on closed intervals are defined as follows,

1) $[a, b] + [c, d] = [a + c, b + d]$

For example, $[2, 5] + [1, 3] = [2 + 1, 5 + 3] = [3, 8]$

2) $[a, b] - [c, d] = [a - d, b - c]$

For example, $[2, 5] - [1, 3] = [2 - 3, 5 - 1] = [-1, 4]$

3) $[a, b] \times [c, d] = \{\min [ac, ad, bc, bd], \max [ac, ad, bc, bd]\}$

For example,

$$[-1, 1] \times [-2, -0.5] = (\min[2, 0.5, -2, -0.5], \max[2, 0.5, -2, -0.5]) = (-2, 2)$$

4) $[a, b]/[c, d] = \left(\min\left[\frac{a}{c}, \frac{a}{d}, \frac{b}{c}, \frac{b}{d}\right], \max\left[\frac{a}{c}, \frac{a}{d}, \frac{b}{c}, \frac{b}{d}\right]\right)$

For example,

$$[4, 10]/[1, 2] = \left(\min\left[\frac{4}{1}, \frac{4}{2}, \frac{10}{1}, \frac{10}{2}\right], \max\left[\frac{4}{1}, \frac{4}{2}, \frac{10}{1}, \frac{10}{2}\right]\right) = (2, 10)$$

real Example 1: Consider fuzzy number A and B defined by,

$$A(x) = \begin{cases} \frac{x+3}{3} & if -3 < x \le 0 \\ \frac{3-x}{3} & if\ 0 < x \le 3 \\ 0 & otherwise \end{cases} \qquad B(x) = \begin{cases} \frac{x-3}{3} & if\ 3 < x \le 6 \\ \frac{9-x}{3} & if\ 6 < x \le 9 \\ 0 & otherwise \end{cases}$$

Calculate fuzzy numbers i) A + B ii) A - B iii) A * B iv) A / B

Solution: We know that α-cut of fuzzy set A as, ${}^{\alpha}A = \{x \in X / A(x) \ge \alpha\}$

$$A(x) \ge \alpha$$
$$\frac{x+3}{3} \ge \alpha$$
$$x+3 \ge 3\alpha$$
$$x \ge 3\alpha - 3 \text{..........(1)}$$

$$A(x) \ge \alpha$$
$$\frac{3-x}{3} \ge \alpha$$
$$3-x \ge 3\alpha$$
$$3-3\alpha \ge x \text{..........(2)}$$

From equation (1) and (2) we get, ${}^{\alpha}A = [3\alpha - 3, 3 - 3\alpha]$

We know that α-cut of fuzzy set B as, ${}^{\alpha}B = \{x \in X / B(x) \ge \alpha\}$

$$B(x) \ge \alpha$$
$$\frac{x-3}{3} \ge \alpha$$
$$x-3 \ge 3\alpha$$
$$x \ge 3\alpha + 3 \text{..........(3)}$$

$$B(x) \ge \alpha$$
$$\frac{9-x}{3} \ge \alpha$$
$$9-x \ge 3\alpha$$
$$9-3\alpha \ge x \text{..........(4)}$$

From equation (3) and (4) we get, ${}^{\alpha}B = [3\alpha + 3, 9 - 3\alpha]$

1. for A+B:

We know that, ${}^{\alpha}A+B = {}^{\alpha}A + {}^{\alpha}B$

$${}^{\alpha}A+B = [3\alpha-3, 3-3\alpha] + [3\alpha+3, 9-3\alpha]$$
$${}^{\alpha}A+B = [3\alpha-3+3\alpha+3, 3-3\alpha+9-3\alpha]$$
$${}^{\alpha}A+B = [6\alpha, 12-6\alpha]$$

i.e. $6\alpha \le x \le 12 - 6\alpha$

$$6\alpha = x$$
$$\alpha = \frac{x}{6}$$

$$12 - 6\alpha = x$$
$$\alpha = \frac{12-x}{6}$$

$$if\ \alpha=0 \Rightarrow x=0$$
$$if\ \alpha=1 \Rightarrow x=6$$

$$if\ \alpha=1 \Rightarrow x=6$$
$$if\ \alpha=0 \Rightarrow x=12$$

$$Hence, \quad (A+B)(x)=\begin{cases} \dfrac{x}{6} & if\ 0<x\leq 6 \\ \dfrac{12-x}{6} & if\ 6<x\leq 12 \\ 0 & otherwise \end{cases}$$

2. for A - B:

We know that, ${}^{\alpha}A-B={}^{\alpha}A-{}^{\alpha}B$

$${}^{\alpha}A-B=[3\alpha-3,3-3\alpha]-[3\alpha+3,9-3\alpha]$$

$${}^{\alpha}A-B=[3\alpha-3-9+3\alpha,3-3\alpha-3\alpha-3]$$

$${}^{\alpha}A-B=[6\alpha-12,-6\alpha]$$

$$i.e.\ 6\alpha-12\leq x\leq -6\alpha$$

$$6\alpha-12=x$$
$$\alpha=\frac{x+12}{6}$$
$$if\ \alpha=0 \Rightarrow x=-12$$
$$if\ \alpha=1 \Rightarrow x=-6$$

$$-6\alpha=x$$
$$\alpha=\frac{-x}{6}$$
$$if\ \alpha=1 \Rightarrow x=-6$$
$$if\ \alpha=0 \Rightarrow x=0$$

$$Hence, \quad (A-B)(x)=\begin{cases} \dfrac{x+12}{6} & if\ -12<x\leq -6 \\ \dfrac{-x}{6} & if\ -6<x\leq 0 \\ 0 & otherwise \end{cases}$$

For A*B:

We know that, ${}^{\alpha}A\times B={}^{\alpha}A\times{}^{\alpha}B$

$${}^{\alpha}A\times B=[3\alpha-3,3-3\alpha]\times[3\alpha+3,9-3\alpha]$$

$${}^{\alpha}A\times B=[\min\{(3\alpha-3)(3\alpha+3),(3\alpha-3)(9-3\alpha),(3-3\alpha)(3\alpha+3),(3-3\alpha)(9-3\alpha)\},$$
$$\max\{(3\alpha-3)(3\alpha+3),(3\alpha-3)(9-3\alpha),(3-3\alpha)(3\alpha+3),(3-3\alpha)(9-3\alpha)\}]$$

α	(3α-3) (3α+3)	(3α-3) (9-3α)	(3-3α) (3α+3)	(3-3α) (9-3α)
0.5	- 6.75	**- 11.25**	6.75	**11.25**

$${}^{\alpha}A\times B=[(3\alpha-3)(9-3\alpha),(3-3\alpha)(9-3\alpha)]$$

$${}^{\alpha}A\times B=[-9\alpha^2+36\alpha-27,9\alpha^2-36\alpha+27]$$

$$i.e.\ -9\alpha^2+36\alpha-27\leq x\leq 9\alpha^2-36\alpha+27$$

$$-9\alpha^2+36\alpha-27=x$$

$$-9\alpha^2-36\alpha=x+27$$

$$\alpha^2-4\alpha=\frac{-x-27}{9}$$

$$\alpha^2-4\alpha+4=\frac{-x-27}{9}+4$$

$$(2-\alpha)^2=\frac{9-x}{9}$$

$$(2-\alpha)=\frac{\sqrt{9-x}}{3}$$

$$2-\frac{\sqrt{9-x}}{3}=\alpha$$

$$\frac{6-\sqrt{9-x}}{3}=\alpha$$

$$if\ \alpha=0\Rightarrow x=-27$$

$$if\ \alpha=1\Rightarrow x=0$$

$$9\alpha^2-36\alpha+27=x$$

$$9\alpha^2-36\alpha=x-27$$

$$\alpha^2-4\alpha=\frac{x-27}{9}$$

$$\alpha^2-4\alpha+4=\frac{x-27}{9}+4$$

$$(2-\alpha)^2=\frac{9+x}{9}$$

$$(2-\alpha)=\frac{\sqrt{9+x}}{3}$$

$$2-\frac{\sqrt{9+x}}{3}=\alpha$$

$$\frac{6-\sqrt{9+x}}{3}=\alpha$$

$$if\ \alpha=1\Rightarrow x=0$$

$$if\ \alpha=0\Rightarrow x=27$$

$$Hence,\quad (A\times B)(x)=\begin{cases}\dfrac{6-\sqrt{9-x}}{3} & if\ -27<x\leq 0\\[2ex] \dfrac{6-\sqrt{9+x}}{3} & if\ 0<x\leq 27\\[2ex] 0 & otherwise\end{cases}$$

4. for A/B:

We know that, $^{\alpha}A/B = {}^{\alpha}A \;/\; {}^{\alpha}B$

$$^{\alpha}A/B = [3\alpha - 3\,,3 - 3\alpha]/[3\alpha + 3\,,9 - 3\alpha]$$

$$^{\alpha}A/B = \left[\min\left(\left(\frac{3\alpha-3}{3\alpha+3}\right),\left(\frac{3\alpha-3}{9-3\alpha}\right),\left(\frac{3-3\alpha}{3\alpha+3}\right),\left(\frac{3-3\alpha}{9-3\alpha}\right)\right),\max\left(\left(\frac{3\alpha-3}{3\alpha+3}\right),\left(\frac{3\alpha-3}{9-3\alpha}\right),\left(\frac{3-3\alpha}{3\alpha+3}\right),\left(\frac{3-3\alpha}{9-3\alpha}\right)\right)\right]$$

α	(3α-3)/ (3α+3)	(3α-3)/ (9-3α)	(3-3α)/ (3α+3)	(3-3α) /(9-3α)
0.5	**- 0.3333**	-0.2	**0.3333**	0.2

$$^{\alpha}A/B = \left[\left(\frac{3\alpha-3}{3\alpha+3}\right),\left(\frac{3-3\alpha}{3\alpha+3}\right)\right]$$

$$i.e.\left(\frac{3\alpha-3}{3\alpha+3}\right) \le x \le \left(\frac{3-3\alpha}{3\alpha+3}\right)$$

$$\frac{3\alpha-3}{3\alpha+3} = x$$

$$3\alpha - 3 = 3\alpha x + 3x$$

$$3\alpha - 3\alpha x = 3 + 3x$$

$$\alpha(1-x) = 1 + x$$

$$\alpha = \frac{1+x}{1-x}$$

$$if\ \alpha = 0 \Rightarrow x = -1$$

$$if\ \alpha = 1 \Rightarrow x = 0$$

$$\frac{3-3\alpha}{3\alpha+3} = x$$

$$3 - 3\alpha = 3\alpha x + 3x$$

$$3 - 3x = 3\alpha x + 3\alpha$$

$$(1-x) = \alpha(1+x)$$

$$\alpha = \frac{1-x}{1+x}$$

$$if\ \alpha = 1 \Rightarrow x = 0$$

$$if\ \alpha = 0 \Rightarrow x = 1$$

$$Hence,\quad (A/B)(x) = \begin{cases} \dfrac{1+x}{1-x} & if\ -1 < x \le 0 \\ \dfrac{1-x}{1+x} & if\ 0 < x \le 1 \\ 0 & otherwise \end{cases}$$

mple 2: Consider fuzzy number A and B defined by,

$$A(x)=\begin{cases}\dfrac{x+4}{4} & if\ -4<x\le 0\\ \dfrac{4-x}{4} & if\ 0<x\le 4\\ 0 & otherwise\end{cases}\qquad B(x)=\begin{cases}\dfrac{x-4}{4} & if\ 4<x\le 8\\ \dfrac{12-x}{4} & if\ 8<x\le 12\\ 0 & otherwise\end{cases}$$

Calculate fuzzy numbers i) A + B ii) A - B iii) A * B iv) A / B

Solution: We know that α-cut of fuzzy set A as, ${}^{\alpha}A=\{x\in X\,/\,A(x)\ge\alpha\}$

$$A(x)\ge\alpha \qquad\qquad A(x)\ge\alpha$$

$$\frac{x+4}{4}\ge\alpha \qquad\qquad \frac{4-x}{4}\ge\alpha$$

$$x+4\ge 4\alpha \qquad\qquad 4-x\ge 4\alpha$$

$$x\ge 4\alpha-4 \text{..........(1)} \qquad\qquad 4-4\alpha\ge x \text{..........(2)}$$

From equation (1) and (2) we get, ${}^{\alpha}A=[4\alpha-4,4-4\alpha]$

We know that α-cut of fuzzy set B as, ${}^{\alpha}B=\{x\in X\,/\,B(x)\ge\alpha\}$

$$B(x)\ge\alpha \qquad\qquad B(x)\ge\alpha$$

$$\frac{x-4}{4}\ge\alpha \qquad\qquad \frac{12-x}{4}\ge\alpha$$

$$x-4\ge 4\alpha \qquad\qquad 12-x\ge 4\alpha$$

$$x\ge 4\alpha+4 \text{..........(3)} \qquad\qquad 12-4\alpha\ge x \text{..........(4)}$$

From equation (3) and (4) we get, ${}^{\alpha}B=[4\alpha+4,12-4\alpha]$

for A+B:

We know that, ${}^{\alpha}A+B={}^{\alpha}A+{}^{\alpha}B$

$${}^{\alpha}A+B=[4\alpha-4,4-4\alpha]+[4\alpha+4,12-4\alpha]$$

$${}^{\alpha}A+B=[4\alpha-4+4\alpha+4,4-4\alpha+12-4\alpha]$$

$${}^{\alpha}A+B=[8\alpha,16-8\alpha]$$

$$i.e.\ 8\alpha\le x\le 16-8\alpha$$

$$8\alpha=x \qquad\qquad 16-8\alpha=x$$

$$\alpha=\frac{x}{8} \qquad\qquad \alpha=\frac{16-x}{8}$$

$if\ \alpha=0 \Rightarrow x=0$

$if\ \alpha=1 \Rightarrow x=8$

$if\ \alpha=1 \Rightarrow x=8$

$if\ \alpha=0 \Rightarrow x=16$

$$\text{Hence,}\quad (A+B)(x)=\begin{cases} \dfrac{x}{8} & if\ 0<x\le 8 \\ \dfrac{16-x}{8} & if\ 8<x\le 16 \\ 0 & otherwise \end{cases}$$

2. for A - B:

We know that, ${}^{\alpha}A-B={}^{\alpha}A\ -{}^{\alpha}B$

$${}^{\alpha}A-B=[4\alpha-4\,,4-4\alpha]-[4\alpha+4\,,12-4\alpha]$$

$${}^{\alpha}A-B=[4\alpha-4-12+4\alpha\,,4-4\alpha-4\alpha-4]$$

$${}^{\alpha}A-B=[8\alpha-16\,,-8\alpha]$$

$$i.e.\ \ 8\alpha-16\ \le\ x\ \le -8\alpha$$

$8\alpha-16\ =x$

$\alpha\ =\dfrac{x+16}{8}$

$if\ \alpha=0 \Rightarrow x=-16$

$if\ \alpha=1 \Rightarrow x=-8$

$-8\alpha\ =x$

$\alpha=\dfrac{-x}{8}$

$if\ \alpha=1 \Rightarrow x=-8$

$if\ \alpha=0 \Rightarrow x=0$

$$\text{Hence,}\quad (A-B)(x)=\begin{cases} \dfrac{x+16}{8} & if\ -16<x\le -8 \\ \dfrac{-x}{8} & if\ -8<x\le 0 \\ 0 & otherwise \end{cases}$$

For A*B:

We know that, ${}^{\alpha}A \times B = {}^{\alpha}A \times {}^{\alpha}B$

$${}^{\alpha}A \times B = [4\alpha-4\,,4-4\alpha]\times[4\alpha+4\,,12-4\alpha]$$

$${}^{\alpha}A \times B = [\min\{(4\alpha-4)(4\alpha+4)\,,(4\alpha-4)(12-4\alpha),(4-4\alpha)(4\alpha+4),(4-4\alpha)(12-4\alpha)\},$$
$$\max\{(4\alpha-4)(4\alpha+4)\,,(4\alpha-4)(12-4\alpha),(4-4\alpha)(4\alpha+4),(4-4\alpha)(12-4\alpha)\}]$$

α	(4α-4) (4α+4)	(4α-4) (12-4α)	(4-4α) (4α+4)	(4-4α) (12-4α)
0.5	- 12	**- 20**	12	**20**

$${}^{\alpha}A \times B = [(4\alpha-4)(12-4\alpha),(4-4\alpha)(12-4\alpha)]$$

$${}^{\alpha}A \times B = [-16\alpha^2+64\alpha-48, 16\alpha^2-64\alpha+48]$$

$$i.e.\ -16\alpha^2+64\alpha-48 \le x \le 16\alpha^2-64\alpha+48$$

$$-16\alpha^2+64\alpha-48 = x$$
$$-16\alpha^2+64\alpha = x+48$$
$$\alpha^2-4\alpha = \frac{-x-48}{16}$$
$$\alpha^2-4\alpha+4 = \frac{-x-48}{16}+4$$
$$(2-\alpha)^2 = \frac{16-x}{16}$$
$$(2-\alpha) = \frac{\sqrt{16-x}}{4}$$
$$2-\frac{\sqrt{16-x}}{4} = \alpha$$
$$\frac{8-\sqrt{16-x}}{4} = \alpha$$
$$if\ \alpha=0 \Rightarrow x=-48$$
$$if\ \alpha=1 \Rightarrow x=0$$

$$16\alpha^2-64\alpha+48 = x$$
$$16\alpha^2-64\alpha = x-48$$
$$\alpha^2-4\alpha = \frac{x-48}{16}$$
$$\alpha^2-4\alpha+4 = \frac{x-48}{16}+4$$
$$(2-\alpha)^2 = \frac{x+16}{16}$$
$$(2-\alpha) = \frac{\sqrt{16+x}}{4}$$
$$2-\frac{\sqrt{16+x}}{4} = \alpha$$
$$\frac{8-\sqrt{16+x}}{4} = \alpha$$
$$if\ \alpha=1 \Rightarrow x=0$$
$$if\ \alpha=0 \Rightarrow x=48$$

$$Hence,\quad (A\times B)(x) = \begin{cases} \dfrac{8-\sqrt{16-x}}{4} & if\ -48 < x \le 0 \\ \dfrac{8-\sqrt{16+x}}{4} & if\ 0 < x \le 48 \\ 0 & otherwise \end{cases}$$

4. for A/B:

We know that, $^{\alpha}A/B = {}^{\alpha}A \; / \; {}^{\alpha}B$

$$^{\alpha}A/B = [4\alpha-4\,,4-4\alpha]/[4\alpha+4\,,12-4\alpha]$$

$$^{\alpha}A/B = \left[\min\left(\frac{4\alpha-4}{4\alpha+4}\right),\left(\frac{4\alpha-4}{12-4\alpha}\right),\left(\frac{4-4\alpha}{4\alpha+4}\right),\left(\frac{4-4\alpha}{12-4\alpha}\right),\max\left(\frac{4\alpha-4}{4\alpha+4}\right),\left(\frac{4\alpha-4}{12-4\alpha}\right),\left(\frac{4-4\alpha}{4\alpha+4}\right),\left(\frac{4-4\alpha}{12-4\alpha}\right)\right]$$

α	(4α - 4)/ (4α + 4)	(4α - 4)/ (12 - 4α)	(4 - 4α)/ (4α + 4)	(4 - 4α) /(12 - 4α)
0.5	**- 0.3333**	- 0.2	**0.3333**	0.2

$$^{\alpha}A/B = \left[\left(\frac{4\alpha-4}{4\alpha+4}\right),\left(\frac{4-4\alpha}{4\alpha+4}\right)\right]$$

$$i.e.\left(\frac{4\alpha-4}{4\alpha+4}\right)\le x \le\left(\frac{4-4\alpha}{4\alpha+4}\right)$$

$$\frac{4\alpha-4}{4\alpha+4}=x$$

$$4\alpha-4=4\alpha x+4x$$

$$4\alpha-4\alpha x=4+4x$$

$$\alpha(1-x)=1+x$$

$$\alpha=\frac{1+x}{1-x}$$

$$if\ \alpha=0\Rightarrow x=-1$$

$$if\ \alpha=1\Rightarrow x=0$$

$$\frac{4-4\alpha}{4\alpha+4}=x$$

$$4-4\alpha=4\alpha x+4x$$

$$4-4x=4\alpha x+4\alpha$$

$$(1-x)=\alpha(1+x)$$

$$\alpha=\frac{1-x}{1+x}$$

$$if\ \alpha=1\Rightarrow x=0$$

$$if\ \alpha=0\Rightarrow x=1$$

$$Hence,\quad (A/B)(x)=\begin{cases}\dfrac{1+x}{1-x} & if\ -1<x\le 0\\[2ex] \dfrac{1-x}{1+x} & if\ 0<x\le 1\\[2ex] 0 & otherwise\end{cases}$$

Examples for Practice

Example 1: Consider fuzzy number A and B defined by,

$$A(x) = \begin{cases} \dfrac{x-1}{3} & \text{if } 1 < x \le 4 \\ \dfrac{7-x}{3} & \text{if } 4 < x \le 7 \\ 0 & \text{otherwise} \end{cases} \qquad B(x) = \begin{cases} \dfrac{x-7}{3} & \text{if } 7 < x \le 10 \\ \dfrac{13-x}{3} & \text{if } 10 < x \le 13 \\ 0 & \text{otherwise} \end{cases}$$

Calculate fuzzy numbers i) A + B ii) A - B iii) A * B iv) A / B

Solution:

$$(A+B)(x) = \begin{cases} \dfrac{x-8}{6} & \text{if } 8 < x \le 14 \\ \dfrac{20-x}{6} & \text{if } 14 < x \le 20 \\ 0 & \text{otherwise} \end{cases} \qquad (A-B)(x) = \begin{cases} \dfrac{x-12}{6} & \text{if } -12 < x \le -6 \\ \dfrac{-x}{6} & \text{if } -6 < x \le 0 \\ 0 & \text{otherwise} \end{cases}$$

$$(A \times B)(x) = \begin{cases} \dfrac{-4+\sqrt{9+x}}{3} & \text{if } 7 < x \le 40 \\ \dfrac{10-\sqrt{9+x}}{3} & \text{if } 40 < x \le 91 \\ 0 & \text{otherwise} \end{cases} \qquad (A/B)(x) = \begin{cases} \dfrac{13x-1}{3(x+1)} & \text{if } \dfrac{1}{13} < x \le \dfrac{2}{5} \\ \dfrac{7-7x}{3(1+x)} & \text{if } \dfrac{2}{5} < x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Example 2: Consider fuzzy number A and B defined by,

$$A(x) = \begin{cases} \dfrac{x+2}{2} & \text{if } -2 < x \le 0 \\ \dfrac{2-x}{2} & \text{if } 0 < x \le 2 \\ 0 & \text{otherwise} \end{cases} \qquad B(x) = \begin{cases} \dfrac{x-2}{2} & \text{if } 2 < x \le 4 \\ \dfrac{6-x}{2} & \text{if } 4 < x \le 6 \\ 0 & \text{otherwise} \end{cases}$$

Calculate fuzzy numbers i) A + B ii) A - B iii) A * B iv) A / B

Solution:

$$(A+B)(x) = \begin{cases} \dfrac{x}{4} & \text{if } 0 < x \le 4 \\ \dfrac{8-x}{4} & \text{if } 4 < x \le 8 \\ 0 & \text{otherwise} \end{cases} \qquad (A-B)(x) = \begin{cases} \dfrac{x+8}{4} & \text{if } -8 < x \le -4 \\ \dfrac{-x}{4} & \text{if } -4 < x \le 0 \\ 0 & \text{otherwise} \end{cases}$$