A Brief Introduction to Legendrian Knots

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Abstract

We start by defining Legendrian knots and isotopies of Legendrian knots. Then we study the classical invariants of Legendrian knots: the topological knot type, the Thurston-Bennequin number, and the rotation number. We then explain how, while sufficient for certain classes of knots in tight contact structures, there are nonisotopic Legendrian knots with equal In particular, we introduce the Chekanov-Eliashberg DGA, and see that its homology, the knot contact homology, is an invariant. We see how this can be computed to distinguish versions of the 5_2 knot with equal classical invariants.

1 Introduction

A *contact structure* on $\mathbb{R}^3$ is a method of placing a plane at every point, such that these planes satisfy certain technical conditions. There are various distinct contact structures we could use, some of them giving very different properties. However, for this paper, we will exclusively consider the standard contact structure: At every point in $\mathbb{R}^3$, we consider the plane spanned by

$$\left\{ \frac{\partial}{\partial y}, \frac{\partial}{\partial x} + y\frac{\partial}{\partial z} \right\}.$$

Visually, we can see that these planes are always tangent to the y-axis, but also twist around once when moving from $y = -\infty$ to $y = \infty$.

Now that we have this contact structure, we can define a *Legendrian knot* as a knot that is at every point tangent to the plane at that point. In particular, if we look at the tangent vector v to K at a point $p = (x, y, z)$, then

$$v = A\frac{\partial}{\partial y} + B\left(\frac{\partial}{\partial x} + y\frac{\partial}{\partial z}\right)$$

for some real numbers A and B. We can equivalently formulate this condition using a parametrization of K as the image of a smooth embedding $\phi : [0,1] \to \mathbb{R}^3$, where $\phi(t) = (x(t), y(t), z(t))$ satisfies

$$z'(t) - y(t)x'(t) = 0.$$

We can now say that two Legendrian knots are equivalent if they can be related by a continuous family of Legendrian knots. This is very similar to the definition of equivalence for smooth knots, only here, we restrict the allowable motions. Therefore, Legendrian equivalence is a finer relation than smooth knot equivalence, and, in fact, we will show that there are many Legendrian inequivalent knots with the same smooth knot type.

2 Projections

Before we can discuss invariants of Legendrian knots, we must first understand the ways to visually represent them. Because Legendrian knots rely on our choice of contact structure, and therefore our choice of coordinates, dealing with projections of Legendrian knots will be more complicated than with smooth knots.

The first method of projecting Legendrian knots is the *front projection*. This is the projection onto the xz-plane. From the differential equation $z'(t) = y(t)x'(t)$, we see that we can recover the y coordinate from the front projection by

$$y(t) = \lim_{t_0 \to t} \frac{z'(t)}{x'(t)}.$$

We find the following two properties of front projections:

i) There may be no vertical tangencies: if $x'(t)$ vanishes, so does $z'(t)$. Instead we allow *cusps*, points where the diagram forms a sharp horizontal point.

ii) At each crossing, the under strand has greater slope than the over strand.

It turns out that these conditions completely characterize front projections: if $(x(t), z(t))$ satisfies these conditions, they are the coordinates of a Legendrian knot. Furthermore, for any Legendrian knot, we can arrange for the set of cusp points to be finite.

We can take a projection of a smooth knot and convert it into the front projection of a Legendrian knot by rotating each crossing until the under strand has greater slope, and then replacing vertical tangencies with cusps. We have therefore proved

Theorem 1. Every smooth knot may be realized as a Legendrian knot.

We now consider the harder to work with, but just as essential, *Lagrangian projection.* This is the projection onto the xy-plane. This time, we must recover the z coordinate by integrating, so up to a constant z_0, we have

$$z(t) = z_0 + \int_0^t y(s)x'(s)\,ds.$$

Now $(x(t), y(t))$ will provide a valid $z(t)$ if and only if

i) $\int_0^1 y(s)x'(s)\,ds = 0$.

ii) $\int_{t_1}^{t_2} y(s)x'(s)\,ds \neq 0$ whenever $(x(t_1), y(t_1)) = (x(t_2), y(t_2))$.

3 Classical Invariants

We now discuss the classical invariants related to Legendrian knots. Given that a Legendrian isotopy between two Legendrian knots is, at its core, an isotopy of the underlying knots, then the first major invariant of the Legendrian knot is the *topological knot type*, here donated as $k(L)$. This is a rather intuitive notion.

The second major invariant is known as the *Thurston-Bennequin invariant*, which essentially measures how much coiling a given Legendrian knot has. Legendrian knots have what is called the *right cusp number*, essentially a measure of how many local right-cusps there are in its knot diagram. These knots also have a quality known as the *writhe* number, which is a measure of counting the knot diagram's crossings, but only counting those crossings that have signs. To calculate the maximal Thurston-Bennequin number, one can simply subtract the *writhe* - *right cusp number.*

This number calculated is the invariant. According to the Knot Atlas, the Thurston-Bennequin invariant is not just this number, but rather the maximum number of a set of two numbers - the two maximal Thurston-Bennequin numbers calculated for both the original knot diagram and its mirror.

Rigorously speaking, following in Etnyre's footsteps, we may define the Thurston-Bennequin invariant by using the idea of a bundle and its trivialization. Here, we denote a bundle as a *vector bundle*, a collection of vector spaces that are parameterized by another space. Let us denote a normal, or perpendicular bundle of a Legendrian knot L by v. A *trivialization* of v is given as identifying v with $L \cdot \mathbb{R}^2$ (I use *cdot* because I could not find a good symbol for the product). Legendrian knots have what are called *line bundles*, which use a Legendrian knot's canonical framing to give a framing of v ove L.

Given some normal bundle, we define a number tw as the twisting of l with respect to a normal bundle's *preassigned* framing. If L is null-homologous, then L has a framing given by a Seifert surface, and the twisting of L with respect to this surface is known as the Thurston-Bennequin invariant of L, and is denoted as $tb(L)$.

We now come to the last classical invariant: *rotation number*. We define this invariant only for null-homologous knots. We assume that L has an embedded, orientable surface. When the contact planes for this knot are restricted only to those planes that intersect this surface create a trivialization by giving a two-dimensional bundle - as Etnyre states, over a surface with boundary, in this case the embedded, orientable surface, any orientable two plane bundle is trivial. Given this trivialization, we acquire another trivialization which is the product of L and $\mathbb{R}^{2.}$ If we denote a non-zero vector field v as one tangent to L, pointing in the direction of the orientation given on L, we can imagine v as a path of vectors in $\mathbb{R}^2$, then we realize there is a *winding number* for these vectors, and this *winding number* is the rotation number of L. Given that we used the orientation of L in determinng *rotation number*, we can say the *rotation number* is dependent upon the orientation of L and will accordingly change signs if there is a change in orientation.

We have, in a non-rigorous way, defined our three classical invariants. We now want to find the calculation of the *rotation number* as an end to this section.

We have to simply find how many times a non-zero tangent vector field v that is in $\mathbb{R}^2$ goes, or winds, around the origin of $\mathbb{R}^2$. This involves making a count of how many times our trivialization and v point in the same direction/ are oriented the same way; this of course creates an intersection between v and our trivialization. We determine the sign at each intersection by figuring out if v passes the trivialization clockwise or counter-clockwise $(1, -1$, respectively). We then notice that the down and up cusps of our diagram also show whether an intersection will be positive or negative, and as this counts the number of times v intersects our trivialization, we just need to divide by two. So the formula is:

$$r(L) = \frac{1}{2}(D - U)$$

We conclude the rough draft here.

4 Chekanov-Eliashberg DGA

This will be added in later, and is not part of the rough draft.

References

[1] J. Etnyre, *Legendrian and Transversal Knots*, from: Handbook of Knot Theory, Elsevier B. V., Amsterdam (2005).

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