An Introduction to Legendrian Knots

Shreyas Casturi, Ana Fishburn

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Contact Structures

Definition

A contact structure on $\mathbb{R}^3$ is a method of assigning a plane to every point, such that these planes satisfy certain technical conditions.

Throughout, we will only consider the standard contact structure, which twists along the y-axis:

Legendrian Knots

Definition

A Legendrian knot K is a smooth knot which is, at every point, parallel to the plane at that point given by the contact structure.

More concretely: if K is the image of $\phi : [0,1] \to \mathbb{R}^3$ given in coordinates by $t \mapsto \phi(x(t), y(t), z(t))$, then K is Legendrian if for all t,

$$z'(t) - y(t)x'(t) = 0.$$

Equivalences of Legendrian Knots

Two Legendrian knots are equivalent if there is a continuous family of Legendrian knots between them. This is similar to the definition of smooth knot equivalence, only with the allowable motions restricted.

Theorem

If two Legendrian knots are Legendrian equivalent, then they are equivalent as smooth knots.

Projections of Legendrian Knots

If we want to have clear visual representations of Legendrian knots, we need diagrams that respect the contact structure.

Definition

The front projection, denoted $\Pi(K)$, of a Legendrian knot K is the projection into the xz-plane.

Definition

The Lagrangian projection, denoted $\pi(K)$, of a Legendrian knot K is the projection into the xy-plane.

Properties of the Front Projection

We solve the equation $z'(t) - y(t)x'(t) = 0$ for y:

$$y(t) = \lim_{s \to t} \frac{z'(s)}{x'(s)}.$$

- No vertical tangencies are allowed: if $z'(t) = 0$, then $x'(t)$ must also be zero. Instead, we allow cusps, where the diagram makes a sharp horizontal point.
- At each crossing, the under strand has greater slope than the over strand.

Converting Smooth Knots into Legendrian Knots

Taking any diagram of a smooth knot, we can get a valid front diagram for a Legendrian knot by the following procedure:

- Rotate the diagram around each crossing until the under strand has greater slope.
- Convert every vertical tangency into a cusp.

Theorem

For any smooth knot K, there is a Legendrian knot K', such that K and K' are equivalent as smooth knots.

Reidemeister Theorem for Front Projections

Theorem

Two front projections represent equivalent Legendrian knots if and only if they can be related by a sequence of the following moves:

Properties of the Lagrangian Projection

This time, we solve the equation $z'(t) - y(t)x'(t) = 0$ for z:

$$z(t) = z_0 + \int_0^t y(s)x'(s)\,ds,$$

so we can always recover the z coordinate (up to translation) from $\pi(K)$.

- $\int_0^1 y(s)x'(s)\,ds = 0$.
- $\int_{t_1}^{t_2} y(s)x'(s)\,ds \neq 0$ whenever $(x(t_1), y(t_1)) = (x(t_2), y(t_2))$.

Classical Invariants

We now consider Legendrian knot invariants, which, similarly to smooth knot invariants, are functions from the set equivalence classes of Legendrian knots.
There are three classical invariants we will be discussing:

- *The Topological Knot Type*
- *The Thurston-Bennequin Number*
- *The Rotation Number*

NOTE: We will consider in the following slides knots that have an *orientation*.

Topological Knot Type

Because two Legendrian equivalent knots are automatically equivalent as smooth knots, we have the following invariant:

Definition

The topological knot type of a Legendrian knot L is $k(L)$, the equivalence class of all knots smoothly equivalent to L.

For example the following two front diagrams give inequivalent Legendrian knots, because one is an unknot while the other is a figure-eight.

Thurston-Bennequin Invariant

Geometrically, the Thurston-Bennequin number of a Legendrian knot L measures how much the contact planes "twist" around L. We write the Thurston-Bennequin number as $tb(L)$. The full definition requires some heavy geometry, so we give effective definitions in terms of the projections of L.

Calculating $tb(L)$ Using Front Projections

For a *front* projection:

- There are two things we must calculate:
  - The *writhe* number of the knot projection.
  - The number of *cusps* the knot projection has.

The formula for a front projection is

$$tb(L) = writhe(\Pi(L)) - \frac{1}{2}(cusps).$$

Infinitely Many Non-equivalent Legendrian Knots

Given a front projection of L, we can always perform the following procedure to decrease its tb number by 1:

Theorem

For any smooth knot K, there are infinitely many non-equivalent Legendrian knots with K as their topological knot type.

Calculating $tb(L)$ Using Lagrangian Projections

For a *Lagrangian* projection, we need only the *writhe* number of the projection.
The formula is simply

$$tb(L) = writhe(\pi(L)).$$

Rotation Number

The rotation number, denoted $r(L)$, of L describes how much L twists inside of the contact planes. Again, the geometric definition is elaborate, so we will describe the methods of effectively computing $r(L)$.

Calculating Rotation Number

With a *front projection*:

- We need the amount of *down* cusps a projection has, denoted as D.
- We also need the amount of *up* cusps a projection has, denoted as U.

With this, our formula is:

$$r(L) = \frac{1}{2}(D - U).$$

NOTE: Swapping the orientation will swap the sign of the rotation number, so $|r(L)|$ is the proper invariant for unoriented Legendrian knots.

Beyond the Classical Invariants

It turns out that these three invariants, $k(L)$, $tb(L)$, and $r(L)$ are not sufficient to classify Legendrian knots. For example, these two Legendrian 5_2 knots have the same tb and r, and some very sophisticated tools are needed to tell them apart.

FIGURE 29. The knot L (left) and L' (right).

References

- Sabloff, Joshua M (2009, November). *What Is... a Legendrian Knot?* Notices of the AMS. Volume 56, Number 10.
- Etnyre, John B (2005). *Legendrian and Tranversal Knots.* Handbook of Knot Theory, Elsevier, pp. 105-185.