A Brief Introduction to Legendrian Knots

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Abstract

We will introduce the concept of a Legendrian knot and then discuss the various attributes that Legendrian knots have. We will also discuss a specific class of knot invariants - the classical invariants - for Legendrian knots and show how these may be calculated.

1 Introduction

A *contact structure* on $\mathbb{R}^3$ is a method of placing a plane at every point, such that these planes satisfy certain technical conditions. There are various distinct contact structures we could use, some of them giving very different properties. However, for this paper, we will exclusively consider the standard contact structure: At every point in $\mathbb{R}^3$, we consider the plane spanned by

$$\left\{ \frac{\partial}{\partial y}, \frac{\partial}{\partial x} + y\frac{\partial}{\partial z} \right\}.$$

Let us show this with an example of the planes:

Visually, we can see that these planes are always tangent to the y-axis, but also twist around once when moving from $y = -\infty$ to $y = \infty$.

Now that we have this contact structure, we can define a *Legendrian knot* as a knot that is at every point tangent to the plane at that point [3]. In particular, if we look at the tangent vector v to K at a point $p = (x, y, z)$, then

$$v = A\frac{\partial}{\partial y} + B\left(\frac{\partial}{\partial x} + y\frac{\partial}{\partial z}\right)$$

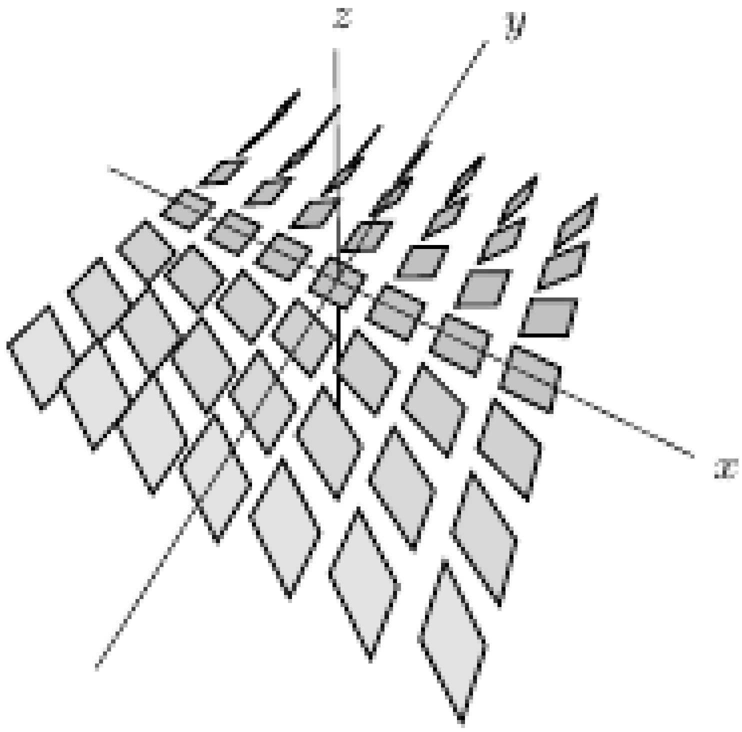

Figure 1: An example of a contact structure in $\mathbb{R}^3$

for some real numbers A and B. We can equivalently formulate this condition using a parametrization of K as the image of a smooth embedding $\phi : [0,1] \to \mathbb{R}^3$, where $\phi(t) = (x(t), y(t), z(t))$ satisfies

$$z'(t) - y(t)x'(t) = 0.$$

We can now say that two Legendrian knots are equivalent if they can be related by a continuous family of Legendrian knots. This is very similar to the definition of equivalence for smooth knots, only here, we restrict the allowable motions. Therefore, Legendrian equivalence is a finer relation than smooth knot equivalence, and, in fact, we will show that there are many Legendrian inequivalent knots with the same smooth knot type.

2 Projections

Before we can discuss invariants of Legendrian knots, we must first understand the ways to visually represent them. Because Legendrian knots rely on our choice of contact structure, and therefore our choice of coordinates, dealing with projections of Legendrian knots will be more complicated than with smooth knots.

The first method of projecting Legendrian knots is the *front projection*. This is the projection onto the xz-plane. From the differential equation $z'(t) = y(t)x'(t)$, we see that we can recover the y coordinate from the front projection by

$$y(t) = \lim_{t_0 \to t} \frac{z'(t)}{x'(t)}.$$

We find the following two properties of front projections:

i) There may be no vertical tangencies: if $x'(t)$ vanishes, so does $z'(t)$. Instead we allow *cusps*, points where the diagram forms a sharp horizontal point.

ii) At each crossing, the under strand has greater slope than the over strand.

It turns out that these conditions completely characterize front projections: if $(x(t), z(t))$ satisfies these conditions, they are the coordinates of a Legendrian knot. Furthermore, for any Legendrian knot, we can arrange for the set of cusp points to be finite.

We can take a projection of a smooth knot and convert it into the front projection of a Legendrian knot by rotating each crossing until the under strand has greater slope, and then replacing vertical tangencies with cusps. We have therefore proved

Theorem 1. Every smooth knot may be realized as a Legendrian knot.

We now consider the harder to work with, but just as essential, *Lagrangian projection*. This is the projection onto the xy-plane. This time, we must recover the z coordinate by integrating, so up to a constant z_0, we have

$$z(t) = z_0 + \int_0^t y(s)x'(s)\, ds.$$

Now $(x(t), y(t))$ will provide a valid $z(t)$ if and only if

i) $\int_0^1 y(s)x'(s)\, ds = 0$.

ii) $\int_{t_1}^{t_2} y(s)x'(s)\, ds \neq 0$ whenever $(x(t_1), y(t_1)) = (x(t_2), y(t_2))$.

3 Classical Invariants

We now discuss the classical invariants related to Legendrian knots.

There are three major invariants we will discuss:

- *The topological knot type*
- *The Thurston-Bennequin number*
- *The rotation number*

3.1 The Topological Knot Type

We consider two *Legendrian knots* as equivalent if there is a a continuous family of *Legendrian knots* between them. This is analogous to smooth knot equivalence, though in our case, the motions/moves allowed to turn one knot into another are restricted, due to the presence of the contact structure framework.

With this idea in mind, we have this theorem:

Theorem 2. If two Legendrian knots are Legendrian equivalent, then they are equivalent as smooth knots.

That is, we may consider their projections not as Lagrangian or front projections, but simply as projections of a smooth knot. If the *Legendrian knots* are equivalent, then we can show that they are equivalent as smooth knots [1].

Therefore, if there is an isotopy between two Legendrian knots is, then there is an isotopy of the underlying knot projections as smooth knots.

This leads to the first major invariant: *the topological knot type*.

For a given *Legendrian knot* L, this invariant is denoted as $k(L)$. This invariant is the equivalence class of all knots that are smoothly equivalent to L.

We may have various invariants for these knots be the same, but the underlying types - *figure-eight* and *unknot* are not the same, and so these two knots cannot be equivalent based on a consideration of their *topological knot type*. One can convince themselves that the equivalence class of all knots smoothly equivalent to a *figure-eight* cannot have the *unknot* in that class.

Let us demonstrate this with a small example.

Let us take a look at a simple example - the *Legendrian unknot* and *Legendrian figure-eight*. Let us look at their front diagrams:

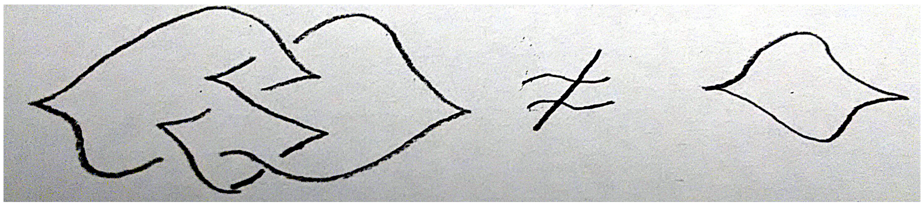

Figure 2: Front projections of non-equivalent Legendrian knots.

We note that as the smooth knot types of these projections are not the same, neither can their *Legendrian knots* be equivalent.

In short, *the topological knot type* is an invariant that uses the smooth knot type of *Legendrian knot projections* to determine equivalence. If we know these types aren't the same, then we can assume that the *Legendrian knots* cannot be the same [1].

3.2 Thurston-Bennequin Invariant

One way in which we can determine if two *Legendrian knots* are equal is by measuring how much their contact planes "coil" around their knots. However, we need a way to actually calculate this degree of coiling. This degree of coiling is called the *Thurston-Bennequin number*, or *invariant*.

Without reverting to a rather geometric definition of this degree, we'll calculate this degree of coiling by taking a look at the twin projections *Legendrian knots* have.

We have one quality of these knot projections called the *right cusp number*, the count of how many cusps there are in a given *Legendrian knot* projection. At the same time, we also have what is known as the *writhe* number, the same number we learned about in MATH 4803 - the total signing, negative or positive, of knot crossings in a given projection. We calculate this number by adding all the positively signed crossings and subtracting from that number the number of negatively signed crossings.

With these two tools, we are ready to calculate the *Thurston-Bennequin number*, hereby denoted as $tb(L)$.

For a given front projection, our formula [1] is:

$$tb(L) = writhe(\Pi(L)) - \frac{1}{2}(cusps).$$

Let us see an example of this calculation:

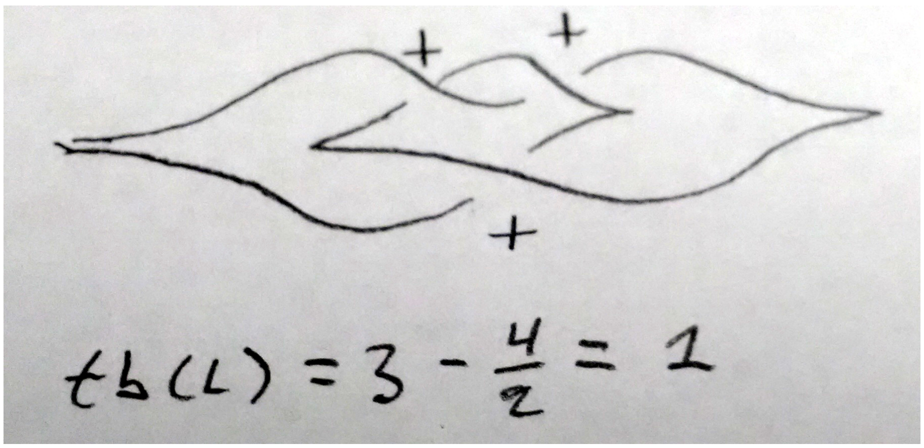

Figure 3: tb calculation for a front projection of a Legendrian knot

Note that the points on the right and left ends that are edge-like and not smooth are the cusps we are counting. In addition, we have two such points inside the knot projection, one on each end. This gives us four cusps, and the rest is easily calculable.

However, given that we have two projections, is it possible there is a *different* formula for calculating $tb(L)$ using the *Lagrangian* projection of L?

Indeed, there is. For a *Lagrangian* projection of L, all we need is the *writhe* number for this given knot [1]:

$$tb(L) = writhe(\pi(L)).$$

Let us see a quick example of this:

Once again, we note that the actual reasons as to why these formulas work require upper-level knowledge of geometry and algebra. Those geometric proofs, however, allow for the simple calculations we see above.

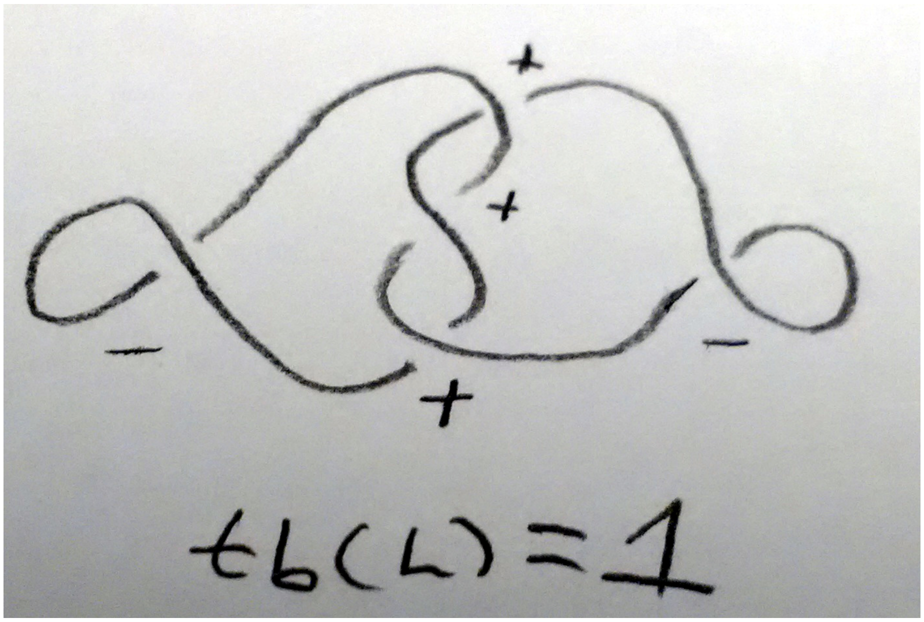

Figure 4: tb calculation for a Lagrangian projection of a Legendrian knot.

3.3 Rotation Number

We take a look at our final classical invariant, the *rotation number*. Whereas $tb(L)$ focused on the degree to which the contact planes "coiled"/"twisted" around our knot L, *rotation number* is the converse quality: the degree to which the *knot* twists inside of the contact planes in $\mathbb{R}^3$.

As with $tb(L)$, there are geometric definitions underpinning the formula we will show. Central to the idea of *rotation number* is the idea of *winding number*: the amount of times that a curve in a plane goes counterclockwise over some arbitrarily defined point. We can reason, intuitively, that the degree of twisting a *Legendrian knot* undergoes around some contact plane is analogous to the *winding number* idea we discussed above.

In any case, what arises from this idea is that we'll need to *orient* our knot projections. For calculating this number, we'll take a look at the *front* projection.

In order to effectively calculate this number, we'll need the amount of *down* cusps that the projection has. We note that, if one is orienting the knot projection, then certain cusps will be oriented *downward* because we travel down them "downwards". We'll denote these cusps as D.

Of course, if there are down cusps, then there are also *up* cusps, cusps that are oriented *upwards* as a result of the orientation of the knot projection. We will denote these cusps as *U*.

We have all we need to calculate rotation number, denoted $r(L)$, for a front projection [1]:

$$r(L) = \frac{1}{2}(D - U).$$

We note that the orientation of the knot can be done in one of two ways, so this means the numbers for D and U will be reversed, potentially changing the sign of *r(L)*. However, the number itself won't change. To work around this, we note that *r(L)* is really the *absolute value* of $\frac{1}{2}(D - U)$.

With this, we have finished a preliminary discussion of classical invariants. However, as one will find out if they choose to study *Legendrian knots* further, these classical invariants will not be enough to distinguish certain *Legendrian knots* from one another.

References

[1] J. Etnyre, *Legendrian and Transversal Knots*, from: Handbook of Knot Theory, Elsevier B. V., Amsterdam (2005).

[2] J. Sabloff, *Augmentations and Rulings of Legendrian Knots*, Int. Math. Res. Not. (2005) 1157-1180.

[3] J. Sabloff, *What Is … a Legendrian Knot?*, AMS Notices, 56 (2009), no. 10, 1282-1284.