An Introduction to Legendrian Knots

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Contact Structures

Definition

A contact structure on $\mathbb{R}^3$ is a method of assigning a plane to every point, such that these planes satisfy certain technical conditions.

Throughout, we will only consider the standard contact structure, which twists along the y-axis:

Legendrian Knots

Definition

A Legendrian knot K is a smooth knot which is, at every point, parallel to the plane at that point given by the contact structure.

We can use coordinates: if K is the image of $t \mapsto (x(t), y(t), z(t))$, then K is Legendrian if for all t,

$$z'(t) - y(t)x'(t) = 0.$$

Equivalences of Legendrian Knots

Two Legendrian knots are equivalent if there is a continuous family of Legendrian knots between them. This is similar to the definition of smooth knot equivalence, only with the allowable motions restricted.

Theorem

If two Legendrian knots are Legendrian equivalent, then they are equivalent as smooth knots.

Projections of Legendrian Knots

If we want to have clear visual representations of Legendrian knots, we need diagrams that respect the contact structure.

Definition

The front projection of a Legendrian knot K is the projection into the xz-plane.

Definition

The Lagrangian projection of a Legendrian knot K is the projection into the xy-plane.

Properties of the Front Projection

To determine what properties front projections must have, look at the formula $x'(t) - y(t)z'(t) = 0$.

- No vertical tangencies are allowed: if $z'(t) = 0$, then $x'(t)$ is also zero. Instead, we allow cusps, where the diagram makes a sharp horizontal point.
- At each crossing, the under strand has greater slope than the over strand.

Converting Smooth Knots into Legendrian Knots

Taking any diagram of a smooth knot, we can get a valid front diagram for a Legendrian knot by the following procedure:

- Rotate the diagram around each crossing until the under strand has greater slope.
- Convert every vertical tangency into a cusp.

Theorem

For any smooth knot K, there is a Legendrian knot K', such that K and K' are equivalent as smooth knots.

Properties of the Lagrangian Projection

On Classical Invariants

We consider the idea of an *knot invariant*, and wish to apply this idea to *Legendrian knots.*

There are three invariants we will be discussing:

- *The Topological Knot Type*
- *The Thurston-Bennequin Number*
- *The Rotation Number*

NOTE: We will consider in the following slides knots that have an *orientation*.

The Topological Knot Type

One invariant (or collection of invariants) we can consider is those invariants for *smooth knots*.
That is, we may consider our projections of the Legendrian knots as projections of smooth knots, and see what invariants hold between these "smooth" knots.
The *topological knot type* is then the type of knot we find when we consider the knots as a *smooth* knots and not as *Legendrian* knots.

The Thurston-Bennequin Invariant

Definition

This invariant measures the degree of twisting (twisting of planes) around our given Legendrian knot, which we denote as L.

The formal definition involves heavy usage of geometrical concepts, so we'll revert to a combinatorial, calculation-based method of obtaining this invariant.

An Example of Calculating TB (Front)

For a *front* projection:

- There are two things we must calculate:
  - The *writhe* number of the knot projection.
  - The number of *cusps* the knot projection has.

The formula for a *front* projection:

$$tb(L) = writhe(\Pi(L)) - \frac{1}{2}(cusps)$$

There are two things we must calculate for a *Legendrian knot*:

An Example of Calculating TB (Lagrangian)

For a *Lagrangian* projection, we need only the *writhe* number of the projection.
The formula is literally:

$$tb(L) = writhe(\pi(L))$$

.

The Rotation Number

Definition

The winding number of a curve around some given point is the number of times the curve travels *counterclockwise* around the point.

Definition

The rotation number is the winding number of a *Legendrian knot*. The point to travel around is the origin of $\mathbb{R}^{\nvDash}$.

As with *tb*, the reasoning of why the rotation number is the winding number involves higher-level geometric concepts, so we'll defer to a calculation to explain the rotation number.

Calculating Rotation Number

With a *front projection*:

- We need the amount of *down* cusps a projection has, denoted as D.
- We also need the amount of *up* cusps a projection has, denoted as U.

With this, our formula is:

$$r(L) = \frac{1}{2}(D - U).$$

With a *Lagrangian projection*:

$$r(L) = winding(\pi(L)).$$

NOTE: The orientation can change the sign of the rotation number.

An Example of Calculating Rotation Number

Insert diagram here.