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Chapter-1

VECTORS...

Physics : The branch of science which deal the nature is known as physics.

There are two types of physics:

① Classical Physics (4 Parts) ② Modern / Quantum

1. **CLASSICAL PHYSICS :** The branch of physics which deals the macro object is known as classical physics.

1 mm To universe $\xrightarrow[\text{macro}]{}$

(i) **Mechanics :** The branch of physics which deal the motion is known as mechanics. There are three types of mechanics physics.

ⓐ Kinetics ⓑ Dynamic ⓒ Statics

(ii) **Thermodynamics :** The branch of physics which deals the heat energy and conversion of heat energy to other energy is called thermodynamics.

(iii) **Optics :** The branch of physics which deal the light is known as optics.

(iv) **Acoustics :** The branch of physics which deal the sound is known as acoustics.

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2. MODERN PHYSICS: The branch of physics which deal the object more near to the speed of light.

PHYSICAL QUANTITIES: The terminology are expression in which we defined the laws of physics is known as Physical Quantities.

Example → : mass, length, time, speed, velocity, force, work, etc.

TYPES OF PHYSICAL QUANTITIES..

A. Fundamental Physical Quantity
B. Derived Physical Quantity

A. Fundamental Physical Quantity
The physical quantity which can not be split into small physical quantities are known as fundamental quantities.

1. Length
2. Mass
3. Time
4. current
5. Amount of Substance
6. Luminious Intensity
7. Temperature

B. Derived Physical Quantity
The physical quantity which can split into small fundamental physical quantities is known as Derived Physical Quantity.

Example → Area, Volume, speed, Velocity, Acceleration, force, Work, etc.

Unit Of A Physical Quantity:

The standard or reference scale by which we measure the physical Quantity is known as Unit.

S.I. unit of Physical Quantities

Fundamental Phy. Quant.	S.I. unit
Length	m
Mass	kg
Time	s
Current	A (ampere)
Amount of substance	mole/mol
Temperature	kelvin (K)
Luminous Intensity	candela (cd)

Types Of Physical Quantities

1. **Scalar Quantities:** The physical quantities which have only magnitude but not direction, are called scalar quantities. Scalers follow the ordinary rules of algebra.

Example: Mass, Length, Time, Speed, Temperature, etc. Distance, Work, Electric Flux, Potential, etc.

Note: Some physical quantities have magnitude as well as direction but they are scalar quantities because they don't follow the law of vector addition.

e.g. Pressure, Current, etc.

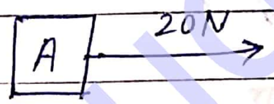
2. __Vector Quantities :__ The physical quantities which have magnitude as well as direction are called vector quantities. Vectors follow the vector law of addition.

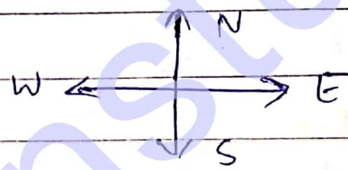
__Eg.__ Displacement, Velocity, Acceleration, Force, Torque, etc.

__REPRESENTATION OF VECTOR :__

To draw a vector quantity in paper we use a line and the length of the line will represent a direction.

__Ex.__ Let a force of 20N is acting on a block.





Let 1cm = 5N
4cm = 20N

$F' = 20N$

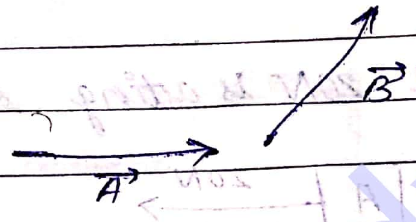
05/Aug/2015 __Note :__ If we shift the vector anywhere in the universe without changing the magnitude & direction the vector will remain same.

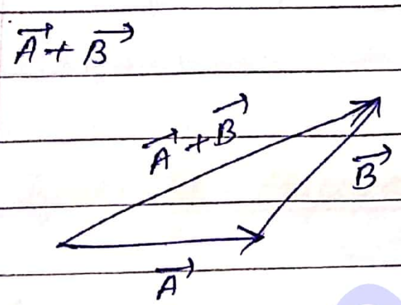
ADDITION OF VECTOR :

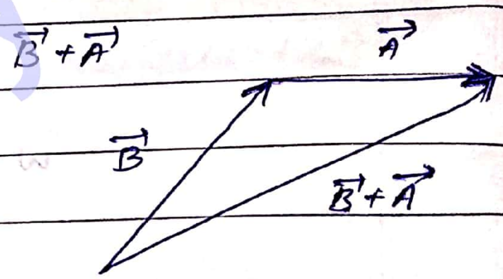
1. Triangle law Of Vector Addition :

It states if two vectors along on a particle at the same time are represented in magnitude and direction by the two sides of a triangle taken in one order. Their resultant vector is represented in magnitude and directed by the third side of the triangle taken in opposite order.

e.g.





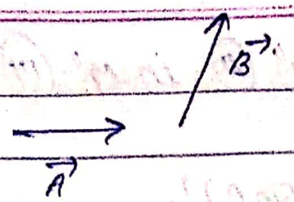


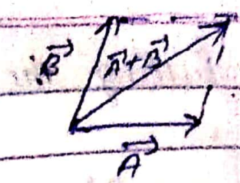
$\because$ $\vec{A}+\vec{B} = \vec{B}+\vec{A}$

$\therefore$ Addition of vector follow commutative law.

2. Parallelogram Law Of Vector Addition :

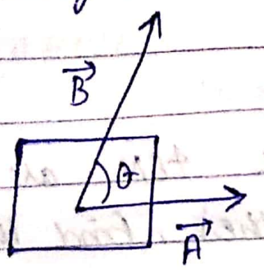
If two vectors acting on a particle at the same time are represented in magnitude and direction by the two sides of parallelogram (||gm) drawn from point their resultant vector is represented in magnitude and direction by the diagonal of the parallelogram drawn from the same point.

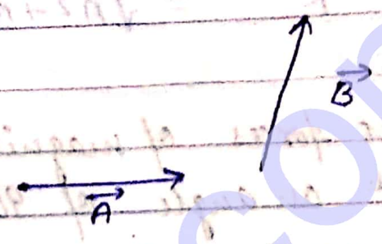
eg. 

$\vec{A} + \vec{B} =$ 

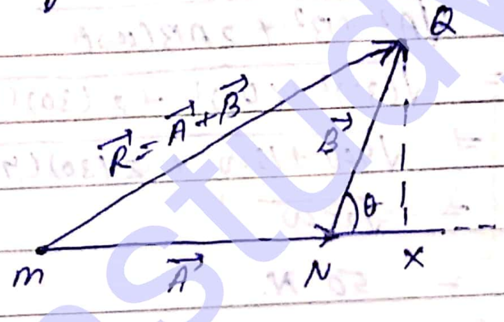
3. ANALYTICAL METHOD OF VECTOR ADDITION :

Let two vectors $\vec{A}$ and $\vec{B}$ are acting on a body at an angle θ with each other.





We are adding $\vec{A}$ and $\vec{B}$ by using Δ law,



$MN = |\vec{A}| = A$

$NQ = |\vec{B}| = B$

$MQ = |\vec{R}| = R$

In ΔQNX, $\cos\theta = \frac{NX}{QN}$

$NX = QN \cos\theta$

$\boxed{NX = B\cos\theta}$ —— (I) $\{\because QN = B\}$

$\sin\theta = \frac{QX}{QN}$

$QX = QN \sin\theta$

$\boxed{QX = B\sin\theta}$ —— (II) $\{\because QN = B\}$

In ΔQMX, Using Pythagoras theorem

$QM^2 = MX^2 + QX^2$

$QM^2 = (MN + NX)^2 + QX^2$ —— (III)

Putting the values of eqⁿ (I) & eqⁿ (II) in eqⁿ (III).

$$R^2 = (A + B\cos\theta)^2 + (B\sin\theta)^2$$
$$R^2 = A^2 + B^2\cos^2\theta + 2AB\cos\theta + B^2\sin^2\theta$$
$$R^2 = A^2 + B^2(\sin^2\theta + \cos^2\theta) + 2AB\cos\theta$$
$$R^2 = A^2 + B^2 + 2AB\cos\theta$$
$$R = \sqrt{A^2 + B^2 + 2AB\cos\theta}$$

Q Two forces of magnitude 30N and 40N are acting at an angle of 90° to each other. Find the magnitude of resultant force.

Sol. $A = 30N$, $B = 40N$, $\theta = 90°$

$$\therefore R = \sqrt{A^2 + B^2 + 2AB\cos\theta}$$
$$R = \sqrt{(30)^2 + (40)^2 + 2(30)(40)\cos 90°}$$
$$R = \sqrt{900 + 1600 + 2(30)(40).0} \quad \{\cos 90° = 0\}$$
$$R = \sqrt{2500}$$
$$R = 50\ N.$$

Q Two equal forces have their resultant equal to either, at what angle they inclined?

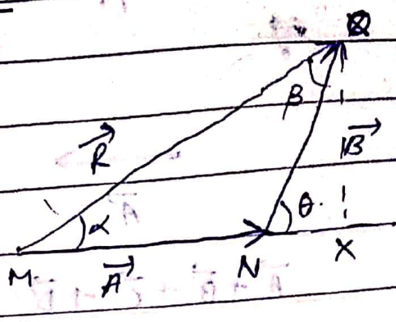
Sol. Let the two forces be $\vec{A}$ and $\vec{B}$.

$$\because |\vec{A}| = |\vec{B}| = |\vec{R}| \quad \text{--- (1)}$$
$$\therefore R = \sqrt{A^2 + B^2 + 2AB\cos\theta}$$
$$A = \sqrt{A^2 + A^2 + 2AA\cos\theta} \quad \{\text{From eq}^n\ (1)\}$$
$$A^2 = 2A^2 + 2A^2\cos\theta$$
$$-1 = 2\cos\theta$$
$$\cos\theta = \frac{-1}{2}$$
$$\theta = 120°$$

DIRECTION OF RESULTANT VECTOR

$NX = B\cos\theta$ —(I)

$QX = B\sin\theta$ —(II)



Let $\vec{R}$ makes an angle α with $\vec{A}$.

In ΔQMX,

$$\tan\alpha = \frac{QX}{MX}$$

$$\tan\alpha = \frac{QX}{MN+NX}$$

$$\boxed{\tan\alpha = \frac{B\sin\theta}{A+B\cos\theta}}$$

[$\because NX = B\cos\theta$, $MN = A$, & $QX = B\sin\theta$]

$$\boxed{\alpha = \tan^{-1}\left(\frac{B\sin\theta}{A+B\cos\theta}\right)}$$

Let $\vec{R}$ makes an angle β with $\vec{B}$

In ΔQMX, $\tan\beta = \frac{A\sin\theta}{B+A\cos\theta}$

$$\boxed{\beta = \tan^{-1}\left(\frac{A\sin\theta}{B+A\cos\theta}\right)}$$

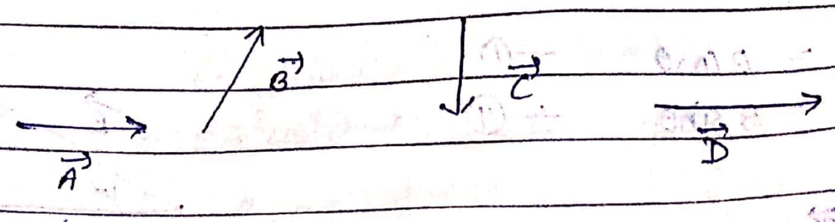
4. POLYGON METHOD :

If we want to add more than two vectors we use the polygon method.

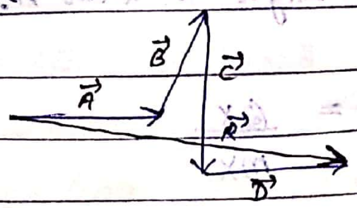
If any number of vectors acting on a particle at the same time are represented in magnitude and direction by various sides of an open polygon taken in same order their resultant is represented in magnitude and direction by the closing side of the polygon Contd.

taken in opposite order.

e.g.



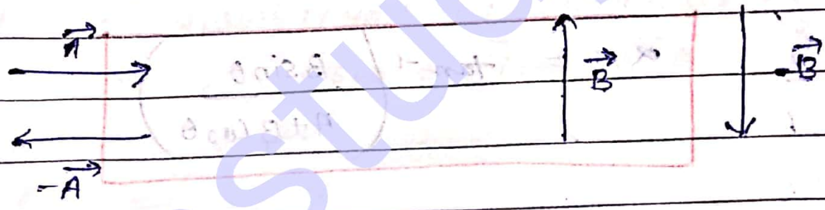
$\vec{A} + \vec{B} + \vec{C} + \vec{D} = \vec{R}$



NEGATIVE VECTORS

The vector having same magnitude and opposite direction is known as negative vector

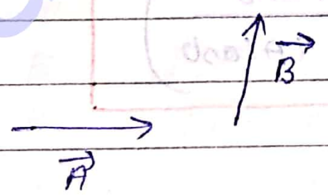
e.g.



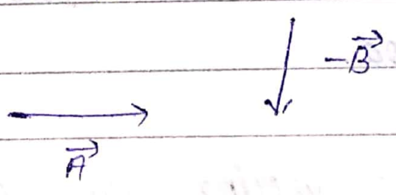
SUBTRACTION OF VECTORS :

The subtraction of vector is possible.

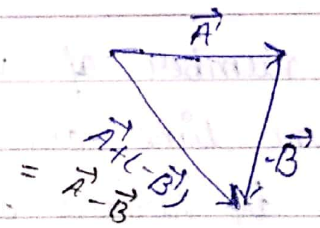
Eg.



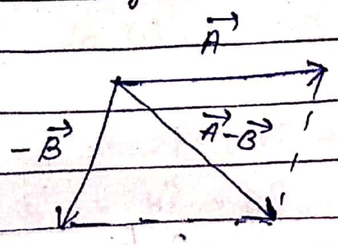
$\vec{A} - \vec{B} = \vec{A} + (-\vec{B})$



Triangle law



Parallelogram method.



Note: (*) $\because \vec{A} - \vec{B} \neq \vec{B} - \vec{A}$

$\therefore$ Subtraction of vectors doesnot follow commutative law.

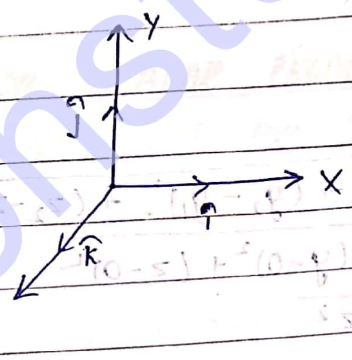
POSITION VECTOR

1. Unit Vector : The vector which has unit magnitude is known as unit vector.

$\hat{i}$ = unit vector along x-axis

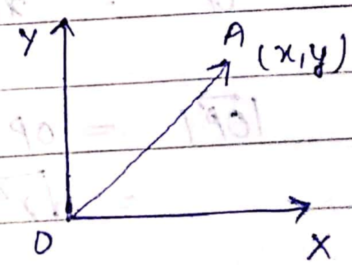
$\hat{j}$ = unit vector along y-axis

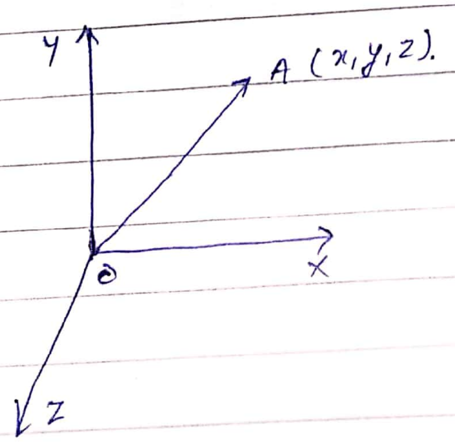
$\hat{k}$ = unit vector along z-axis



Let the coordinate of point A is (x, y).
Then, the position vector $\vec{OA}$

$= x\hat{i} + y\hat{j}$





$\vec{OA} = x\hat{i} + y\hat{j} + z\hat{k}$

ADDITION OF VECTOR

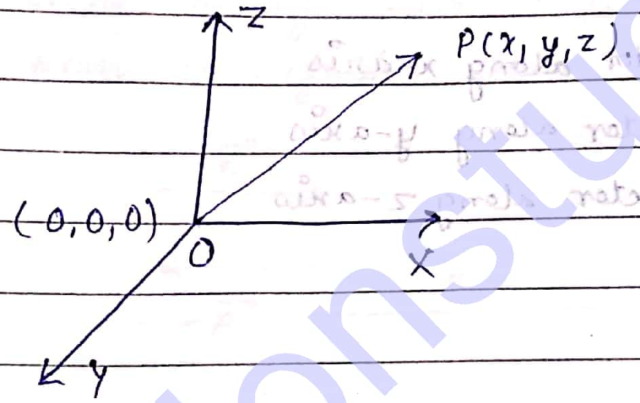
If $\vec{A} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$

$\vec{B} = x_2\hat{i} + y_2\hat{j} + z_2\hat{k}$

$$\boxed{\vec{A}+\vec{B} = (x_1+x_2)\hat{i} + (y_1+y_2)\hat{j} + (z_1+z_2)\hat{k}}$$

SUBTRACTION OF VECTOR

$$\boxed{\vec{A}-\vec{B} = (x_1-x_2)\hat{i} + (y_1-y_2)\hat{j} + (z_1-z_2)\hat{k}}$$



$OP = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2}$

$OP = \sqrt{(x-0)^2 + (y-0)^2 + (z-0)^2}$

$OP = \sqrt{x^2+y^2+z^2}$

$\vec{OP} = x\hat{i} + y\hat{j} + z\hat{k}$

$|\vec{OP}| = OP$

$= \sqrt{x^2+y^2+z^2}$

Q If $\vec{A} = 2\hat{i} + 3\hat{j} + 4\hat{k}$ and $\vec{B} = 4\hat{i} + 8\hat{j} + 7\hat{k}$. Find modulus $|\vec{A}+\vec{B}|$ and $|\vec{A}-\vec{B}|$.

$$\vec{A} = 2\hat{i} + 3\hat{j} + 4\hat{k}$$
$$\vec{B} = 4\hat{i} + 8\hat{j} + 7\hat{k}$$

$$\vec{A}+\vec{B} = 6\hat{i} + 11\hat{j} + 11\hat{k} \Rightarrow |\vec{A}+\vec{B}| = \sqrt{(6)^2+(11)^2+(11)^2}$$
$$|\vec{A}+\vec{B}| = \sqrt{36+121+121} = \sqrt{278}$$
$$\vec{A}-\vec{B} = -2\hat{i} - 5\hat{j} - 3\hat{k}$$

$$|\vec{A}-\vec{B}| = \sqrt{(-2)^2+(-5)^2+(-3)^2}$$
$$= \sqrt{4+25+9}$$
$$= \sqrt{38}$$

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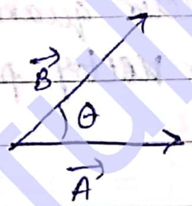
MULTIPLICATION OF VECTOR

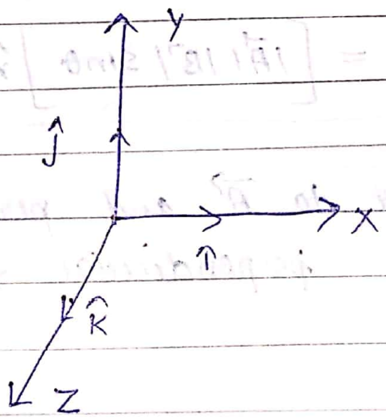
(1) DOT PRODUCT OR SCALAR PRODUCT

If the product of two vectors is a scalar quantity. This product is known as scalar product or dot product.



$$\vec{A}\cdot\vec{B} = \text{scalar}$$
$$\vec{A}\cdot\vec{B} = (\text{magnitude})$$

$$\boxed{\vec{A}\cdot\vec{B} = |\vec{A}|\,|\vec{B}|\cos\theta}$$



$$\hat{i}\cdot\hat{i} = |\hat{i}|\,|\hat{i}|\cos 0$$
$$\hat{i}\cdot\hat{i} = 1\cdot1\cdot1$$
$$\boxed{\hat{i}\cdot\hat{i} = 1}$$

$$\hat{j}\cdot\hat{j} = |\hat{j}|\,|\hat{j}|\cos 0$$
$$\hat{j}\cdot\hat{j} = 1\cdot1\cdot1$$
$$\boxed{\hat{j}\cdot\hat{j} = 1}$$

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$\hat{k}.\hat{k} = |\hat{k}||\hat{k}| \cos 0$

$= 1 \times 1 \times 1$

$\boxed{\hat{k}.\hat{k} = 1}$

$\hat{i}.\hat{j} = |\hat{i}||\hat{j}| \cos 90^\circ$

$\hat{i}.\hat{j} = 1 \times 1 \times 0$

$\boxed{\hat{i}.\hat{j} = 0}$

Similarly $\boxed{\hat{j}.\hat{k} = 0}$ & $\boxed{\hat{k}.\hat{i} = 0}$

Q. If $\vec{A} = 2\hat{i} + 3\hat{j} + 4\hat{k}$ & $\vec{B} = 4\hat{i} + 7\hat{j} + 8\hat{k}$. Find $\vec{A}.\vec{B}$

Sol. →

$\vec{A} = 2\hat{i} + 3\hat{j} + 4\hat{k}$

$\vec{B} = 4\hat{i} + 7\hat{j} + 8\hat{k}$

$\vec{A}.\vec{B} = (2\hat{i} + 3\hat{j} + 4\hat{k})(4\hat{i} + 7\hat{j} + 8\hat{k})$

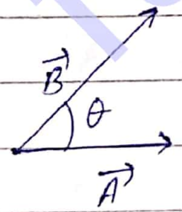
$= 8(\hat{i}.\hat{i}) + 21(\hat{j}.\hat{j}) + 32(\hat{k}.\hat{k}) \Rightarrow 8 \times 1 + 21 \times 1 + 32 \times 1$

$= 8 + 21 + 32$

$\Rightarrow \boxed{\vec{A}.\vec{B} = 61 \text{ units}}$ 'scalar'

② CROSS PRODUCT OR VECTOR PRODUCT :

If the product of two vectors is a vector quantity. This product is known as cross product or vector product.



$\vec{A} \times \vec{B} = \text{vector}$

$\vec{A} \times \vec{B} = \text{magnitude} + \text{direction}$

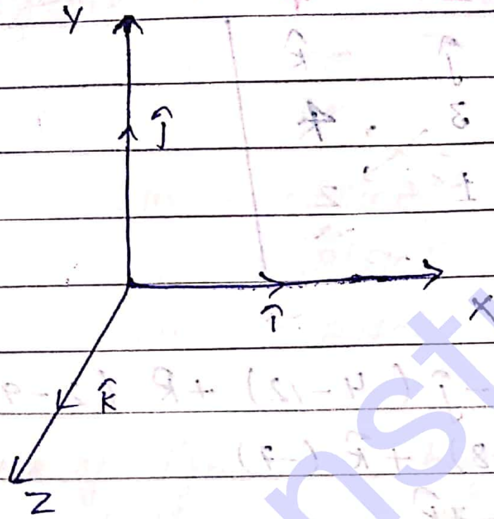
$$\boxed{\vec{A} \times \vec{B} = \left[|\vec{A}||\vec{B}| \sin\theta \right] \hat{n}}$$

Note: $(\vec{A} \times \vec{B})$ is perpendicular to $\vec{A}$ and perpendicular to $\vec{B}$. So, unit vector $(\hat{n})$, is perpendicular to $\vec{A}$ and $\perp$ to $\vec{B}$

To find the direction of $(\vec{A} \times \vec{B})$, we use the Right Hand Thumb Rule

RIGHT HAND THUMB RULE :

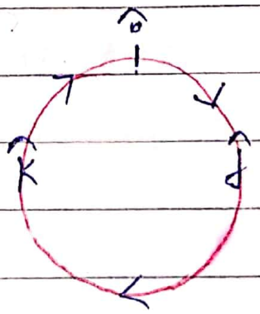
According to this rule, stretch your right hand finger towards the first ~~fin~~ vector $\vec{A}$. Curl your right hand fingers towards second vector $\vec{B}$. The direction of thumb will give the direction of $\vec{A} \times \vec{B}$.



$$\hat{i} \times \hat{j} = [|\hat{i}|\,|\hat{j}| \sin 90^\circ]\, \hat{k}$$
$$\hat{i} \times \hat{j} = [1 \times 1 \times 1]\, \hat{k}$$
$$\hat{i} \times \hat{j} = 1 . \hat{k}$$
$$\boxed{\hat{i} \times \hat{j} = \hat{k}}$$

$$\hat{j} \times \hat{k} = [|\hat{j}|\,|\hat{k}| \sin 90^\circ]\, \hat{i}$$
$$= [1 \times 1 \times 1]\, \hat{i}$$
$$= 1 . \hat{i}$$
$$\boxed{\hat{j} \times \hat{k} = \hat{i}}$$

$$\hat{k} \times \hat{i} = [|\hat{k}|\,|\hat{i}| \sin 90^\circ]\, \hat{j}$$
$$= [1 \times 1 \times 1]\, \hat{j}$$
$$= 1 . \hat{j}$$
$$\boxed{\hat{k} \times \hat{i} = \hat{j}}$$



Similarly,

$$\hat{j} \times \hat{i} = -\hat{k}$$
$$\hat{k} \times \hat{j} = -\hat{i}$$
$$\hat{i} \times \hat{k} = -\hat{j}$$
$$\hat{i} \times \hat{i} = [|\hat{i}|\,|\hat{i}| \sin 0]\, \hat{n} = 0 \times \hat{n}$$
$$\hat{i} \times \hat{i} = \vec{0} \quad = \text{Zero vector}$$

Q. If $\vec{A} = 2\hat{i} + 3\hat{j}$ and $\vec{B} = 3\hat{i} + 4\hat{j}$, find $\vec{A} \times \vec{B}$

$$\vec{A} \times \vec{B} = (2\hat{i} + 3\hat{j}) \times (3\hat{i} + 4\hat{j})$$
$$= 6(\hat{i} \times \hat{i}) + 8(\hat{i} \times \hat{j}) + 9(\hat{j} \times \hat{i}) + 12(\hat{j} \times \hat{j})$$
$$= 6.(\vec{0}) + 8.\hat{k} + 9.(-\hat{k}) + 12.(\vec{0})$$
$$= 8\hat{k} - 9\hat{k}$$
$$\vec{A} \times \vec{B} = -\hat{k}$$

Q. If $\vec{A} = 2\hat{i} + 3\hat{j} + 4\hat{k}$ and $\vec{B} = 3\hat{i} + \hat{j} + 2\hat{k}$, find $\vec{A} \times \vec{B}$

$$\vec{A} \times \vec{B} = \begin{array}{ccc} + & - & + \\ \end{array}$$
$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 4 \\ 3 & 1 & 2 \end{vmatrix}$$

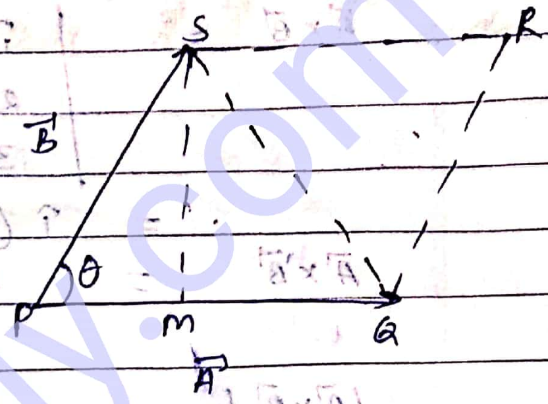
$$\vec{A} \times \vec{B} = \hat{i}(6-4) - \hat{j}(4-12) + \hat{k}(2-9)$$
$$= 2\hat{i} - \hat{j}(-8) + \hat{k}(-7)$$
$$\vec{A} \times \vec{B} = 2\hat{i} + 8\hat{j} - 7\hat{k}$$

Application Of Vector Product

① Area Of Parallelogram

2 Marks

Let $\vec{A}$ and $\vec{B}$ are representing the two sides of parallelogram



Area of ||gm = height × base

= SM × PQ.

In ΔSMP,

$$\sin\theta = \frac{SM}{SP}$$

$SM = SP\sin\theta$

$SM = |\vec{B}|\sin\theta$

$SM = B\sin\theta \longleftarrow ①$

Area of ||gm = height × base.

Area of ||gm = SM × PQ.

Area of ||gm = $(B\sin\theta) \times |\vec{A}|$

$= AB\sin\theta$.

$$\boxed{\text{Area of ||gm} = |\vec{A} \times \vec{B}|}$$

$$\boxed{\text{Area of } \Delta PQS = \text{Area of } \Delta QRS = \frac{1}{2}|\vec{A} \times \vec{B}|}$$

Cross Product

Sec. B (2010)

Q. If sides of ‖gm are represented by the vectors $2\hat{i}+4\hat{j}+5\hat{k}$ and $3\hat{i}+2\hat{j}+6\hat{k}$, find the area of the ‖gm.

R.V.

$\vec{A} = 2\hat{i}+4\hat{j}+5\hat{k}$

$\vec{B} = 3\hat{i}+2\hat{j}+6\hat{k}$

$$\vec{A}\times\vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 4 & 5 \\ 3 & 2 & 6 \end{vmatrix}$$

$= \hat{i}(24-10) - \hat{j}(12-15) + \hat{k}(4-12)$

$\vec{A}\times\vec{B} = 14\hat{i} + 3\hat{j} - 8\hat{k}$

$|\vec{A}\times\vec{B}| = \sqrt{(14)^2+(3)^2+(8)^2}$

$= \sqrt{196+9+64}$

$= \sqrt{260+9}$

$|\vec{A}\times\vec{B}| = \sqrt{269}$

Area of ‖gm $= |\vec{A}\times\vec{B}|$

$= \sqrt{269}$ units

Q. If $\vec{A} = 2\hat{i}+3\hat{j}+8\hat{k}$ and $\vec{B} = 4\hat{i}+4\hat{j}+4\hat{k}$ represent 2 sides ‖gm. Find the area of ‖gm.

$$\vec{A}\times\vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 8 \\ 4 & 4 & 4 \end{vmatrix}$$

$= \hat{i}(12-32) - \hat{j}(8-32) + \hat{k}(8-12)$

$= -20\hat{i} + 24\hat{j} - 4\hat{k}$

* $\vec{A}+\vec{0}=\vec{A}$ (additive Identity)

$$24 \times 24 = 576$$

$$|\vec{A}\times\vec{B}| = \sqrt{400+576+16}$$

$$= \sqrt{976+16}$$

$$|\vec{A}\times\vec{B}| = \sqrt{992} \text{ unit}$$

Area of ||gm $= \sqrt{992}$ unit

Sec.B. **Q.** If $\vec{A}$, $\vec{B}$ and $\vec{C}$ are three vectors & none of them have zero magnitude. Given that $\vec{A}\times\vec{B}=\vec{0}$ and $\vec{B}\times\vec{C}=\vec{0}$.

Proof that $\vec{A}\times\vec{C}=\vec{0}$.

$$\vec{A}\times\vec{B}=\vec{0}$$

$$|\vec{A}||\vec{B}|\sin\theta = \vec{0}$$

Sol. Given: $\vec{A}\times\vec{B}=\vec{0}$

$$|\vec{A}\times\vec{B}| = 0$$

$$|\vec{A}||\vec{B}|\sin\theta = 0$$

$$AB\sin\theta = 0$$

A/Q $A\neq 0$, $B\neq 0$.

$$\sin\theta = 0$$

$$\boxed{\theta = 0^\circ}$$

i.e. $\vec{A}$ and $\vec{B}$ are parallel to each other. —— (A)

$$\vec{B}\times\vec{C}=\vec{0} \text{ (given)}$$

$$|\vec{B}\times\vec{C}| = 0$$

$$|\vec{B}||\vec{C}|\sin\theta = 0$$

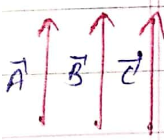
$$BC\sin\theta = 0$$

A/Q $|\vec{B}|\neq 0$, $|\vec{C}|\neq 0$

$$\sin\theta = 0$$

$$\boxed{\theta = 0^\circ}$$

i.e. $\vec{B}$ and $\vec{C}$ are also parallel to each other. —— (B)



From eqn (A) & (B) it is clear $\vec{A}$ and $\vec{C}$ are parallel to each other.

Hence proved.

Q. Find the vector having magnitude 6 and ⊥ to the vector $\vec{A} = 2\hat{i}+2\hat{j}+\hat{k}$ and $\vec{B} = 2\hat{i}-2\hat{j}+3\hat{k}$. Find the angle between $\vec{A}$ and $\vec{B}$

$\vec{A} = 2\hat{i}+2\hat{j}+\hat{k}$, $\vec{B} = 2\hat{i}-2\hat{j}+3\hat{k}$

$$\vec{A}\times\vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 2 & 1 \\ 2 & -2 & 3 \end{vmatrix}$$

$$= \hat{i}(6+2) - \hat{j}(6-2) + \hat{k}(-4-4)$$

$$= 8\hat{i} - 4\hat{j} - 8\hat{k}$$

$$|\vec{A}\times\vec{B}| = \sqrt{64+16+64} = \sqrt{144} = 12.$$

$\because$ $\vec{A}\times\vec{B}$ ⊥ to $\vec{A}$ and ⊥ to $\vec{B}$

Unit ⊥ to $\vec{A}$ and ⊥ to $\vec{B}$ $= \dfrac{\vec{A}\times\vec{B}}{|\vec{A}\times\vec{B}|}$

$$= \frac{8\hat{i}-4\hat{j}-8\hat{k}}{12}$$

$$\hat{n} = \frac{2}{3}\hat{i} - \frac{\hat{j}}{3} - \frac{2}{3}\hat{k}.$$

The Vector ⊥ $\vec{A}$ and ⊥ $\vec{B}$ and having magnitude 6 is

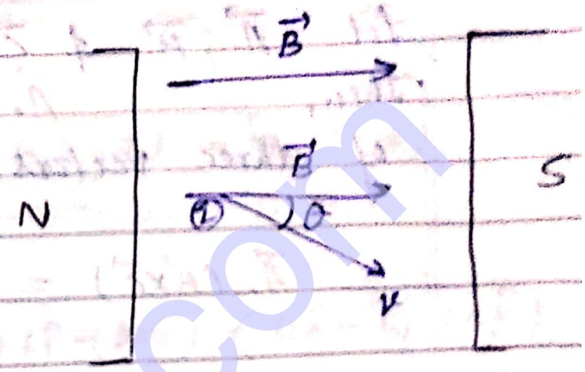
$$\vec{X} = 6\,(\text{Unit Vector})$$

$$\vec{X} = \cancel{6}^{\,2}\left[\frac{2\hat{i}-\hat{j}-2\hat{k}}{\cancel{3}}\right]$$

$$\boxed{\vec{X} = 4\hat{i} - 2\hat{j} - 4\hat{k}}$$

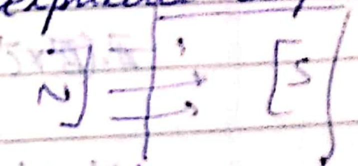
APPLICATION OF VECTOR PRODUCT

① When a charged particle moves in magnetic field, it experience a force known as Lorentz force. And we can find the direction of Lorentz force by using Right Hand Thumb Rule.



② When a current carrying wire is placed in magnetic field, the wire will expirience a force According to the relation,

$$\boxed{\vec{F} = I(\vec{l} \times \vec{B})}$$

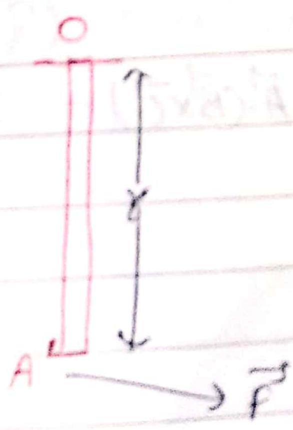


We can find the direction of $\vec{F}$ by using Right Hand Thumb Rule.

③ Let a rod OA is rotating about point O. A force $\vec{F}$ is acting at point A. This force will try to rotate the rod and produce a torque.

$$\vec{OA} = \vec{r}$$

$$\boxed{\vec{\tau} = \vec{r} \times \vec{F}}$$



The direction of torque can be obtained by using Right Hand Thumb Rule.

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1 Ques will be Asked

SCALAR TRIPLE PRODUCT

If the product of 3 vectors is a scalar quantity. This product is known scalar triple product.

Let $\vec{A}$, $\vec{B}$ & $\vec{C}$ are three vector quantities. Then, $\vec{A}\cdot(\vec{B}\times\vec{C})$ is scalar triple product of three vectors.

$$\vec{A}\cdot(\vec{B}\times\vec{C}) = \vec{B}\cdot(\vec{A}\times\vec{C}) = \vec{C}\cdot(\vec{A}\times\vec{B})$$

Q If $\vec{A} = 2\hat{i}+3\hat{j}+4\hat{k}$, $\vec{B} = 4\hat{i}+6\hat{j}+8\hat{k}$ & $\vec{C} = 2\hat{i}+\hat{j}+\hat{k}$.

$$\vec{A}\cdot(\vec{B}\times\vec{C}) = (2\hat{i}+3\hat{j}+4\hat{k})\cdot\left[(4\hat{i}+6\hat{j}+8\hat{k})\times(2\hat{i}+\hat{j}+\hat{k})\right]$$

$$\vec{B}\times\vec{C} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & 6 & 8 \\ 2 & 1 & 1 \end{vmatrix}$$

$$= \hat{i}(6-8) - \hat{j}(4-16) + \hat{k}(4-12)$$
$$= -2\hat{i} + 12\hat{j} - 8\hat{k}.$$

$$\vec{A}\cdot(\vec{B}\times\vec{C}) = (2\hat{i}+3\hat{j}+4\hat{k})\cdot(-2\hat{i}+12\hat{j}-8\hat{k})$$
$$= -4+36-32$$
$$\vec{A}\cdot(\vec{B}\times\vec{C}) = 0$$

Q. If $\vec{A} = 4\hat{i} + 6\hat{j} + 8\hat{k}$, $\vec{B} = \hat{i} + \hat{j} + \hat{k}$, $\vec{C} = 3\hat{i} + 3\hat{j} + 3\hat{k}$

Find $\vec{A} \cdot (\vec{B} \times \vec{C})$.

Sol. $\vec{B} \times \vec{C} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ 3 & 3 & 3 \end{vmatrix}$

$$\hat{i}(3-3) - \hat{j}(3-3) + \hat{k}(3-3)$$

$$\vec{B} \times \vec{C} = 0\hat{i} - 0\hat{j} + 0\hat{k}.$$

$$\vec{A} \cdot (\vec{B} \times \vec{C}) = (4\hat{i} + 6\hat{j} + 8\hat{k})(0\hat{i} - 0\hat{j} + 0\hat{k})$$

$$\vec{A} \cdot (\vec{B} \times \vec{C}) = 0$$

Q. $\vec{A} = 3\hat{i} + 3\hat{j} + 3\hat{k}$

$\vec{B} = 4\hat{i} + 4\hat{j} + 4\hat{k}$

$\vec{C} = 2\hat{i} + \hat{j} + \hat{k}$

Sol. $\vec{B} \times \vec{C} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & 4 & 4 \\ 2 & 1 & 1 \end{vmatrix}$

$$\vec{B} \times \vec{C} = \hat{i}(4-4) - \hat{j}(4-8) + \hat{k}(4-8)$$

$$= 0\hat{i} + 4\hat{j} - 4\hat{k}.$$

$$\vec{A} \cdot \vec{B} \times \vec{C} = (3\hat{i} + 3\hat{j} + 3\hat{k})(4\hat{j} - 4\hat{k})$$

$$= 12\hat{j} - 12\hat{k}$$

$$= 0$$

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2nd Method 2marks →

$$\vec{A}\cdot(\vec{B}\times\vec{C}) = \begin{vmatrix} 3 & 3 & 3 \\ 4 & 4 & 4 \\ 2 & 1 & 1 \end{vmatrix}$$

$$= 3(4-4) - 3(4-8) + 3(4-8)$$

$$= 0 + 12 - 12$$

$$\vec{A}\cdot(\vec{B}\times\vec{C}) = 0$$

Q. If $\vec{A} = 2\hat{i} + 3\hat{j} + 4\hat{k}$, $\vec{B} = 4\hat{i} + 2\hat{j} + \hat{k}$ and $\vec{C} = 3\hat{i}+2\hat{j}+2\hat{k}$
find $\vec{A}\cdot(\vec{B}\times\vec{C})$.

Sol. $\vec{A}\cdot(\vec{B}\times\vec{C}) = \begin{vmatrix} 2 & 3 & 4 \\ 4 & 2 & 1 \\ 3 & 2 & 2 \end{vmatrix}$

$$= 2(4-2) - 3(8-3) + 4(8-6)$$

$$= 2\times2 - 3\times5 + 4\times2$$

$$= 4 - 15 + 8$$

$$= 12 - 15$$

$$\vec{A}\cdot(\vec{B}\times\vec{C}) = -3$$

Q. If $\vec{A} = 3\hat{i} + 7\hat{j} + 8\hat{k}$, $\vec{B} = \hat{i}+\hat{j}+7\hat{k}$ & $\vec{C} = 3\hat{i}+\hat{j}+4\hat{k}$.
find $\vec{C}\cdot(\vec{A}\times\vec{B})$.

$\therefore \vec{A}\cdot(\vec{B}\times\vec{C}) = \vec{C}\cdot(\vec{A}\times\vec{B})$

Sol $\Rightarrow \vec{A}\cdot(\vec{B}\times\vec{C}) = \begin{vmatrix} 3 & 7 & 8 \\ 1 & 1 & 7 \\ 3 & 1 & 4 \end{vmatrix}$

$$= 3(4-7) - 7(4-21) + 8(1-3)$$

$$= +3\times-3 - 7\times-17 + 8\times-2$$

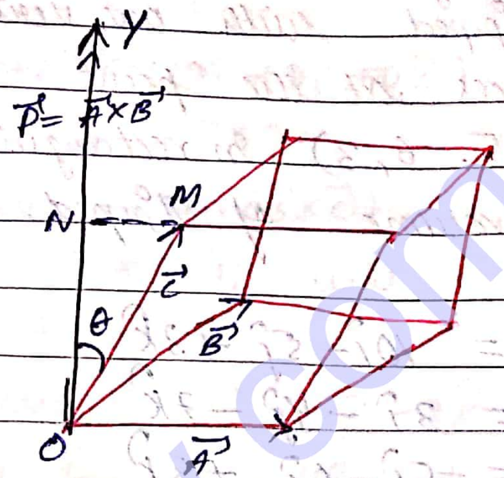
$$\vec{C}\cdot(\vec{A}\times\vec{B}) = -9 + 119 - 16 = 25 + 69 = +94$$

VOLUME OF PARALLELOPIPED

(Derivation will be ask. in exam)

If $\vec{A}, \vec{B}$ and $\vec{C}$ are representing the three sides of parallelopiped as shown in figure.

Let vector $\vec{P} = \vec{A} \times \vec{B}$.
It is clear $\vec{P}$ is perpendicular to $\vec{A}$ and perpendicular to $\vec{B}$ and the direction of $\vec{P}$ is represented by OY.



Area of the base of ||piped $= |\vec{A} \times \vec{B}| = |\vec{P}|$

Perpendicular distance between two bases of ||piped $= ON$.

In Δ MNO, $\cos\theta = \dfrac{ON}{|\vec{C}|}$

$$ON = |\vec{C}| \cos\theta$$

Volume of ||piped = (Area of base) x perpendicular distance b/w two ba

$$V = |\vec{P}|\ |\vec{C}| \cos\theta$$

$$\boxed{V = \vec{P} \cdot \vec{C}}$$

$$\boxed{V = (\vec{A} \times \vec{B}) \cdot \vec{C}}$$

11/Aug/15

Volume of ||^{0}piped, $V = (\vec{A}\times\vec{B})\cdot\vec{C}$
$= (\vec{B}\times\vec{C})\cdot\vec{A}$
$= (\vec{C}\times\vec{A})\cdot\vec{B}$

* If we interchange the dot & cross in scalar triple product the magnitude will remain same

Volume $= \vec{A}\cdot(\vec{B}\times\vec{C}) = \vec{B}\cdot(\vec{C}\times\vec{A}) = \vec{C}(\vec{A}\times\vec{B})$

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2010

Q ① R.V. In ||^{0}piped with one vertex at the origin has 3 vertices at the point $(10,-5,3)$, $(3,-4,-7)$ and $(-5,-6,3)$. In rectangular coordinates x, y z. Find the volume of ||^{0}piped.

Sol.

$\vec{OA} = 10\hat{i} - 5\hat{j} + 3\hat{k}$
$\vec{OB} = 3\hat{i} - 4\hat{j} - 7\hat{k}$
$\vec{OC} = -5\hat{i} - 6\hat{j} + 3\hat{k}$

$$\vec{OB}\times\vec{OC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -4 & -7 \\ -5 & -6 & 3 \end{vmatrix}$$

$= \hat{i}(-12-42) - \hat{j}(9-35) + \hat{k}(-18-20)$
$= -54\hat{i} + 26\hat{j} - 38\hat{k}$.

$\vec{OA}\cdot(\vec{OB}\times\vec{OC}) = (10\hat{i} - 5\hat{j} + 3\hat{k})(-54\hat{i} + 26\hat{j} - 38\hat{k})$
$= -540 - 130 - 114$
$= -784$

$\therefore$ Volume of ||^{0}piped $= \vec{OA}\cdot(\vec{OB}\times\vec{OC})$
$= -784$ units
$= 784$ units

Q. (2) A $\parallel^{o}$piped has the edge described by the vectors.
$\vec{A} = \hat{i} + 2\hat{j}$, $\vec{B} = 4\hat{j}$ and $\vec{C} = \hat{j} + 3\hat{k}$.
Compute its volume.

$$\vec{B} \times \vec{C} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 4 & 0 \\ 0 & 1 & 3 \end{vmatrix}$$

$$= \hat{i}(12) - \hat{j}(0) + \hat{k}(0)$$

$$\vec{B} \times \vec{C} = 12.$$

$$\vec{A} \cdot (\vec{B} \times \vec{C}) = (\hat{i} + 2\hat{j}) \cdot (12\hat{i} + 0\hat{j} + 0\hat{k})$$

$$= 12$$

$\therefore$ Volume of $\parallel^{o}$piped $= \vec{A} \cdot (\vec{B} \times \vec{C})$
$= 12$ unit

Q (3) Find n so that the vectors $2\hat{i} + 3\hat{j} - 2\hat{k}$, $5\hat{i} + n\hat{j} + \hat{k}$ and $-\hat{i} + 2\hat{j} + 3\hat{k}$ may form a $\parallel^{o}$piped of volume. 52 units.

Sol
Let $\vec{A} = 2\hat{i} + 3\hat{j} - 2\hat{k}$, $\vec{B} = 5\hat{i} + n\hat{j} + \hat{k}$
$\vec{C} = -\hat{i} + 2\hat{j} + 3\hat{k}$

$$\vec{B} \times \vec{C} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 5 & n & 1 \\ -1 & 2 & 3 \end{vmatrix}$$

$$= \hat{i}(3n - 2) - \hat{j}(15 + 1) + \hat{k}(10 + n)$$

$$= \hat{i}(3n - 2) - 16\hat{j} + \hat{k}(10 + n)$$

$$\vec{A} \cdot (\vec{B} \times \vec{C}) = (2\hat{i} + 3\hat{j} - 2\hat{k}) \cdot [(3n - 2)\hat{i} - 16\hat{j} + (10 + n)\hat{k}]$$

$$= 6n - 4 - 48 - 20 - 2n$$

$$= 4n - 72.$$

$\therefore$ Volume $= 52$

$4(n - 18) = 52$ 13

$n = 13 + 18$

K.U. 2003

(4) If the sides of $\parallel^{o}$pipe are given by vectors

$\vec{A} = 5\hat{i} - 7\hat{j} + 3\hat{k}$, $\vec{B} = -4\hat{i} + 7\hat{j} - 8\hat{k}$

$\vec{C} = 2\hat{i} - 3\hat{j}$.

Find the volume

$$\vec{B} \times \vec{C} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -4 & 7 & -8 \\ 2 & -3 & 0 \end{vmatrix}$$

$$= \hat{i}(0 - 24) - \hat{j}(0 + 16) + \hat{k}(12 - 14)$$

$$= -24\hat{i} - 16\hat{j} - 2\hat{k}$$

$$\vec{A} \cdot (\vec{B} \times \vec{C}) = (5\hat{i} - 7\hat{j} + 3\hat{k})(-24\hat{i} - 16\hat{j} - 2\hat{k})$$

$$= -120 + 112 - 6$$

$$= 14 \text{ units}$$

Note: If three vectors $\vec{A}, \vec{B}$ & $\vec{C}$ are coplanar then, the volume of the $\parallel^{o}$piped formed by these vectors will be zero.

$$\vec{A} \cdot (\vec{B} \times \vec{C}) = 0.$$

∵ Breadth is zero.

K.U. (5) Proof that the vectors $\hat{i} - 2\hat{j} + 3\hat{k}$, $-2\hat{i} + 3\hat{j} - 4\hat{k}$ and $-\hat{j} + 2\hat{k}$ are coplanar.

R.U.

If these vectors are coplanar their scalar triple product will be zero.

Let $\vec{A} = \hat{i} - 2\hat{j} + 3\hat{k}$

$\vec{B} = -2\hat{i} + 3\hat{j} - 4\hat{k}$

$\vec{C} = -\hat{j} + 2\hat{k}$

$$\vec{A}\cdot(\vec{B}\times\vec{C}) = \begin{array}{ccc} \hat{i} & \hat{j} & \hat{k} \\ \end{array}$$

$$\vec{A}\cdot(\vec{B}\times\vec{C}) = \begin{vmatrix} 1 & -2 & 3 \\ -2 & 3 & -4 \\ 0 & -1 & 2 \end{vmatrix}$$

$= 1(6-4) + 2(-4+0) + 3(+2-0)$

$= 2 - 8 + 6$

$\vec{A}\cdot(\vec{B}\times\vec{C}) = 0$

Q. (6) Find the constant 'a' such that the vectors $2\hat{i}-\hat{j}+\hat{k}$, $\hat{i}+2\hat{j}-3\hat{k}$ and $3\hat{i}+a\hat{j}+5\hat{k}$ are coplanar.

S.U.

If these vectors are coplanar. Then, their scalar triple product will be zero.

Let $\vec{A} = 2\hat{i}-\hat{j}+\hat{k}$

$\vec{B} = \hat{i}+2\hat{j}-3\hat{k}$

$\vec{C} = 3\hat{i}+a\hat{j}+5\hat{k}$

$$\vec{A}\cdot(\vec{B}\times\vec{C}) = \begin{vmatrix} 2 & -1 & 1 \\ 1 & 2 & -3 \\ 3 & a & 5 \end{vmatrix} = 0.$$

$2(10+3a) + 1(5+9) + 1(a-6) = 0.$

$20 + 6a + 14 + a - 6 = 0$

$7a = -28$

$a = -4.$

Vector Triple Product

If $\vec{A}$, $\vec{B}$ and $\vec{C}$ are three vectors then $\vec{A}\times(\vec{B}\times\vec{C})$ is the vector triple product of three vectors.

$$\underbrace{\vec{A}}_{\text{Vector}}\times\underbrace{(\vec{B}\times\vec{C})}_{\text{Vector}} = \text{vector}$$

Q.① If $\vec{A} = 2\hat{i}+3\hat{j}+4\hat{k}$
$\vec{B} = 4\hat{i}+\hat{j}+2\hat{k}$
$\vec{C} = 3\hat{i}+\hat{j}+2\hat{k}$.
Find $\vec{A}\times(\vec{B}\times\vec{C})$.

Solution $\vec{B}\times\vec{C} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & 1 & 2 \\ 3 & 1 & 2 \end{vmatrix}$

$= \hat{i}(2-2)-\hat{j}(8-6)+\hat{k}(4-3)$
$= -2\hat{j}+\hat{k}$

$\vec{A}\times(\vec{B}\times\vec{C}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 4 \\ 0 & -2 & 1 \end{vmatrix}$

$= \hat{i}(3+8)-\hat{j}(2-0)+\hat{k}(-4)$

$\vec{A}\times(\vec{B}\times\vec{C}) = 11\hat{i}-2\hat{j}-4\hat{k}$

Q (2) If $\vec{A} = 4\hat{i} + 4\hat{j} + \hat{k}$. find $\vec{A} \times (\vec{B} \times \vec{C})$

$\vec{B} = 2\hat{i} + \hat{j} + \hat{k}$

$\vec{C} = 7\hat{i} + \hat{j} + 3\hat{k}$

$$\vec{B} \times \vec{C} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & 1 \\ 7 & 1 & 3 \end{vmatrix}$$

$$= \hat{i}(3-1) - \hat{j}(6-7) + \hat{k}(2-7)$$

$$= 2\hat{i} + \hat{j} - 5\hat{k}.$$

$$\vec{A} \times (\vec{B} \times \vec{C}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & 4 & 1 \\ 2 & 1 & -5 \end{vmatrix}$$

$$= \hat{i}(-20-1) - \hat{j}(-20-2) + \hat{k}(4-8$$

$$= -21\hat{i} + 22\hat{j} - 4\hat{k}$$

Uniform Circular Motion

If an object is revolving in a circular orbit this motion is known as circular motion.

In circular motion, the direction of velocity is changing every point. It means the velocity is changing at each instant.
So, every circular motion is Accelerated Motion.

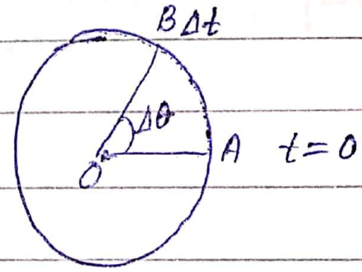
Uniform Circular Motion...
It is that circular motion in which the magnitude of velocity remains constant.

Non-Uniform Circular Motion...
If the magnitude & direction of velocity both are changing w.r.t. time. This circular motion is known as Non-Uniform Circular Motion.

ANGULAR DISPLACEMENT

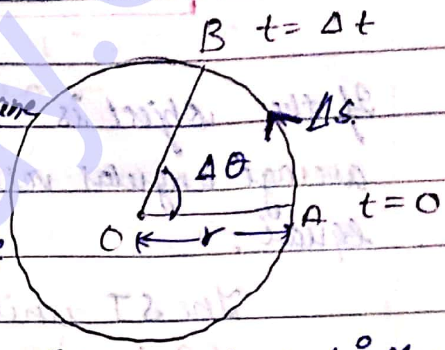
The angle subtended by the object at the centre is known as angular displacement.

The SI unit of angular displacement is radian (Rad)



Let $t = 0$, the object is at point A. After time Δt, the object is at point B. The angular displacement during this time is $\Delta\theta$.

Angular Displacement is a scalar physical quantity but when an angle is very small we can take angular displacement as a vector quantity.
And the direction of angular displacement can be obtained by using Right Hand Thumb Rule.



12-Aug-2015

Anticlockwise – onto the plane

If the
① Circular motion is in Anti-clockwise direction, the angular displacement will be $\perp$ to the plane of motion and radially outwards.

② Circulation motion is clockwise, the direction of angular displacement is $\perp$ to the plane of motion and radially inwards.

ANGULAR VELOCITY.

Time rate of change of angular displacement is known as angular velocity.

Let the object is at point A at $t = 0$. After time Δt, the object reach at point B.
Av. Angular Velocity, $\bar{\omega} = \dfrac{\Delta\theta}{\Delta t}$

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Instantaneous Angular Velocity:

If time is infinitely small, average angular velocity will convert into Instantaneous Angular Velocity.

$$\bar{\omega} = \frac{\Delta\theta}{\Delta t}$$

$$\omega = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t}$$

$$\boxed{\omega = \frac{d\theta}{dt}}$$

If the object is moving with constant angular velocity the average angular velocity and instantaneous angular velocity will be equal:

The SI unit of angular velocity is radian per second (Rad s^{-1})

ω is a vector quantity and the direction of ω is same as the direction of angular displacement.

LINEAR VELOCITY..

The rate of change of linear displacement is known as linear velocity.

Av. Linear Velocity, $\bar{V} = \frac{\Delta s}{\Delta t}$.

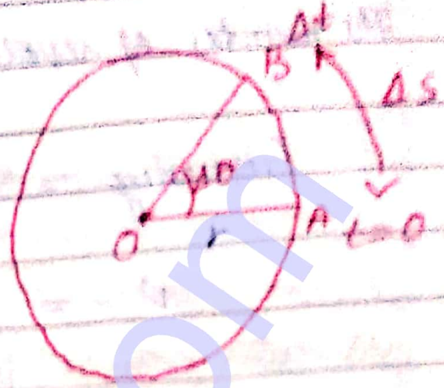
Instantaneous Linear Velocity:

$$V = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t}$$

$$\boxed{V = \frac{ds}{dt}}$$

Relation between Linear and Angular Velocity :

We know that Angle = $\frac{\text{Arc}}{\text{Radius}}$



$$\Delta\theta = \frac{\Delta s}{r} \quad \text{—} ①$$

$$\boxed{\Delta s = r \times \Delta\theta} \quad \text{—} ②$$

Dividing eqⁿ ② by Δt.

$$\frac{\Delta s}{\Delta t} = r \times \frac{\Delta\theta}{\Delta t} \quad \text{—} ③$$

$$\lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = r \times \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t}$$

$$\frac{ds}{dt} = r \times \frac{d\theta}{dt}$$

$$\boxed{v = r\omega} \quad (A)$$

Linear Velocity = Radius × Angular Velocity

TIME PERIOD :

The time taken by the object to complete 1 revolution is known as time period (ΔT).

$$\therefore \omega = \frac{d\theta}{dt}$$

In one revolution,

$$d\theta = 2\pi \text{ rad}$$
$$dt = T$$

$$\Rightarrow \boxed{\omega = \frac{2\pi}{T}} \quad (B)$$

FREQUENCY

The number of revolutions per second is known as frequency

$$\text{Frequency} = \frac{1}{\text{Time-period}}$$

$$\nu = \frac{1}{T}$$

$\because \omega = \frac{2\pi}{T}$

$$\boxed{\omega = 2\pi\nu}$$

Q. Find the angular velocity of minute hand, hour hand & second hand

Time period of minute hand = 60 min
= 60×60 sec.

$\because \omega = \frac{d\theta}{dt}$.

$\because \omega = \frac{2\pi}{60\times60} \frac{\text{rad}}{\text{sec}}$

$\omega = \frac{\pi}{1800} \text{ rad sec}^{-1}$

×60×60

Time period of hour hand = 12 hours
= 12×60×60 sec.

$\because \omega = \frac{d\theta}{dt}$

$\omega = \frac{2\pi}{12\times60\times60}$

$\omega = \frac{\pi}{12\times1800} = \frac{\pi}{21600} \text{ rad sec}^{-1}$

Time period of ~~hour~~ second hand $= 1$ minute
$= 60$ sec.

$$\because \omega = \frac{2\pi}{T}$$

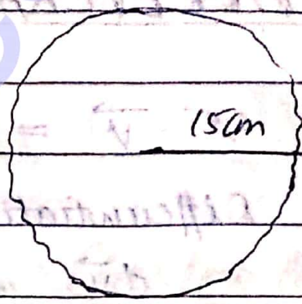
$$\omega = \frac{2\pi}{60} = \frac{\pi}{30} \text{ rad sec}^{-1}$$

Q. If the length of minute hand is 15 cm, find the linear velocity of tip of minute hand.

$V = r\omega$

Angular velocity for minute hand

$$= \frac{\pi}{1800} \text{ rad s}^{-1}$$



$r = 15$ cm

$r = 0.15$ m

$$\because \quad v = r\omega$$

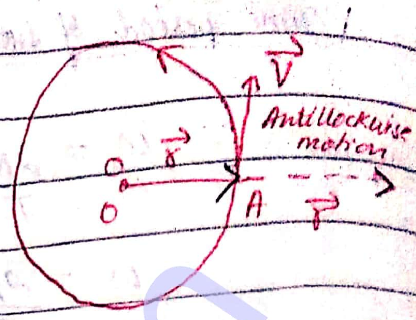
$$= \frac{15}{100} \times \frac{\pi}{1800}$$

$$v = \frac{5\pi}{6} \times 10^{-4} \text{ ms}^{-1}.$$

$$v = 0.83\pi \times 10^{-4} \text{ ms}^{-1}$$

$$v = \frac{\pi}{120} \times 10^{-2} \text{ ms}^{-1}$$

Vector form of $v = r\omega$



$$\vec{V} = \vec{\omega} \times \vec{r}$$

(Right Hand Thumb Rule)

a. $\vec{\omega}$ is $\perp$ to the plane and radially outward

b. Linear velocity $\vec{v}$ tangential.

c. radius vector $\vec{r}$ = radial

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ACCELERATION IN CIRCULAR MOTION:

$\therefore$ $\vec{V} = \vec{\omega} \times \vec{r}$ ——(1)

Differentiating eqn(1) w.r.t. 't'

$$\frac{d\vec{v}}{dt} = \frac{d}{dt}(\vec{\omega} \times \vec{r})$$

$$\frac{d\vec{v}}{dt} = \frac{d\vec{\omega}}{dt} \times \vec{r} + \vec{\omega} \times \frac{d\vec{r}}{dt}$$

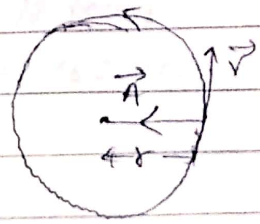
$$\boxed{\vec{A} = \vec{\alpha} \times \vec{r} + \vec{\omega} \times \vec{V}}$$

$$\vec{\alpha} = \text{angular acceleration} = \frac{d\vec{\omega}}{dt}$$

ase 1: If the circular motion is uniform circular motion.

$\vec{\omega}$ = constant

$$\alpha = \frac{d\vec{\omega}}{dt} = 0.$$



$$\vec{A} = \vec{\alpha} \times \vec{r} + \vec{\omega} \times \vec{V}$$

$$\boxed{\vec{A} = \vec{\omega} \times \vec{V}}$$ [uniform circular motion]

→ The direction of $\vec{\omega} \times \vec{v}$ is towards the centre. So, this acceleration is known as centripetal acceleration which acts towards the centre.

Magnitude of Centripetal Acceleration

$$\vec{A} = \vec{\omega} \times \vec{v}$$

$$A = \omega v \sin 90^\circ$$

$$\boxed{A = \omega v}$$

$\because \; v = r\omega$

$$\Rightarrow \boxed{A = r\omega^2}$$

$\because \; \omega = \frac{v}{r}$.

$$\Rightarrow \boxed{A = \frac{v^2}{r}}$$

CENTRIPETAL FORCE. (Uniform Circular Motion):

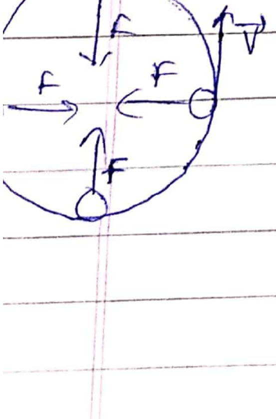
According to Newton's second law of ~~law~~ motion,

$$\vec{F} = m \times \vec{a}$$

Direction of $\vec{F}$ is along the direction of $\vec{a}$.
It means a force act in uniform circular motion which is towards the centre and this force is known as centripetal force.

$$F = m \times (\omega v) \qquad [\because A = \omega v]$$

$$\boxed{F = m\omega v}$$



$$F = m \times (r\omega^2) \qquad [\because A = r\omega^2]$$

$$\boxed{F = mr\omega^2}$$

$$F = m \times (v^2/r) \qquad \left[\because A = \frac{v^2}{r}\right]$$

$$\boxed{F = \frac{mv^2}{r}}$$

* When centripetal force becomes zero, the circular motion converts into straight-line motion.

e.g.

① The gravitational force between Earth and Sun provide centripetal force to revolve the Earth around the sun.

② In Bohr's Atomic model, the electrostatic force between the nucleus & electron provide necessary centripetal force to revolve the electron around the nucleus.

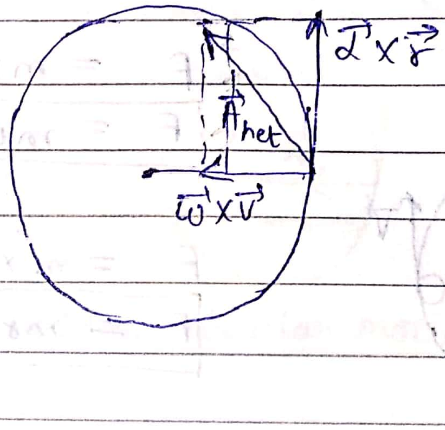
NON-UNIFORM CIRCULAR MOTION:

ase 2 Acceleration in Non-Uniform circular Motion

$$\vec{A} = \underbrace{\vec{\alpha}\times\vec{r} + \vec{\omega}\times\vec{v}}_{\text{same direction}}$$

① The direction of $\vec{\alpha}\times\vec{r}$ is along the tangent.
② The direction of $\vec{\alpha}$ is along the direction of $\vec{\omega}$.
③ The direction of $\vec{\omega}\times\vec{v}$ is towards the centre.

Non-Uniform C.M.



$$|\vec{\alpha}\times\vec{r}| = \alpha\times r\times\sin 90^\circ$$
$$= \alpha\times r$$

$$|\vec{\omega}\times\vec{v}| = \omega\times v\times\sin 90^\circ$$
$$= \omega\times v$$

The resultant of two accelerations

$$R = \sqrt{A^2 + B^2 + 2AB\cos\theta}$$

$$A = \sqrt{(\alpha \times r)^2 + (\omega \times v)^2 + 2(\alpha \times r)(\omega \times v)\cos 90^\circ}$$

$$A = \sqrt{(\alpha \times r)^2 + (\omega \times v)^2}$$

$$\boxed{A = \sqrt{\alpha^2 r^2 + \omega^2 v^2}}$$

Q. An object of mass 2kg is moving in a circle of radius 10m and completes 120 revolutions per minute. Find centripetal acceleration and centripetal force.

Sol. 1 minute = 60 s = 120 revolution

1 s = 2 revolution.

ν, Frequency = 2 Hz.

Angular freq, $\omega = 2\pi\nu$

$= 2\pi \times 2$

$= 4\pi$ rad/sec.

Centripetal acc., $A = r\omega^2$

$= 10(4\pi)^2$

$= 160\pi^2$ $\quad (\pi^2 = 10)$

$A = 1600$ m-sec^{-2}

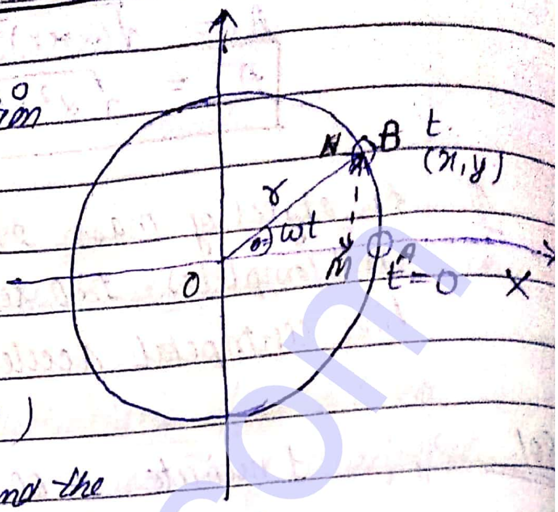
Centripetal force, $F = m \times A$

$= 2 \times 1600$

$F = 3200$ N.

Circular Motion In Vector form :

Let in a uniform circular motion the object is at point A at $t=0$.
After time t, the object reach at point B.
The coordinate of B is (X, Y) and radius of circle 'r' and the angular velocity of object is ω.



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$$\text{Angular Velocity} = \frac{\text{Angular Displacement}}{\text{Time}}$$

$$\omega = \frac{\theta}{t}$$

$$\theta = \omega t$$

In Δ OMB, $\cos\theta = \frac{OM}{OB} \Rightarrow OM = OB\cos\theta$

$$OM = r\cos\theta \quad \text{——(i)}$$

$$\sin\theta = \frac{BM}{OB} \Rightarrow BM = OB\sin\theta$$

$$BM = r\sin\theta \quad \text{——(ii)}$$

$$\vec{OM} = (r\cos\theta)\hat{i}$$
$$\vec{MB} = (r\sin\theta)\hat{j}$$

Using Δ law in ΔOMB,

$$\vec{OB} = \vec{OM} + \vec{MB}$$

$$\boxed{\vec{r} = (r\cos\theta)\hat{i} + (r\sin\theta)\hat{j}}$$

General Eqn of circle,

$$\vec{r} = (r\cos\theta)\hat{i} + (r\sin\theta)\hat{j}$$
$$\vec{r} = r\cos\omega t\,\hat{i} + r\sin\omega t\,\hat{j}$$

The Equation of position vector,

$$\vec{r} = (r\cos\omega t)\hat{i} + (r\sin\omega t)\hat{j}$$

$$\vec{v} = \frac{d\vec{r}}{dt}$$

$$\vec{v} = \frac{d}{dt}\left[r\cos\omega t\,\hat{i} + r\sin\omega t\,\hat{j}\right]$$

$$\vec{v} = r(-\sin\omega t).\omega\,\hat{i} + r(\cos\omega t).\omega\,\hat{j}$$

$$\boxed{\vec{v} = -r\omega\sin\omega t\,\hat{i} + r\omega\cos\omega t\,\hat{j}}$$

Magnitude of Velocity,

$$|\vec{v}| = \sqrt{(-r\omega\sin\omega t)^2 + (r\omega\cos\omega t)^2}$$

$$|\vec{v}| = r\omega\sqrt{\sin^2\omega t + \cos^2\omega t}$$

$$|\vec{v}| = r\omega.(1)$$

$$\boxed{|\vec{v}| = r\omega}$$

Uniform circular motion, $\therefore \omega = \text{constant}$

Expression for Acceleration :

$\vec{V} = -r\omega \sin\omega t\, \hat{i} + r\omega \cos\omega t\, \hat{j}$

$\because \vec{a} = \frac{d\vec{V}}{dt}$

$= \frac{d}{dt}[-r\omega \sin\omega t\, \hat{i} + r\omega \cos\omega t\, \hat{j}]$

$= -r\omega . \cos\omega t . \omega . \hat{i} + r\omega . (-\sin\omega t) . \omega . \hat{j}$

$\vec{a} = -r\omega^2 \cos\omega t\, \hat{i} - r\omega^2 \sin\omega t\, \hat{j}$

$\vec{a} = -[r\omega^2 \cos\omega t\, \hat{i} + r\omega^2 \sin\omega t\, \hat{j}]$

$\vec{a} = -\omega^2 [r\cos\omega t\, \hat{i} + r\sin\omega t\, \hat{j}]$

$\boxed{\vec{a} = -\omega^2 . \vec{r}}$

It is clear from the expression that $\vec{a}$ is opposite to the direction of $\vec{r}$ i.e. $\vec{a}$ will act towards the centre and this acceleration is known as centripetal acceleration.

$\therefore \vec{a} = -\omega^2 [r\cos\omega t\, \hat{i} + r\sin\omega t\, \hat{j}]$

$\vec{a} = -\omega^2 r \cos\omega t\, \hat{i} - \omega^2 r \sin\omega t\, \hat{j}$

$|\vec{a}| = \sqrt{(-\omega^2 r \cos\omega t)^2 + (-\omega^2 r \sin\omega t)^2}$

$|\vec{a}| = \omega^2 r \sqrt{\cos^2\omega t + \sin^2\omega t} \Rightarrow |\vec{a}| = \omega^2 r \;(1)$

$\boxed{|\vec{a}| = \omega^2 r}$ $[\because \sin^2 x + \cos^2 x = 1]$

A particle moves along the curve $x = 2\sin 3t$, $y = 2\cos 3t$ and $z = 8t$.

Find its velocity and acceleration at any time $t = 0$.

Find the magnitude of velocity and acceleration

⇒ The position vector,

$$\vec{r} = (2\sin 3t)\hat{i} + (2\cos \omega t)\hat{j} + (8t)\hat{k}$$

$$\vec{v} = \frac{d\vec{r}}{dt}$$

$$= \frac{d}{dt}\left[2\sin\omega t\,\hat{i} + 2\cos\omega t\,\hat{j} + 8t\,\hat{k}\right]$$

$$= 2\cos\omega t.\omega\hat{i} + 2(-\sin\omega t).\omega\hat{j} + 8\hat{k}$$

$$\vec{v} = 2\omega\cos\omega t\,\hat{i} - 2\omega\sin\omega t\,\hat{j} + 8\hat{k} \quad \text{--- (I)}$$

$$|\vec{v}| = \sqrt{(2\omega\cos\omega t)^2 + (-2\omega\sin\omega t)^2 + (8)^2}$$

$$= \sqrt{4\omega^2\cos^2\omega t + 4\omega^2\sin^2\omega t + 64}$$

$$|\vec{v}| = \sqrt{4\omega^2 + 64}$$

$$|\vec{v}| = 2\sqrt{\omega^2 + 16}$$

$$= 2\sqrt{9+16}$$

$$= 2\sqrt{25}$$

$$|\vec{v}| = 10$$

$$\vec{a} = \frac{d\vec{v}}{dt}$$

$$\vec{a} = \frac{d}{dt}\left[2\omega\cos\omega t\,\hat{i} - 2\omega\sin\omega t\,\hat{j} + 8\hat{k}\right]$$

$$\vec{a} = 2\omega(-\sin\omega t).\omega\hat{i} - 2\omega\cos\omega t.\omega\hat{j} + 8(0)$$

$$\vec{a} = -2\omega^2\sin\omega t\,\hat{i} - 2\omega^2\cos\omega t\,\hat{j} \quad \text{--- (II)}$$

$$|\vec{a}| = \sqrt{(-2\omega^2\sin\omega t)^2 + (-2\omega^2\cos\omega t)^2}$$

$$= 2\omega^2(1)$$

$$= 2(3)^2$$

$$|\vec{a}| = 18 \text{ unit}$$

Alternate: From eqn (I), At $t=0$ $\vec{v} = 6\hat{i} + 8\hat{k}$

from eqn (II), $\vec{a} = -18\hat{j}$

Magnitude of $|\vec{v}| = \sqrt{(6)^2+(8)^2} = \sqrt{36+64} = 10$ unit

Magnitude of acc., $|\vec{a}| = \sqrt{(-18)^2} = 18$ unit

Q. A particle moves along a curve $x = e^{-t}$, $y = 9\cos 3t$, $z = 2\sin 3t - 5$. Find the magnitude of velocity and acceleration at $t = 0$.

The position vector, $\vec{r} = e^{-t}\hat{i} + 2\cos 3t\,\hat{j} + (2\sin 3t - 5)\hat{k}$

$\therefore \vec{v} = \dfrac{d\vec{r}}{dt}$

$= \dfrac{d}{dt}\left[e^{-t}\hat{i} + 2\cos 3t\,\hat{j} + (2\sin 3t - 5)\hat{k}\right]$

$\vec{v} = -e^{-t}\hat{i} + 2(-\sin 3t).3\hat{j} + 6\cos 3t\,\hat{k}$

$\vec{a} = \dfrac{d\vec{v}}{dt}$

$= \dfrac{d}{dt}\left[-e^{-t}\hat{i} + 6(-\sin 3t)\hat{j} + 6\cos 3t\,\hat{k}\right]$

$\vec{a} = e^{-t} - 18\cos 3t\,\hat{j} + (-18\sin 3t)\hat{k}$

At $t = 0$, $\vec{v} = -1\hat{i} + 6(0)\hat{j} + 6(1)\hat{k}$

$\vec{v} = -\hat{i} + 6\hat{k}$

At $t = 0$, $\vec{a} = \hat{i} - 18\hat{j}$

$|\vec{v}| = \sqrt{(-1)^2 + (6)^2}$

$= \sqrt{37}$ unit

$|\vec{a}| = \sqrt{(1)^2 + (-18)^2}$

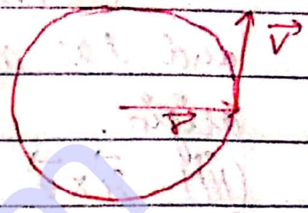
$= \sqrt{1 + 324}$

$|\vec{a}| = \sqrt{325}$ unit

V.Q. Most Important.

Q. Derive the condition for 2 vectors to be perpendicular to each other. If $\vec{A}$ has constant magnitude and $\left|\frac{d\vec{A}}{dt}\right| \neq 0$. Prove that $\vec{A}$ is $\perp$ to $\frac{d\vec{A}}{dt}$.

Sol. $\vec{A}$ is a constant vector.



$\vec{A}\cdot\vec{A} = |\vec{A}|\,|\vec{A}|\cos 0$

$\vec{A}\cdot\vec{A} = |\vec{A}|\,|\vec{A}| \times 1$

$\vec{A}\cdot\vec{A} = A\cdot A$

$\vec{A}\cdot\vec{A} = A^2$

$\boxed{\vec{A}\cdot\vec{A} = \text{constant}}$ ——(i)

$\vec{V} = \frac{d\vec{r}}{dt}$

$\vec{r} \perp \vec{V}$

$\vec{r} \perp \frac{d\vec{r}}{dt}$

Differentiating eqn (i) w.r.t. t.

$$\frac{d}{dt}(\vec{A}\cdot\vec{A}) = 0$$

$$\frac{d\vec{A}}{dt}\cdot\vec{A} + \vec{A}\cdot\frac{d\vec{A}}{dt} = 0 \quad \text{——(ii)}$$

Dot Product follow commutative law.

$$\Rightarrow \frac{d\vec{A}}{dt}\cdot\vec{A} = \vec{A}\cdot\frac{d\vec{A}}{dt}.$$

Putting in eqn (ii)

$$\vec{A}\cdot\frac{d\vec{A}}{dt} + \vec{A}\cdot\frac{d\vec{A}}{dt} = 0.$$

$$2\vec{A}\cdot\frac{d\vec{A}}{dt} = 0.$$

$$\vec{A}\cdot\frac{d\vec{A}}{dt} = 0$$

$\because \vec{A} \neq 0$ & $\left|\frac{d\vec{A}}{dt}\right| \neq 0$

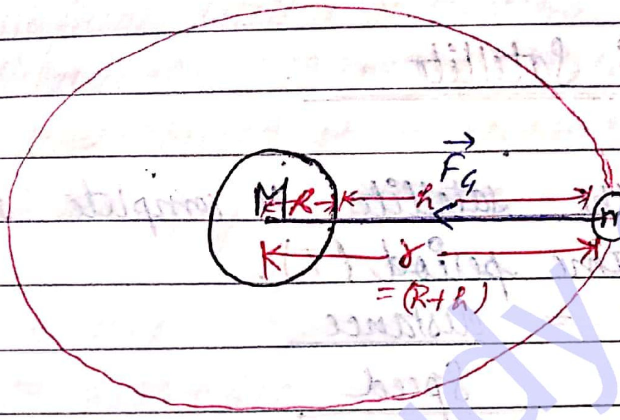
Hence, $\vec{A}$ is $\perp$ to $\frac{d\vec{A}}{dt}$.

4/11c -10^6 K   Nuclear fusion → Solar energy

SATELLITE..

The heavenly body which revolve around the planet is known as satellite

Let a satellite of mass 'm' is revolving around the planet of mass 'M' in a radius (circular orbit) of radius 'r' as shown in figure.



The gravitational force between planet and satellite will provide centripetal force to revolve the satellite around the planet

Centripetal Force = Gravitational Force

$$\frac{mv^2}{r} = \frac{GMm}{r^2}$$

$$\boxed{v = \sqrt{\frac{GM}{r}}} \quad \text{—— (I)}$$

Gravitational Acceleration At the Surface Of Planet:

$$g = \frac{GM}{R^2} \qquad \boxed{\because mg = F_g}$$

where, R = Radius of planet.

$$\boxed{GM = gR^2} \quad \text{—— (II)}$$

Putting value of eqⁿ (2) in eqⁿ (1)

$$v = \sqrt{\frac{gR^2}{r}}$$

$$\boxed{v = R\sqrt{\frac{g}{r}}}$$

$$\Rightarrow \quad v = R\sqrt{\frac{g}{R+h}} \qquad [\because r = R+h]$$

Time period Of Satellite

Time taken by the satellite to complete 1 revolution is known as time period (T).

$$\text{Time} = \frac{\text{distance}}{\text{Speed}}$$

$$\boxed{T = \frac{2\pi r}{v}}$$

We know that $v = \sqrt{\frac{GM}{r}}$

$$T = \frac{2\pi r}{\sqrt{GM/r}}$$

$$T = \frac{2\pi r^{1/2+1}}{\sqrt{GM}}$$

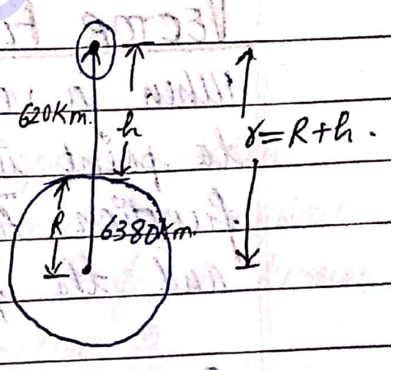
$$\boxed{T = 2\pi\left[\frac{r^3}{GM}\right]^{1/2}}$$

We know $GM = gR^2$

$$T = 2\pi \left[\frac{r^3}{gR^2}\right]^{1/2}$$

$$\boxed{T = \frac{2\pi}{R}\left[\frac{r^3}{g}\right]^{1/2}}$$

Q. An artificial satellite is revolving around the earth at a distance of 620 km. Calculate the minimum velocity and time period of revolution if radius of earth 6380 km. & $g = 9.8\ ms^2$.



Sol. $\therefore r = R+h$

$= 6380 + 620$

$= 7000$ km

$r = 1000 \times 7000$

$r = 7 \times 10^6 m$

$$\therefore v = R\sqrt{\frac{g}{r}}$$

$$= 6380\sqrt{\frac{9.8}{7\times10^6}}$$

$$v = \frac{6380}{10^3}\sqrt{\frac{9.8}{7}}$$

$$T = \frac{2\pi r}{v}$$

$$= \frac{2\pi \times 7\times10^6}{\frac{6380}{10^3}\times\sqrt{\frac{9.8}{7}}}$$

VECTOR Field &

Scalar field..

x,y → Vector By differentiation

Field of A Physical Quantity : A physical quantity can be expressed as a continuous function of position of a point in a region of space.
This function is called point function and the region in which it specifies is known as field.

VECTOR Field..

When a vector physical quantity is expressed from point to point in a region of space by a continuous vector function $\vec{A}(x,y,z)$. Then, the region is a vector field and the function $\vec{A}$ is known as vector point function.

SCALAR Field..

When a scalar physical quantity is expressed from point to point in a region of space by a continuous point function $\phi(x,y,z)$. Then, the region is a scalar field and the function ϕ is known as scalar point function.

R.V.

GRADIENT..

If ϕ is a scalar function of position in space (x,y,z) Then, its partial derivatives along the three orthogonal axis are

$$\frac{\partial \phi}{\partial x}, \quad \frac{\partial \phi}{\partial y}, \quad \frac{\partial \phi}{\partial z}$$

Then, gradient ϕ, $= \left(\frac{\partial \phi}{\partial x}\right)\hat{i} + \left(\frac{\partial \phi}{\partial y}\right)\hat{j} + \left(\frac{\partial \phi}{\partial z}\right)\hat{k}$

$$\text{grad}(\phi) = \hat{i}\frac{\partial \phi}{\partial x} + \hat{j}\frac{\partial \phi}{\partial y} + \hat{k}\frac{\partial \phi}{\partial z} \quad \text{—— (I)}$$

In vector alzebra, an operator known as $\vec{\nabla}$ (del/nebla) played an important role to define gradient

$$\vec{\nabla} = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

$$\vec{\nabla}.\phi = \hat{i}\frac{\partial \phi}{\partial x} + \hat{j}\frac{\partial \phi}{\partial y} + \hat{k}\frac{\partial \phi}{\partial z} \quad \text{—— (II)}$$

From eqn (I) & (II), it is clear that

$$\vec{\nabla}.\phi = \text{grad}(\phi)$$

The del operator $(\vec{\nabla})$ is a vector operator when operated with a scalar it convert a scalar physical quantity to vector physical quantity & $\vec{\nabla}.\phi$ is known as gradient of the scalar point function ϕ.

Q. If $\phi(x,y,z) = 3x^2y - y^3z^2$, find the value of grad$(\phi)$ at $(1,-2,-1)$

R.V.

$$\phi = 3x^2y - y^3z^2$$

$$\text{grad}\,\phi = \vec{\nabla}.\phi$$

$$\text{grad}\,\phi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(3x^2y - y^3z^2)$$

$$\text{grad}\,\phi = \hat{i}\frac{\partial}{\partial x}(3x^2y - y^3z^2) + \hat{j}\frac{\partial}{\partial y}(3x^2y - y^3z^2) + \hat{k}\frac{\partial}{\partial z}(3x^2y - y^3z^2)$$

$$\text{grad}\,\phi = \hat{i}[3y.(2x)+0]+\hat{j}[3x^2-3y^2z^2]+\hat{k}[0-2zy^3]$$

$$\text{grad}\,\phi = (6xy)\hat{i}+(3x^2-3y^2z^2)\hat{j} - 2zy^3.\hat{k}$$

grad ϕ at $(1,-2,-1)$

$$\text{grad}\,\phi = (6\times1\times-2)\hat{i}+[3(1)^2-3(-2)^2(-1)^2]\hat{j} - 2(-2)^3(-1)\hat{k}$$

$$\Rightarrow \text{grad}\,\phi = -12\hat{i} - 9\hat{j} - 16\hat{k}$$

Q. If $\phi(x,y,z) = 6x^2y+2y^2z+2x^2zy$. Find the value of gradient ϕ at $(2,-2,-3)$

$$\phi = 6x^2y+2y^2z+2x^2zy$$

$$\text{grad}\,\phi = \vec{\nabla}.\phi$$

$$\text{grad}\,\phi = \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)(6x^2y+2y^2z+2x^2zy)$$

$$\text{grad}\,\phi = \hat{i}\frac{\partial}{\partial x}(6x^2y+2y^2z+2x^2zy)+\hat{j}\frac{\partial}{\partial y}(6x^2y+2y^2z+2x^2zy)$$

$$+\hat{k}\frac{\partial}{\partial y}(6x^2y+2y^2z+2x^2zy).$$

$$\text{grad}\,\phi = \hat{i}(12xy+4xzy)+\hat{j}(6x^2+4yz+2x^2z)+$$

$$\hat{k}(2y^2+2x^2y).$$

grad ϕ at $(2,-2,-3)$.

$$\text{grad}\,\phi = \hat{i}(12\times-4+4\times2\times-2\times-3)+\hat{j}(6(4)4\times-2\times-3+2\times4\times-3)$$

$$+\hat{k}(2\times4+2\times4\times-2)$$

$$= \hat{i}(-48+48)+\hat{j}(-24-24)+\hat{k}(-8)$$

$$= +24\hat{j} - 8\hat{k}$$

16
3
48
72

Q. If $\vec{r}$ is the position vector of a point, find grad r.

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$
$$r = \sqrt{x^2+y^2+z^2}$$
$$\text{grad } r = \vec{\nabla}, r$$
$$\text{grad } r = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\left(\sqrt{x^2+y^2+z^2}\right)$$
$$\text{grad } r = \hat{i}\frac{\partial}{\partial x}(x^2+y^2+z^2)^{1/2} + \hat{j}\frac{\partial}{\partial y}(x^2+y^2+z^2)^{1/2} + \hat{k}\frac{\partial}{\partial z}(x^2+y^2+z^2)^{1/2}.$$
$$\text{grad } r = \hat{i}.\frac{1}{2}(x^2+y^2+z^2)^{-1/2}.2x + \hat{j}.\frac{1}{2}(x^2+y^2+z^2)^{-1/2}(2y) + \hat{k}\frac{1}{2}(x^2+y^2+z^2)^{-1/2}(2z)$$
$$\text{grad } r = \hat{i}(x^2+y^2+z^2)^{-1/2}.x + \hat{j}(x^2+y^2+z^2)^{-1/2}y + \hat{k}(x^2+y^2+z^2)^{-1/2}$$
$$\text{grad } r = (x^2+y^2+z^2)^{-1/2}(x\hat{i} + y\hat{j} + z\hat{k})$$
$$= (r^2)^{-1/2}, \vec{r}$$
$$\text{grad } r = \frac{\vec{r}}{r}$$

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Imp Q. If the $\vec{r}$ is the position vector of a point. Then, find grad $\left(\frac{1}{r}\right)$ and grad $(\log r)$.

Sol. (i) $\because \vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$r = \sqrt{x^2+y^2+z^2}$

$$\text{grad}\left(\frac{1}{r}\right) = \vec{\nabla}\frac{1}{r}.$$

$$\text{grad}\left(\frac{1}{r}\right) = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right)$$

$$= \hat{i}\frac{\partial}{\partial x}\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right) + \hat{j}\frac{\partial}{\partial y}\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right) + \hat{k}\frac{\partial}{\partial z}\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right)$$

$$= \hat{i}\left(\frac{-1}{2}(x^2+y^2+z^2)^{-3/2}\cdot 2x\right) + \hat{j}\left[\frac{-1}{2}(x^2+y^2+z^2)^{-3/2}\cdot 2y\right]$$
$$+ \hat{k}\left[\frac{-1}{2}(x^2+y^2+z^2)^{-3/2}\cdot 2z\right]$$

$$= -(x^2+y^2+z^2)^{-3/2}(x\hat{i}+y\hat{j}+z\hat{k}).$$

$$= -(x^2+y^2+z^2)^{-3/2}\,\vec{r}$$

$$= -(r^2)^{-3/2}\,\vec{r} \quad = -(r^{-3})\cdot\vec{r}$$

$$\text{grad}\left(\frac{1}{r}\right) = \frac{-\vec{r}}{r^3}$$

(ii) $$\text{grad}(\log r) = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\left(\log\sqrt{x^2+y^2+z^2}\right)$$

$$= \hat{i}\frac{\partial}{\partial x}\left(\log\sqrt{x^2+y^2+z^2}\right) + \hat{j}\frac{\partial}{\partial y}\left(\log\sqrt{x^2+y^2+z^2}\right) +$$
$$\hat{k}\frac{\partial}{\partial x}\left(\log\sqrt{x^2+y^2+z^2}\right)$$

$$= \hat{i}\left[\frac{1}{2}\frac{\partial}{\partial x}\log(x^2+y^2+z^2)\right] + \hat{j}\left[\frac{1}{2}\frac{\partial}{\partial y}\log(x^2+y^2+z^2)\right] + \hat{k}\left[\frac{1}{2}\frac{\partial}{\partial z}\log(x^2+y^2+z^2)\right]$$

$$= \hat{i}\,\frac{1}{2}\cdot\frac{1}{x^2+y^2+z^2}\cdot 2x + \hat{j}\,\frac{1}{2}\cdot\frac{1}{x^2+y^2+z^2}\cdot 2y + \hat{k}\cdot\frac{1}{2}\cdot\frac{1}{x^2+y^2+z^2}\cdot 2z$$

$$= \frac{1}{x^2+y^2+z^2}\left[x\hat{i}+y\hat{j}+z\hat{k}\right]$$

$$= \frac{1}{r^2}\cdot\vec{r}$$

$$\text{grad}(\log r) = \frac{\vec{r}}{r^2}$$

Q. Proof that grad $r^n = n r^{n-2}\vec{r}$ where $\vec{r}$ is the position vector of any point.

1. $\because$ $\vec{r} = x\hat{i}+y\hat{j}+z\hat{k}$

$$r = \sqrt{x^2+y^2+z^2}$$

$$\text{grad}(r^n) = \vec{\nabla} r^n$$

$$= \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\left(\sqrt{x^2+y^2+z^2}\right)^n$$

$$= \hat{i}\frac{\partial}{\partial x}\left(\sqrt{x^2+y^2+z^2}\right)^n + \hat{j}\frac{\partial}{\partial y}(x^2+y^2+z^2)^{n/2} + \hat{k}\frac{\partial}{\partial z}(x^2+y^2+z^2)^{n/2}$$

$$= \hat{i}\,\frac{1}{(x^2+y^2+z^2)^{-(n/2-1)}}\cdot\frac{n}{2}\cdot 2x + \hat{j}\,\frac{1}{(x^2+y^2+z^2)^{-(n/2-1)}}\cdot\frac{n}{2}\cdot 2y + \hat{k}\,\frac{1}{(x^2+y^2+z^2)^{-(n/2-1)}}\cdot\frac{n}{2}\,2z$$

$$= n(x^2+y^2+z^2)^{\frac{n}{2}-1}\,[x\hat{i}+y\hat{j}+z\hat{k}]$$

$$= n\,(r^2)^{\frac{n-2}{2}}\,[\vec{r}]$$

$$\text{grad}(r^n) = n\,(r)^{n-2}\,\vec{r}$$

Divergence:

If we want to convert a vector quantity to a scalar quantity we will find the divergence of vector quantity

∵ (del/nebla) $\vec{\nabla} = \hat{i}\frac{\delta}{\delta x} + \hat{j}\frac{\delta}{\delta y} + \hat{k}\frac{\delta}{\delta z}$

If $\vec{F}$ is a vector quantity, $\vec{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$

Then,

$$\vec{\nabla}.\vec{F} = \text{div}.\vec{F}$$

$$\text{div}.\vec{F} = \vec{\nabla}.\vec{F}$$

$$\text{div}.\vec{F} = \left(\hat{i}\frac{\delta}{\delta x} + \hat{j}\frac{\delta}{\delta y} + \hat{k}\frac{\delta}{\delta z}\right)(F_1\hat{i}+F_2\hat{j}+F_3\hat{k})$$

$$\boxed{\text{div}.\vec{F} = \frac{\delta F_1}{\delta x} + \frac{\delta F_2}{\delta y} + \frac{\delta F_3}{\delta z}}$$

Q. If $\vec{F} = y\hat{i} + (x^2+y^2)\hat{j} + (yz+zx)\hat{k}$, calculate divergence of F, div $\vec{F}$.

Sol.

$$\text{div}.\vec{F} = \frac{\delta F_1}{\delta x} + \frac{\delta F_2}{\delta y} + \frac{\delta F_3}{\delta z}$$

$$= \frac{\delta}{\delta x}(y) + \frac{\delta}{\delta y}(x^2+y^2) + \frac{\delta}{\delta z}(yz+zx)$$

$$= 0 + (x^2_{.}0 + 2y) + (y+x)$$

$$\text{div}.\vec{F} = x+3y$$

Q. If $\vec{A} = 3x^2\hat{i} + 5xy^2\hat{j} + xyz^2\hat{k}$. Find div. $\vec{A}$ at (1, 2, 3).

Sol. $\text{div}.\vec{A} = \frac{\delta A_1}{\delta x} + \frac{\delta A_2}{\delta y} + \frac{\delta A_3}{\delta z}$

$= \frac{\delta}{\delta x}(3x^2) + \frac{\delta}{\delta y}(5xy^2) + \frac{\delta}{\delta z}(xyz^2)$

$= 6x + 10xy + 2xyz.$

$(\text{div}.\vec{A})_{(1,2,3)} = 6(1) + 10(1)(2) + 2(1)(2)(3)$

$= 6 + 20 + 12$

$\text{div}.\vec{A} = 38$ (unit)

Q. If $\vec{A} = x^2z\,\hat{i} + (-2y^3z^2)\hat{j} + (xy^2z)\hat{k}$. Find div. $\vec{A}$ at (1, -1, 1)

Sol $\text{div}.\vec{A} = \frac{\delta}{\delta x}A_1 + \frac{\delta}{\delta y}A_2 + \frac{\delta}{\delta z}A_3$

$= \frac{\delta}{\delta x}(x^2z) + \frac{\delta}{\delta y}(-2y^3z^2) + \frac{\delta}{\delta z}(xy^2z)$

$= 2xz - 6y^2z^2 + xy^2$

$(\text{div}.\vec{A})_{(1,-1,1)} = 2\times1\times1 - 6\times(1)(1) + (1)(1)$

$= 2 - 6 + 1$

$= -3.$

V. Q. If $\vec{R}$ is the position vector. Proof div $(\vec{r}) = 3$.

$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$.

$\text{div}.\vec{R} = \frac{\delta}{\delta x}r_1 + \frac{\delta}{\delta y}r_2 + \frac{\delta}{\delta z}r_3$

$= \frac{\delta}{\delta x}x + \frac{\delta y}{\delta y} + \frac{\delta z}{\delta z}$

$= 1 + 1 + 1$

$= 3$

Divergence of a position vector is

* zero   * 3   * 1   * 3^2

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2007 Q. R.V. Proof that $\text{div}\, r^n\vec{r} = (3+n)r^n$.

$$\text{div}(r^n\vec{r}) = \vec{\nabla}\cdot(r^n\vec{r})$$

$$= \left(\hat{i}\frac{\delta}{\delta x} + \hat{j}\frac{\delta}{\delta y} + \hat{k}\frac{\delta}{\delta z}\right)\left(\left(\sqrt{x^2+y^2+z^2}\right)^n(x\hat{i}+y\hat{j}+z\hat{k})\right)$$

$$= \frac{\delta}{\delta x}\left[(x^2+y^2+z^2)^{n/2}\cdot x\right] + \frac{\delta}{\delta y}\left[(x^2+y^2+z^2)^{n/2}\cdot y\right] + \frac{\delta}{\delta z}\left[(x^2+y^2+z^2)^{n/2}\cdot z\right]$$

$$= \frac{n}{2}(x^2+y^2+z^2)^{\frac{n}{2}-1}\cdot 2x\cdot x + (x^2+y^2+z^2)^{n/2} + \frac{n}{2}(x^2+y^2+z^2)^{\frac{n}{2}-1}\cdot 2y\cdot y + (x^2+y^2+z^2)^{n/2} + \frac{n}{2}(x^2+y^2+z^2)^{\frac{n}{2}-1}\cdot 2z\cdot z + (x^2+y^2+z^2)^{n/2}$$

$$= n(x^2+y^2+z^2)^{\frac{n}{2}-1}\left[x^2+y^2+z^2\right] + 3(x^2+y^2+z^2)^{n/2}$$

$$= n(x^2+y^2+z^2)^{n/2} + 3(x^2+y^2+z^2)^{n/2}$$

$$= (3+n)\left[r^2\right]^{n/2}$$

$$\text{div}(r^n\vec{r}) = (3+n)r^n$$ Hence proved

K.U. Q. Show that if $\vec{\omega}$ is a constant vector and $\vec{V} = \vec{\omega}\times\vec{r}$, then $\text{div}\,\vec{V} = 0$

Sol. $\vec{\omega} = \omega_1\hat{i} + \omega_2\hat{j} + \omega_3\hat{k}$ [ω_1, ω_2 & ω_3 are constants]

$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

curl grad ϕ = 0

$\because \vec{V} = \vec{\omega} \times \vec{r}$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \omega_1 & \omega_2 & \omega_3 \\ x & y & z \end{vmatrix}$$

$$\vec{V} = \hat{i}(\omega_2 z - \omega_3 y) - \hat{j}(\omega_1 z - \omega_3 x) + \hat{k}(\omega_1 y - \omega_2 x)$$

$$\text{div } \vec{V} = \left(\hat{i}\frac{\delta}{\delta x} + \hat{j}\frac{\delta}{\delta y} + \hat{k}\frac{\delta}{\delta z}\right) \cdot \left[\hat{i}(\omega_2 z - \omega_3 y) - \hat{j}(\omega_1 z - \omega_3 x) + \hat{k}(\omega_1 y - \omega_2 x)\right]$$

$$= \frac{\delta}{\delta x}(\omega_2 z - \omega_3 y) - \frac{\delta}{\delta y}(\omega_1 z - \omega_3 x) + \frac{\delta}{\delta z}(\omega_1 y - \omega_2 x)$$

$$= 0 - 0 + 0$$

$$\text{div}(\vec{V}) = 0.$$

CURL

If the cross product of del operator with a vector function is known as curl of the vector function.

$$\vec{\nabla} \times \vec{F} = \text{Vector}$$

$$\therefore \text{curl } \vec{F} = \text{Vector}$$

Q. If $\vec{F} = y\hat{i} + (x^2+y^2)\hat{j} + (yz+zx)\hat{k}$. Find curl $\vec{F}$.

Sol. →

$$\text{Curl } \vec{F} = \vec{\nabla} \times \vec{F}$$

$$= \left(\hat{i}\frac{\delta}{\delta x} + \hat{j}\frac{\delta}{\delta y} + \hat{k}\frac{\delta}{\delta z}\right) \times \left[y\hat{i} + (x^2+y^2)\hat{j} + (yz+zx)\hat{k}\right]$$

$$\text{curl } \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \delta/\delta x & \delta/\delta y & \delta/\delta z \\ y & x^2+y^2 & yz+zx \end{vmatrix}$$

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$$\text{curl}\,\vec{F} = \hat{i}\left[\frac{\delta}{\delta y}(yz+zx) - \frac{\delta}{\delta z}(x^2+y^2)\right] - \hat{j}\left[\frac{\delta}{\delta x}(yz+zx) - \frac{\delta}{\delta z}(y)\right]$$

$$+\hat{k}\left[\frac{\delta}{\delta x}(x^2+y^2) - \frac{\delta}{\delta y}(y)\right]$$

$$= \hat{i}[z-0] - \hat{j}[0+z-0] + \hat{k}[2x-1]$$

$$\text{curl}\,\vec{F} = z\hat{i} - z\hat{j} + (2x-1)\hat{k}$$

Q. If $\vec{A} = 2xz^2\hat{i} - yz\hat{j} + 3xz^3\hat{k}$, find curl $\vec{A}$ at $(1,1,1)$

Sol. $\text{curl}\,\vec{A} = \vec{\nabla} \times \vec{A}$

$$= \left(\hat{i}\frac{\delta}{\delta x} + \hat{j}\frac{\delta}{\delta y} + \hat{k}\frac{\delta}{\delta z}\right) \times \left(2xz^2\hat{i} - yz\hat{j} + 3xz^3\hat{k}\right)$$

$$= \begin{vmatrix} \overset{+}{\hat{i}} & \overset{-}{\hat{j}} & \overset{+}{\hat{k}} \\ \delta/\delta x & \delta/\delta y & \delta/\delta z \\ 2xz^2 & -yz & 3xz^3 \end{vmatrix}$$

$$= \hat{i}\left(\frac{\delta}{\delta y}(3xz^3) + \frac{\delta}{\delta z}(yz)\right) - \hat{j}\left(\frac{\delta}{\delta x}(3xz^3) - \frac{\delta}{\delta z}(2xz^2)\right)$$

$$+\hat{k}\left(\frac{\delta}{\delta x}(3yz^3) - \frac{\delta}{\delta y}(2xz^2)\right)$$

$$= y\hat{i} - \hat{j}(3z^3 - 4xz) + \hat{k}(z^3 - 4xz)$$

$$= \hat{i} - \hat{j}(3-4) + \hat{k}(0-0)$$

$$= \hat{i} + \hat{j} + 0\hat{k}$$

Q. In the above ques, find curl curl $\vec{A}$ at (1,1,1)

$\because$ curl $\vec{A} = y\hat{i} - \hat{j}(3z^3 - 4xz)$

$$\text{curl}(\text{curl}\vec{A}) = \vec{\nabla} \times \text{curl}\vec{A}$$

$$= \left(\hat{i}\frac{\delta}{\delta x} + \hat{j}\frac{\delta}{\delta y} + \hat{k}\frac{\delta}{\delta z}\right) \times \left[y\hat{i} - \hat{j}(3z^3 - 4xz)\right]$$

$$= \begin{vmatrix} \overset{+}{\hat{i}} & \overset{-}{\hat{j}} & \overset{+}{\hat{k}} \\ \delta/\delta x & \delta/\delta y & \delta/\delta z \\ y & -(3z^3 - 4xz) & 0 \end{vmatrix}$$

$$= \hat{i}\left[\frac{\delta}{\delta y}(0) - \frac{\delta}{\delta z}(-3z^3 + 4xz)\right] - \hat{j}\left[\frac{\delta}{\delta x}(0) + \frac{\delta}{\delta z}y\right] + \hat{k}\left[\frac{\delta}{\delta x}(-3z^3 + 4xz) - \frac{\delta y}{\delta y}\right]$$

$$= \hat{i}(9z^2 - 4z) - \hat{j}(0) + \hat{k}[4z - 1]$$

$$= \hat{i}[9(1)^2 - 4(1)] + \hat{k}[4(1) - 1]$$

$$= 5\hat{i} + 3\hat{k}$$

Note: If $\vec{A}$ is irrotational. Then curl $\vec{A} = 0$.
for ideal liquid, curl $\vec{A} = 0$.

Laplacian Operator

If ϕ is a scalar quantity. Then, grad ϕ is a vector quantity.

$$\text{div grad}\,\phi = \vec{\nabla}\cdot(\vec{\nabla}\phi)$$

$$\text{div}(\text{grad}\,\phi) = \left(\hat{i}\frac{\delta}{\delta x}+\hat{j}\frac{\delta}{\delta y}+\hat{k}\frac{\delta}{\delta z}\right)\cdot\left(\hat{i}\frac{\delta\phi}{\delta x}+\hat{j}\frac{\delta\phi}{\delta y}+\hat{k}\frac{\delta\phi}{\delta z}\right)$$

$$\text{div}.(\text{grad}\,\phi) = \frac{\delta}{\delta x}\left(\frac{\delta\phi}{\delta x}\right)+\frac{\delta}{\delta y}\left(\frac{\delta\phi}{\delta y}\right)+\frac{\delta}{\delta z}\left(\frac{\delta\phi}{\delta z}\right)$$

$$= \frac{\delta^2\phi}{\delta x^2}+\frac{\delta^2\phi}{\delta y^2}+\frac{\delta^2\phi}{\delta z^2}.$$

$$\text{div. grad}\,\phi = \left(\frac{\delta^2}{\delta y^2}+\frac{\delta^2}{\delta y^2}+\frac{\delta^2}{\delta z^2}\right)\phi \quad \text{—— (I)}$$

$$\vec{\nabla}.\vec{\nabla} = \left(i\frac{\delta}{\delta x}+\hat{j}\frac{\delta}{\delta y}+\hat{k}\frac{\delta}{\delta z}\right)\left(\hat{i}\frac{\delta}{\delta x}+\hat{j}\frac{\delta}{\delta y}+\hat{k}\frac{\delta}{\delta z}\right)$$

$$\nabla^2 = \frac{\partial^2}{\delta x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2} \quad \text{—— (II)}$$

Putting eq (II) in eqn (I).

$$\text{div. grad}\,\phi = \nabla^2\phi$$

where

$$\nabla^2 = \left(\frac{\partial^2}{\partial x^2}+\frac{\delta^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)$$ is known as

Laplacian operator.

$$\text{div.grad}\,\phi = \nabla^2\phi.$$ (scalar)

Q. If $\phi = x^2+y^2$; find $\nabla^2\phi$.

$\nabla^2\phi$ = div grad ϕ

Sol. (i) div grad ϕ $= \vec{\nabla} \cdot \text{grad}\,\phi$

$= \vec{\nabla} \cdot (\vec{\nabla}\phi)$

$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\left(\hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}\right)$

$= \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}$

or.

$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\left[\hat{i}\frac{\partial}{\partial x}(x^2+y^2) + \hat{j}\frac{\partial}{\partial y}(x^2+y^2) + \hat{k}\frac{\partial}{\partial z}(x^2+y^2)\right]$

$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\left[\hat{i}(2x) + \hat{j}(2y) + 0\hat{k}\right]$

$= \frac{\partial}{\partial x}(2x) + \frac{\partial}{\partial y}(2y) + 0$

$= 2 + 2 + 0$

div(grad ϕ) $= 4$.

Sol. (ii) $\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}$

$= \frac{\partial^2(x^2+y^2)}{\partial x^2} + \frac{\partial^2(x^2+y^2)}{\partial y^2} + \frac{\partial^2(x^2+y^2)}{\partial z^2}$

$= 2 + 2 + 0$

$\nabla^2\phi = 4$.

R.V.

Q If $\phi = 3x^2y + y^3x^2 + 4xz^3y$. Find $\nabla^2\phi$

Sol. $\nabla^2\phi$ = div grad ϕ.

$$= \frac{\partial^2}{\partial x^2}(3x^2y + y^3x^2 + 4xz^3y) + \frac{\partial^2}{\partial y^2}(3x^2y + y^3x^2 + 4xz^3y)$$

$$+ \frac{\partial^2}{\partial z^2}(3x^2y + y^3x^2 + 4xz^3y).$$

$$= 6y + 2y^3 + 0 + 0 + 6yx^2 + 0 + 0 + 0 + 24xzy.$$

$$= 6y + 2y^3 + 6yx^2 + 24xyz$$

Note: A vector $\vec{F}$ is said to be solenoidal if div $\vec{F} = 0$.

Q. Find the value of 'a' for which the vector $\vec{A} = (x+3y)\hat{i} + (y-2z)\hat{j} + (x+az)\hat{k}$ is solenoidal.

R.V.

Sol. If $\vec{A}$ is solenoidal, then div $\vec{A} = 0$.

div $\vec{A} = 0$.

$$\vec{\nabla} \cdot \vec{A} = 0$$

$$\left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\left[(x+3y)\hat{i} + (y-2z)\hat{j} + (x+az)\hat{k}\right] = 0.$$

$$\frac{\partial}{\partial x}(x+3y) + \frac{\partial}{\partial y}(y-2z) + \frac{\partial}{\partial z}(x+az) = 0.$$

$$1 + 1 + a = 0$$

$$\boxed{a = -2}$$

Applications of Divergence

Flux = Velocity × Area
= Area × Force.

Divergence Of A Vector field..

The divergence of a vector field at any point is defined as the amount of magnetic flux per unit volume diverging from that point.
(Rate of flow of flux or a fluid)

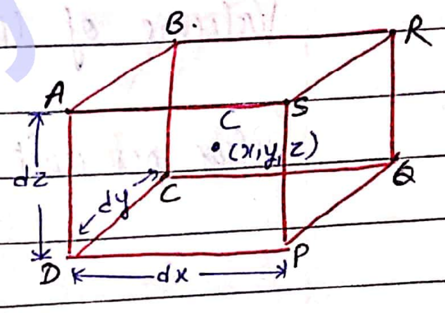
Derivation of Expression

Let us consider a small ||piped whose centre C at point (x, y, z) and whose sides are dx, dy and dz.
Velocity of the fluid at point C is given by

$$\vec{V} = V_x\hat{i} + V_y\hat{j} + V_z\hat{k}$$

x comp. of velocity at ABCD,

$$= V_x - \left(\frac{\partial V_x}{\partial x}\right)\frac{dx}{2}$$



Net Flux passing through ABCD

$$= \left[V_x - \left(\frac{\partial V_x}{\partial x}\right)\frac{dx}{2}\right] dy\,dz$$

x-comp. of Velocity of the fluid at PQRS,

$$= V_x + \left(\frac{\partial V_x}{\partial x}\right)\frac{dx}{2}$$

Net flux leaving through the face PQRS,

$$= \left[V_x + \left(\frac{\partial V_x}{\partial x}\right)\frac{dx}{2}\right] dy\,dz$$

Net flux inside the ||piped due to face ABCD and PQRS.

$$= \left[V_x + \frac{\partial V_x}{\partial x}\cdot\frac{dx}{2}\right] dy\,dz - \left[V_x - \frac{\partial V_x}{\partial x}\cdot\frac{dx}{2}\right] dy\,dz$$

$$= \frac{\partial V_x}{\partial x}\cdot\frac{dx}{2}\,dy\,dz + \frac{\partial V_x}{\partial x}\,\frac{dx}{2}\,dy\,dz$$

$$= \frac{\partial V_x}{\partial x} dx dy dz.$$

Net flux inside the ||piped due to other four faces are:

$$\frac{\partial V_y}{\partial y} dx dy dz \quad \& \quad \frac{\partial V_z}{\partial z} dx dy dz$$

Total flux inside the ||piped $= \frac{\partial V_x}{\partial x} dx dy dz + \frac{\partial V_y}{\partial y} dx dy dz + \frac{\partial V_z}{\partial z} dx dy dz$

$$= \left[\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}\right] dx dy dz.$$

Volume of the parallelopiped $= dx dy dz.$

Flux per unit volume of the ||piped $= \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}$

$$= \vec{\nabla} \cdot (V_x\hat{i} + V_y\hat{j} + V_z\hat{k})$$

$$= \vec{\nabla} \cdot \vec{V}$$

$$= \text{div}\,\vec{V}$$

NOTE: ① If $\text{div}\,\vec{V} > 0$, it means the point is behaving as a source or fluid is expanding and density is decreasing.

② If $\text{div}\,\vec{V} < 0$, it means the fluid is contracting and density is increasing or the point is behaving as a sink (absorber)

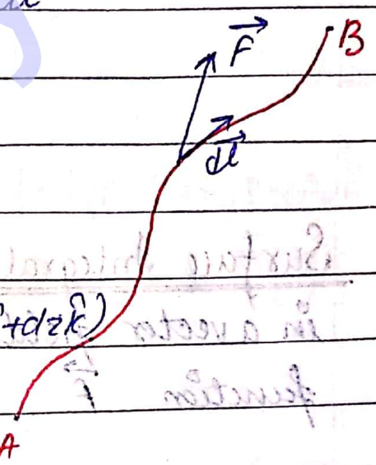
③ If $\text{div}\,\vec{V} = 0$, it means the density is neither increasing nor decreasing or the point is neither behaving as a source nor a sink. [Incompressible]

④ If the divergence of a vector at a point is zero. The vector function is said to be solenoidal.

Line Integral

Let $\vec{dl}$ is the line element of length dl at a point on a smooth curve AB drawn in a vector field. $\vec{F}$ is a continuous vector point function.

Line integral of $\vec{F}$ over AB $= \int_A^B \vec{F}\cdot\vec{dl}$



$\because \vec{F} = F_x\hat{i} + F_y\hat{j} + F_z\hat{k}$

$\vec{dl} = dx\hat{i} + dy\hat{j} + dz\hat{k}$

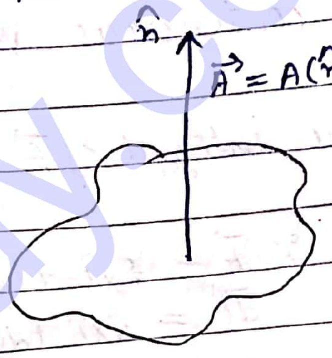
$$= \int_A^B (F_x\hat{i}+F_y\hat{j}+F_z\hat{k})\cdot(dx\hat{i}+dy\hat{j}+dz\hat{k})$$

$$= \int_A^B (F_x dx + F_y dy + F_z dz)$$

Surface Integral (Flux) (vectorfield)

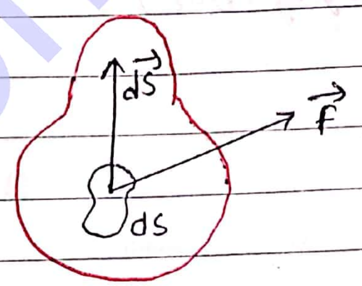
Note: Area is a scalar quantity but in some problems we take area as vector. And in this condition this is known as area vector. The magnitude of area vector is equals to the area of the surface and the direction of area vector is perpendicular to the plane and radially outwards.

$$\vec{A} = A\,(\hat{n})$$



$\hat{n} \perp$ plane (radially outwards)

Surface Integral: Imagine a smooth surface 'S' drawn in a vector field and a continuously varying vector function $\vec{F}$



$\iint_S \vec{F}\cdot\vec{ds}$ = Surface integral of $\vec{F}$ over $\vec{S}$

Let $\vec{F} = f_x\hat{i} + f_y\hat{j} + f_z\hat{k}$

$\vec{ds} = dx\hat{i} + dy\hat{j} + dz\hat{k}$

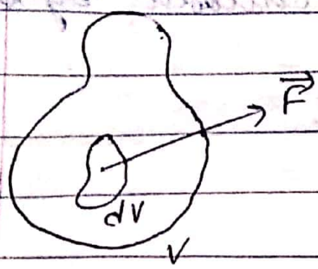
Surface integral of $\vec{F}$ over $\vec{S}$ $= \iint_S \vec{F}\cdot\vec{ds}$

$$= \iint_S (f_x\hat{i} + f_y\hat{j} + f_z\hat{k})(dx\hat{i} + dy\hat{j} + dz\hat{k})$$

$$= \iint_S (f_x dx + f_y dy + f_z dz)$$

Volume Integral

Let we have a surface enclosing volume V and $\vec{F}$ is a vector function at a point in a small element $d\vec{V}$ in the ~~rigon~~ region.



Volume integral of $\vec{F}$ over volume

$$V = \iiint_V \vec{F}\cdot dV$$

If $\vec{F} = F_x\hat{i} + F_y\hat{j} + F_z\hat{k}$

$dV = dx\,dy\,dz.$

Volume integral of $\vec{F}$ over $V = \iiint_V (F_x\hat{i} + F_y\hat{j} + F_z\hat{k})\,dx\,dy\,dz$

$$\iint_S \vec{E}\cdot\vec{ds} = \frac{q}{\varepsilon_0}$$

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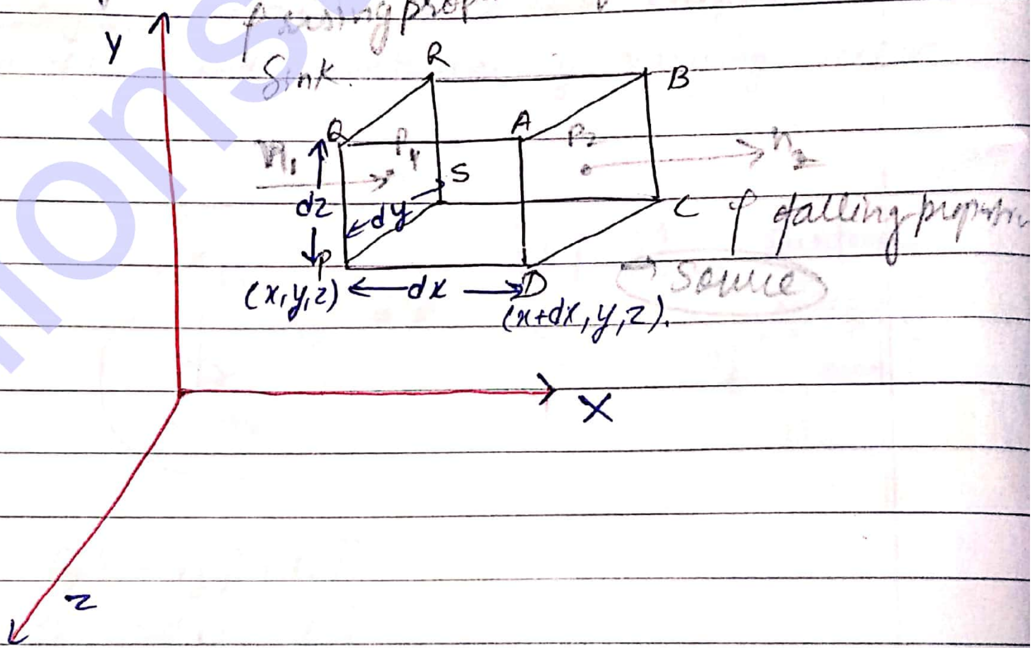
Gauss's Theorem Of Divergence

Q. The normal surface integral of function (vector field) $\vec{F}$ over the boundary of closed surface S (flux across S) is equal to the volume integral of the divergence of the function taken over the volume V enclosed by the surface.

$$\text{Flux} = \iint_S \vec{F}\cdot\vec{ds} = \iiint_V (\text{div}\vec{F})\, dV$$

PROOF: Let the surface S enclosing a volume V is divided up into a very large number of ||$^{\circ}$piped adjoining each other and one of them is shown in fig.



Let the vector point function defined in the region is given by

$$\vec{F} = F_x\hat{i} + F_y\hat{j} + F_z\hat{k}$$

— F_x (P) dy dz.

Flux entering through the face PQRS, $= F_x (dy \times dz)$

The x-component of $\vec{F}$ in face ABCD,

$$= F_x + \left(\frac{\partial F_x}{\partial x}\right) dx.$$

Flux leaving the face ABCD,

$$= \boxed{F_x + \left(\frac{\partial F_x}{\partial x}\right) dx}\, dy\, dz.$$

Net flux through the two faces ABCD and PQRS.

$$= \left[F_x + \left(\frac{\partial F_x}{\partial x}\right) dx\right] dy\,dz - F_x\, dy\, dz$$

$$= F_x dy\,dz + \left(\frac{\partial F_x}{\partial x}\right) dx\,dy\,dz - F_x\,dy\,dz$$

$$= \frac{\partial F_x}{\partial x}\, dx\,dy\,dz.$$

Net flux through the remaining four faces will be

$$\frac{\partial F_y}{\partial y} dx\,dy\,dz \quad \& \quad \frac{\partial F_z}{\partial z} dx\,dy\,dz.$$

Total flux passing through the ||$^{\circ}$piped,

$$\text{Flux} = \iint_{\substack{\text{surface of}\\ ||^{\circ}\text{piped}}} \vec{F}\cdot\vec{ds} = \frac{\partial F_x}{\partial x} dx\,dy\,dz + \frac{\partial F_y}{\partial y} dx\,dy\,dz + \frac{\partial F_z}{\partial z} dx\,dy\,dz$$

$$\iint_S \vec{F}\cdot\vec{ds} = \left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}\right) dx\,dy\,dz$$

$$= \left\{\vec{\nabla}\cdot(F_x\hat{i} + F_y\hat{j} + F_z\hat{k})\right\} dx\,dy\,dz.$$

$$= (\vec{\nabla}\cdot\vec{F})\, dx\,dy\,dz.$$

$$= (\operatorname{div}\vec{F})\, dx\,dy\,dz.$$

$$\boxed{\text{Flux} = \iint_{\substack{\text{surface of}\\ ||^{\circ}\text{piped}}} \vec{F}\cdot\vec{ds} = (\operatorname{div}\vec{F})\cdot dV} \quad \text{—— (1)}$$

Contd.

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We are adding all the surface integrals of ||$^{\circ}$piped to get the total surface integral and total volume integral

After adding all the fluxes we get total flux associated with the surface S having volume V.

$$\text{flux} = \iint_S \vec{F}\cdot\vec{ds} = \iiint_V (\text{div}\vec{F})\cdot dV$$

Hence Proved.

25/Aug/2015

1. The electric field $\vec{E}$ in a certain space is given by $E_x = ax + by + c$, $E_y = 0$, $E_z = 0$. Using Gauss's theorem: Evaluate the charge enclosed in a cube with one of its side of length L along the x-axis, and the two faces of cube ⊥ the x-axis lying in the plane $x = L$ to $x = L + L$.

According to Gauss's theorem of electrostatics

$$\oint \vec{E}\cdot\vec{ds} = \frac{q}{\varepsilon_0} \quad \text{—}(1)$$

Accord. to Gauss's theorem of divergence,

$$\oint \vec{E}\cdot\vec{ds} = \iiint_V (\text{div}\,\vec{E})\,dV$$

$$\oint \vec{E}\cdot\vec{ds} = \iiint_V \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(E_x\hat{i} + E_y\hat{j} + E_z\hat{k})\,dV$$

$$= \iiint_V \left(\frac{\partial}{\partial x}E_x + \frac{\partial}{\partial y}E_y + \frac{\partial}{\partial z}E_z\right)dV$$

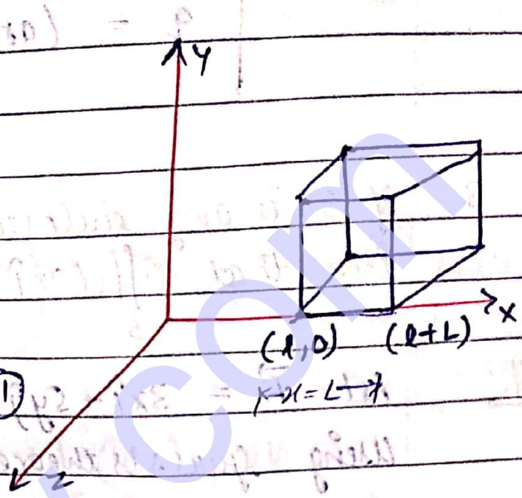
Projection of vector A on B :

$$= \iiint_V \left(\frac{\partial}{\partial x}(ax+by+c) + 0 + 0 \right) dV$$ {Given, $E_y=0$, $E_z=0$, $E_x = ax+by+c$}

$$= \iiint_V a\, dV$$

$$= a \iiint_V dV$$

$$= a \iiint_V L^3$$



$$\oint \vec{E}\cdot\vec{ds} = a \times L^3 \quad \text{--- (11)}$$

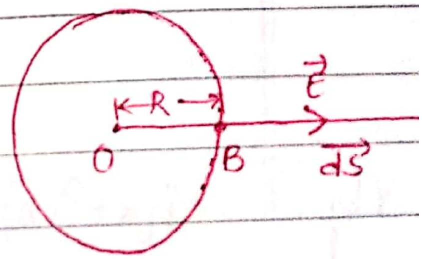
$$\frac{q}{\varepsilon_0} = aL^3$$

$$\boxed{q = aL^3 \varepsilon_0}$$

2. The electric field at any point in a given space is directed radially outwards along the line joining the point to a fixed point, $r=0$ and is given by $ar + br^2$, calculate the charge enclosed in a sphere of radius R with the fixed point $r=0$ at the centre.

Electric field intensity at the surface of sphere, $r = R$

$$E = ar + br^2 \quad \text{--- (1)}$$



Total flux associate with the sphere,

$$\text{Flux} = \iint_S \vec{E}\cdot\vec{ds}$$

$$= \iint_S (ar + br^2)\, ds \qquad \{\theta = 0°\}$$

$$= (ar + br^2) \cdot 4\pi r^2$$

Using Gauss theorem of electrostatics:

$$(ar+br^2)\,4\pi r^2 = \frac{q}{\varepsilon_0}$$

$$\boxed{q = (ar+br^2)\,4\pi r^2 \varepsilon_0}$$

3. If S is any enclosed surface enclosing a volume V. Prove that $\iint_S (3x\hat{i}+5y\hat{j}+7z\hat{k})\,\vec{ds} = 15V$

Sol. Let $\vec{F} = 3x\hat{i}+5y\hat{j}+7z\hat{k}$

Using Gauss's theorem of divergence,

$$\iint_S \vec{F}\cdot\vec{ds} = \iiint_V (\vec{\nabla}\cdot\vec{F})\,dV$$

$$= \iiint_V \left[\vec{\nabla}\cdot(3x\hat{i}+5y\hat{j}+7z\hat{k})\right]dV$$

$$= \iiint_V \left[\left(\frac{\partial}{\partial x}\hat{i}+\frac{\partial}{\partial y}\hat{j}+\frac{\partial}{\partial z}\hat{k}\right)(3x\hat{i}+5y\hat{j}+7z\hat{k})\right]dV$$

$$= \iiint_V \left[\frac{\partial}{\partial x}(3x)+\frac{\partial}{\partial y}(5y)+\frac{\partial}{\partial z}(7z)\right]dV$$

$$= \iiint_V (3+5+7)\,dV$$

$$= 15\iiint_V dV$$

$$= 15V$$

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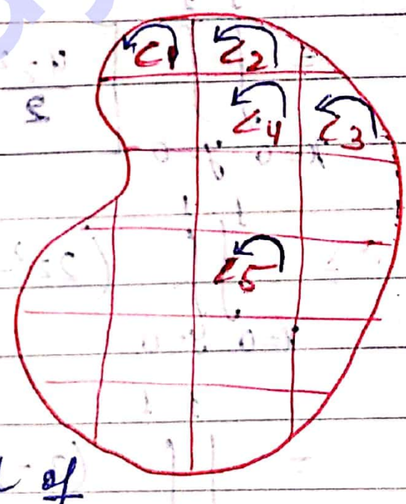
R.U.

Stoke's Theorem..

The line integral of a vector field $\vec{F}$ around any closed curve C is equal to the normal surface integral of the curl of $\vec{F}$ taken over any surface 'S' of which curve C forms the boundary.

$$\int_C \vec{F}.\vec{dl} = \iint_S (Curl \vec{F})\vec{ds}$$

Proof → Let a closed surface forming the boundary of the surface 'S' is divided up into a very large number of small loops $C_1, C_2, ------ C_n$ having area $ds_1, ds_2, ---- ds_n$.



Considering one such loop C_1 enclosing area ds_1

Line integral of $\vec{F}$ over C_1

$$\int_{loop\, C_1} \vec{F}.\vec{dl} = (Curl \vec{F})\vec{ds_1} \quad \text{—} ①$$

After finding the line integral of small loops, we are adding all the line integrals of the small loops. we get line integral of entire loop.

$$\int_C \vec{F}.\vec{dl} = \iint_S (Curl. \vec{F})\vec{ds}$$

Hence Proved

Imp Q. R.V. If $\vec{A} = -y\hat{i} + x\hat{j}$. Then calculate

(I) curl of $\vec{A}$

(II) $\oint_C \vec{A}\cdot\vec{dl}$ for a closed curve $x^2+y^2=r^2,\ z=0$

(3) Hence verify Stoke's theorem.

(I) $\vec{A} = -y\hat{i} - x\hat{j}$

$$\text{curl}\,\vec{A} = \vec{\nabla}\times\vec{A}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -y & x & 0 \end{vmatrix}$$

$$= \hat{i}\left[-\frac{\partial}{\partial z}(x)\right] - \hat{j}\left[\frac{\partial}{\partial x}(0) - \frac{\partial}{\partial z}(-y)\right] + \hat{k}\left[\frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y)\right]$$

$$\text{Curl}\,\vec{A} = 2\hat{k}$$

(II) $\oint_C \vec{A}\cdot\vec{dl} = \int(-y\hat{i}+x\hat{j})(dx\hat{i}+dy\hat{j})$ [$\because z=0$]

$$\int \vec{A}.\vec{dl} = \int(-y\,dx + x\,dy) \quad \text{—— (I)}$$

The polar equation of the circle, $x = r\cos\theta$, $y = r\sin\theta$.

$$x = r\cos\theta$$

$$dx = -(r\sin\theta)\,d\theta \quad \text{—— (II)} \qquad [\text{Diff. w.r.t. } \theta]$$

$$y = r\sin\theta$$

$$dy = r\cos\theta\,d\theta \quad \text{—— (III)}$$

Putting eq^n (II) & (III) in eq^n (I).

$$\int \vec{A}.\vec{dl} = \int\left[+r\sin\theta.(+r\sin\theta)d\theta + r\cos\theta . r\cos\theta\,d\theta\right]$$

$$= \int r^2\sin^2\theta\,d\theta + r^2\cos^2\theta\,d\theta$$

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$= \int r^2 \ (\sin^2\theta + \cos^2\theta)\, d\theta$

$= r^2 \int d\theta$ [$\because \sin^2\theta + \cos^2\theta = 1$]

$= r^2 \cdot \theta$

$= r^2 \ (2\pi)$

$\int_C \vec{A} \cdot \vec{dl} = 2\pi r^2$ → (IV)

(III) According to stoke's theorem,

$$\int_C \vec{A} \cdot \vec{dl} = \iint_S (\text{curl} \vec{A}) \vec{ds}$$

L.H.S. $\int_S \vec{A} \cdot \vec{dl} = 2\pi r^2$ [from eqn (IV)]

R.H.S. $= \iint_S (\text{curl} \vec{A}) \vec{ds}$

$= \iint_S (2\hat{k}) \vec{ds}$

$= \iint_S (2\hat{k}) (dx\,dy\,\hat{k})$ [Direcn of Area is $\perp$ outwards to the plane]

$= \iint_S 2dx\,dy$

$= 2 \iint_S dx\,dy$ $= 2 \iint_S ds$

$= 2S$

R.H.S. $= 2\pi r^2$

The value of left hand side of Stroke's theorem is equal to the value of R.H.S. of Stroke's theorem.

Hence Stroke's theorem verified.

Q① If $\vec{F} = x(x\hat{i} + y\hat{j})$. Then, calculate
(i) curl $\vec{F}$
(ii) Verify the Stoke's Theorem for $\vec{F}$ integrated around the square, in the plane $z=0$ whose sides are along the lines $x=0$, $y=0$, $x=a$, $y=a$.

Most Imp. RV.

Sol. ⇒ (i) $\vec{F} = x^2\hat{i} + xy\hat{j}$

$$\text{curl}\,\vec{F} = \vec{\nabla}\times\vec{F}$$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\times\left(x^2\hat{i} + xy\hat{j}\right)$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2 & xy & 0 \end{vmatrix}$$

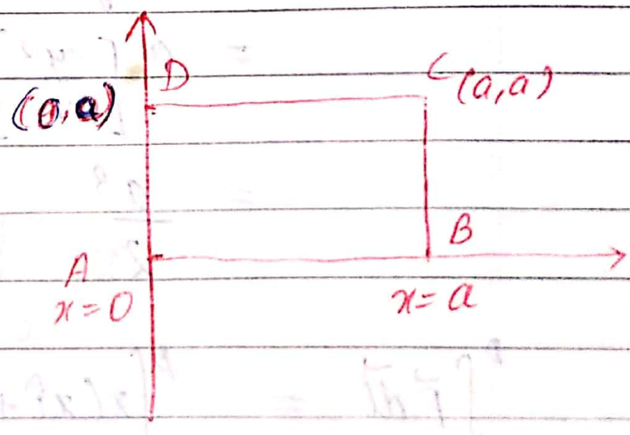
$$= \hat{i}\left[-\frac{\partial}{\partial z}(xy)\right] - \hat{j}\left[-\frac{\partial}{\partial z}(x^2)\right] + \hat{k}\left[\frac{\partial}{\partial x}(xy) - \frac{\partial}{\partial y}(x^2)\right]$$

$$= -(0)\hat{i} - 0\hat{j} + y\hat{k} - 0\hat{k}$$

$$\text{curl}\,\vec{F} = y\hat{k}$$

(ii) Using stoke's theorem,

$$\int \vec{F}\cdot\vec{dl} = \iint_S (\text{curl}\,\vec{F})\,\vec{ds}.$$



L.H.S. $\displaystyle\int_{ABCD} \vec{F}\cdot\vec{dl} = \int_{ABCD} (x^2\hat{i} + xy\hat{j})\,\vec{dl}$

$$= \int_A^B \vec{F}\cdot\vec{dl} + \int_B^C \vec{F}\cdot\vec{dl} + \int_C^D \vec{F}\cdot\vec{dl} + \int_D^A \vec{F}\cdot\vec{dl}$$

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$$\int_A^B \vec{F}\cdot\vec{dl} = \int_A^B x(x\hat{i}+y\hat{j})\cdot dx\hat{i}$$

$$= \int_A^B x^2 dx$$

$$= \int_{x=0}^{x=a} x^2 dx.$$

$$= \left[\frac{x^3}{3}\right]_0^a$$

$$\int_A^B \vec{F}\cdot\vec{dl} = \frac{a^3}{3}$$

$$\int_B^C \vec{F}\cdot\vec{dl} = \int_B^C (x^2\hat{i}+xy\hat{j})\, dy\hat{j}$$

$$= \int_{y=0}^{a} x^2 dy(0) + xydy.$$

$$= \int_{y=0}^{a} x(ydy) \qquad [\because x=a \text{ constant}]$$

$$= a\int_{y=0}^{a} ydy$$

$$= a\left[\frac{y^2}{2}\right]_0^a$$

$$= \frac{a^3}{2}$$

$$\int_C^D \vec{F}\cdot\vec{dl} = \int_C^D x(x\hat{i}+y\hat{j})\, dx\hat{i}$$

$$= \int_a^0 x(xdx)$$

$$= \left[\frac{x^3}{3}\right]_a^0 = \frac{-a^3}{3}$$

$$\int_D^A \vec{F}\cdot\vec{dl} = \int_D^A x(x\hat{i}+y\hat{j})\,dy\,\hat{j}$$

$$= \int_{y=a}^{0} xy\,dy.$$

$$\cancel{= \left[\frac{xy^2}{2}\right]_a^0} \Rightarrow \because x=0$$

$$\cancel{= -\frac{a^2}{2}} \quad \int_D^A \vec{F}\cdot\vec{dl} = 0.$$

$$\int_{ABCD} \vec{F}\cdot\vec{dl} = \int_A^B \vec{F}\cdot\vec{dl} + \int_B^C \vec{F}\cdot\vec{dl} + \int_C^D \vec{F}\cdot\vec{dl} + \int_D^A \vec{F}\cdot\vec{dl}$$

$$= \frac{a^3}{3} + \frac{a^3}{2} - \frac{a^3}{3} + 0.$$

$$= \frac{a^3}{2}$$

R.H.S. $\iint_S (\text{curl}\,\vec{F})\cdot\vec{ds} = \iint_S (y\hat{k})(ds\,\hat{k})$

$$= \iint_S y\,ds.$$

$$= \int_{x=0}^{a}\int_{y=0}^{a} y\,dx\,dy$$

$$= \int_{x=0}^{a} \left[\frac{y^2}{2}\right]_0^a dx.$$

$$= \frac{a^2}{2}\int_{x=0}^{a} dx.$$

$$\iint (\text{curl}\,\vec{F})\cdot\vec{ds} = \frac{a^2}{2}[x]_0^a = \frac{a^3}{2}$$

L.H.S. = R.H.S.

i.e. $\int_{a} \vec{F}.\vec{dl} = \iint_{S} (\text{curl}\,\vec{F})\, \vec{ds}$

Hence, Stoke's verified.

R.U.

GREEN'S THEOREM

First Form of Green's Theorem.

$$\iiint_V (\phi \nabla^2 \psi + \vec{\nabla}\phi . \vec{\nabla}\psi)\, dV = \iint_S (\phi \vec{\nabla}\psi)\, \vec{ds}$$

Second Form of Green's Theorem.

$$\iiint_V (\phi \nabla^2 \psi - \psi \nabla^2 \phi)\, dV = \iint_S (\phi \vec{\nabla}\psi - \psi \vec{\nabla}\phi)\, \vec{ds}$$

proof: We know the Gauss's divergence theorem

$$\iiint_V (\text{div}\,\vec{A})\, dV = \iint_S \vec{A}\,\vec{ds} \quad \text{—— (I)}$$

Let ϕ and ψ are two scalar function. And

Let $\vec{A} = \phi . \vec{\nabla} \psi$ —— (II)

$$\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k} = \phi\left(\hat{i}\frac{\partial \psi}{\partial x} + \hat{j}\frac{\partial \psi}{\partial y} + \hat{k}\frac{\partial \psi}{\partial z}\right)$$

Comparing the RHS and LHS,

$$A_x = \phi \frac{\partial \psi}{\partial x} \quad \text{—— (III)}$$

$$A_y = \phi \frac{\partial \psi}{\partial y} \quad \text{—— (IV)}$$

$$A_z = \phi \frac{\partial \psi}{\partial z} \quad \text{—— (V)}$$

$$\operatorname{div}\vec{A} = \vec{\nabla}\cdot\vec{A}$$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot\left(A_x\hat{i} + A_y\hat{j} + A_z\hat{k}\right)$$

$$\operatorname{div}\vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \quad \text{—— (VI)}$$

Putting eqn (III), (IV), (V) in eqn (VI)

$$\operatorname{div}\vec{A} = \frac{\partial}{\partial x}\left(\phi\frac{\partial\psi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\phi\frac{\partial\psi}{\partial y}\right) + \frac{\partial}{\partial z}\left(\phi\frac{\partial\psi}{\partial z}\right) \quad \text{—— (VII)}$$

$$\operatorname{div}\vec{A} = \left(\phi\frac{\partial^2\psi}{\partial x^2} + \frac{\partial\phi}{\partial x}\frac{\partial\psi}{\partial x}\right) + \left(\phi\frac{\partial^2\psi}{\partial y^2} + \frac{\partial\phi}{\partial y}\frac{\partial\psi}{\partial y}\right) + \left(\phi\frac{\partial^2\psi}{\partial z^2} + \frac{\partial\phi}{\partial z}\frac{\partial\psi}{\partial z}\right)$$

$$\operatorname{div}\vec{A} = \phi\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\psi + \left(\frac{\partial\phi}{\partial x}\frac{\partial\psi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{\partial\psi}{\partial y} + \frac{\partial\phi}{\partial z}\frac{\partial\psi}{\partial z}\right)$$

$$\operatorname{div}\vec{A} = \phi\left(\nabla^2\psi\right) + \vec{\nabla}\phi\cdot\vec{\nabla}\psi \quad \text{—— (VIII)}$$

Putting eqn (VIII) in eqn (I).

$$\iiint_V (\operatorname{div}\vec{A})\, dV = \iint_S \vec{A}\, \vec{ds}$$

$$\boxed{\iiint_V \left(\phi\nabla^2\psi + \vec{\nabla}\phi\cdot\vec{\nabla}\psi\right) dV = \iint_S \left(\phi\vec{\nabla}\psi\right)\vec{ds}} \quad \text{—— (IX)}$$

This is Green's first form

Interchanging ϕ and ψ in eq^n (IX).

$$\iiint_V \left(\psi \nabla^2 \phi + \vec{\nabla}\psi \cdot \vec{\nabla}\phi\right) dV = \iint_S \psi \cdot \vec{\nabla}\phi \, ds \quad \text{—— (X)}$$

Subtracting eq^n (X) from (IX)

$$\boxed{\iiint_V \left(\phi \nabla^2 \psi - \psi \nabla^2 \phi\right) dV = \iint_S \left(\phi \vec{\nabla}\psi - \psi \vec{\nabla}\phi\right) ds}$$

This is II form of green's theorem.