An Introduction to Legendrian Knots

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Contact Structures

Definition

A contact structure on $\mathbb{R}^3$ is a method of assigning a plane to every point, such that these planes satisfy certain technical conditions.

Throughout, we will only consider the standard contact structure, which twists along the y-axis:

Legendrian Knots

Definition

A Legendrian knot K is a smooth knot which is, at every point, parallel to the plane at that point given by the contact structure.

More concretely: if K is the image of $\phi : [0, 1] \to \mathbb{R}^3$ given in coordinates by $t \mapsto \phi(x(t), y(t), z(t))$, then K is Legendrian if for all t,

$$z'(t) - y(t)x'(t) = 0.$$

Equivalences of Legendrian Knots

Two Legendrian knots are equivalent if there is a continuous family of Legendrian knots between them. This is similar to the definition of smooth knot equivalence, only with the allowable motions restricted.

Theorem

If two Legendrian knots are Legendrian equivalent, then they are equivalent as smooth knots.

Projections of Legendrian Knots

If we want to have clear visual representations of Legendrian knots, we need diagrams that respect the contact structure.

Definition

The front projection, denoted $\Pi(K)$, of a Legendrian knot K is the projection into the xz-plane.

Definition

The Lagrangian projection, denoted $\pi(K)$, of a Legendrian knot K is the projection into the xy-plane.

Properties of the Front Projection

We solve the equation $z'(t) - y(t)x'(t) = 0$ for y:

$$y(t) = \lim_{s \to t} \frac{z'(s)}{x'(s)}.$$

- No vertical tangencies are allowed: if $z'(t) = 0$, then $x'(t)$ must also be zero. Instead, we allow cusps, where the diagram makes a sharp horizontal point.
- At each crossing, the under strand has greater slope than the over strand.

Converting Smooth Knots into Legendrian Knots

Taking any diagram of a smooth knot, we can get a valid front diagram for a Legendrian knot by the following procedure:

- Rotate the diagram around each crossing until the under strand has greater slope.
- Convert every vertical tangency into a cusp.

Theorem

For any smooth knot K, there is a Legendrian knot K', such that K and K' are equivalent as smooth knots.

Reidemeister Theorem for Front Projections

Theorem

Two front projections represent equivalent Legendrian knots if and only if they can be related by a sequence of the following moves:

Properties of the Lagrangian Projection

This time, we solve the equation $z'(t) - y(t)x'(t) = 0$ for z:

$$z(t) = z_0 + \int_0^t y(s)x'(s)\,ds,$$

so we can always recover the z coordinate (up to translation) from $\pi(K)$.

- $\int_0^1 y(s)x'(s)\,ds = 0$.
- $\int_{t_1}^{t_2} y(s)x'(s)\,ds \neq 0$ whenever $(x(t_1), y(t_1)) = (x(t_2), y(t_2))$.