An Introduction to Legendrian Knots

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Contact Structures

Definition

A contact structure on $\mathbb{R}^3$ is a method of assigning a plane to every point, such that these planes satisfy certain technical conditions.

Throughout, we will only consider the standard contact structure, which twists along the y-axis:

Legendrian Knots

Definition

A Legendrian knot K is a smooth knot which is, at every point, parallel to the plane at that point given by the contact structure.

We can use coordinates: if K is the image of $t \mapsto (x(t), y(t), z(t))$, then K is Legendrian if for all t,

$$z'(t) - y(t)x'(t) = 0.$$

Equivalences of Legendrian Knots

Two Legendrian knots are equivalent if there is a continuous family of Legendrian knots between them. This is similar to the definition of smooth knot equivalence, only with the allowable motions restricted.

Theorem

If two Legendrian knots are Legendrian equivalent, then they are equivalent as smooth knots.

Projections of Legendrian Knots

If we want to have clear visual representations of Legendrian knots, we need diagrams that respect the contact structure.

Definition

The front projection of a Legendrian knot K is the projection into the xz-plane.

Definition

The Lagrangian projection of a Legendrian knot K is the projection into the xy-plane.

Properties of the Front Projection

To determine what properties front projections must have, look at the formula $x'(t) - y(t)z'(t) = 0$.

- No vertical tangencies are allowed: if $z'(t) = 0$, then $x'(t)$ is also zero. Instead, we allow cusps, where the diagram makes a sharp horizontal point.
- At each crossing, the under strand has greater slope than the over strand.

Converting Smooth Knots into Legendrian Knots

Taking any diagram of a smooth knot, we can get a valid front diagram for a Legendrian knot by the following procedure:

- Rotate the diagram around each crossing until the under strand has greater slope.
- Convert every vertical tangency into a cusp.

Theorem

For any smooth knot K, there is a Legendrian knot K', such that K and K' are equivalent as smooth knots.

Properties of the Lagrangian Projection