Structure Preserving Discretizations using Retraction Maps

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Abstract

Mechanical systems are most often described by a set of continuous-time, nonlinear, second-order differential equations (SODEs) of a particular structure governed by the covariant derivative. The digital implementation of controllers for such systems requires a discrete model of the system and hence requires numerical discretization schemes. Feedback linearizability of such sampled systems, however, depends on the discretization scheme employed.

In this thesis, we utilize retraction maps and their lifts to construct feedback linearizable discretizations for SODEs which can be applied to many mechanical systems.

Keywords: geometric integrators, retraction maps, discrete systems, feedback linearization, mechanical systems

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$\mathcal{R}$	retraction map
$\mathcal{D}$	discretization map

Chapter 1

Retraction Maps

1.1 Introduction

The notion of a retraction map is fundamental in research areas like optimization theory, machine learning, numerical analysis, and in this context, geometric integrators.
Many mechanical systems usually evolve on manifolds, which naturally requires some method of discretely approximating the dynamics on the manifold (i.e., the geodesic).

In Riemannian geometry, this idea is given by the exponential map. On a Riemannian manifold (M, g), we define $\exp : T_xM \longrightarrow M$ as the exponential map at the point x. For instance, if $\gamma : [0, 1] \longrightarrow M$ is a unique geodesic on M, and $\gamma(0) = x$, then $\exp_x(v) = \gamma(1)$, where $v \in T_xM$ is the initial velocity of the geodesic at x such that $\dot{\gamma}(0) = v$.

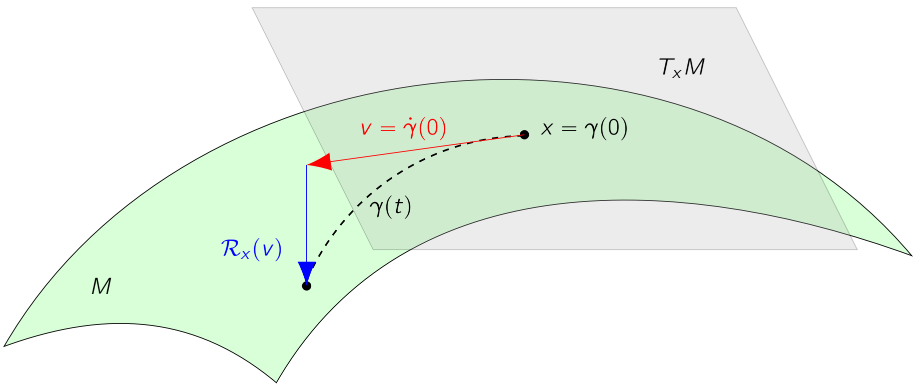

Figure 1.1: Retraction maps: A visualization

Let M be an n dimensional manifold, and TM be its tangent bundle.

Definition 1.1.1. *We define a* **retraction map** *on a manifold M as a smooth map $\mathcal{R} : TM \to M$, such that if $\mathcal{R}_x$ be the restriction of $\mathcal{R}$ to T_xM, then the following properties are satisfied:*

1. *$\mathcal{R}_x(0_x) = x$ where 0_x is the zero element of T_xM.*
2. *$D\mathcal{R}_x(0_x) = T_{0_x}\mathcal{R}_x = Id_{T_xM}$, where Id_{T_xM} is the identity mapping on T_xM.*

Here, the first property is trivial, whereas the second property is known as the **local rigidity condition** since, given $v \in T_xM$, the curve $\gamma_v(t) = \mathcal{R}_x(tv)$ has initial velocity v at x. Hence,

$$\dot{\gamma}_v(t) = \langle \mathrm{D}\mathcal{R}_x(tv), v\rangle \implies \dot{\gamma}_v(0) = \mathrm{Id}_{T_xM}(v) = v$$

1.2 Discretization maps

Definition 1.2.1. *A map $\mathcal{D} : U \subset TM \longrightarrow M \times M$ given by*

$$\mathcal{D}(x, v) = \left(\mathcal{R}_x^1(v), \mathcal{R}_x^2(v)\right)$$

where U is the open neighborhood of the zero section $0_x \in TM$, is called a **discretization map** *on M, if the following properties are satisfied:*

1. $\mathcal{D}(x, 0_x) = (x, x)$
2. $T_{0_x}\mathcal{R}_x^2 - T_{0_x}\mathcal{R}_x^1 = Id_{T_xM}$*, which is the identity map on T_xM for any $x \in M$.*

Using this definition, one can prove (not included here) that the discretization map $\mathcal{D}$ is a local diffeomorphism around the zero section $0_x \in TM$. This is a crucial property for the construction of geometric integrators, since we need to be able to define $\mathcal{D}^{-1}(x_k, x_{k+1})$.

Thus, given a vector field $X \in \mathfrak{X}(M)$ on M, i.e., $X : M \longrightarrow TM$ such that $\tau_M \circ X = \mathrm{Id}_M$, where $\tau_M : TM \longrightarrow M$ is the canonical projection on the tangent bundle, we can approximate the integral curve by the following first-order discrete equation:

$$hX(\tau_M(\mathcal{D}^{-1}(x_k, x_{k+1}))) = \mathcal{D}^{-1}(x_k, x_{k+1})$$

Hence, given an initial condition x_0, we may be able to solve the discrete equation iteratively to obtain the sequence $\{x_k\}$ which is indeed an approximation of $\{x(kh)\}$, where $x(t)$ is the integral curve of X with initial condition x_0 and time-step h.

1.2.1 Examples

We consider a few examples of discretization maps on $\mathbb{R}^n$:

Discretization map $\mathcal{D}$	Scheme	Order
$\mathcal{D}(x, v) = (x, x + v)$	Forward Euler $x_{k+1} = x_k + hX(x_k)$	$\mathcal{O}(h)$
$\mathcal{D}(x, v) = (x - v, x)$	Backward Euler $x_k = x_{k+1} - hX(x_{k+1})$	$\mathcal{O}(h)$
$\mathcal{D}(x, v) = \left(x - \frac{v}{2}, x + \frac{v}{2}\right)$	Symmetric Euler $x_{k+1} = x_k + hX\left(\frac{x_k + x_{k+1}}{2}\right)$	$\mathcal{O}(h^2)$

1.3 Lifts of discretization maps

As mentioned before, discretization maps are diffeomorphisms around the zero section $0_x \in TM$. This is useful because typically when studying mechanical systems, we would like to define the discretization map on the tangent bundle TM (for Lagrangian frameworks) or the cotangent bundle T^*M (for Hamiltonian frameworks), in order to generate geometric integrators on the manifold.

Thus, since discretization maps can be defined on different manifolds, we denote $\mathcal{D}^{TM} : TM \longrightarrow M \times M$ as a discretization map on M.

1.3.1 Tangent Lifts of Discretization Maps

Given a smooth map $\varphi : M \longrightarrow N$ between two n-dimenstional manifolds M and N, we can define the **tangent lift** of φ as the map $T\varphi : TM \longrightarrow TN$ such that

$$T\varphi(v_x) = T_x\varphi(v_x) \in T_{\varphi(x)}N$$

where $v_x \in T_xM$ and $T_x\varphi$ is the tangent map of φ, whose matrix is the Jacobian at $x \in M$, in a local chart.

Proposition 1.3.1. *Let M and N be two n-dimensional manifolds, and $\varphi : M \longrightarrow N$ be a smooth map (diffeomorphism). For a given discretization map $\mathcal{D}^{TM}$ on M, the map $\mathcal{D}_\varphi := (\varphi \times \varphi) \circ \mathcal{D}^{TM} \circ T\varphi^{-1}$ is a discretization map on N i.e., $\mathcal{D}_\varphi \equiv \mathcal{D}^{TN} : TN \longrightarrow N \times N$.*

Proof. For any given $y \in N$, we have that

$$\begin{aligned}
\mathcal{D}_\varphi(y, 0_y) &= ((\varphi \times \varphi) \circ \mathcal{D}^{TM} \circ T\varphi^{-1})(y, 0_y) \\
&= ((\varphi \times \varphi) \circ \mathcal{D}^{TM} \circ T\varphi^{-1})(\varphi(x), 0_{\varphi(x)}) \\
&= (\varphi \times \varphi) \circ \mathcal{D}^{TM}(x, 0_x) \\
&= (\varphi \times \varphi)(x, x) = (y, y)
\end{aligned}$$

which proves the first condition. For the second condition, let $v_y \in T_yN$, be a given vector.

$$\begin{aligned}
(T_{0_x}\mathcal{R}^2_{x,\varphi} - T_{0_x}\mathcal{R}^1_{x,\varphi})(y, u_y) &= \left.\frac{d}{ds}\right|_{s=0} \big(\mathcal{R}^2_{x,\varphi}(y, su_y) - \mathcal{R}^1_{x,\varphi}(y, su_y)\big) \\
&= \left.\frac{d}{ds}\right|_{s=0} \big(\varphi \circ \mathcal{R}^1_x \circ T\varphi^{-1}(y, su_y)\big) - \big(\varphi \circ \mathcal{R}^2_x \circ T\varphi^{-1}(y, su_y)\big) \\
&= T_y\varphi\left(\left.\frac{d}{ds}\right|_{s=0} \big[\mathcal{R}^1_x(t(T\varphi^{-1}(y, u_y)))\big] - \big[\mathcal{R}^2_x(t(T\varphi^{-1}(y, u_y)))\big]\right) \\
&= T_y\varphi(T_y\varphi^{-1}(y, u_y)) = (y, u_y)
\end{aligned}$$

This satisfies the local rigidity condition. $\square$

The above proposition can be visualized as shown below in Figure 3.1.

$$\begin{tikzcd}
TM \arrow[r, "T\varphi"] \arrow[d, "\mathcal{D}^{TM}"'] & TN \arrow[d, "\mathcal{D}^{TN}"] \\
M \times M \arrow[r, "\varphi \times \varphi"'] & N \times N
\end{tikzcd}$$

Figure 3.1: $\mathcal{D}^{TM}$ and $\mathcal{D}^{TN}$ commute as shown

Appendix A

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