A Brief Introduction to Legendrian Knots

Shreyas Casturi
Ana Fishburn

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Abstract

We start by defining Legendrian knots and isotopies of Legendrian knots. Then we study the classical invariants of Legendrian knots: the topological knot type, the Thurston-Bennequin number, and the rotation number. We then explain how, while sufficient for certain classes of knots in tight contact structures, there are nonisotopic Legendrian knots with equal classical invariants.

In particular, we introduce the Chekanov-Eliashberg DGA, and see that its homology, the knot contact homology, is an invariant. We see how this can be computed to distinguish versions of the 5_2 knot with equal classical invariants.

1 Introduction

A *contact structure* on $\mathbb{R}^3$ is a method of placing a plane at every point, such that these planes satisfy certain technical conditions. There are various distinct contact structures we could use, some of them giving very different properties. However, for this paper, we will exclusively consider the standard contact structure: At every point in $\mathbb{R}^3$, we consider the plane spanned by

$$\left\{\frac{\partial}{\partial y}, \frac{\partial}{\partial x} + y\frac{\partial}{\partial z}\right\}.$$

Visually, we can see that these planes are always tangent to the y-axis, but also twist around once when moving from $y = -\infty$ to $y = \infty$.

Now that we have this contact structure, we can define a *Legendrian knot* as a knot that is at every point tangent to the plane at that point. In particular, if we look at the tangent vector v to K at a point $p = (x, y, z)$, then

$$v = A\frac{\partial}{\partial y} + B\left(\frac{\partial}{\partial x} + y\frac{\partial}{\partial z}\right)$$

for some real numbers A and B. We can equivalently formulate this condition using a parametrization of K as the image of a smooth embedding $\phi : [0, 1] \to \mathbb{R}^3$, where $\phi(t) = (x(t), y(t), z(t))$ satisfies

$$z'(t) - y(t)x'(t) = 0.$$

We can now say that two Legendrian knots are equivalent if they can be related by a continuous family of Legendrian knots. This is very similar to the definition of equivalence for smooth knots, only here, we restrict the allowable motions. Therefore, Legendrian equivalence is a finer relation than smooth knot equivalence, and, in fact, we will show that there are many Legendrian inequivalent knots with the same smooth knot type.

2 Projections

Before we can discuss invariants of Legendrian knots, we must first understand the ways to visually represent them. Because Legendrian knots rely on our choice of contact structure, and therefore our choice of coordinates, dealing with projections of Legendrian knots will be more complicated than with smooth knots.

The first method of projecting Legendrian knots is the *front projection*. This is the projection onto the xz-plane. From the differential equation $z'(t) = y(t)x'(t)$, we see that we can recover the y coordinate from the front projection by

$$y(t) = \lim_{t_0 \to t} \frac{z'(t)}{x'(t)}.$$

We find the following two properties of front projections:

i) There may be no vertical tangencies: if $x'(t)$ vanishes, so does $z'(t)$. Instead we allow *cusps*, points where the diagram forms a sharp horizontal point.

ii) At each crossing, the under strand has greater slope than the over strand.

It turns out that these conditions completely characterize front projections: if $(x(t), z(t))$ satisfies these conditions, they are the coordinates of a Legendrian knot. Furthermore, for any Legendrian knot, we can arrange for the set of cusp points to be finite.

We can take a projection of a smooth knot and convert it into the front projection of a Legendrian knot by rotating each crossing until the under strand has greater slope, and then replacing vertical tangencies with cusps. We have therefore proved

Theorem 1. Every smooth knot may be realized as a Legendrian knot.

We now consider the harder to work with, but just as essential, *Lagrangian projection*. This is the projection onto the xy-plane. This time, we must recover the z coordinate by integrating, so up to a constant z_0, we have

$$z(t) = z_0 + \int_0^t y(s)x'(s)\, ds.$$

Now $(x(t), y(t))$ will provide a valid $z(t)$ if and only if

i) $\int_0^1 y(s)x'(s)\, ds = 0$.

ii) $\int_{t_1}^{t_2} y(s)x'(s)\, ds \neq 0$ whenever $(x(t_1), y(t_1)) = (x(t_2), y(t_2))$.

3 Classical Invariants

4 Chekanov-Eliashberg DGA

References

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