Network Model
Of
The Constant Phase Element

Project Report

Submitted in partial Requirement for the degree of

BACHELOR OF ENGINEERING

In

INSTRUMENTATION ENGINEERING

Guided By

Dr. (Mrs.) J.M.Nair

By
Shreyas S. Bhandare
Vaibhav P. Deorukhkar
Rohan R. Dotihal
Mihir V. Ghangrekar

VIVEKANAND EDUCATION SOCIETY'S INSTITUTE OF TECHNOLOGY
COLLECTOR'S COLONY, CHEMBUR,
MUMBAI – 400074
Department of Instrumentation Engineering
University of Mumbai
Academic Year 2012-13

Vivekanand Education Society's

INSTITUTE OF TECHNOLOGY

HAMC, Collector's Colony, Chembur, Mumbai -74.

CERTIFICATE OF APPROVAL OF PROJECT WORK

This is to certify that Mr.______________________________________ has satisfactorily carried out the project work entitled **NETWORK MODEL OF THE CONSTANT PHASE ELEMENT** in partial fulfilment of the B.E Degree in **INSTRUMENTATION ENGINEERING** of The University of Mumbai, Maharashtra state during the academic year 2012-13.

PRINCIPAL

HEAD OF DEPT.

PROJECT EXAMINER

PROJECT GUIDE

APPROVAL SHEET

DEPARTMENT OF INSTRUMENTATION ENGINEERING
V.E.S. INSTITUTE OF TECHNOLOGY
UNIVERSITY OF MUMBAI

PROJECT NAME: Network Model of the Constant Phase Element

PROJECT REPORT BY: Shreyas S. Bhandare
Vaibhav P. Deorukhkar
Rohan R. Dotihal
Mihir V. Ghangrekar

SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OFBACHELOR OF ENGINEERINGININSTRUMENTATION ENGINEERING 2012-2013

INTERNAL GUIDE: **DR. (MRS.) J.M.NAIR** ________________________

EXTERNAL GUIDE : ________________________

INTERNAL EXAMINER: ________________________

EXTERNAL EXAMINER: ________________________

H.O.D.: **DR. P.P. VAIDYA** ________________________

PRINCIPAL: **DR. (MRS.) J. M. NAIR** ________________________

ACKNOWLEDGEMENTS

Throughout the course of this project, we gathered valuable insight and knowledge which would have not been possible without the help of these people who have shown immense faith in our abilities.

We would like to thank our project guide **Dr. (Mrs.) J.M Nair** for her constant encouragement, innovative ideas, honest feedback and words of motivation. Crucial points and important enhancements to the project would not have been possible without his constant support.

We are thankful to our Head of Department **Dr.P.P. Vaidya** for giving us this wonderful opportunity to work on this project under the Department of Instrumentation. We are also thankful to **Ms. Nidhi Saxena**, student of ME Instrumentation & control for her continuous guidance.

We take this opportunity to appreciate the valuable insight and ideas provided by **Dr. Shantanu Das**, Scientist, Reactor Control Division, BARC, Mumbai.

We would like to thank the Instrumentation department for their support and appreciation. Finally, we express a deep gratitude to all our lab staff, classmates and friends who have provided us with constant encouragement and support throughout the development of this project.

Abstract

Fractional calculus is three centuries old as the conventional calculus, but not very popular among science and/or engineering community. The beauty of this subject is that fractional derivatives (and integrals) are not a local (or point) property (or quantity). Thereby this considers the history and non-local distributed effects. In other words, perhaps this subject translates the reality of nature better!

Analysis of fractal systems (i.e. systems described by fractional differential equations) necessitates creating an analog model of a crucial subsystem called Constant Phase Element (CPE).

This Project describes a possible realization of constant phase element model that is quite simple and in spite of its simplicity makes it possible to simulate the properties of ideal CPEs. The CPE model hardly requires special components and can be easily implemented.

The project also deals with the effect of the component tolerances on the resultant responses of the model and describes several typical model applications.

The applications typically include fractal systems and fractional order PD controllers which has to be designed using CPE.

The phenomenons described by such equations are frequent for instance in biochemistry or biomedicine, in electrochemistry, in modern control techniques, in acoustic or optical signal processing and in many other practical domains.

The solution of many practical problems can be simplified with the help of electrical or electronic analog models. Formulation of effective models of fractional systems is the substance of this project.

Thus the project work deals with the actual software and hardware implementation of the realization of CPE for $\Phi = -30°$ and $\Phi = -40°$.

TABLE OF CONTENTS

INTRODUCTION

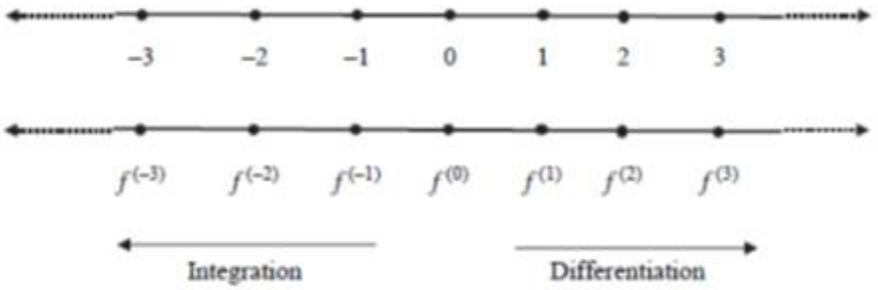
1.1 Introduction to Fractional Calculus

Fractional calculus is three centuries old as the conventional calculus, but not very popular among science and/or engineering community. The beauty of this subject is that fractional derivatives (and integrals) are not a local (or point) property (or quantity). Thereby this considers the history and non-local distributed effects. In other words, perhaps this subject translates the reality of nature better![2]

$$\frac{D^n f(x)}{Dx^n}$$

What would the result be if $n = ½$?

Simple number line representation to show how differentigrals are achieved:



Number line and Interpolation of the same to differintegrals of fractional calculus

$$\leftarrow \ldots, \int dt \int dt \int f\,dt, \int dt \int f\,dt, \int f\,dt, f$$

Writing the same in differintegral notation as represented in number line we have

$$\leftarrow \ldots \frac{d^{-3}f}{dt^{-3}}, \frac{d^{-2}f}{dt^{-2}}, \frac{d^{-1}f}{dt^{-1}} f, \frac{df}{dt}, \frac{d^2 f}{dt^2}, \frac{d^3 f}{dt^3}, \ldots \rightarrow$$

If we want derivative of an order of 2.5, we can differentiate the function 3 times and then integrate 0.5 times. Similarly we can achieve the same result by integrating the function 0.5 times and differentiating it 3 times. Thus the order in which integration or differentiation is performed does not matter.[2]

1.2 A Thought Experiment to understand fractance

From an aircraft, we can see the city roads and observe the vehicular traffic movement. The vehicle seems to move in a straight line. Therefore, as an observer, we draw the velocity curve by simple one-order integer derivative of displacement and find that it maps a straight line. In Fig given below, the pair of straight line gives the velocity trajectory of the upstream vehicle and downstream vehicle, as observed in macroscopic scale. [2]

The same vehicle when looked with enlarged view tells us its continuous movement but to avoid road heterogeneity it travels in zigzag fashion. The curve in the lower frame of Figure maps this picture. Here the scale is enlarged. The velocities for upstream and downstream vehicles are not pair of straight lines, but follow a continuous, nowhere differentiable curve. So will the dx/dt give the true picture of velocity or will it be d1+αx/dt1+α, where $0 < \alpha < 1$, give the representation of the actual zigzag pattern is the thought experiment. Now the question about the dimensions of velocity, in the thought experiment when defined as fractional derivative of displacement, is the matter of another thought. In the present understanding, as per uniform time scales, the quantity dx/dt is velocity, and d^2x/dt^2 is the acceleration; however, the quantification of $d^{1.23}x/dt^{1.23}$ is hard to visualize. [2]

This fractional differentiation is in between velocity and acceleration, perhaps a velocity in some transformed time scale, which is non-uniform-enriching thought for physical understanding of fractional quantities. The nature of zigzag pattern shown is somewhat called fractal curve, actually a continuous and nowhere differentiable function. The relation of fractal dimensions and fractional calculus is an evolving field of science at present. The macroscopic view presented above gives a thought of explanation of discontinuity and singularity formations in nature, in classical integer order calculus. Can fractional calculus be an aid for explanation of discontinuity formation and singularity formation is an enriching thought experiment. [2]

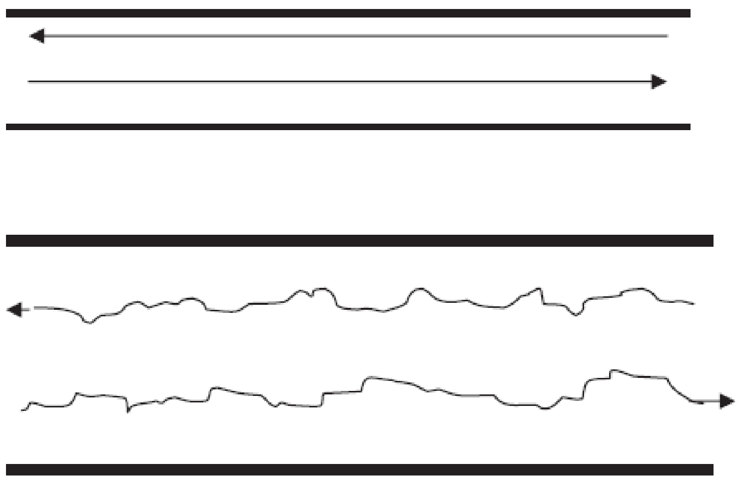


Fig.1 Macroscopic and Microscopic view of vehicle moving on road

1.3 Observation of Fractional Calculus in Physical System

Many systems are known to display fractional order dynamics. Probably the first physical system to be widely recognized as one demonstrating fractional behaviour is the semi-infinite lossy (*RC*) transmission line. The current into the line is equal to the half derivative of the applied voltage. That is, impedance is

$$V(s) = \frac{1}{\sqrt{s}} I(s)$$

Some of the other examples in which fractional calculus is observed are,

1) Diffusion model in electrochemistry:

The important aspect is to find the concentration of electro active species near the electrode surface. The characteristic describing function was found to be,

$$\boxed{m(t) = {}_0D_t^{-0.5} i(t)}$$

Where,

i(t) : Current generated by the charge passed.

m(t) : Characteristic intermediate between i(t) and charge passed q(t)

This equation is uses half integral of current.

2) Viscoelasticity (Stress–Strain)

Stress relaxation and creep behaviour in stress–strain relationship are well described by fractional order models. A stress–strain law for viscoelastic materials is described as,

$$\boxed{\varepsilon(t) = \frac{1}{K} D_t^{-v} \sigma(t)}$$

Where,
ε: strain, σ: stress, K: material constant

When v= 0, : Elastic solid material
v= 1, : Viscous Liquid material

When ‘v’ is any fraction, the material is Viscoelastic.

3) Temperature–Heat Flux Relationship for Heat Flowing in Semi-infinite Conductor

Heat flow in semi-infinite conductor being one-dimensional, we can now relate the conduction heat flux through semi-infinite thermocouple wire to the body temperature at the origin by using the derived equation:

$$\frac{k \times_a D_t^{1/2} \times [T_{surf}(t) - T_0]}{\sqrt{(\alpha)}}$$

Where,

k = co-eff. of heat conduction,

α = ratio of co-efficient of heat conduction to the product of specific heat and density of the conductor.

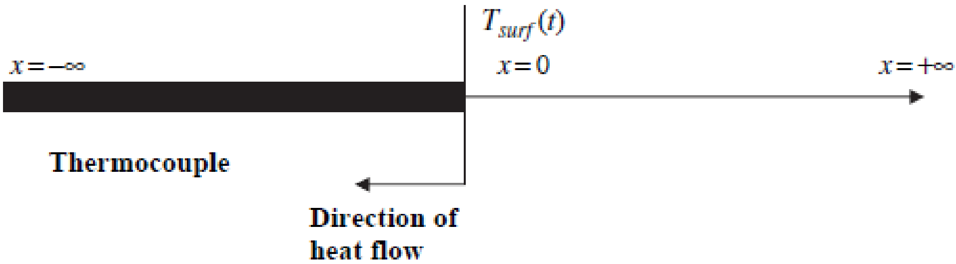


Fig.2 Heat flow in semi-infinite wire thermocouple

In a Single Thermocouple Junction Temperature in Measurement of Heat Flux using the equation :

$$T_b(t) = \frac{D_t^{-1}(Q_i(t) - Q_1(t) - Q_2(t))}{mc}$$

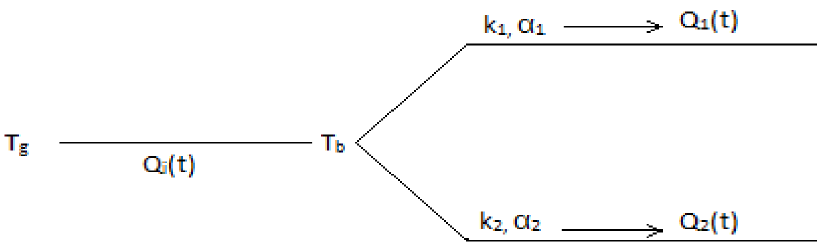


Fig.3 Thermocouple junction for temperature (heat flux) measurement

$T_b = [T_{surf}(t) - T_0]$; $Q_i(t)$ = heat flux from input to tip; $Q_1(t)$ = heat flux at tip1; $Q_2(t)$ = heat flux at tip 2.

1.4 The Constant Phase Element

The term CPE corresponds to **C**onstant **P**hase **E**lement. Analysis of fractal systems necessitates creating an electrical analog model of a crucial subsystem called Constant Phase Element. [1]

In other words The Constant Phase Element (CPE) is a non-intuitive circuit element that was discovered (or *invented*) while looking at the response of real-world systems. [4]

During the investigations in many scientific disciplines, it proved that the mentioned classical methods do not suffice. There exist phenomena that are not accurately depicted and therefore it is necessary to apply differential equations with derivatives and integrals of non-integer orders. These are fractional differential equations and the question is how to solve them. [1]

The phenomena described by such equations are frequent for instance in biochemistry or biomedicine, in electrochemistry, in modern control technique, in acoustic or optical signal processing and in many other practical domains. [1]

In CPE the phase angle is constant, independent of the frequency within wide range of frequency band. [1]

What is the need of CPE?

Implementation of fractance:

Implementation of fractance is the most important reason for CPE need. The fractance can be implemented only if we have CPE which gives us approximated fractance. [1]

Solution of fractional differential equations:

If the fractional equations are sufficiently simple then it is possible to solve them with the help of special functions. In case of linear systems the solution may be based very effectively on Laplace transformation, namely on numerical methods of inversion. For solution of nonlinear systems there exist numerical methods of integration. The solution of many practical problems can be simplified with the help of electrical or electronic analog models which are nothing but CPE. [2]

LITERATURE REVIEW

2.1 IDEAL CPE

The impedance of an ideal CPE is defined as,

$$\boxed{\mathbf{Z(s) = Ds^{\alpha}}}$$

For s=jw,

$$Z'(jw) = D(jw)^{\alpha} = Dw^{\alpha}j^{\alpha} = Dw^{\alpha}e^{i\Phi} = Dw^{\alpha}(\cos\varphi + j\sin\varphi)$$

$\varphi = \alpha\pi/2$ for φ in radians

$\varphi = 90\alpha$ for φ in degrees

The exponent α decides the character of the impedance Z(s). If $\alpha = +1$, it is a classical inductive reactance, $\alpha = 0$ means a real resistance or conductance, $\alpha = -1$ represents a classical capacitive reactance.

The values $0 < \alpha < 1$ corresponds to a fractal inductor, the value $-1 < \alpha < 0$ to a fractal capacitor. This project is devoted predominantly to the fractal capacitor.

The modulus of impedance Z(jw) depends on frequency according to the magnitude of α. Its value in decibels varies with 20α decibels per decade of frequency and in correspondence with the sign of α, the modulus increases or decreases.

At w= 1 the modulus equals D with no respect to α. Argument φ of the impedance is constant, frequency independent.

It is evident that the properties of ideal CPE cannot be realized with classical electrical networks containing a finite number of discrete R, C, L components. On the other hand, it is possible to build networks that approximate the CPE in a reasonable way. The design starts with s^{α} approximation by an infinite series or chain fraction and necessitates partial fraction decomposition together with calculation of denominator roots and even demands optimization steps. The resultant network is very complicated, may demand components with negative parameters or a number of active elements. Therefore, there is a need to find a simple design of a reasonably accurate CPE model.

2.2 BASIC PARALLEL NETWORK MODEL

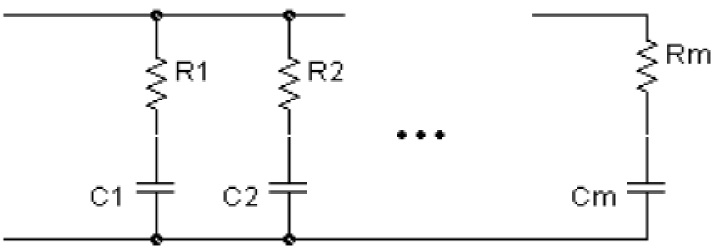


Fig.4 Basic network model.

The network consists of altogether m series RC branches connected in parallel. The resistances and Capacitances in branches form a geometric sequence. [1]

$R_k=R_1a^{k-1}$, $C_k=C_1b^{k-1}$, k=1,2,....,m

here, $0<a<1$ $0<b<1$

Input impedance of the network $Z(jw)=1/Y(jw)$, where $Y(jw)$ is input admittance equal to the sum of admittances of individual branches [1]

$$Y(j\omega)=\sum_{k=1}^{m}\frac{j\omega b^{k-1}C_1}{1+j\omega(ab)^{k-1}R_1C_1}.$$

The modulus of input impedance in decibels equals

$$Z_{dB}(\omega)=20\log|Z(j\omega)|$$ [1]

and argument in degrees

$$\varphi(\omega)=\frac{180}{\pi}arctg\frac{imag(Z(j\omega))}{real(Z(j\omega))}.$$

A typical example of both characteristics for the case R = 1, C = 1, m = 10, a = 0.25, b = 0.48 is shown in Fig. 5

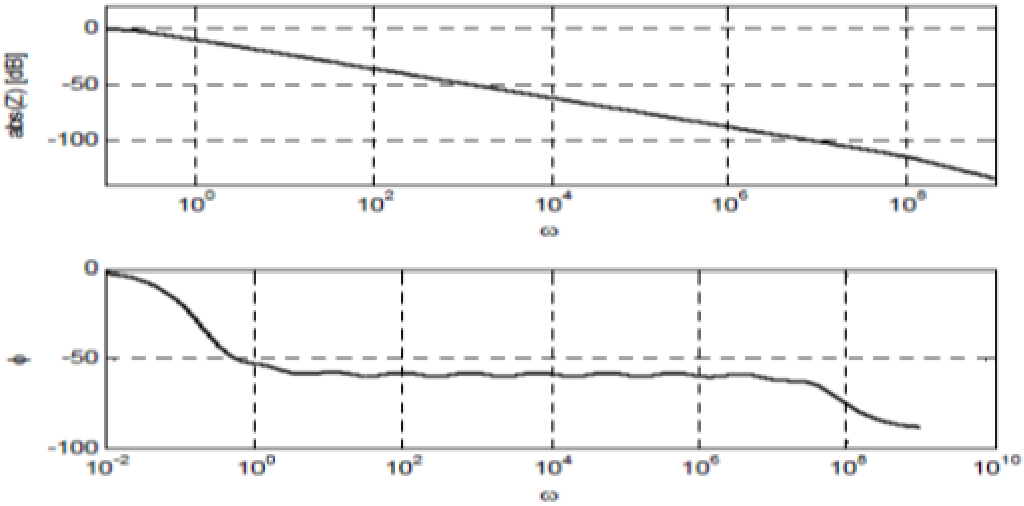


Fig. 5 Modulus and argument as function of frequency.

The figures show that under the presented conditions the network is able to simulate the ideal CPE at least in a limited band of frequencies approximately from $\omega d = 10^1$ to $\omega h = 10^7$. The modulus decreases with the rate -13 dB per decade and for $\omega = 1$ it equals -10 dB ($D = 10^{-0.5} = 0.316$). The argument has average value $\varphi_{av}= -59°$ and oscillates around it with amplitude $\Delta\varphi = \pm1°$.

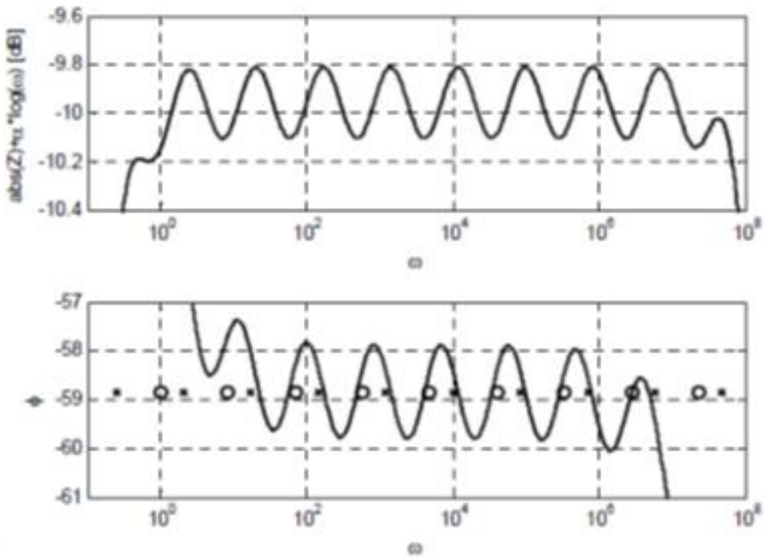


Fig. 6 Details of modulus and phase.

The modulus response in the original scale does not clearly show the ripple around the average curve. After adding the component with corresponding slope we get clearer idea about the real situation. The detailed shapes of both the responses are shown in Fig. 6

The a and b parameters, as will be shown later, determine φ_{av}, $\Delta\varphi$, ω_d, ω_h and D. Under the supposition of a sufficient number m of sections it is possible to derive easily the basic relations for the network properties in the frequency range of interest. The reasoning starts with the asymptotic Bode characteristics.

The modulus characteristic of the k-th section consists of a part with the slope of 20dB/decade and of another part with zero slope. The breaking point lies at frequency equal to to $w_k = z_k = 1/R_kC_k$. The breaking point of the following section lies at $w_{k+1} = w_k/(ab)$. At the logarithmic scale the breaking frequencies are equidistant. The just described situation is periodically repeated.

The resultant admittance equals the sum of individual admittances.

$$Y(s) = \sum_{k=1}^{m} \frac{sb^{k-1}C_1}{1 + s(ab)^{k-1}R_1C_1} .$$

The input impedance Z(s) = 1/Y(s) has zeros at the breaking points

$$z_k = \frac{1}{R_1C_1(ab)^{k-1}}, \; k = 1, \ldots, m$$

and poles

$$p_k = \frac{z_k}{b}, \quad z_{k+1} = \frac{p_k}{a}.$$

An example of zeroes and poles distribution is shown in figure (6), zeroes denoted by circle and poles by crosses.

For s = *jw*

$$Y(j\omega) = \sum_{k=1}^{m} \frac{j\omega b^{k-1} C_1}{1 + j\omega (ab)^{k-1} R_1 C_1}, \quad Z(j\omega) = 1 / Y(j\omega).$$

Argument φ(w) of *Z*(jw) in degrees is

$$\varphi(\omega) = \frac{180}{\pi} arctg(imag(Z) / real(Z))$$

Its average value can be obtained as

$$\varphi_{av} = 90\alpha = 90 \frac{\log a}{\log a + \log b} = 90 \frac{\log a}{\log(ab)}$$

The phase characteristic passes values φ_{av} *in* the points of

$$\omega_{av}(k) = z_k \left(\frac{a}{b}\right)^{1/4}$$

and reaches its extremes at

$$\omega_{\varphi \min}(k) = z_k \sqrt{a} \quad \text{or} \quad \omega_{\varphi \max}(k) = p_k \sqrt{b} = \frac{z_k}{\sqrt{b}}.$$

Evidently it holds

$$\omega_{av} = \sqrt{\omega_{\varphi \min} \omega_{\varphi \max}}$$

the frequencies w_{av} are in the middle (on logarithmic scale) between the frequencies of extremes. The values $w\varphi$min and $w\varphi$max are simultaneously frequencies where the modulus has its average value equal to that of the ideal CPE.

2.3 The Principle of the Optimal Model

The preceding section 2.3 presented a model with characteristics simulating well those of an ideal CPE. The model however necessitates a relatively high number of sections to secure a good approximation in the given frequency range. Fig. 7 demonstrates that if $m < 10$ the results are not satisfactory at all. [1]

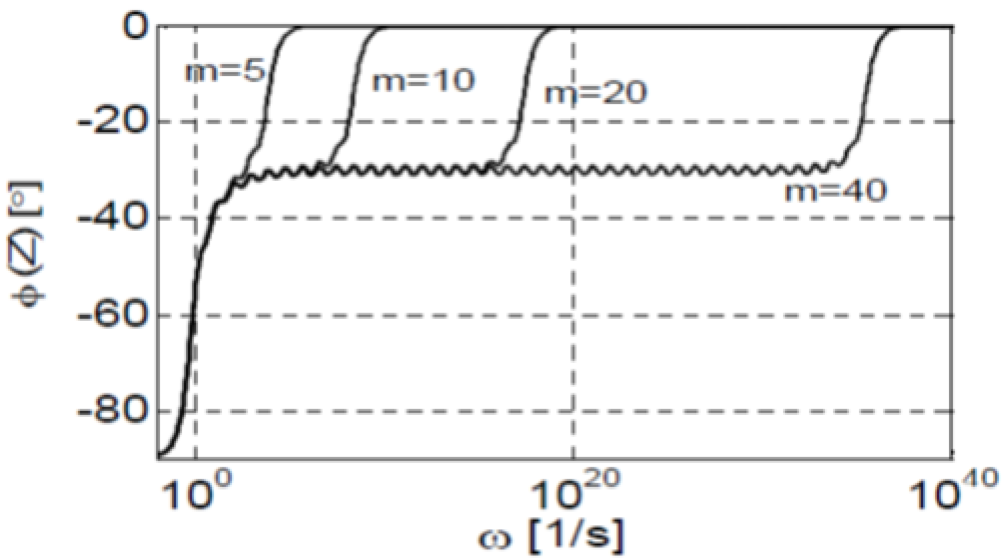


Fig. 7 Phase responses with different number of sections.

We shall show a possibility how to reduce the complexity of the model and preserve at the same time its required good qualities. [1]

Fig. 8a shows that at a certain frequency different sections contribute to the total input current (and to resultant input admittance) with different weight. The resultant input current is determined by a limited number of sections. Fig. 8b and 8c make it clear that at both ends of the frequency range the normal operating conditions are not satisfied since the necessary sections are missing. At the lower end they are the sections with indexes $k < 1$, at the upper end the sections $k > m$. For the model to operate correctly it is necessary to substitute the missing sections (we suppose an infinite number of them at both ends of the frequency range) by a simple network. The missing sections at the lower end have large time constants. Their capacitances can thus be neglected and the conductances added. As the values of conductances form a geometric sequence the infinite sum of them can be substituted with a single

$$G_p = \frac{1}{R_1} \sum_{k=1}^{\infty} a^k = \frac{1}{R_1} \frac{a}{1-a}$$

A single resistor at lower end and single capacitor at upper end is given by

$$R_p = R_1 \frac{1-a}{a}. \qquad C_p = C_1 \frac{b^m}{1-b}.$$

[1]

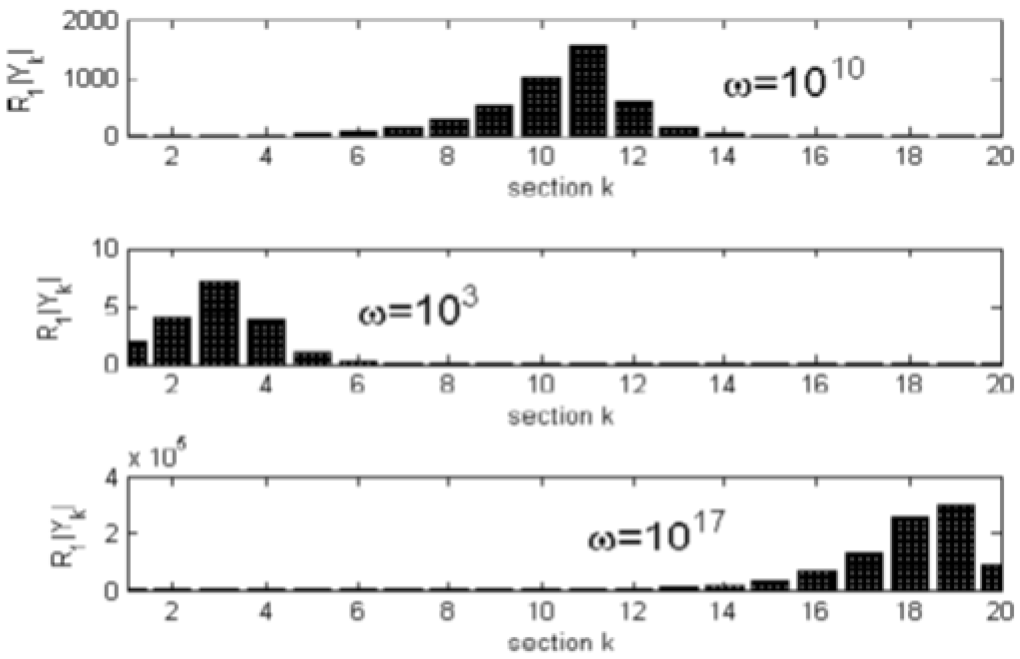


Fig. 8 Relative contribution of individual sections to the total current: a. Middle frequency, b. Low frequency, c. High frequency

The resultant electrical scheme of the model is shown in fig. 9

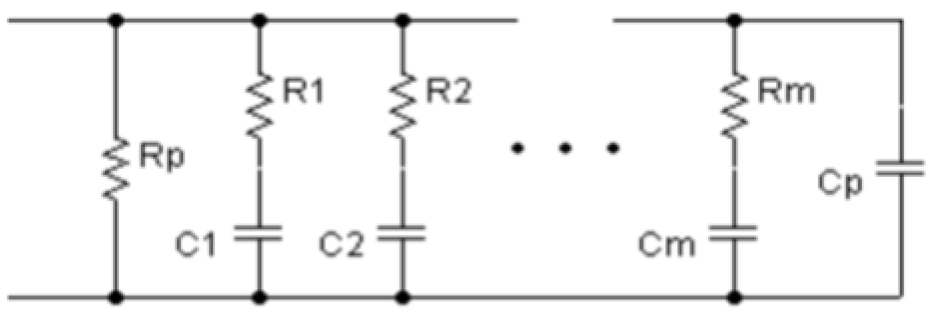


Fig. 9 Resultant scheme with correcting elements R_p and C_p.

The substitute is not perfect since the original parallel combinations of series *RC* sections were replaced by two single components. The resultant characteristics are however much better than those of the original model without the corrective elements.

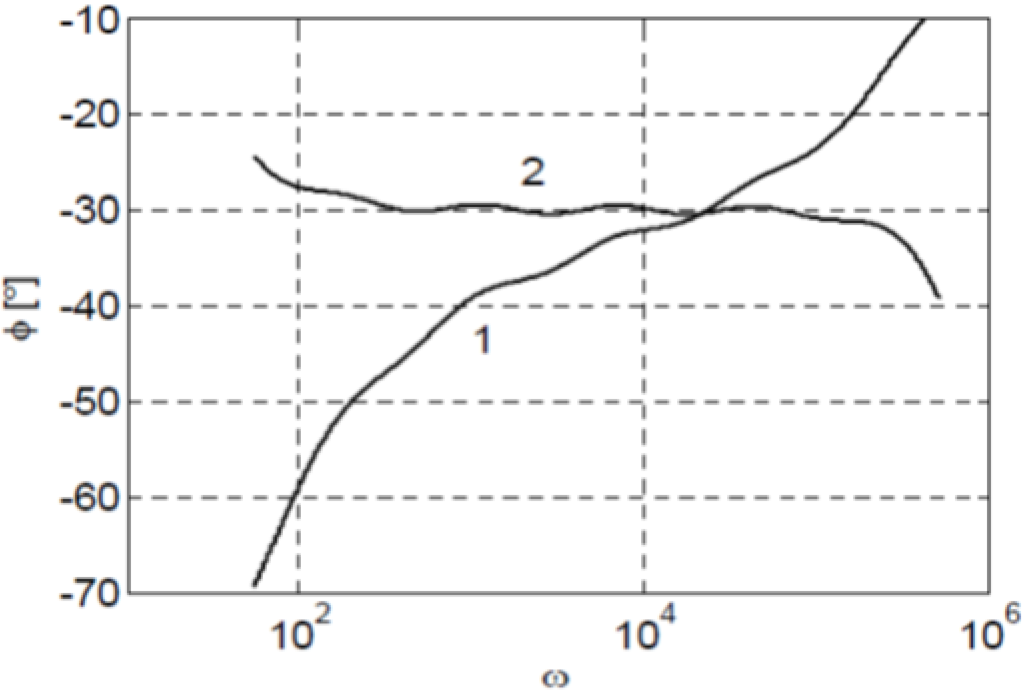


Fig. 10 Comparison of phase responses; uncorrected model (curve 1), model with correction elements (curve 2).

Fig. 10 compares the phase responses of the model with m = 5, φ = -30° without and with correction. Obviously, the original model is not applicable while that with correcting elements is quite good. The phase is virtually constant over the frequency range covering nearly 3 decades. The model contains only 6 resistors and 6 capacitors.

The frequency band may be easily extended by adding further sections and recalculating the capacitance *Cp*.

	ideal	uncorrected	corrected
1	-0.54288	0	-0.49213
2	-3.3931	-2.5885	-3.3760
3	-21.207	-18.806	-21.198
4	-132.55	-127.18	-132.57
5	-828.42	-866.16	-832.57
6	-5177.7		-8814.8

The effect of correcting elements can be understood even from Tab. 1, showing the positions of poles of input impedance (the positions of zeroes are not affected). The table confirms that the correction substantially contributes to the ideal pole positions

2.3.1 Design Procedure

The graph in Fig. 11, obtained as a result of computer simulations, shows how phase φ and ripple *Δφ* depend on basic *a* and *b* parameters. It is evident that at least between *φ* = -30° and *φ* = -60° the ripple is connected with the product *ab*. In the figure, three important cases are depicted: for *Δφ* = ±0.5° the product *ab* = 0.160 (asterisks in Fig. 8), for *Δ*φ = ±1° , *ab* = 0.12 (crosses) and for *Δφ* = ±2° *ab* = 0.08 (dots). [1]

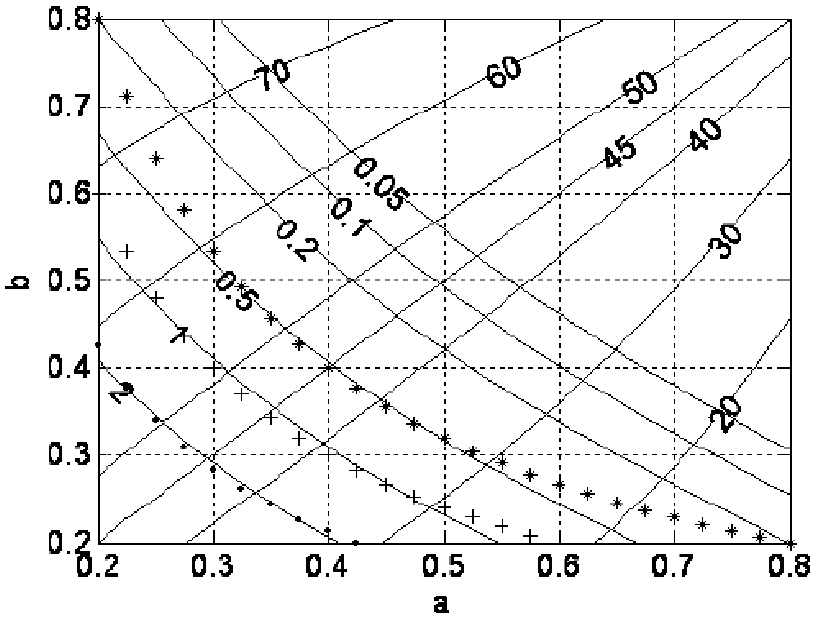


Fig.11 Dependence of phase, *φ* and ripple *Δφ* on parameters a and b.

The design of the model starts with the given $\tau_1 = R_1C_1$, φ_{av}, $\Delta\varphi$, m, Dp.

Time constant τ_1 together with the number m approximately determines both end frequencies

$$\omega_d = \frac{1}{\tau_1}, \quad \omega_h = \frac{1}{\tau_1 (ab)^m} = \frac{\omega_d}{(ab)^m}.$$

Allowable ripple $\Delta\varphi$ leads to

$$ab \cong \frac{0.24}{1+\Delta\varphi}$$

Where Δφ is in degrees. Starting with

$$\alpha = \varphi / 90,$$

$$\log a = \alpha \log(ab), \quad a = 10^{\log a}, \quad b = ab / a.$$

[1]

We determine the values of resistors and capacitors in sections

$$R_k = R_1 a^{k-1}, \quad k = 1, 2, \ldots m$$

$$C_k = C_1 b^{k-1}, \quad k = 1, 2, \ldots m$$

The correction elements are

$$R_p = R_1 \frac{1-a}{a}, \quad C_p = C_1 \frac{b^m}{1-b}.$$

For the chosen values of R_1 and C_1 we get the input admittance

$$Y(j\omega_{av}) = \frac{1}{R_p} + j\omega_{av} C_p + \sum_{k=1}^{m} \frac{j\omega_{av} C_k}{1 + j\omega_{av} R_k C_k}$$

at the some of the frequencies of phase extremes,

$$\omega_{av} = z_k \sqrt{a} = \frac{1}{R_1 C_1 (ab)^{k-1}} \sqrt{a},$$

$$k = \text{int}(m/2)$$

Since the slope of the modulus is proportional to α we get the modulus D at $w = 1$ as

$$D = Z_{av} \omega_{av}^{-\alpha}$$

Where,

$$Z_{av} = \frac{1}{|Y(j\omega_{av})|}$$

is modulus at $w = w_{av}$.

The obtained value of *D* will generally differ from the required *Dp*. Therefore, all values of resistances in sections *Rk* and *Rp* have to be multiplied by ratio *Dp / D* and all capacitances divided by the same ratio. Time constant $\tau_1 = R_1C_1$ and limits of the frequency range remain unchanged. [1]

A fractance element was developed using this method for Φ = -30°, Φ = -40° and Φ = -60°

2.4 Alternative Series Model

For some applications, another model in Fig. 9 may be more advantageous. Its input impedance is given by, [1]

$$Z(j\omega) = R_s + \sum_{k=1}^{m} \frac{R_k}{1 + j\omega R_k C_k} + \frac{1}{j\omega C_s},$$

With resistances and capacitances in individual parallel RC circuits [1]

$$R_k = R_1 a^{k-1}, \; C_k = C_1 b^{k-1},$$

And correcting elements

$$R_s = R_1 \frac{a^m}{1-a}, \; C_s = C_1 \frac{1-b}{b}.$$

The argument response of this model reasonably corresponds to that of the previous variant. [1].

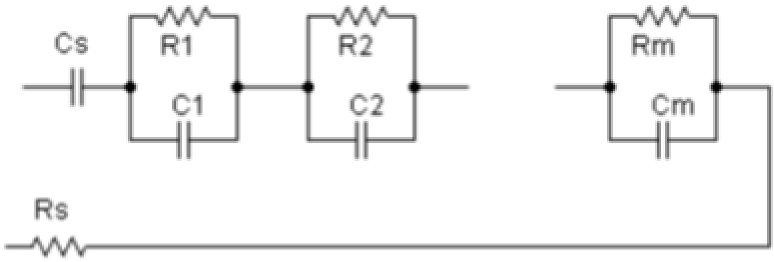


Fig.12 Scheme of a series model.

2.5 Effect of Component Tolerances on the Model Responses

When realizing the network model in practice one necessarily meets the problem with component tolerances. The calculated component values will unavoidably differ from the stock values delivered in standard series and the influence of environmental and temperature changes has also to be taken into consideration. [1]

For objective evaluation of such effects, the calculation of sensitivities serves best. [1]

Absolute sensitivity of the input impedance $Z_{in} = U_{in}/I_{in}$ to the variation of admittance Y_q is defined as [1]

$$S(Z_{in}, Y_q) = \frac{\partial Z_{in}}{\partial Y_q} \quad .$$

The so-called semi-relative sensitivity (change of Z_{in} caused by a relative change of Y_q) is

$$S_{\mathrm{r}}(Z_{in}, Y_q) = \frac{\partial Z_{in}}{\dfrac{\partial Y_q}{Y_q}} = Y_q S(Z_{in}, Y_q)$$

The partial derivative can be obtained as negatively taken product of transfer impedance from input to the terminals of Y_q and transfer impedance taken in reverse direction. Due to the reciprocity of the CPE model, the two named transfer impedances equal each other and their product equals the transfer impedance Z_t from input to Y_q squared.[1]

$$S_{Y_q}(Z_{in}, Y_q) = \frac{\partial Z_{in}}{\dfrac{\partial Y_q}{Y_q}} =$$

$$= Y_q S(Z_{in}, Y_q) = -Y_q (Z_t)^2$$

and the sensitivity of phase to Y_q will be

$$S_{Y_q}(\varphi, Y_q) = \frac{\partial \varphi}{\dfrac{\partial Y_q}{Y_q}} = imag\left(\frac{S_{Y_q}(Z_{in}, Y_q)}{Z_{in}}\right).$$

Fig. 13 shows an example of semi-relative sensitivities of phase to component values for the case m = 5, $R_1 = 10$ kΩ, $C_1 = 1$ μF. Evidently, a 10% tolerance of any component will not cause (in the frequency band of interest) greater phase error than about 1 degree.

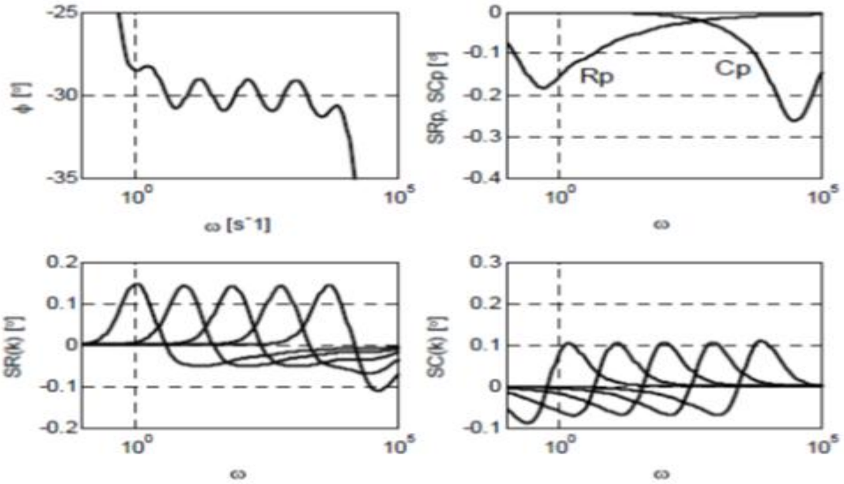


Fig. 13 Changes of phase φ caused by component variations by 1%. a. Phase, b. Variation of R_p, C_p. c. Variation of R_k. d. Variation of C_k.

CPE DESIGN AND SIMULATION

3.1 CPE Design for m = 1 to 10 ($\Phi = -30°$)

R1 = 10kΩ
C1 = 1µF
τ = 10ms
$\Phi = -30°$
$D_p = 10^4$
m = 5
$\Delta\Phi = 0.5$

$$ab = \frac{0.24}{1 + \Delta\Phi} = 0.16$$

$$a = 10^{\alpha \log(ab)} = 0.54288$$

$$b = \frac{ab}{a} = 0.29472$$

$$R_p = \frac{R_1(1 - a)}{a} = 8420.1\ \Omega$$

$$C_p = \frac{C_1 b^m}{1 - b} = 3.1529\text{nF}$$

$$\omega_{av} = (a/b)^{0.25} \times \frac{1}{R_1 C_1 (ab)^2} = 4550.8 \text{ rad/sec}$$

For m = 5

a = 0.54288

b = 0.29472

K = 1;	99.95µ + j (2.196µ)
K = 2;	181µ + j (24.9µ)
K = 3;	195.3µ + j (167.6µ)
K = 4;	20.9µ + j (112.5µ)
K = 5;	994n + j (34.28µ)

Therefore,

$$\sum_{k=1}^{m} \frac{j\omega_{av} C_k}{1 + j\omega_{av} R_k C_k} = 498.14\ \mu + j(341.47\ \mu)\ [1]$$

$Y(j\omega_{av}) = 616.90\mu + j\ (355.81\mu)$

$|Y(j\omega_{av})| = 712.156\mu$

$Z_{av} = [Y(j\omega_{av})]^{-1} = 1.404k\Omega$

$$D = Z_{av}\,\omega_{av}{}^{-\alpha} \quad [1]$$

$D = 23276.20$

$D_p/D = 0.4296$

New values of $R_1, R_2, \ldots, R_{10}$ & $C_1, C_2, \ldots, C_{10}$

$R_1 = 4296\Omega$	$C_1 = 2.33\mu F$
$R_2 = 2333\Omega$	$C_2 = 686nF$
$R_3 = 1266\Omega$	$C_3 = 202nF$
$R_4 = 687.7\Omega$	$C_4 = 59.6nF$
$R_5 = 373.3\Omega$	$C_5 = 17.60nF$

$R_P = 3619\Omega$	$C_P = 7.34nF$

$w_d = 1/R_1C_1 = 99.86$ rad/sec $= 15.89$Hz

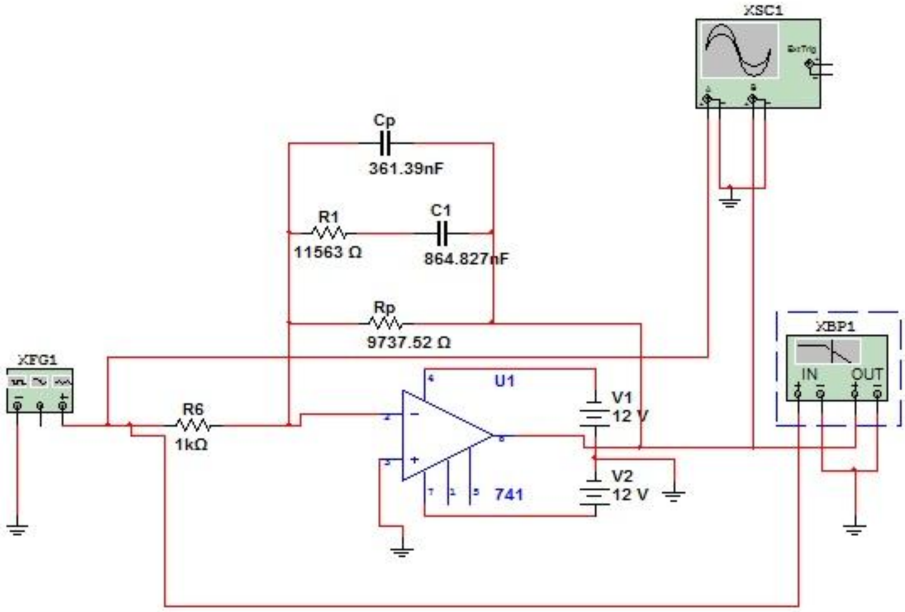
$w_h = w_d/(ab)^m = 952.22 \times 10^3$ rad/sec $= 152$kHz

Expected Observations from the calculations:

Thus, the phase response should be constant at -30° within frequency range of approximately 16Hz to 150 kHz.

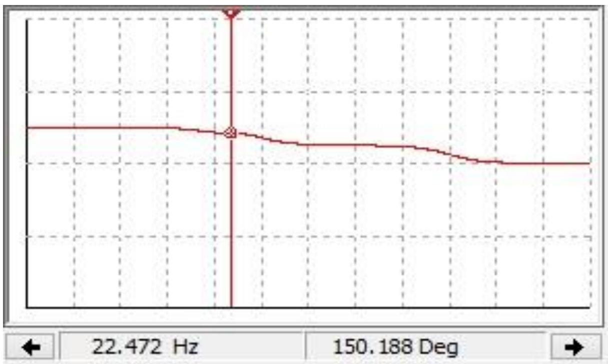
Magnitude response should show decrease in the gain with the constant slope of approximately $-20x\alpha = 6.67$dB/decade within frequency range of approximately 16Hz to 150 kHz.

Φ = -30° m =1



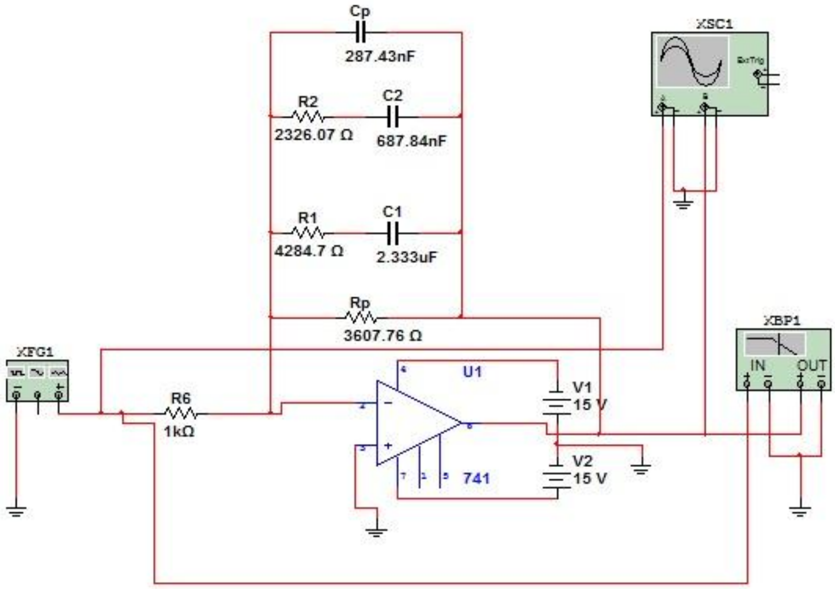
Circuit diagram

Magnitude plot



Phase plot

Φ = -30° m =2

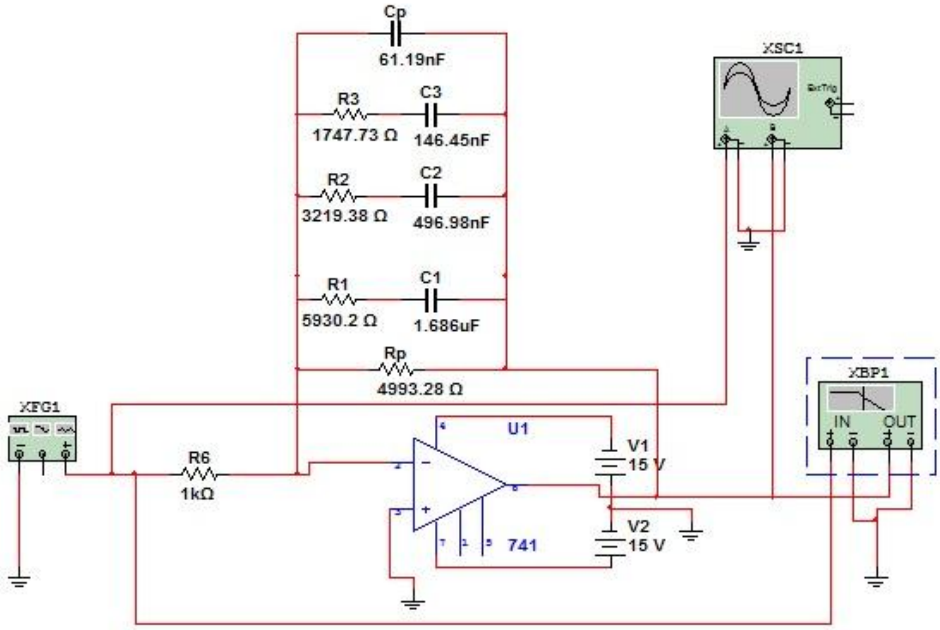


Circuit diagram

Magnitude plot

Phase plot

Φ = -30° m =3

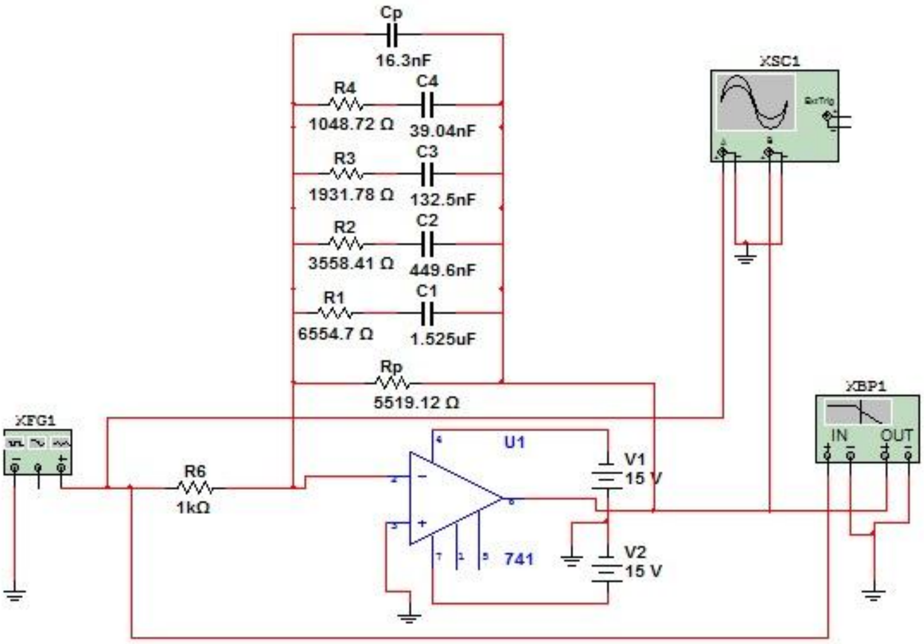


Circuit diagram

Magnitude plot

Phase plot

Φ = -30° m =4

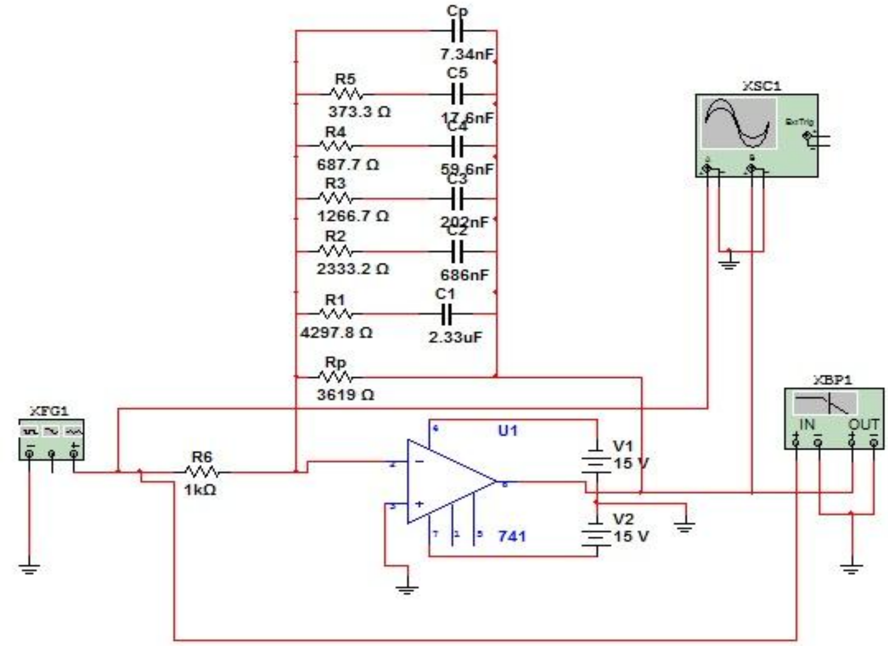


Circuit diagram

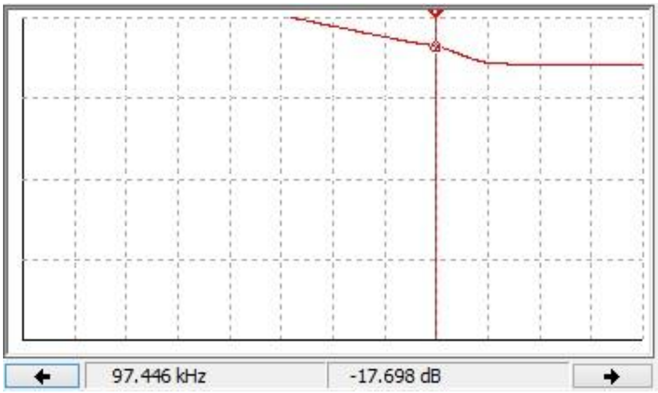
Magnitude Plot

Phase plot

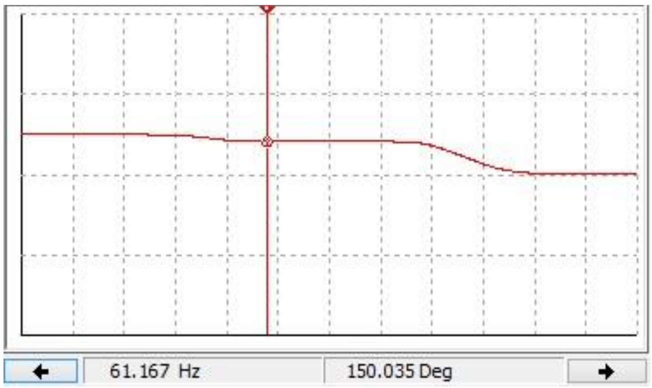
Φ = -30° m =5



Circuit diagram

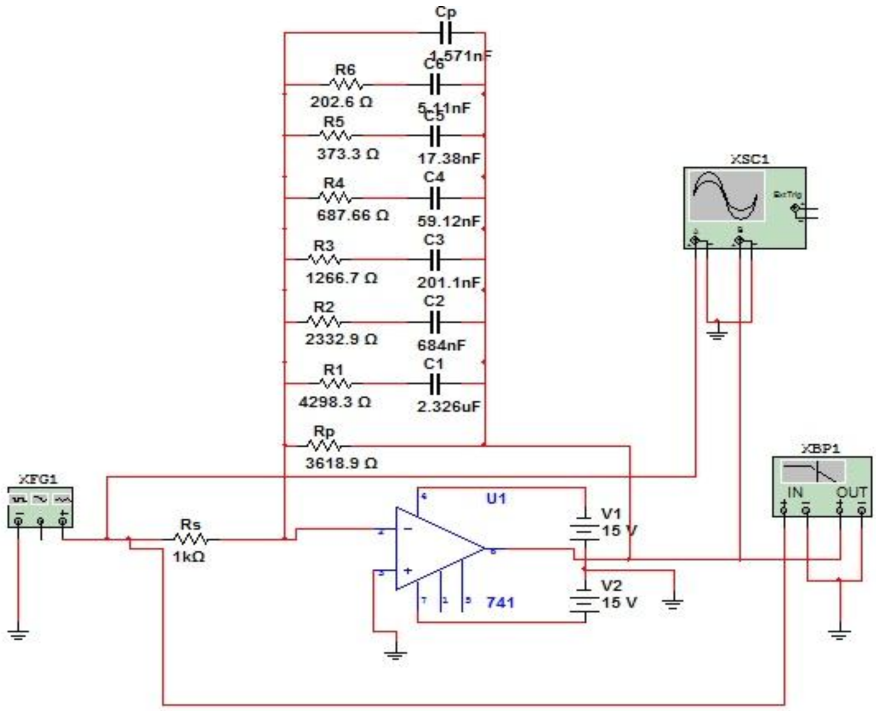


Magnitude plot



Phase plot

Φ = -30° m =6

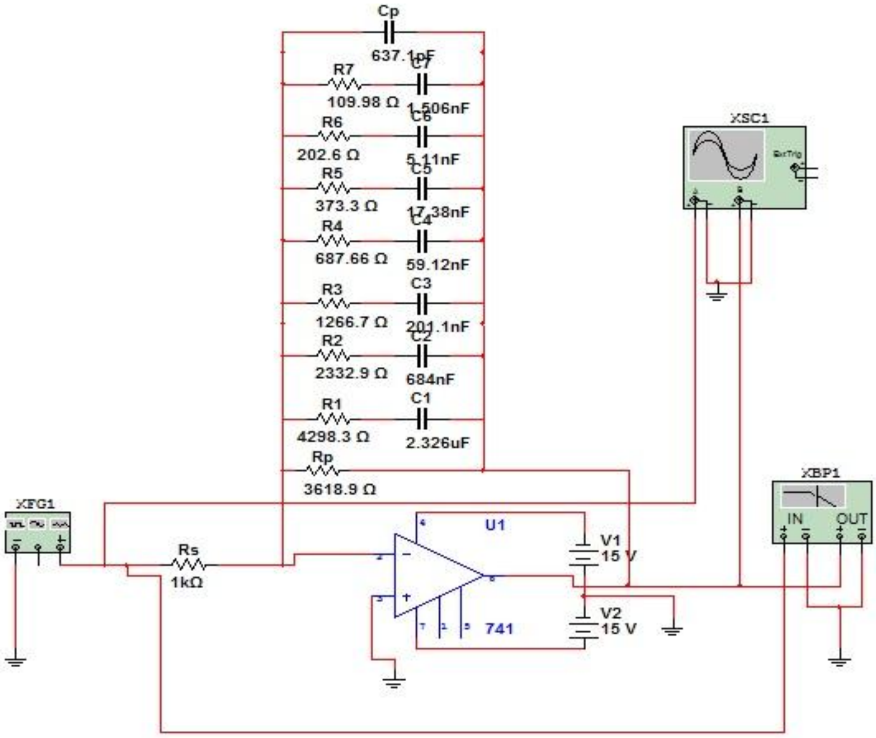


Circuit diagram

Magnitude plot

Phase plot

Φ = -30° m =7

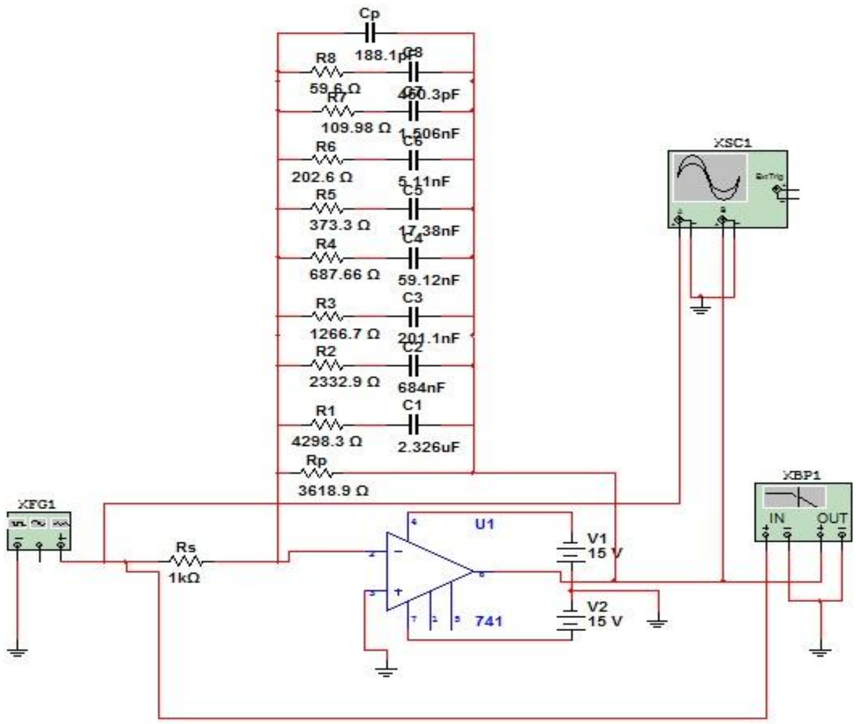


Circuit diagram

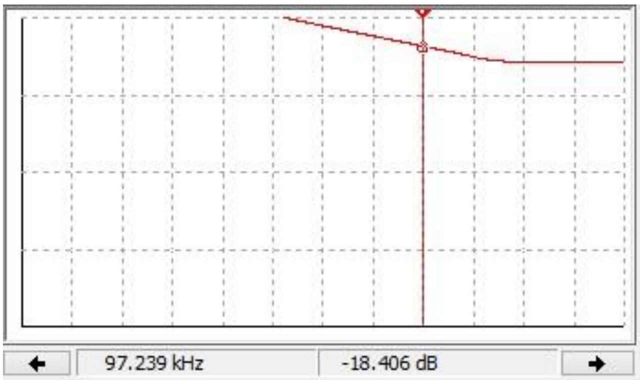
Magnitude plot

Phase plot

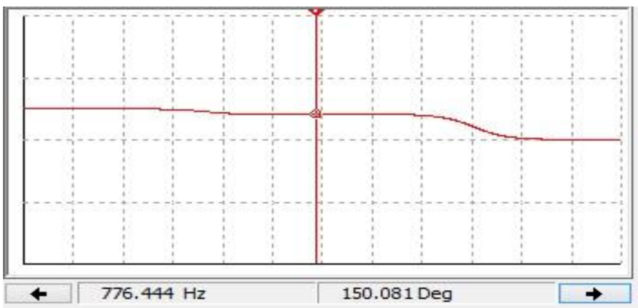
Φ = -30° m =8



Circuit diagram

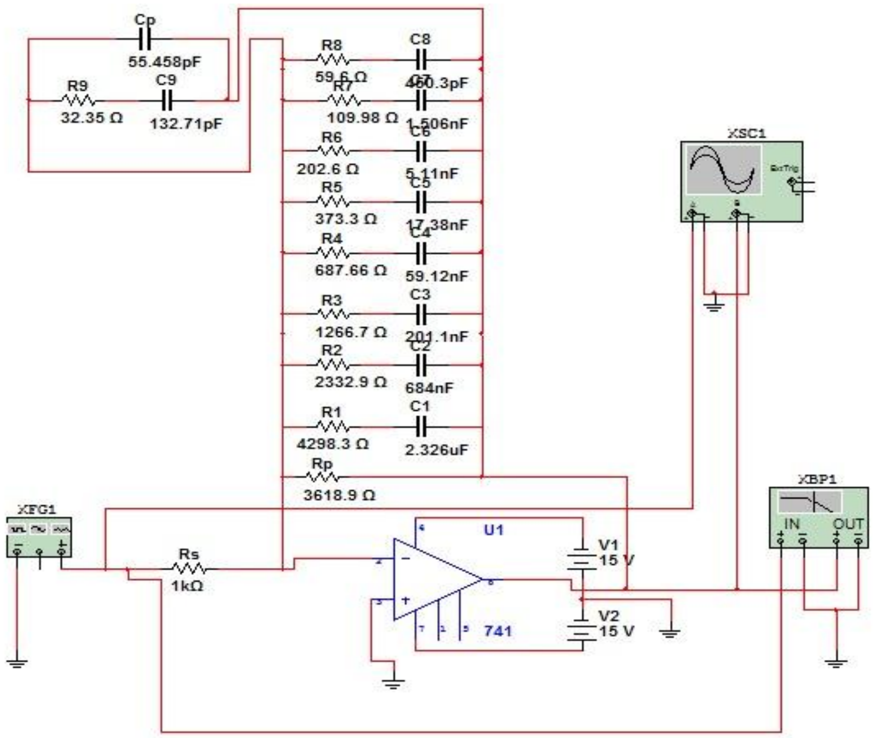


Magnitude plot

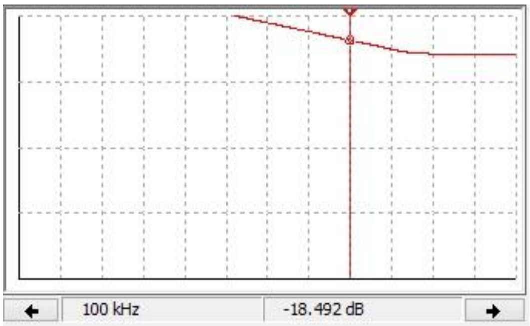


Phase plot

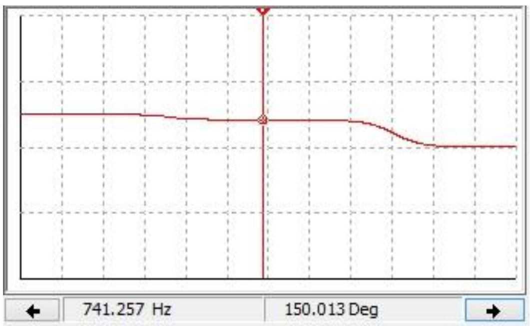
Φ = -30° m =9



Circuit diagram

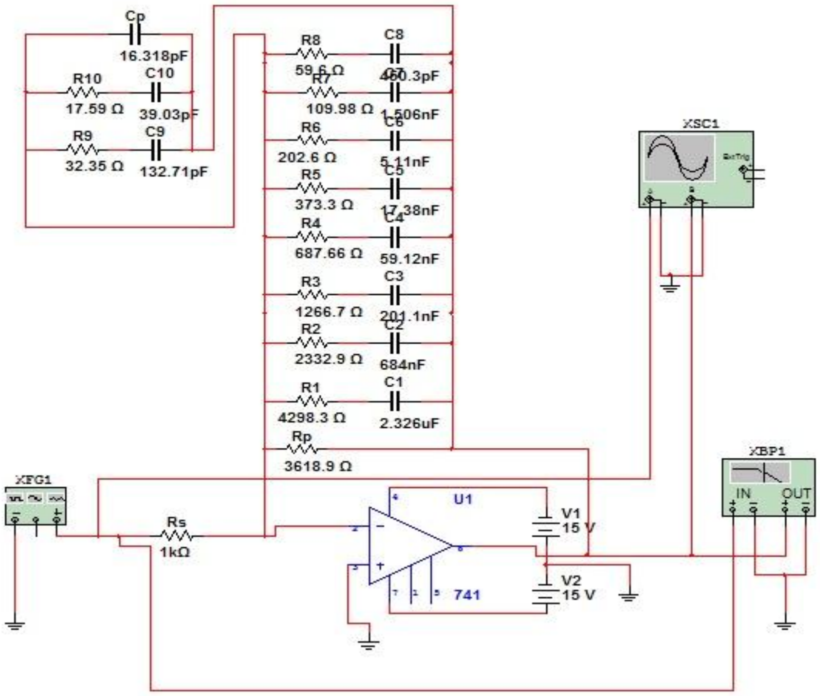


Magnitude plot

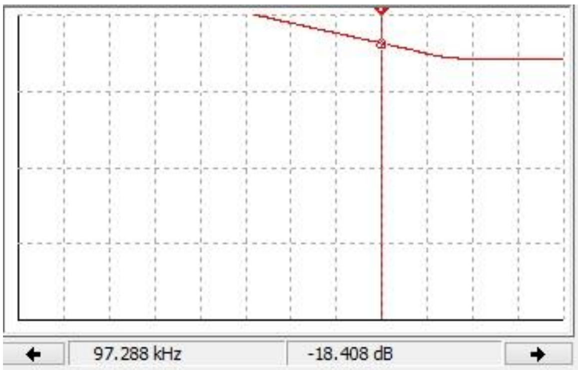


Phase plot

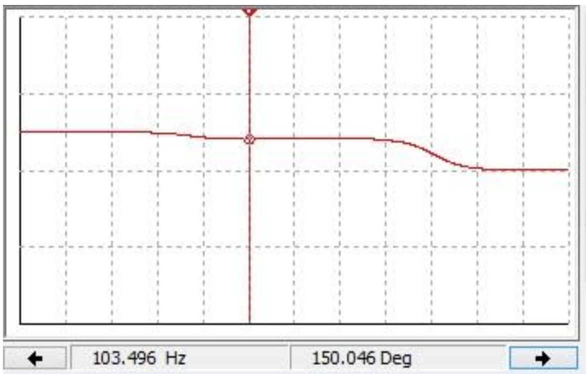
Φ = -30° m =10



Circuit diagram



Magnitude plot



Phase plot

CPE simulations results:

- It was found that as the 'm' i.e. number of rungs increases, the frequency range for Constant phase also increases up to certain values. After m=7, the additional rungs provides only the phase accuracy.

- When you go for higher values of 'm', the value of capacitors and resistors that we get from calculations are not feasible enough for implementation.

- Thus we need to select appropriate 'm' value to implement CPE on hardware.

3.2 CPE Design for m =5 (Φ = -40°)

Values obtained from the calculations are,

R_1 = 2268.1Ω	C_1 = 4.40µF
R_2 = 1004.4Ω	C_2 = 1.59µF
R_3 = 444.83Ω	C_3 = 575.45nF
R_4 = 196.99Ω	C_4 = 207.92nF
R_5 = 87.24Ω	C_5 = 75.12nF

R_P =2853.38Ω	C_P = 42.48nF

$w_d = 1/R_1C_1$ = 100.2 rad/sec = 15.91Hz

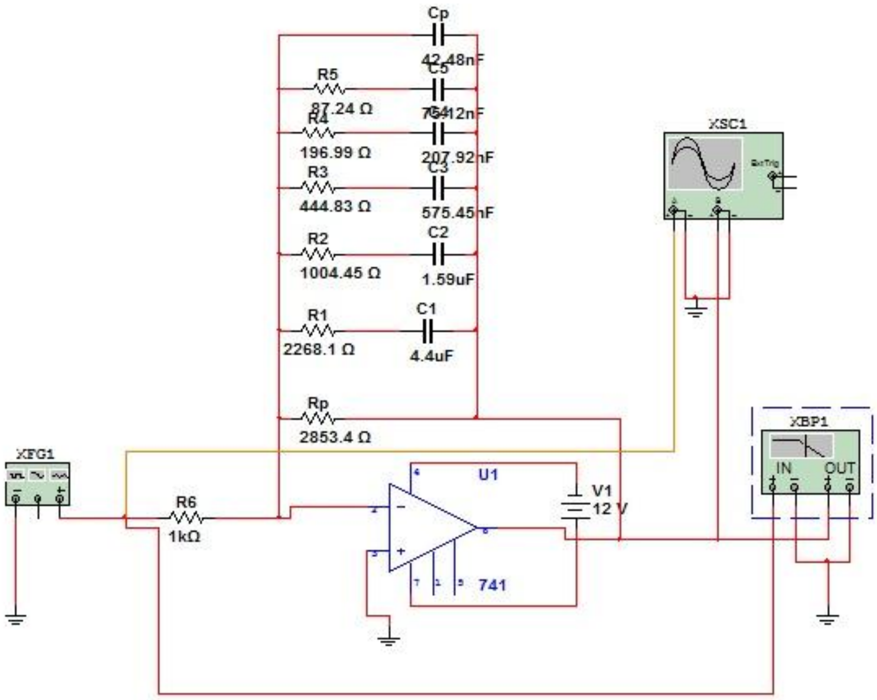
$w_h = w_d/(ab)^m$ = 955.6 x10^3 rad/sec = 151kHz

Expected Observations from the calculations:

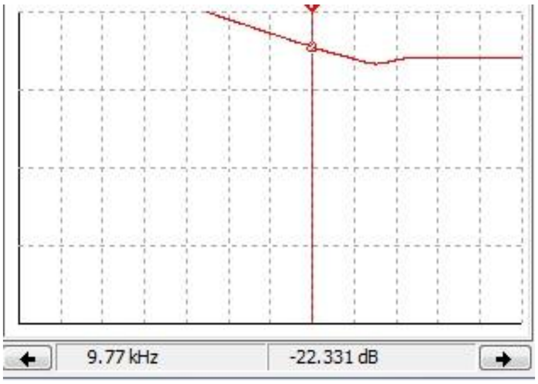
Thus, the phase response should be constant at -40° within frequency range of approximately 16Hz to 150 kHz.

Magnitude plot should show decrease in the gain with the constant slope of approximately -20 x α = 8.89 dB/decade within frequency range of approximately 16Hz to 150 kHz.

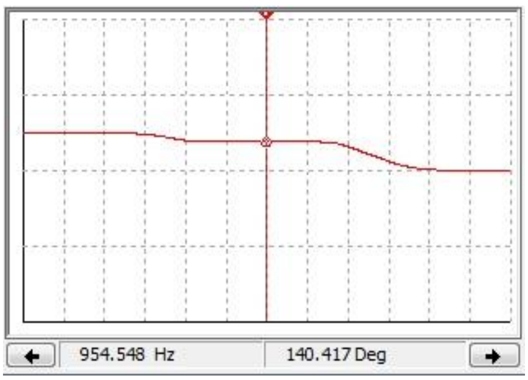
Φ = -40° m =5



Circuit diagram



Magnitude plot



Phase plot

3.3 CPE Design for m =5 (Φ = -60°)

Values obtained from the calculations are,

$R_1 = 961.6\Omega$	$C_1 = 10.39\mu F$
$R_2 = 283.4\Omega$	$C_2 = 5.644\mu F$
$R_3 = 66.02\Omega$	$C_3 = 3.064\mu F$
$R_4 = 24.61\Omega$	$C_4 = 1.663\mu F$
$R_5 = 7.254\Omega$	$C_5 = 903.1nF$

$R_P = 2301.15\Omega$	$C_P = 1.072\mu F$

$w_d = 1/R_1C_1 = 100$ rad/sec = 15.91Hz

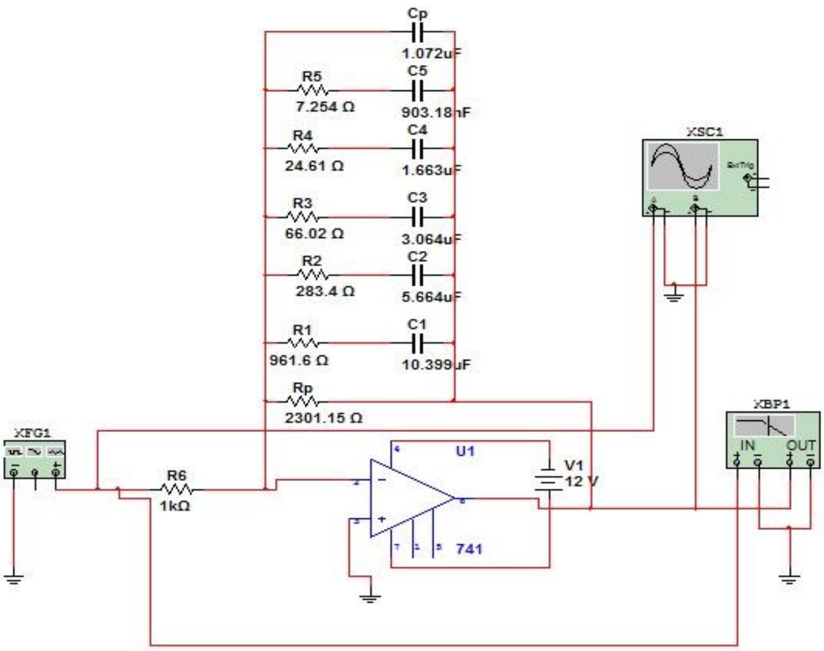
$w_h = w_d/(ab)^m = 953.7 \text{ x}10^3$ rad/sec = 151kHz

Expected Observations from the calculations:

Thus, the phase response should be constant at -60° within frequency range of approximately 16Hz to 150 kHz.

Magnitude plot should show decrease in the gain with the constant slope of approximately -20 x α = 13.33dB/decade within frequency range of approximately 16Hz to 150 kHz.

Φ = -60° m =5



Circuit Diagram

Magnitude Plot

Phase Plot

HARDWARE IMPLEMENTATION

4.1 Important Steps

I. Adjusted the resistor (Potentiometer) values according to the available capacitors by time constant method.

II. Mounted the circuit on breadboard carefully.

III. Proper biasing voltages to the 741 Operational amplifier was given.

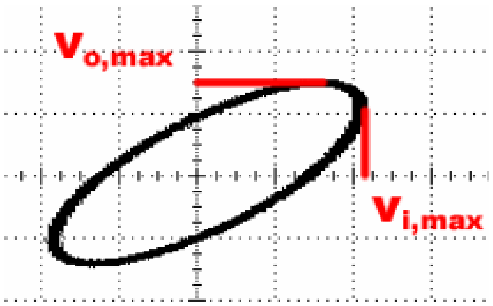
IV. Provided proper grounding to the circuit.

V. Checked if there are any loose connections.

VI. Checked the DC power source, Function generator and both the CRO terminals for proper working.

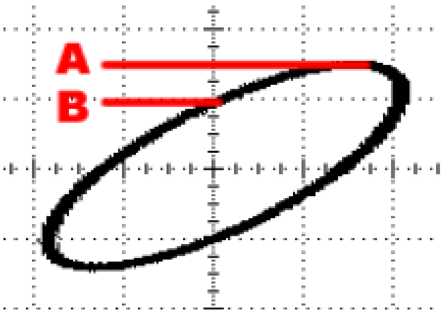
VII. Procedure to check the output on the CRO by lissajous patterns. [3]

Measurement of voltage gain on X-Y mode: [3]



Voltage Gain = 20log($V_{0,max}/V_{i,max}$)

Measurement of phase shift on X-Y mode: [3]



Phase Shift = $\sin^{-1}$(B/A)

4.2 Hardware Implementation for m = 5, ϕ = - 30°

Implement the designed CPE for Φ = -30° and m = 5 on the hardware by connecting it in the feedback path of an opamp to make the integrator circuit.

Apparatus used is CRO, DC power supply, Function generator, circuit mounted on breadboard or working PCB, connecting wires, probes, Digital Multimeter etc.

Observation Table:

Sr. No	Frequency (Hz)	A	B	$V_{o,max}$	$V_{i,max}$	Gain (db)	Phase Shift (◦)
1	5	6.2	2	3.1	0.95	10.27	-18.87
2	7	6.3	2.4	3.1	1.05	9.40	-22.39
3	**10**	**5.9**	**2.6**	**2.95**	**1.15**	**8.12**	**-26.30**
4	16	5.1	2.4	2.5	1.2	6.37	-28.07
5	30	4.2	2	2.1	1.2	4.86	-28.62
6	50	3.5	1.75	1.75	1.15	3.64	-30
7	**100**	**2.75**	**1.35**	**1.4**	**1.1**	**2.09**	**-29.40**
8	200	2.2	1.05	1.1	1.1	0	-28.50
9	300	3.9	1.9	1.9	2.25	-1.46	-29.15
10	500	3.25	1.6	1.6	2.25	-2.96	-29.49
11	700	2.75	1.3	1.4	1.25	-4.12	-28.21
12	**1k**	**2.25**	**1.1**	**1.25**	**2.3**	**-5.29**	**-29.26**
13	2k	4	1.9	2	4.6	-7.23	-28.35
14	3k	2.6	0.8	0.8	2.25	-8.98	-30
15	5k	1.5	0.7	0.75	2.3	-9.73	-27.81
16	10k	1.4	0.4	0.7	2.3	-10.33	-16.6
17	50k	2.7	0.3	1.3	4.3	-10.39	-6.37
18	100k	2.6	0.4	1.25	4.3	-10.73	-8.84

Results:

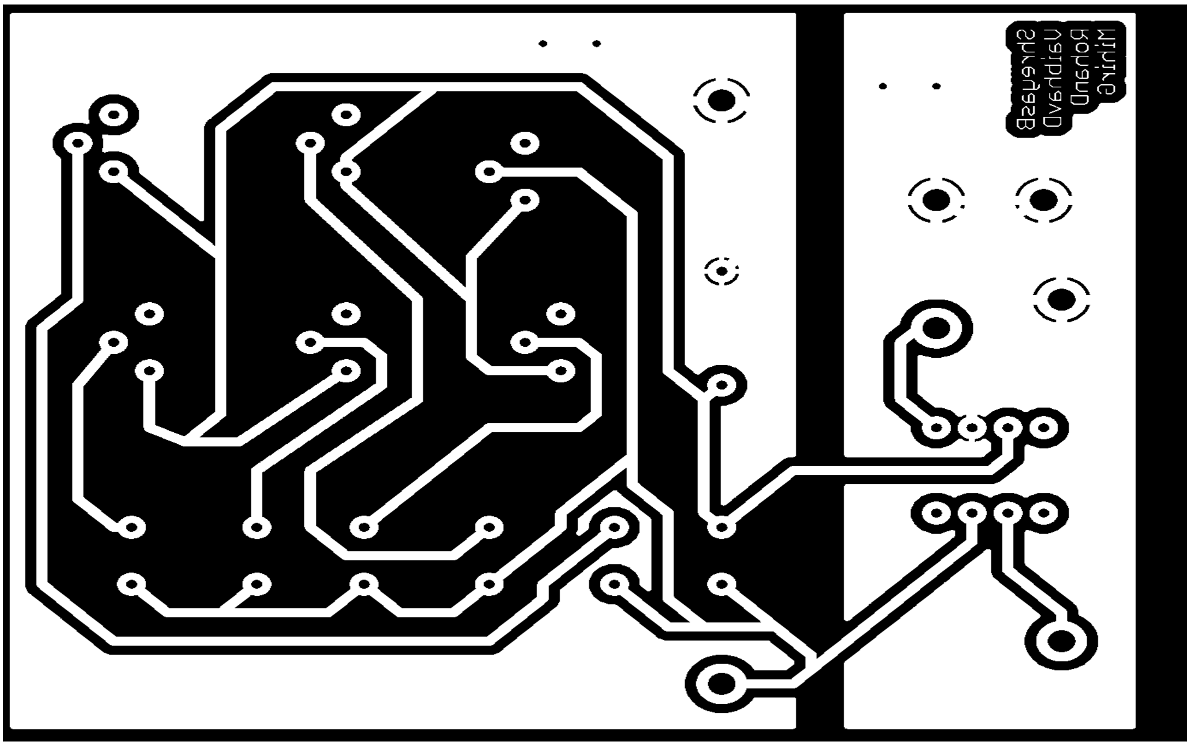
- The CPE is designed for m = 5, Φ = 30°. The tolerable deviation is ±0.5°
- 'm' corresponds to the number of rungs in the ladder circuit and Φ refers to the phase angle at which the phase should be constant.
- The ideal working range of frequency from the calculations is 16Hz to 151kHz according to the design.
- The actual working range of frequency from the calculations is 16Hz to 10kHz according to the design.

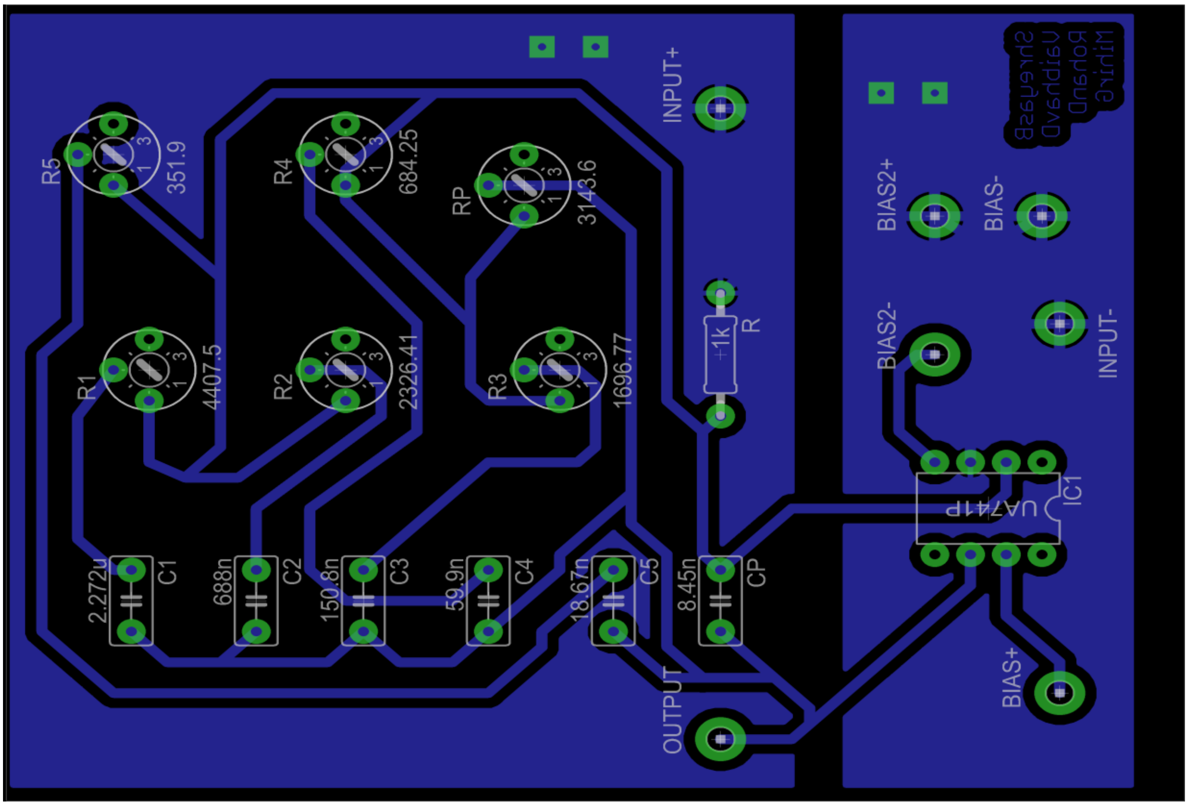
- In the magnitude plot, the magnitude drops at the rate of 6.67dB/decade in the working range of frequency.
- In the phase plot, the phase remains constant at approximately 150°.
- The data mentioned above defines the working constraint of the CPE.

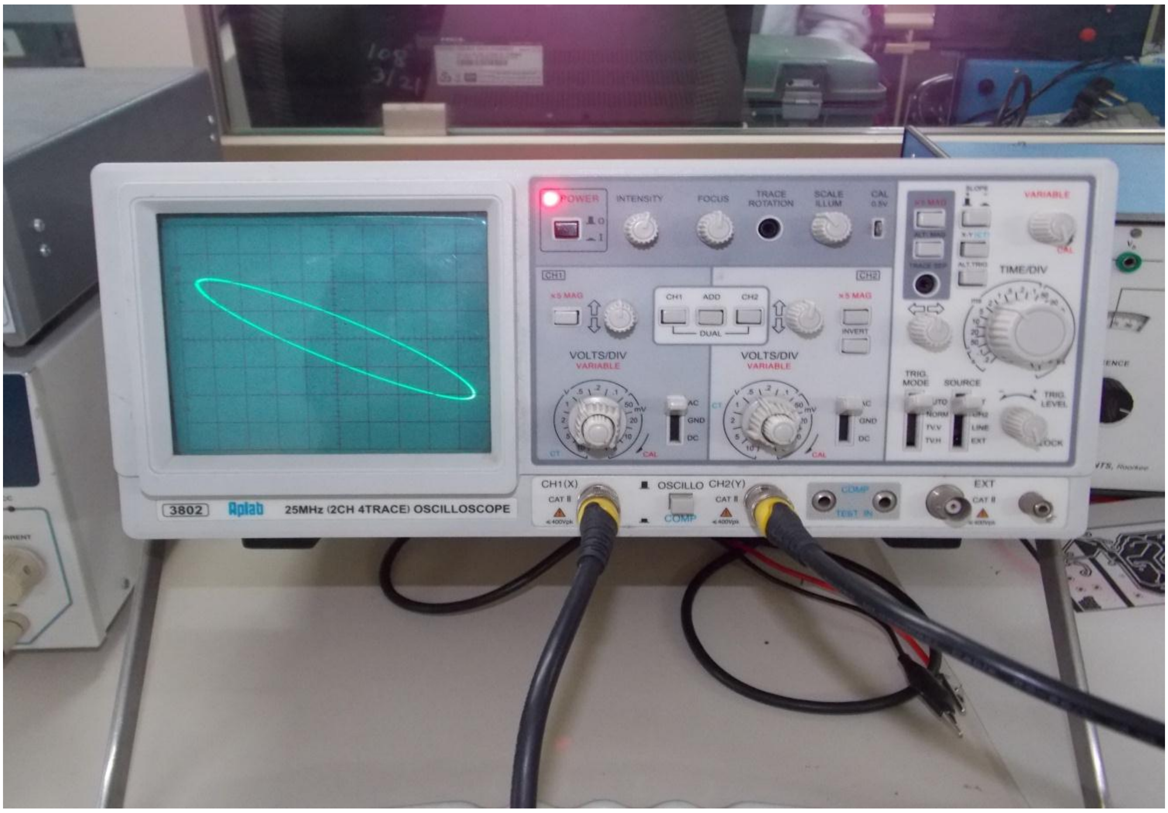
The resistor values are approximated using Time Constant Method. Adjusted values are as follows:

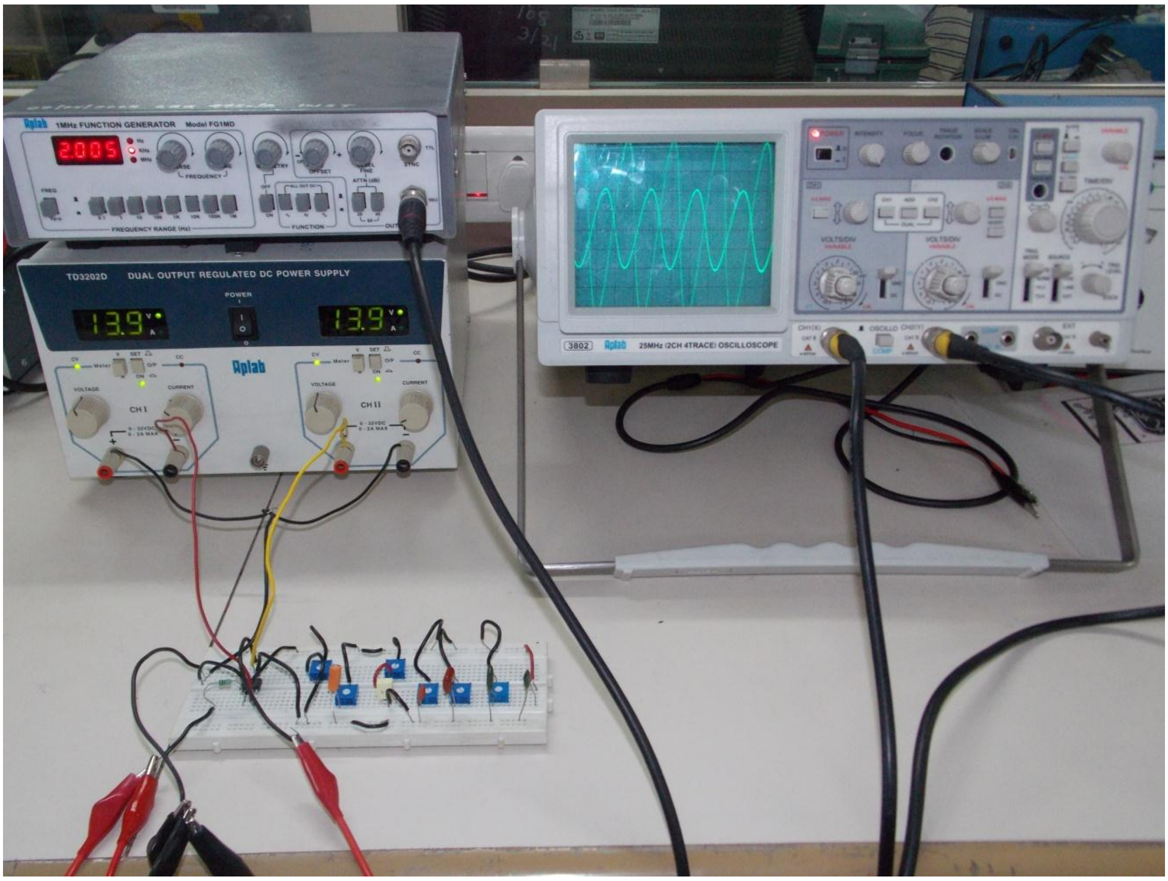
R_P	**3619 Ω**	**C_P**	**9.81 nF**
R_1	**4510.75 Ω**	**C_1**	**2.22 µF**
R_2	**2323.04 Ω**	**C_2**	**689 nF**
R_3	**1068.81 Ω**	**C_3**	**239.4 nF**
R_4	**679.71 Ω**	**C_4**	**60.3 nF**
R_5	**342.36 Ω**	**C_5**	**19.19 nF**

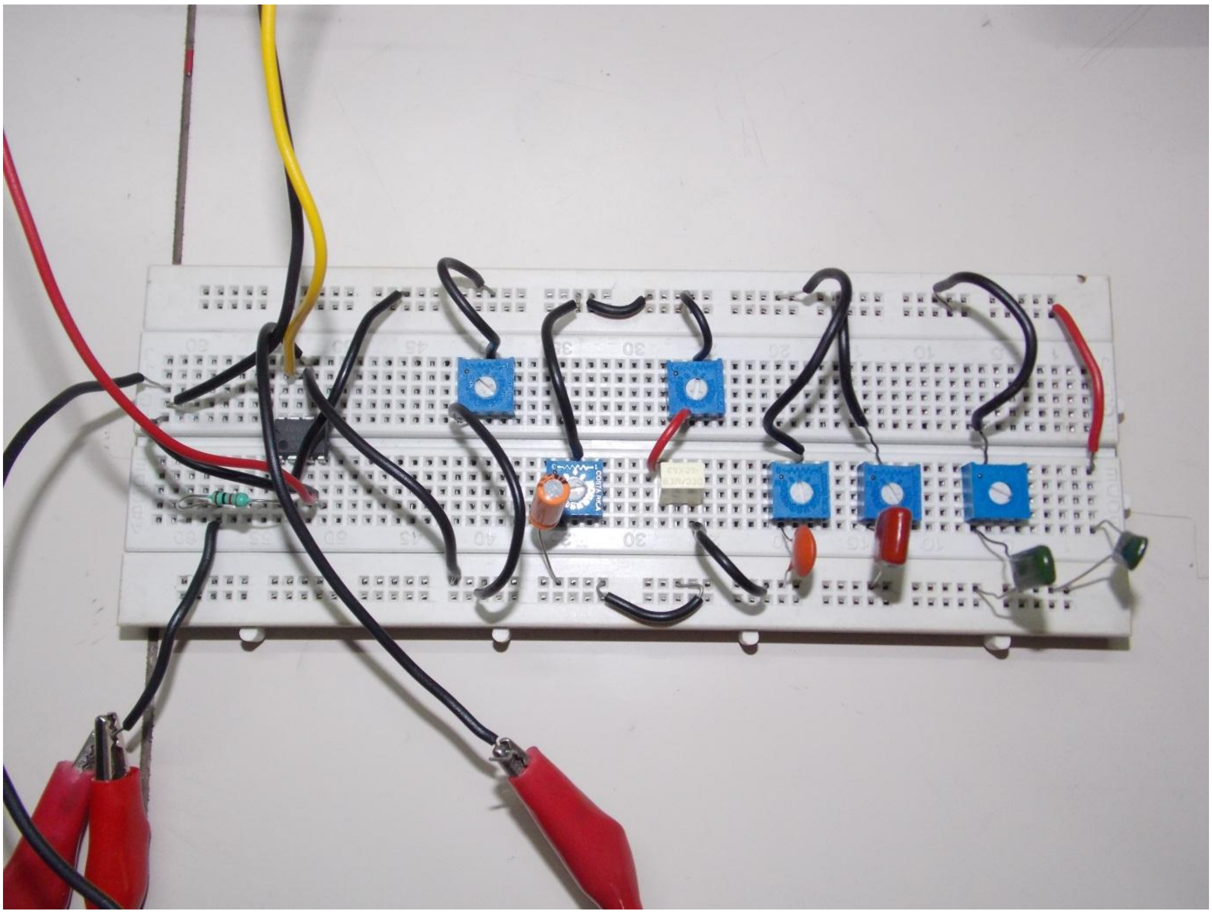
4.3 PCB layout for hardware

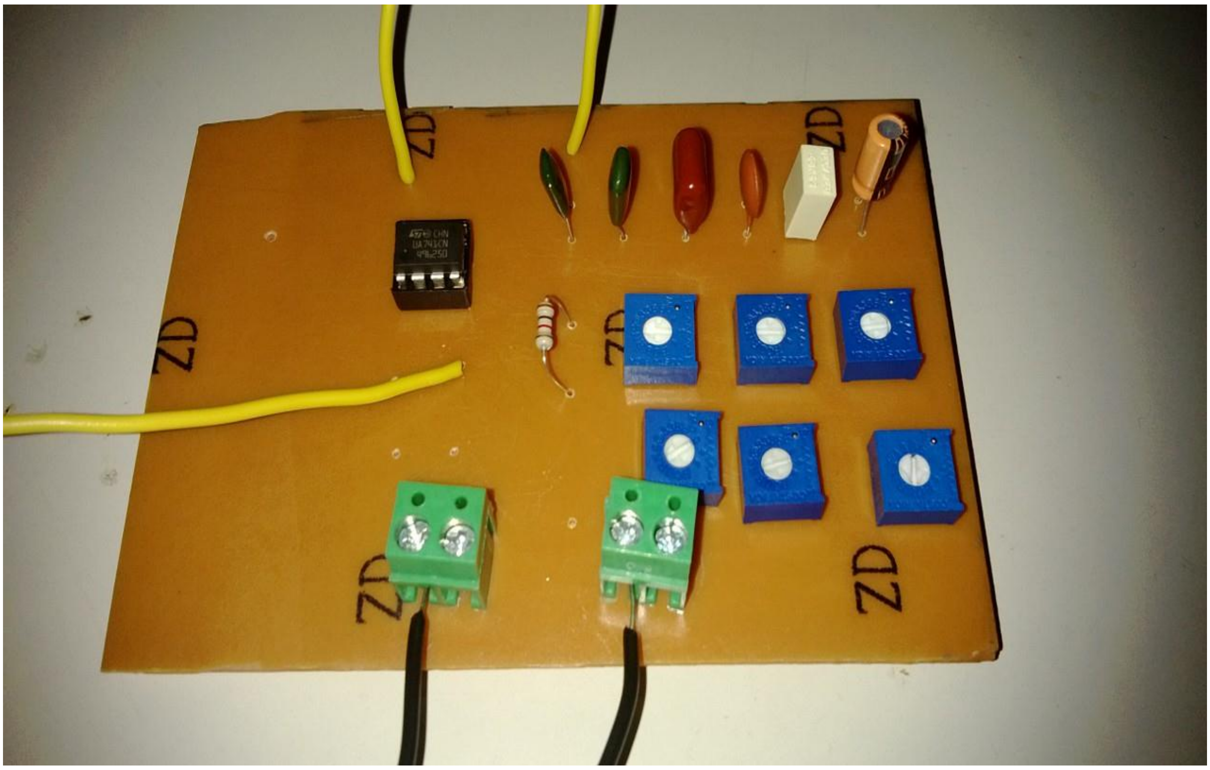





POWER
INTENSITY
FOCUS
TRACE ROTATION
SCALE ILLUM
VOLTS/DIV
VARIABLE
TIME/DIV
TRIG. MODE
SOURCE
TRIG. LEVEL
CH1(X)
OSCILLO
CH2(Y)
EXT
3802
25MHz (2CH 4TRACE) OSCILLOSCOPE


1MHz FUNCTION GENERATOR Model FG1MD
2.005
TD3202D DUAL OUTPUT REGULATED DC POWER SUPPLY
13.9
13.9
POWER
CH I
CH II
3802
25MHz (2CH 4TRACE) OSCILLOSCOPE




ZD
ZD
ZD
ZD
ZD
ZD

FUTURE SCOPE

5. FUTURE SCOPE

A good CPE model can be used for simulation and experimental verification of properties of various fractal systems. In connection with operational amplifiers, it makes it possible to build analog networks described by fractional differential equations. The simplest networks of this kind are fractional differentiator and fractional integrator. [1]

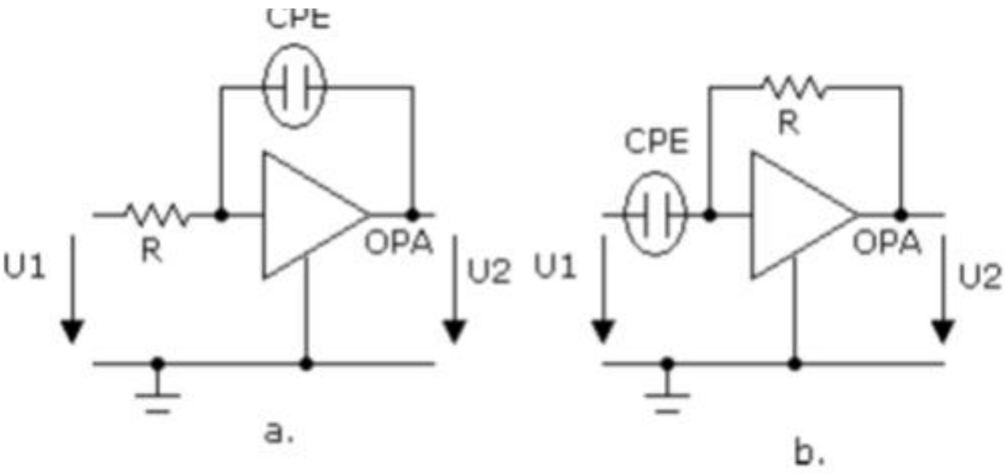


Fig. 14 Fractional analog integrator and differentiator.

Voltage transfer factor of the fractional integrator is given by

$$K(s) = -\frac{D}{s^{|\alpha|}R}, \quad -1 < \alpha < 0$$

While that of the differentiator

$$K(s) = -s^{|\alpha|}\frac{R}{D}, \quad -1 < \alpha < 0.$$

With the help of these simple circuits it is possible to perform analog operations with electric signals. Fig. 12 compares as an example the “ideal” fractional derivatives of signal Asint, $0 < t < \pi$, for $\alpha = -0.3$, $\alpha = -0.5$, $\alpha = -0.7$, $\alpha = -1$ (classical 1st derivative Acost) with the output of the analog differentiator (m = 4, time constant $R_1C_1 = 10$ s, denoted with crosses). All the waveforms were obtained by numerical inversion of the corresponding Laplace transforms under zero initial conditions. Obviously, the used relatively simple CPE model delivers very good results. [1]

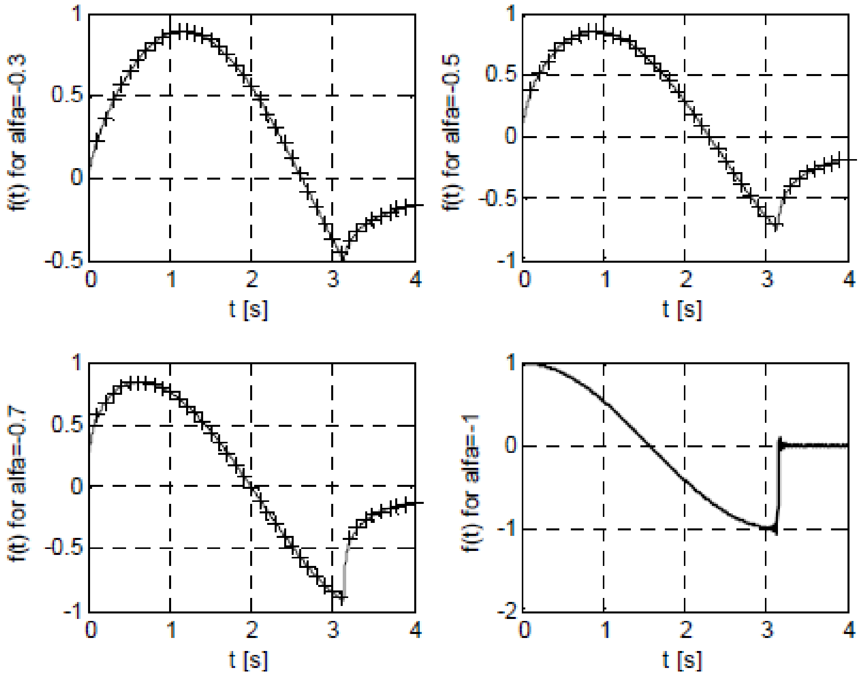


Fig. 12. Fractional derivatives of the signal $A\sin t$, $0 < t < \pi$.

Fig. 16 shows an analog model of a fractional regulation system with transfer factor

$$K_u(s) = \frac{1}{1 + 0.5s^{0.9} + 0.8s^{2.2}} =$$

$$= \frac{-1}{\dfrac{R_1}{R_{f2}} + \dfrac{R_1 R_3}{R_{f1} D_3} s^{-\alpha_3} + \dfrac{R_1 R_2 R_3}{D_1 D_2 D_3} s^{-(\alpha_1 + \alpha_2 + \alpha_3)}}$$

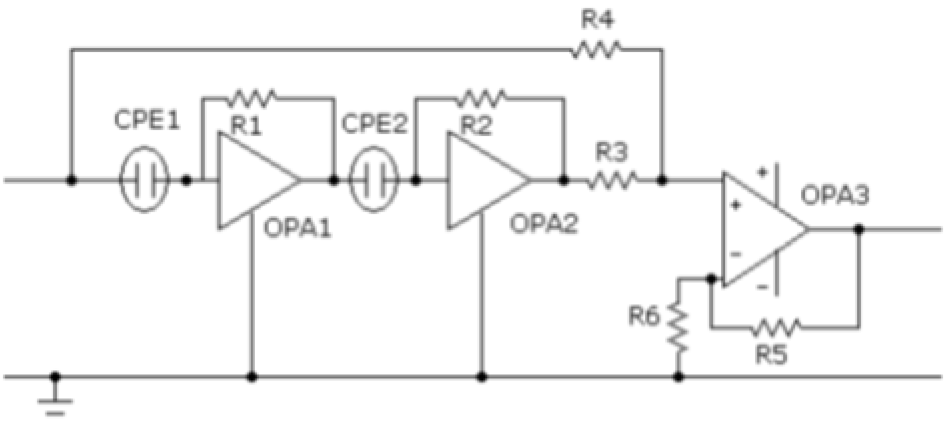


Fig. 16 **Analog model of the controller (34).**

Parameters of individual CPEs used in the scheme are for instance α_1= -1 (the first stage is a classical integrator with the capacitor C= 1/D_1), α_2 = -0.3, φ_2= -27°, α_3 = -0.9, Φ_3 = -81°, R_1= R_{f2}.

Optimal PD controller should have transfer factor $K_c(s) = 20.5+3.7343s^{1.15}$
It is possible model is in Fig. 16. It can be implemented by using two CPEs of the order 0.5 and 0.65.

Transfer factor of the controller in Fig. 16

$$K_c(s) = \frac{R_3 + R_4 \dfrac{R_1 R_2}{D_1 D_2} s^{|\alpha_1+\alpha_2|}}{R_3 + R_4}\left(1+\frac{R_5}{R_6}\right),$$

$$-1 < \alpha_1 < 0, \quad -1 < \alpha_2 < 0 .$$

In this project relatively simple yet still acceptably faithful network model of the CPE was developed, designed and implemented on simulator as well as on hardware.

The simulation and Hardware results approximately match and are verified by reliable measuring instruments.

The components required are not of high cost and rare but they are easily available and are shelf-passive resistors and capacitors i.e. components doesn't require special selection.

Contrary to other known models its design does not need complicated optimization steps.

Sensitivity analysis shows that standard component tolerances do not affect the properties of the model too much.

6. APPENDIX

National Instrument's Multisim version 11:

Multisim is an industry-standard, best-in-class SPICE simulation environment. It is the cornerstone of the NI circuits teaching solution to build expertise through practical application in designing, prototyping, and testing electrical circuits.

The Multisim design approach helps you save prototype iterations and optimize printed circuit board (PCB) designs earlier in the process.

Multisim was originally called Electronics Workbench and created by a company called Interactive Image Technologies. At the time it was mainly used as an educational tool to teach electronics technician and electronics engineering programs in colleges and universities. National Instruments has maintained this educational legacy, with a specific version of Multisim with features developed for teaching electronics.

In 1999, Multisim was integrated with Ultiboard after the original company merged with Ultimate Technology, a PCB layout software company.

In 2005, Interactive Image Technologies was acquired by National Instruments Electronics Workbench Group and Multisim was renamed to NI Multisim.

Eagle version 6.4.0:

EAGLE (Easily Applicable Graphical Layout Editor) by Cadsoft is a flexible and expandable EDA schematic capture, PCB layout, autorouter and CAM program widely used since 1988. EAGLE is popular among hobbyists because of its freeware license and rich availability of component libraries on the web.

7. References

[1] **'Network Model of the CPE'** by Juraj VALSA, Petr DVOŘÁK,

Martin FRIEDL, 2011.

[2] **'Functional Fractional Calculus for System Identification and Controls'** by

Dr. Shantanu Das, Scientist, Reactor Control Division, BARC, Mumbai, 2008.

[3] **'Using X-Y Mode',** an instruction guide for the Cathode Ray Oscilloscope, written by

Elliott Wood,

[4] **Websites**: https://en.wikipedia.org/wiki/Constant_phase_element

https://en.wikipedia.org/wiki/Fractional_calculus