Aggregate Play and Welfare in Strategic Interaction on Networks

Karan N. Chadha [*] Ankur A. Kulkarni [†]

Abstract

Abstract Goes here.

1 Introduction

We consider the point of view of a system planner and study how do network effects affect aggregate play. In particular, we are interested in deriving bounds on the maximum aggregate play for the strategic interaction game with local contributions. This is of interest in the cases where more aggregate play is better, examples being research collaboration between firms.

The secondary aim is to find which equilibria are stable with respect to parameter uncertainty. Many games are modelled with $\delta = 1$, but in reality it may not be exactly 1 but close to 1. We characterize in such cases which configurations would be aggregate play maximizing equilibria even for $\delta \neq 1$. The results in [6] state that for $\delta = 1$, the aggregate play maximizing equilibria are characteristic vectors of maximum independent sets. We show that these equilibria are not robust to parametric perturbation. We introduce a new class of equilibria called *Independent Clique Equilibria* (ICE) which are robust to parametric uncertainty and aggregate play maximizing.

2 Model

We consider the model of strategic interaction over networks as discussed in [3]. A network $G = (V, E)$ is an ordered pair of a set of nodes (V) and a set of links $(E \subseteq V \times V)$. Two nodes $i, j \in V$ are said to be *neighbours* of each other or *adjacent* to each other if $(i, j) \in E$. The neighbourhood of a node i in network G, denoted by $N_G(i)$, is the set of all nodes which are adjacent to i. A path from node i to node j in a network is an alternating sequence of distinct nodes and links $v_0(= i), e_0, v_1, e_1, \ldots, v_n(= j)$ which begins with i and ends with j. Here, the link e_i is such that its endpoints are v_i and v_{i+1}. For a network G on n nodes, let $A \in \{0,1\}^{n\times n}$ denote its adjacency matrix: i.e.,

$$a_{ij} \coloneqq A(i,j) = \begin{cases} 1, & \text{if } (i,j) \in E(G) \\ 0, & \text{otherwise} \end{cases}$$

Now we define the concept of indepdendent sets of networks, which are essential to give characterizations of aggregate play maximizing equilibria. An *independent set* of a network is defined as a set of nodes such that none of them are neighbours. A *maximal independent set* is

[*]Karan is with the Electrical Engineering Department at the Indian Institute of Technology Bombay, Mumbai, India 400076. Email : karanchadhaiitb@gmail.com

[†]Ankur is with the Systems and Control Engineering group at the Indian Institute of Technology Bombay, Mumbai, India 400076. Email : kulkarni.ankur@iitb.ac.in

defined as an independent set S such that all the nodes which are not in S have at least one neighbour in S, i.e. $|N_G(i) \cap S| \geq 1 \; \forall i \notin S$. A *maximum* independent set is an independent set which has the highest cardinality amongst all independent sets. The cardinality of the maximum indepdendent set is denoted by $\alpha(G)$ and is also called the independence number of the network G.

We consider the set of agents to be the set of nodes V of a network G with $|V| = n$. Each agent simultaneously choses an action $x_i \geq 0$. We denote the vector of actions of all agents as $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and by $\mathbf{x}_{-i}$ the actions of all agents except i. A payoff parameter δ determines how much the action of an agent affects the payoff of his neighbour. The payoff of an agent depends on its own action and those of its neighbours. Effectively, the payoff function of agent i depends on its action x_i, the action of others $\mathbf{x}_{-i}$, a payoff parameter δ and the network G which defines the interaction structure of the game. Let it be denoted by $U_i(x_i, \mathbf{x}_{-i}; G, \delta)$. Actions are chosen by the agents to maximize individual payoffs. We denote the *best reply* of agent i by $x_i = f_i(\mathbf{x}_{-i}; G, \delta)$, i.e. it represents the actions which maximize $U_i(\cdot, \mathbf{x}_{-i}; G, \delta)$ given $\mathbf{x}_{-i}, G$ and δ. We consider Nash equilibrium outcomes $\mathbf{x}^*$ for which $x_i^* = f_i(\mathbf{x}_{-i}^*; G, \delta)$ for all agents $i \in V$, i.e. no agent is better off chosing an action other than x_i^*, given that all agents choose $\mathbf{x}_{-i}^*$.

We consider games with max-linear best replies. Examples of such games have been studied in [2], where they study the problem of private provision of public goods and in [1] where they study a game of strategic complements with linear quadratic payoffs. In our case, we consider a general payoff of the form

$$U_i(x_i, \mathbf{x}_{-i}; G, \delta) = b\left(x_i + \delta \sum_{j \in V} a_{ij} x_j\right) - c x_i,$$

where $b(\cdot)$ is a differentiable strictly concave increasing benefit function and $c > 0$ is the marginal cost. On maximizing $U_i(x_i, \mathbf{x}_{-i}; G, \delta)$ with respect to x_i under the constraint $x_i \geq 0 \; \forall i \in V(G)$, we get

$$f_i(\mathbf{x}_{-i}; G, \delta) = \begin{cases} e^* - \delta \sum_{j \in V} a_{ij} x_j, & \text{if } \delta \sum_{j \in V} a_{ij} x_j < e^*, \\ 0, & \text{if } \delta \sum_{j \in V} a_{ij} x_j \geq e^*, \end{cases} \tag{1}$$

where e^* is such that the marginal benefit is equal to the marginal cost at e^*, i.e. $b'(e^*) = c$. The best response can be succinctly written as

$$f_i(\mathbf{x}_{-i}; G, \delta) = \max\{0, e^* - \delta \sum_{j \in V} a_{ij} x_j\} \tag{2}$$

Hereon, we denote the game we consider as *SILC* (Strategic interaction with local contribution). Formally, it is restated below.

- Agents : V, the set of nodes of a network G.
- Actions : $x_i \geq 0$ denotes the action of agent i, chosen to maximize the utility.
- Utility : it depends on one's own action x_i, the action of others $\mathbf{x}_{-i}$, a payoff parameter δ and the network G which defines the interaction structure of the game,

  $$U_i(x_i, \mathbf{x}_{-i}; G, \delta) = b\left(x_i + \delta \sum_{j \in V} a_{ij} x_j\right) - c x_i,$$

  where $b(\cdot)$ is a differentiable strictly concave increasing benefit function and $c > 0$ is the marginal cost. Here, e^* is the action such that marginal benifit is equal to marginal cost, i.e. $b'(e^*) = c$.

2.1 Characterization of Nash equilibria of the game

To give a characterization of the Nash equilibria, we define $\mathcal{C}_i(\mathbf{x}) := x_i + \delta \sum_{j \in V} a_{ij} x_j = x_i + \delta \sum_{j \in N_G(i)} x_j$ which denotes *discounted sum of closed neighbourhood* of i with respect to $\mathbf{x}$. Let $\mathcal{C}(\mathbf{x}) = (\mathcal{C}_1(\mathbf{x}), \mathcal{C}_2(\mathbf{x}), \ldots, \mathcal{C}_n(\mathbf{x}))$. Then, $\mathcal{C}(\mathbf{x}) = (I + \delta A)\mathbf{x}$ where I denotes the $n \times n$ identity matrix and A denotes the adjacency matrix of the network G. $\mathbf{x}$ is an equilibrium of the above game if $x_i = f_i(\mathbf{x}_{-i}; G, \delta)$ $\forall i \in V$, i.e. for any $i \in V$, we have

1. $x_i = 0$ and $\mathcal{C}_i(\mathbf{x}) \geq e^*$.
2. $x_i > 0$ and $\mathcal{C}_i(\mathbf{x}) = e^*$

These conditions can be equivalently written as

$$x_i \geq 0,\ \mathcal{C}_i(\mathbf{x}) \geq e^*,\ x_i(\mathcal{C}_i(\mathbf{x}) - e^*) = 0 \quad \forall i \in V.$$

In the vector form these conditions are equivalent to

$$\mathbf{x} \geq 0,\ \mathcal{C}(\mathbf{x}) \geq e^*\mathbf{e},\ \mathbf{x}^\top(\mathcal{C}(\mathbf{x}) - e^*\mathbf{e}) = 0, \tag{3}$$

where $\mathbf{e}$ is the vector of 1's, i.e. $\mathbf{e} = (1, 1, \ldots, 1)$.

We also introduce the notion of a Linear Complementarity Problem, which has been extensively studied in operations research literature [5]. Given any matrix $M \in \mathbb{R}^{n \times n}$ and a vector $q \in \mathbb{R}^n$, LCP(M, q) is the following problem:

$$\text{Find } \mathbf{x} \in \mathbb{R}^n \text{ such that } \ \mathbf{x} \geq 0,\ y = M\mathbf{x} + q \geq 0,\ y^\top \mathbf{x} = 0.$$

Let SOL(LCP(M, q)) denote the set of $\mathbf{x}$ such that $\mathbf{x}$ solves the LCP(M, q). Specifically, for $M = I + \delta A$ and $q = -\mathbf{e}$, LCP$(I + \delta A, -\mathbf{e})$ is given by

$$\text{Find } \mathbf{x} \in \mathbb{R}^n \text{ such that } \ \mathbf{x} \geq 0,\ (I + \delta A)\mathbf{x} \geq \mathbf{e},\ ((I + \delta A)\mathbf{x} - \mathbf{e})^\top \mathbf{x} = 0.$$

We denote LCP$(I + \delta A, -\mathbf{e})$ by LCP$_\delta(G)$ and its set of solutions by SOL$_\delta(G)$. The relation between Nash equilibria of *SILC* and the solutions of LCP$_\delta(G)$ are given by the following theorem.

Theorem 2.1. *An action profile* $\mathbf{x}$ *is a Nash equilibrium of SILC if and only if*

$$\frac{1}{e^*}\mathbf{x} \in \mathrm{SOL}_\delta(G).$$

Proof. Let $\mathbf{x}$ be a Nash equilibrium of *SILC*. Then, it satisfies the conditions given in (3), which are equivalent to

$$\mathbf{x} \geq 0,\ \frac{1}{e^*}\mathcal{C}(\mathbf{x}) \geq \mathbf{e},\ \frac{1}{e^*}\mathbf{x}^\top(\frac{1}{e^*}\mathcal{C}(\mathbf{x}) - \mathbf{e}) = 0. \tag{4}$$

Also,

$$\frac{1}{e^*}\mathcal{C}(\mathbf{x}) = \frac{1}{e^*}(x_i + \delta \sum_{j \in N_G(i)} x_j) = \mathcal{C}(\frac{1}{e^*}\mathbf{x}).$$

Thus, (4) becomes

$$\mathbf{x} \geq 0,\ \mathcal{C}(\frac{1}{e^*}\mathbf{x}) \geq \mathbf{e},\ \frac{1}{e^*}\mathbf{x}^\top(\mathcal{C}(\frac{1}{e^*}\mathbf{x}) - \mathbf{e}) = 0.$$

Using $\mathcal{C}(\mathbf{x}) = (I + \delta A)\mathbf{x}$, we get

$$\frac{1}{e^*}\mathbf{x} \geq 0,\ (I + \delta A)\frac{1}{e^*}\mathbf{x} \geq \mathbf{e},\ \frac{1}{e^*}\mathbf{x}^\top((I + \delta A)\frac{1}{e^*}\mathbf{x} - \mathbf{e}) = 0, \tag{5}$$

Using the definition of LCP$_\delta(G)$, $\mathbf{x}$ is a Nash equilibrium of *SILC* only if $\frac{1}{e^*}\mathbf{x}$ solves LCP$_\delta(G)$.

To prove the other way around, we assume that $\frac{1}{e^*}\mathbf{x} \in \mathrm{SOL}_\delta(G)$. Then, the conditions given in (5) are satisfied. On multiplying by e^* throughout and replacing $(I + \delta A)\mathbf{x}$ by $\mathcal{C}(\mathbf{x})$, we get

$$\mathbf{x} \geq 0,\ \mathcal{C}(\mathbf{x}) \geq e^*\mathbf{e},\ \mathbf{x}^\top(\mathcal{C}(\mathbf{x}) - e^*\mathbf{e}) = 0.$$

Thus, if $\frac{1}{e^*}\mathbf{x} \in \mathrm{SOL}_\delta(G)$, then the action profile $\mathbf{x}$ is a Nash equilibrium of *SILC*. ■

For a vector x indexed by $V(G)$, let $\sigma(x)$ be its support, i.e.

$$\sigma(x) := \{i \in V(G) | x_i > 0\}.$$

Let $\mathbf{1}_S$ denote the characteristic vector of set S, i.e.

$$\mathbf{1}_S(i) := (\mathbf{1}_S)_i = \begin{cases} 1, & \text{if } i \in S \\ 0, & \text{otherwise.} \end{cases}$$

3 Aggregate play and welfare

In this section, we first find tight lower and upper bounds on the maximum aggegate play in terms of the properties of just the network for δ in a suitable interval. Then, we find approximations of the aggregate welfare in terms of the aggregate play. Using these approximations and the bounds on maximum aggegate play, we find bounds on the maximum aggregate welfare of *SILC*.

3.1 Lower bound on aggregate play

In this section, we discuss a lower bound on the maximum aggregate play which holds true for all networks and all values of δ. To find such a lower bound, we use Proposition 8 of [3], which says that the maximum aggregate play is decreasing in δ for a fixed network. Formally, the theorem which we use is stated below:

Theorem 3.1. *Let $x^*(\delta, G)$ be the aggregate play maximizing equilibrium of the SILC with parameter δ and network G. If $\delta' \geq \delta$,*

$$\sum_{i=1}^{n} x_i^*(\delta', G) \leq \sum_{i=1}^{n} x_i^*(\delta, G)$$

Theorem 3.2. *For SILC on any network G and any value of $\delta \leq 1$, the maximum aggregate play is at least $e^*\alpha(G)$, where $\alpha(G)$ is the independence number of the graph.*

Proof. From Theorem 3.1, the maximum aggregate play for any game with $\delta \leq 1$ will be at least as much as the maximum aggregate play in the case when $\delta = 1$. For $\delta = 1$, we know from [6] that the maximum aggregate play is $e^*\alpha(G)$. Thus, the maximum aggregate play for $\delta \leq 1$ at least $e^*\alpha(G)$. ■

3.2 Upper bounds on aggregate play

In this section, we give a general form of the aggregate play maximizing solution for any network. We also provide an absolute upper bound on the maximum aggregate play depending on the independence number of the network. For this purpose, we extensively use the results of [4], wherein the concept of independent clique solutions (ICS), which are a special class of solutions of LCP$(I + \delta A, -\mathbf{e})$ was introduced. We define indepdendent clique equilibria (ICE) which on normalization by e^* give independent clique solutions. We show that for suitable conditions on δ, the aggregate play maximizing equilbirium is an ICE. Then, we provide an absolute upper bound for the maximum aggregate play for these solutions.

First we reproduce the definition of ICS from [4]:

Definition 3.1. *Two cliques in a network are said to be independent if no node of one clique has any node of the other clique as its neighbour.*

Definition 3.2. *An Independent Clique Solution (ICS) is a vector in $\mathbb{R}^{|V|}$ which is a solution of the LCP$_\delta(G)$ for some $\delta < 1$ and its support is a union of independent cliques.*

An algorithm to find these solutions has been given in [4]. The input to this algorithm is the network which underlies the game, a maximum independent set of the network and the substitutability factor δ. Since a maximum independent set always exists for any graph, ICSs of LCP$_\delta(G)$ exist for any G and suitable δ. The algorithm is explained in words in Algorithm 1.

Algorithm 1 Algorithm to find ICS

Input: $G(V, E)$, S (a maximum independent set of V)
Output: x(an ICS of LCP$_\delta(G)$)
for $i \in S$ **do**
  Find neighbours of i for which LCP conditions are not satisfied, say C_i.
  for all $j \in C_i \cup \{i\}$ **do**
    $x_j = \frac{1}{1+(|C_i|)\delta}$
  end for
  Remove the nodes $C_i \cup \{i\}$ and their neighbours from the graph and apply the same algorithm on a smaller graph.
end for
return x

The results of [4] dictate that if the support of an ICS is the set of independent cliques given by $\mathcal{K} = \{C_1, C_2, \ldots, C_{|\mathcal{K}|}\}$, then the ICS supported on $\mathcal{K}$, if it exists, is given by

$$x_i = \begin{cases} \frac{1}{1+(|C_j|-1)\delta}, & \text{if } i \in C_j,\ C_j \in \mathcal{K} \\ 0, & \text{otherwise.} \end{cases}$$

Next, we define a special class of equilibria which we call independent clique equilibria.

Definition 3.3. *An Independent Clique Equilibrium (ICE) of SILC is an action profile $\mathbf{x} \in \mathbb{R}^{|V|}$ such that $\frac{1}{e^*}\mathbf{x}$ is an ICS.*

Thus, given the support as a union of independent cliques $\mathcal{K} = \{C_1, C_2, \ldots, C_{|\mathcal{K}|}\}$, the only possible ICE is given by

$$x_i = \begin{cases} \frac{1}{1+(|C_j|-1)\delta} e^*, & \text{if } i \in C_j,\ C_j \in \mathcal{K} \\ 0, & \text{otherwise.} \end{cases}$$

Example 3.1. An example of a graph with an independent clique equilbirium is shown in Figure 1. The support of the ICE is shown as shaded nodes. Specifically, the ICE is given by

$$\mathbf{x} = \left(\frac{e^*}{1+\delta}, \frac{e^*}{1+\delta}, 0, \frac{e^*}{1+2\delta}, \frac{e^*}{1+2\delta}, \frac{e^*}{1+2\delta}, 0, e^*, 0, 0, \frac{e^*}{1+\delta}, \frac{e^*}{1+\delta} \right)$$

□

We now prove that under suitable condtions on δ, the maximum aggregate play is achieved by an ICE. We also define $\eta(G)$,

$$\eta(G) = \max \left\{ \frac{\omega(G) - 3 + \sqrt{(\omega(G)-3)^2 + 4(\omega(G)-1)}}{2(\omega(G)-1)}, \frac{\alpha(G)(\omega(G)-1) - \omega(G)}{\alpha(G)(\omega(G)-1)} \right\} \tag{6}$$

which is used to define a limit on δ above which, the following results hold

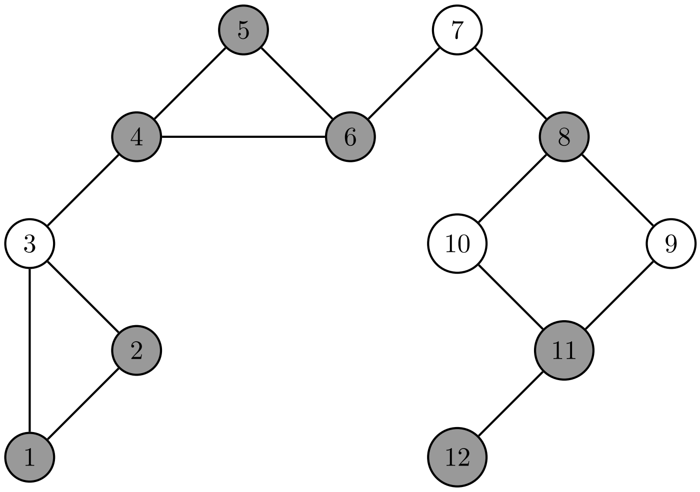

Figure 1: Example of an independent clique equilibrium (ICE). The support is the shaded nodes.

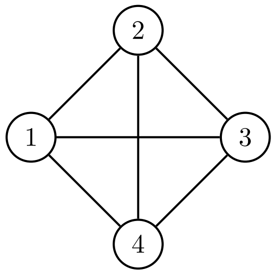

Figure 2: Example of a complete support equilibrium.

Theorem 3.3. *For any network G, if $\delta \geq \eta(G)$, then the maximum aggregate play amongst the Nash equilbira of SLIC is achieved by an independent clique equilibrium.*

Proof. From Theorem 2.1, we know that $\mathbf{x}$ is a Nash equilibrium of *SILC* if and only if $\frac{1}{e^*}\mathbf{x}$ is a solution of the Linear Complementarity Problem, LCP$(I + \delta A, -\mathbf{e})$. From Theorem 4.16 of [4], we know that the ℓ_1 norm maximizing solution of LCP$(I + \delta A, -\mathbf{e})$ is an indepdendent clique solution (say $\mathbf{y}$). Hence, the aggregate play maximizing Nash equilibrium of *SILC* is $e^*\mathbf{y}$, an ICE. ■

Example 3.2. In this example, we illustrate independent clique equilibria and how they differ from those equilibria in the case when $\delta = 1$. We consider a complete network (Figure 2) to illustrate this. Consider a maximum independent set of the graph in Figure 2, say $S = \{1\}$. Then, $e^*\mathbf{1}_S = \big(e^*, 0, 0, 0\big)$ is an equilibrium for the *SILC* with $\delta = 1$.

When $\delta < 1$, $e^*\mathbf{1}_S$ is no longer an equilibrium of the game. This is because, for every node other than $\{1\}$, the net play observed is δe^* and since $\delta e^* < e^*$ the equilbrium conditions are not satisfied. Using Algorithm 1 with the input maximum independent set being S, an equilibrium for *SILC* with $\delta < 1$ is given by $\mathbf{x} = \left(\frac{e^*}{1+3\delta}, \frac{e^*}{1+3\delta}, \frac{e^*}{1+3\delta}, \frac{e^*}{1+3\delta}\right)$.

We now provide arguments as to why all the nodes contributing equally is the only equilibrium. As δ changes from 1 to less than 1, the free rider nodes for whom the LCP conditions were just satisfied, i.e. the nodes which didn't observe a surplus in the sum of closed neighbourhood, start contributing positively. This provides incentive to node $\{1\}$ to contribute lesser than e^*. This establishes the fact that each node contributes a positive amount. Thus, equilibrium conditions dictate that the discounted sum of closed neighbourhood ($\mathcal{C}_i(\mathbf{x})$) for each node must be e^*. Since the game as seen by nodes $\{2\}, \{3\}$ and $\{4\}$ is the same, we assume they will contribute

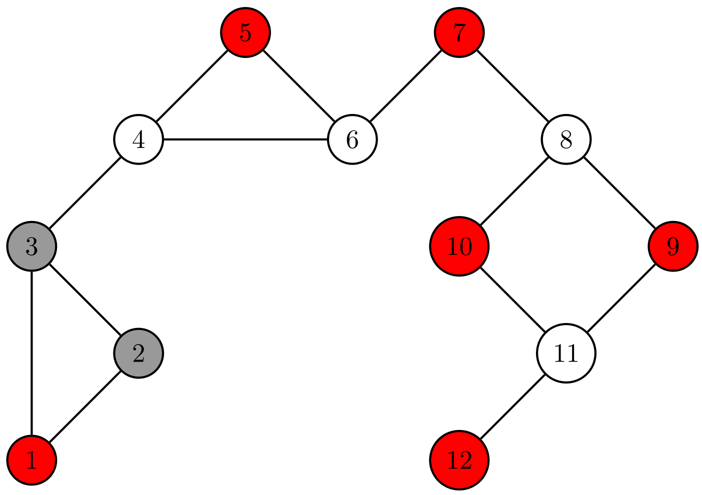

Figure 3: Example of an independent clique equilibrium (ICE). The support is the colored nodes and the red nodes denote the maximum independent set which when input to Algorithm 1 generates this ICE.

the same amount in equilibrium (say y). For the equilibrium conditions of $\{1\}$ to be satisfied, we get $x_1 = e^* - 3\delta y$.

Let $x_1 > y$. Then, we have $y < \frac{1}{1+3\delta}$ giving $\mathcal{C}_2(\mathbf{x}) = y + \delta(2y + e^* - 3\delta y) = y(1 + 2\delta - 3\delta^2) + \delta e^* < e^*$. Thus, the equilibrium conditions are not satisfied for $\{2\}$ and similarly for $\{3\}$ and $\{4\}$. Thus, we must have $x_1 \leq y$. Let $x_1 < y$. Then, we have $y > \frac{1}{1+3\delta}$ giving $\mathcal{C}_2(\mathbf{x}) = y(1+2\delta-3\delta^2) + \delta e^* > e^*$. But this provides incentive to $\{2\}$ (simlarly to $\{3\}$ and $\{4\}$) to contribute less than y. Thus, $x_1 < y$ cannot be possible under equilibrium either, which gives the only possible condition for equilibrium to be $x_1 = y$ which gives the equilbrium as $\mathbf{x}$.

We also note that the aggregate play of $\mathbf{x}$ is $\frac{4e^*}{1+3\delta}$, which is greater than $4e^*$ (the aggregate play of $e^*\mathbf{1}_S$).

□

Example 3.3. In this example, we illustrate independent clique equilibria and how they differ from those equilibria in the case when $\delta = 1$. We consider an example network in Figure 3 to illustrate this. Consider a maximum independent set of the network which is formed by the nodes colored in red, $S = \{1, 5, 7, 9, 10, 12\}$. Then, $e^*\mathbf{1}_S = \left(e^*, 0, 0, 0, e^*, 0, e^*, 0, e^*, e^*, 0, e^*\right)$ is an equilibrium for the *SILC* with $\delta = 1$.

When $\delta < 1$, $e^*\mathbf{1}_S$ is no longer an equilibrium of the game. This is because, for nodes $\{2\}, \{3\}$ and $\{4\}$, the net play observed is δe^* and since $\delta e^* < e^*$ the equilbrium conditions are not satisfied. Using Algorithm 1 with the input maximum independent set being S, an equilibrium for *SILC* with $\delta < 1$ is given by $\mathbf{x} = \left(\frac{e^*}{1+2\delta}, \frac{e^*}{1+2\delta}, \frac{e^*}{1+2\delta}, 0, e^*, 0, e^*, 0, e^*, e^*, 0, e^*\right)$.

Since the all the nodes connecting the network restricted to $\{1, 2, 3\}$ to the other parts of the network contribute 0, we can consider the network restricted to $\{1, 2, 3\}$ independent of the remaining network. While shifting from $\delta = 1$ to $\delta < 1$, the nodes which have only one neighbour in the maximum independent set are the ones for which the equilibrium conditions are now not satisfied. This provides them incentive to contribute positively and the support of the solution blows up from indepedent set to independent cliques. Using arguments similar to example 3.2, it can be seen that all nodes in each independent clique contributing equally is the only solution.

We also note that in the blowing up process we never decrease the aggregate play because $\frac{ne^*}{1+(n-1)\delta}$ is the contribution of a clique of size n obtained by blowing up the support from a node in the maximum independent set which is at least e^* for $n \geq 1$. Thus, the change in aggregate play on such a replacement is $\frac{ne^*}{1+(n-1)\delta} - e^*$, which is non-negative. $\square$

We now find an absolute upper bound on the maximum aggregate play for $\delta \geq \eta(G)$.

Theorem 3.4. *For any network G, if $\delta \geq \eta(G)$, then the maximum aggregate play amongst the Nash equilbira of SLIC is at most $e^* \left(\alpha(G) + 1 + \frac{1}{\alpha(G)-1}\right)$ and it is at least $e^*\alpha(G)$.*

3.3 Bounds on aggregate welfare

In this section, we first relate the aggregate welfare of the *SILC* game to aggregate play by defining a term called concavity of $b(\cdot)$. Then, we use the bounds on aggregate play found in the previous sections to find bounds on the aggregate welfare.

The aggregate welfare is defined as the sum of utilities of all agents, i.e.

$$W(\mathbf{x}; \delta, G) \coloneqq \sum_{i\in V} U_i(\mathbf{x}; \delta, G) = \sum_{i\in V} (b(\mathcal{C}_i(\mathbf{x})) - cx_i),$$

where $W(\mathbf{x}; \delta, G)$ denotes the aggregate welfare of the agents in *SILC* with action profile $\mathbf{x}$, substitutability δ and the interaction structure being a network G. To relate the aggregate welfare to aggregate play, we define the concavity σ_b of the payoff function similar to that defined in [2] as follows.

$$\sigma_b \coloneqq \frac{b(e^* + \delta(n-1)e^*) - b(e^*)}{c(n-1)e^*\delta}, \tag{7}$$

where n is the number of agents. It is the slope of the secant between e^* and $e^* + \delta(n-1)e^*$ normalized by c so that $\sigma_b < 1$. Note that since $b(\cdot)$ is increasing, we have $\sigma_b \geq 0$. By strict concavity of $b(\cdot)$, we have $\frac{b(e^*+\delta(n-1)e^*)-b(e^*)}{\delta(n-1)e^*} < b'(e^*) = c$, where the last equality follows from the definition of e^*. Thus, we have $0 \leq \sigma_b < 1$.

Our aim is to find approximations of aggregate welfare in terms of aggregate play and then bound the welfare using bounds on aggregate play. The following inequality is central to our analysis:

$$b(e^*) + c\sigma_b(\mathcal{C}_i(\mathbf{x}) - e^*) \leq b(\mathcal{C}_i(\mathbf{x})) \leq b(e^*) + c(\mathcal{C}_i(\mathbf{x}) - e^*), \tag{8}$$

for all $i \in V$, where $\mathbf{x}$ is an equilibrium of *SILC*. To prove,the first inequality in (8), we note that since $\mathbf{x}$ is an equilibrium, $\mathcal{C}_i(\mathbf{x}) \leq e^* + \delta(n-1)e^*$. Since $b(\cdot)$ is a concave function, we have

$$\frac{b(e^* + \delta(n-1)e^*) - b(e^*)}{(n-1)e^*\delta} \leq \frac{b(\mathcal{C}_i(\mathbf{x})) - b(e^*)}{\mathcal{C}_i(\mathbf{x}) - e^*},$$

which implies $b(e^*) + c\sigma_b(\mathcal{C}_i(\mathbf{x}) - e^*) \leq b(\mathcal{C}_i(\mathbf{x}))$. Also, we know that $c = b'(e^*)$ is the slope of the tangent at e^* and since $b(\cdot)$ is concave, the tangent lies above the secant giving,

$$\frac{b(\mathcal{C}_i(\mathbf{x})) - b(e^*)}{\mathcal{C}_i(\mathbf{x}) - e^*} < b'(e^*) = c,$$

which implies $b(\mathcal{C}_i(\mathbf{x})) \leq b(e^*) + c(\mathcal{C}_i(\mathbf{x}) - e^*)$. Thus, we have proved inequalities stated in (8). Taking limits as σ_b tends to 1 in (8), we get

$$\lim_{\sigma_b \to 1} b(\mathcal{C}_i(\mathbf{x})) = b(e^*) + c(\mathcal{C}_i(\mathbf{x}) - e^*). \tag{9}$$

We now find approximations of the total welfare for a given Nash equilibrium using expressions involving the total play.

Theorem 3.5. *Consider a strategic interactions game on a network G with minimum and maximum degree being d_{min} and d_{max} respectively and substitutability factor $\delta \geq \eta(G)$. Let $\mathbf{x}$ be a Nash equilibrium for such a game. Then, we have*

$$n(b(e^*) - ce^*) + c((d_{min}\delta + 1)\sigma_b - 1)\sum_{i\in V} x_i \leq W(\mathbf{x};\delta,G), \tag{10}$$

$$W(\mathbf{x};\delta,G) \leq n(b(e^*) - ce^*) + cd_{max}\delta\sigma_b\sum_{i\in V} x_i. \tag{11}$$

Using Theorem 3.5 and Theorem 3.4, we find bounds on the maximum aggregate welfare. Let $W^*(\delta, G)$ denote the maximum aggregate welfare.

Theorem 3.6. *Consider a strategic interactions game on a network G with minimum and maximum degree being d_{min} and d_{max} respectively and substitutability factor $\delta \geq \eta(G)$. Then, we have the following:*

(a) For $\sigma_b \in (0,1)$, we have

$$W^*(\delta,G) \leq n(b(e^*) - ce^*) + cd_{max}\delta\sigma_b e^*\left(\alpha(G) + 1 + \frac{1}{\alpha(G)-1}\right). \tag{12}$$

(b) For $\sigma_b \leq \frac{1}{1+d_{min}\delta}$, we have

$$W^*(\delta,G) \geq n(b(e^*) - ce^*) + c((d_{min}\delta + 1)\sigma_b - 1)e^*\left(\alpha(G) + 1 + \frac{1}{\alpha(G)-1}\right). \tag{13}$$

(c) For $\sigma_b \geq \frac{1}{1+d_{min}\delta}$, we have

$$W^*(\delta,G) \geq n(b(e^*) - ce^*) + c((d_{min}\delta + 1)\sigma_b - 1)e^*\alpha(G). \tag{14}$$

Proof. Let the aggregate play maximizing ICE be given by $\mathbf{x}^*$.

(a) We maximize the RHS of (11) over $\mathbf{x}$ to get

$$\begin{aligned} W(\mathbf{x};\delta,G) &\leq n(b(e^*) - ce^*) + cd_{max}\delta\sigma_b\sum_{i\in V} x_i^* \\ &\leq n(b(e^*) - ce^*) + cd_{max}\delta\sigma_b e^*\left(\alpha(G) + 1 + \frac{1}{\alpha(G)-1}\right) \end{aligned} \tag{15}$$

Since (15) holds for every $\mathbf{x}$ such that $\frac{\mathbf{x}}{e^*} \in \mathrm{SOL}_\delta(G)$, we have

$$W^*(\delta,G) \leq n(b(e^*) - ce^*) + cd_{max}\delta\sigma_b e^*\left(\alpha(G) + 1 + \frac{1}{\alpha(G)-1}\right). \tag{16}$$

(b) Since (10) holds for all $\mathbf{x}$, we consider this equation for $\mathbf{x}^*$, i.e.

$$W^*(\delta,G) \geq W(\mathbf{x}^*;\delta,G) \geq n(b(e^*) - ce^*) + c((d_{min}\delta + 1)\sigma_b - 1)\sum_{i\in V} x_i^* \tag{17}$$

For $\sigma_b \leq \frac{1}{1+d_{min}\delta}$, to minimize the lower bound, we need to consider the maximum value of $\sum_{i\in V} x_i^*$. Thus, we have for $\sigma_b \leq \frac{1}{1+d_{min}\delta}$

$$W^*(\delta,G) \geq n(b(e^*) - ce^*) + c((d_{min}\delta + 1)\sigma_b - 1)e^*\left(\alpha(G) + 1 + \frac{1}{\alpha(G)-1}\right). \tag{18}$$

(c) For $\sigma_b \geq \frac{1}{1+d_{min}\delta}$, to minimize the lower bound, we need to consider the minimum value of $\sum_{i \in V} x_i^*$. Thus, we have for $\sigma_b \geq \frac{1}{1+d_{min}\delta}$

$$W^*(\delta, G) \geq n(b(e^*) - ce^*) + c((d_{min}\delta + 1)\sigma_b - 1)e^*\alpha(G). \tag{19}$$

■

The bounds given in (15), (31) and (32) give good approximations to the maximum total welfare so as to help a designer define the structure of interactions to get desirable output. Taking the limit as concavity (σ_b) tends to 1 in (15) and (32), we have

$$\lim_{\sigma_b \to 1} W^*(\delta, G) \geq n(b(e^*) - ce^*) + cd_{min}\delta e^*\alpha(G). \tag{20}$$

and

$$\lim_{\sigma_b \to 1} W^*(\delta, G) \geq n(b(e^*) - ce^*) + cd_{max}\delta e^* \left(\alpha(G) + 1 + \frac{1}{\alpha(G) - 1}\right). \tag{21}$$

Next we consider the special case in which the network given is regular. A regular network is one in which all nodes have the same degree, i.e. $d_{max} = d_{min}$. In such a case, we can give an exact expression for the total welfare as $\sigma_b \to 1$ for any given Nash equilibrium.

Theorem 3.7. *For a strategic interactions game on a regular network G with degree d and substitutability factor δ, if $\mathbf{x}$ is an equilibrium profile, keeping $b(e^*)$ and $b'(e^*)$ fixed, we have that*

$$\lim_{\sigma_b \to 1} W(\mathbf{x}; \delta, G) = n(b(e^*) - ce^*) + cd\delta e^* \sum_{i \in V} x_i.$$

It is also worth noting that for regular networks, in the limit as $\sigma_b \to 1$, the maximum total welfare is achieved by the same equilibrium which achieves the maximum aggregate play. Let $\mathbf{x}^*$ be the play maximizing ICE profile. Then, taking the limit as $\sigma_b \to 1$, we have

$$\lim_{\sigma_b \to 1} W^*(\delta, G) = \lim_{\sigma_b \to 1} W(\mathbf{x}^*; \delta, G) = n(b(e^*) - ce^*) + cd\delta e^* \sum_{i \in V} x_i^*. \tag{22}$$

To prove this, note that we have

$$W(\mathbf{x}^*; \delta, G) \leq W^*(\delta; G) \leq n(b(e^*) - ce^*) + cd\delta e^* \sum_{i \in V} x_i^*.$$

The first inequality follows from the definition of W^* and the second one follows since the maximum of upper bound in (11) is achieved by $\mathbf{x}^*$ (the play maximizing ICE) by Theorem 3.3. Taking limits as $\sigma_b \to 1$, we have that $\lim_{\sigma_b \to 1} W(\mathbf{x}^*; \delta, G) = n(b(e^*) - ce^*) + cd\delta e^* \sum_{i \in V} x_i^*$. Thus, (22) is proved.

3.4 Networks with unique maxiumum independent sets

In this section, we consider the special case of networks for which a unique maximum independent set exists. We show that for a network with a unique maximum independent sets, for suitable conditions on δ, the characteristic vector of the unique maximum independent set is an aggregate play maximizing equilibrium. We extensively use the results of [4] for this purpose.

Theorem 3.8. *If G has a unique maximum independent set (S), for $\delta \geq \eta(G)$ then the maximum aggregate play amongst the Nash equilbira of SILC is $e^*\alpha(G)$ and is achieved by the characteristic vector of S.*

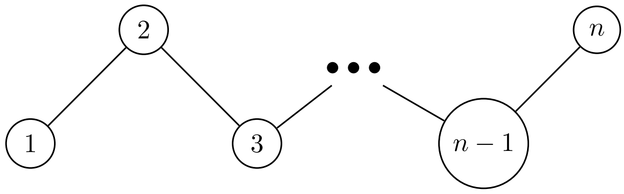

Figure 4: A line network with n nodes.

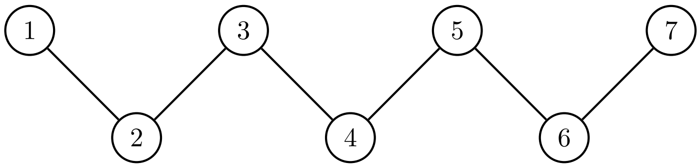

Figure 5: A line network with 7 nodes for Example 4.4

Proof. From Theorem 2.1, we know that $\mathbf{x}$ is a Nash equilibrium of *SILC* if and only if $\frac{1}{e^*}\mathbf{x}$ is a solution of the Linear Complementarity Problem, LCP$(I + \delta A, -\mathbf{e})$. From Corollary 4.17 of [4], we know that the ℓ_1 norm maximizing solution of LCP$(I + \delta A, -\mathbf{e})$ is the characteristic vector of the unique maximum independent set ($\mathbf{1}_S$). Hence, the aggregate play maximizing Nash equilbirium of *SILC* is $e^*\mathbf{1}_S$ and the maximum aggregate play is given by $e^*\alpha(G)$. ■

In this case, the approximations to aggregate welfare can be found by substituting $\sum_{i \in V} x_i = e^*\alpha(G)$ in (10) and (30). The approximations are given by

$$n(b(e^*) - ce^*) + c((d_{min}\delta + 1)\sigma_b - 1)e^*\alpha(G) \leq W(\mathbf{x}; \delta, G), \tag{23}$$

$$W(\mathbf{x}; \delta, G) \leq n(b(e^*) - ce^*) + cd_{max}\delta\sigma_b e^*\alpha(G). \tag{24}$$

4 Approximations of welfare for tree-networks

To find the upper bounds on the maximum aggregate welfare for tree networks, we first consider the case of underlying structure being line networks and find bounds on aggregate play for them. A *line network (chain)* is a network whose nodes can be listed as $1, 2, \ldots, n$ such that the links are $\{i, i+1\}$ for $i = 1, 2, \ldots, n-1$ (Figure 4). Then we consider *starlike* trees which are defined as trees with exactly one node of degree strictly greater than 2 (Figure 6). Using these as the building blocks, we give an expression for a tight upper bound on maximum aggregate play for general trees based on only the total number of nodes and degrees of each node for $\delta \geq \frac{1}{2}$. On getting the bounds on maximum aggregate play for trees, we find approximations to aggregate welfare using Theorem 3.5

4.1 Line networks

In this section, we consider the case of a line network with n nodes and find a tight upper bound on the maximum aggregate play. A sample of such a network is shown in Figure 4. We also show that when n is odd, i.e. the line network has an odd number of nodes, the upper bound is achieved.

Theorem 4.1. *If G is a line network with n nodes and $\delta \geq \frac{1}{2}$, the aggregate play of any Nash equilibrium (say $\mathbf{x}$) of SILC is at most $\frac{n+1}{2}e^*$, i.e. $\sum_{i=1}^{n} x_i \leq \frac{n+1}{2}e^*$. Moreover, the bound is achieved for line networks with odd number of nodes.*

Agent ↓	$\mathbf{x}^I$	$\mathbf{x}^{II}$	$\mathbf{x}^{III}$	$\mathbf{x}^{IV}$	$\mathbf{x}^V$
1	e^*	$-\frac{-\delta^3+2\,\delta^2+\delta-1}{2\,\delta^4-4\,\delta^2+1}$	$\frac{1}{1+\delta}e^*$	$\frac{1}{-\delta^2+\delta+1}e^*$	$\frac{1}{1+\delta}e^*$
2	0	$-\frac{-2\,\delta^3+\delta^2+2\,\delta-1}{2\,\delta^4-4\,\delta^2+1}$	$\frac{1}{1+\delta}e^*$	$\frac{1-\delta}{-\delta^2+\delta+1}e^*$	$\frac{1}{1+\delta}e^*$
3	e^*	$\frac{(\delta-1)\left(\delta^2+\delta-1\right)}{2\,\delta^4-4\,\delta^2+1}$	0	$\frac{1-\delta}{-\delta^2+\delta+1}e^*$	0
4	0	$-\frac{2\,\delta-1}{2\,\delta^4-4\,\delta^2+1}$	e^*	$\frac{1}{-\delta^2+\delta+1}e^*$	$\frac{1}{-\delta^2+\delta+1}e^*$
5	e^*	$\frac{(\delta-1)\left(\delta^2+\delta-1\right)}{2\,\delta^4-4\,\delta^2+1}$	0	0	$\frac{1-\delta}{-\delta^2+\delta+1}e^*$
6	0	$-\frac{-2\,\delta^3+\delta^2+2\,\delta-1}{2\,\delta^4-4\,\delta^2+1}$	$\frac{1}{1+\delta}e^*$	$\frac{1}{1+\delta}e^*$	$\frac{1-\delta}{-\delta^2+\delta+1}e^*$
7	e^*	$-\frac{-\delta^3+2\,\delta^2+\delta-1}{2\,\delta^4-4\,\delta^2+1}$	$\frac{1}{1+\delta}e^*$	$\frac{1}{1+\delta}e^*$	$\frac{1}{-\delta^2+\delta+1}e^*$
Validity →	$\delta \in [0.5, 1]$	$\delta \in [\frac{\sqrt{5}-1}{2}, 1] \cup [0, 0.5]$	$\delta \in [\frac{\sqrt{5}-1}{2}, 1]$	$\delta \in [\frac{1}{\sqrt{2}}, 1]$	$\delta \in [\frac{1}{\sqrt{2}}, 1]$
Total →	$4e^*$	$\frac{8\delta^3-6\delta^2-12\delta+7}{2\,\delta^4-4\,\delta^2+1}e^*$	$e^* + \frac{4}{1+\delta}e^*$	$\frac{6+4\delta-4\delta^2}{1+\delta-\delta^2}e^*$	$\frac{6+4\delta-4\delta^2}{1+\delta-\delta^2}e^*$

Table 1: A table showing all equilbira for a line network of 7 nodes.

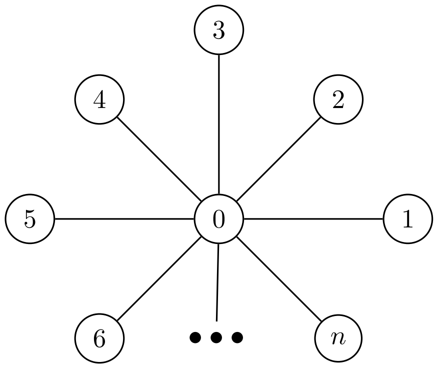

Figure 6: A star with n peripheral nodes.

We illustrate all equilibria of a line network of 7 nodes with an example given in Figure 5.

Example 4.4. As an example, we consider the network in Figure 5 which is a line network with 7 nodes. Let $\mathbf{x} = (x_1, x_2, \ldots, x_7)$ denote a vector of plays. example 4.4 shows all the equilibria and corresponding aggregate play of each of them.

From Theorem 4.1, the aggregate play is upper bounded by $\frac{n+1}{2}e^* = 4e^*$, for $\delta \geq \frac{1}{2}$. *Equilibrium I* has aggregate play $4e^*$ and hence it is the aggregate play maximizing equilibrium. From Proposition 2 of [3], *SILC* has a unique Nash equilibrium for $\delta \leq \frac{1}{|\lambda_{min}(A)|}$ and $\frac{1}{|\lambda_{min}(A)|} = 0.5412$ in our case. Thus, for $\delta \leq 0.5412$, we have a unique equilibrium. Specifically, for $\delta \leq \frac{1}{2}$, the only equilibrium is *Equilibrium II* and hence it is aggregate play maximizing. □

4.2 Stars and Starlike Networks

In this section, we consider the case when the interaction structure is a star-like network. First we show that, for a star with a central node (degree ≥ 2) and n peripheral nodes (as shown in Figure 6), there is a unique equilibrium.

Theorem 4.2. *If G is a star network with n (≥ 3) peripheral nodes and $\delta \geq \frac{1}{n}$, SILC has only one equilbrium in which all the peripheral nodes play action e^* and the central node free-rides.*

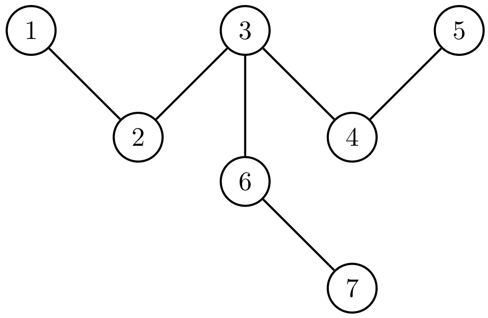

Figure 7: A starlike networks with 7 nodes for Example 4.5

Agent ↓	$\mathbf{x}^{I}$	$\mathbf{x}^{II}$	$\mathbf{x}^{III}$
1	e^*	$\frac{\delta+1}{2\delta+1}$	$\frac{1}{1+\delta}e^*$
2	0	$\frac{1}{2\delta+1}$	$\frac{1}{1+\delta}e^*$
3	e^*	$\frac{1-\delta}{2\delta+1}$	0
4	0	$\frac{1}{2\delta+1}$	$\frac{1}{1+\delta}e^*$
5	e^*	$\frac{\delta+1}{2\delta+1}$	$\frac{1}{1+\delta}e^*$
6	0	$\frac{1}{2\delta+1}$	$\frac{1}{1+\delta}e^*$
7	e^*	$\frac{\delta+1}{2\delta+1}$	$\frac{1}{1+\delta}e^*$
Validity →	$\delta \in [0.5, 1]$	$\delta \in [0, 1]$	$\delta \in [\frac{1}{2}, 1]$
Total →	$4e^*$	$\frac{2\delta+7}{2\delta+1}e^*$	$\frac{6}{1+\delta}e^*$

Table 2: A table showing all equilbira for a starline network of 7 nodes.

Next, we consider a starlike network structure with peripherals coming out of the central node (degree ≥ 3) themselves being chains instead of just single nodes. We consider m chains $S_1, S_2, \ldots, S_m$ with $n_1, n_2, \ldots, n_m$ nodes respectively having one leaf node connected to a central node c and the other leaf node free. We find a tight upper bound on the maximum aggregate play for such networks and also show that when all of $n_1, n_2, \ldots, n_m$ are odd, i.e. all peripheral chains have an odd number of nodes, the upper bound is achieved.

Theorem 4.3. *If G is a star network having $deg_G(c) = m(\geq 3)$ chains connected at a central node c and for $\delta \geq \frac{1}{2}$, the aggregate play of any Nash equilibrium (say $\mathbf{x}$) of SILC is at most $\frac{|V(G)|+m-1}{2}e^*$ i.e. $\sum_{i=1}^{n} x_i \leq \frac{|V(G)|+m-1}{2}e^*$. Moreover, the bound is achieved in the case of all chains are of odd length.*

Example 4.5. As an example, we consider the network in Figure 7 which is a line network with 7 nodes. Let $\mathbf{x} = (x_1, x_2, \ldots, x_7)$ denote a vector of plays. example 4.5 shows all the equilibria and corresponding aggregate play of each of them.

From Theorem 4.3, the aggregate play is upper bounded by $\frac{|V(G)|+m-1}{2} = \frac{7+3-1}{2} = 4.5e^*$, for $\delta \geq \frac{1}{2}$. On comparing the aggregate play of the three equilibria for $\delta \geq \frac{1}{2}$, we get *Equilibrium I* is the aggregate play maximizing equilibrium. From Proposition 2 of [3], *SILC* has a unique Nash equilibrium for $\delta \leq \frac{1}{|\lambda_{min}(A)|}$ and $\frac{1}{|\lambda_{min}(A)|} = 0.5$ in our case. Thus, for $\delta \leq 0.5$, we have a unique equilibrium. Speicifically, for $\delta \leq \frac{1}{2}$, the only equilibrium is *Equilibrium II* and hence it is the aggregate play maximizing equilibrium. $\square$

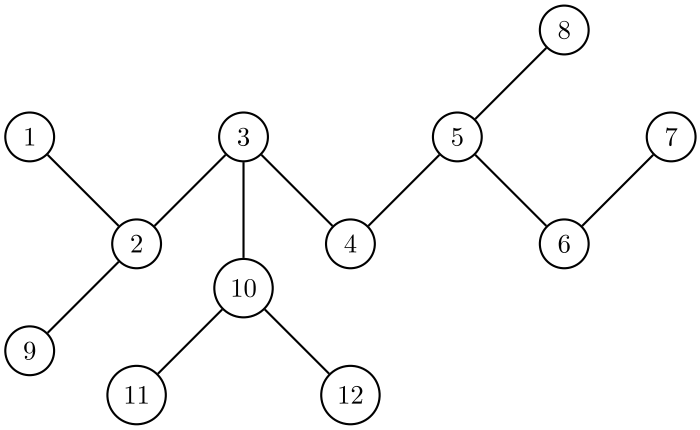

Figure 8: An example tree T

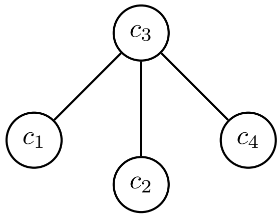

Figure 9: Tree T_C corresponding to the tree in Figure 8 with $c_1 = 2, c_2 = 10, c_3 = 3, c_4 = 5$

4.3 Trees

For a general tree T, we define the centers of the tree as those nodes with degree ≥ 3. Let the centers of a network T be $\mathcal{C}_T = \{c_1, c_2, \ldots, c_t\}$ with degree of center c_i in tree T denoted by d_i^T. Let center c_i have m_i chains attached which terminate in leaves and $d_i - m_i$ paths connecting other center nodes. Construct a network T_C such that $V(T_C) = \mathcal{C}_T$ and (c_i, c_j) is a link in T_C if and only if there is a path P_{ij} between c_i, c_j such that there is no other center in P_{ij}. Let $n_{join}^{ij} = |V(P_{ij})| - 2$. In the next theorem, we give a bound on the aggregate play of any Nash equilibrium in of the game *SILC* with $G = T$, a tree.

To illustrate the concept of a tree (say T), its centers ($\mathcal{C}_T$) and the new network constructed T_C, we provide an example tree shown in Figure 8. The central nodes are $\mathcal{C} = \{2, 10, 3, 5\}$. Let $c_1 = 2, c_2 = 10, c_3 = 3, c_4 = 5$. Then, $d_1^T = 3, d_2^T = 3, d_3^T = 3, d_4^T = 3$ and $m_1 = 2, m_2 = 2, m_3 = 0, m_4 = 2$. Also, T_C is as drawn in Figure 9 and $n_{join}^{13} = 0, n_{join}^{23} = 0, n_{join}^{43} = 1$.

Theorem 4.4. *Let G be a tree (T) with $\mathcal{C}_T = \{c_1, c_2, \ldots, c_t\}$ as the set of centers ($\mathcal{C}_T \neq \emptyset$) and let $\delta \geq \frac{1}{2}$. Then, the aggregate play of any Nash equilibrium (say $\mathbf{x}$) of SILC is at most $\frac{|V(G)| + \sum_{c_i \in \mathcal{C}_T} (d_i^T - d_i^{T_C}) - 1}{2}$,i.e. $\sum_{i \in V} x_i \leq \frac{|V(G)| + \sum_{c_i \in \mathcal{C}_T} (d_i^T - d_i^{T_C}) - 1}{2}$.*

Remark : The bound given in Theorem 4.4 is tight in the case when all paths between centrers P_{ij} have odd number of nodes and all chains emanating from each center nodes which terminate into leaves also have an odd number of nodes. □

Example 4.6. As an example, we consider the network in Figure 10 which is a line network with 7 nodes. Let $\mathbf{x} = (x_1, x_2, \ldots, x_7)$ denote a vector of plays. example 4.6 shows all the equilibria and corresponding aggregate play of each of them.

Note that for this example, the tree T_C is given by Figure 11. From Theorem 4.4, the aggregate play is at most $\frac{|V(G)| + \sum_{c_i \in \mathcal{C}_T} (d_i^T - d_i^{T_C}) - 1}{2} = \frac{7+2+2-1}{2} = 5e^*$, for $\delta \geq \frac{1}{2}$. On comparing the aggregate play of the three equilibria for $\delta \geq \frac{1}{2}$, we get *Equilibrium I* is the aggregate play

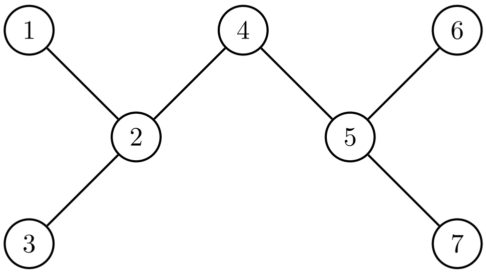

Figure 10: A tree with tight bounds for Example 4.6

Agent ↓	$\mathbf{x}^{I}$	$\mathbf{x}^{II}$
1	e^{*}	$\frac{\delta^2+\delta-1}{4\,\delta^2-1}$
2	0	$\frac{3\,\delta-1}{4\,\delta^2-1}$
3	e^{*}	$\frac{\delta^2+\delta-1}{4\,\delta^2-1}$
4	e^{*}	$-\frac{2\,\delta^2-2\,\delta+1}{4\,\delta^2-1}$
5	0	$\frac{3\,\delta-1}{4\,\delta^2-1}$
6	e^{*}	$\frac{\delta^2+\delta-1}{4\,\delta^2-1}$
7	e^{*}	$\frac{\delta^2+\delta-1}{4\,\delta^2-1}$
Validity →	$\delta \in [\frac{1}{3}, 1]$	$\delta \in [0, \frac{1}{3}]$
Total →	$5e^{*}$	$\frac{2\delta^2+12\delta-7}{4\delta^2-1}e^{*}$

Table 3: A table showing all equilbira for a tree network of 7 nodes.

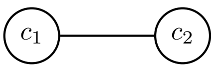

Figure 11: Tree T_C corresponding to the tree in Figure 10 with $c_1 = 2, c_2 = 5$

maximizing equilibrium. From Proposition 2 of [3], *SILC* has a unique Nash equilibrium for $\delta \leq \frac{1}{|\lambda_{min}(A)|}$ and $\frac{1}{|\lambda_{min}(A)|} = 0.5$ in our case. Thus, for $\delta \leq 0.5$, we have a unique equilibrium. Speicifically, for $\delta \leq \frac{1}{3}$, the only equilibrium is *Equilibrium II* and hence it is aggregate play maximizing. Also, for $\delta \in [\frac{1}{3}, \frac{1}{2}]$, the only equilibrium is *Equilibrium I* and thus it is aggregate play maximizing.

□

In the next theorem, using the bounds on aggregate play found for trees, we find bounds on the aggregate welfare for trees.

Theorem 4.5. *Consider a strategic interactions game on a tree T with maximum degree d_{max}, centers $\mathcal{C}_T = \{c_1, c_2, \ldots, c_t\}$ $((\mathcal{C}_T \neq \emptyset))$ and substitutability factor $\delta \geq \frac{1}{2}$. Then, we have the following:*

(a) For $\sigma_b \in (0,1)$, we have

$$W^*(\delta, G) \leq n(b(e^*) - ce^*) + cd_{max}\delta\sigma_b e^* \left(\frac{|V(G)| + \sum_{c_i \in \mathcal{C}_T}(d_i^T - d_i^{T_C}) - 1}{2} \right).$$

(b) For $\sigma_b \leq \frac{1}{1+\delta}$, we have

$$W^*(\delta, G) \geq n(b(e^*) - ce^*) + c((\delta+1)\sigma_b - 1)e^* \left(\frac{|V(G)| + \sum_{c_i \in \mathcal{C}_T}(d_i^T - d_i^{T_C}) - 1}{2} \right).$$

(c) For $\sigma_b \geq \frac{1}{1+\delta}$, we have

$$W^*(\delta, G) \geq n(b(e^*) - ce^*) + c((\delta+1)\sigma_b - 1)e^* \alpha(G).$$

5 Conclusion

A Proofs

A.1 Proof of Theorem 3.4

From Theorem 3.1, the maximum aggregate play for any game with $\delta \leq 1$ will be at least as much as that in the case when $\delta = 1$. For $\delta = 1$, we know from [6] that the maximum aggregate play is $e^*\alpha(G)$. Thus, the maximum aggregate play for $\delta \leq 1$ at least $e^*\alpha(G)$.

From Theorem 3.3, we know that the maximum aggregate play is achieved by an indepdendent clique equilibrium. We find an upper bound on the aggregate play of any indepdendent clique equilibrium to get a upper bound on the maximum aggregate play. Let $\mathbf{x}$ be an independent clique equilibrium such that $\sigma(\mathbf{x}) = \cup_{i=1}^{\alpha(G)} C_i$, where C_i are indepdendent cliques with $|V(C_i)| = n_i$. Then, the aggregate play of $\mathbf{x}$ is given by

$$\sum_{i \in V} x_i = \sum_{i=1}^{\alpha(G)} \frac{n_i}{1 + (n_i - 1)\delta} e^*. \tag{25}$$

Each term in the RHS of (25) is increasing in n_i and decreasing in δ. Thus, to find the maximum value, we take all $n_i = \omega(G)$, which is the size of the largest clique in the network. We know that $\delta \geq \frac{\alpha(G)(\omega(G)-1)-\omega(G)}{\alpha(G)(\omega(G)-1)}$, so we take $\delta = \frac{\alpha(G)(\omega(G)-1)-\omega(G)}{\alpha(G)(\omega(G)-1)}$ to find the bound.

Substituting in (25), we get

$$\begin{aligned}
\sum_{i\in V} x_i &= \sum_{i=1}^{\alpha(G)} \frac{n_i}{1+(n_i-1)\delta} e^* \\
&\le \frac{\alpha(G)\omega(G)}{1+(\omega(G)-1)\frac{\alpha(G)(\omega(G)-1)-\omega(G)}{\alpha(G)(\omega(G)-1)}} e^* \\
&= \frac{\alpha(G)\omega(G)}{\frac{\alpha(G)+\alpha(G)\omega(G)-\alpha(G)-1}{\alpha(G)}} e^* \\
&= \frac{\alpha^2(G)}{\alpha(G)-1} e^*.
\end{aligned}$$

Thus, we have $\sum_{i\in V} x_i \le \left(\alpha(G)+1+\frac{1}{\alpha(G)-1}\right) e^*$. ■

A.2 Proof of Theorem 3.5

We first prove (10). From the first inequality in (8), we have

$$\begin{aligned}
W(\mathbf{x};\delta,G) &= \sum_{i\in V}(b(\mathcal{C}_i(\mathbf{x})) - cx_i) \\
&\ge \sum_{i\in V}(b(e^*) - ce^* + c(\sigma_b \mathcal{C}_i(\mathbf{x}) - x_i)) \\
&= n(b(e^*) - ce^*) + c\left(\delta\sigma_b \sum_{i\in V}\sum_{j\in V} a_{ij}x_j - (1-\sigma_b)\sum_{i\in V} x_i\right) \\
&\ge n(b(e^*) - ce^*) + c(d_{min}\delta\sigma_b - (1-\sigma_b))\sum_{i\in V} x_i \qquad (26) \\
&\ge n(b(e^*) - ce^*) + c((d_{min}\delta+1)\sigma_b - 1)\sum_{i\in V} x_i.
\end{aligned}$$

Here, (26) follows from the fact that $\sum_{i\in V}\sum_{j\in V} a_{ij}x_j = \sum_{i\in V} d_i x_i \ge d_{min}\sum_{i\in V} x_i$, where d_i denotes the degree of node i. Thus, (10) is proved.

Next, we prove (11) using the second inequality in (8).

$$\begin{aligned}
W(\mathbf{x};\delta,G) &= \sum_{i\in V}(b(\mathcal{C}_i(\mathbf{x})) - cx_i) \\
&\le \sum_{i\in V}(b(e^*) - ce^* + c(\mathcal{C}_i(\mathbf{x}) - x_i)) \\
&= n(b(e^*) - ce^*) + c\delta\sum_{i\in V}\sum_{j\in V} a_{ij}x_j \\
&\le n(b(e^*) - ce^*) + c(d_{max}\delta\sigma_b)\sum_{i\in V} x_i. \qquad (27)
\end{aligned}$$

Here, (27) follows from $\sum_{i\in V}\sum_{j\in V} a_{ij}x_j = \sum_{i\in V} d_i x_i \le d_{max}\sum_{i\in V} x_i$, where d_i denotes the degree of node i. Thus, (10) is proved. ■

A.3 Proof of Theorem 3.7

Taking the limit as $\sigma_b \to 1$ in (10) we get,

$$\lim_{\sigma_b\to 1} W(\mathbf{x};\delta,G) \ge n(b(e^*) - ce^*) + cd\delta\sum_{i\in V} x_i$$

and taking the limit as $\sigma_b \to 1$ in (11) we get,

$$\lim_{\sigma_b \to 1} W(\mathbf{x}; \delta, G) \leq n(b(e^*) - ce^*) + cd\delta \sum_{i \in V} x_i.$$

Thus, we have

$$\lim_{\sigma_b \to 1} W(\mathbf{x}; \delta, G) = n(b(e^*) - ce^*) + cd\delta e^* \sum_{i \in V} x_i. \blacksquare$$

A.4 Proof of Theorem 4.1

We will prove this by induction. First, for the base case, note that for a line network with just 1 node, the only equilibrium is itself contributing 1, making the aggregate play $e^* = \frac{1+1}{2}e^* = \frac{n+1}{2}e^*$.

Let the aggregate play be upper bounded by $\frac{k+1}{2}e^*$ for all line networks with number of nodes equal to $k < n$. Let P_n denote the line network with n nodes. Let $\mathbf{x}$ represent an equilibrium of *SILC* with $G = P_n$ and $\delta \geq \frac{1}{2}$.

Case 1: $\sigma(\mathbf{x}) = V$.

If an equilibrium has full support the discounted sum of plays of the closed neighbourhood of each node is one, i.e. $\mathcal{C}_i(\mathbf{x}) = e^*$. Then,

$$\begin{aligned}
\sum_{i=1}^{n} \mathcal{C}_i(x) &= \sum_{i=1}^{n} x_i + \delta(\sum_{i=1}^{n} x_i) + \delta(\sum_{i=1}^{n} x_i) - \delta \quad \underbrace{(x_1 + x_n)}_{\text{1 and n are leaves}} \\
ne^* + \delta(x_1 + x_n) &= (2\delta + 1)\sum_{i=1}^{n} x_i \\
ne^* + 2\delta &\geq (2\delta + 1)\sum_{i=1}^{n} x_i \\
\frac{n-1}{2\delta+1}e^* + 1e^* &\geq \sum_{i=1}^{n} x_i \\
\frac{n+1}{2}e^* &\geq \sum_{i=1}^{n} x_i \ (\text{since } \delta \geq \frac{1}{2})
\end{aligned}$$

Thus, the aggregate play of an equilibrium with full support is less than or equal $\frac{n+1}{2}e^*$.

Case 2: $\sigma(\mathbf{x}) \subset V$, a proper subset.

If the $(r+1)^{th}$ node is assumed to contribute zero, we get two subnetworks P_r and P_{n-1-r}. By induction hypothesis, aggregate play of any equilibrium in P_r is less than or equal to $\frac{r+1}{2}e^*$ and that of any equilibrium in P_{n-1-r} is less than or equal to $\frac{n-r}{2}e^*$. Thus the total aggregate play is less than or equal to $\frac{r+1}{2}e^* + \frac{n-r}{2}e^* = \frac{n+1}{2}e^*$. By the principle of mathematical induction, the first part of the theorem is proved.

For the second part, consider a line network with $2k+1$ nodes and $x_{2i+1} = e^* \ \forall i \in \{0, 1, \ldots k\}$ is equilibrium with aggregate play $k + 1 = \frac{2k+1+1}{2}e^*$. Hence, we see that the bound is achieved for line networks with odd number of nodes. $\blacksquare$

A.5 Proof of Theorem 4.2

Consider the star network G with n peripheral nodes and let $\mathbf{x}$ be an equilbirium of the *SILC* with $\delta \geq \frac{1}{n}$.

Case 1: $\sigma(\mathbf{x}) = V$.
Then, on solving the $(I + \delta A)\mathbf{x} = \mathbf{e}e^*$, we get

$$x_0 = \frac{n\delta - 1}{n\delta^2 - 1}e^* \quad \text{and} \quad x_1 = x_2 = \cdots = x_n = \frac{\delta - 1}{n\delta^2 - 1}e^*$$

Since x_0 and $x_i, i \neq 0$ have opposie signs for all values of δ with $\delta \geq \frac{1}{n}$, such an equilibrium cannot exist.

Case 2: $\sigma(\mathbf{x}) \subset V$, a proper subset
Note that, any vector $\mathbf{x}$ with $x_i = 0$ where i is a peripheral node cannot be a Nash equilbirium. Let $x_0 = 0$, where 0 is the central node. Then the candidate equilibrium $\mathbf{x}$ is $x_0 = 0$ and $x_i = 1\ \forall i \in \{1, 2, \ldots, n\}$. For this $\mathbf{x}$, $\mathcal{C}_0(\mathbf{x}) = n\delta e^* \geq e^*$ for $\delta \geq \frac{1}{n}$. Hence, $\mathbf{x}$ is an equilibrium.

Hence, the only equilbirium in a star network is one in which all the peripheral nodes play e^* and the central node free-rides.

A.6 Proof of Theorem 4.3

We will prove this by induction on the number of nodes. First, for the base case, consider a star network with 3 peripheral nodes ($m = 3$). The only equilibrium is when each peripheral node contributes e^* and the central node contributes 0 for $\delta \geq \frac{1}{2}$, and its aggregate play is $3e^* = \frac{4+3-1}{2}e^*$. Hence, our bound holds for the base case.

Consider the star G having line networks $S_1, S_2, \ldots, S_m a$ with $n_1, n_2, \ldots, n_m$ nodes respectively. Let $\mathbf{x}$ be an equilibrium of *SILC* with the network G.

Induction Hypothesis For all stars G' with $m'(\geq 3)$ line networks connected at a central node c' and for $\delta \geq \frac{1}{2}$ with $|V(G')| < |V(G)|$, the aggregate play of a Nash equilibrium is at most $\frac{|V(G')|+m'-1}{2}$.

Case 1: $\sigma(\mathbf{x}) = V$.
If an equilibrium has full support the discounted sum of plays of the closed neighbourhood of each node is e^*, i.e. $(\mathcal{C}_i(\mathbf{x})) = e^*$. Then,

$$\begin{aligned}
\sum_{i=1 \in V} \mathcal{C}_i(\mathbf{x}) &= \sum_{i \in V} x_i + 2\delta(\sum_{i \in V} x_i) + \delta(x_c) - \delta(\sum_{i=1}^{m} x_l^i) \quad \text{(where} \\
&\qquad x_l^i \text{ is the action of the leaf corresponding to line network } S_i) \\
|V(G)|e^* + m\delta e^* &\geq (2\delta + 1) \sum_{i \in V} x_i \qquad \text{(taking worst case values of } x_c \text{ and } x_l^i) \\
\frac{|V(G)| + m\delta}{2\delta + 1}e^* &\geq \sum_{i \in V} x_i \\
\frac{|V(G)| + \frac{m}{2}}{2}e^* &\geq \sum_{i \in V} x_i \qquad \text{(maximized at } \delta = 0.5 \text{ since each} \quad n_i \geq 1) \\
\frac{|V(G)| + m - 1}{2}e^* &\geq \sum_{i \in V} x_i
\end{aligned}$$

Thus, the aggregate play of an equilibrium with full support is less than or equal to $\frac{|V(G)|+m-1}{2}e^*$.

Case 2: $\sigma(\mathbf{x}) \subset V$, a proper subset.
We divide this case into three separate cases.
Case 2a: $x_c = 0$.

On restricting the network to the rest of the nodes, we get m disjoint line networks of lengths $n_1, n_2, \ldots, n_m$. Applying Theorem 4.1 to each line network and adding the upper bounds, we get the upper bound as $\frac{\sum_{i=1}^{m} n_i + m}{2} e^* = \frac{|V(G)|+m-1}{2} e^*$.

Case 2b: $x_k = 0$ where k is the $(r+1)^{th}$ node from c, where $n_m - 1 \geq r \geq 1$.
On restricting the network to the rest of the nodes, we get a star with m line networks and $|V(G)| - n_m + r$ nodes and a line network of length $n_m - 1 - r$. By the induction hypothesis, since the new star has less number of nodes than G, and by Theorem 4.1, we get the upper bound as $\frac{|V(G)|-n_m+r+m-1}{2} e^* + \frac{n_m-1-r+1}{2} e^* = \frac{|V(G)|+m-1}{2} e^*$.

Case 2c: The neighbour of the central node in S_m contributes 0.
On restricting the network to the rest of the nodes, we get a star with $m-1$ peripheral line networks with $|V(G)| - n_m$ nodes and a line network with $n_m - 1$ nodes. For $m \geq 4$, by the induction hypothesis, since the new star-like network has fewer nodes than G, and by Theorem 4.1, we get the upper bound as $\frac{|V(G)|+(m-1)-1}{2} e^* + \frac{n_m-1+1}{2} e^* = \frac{|V(G)|+m-2}{2} e^* \leq \frac{|V(G)|+m-1}{2} e^*$. For $m = 3$, we get two disjoint line networks with $|V(G)| - n_3$ and $n_3 - 1$ nodes respectively, for which using Theorem 4.1 gives the upper bound as $\frac{|V(G)|-n_3+1}{2} e^* + \frac{n_3-1+1}{2} e^* = \frac{|V(G)|+1}{2} e^* < \frac{|V(G)|-1+m}{2} e^*$.

Hence, the aggregate play is at most $\frac{|V(G)|+m-1}{2} e^*$ for all equilibria. For the achievability of the bound, consider $\mathbf{x}$ defined as follows with line networks of length $2k_1+1, 2k_2+1, \ldots, 2k_m+1$.

$$
\begin{aligned}
x_c &= 0, \\
x_i^j &= \begin{cases} e^*, & \text{if } j \text{ is odd,} \\ 0, & \text{otherwise,} \end{cases}
\end{aligned}
$$

where x_i^j is the contribution of the j^{th} node of the line network S_i when we start counting from the leaf. It can be seen that $||\mathbf{x}||_1 = \sum_{i=1}^{m} k_i + m = \frac{\sum_{i=1}^{m} n_i + m}{2} e^* = \frac{|V(G)|-1+m}{2} e^*$. ■

A.7 Proof of Theorem 4.4

We will prove this by induction on the number of nodes. First, for the base case, consider a star network with 3 peripheral nodes ($m_1 = 3$), i.e. $\mathcal{C}_T = c_1$, $d_1^T = 3$ and $d_1^{T_C} = 0$. Using Theorem 4.3, its aggregate play $\leq 3 = \frac{4+3-1}{2}$. Hence, our bound holds for the base case.

Let G be a tree (T) with $\mathcal{C}_T = \{c_1, c_2, \ldots, c_t\}$ as the set of centers $(\mathcal{C}_T \neq \emptyset)$. Let P_{ij} denote the path connecting c_i and c_j with $|V(P_{ij})| = n_{join}^{ij}$. Let center c_i have line networks $S_1^i, S_2^i, \ldots, S_{m_i}^i$ of length $n_1^i, n_2^i, \ldots, n_{m_i}^i$ connected to it which terminate at leaves.
Induction Hypothesis For all trees T' with centers $\mathcal{C}_{T'} = \{c'_1, c'_2, \ldots, c'_{t'}\}$ as the set of centers $(\mathcal{C}_T \neq \emptyset)$, and for $\delta \geq \frac{1}{2}$ with $|V(G')| < |V(G)|$, the aggregate play of a Nash equilibrium is at most $\frac{|V(T')| + \sum_{c'_i \in \mathcal{C}_{T'}} (d_i^{T'} - d_i^{T'_C}) - 1}{2}$.

Case 1: $\sigma(\mathbf{x}) = V$.
If an equilibrium has full support the discounted sum of plays of the closed neighbourhood of each node is one, i.e. $(\mathcal{C}_i(\mathbf{x})) = 1$. Then,

$$\sum_{i\in V} C_i(x) = \sum_{i\in V} x_i + 2\delta(\sum_{i\in V} x_i) + \delta(\sum_{c_j\in\mathcal{C}_T}(d_j-2)x_{c_j}) - \delta(\sum_{c_j\in\mathcal{C}_T}\sum_{i=1}^{m_j} x^j_{i,l})$$

(where $x^j_{i,l}$ is the play of the leaf of path S^j_i)

$$|V(T)| + \delta(\sum_{c_j\in\mathcal{C}_T}\sum_{i=1}^{m_j} x^j_{l,i}) = (2\delta+1)\sum_{i\in V} x_i + \delta(\sum_{c_j\in\mathcal{C}_T}(d_j-2)x_{c_j})$$

$$|V(T)| + \delta(\sum_{c_j\in\mathcal{C}_T} m_j) \geq (2\delta+1)\sum_{i\in V} x_i \quad \text{(putting worst case values of } x_{c_j} \text{ and } x^j_{l,i})$$

$$\frac{|V(T)| + \delta(\sum_{c_j\in\mathcal{C}_T} m_j)}{2\delta+1} \geq \sum_{i\in V} x_i$$

$$\frac{|V(T)| + \frac{\sum_{c_j\in\mathcal{C}_T} m_j}{2}}{2} \geq \sum_{i\in V} x_i \quad \text{(maximized at a = 0.5 since each } n_i \geq 1)$$

$$\frac{|V(T)| + \sum_{c_j\in\mathcal{C}_T} m_j - 1}{2} \geq \sum_{i\in V} x_i$$

Thus, the aggregate play of an equilibrium with full support is at most $\frac{|V(G)|+\sum_{c_i\in\mathcal{C}_T}(d^T_i - d^{T_C}_i)-1}{2}$.
Case 2: $\sigma(\mathbf{x}) \subset V$, a proper subset
We divide this case into five separate cases.
Case 2a: $x_{c_j} = 0$ for some $c_j \in \mathcal{C}_T$.
On restricting the network to the rest of the nodes, we get m_j line networks of lengths $n^j_1, n^j_2, \ldots, n^j_{m_j}$ and $d^{T_C}_j$ disjoint trees $T^1, \ldots, T^{d^{T_C}_j}$ with centers $\mathcal{C}_{T^1}, \ldots, \mathcal{C}_{T^{d^{T_C}_j}}$ such that $\bigcup_{i=1}^{d^{T_C}_j} \mathcal{C}_{T^i} \subset \mathcal{C}_T \backslash c_j$ and $\sum_{i=1}^{d^{T_C}_j} |V(T^i)| = |V(T)| - (1 + \sum_{i=1}^{m_j} n^j_i)$. Also, $d^{T^k}_i \leq d^T_i$ and for $c_i \in \mathcal{C}_{T^k}$

$$d^{T^k_C}_i = \begin{cases} d^{T_C}_i, & \text{if} \quad (c_i, c_j) \notin E(T_C), \\ d^{T_C}_i - 1, & \text{if} \quad (c_i, c_j) \in E(T_C), \end{cases}$$

By the induction hypothesis, Theorem 4.1 and adding all the bounds we get $\sum_{i\in V} x_i \leq$ $\sum_{i=1}^{d^{T_C}_j} \frac{|V(T^i)|+\sum_{c_s\in\mathcal{C}_{T^k}}(d^{T^k}_s - d^{T^k_C}_s)-1}{2} + \frac{\sum_{i=1}^{m_j} n^j_i + \overbrace{m_j}^{d^T_j - d^{T_C}_j}}{2} \leq \frac{|V(G)|+\sum_{c_i\in\mathcal{C}_T}(d^T_i - d^{T_C}_i)-1}{2}$.

Case 2b: The $(r+1)^{th}$ node from c_j in path S^j_i contributes 0, where $n_{m_j} - 1 \geq r \geq 1$, i.e. $x^j_{i,r+1} = 0$
On restricting the network to the rest of the nodes, we get a path of length $n_{m_j} - 1 - r$ and a tree T' with centers $\mathcal{C}_{T'} = \mathcal{C}_T$ and $|V(T')| = |V(T)| - (n_{m_j} - r)$ and degrees of all centers remain the same. By the induction hypothesis and Theorem 4.1, we get $\sum_{i\in V} x_i \leq$ $\frac{|V(T')|+\sum_{c_s\in\mathcal{C}_T}(d^T_s - d^{T_C}_s)-1}{2} + \frac{n_{m_j}-1-r+1}{2} \leq \frac{|V(G)|+\sum_{c_i\in\mathcal{C}_T}(d^T_i - d^{T_C}_i)-1}{2}$.

Case 2c: The neighbour of a central node c_j in S^j_i contributes 0, i.e. $x^j_{i,1} = 0$.
On restricting the network to the rest of the nodes, we get a path of length $n_{m_1} - 1$ and a tree T' with centers $\mathcal{C}_{T'} = \mathcal{C}_T$, $|V(T')| = |V(T)| - (n_{m_j})$, $d^{T'_C}_j = d^{T_C}_j - 1$ and degrees of all

other centers remain the same. By the induction hypothesis and Theorem 4.1, we get the upper bound as $\sum_{i\in V} x_i \leq \frac{|V(T')|+\sum_{c_s\in\mathcal{C}_T}(d_s^{T'}-d_s^{T'_C})-1}{2} + \frac{n_{m_j}-1+1}{2} = \frac{|V(G)|+\sum_{c_i\in\mathcal{C}_T}(d_i^T-d_i^{T_C})-1-1}{2} < \frac{|V(G)|+\sum_{c_i\in\mathcal{C}_T}(d_i^T-d_i^{T_C})-1}{2}$.

Case 2d: The $(r+1)^{th}$ node from c_j in P_{join}^{ij} contributes 0, where $n_{join}^{ij}-1 > r \geq 1$.
On restricting the network to the rest of the nodes, we get two trees T^i and T^j with centers $\mathcal{C}_{T^i}, \mathcal{C}_{T^j}$ satisfying $\mathcal{C}_{T^i} \cup \mathcal{C}_{T^j} = \mathcal{C}_T$ and $|V(T^i)| + |V(T^j)| = |V(T)| - 1$ and degrees of all centers remain the same. By Induction Hypothesis, we get $\sum_{i\in V} x_i \leq \frac{|V(T^i)|+\sum_{c_s\in\mathcal{C}_{T^i}}(d_s^{T^i}-d_s^{T^i_C})-1}{2} + \frac{|V(T^j)|+\sum_{c_s\in\mathcal{C}_{T^j}}(d_s^{T^j}-d_s^{T^j_C})-1}{2} < \frac{|V(G)|+\sum_{c_i\in\mathcal{C}_T}(d_i^T-d_i^{T_C})-1}{2}$.

Case 2e: The neighbouring node of c_j in P_{join}^{ij} contributes 0.
On restricting the network to the rest of the nodes, we get two trees T^i and T^j with centers $\mathcal{C}_{T^i}, \mathcal{C}_{T^j}$ satisfying $\mathcal{C}_{T^i} \cup \mathcal{C}_{T^j} = \mathcal{C}_T$ and $|V(T^i)| + |V(T^j)| = |V(T)| - 1$, $d_j^{T^j_C} = d_j^{T_C} - 1$ and degrees of all other centers remain the same. By Induction hypothesis, we get the upper bound as $\sum_{i\in V} x_i \leq \frac{|V(T^i)|+\sum_{c_s\in\mathcal{C}_{T^i}}(d_s^{T^i}-d_s^{T^i_C})-1}{2} + \frac{|V(T^j)|+\sum_{c_s\in\mathcal{C}_{T^j}}(d_s^{T^j}-d_s^{T^j_C})-1}{2} = \frac{|V(G)|+\sum_{c_i\in\mathcal{C}_T}(d_i^T-d_i^{T_C})-1}{2}$.

Hence, the aggregate play is at most $\frac{|V(G)|+\sum_{c_i\in\mathcal{C}_T}(d_i^T-d_i^{T_C})-1}{2}$ for all equilibria. ■

A.8 Proof of Theorem 4.5

Let the aggregate play maximizing ICE be given by $\mathbf{x}^*$.

(a) We maximize the RHS of (11) over $\mathbf{x}$ to get

$$\begin{aligned} W(\mathbf{x};\delta,G) &\leq n(b(e^*)-ce^*) + cd_{max}\delta\sigma_b \sum_{i\in V} x_i^* \\ &\leq n(b(e^*)-ce^*) + cd_{max}\delta\sigma_b e^* \left(\frac{|V(G)|+\sum_{c_i\in\mathcal{C}_T}(d_i^T-d_i^{T_C})-1}{2}\right) \end{aligned} \tag{28}$$

Since (28) holds for every $\mathbf{x}$ such that $\frac{\mathbf{x}}{e^*} \in \mathrm{SOL}_\delta(G)$, we have

$$W^*(\delta,G) \leq n(b(e^*)-ce^*) + cd_{max}\delta\sigma_b e^* \left(\frac{|V(G)|+\sum_{c_i\in\mathcal{C}_T}(d_i^T-d_i^{T_C})-1}{2}\right). \tag{29}$$

(b) Since (10) holds for all $\mathbf{x}$, we consider this equation for $\mathbf{x}^*$, i.e.

$$W^*(\delta,G) \geq W(\mathbf{x}^*;\delta,G) \geq n(b(e^*)-ce^*) + c((\delta+1)\sigma_b-1)\sum_{i\in V} x_i^* \tag{30}$$

For $\sigma_b \leq \frac{1}{1+\delta}$, to minimize the lower bound, we need to consider the maximum value of $\sum_{i\in V} x_i^*$. Thus, we have for $\sigma_b \leq \frac{1}{1+\delta}$

$$W^*(\delta,G) \geq n(b(e^*)-ce^*) + c((\delta+1)\sigma_b-1)e^* \left(\frac{|V(G)|+\sum_{c_i\in\mathcal{C}_T}(d_i^T-d_i^{T_C})-1}{2}\right). \tag{31}$$

(c) For $\sigma_b \geq \frac{1}{1+\delta}$, to minimize the lower bound, we need to consider the minimum value of $\sum_{i\in V} x_i^*$. Thus, we have for $\sigma_b \geq \frac{1}{1+\delta}$

$$W^*(\delta,G) \geq n(b(e^*)-ce^*) + c((\delta+1)\sigma_b-1)e^*\alpha(G). \tag{32}$$

References

[1] C. Ballester, A. Calvó-Armengol, and Y. Zenou. Who's who in networks. wanted: The key player. *Econometrica*, 74(5):1403–1417, 2006.

[2] Y. Bramoullé and R. Kranton. Public goods in networks. *Journal of Economic Theory*, 135(1):478–494, 2007.

[3] Y. Bramoullé, R. Kranton, and M. D'Amours. Strategic interaction and networks. *American Economic Review*, 104(3):898–930, March 2014.

[4] K. N. Chadha and A. A. Kulkarni. On Independent Cliques and Linear Complementarity Problems. *ArXiv e-prints*, page arXiv:1811.09798, Nov. 2018.

[5] R. W. Cottle and G. B. Dantzig. Complementary pivot theory of mathematical programming. *Linear Algebra and its applications*, 1:103–125, 1968.

[6] P. Pandit and A. A. Kulkarni. Refinement of the equilibrium of public goods games over networks: Efficiency and effort of specialized equilibria. *(in press, available online) Journal of Mathematical Economics*, 2018.