Structure Preserving Discretizations using Retraction Maps

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Abstract

Mechanical systems are most often described by a set of continuous-time, nonlinear, second-order differential equations (SODEs) of a particular structure governed by the covariant derivative. The digital implementation of controllers for such systems requires a discrete model of the system and hence requires numerical discretization schemes. Feedback linearizability of such sampled systems, however, depends on the discretization scheme employed.

In this thesis, we utilize retraction maps and their lifts to construct feedback linearizable discretizations for SODEs which can be applied to many mechanical systems.

Keywords: geometric integrators, retraction maps, discrete systems, feedback linearization, mechanical systems

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M	manifold
γ	curve/geodesic on M
g	metric tensor on M
$\exp$	exponential map on M
TM	tangent bundle of M
$\mathfrak{X}(M)$	space of vector fields on M
X	vector field on M
TTM	double tangent bundle of M
φ	diffeomorphism
$T\varphi$	tangent lift of φ
$\mathcal{R}$	retraction map
$\mathcal{D}$	discretization map
τ_M	canonical projection map from TM
κ_M	canonical involution map on TTM
$f \circ g$	composition of f and g
$\mathbb{I}$	identity map
n	default dimension of M
$\langle \cdot, \cdot \rangle$	inner product
∂_x	partial derivative with respect to x
h	step-size of discretization

Chapter 1

Introduction

1.1 Retraction Maps

The notion of a retraction map is fundamental in research areas like optimization theory, machine learning, numerical analysis, and in this context, geometric integrators.
Many mechanical systems usually evolve on manifolds, which naturally requires some method of discretely approximating the dynamics on the manifold (i.e., the geodesic).

In Riemannian geometry, this idea is given by the exponential map. On a Riemannian manifold (M, g), we define $\exp_x : T_xM \longrightarrow M$ as the exponential map at the point x. For instance, if $\gamma : [0,1] \longrightarrow M$ is a unique geodesic on M, and $\gamma(0) = x$, then $\exp_x(v) = \gamma(1)$, where $v \in T_xM$ is the initial velocity of the geodesic at x such that $\dot{\gamma}(0) = v$.

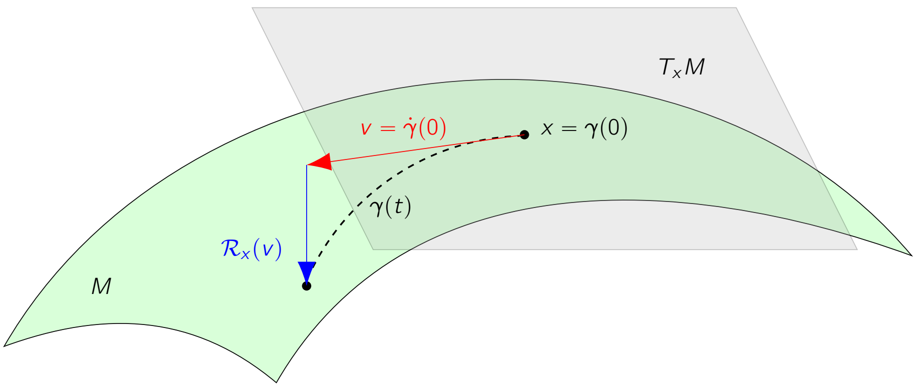

Figure 1.1: Retraction maps: A visualization

Let M be an n dimensional manifold, and TM be its tangent bundle.

Definition 1.1.1. *We define a* **retraction map** *on a manifold M as a smooth map $\mathcal{R} : TM \to M$, such that if $\mathcal{R}_x$ be the restriction of $\mathcal{R}$ to T_xM, then the following properties are satisfied:*

1. *$\mathcal{R}_x(0_x) = x$ where 0_x is the zero element of T_xM.*
2. *$D\mathcal{R}_x(0_x) = T_{0_x}\mathcal{R}_x = \mathbb{I}_{T_xM}$, where $\mathbb{I}_{T_xM}$ is the identity mapping on T_xM.*

Here, the first property is trivial, whereas the second property is known as the **local rigidity condition** since, given $v \in T_xM$, the curve $\gamma_v(t) = \mathcal{R}_x(tv)$ has initial velocity v at x. Hence,

$$\dot{\gamma}_v(t) = \langle \mathrm{D}\mathcal{R}_x(tv), v\rangle \implies \dot{\gamma}_v(0) = \mathbb{I}_{T_xM}(v) = v$$

1.2 Discretization maps

Definition 1.2.1. *A map $\mathcal{D} : U \subset TM \longrightarrow M \times M$ given by*

$$\mathcal{D}(x, v) = \left(\mathcal{R}^1_x(v), \mathcal{R}^2_x(v)\right)$$

where U is the open neighborhood of the zero section $0_x \in TM$, is called a **discretization map** *on M, if the following properties are satisfied:*

1. *$\mathcal{D}(x, 0_x) = (x, x)$*
2. *$T_{0_x}\mathcal{R}^2_x - T_{0_x}\mathcal{R}^1_x = \mathbb{I}_{T_xM}$, which is the identity map on T_xM for any $x \in M$.*

Using this definition, one can prove (not included here) that the discretization map $\mathcal{D}$ is a local diffeomorphism around the zero section $0_x \in TM$. This is a crucial property for the construction of geometric integrators, since we need to be able to define $\mathcal{D}^{-1}(x_k, x_{k+1})$.

Thus, given a vector field $X \in \mathfrak{X}(M)$ on M, i.e., $X : M \longrightarrow TM$ such that $\tau_M \circ X = \mathbb{I}_M$, where $\tau_M : TM \longrightarrow M$ is the canonical projection on the tangent bundle, we can approximate the integral curve by the following first-order discrete equation:

$$hX(\tau_M(\mathcal{D}^{-1}(x_k, x_{k+1}))) = \mathcal{D}^{-1}(x_k, x_{k+1})$$

Hence, given an initial condition x_0, we may be able to solve the discrete equation iteratively to obtain the sequence $\{x_k\}$ which is indeed an approximation of $\{x(kh)\}$, where $x(t)$ is the integral curve of X with initial condition x_0 and time-step h.

1.2.1 Examples

We consider a few examples of discretization maps on $\mathbb{R}^n$:

Discretization map $\mathcal{D}$	Scheme	Order
$\mathcal{D}(x, v) = (x, x + v)$	Forward Euler $x_{k+1} = x_k + hX(x_k)$	$\mathcal{O}(h)$
$\mathcal{D}(x, v) = (x - v, x)$	Backward Euler $x_k = x_{k+1} - hX(x_{k+1})$	$\mathcal{O}(h)$
$\mathcal{D}(x, v) = \left(x - \frac{v}{2}, x + \frac{v}{2}\right)$	Symmetric Euler $x_{k+1} = x_k + hX\left(\frac{x_k + x_{k+1}}{2}\right)$	$\mathcal{O}(h^2)$

Table 2.1: Examples of discretization maps

1.3 Lifts of discretization maps

As mentioned before, discretization maps are diffeomorphisms around the zero section $0_x \in TM$. This is useful because typically when studying mechanical systems, we would like to define the discretization map on the tangent bundle TM (for Lagrangian frameworks) or the cotangent bundle T^*M (for Hamiltonian frameworks), in order to generate geometric integrators on the manifold.

Thus, since discretization maps can be defined on different manifolds, we denote $\mathcal{D}^{TM} : TM \longrightarrow M \times M$ as a discretization map on M.

1.3.1 Tangent Lifts

Given a smooth map $\varphi : M \longrightarrow N$ between two n-dimenstional manifolds M and N, we can define the **tangent lift** of φ as the map $T\varphi : TM \longrightarrow TN$ such that

$$T\varphi(v_x) = T_x\varphi(v_x) \in T_{\varphi(x)}N$$

where $v_x \in T_xM$ and $T_x\varphi$ is the tangent map of φ, whose matrix is the Jacobian at $x \in M$, in a local chart.

Proposition 1.3.1. *Let M and N be two n-dimensional manifolds, and $\varphi : M \longrightarrow N$ be a smooth map (diffeomorphism). For a given discretization map $\mathcal{D}^{TM}$ on M, the map $\mathcal{D}_\varphi := (\varphi \times \varphi) \circ \mathcal{D}^{TM} \circ T\varphi^{-1}$ is a discretization map on N i.e., $\mathcal{D}_\varphi \equiv \mathcal{D}^{TN} : TN \longrightarrow N \times N$.*

Proof. For any given $y \in N$, we have that

$$\begin{aligned}
\mathcal{D}_\varphi(y, 0_y) &= ((\varphi \times \varphi) \circ \mathcal{D}^{TM} \circ T\varphi^{-1})(y, 0_y) \\
&= ((\varphi \times \varphi) \circ \mathcal{D}^{TM} \circ T\varphi^{-1})(\varphi(x), 0_{\varphi(x)}) \\
&= (\varphi \times \varphi) \circ \mathcal{D}^{TM}(x, 0_x) \\
&= (\varphi \times \varphi)(x, x) = (y, y)
\end{aligned}$$

which proves the first condition. For the second condition, let $v_y \in T_yN$, be a given vector.

$$\begin{aligned}
(T_{0_x}\mathcal{R}^2_{x,\varphi} - T_{0_x}\mathcal{R}^1_{x,\varphi})(y, u_y) &= \left.\frac{d}{ds}\right|_{s=0} \left(\mathcal{R}^2_{x,\varphi}(y, su_y) - \mathcal{R}^1_{x,\varphi}(y, su_y)\right) \\
&= \left.\frac{d}{ds}\right|_{s=0} \left(\varphi \circ \mathcal{R}^1_x \circ T\varphi^{-1}(y, su_y)\right) - \left(\varphi \circ \mathcal{R}^2_x \circ T\varphi^{-1}(y, su_y)\right) \\
&= T_y\varphi\left(\left.\frac{d}{ds}\right|_{s=0} \left[\mathcal{R}^1_x(t(T\varphi^{-1}(y, u_y)))\right] - \left[\mathcal{R}^2_x(t(T\varphi^{-1}(y, u_y)))\right]\right) \\
&= T_y\varphi(T_y\varphi^{-1}(y, u_y)) = (y, u_y)
\end{aligned}$$

Thus, both the conditions from Definition 1.1.1 are satisfied. ☐

The above proposition can be visualized as shown below in Figure 3.1.

$$\begin{tikzcd}
TM \arrow[r, "T\varphi"] \arrow[d, "\mathcal{D}^{TM}"'] & TN \arrow[d, "\mathcal{D}^{TN}"] \\
M \times M \arrow[r, "\varphi \times \varphi"'] & N \times N
\end{tikzcd}$$

Figure 3.1: $\mathcal{D}^{TM}$ and $\mathcal{D}^{TN}$ commute as shown

Now, if we suitably lift the discretization map $\mathcal{D} : TM \longrightarrow M \times M$, we can get a discretization map on TM, i.e., we can define $\mathcal{D}^{TTM} : TTM \longrightarrow TM \times TM$ as a discretization map on TM. This construction will provide the geometric framework for integrators for second-order differential equations (SODEs) on manifolds, and consequently, for mechanical systems.

Let M be an n-dimensional manifold, and $\tau_M : TM \longrightarrow M$ be the canonical projection on the tangent bundle. We denote TTM as the **double tangent bundle** of M.

We note that the manifold TTM naturally accepts two different vector bundle structures:

1. The canonical vector bundle with projection $\tau_{TM} : TTM \longrightarrow TM$.
2. The vector bundle given by the projection of the tangent map $T\tau_M : TTM \longrightarrow TM$.

Thus, we denote the canonical involution map $\kappa_M : TTM \longrightarrow TTM$ which is a vector bundle isomorphism, over the identity of TM between the above two vector bundle structures.

This can be seen here: Let (x, v) be the canonical coordinates on TM, and $(x, v, \dot{x}, \dot{v})$ are the corresponding canonical fibered coordinates on TTM. Then,

$$\kappa_M(x, v, \dot{x}, \dot{v}) = (x, \dot{x}, v, \dot{v})$$

Remark 1.3.1. *Why do we need this? Remember that the tangent lift of a vector field X on M does not define a vector field on TM. It is necessary to consider the composition $\kappa_M \circ TX$ to obtain a vector field on TM, and this is called the* **complete lift** *X^c of the vector field X. Hence, a similar technique must be used to lift a discretization map from TM to TTM.*

$$\begin{tikzcd}
TTM \arrow[r, "\mathcal{D}^{TTM}"] \arrow[d, leftrightarrow, "\kappa_M"', "\kappa_M"] & TM \times TM \arrow[d, equal] \\
TTM \arrow[r, "T\mathcal{D}^{TM}"] \arrow[d, "\tau_{TM}"'] & T(M \times M) \arrow[d, "\tau_{M\times M}"] \\
TM \arrow[r, "\mathcal{D}^{TM}"] & M \times M
\end{tikzcd}$$

Figure 3.2: Tangent lift structure of discretization maps

Using the above construction, we can now define the tangent lift of a discretization map.

Proposition 1.3.2. *If $\mathcal{D}^{TM} : TM \longrightarrow M \times M$ is a discretization map on M, then the map defined by $\mathcal{D}^{TTM} = T\mathcal{D}^{TM} \circ \kappa_M$ is a discretization map on TM.*

Proof. For $(x, v, \dot{x}, \dot{v}) \in TTM$, we have that

$$T\mathcal{D}^{TM}(x, v, \dot{x}, \dot{v}) = \left(\mathcal{D}^{TM}(x, v), D_{(x,v)}\mathcal{D}^{TM}(x, v)(\dot{x}, \dot{v})^T\right)$$

and

$$\mathcal{D}^{TTM}(x, \dot{x}, v, \dot{v}) = (\mathcal{D}^{TM}(x, v), D_{(x,v)}\mathcal{D}^{TM}(\dot{x}, \dot{v})^T)$$

Using the properties defined in Definition (1.1.1),

1. We know that $\mathcal{D}^{TM}(x,0) = (x,x)$ for all $x \in M$. Thus,

$$\begin{aligned}\mathcal{D}^{TTM}(x,\dot{x},0,0) &= \left(\mathcal{D}^{TM}(x,0), D_{(x,0)}\mathcal{D}^{TM}(\dot{x},0)\right)\\ &= (x,x,\dot{x},\dot{x}) \equiv (x,\dot{x},x,\dot{x})\end{aligned}$$

   where we trivially identify $T(M \times M) \equiv TM \times TM$.

2. For the rigidity property, we know that

$$\mathcal{D}^{TTM}(x,\dot{x},v,\dot{v}) = \Big((T\mathcal{R}^1)_{(x,\dot{x})}(v,\dot{v}), (T\mathcal{R}^2)_{(x,\dot{x})}(v,\dot{v})\Big)$$

   So, we need to compute

$$T_{(0,0)_{(x,\dot{x})}}(T\mathcal{R}^a)_{(x,\dot{x})}(x,\dot{x}) : T_{(x,\dot{x})}TM \longrightarrow T_{(x,\dot{x})}TM$$

   for $a = 1,2$, to prove that the map $T(T\mathcal{R}^2)_{(x,\dot{x})} - T(T\mathcal{R}^1)_{(x,\dot{x})}$ is the identity map at the zero section $(0,0)_{(x,\dot{x})}$, from $T_{(x,\dot{x})}TM$ to itself.

   We can calculate

$$\left.\frac{d}{ds}\right|_{s=0} (\mathcal{R}^a_x(sv), \partial_x\mathcal{R}^a_x(sv)\dot{x} + \partial_v\mathcal{R}^a_x(sv)s\dot{v})$$

   At $(x,\dot{x},0,0)$, the map $T_{(0,0)_{(x,\dot{x})}}(T\mathcal{R}^a)_{(x,\dot{x})}$ is thus given by:

$$\begin{pmatrix} \partial_{v^j}(\mathcal{R}^a)^i(x,0) & 0 \\ \partial_{x^k}\partial_{v^j}(\mathcal{R}^a)^i(x,0)\dot{x}^k & \partial_{v^j}(\mathcal{R}^a)^i(x,0)\end{pmatrix}$$

   Thus, using the properties of the discretization map $\mathcal{D}$, we have the Jacobian matrix of $(T\mathcal{R}^2)_{(x,\dot{x})} - (T\mathcal{R}^1)_{(x,\dot{x})}$ at $(0,0)_{(x,\dot{x})}$ as:

$$\begin{pmatrix} \partial_v(\mathcal{R}^2 - \mathcal{R}^1)(x,0) & 0 \\ \partial_x(\partial_v(\mathcal{R}^2 - \mathcal{R}^1)(x,0))\dot{x} & \partial_v(\mathcal{R}^2 - \mathcal{R}^1)(x,0)\end{pmatrix} = \mathbb{I}_{2n\times 2n}$$

   since $\partial_v(\mathcal{R}^2 - \mathcal{R}^1)(x,0) = \mathbb{I}_{n\times n}$ which also implies $\partial_x(\partial_v(\mathcal{R}^2 - \mathcal{R}^1))(x,0) = 0$

□

1.3.2 Example

Let us consider the midpoint rule as an example. Thus, if M is a vector space, $\mathcal{D} : TM \longrightarrow M \times M$ is the discretization map given by $\mathcal{D}(x,v) = \left(x - \frac{1}{2}v, x + \frac{1}{2}v\right)$. We can also compute the inverse map as $\mathcal{D}^{-1}(x_k, x_{k+1}) = \left(\dfrac{x_k + x_{k+1}}{2}, x_{k+1} - x_k\right)$.

To define the tangent lift of $\mathcal{D}$, denoted by $\mathcal{D}^{TTM} : TTM \longrightarrow TM \times TM$, we need to compute the Jacobian of $\mathcal{D}$,

$$D_{(x,v)}\mathcal{D} = \begin{pmatrix} \mathbb{I} & -\frac{1}{2}\mathbb{I} \\ \mathbb{I} & \frac{1}{2}\mathbb{I} \end{pmatrix}$$

which yields the tangent lift of $\mathcal{D}$ as:

$$\begin{aligned} \mathcal{D}^{TTM}(x, \dot{x}, v, \dot{v}) &= (T\mathcal{D} \circ \kappa_M)\,(x, \dot{x}, v, \dot{v}) = T\mathcal{D}(x, v; \dot{x}, \dot{v}) \\ &= \left(x - \frac{1}{2}v, x + \frac{1}{2}v; \dot{x} - \frac{1}{2}\dot{v}, \dot{x} + \frac{1}{2}\dot{v}\right) \\ &\equiv \left(x - \frac{1}{2}v, \dot{x} - \frac{1}{2}\dot{v}; x + \frac{1}{2}v, \dot{x} + \frac{1}{2}\dot{v}\right) \end{aligned}$$

We can also obtain the inverse map of $\mathcal{D}^{TTM}$ as

$$\left(\mathcal{D}^{TTM}\right)^{-1}(x_k, v_k; x_{k+1}, v_{k+1}) = \left(\frac{x_k + x_{k+1}}{2}, \frac{v_k + v_{k+1}}{2}; x_{k+1} - x_k, v_{k+1} - v_k\right)$$

1.4 Generalizing construction of discretization maps

In Proposition 1.3.1, we have seen how a discretization map $\mathcal{D}_\varphi$ can be constructed on a manifold N given a diffeomorphism $\varphi : M \longrightarrow N$ and a discretization map $\mathcal{D}^{TM}$ on M. This construction can be extended to TTN as well.

This is useful because, often we deal with *change of coordinates* in mechanical systems, which may simplify, or attribute more meaning to the system. Hence, if we choose to define a discretization scheme on the new coordinates, we must be able to lift it back to the original manifold.

$$\begin{tikzcd}
TM \arrow[r, equal] \arrow[dddr, bend right, "\mathcal{D}^{TM}"'] & TM \arrow[r, "T\varphi"] & TN \arrow[r, equal] & TN \arrow[dddl, bend left, "\mathcal{D}^{TN}"] \\
TTM \arrow[u, "T\tau_M"] \arrow[r, leftrightarrow, "\kappa_M"] \arrow[d, "T\mathcal{D}^{TM}"'] & TTM \arrow[u, "\tau_{TM}"] \arrow[r, "TT\varphi"] \arrow[d, "\mathcal{D}^{TTM}"'] & TTN \arrow[u, "\tau_{TN}"'] \arrow[r, leftrightarrow, "\kappa_N"] \arrow[d, "\mathcal{D}^{TTN}"] & TTN \arrow[u, "T\tau_N"'] \arrow[d, "T\mathcal{D}^{TN}"] \\
T(M \times M) \arrow[r, equal] & TM \times TM \arrow[r, "T\varphi \times T\varphi"] \arrow[d, "\tau_{M\times M}"'] & TN \times TN \arrow[r, equal] \arrow[d, "\tau_{N\times N}"] & T(N \times N) \\
 & M \times M \arrow[r, "\varphi \times \varphi"] & N \times N &
\end{tikzcd}$$

Figure 4.1: $\mathcal{D}^{TTM}$ and $\mathcal{D}^{TTN}$ commute as shown.

Let M and N be two n-dimensional manifolds, and $\varphi : M \longrightarrow N$ be a diffeomorphism, which denotes some *change of coordinates*. The questions of importance is the following:

If we wish to have a discretization map $\mathcal{D}^{TN}$ on N, how do we obtain the discretization on the original tangent space TM, i.e., $\mathcal{D}^{TTM}$?

The double commutator in Figure 4.1 explains the procedure as follows:

1. Start with a required discretization map $\mathcal{D}^{TN}$ on N, and a given $\varphi : M \longrightarrow N$.
2. Lift it back (refer Fig. 3.1) to TM to obtain $\mathcal{D}^{TM}$ using:

$$\mathcal{D}^{TM} = (\varphi \times \varphi)^{-1} \circ \mathcal{D}^{TN} \circ T\varphi$$

3. Obtain D^{TTM} by the tangent lift (refer Fig. 3.2) of $\mathcal{D}^{TM}$, i.e.,

$$\mathcal{D}^{TTM} = T\mathcal{D}^{TM} \circ \kappa_M$$

Can we construct $\mathcal{D}^{TTM}$ from $\mathcal{D}^{TTN}$?

Proposition 1.4.1. *Let M and N be n dimensional manifolds and $\varphi(x) = \tilde{x}$, where φ is a diffeomorphism and $x \in M, \tilde{x} \in N$. Let TM and TN be the tangent bundles of M and N, respectively. By definition, if $(x, \dot{x}) \in TM$ and $(\tilde{x}, \dot{\tilde{x}}) \in TN$, then $T\varphi(x, \dot{x}) = (\tilde{x}, \dot{\tilde{x}})$ through the same diffeomorphism. For a given discretization map $\mathcal{D}^{TTM}$ on TM, $\mathcal{D}^{TTN} := (T\varphi \times T\varphi) \circ \mathcal{D}^{TTM} \circ TT\varphi^{-1}$ is a discretization map on TN (refer Figure 4.1).*

Proof. For any given $(\tilde{x}, \dot{\tilde{x}}) \in TN$, we have that:

$$\begin{aligned}
\mathcal{D}^{TTN}(\tilde{x}, \dot{\tilde{x}}, 0, 0) &= \left((T\varphi \times T\varphi) \circ \mathcal{D}^{TTM} \circ TT\varphi^{-1}\right)(\tilde{x}, \dot{\tilde{x}}, 0, 0) \\
&= (T\varphi \times T\varphi) \circ \mathcal{D}^{TTM}(x, \dot{x}, 0, 0) \\
&= (T\varphi \times T\varphi)(x, \dot{x}, x, \dot{x}) = (\tilde{x}, \dot{\tilde{x}}, \tilde{x}, \dot{\tilde{x}})
\end{aligned}$$

which proves the first condition in (1.1.1).

Now, for coordinates $(\tilde{x}, \dot{\tilde{x}}, \tilde{y}, \dot{\tilde{y}}) \in TTN$,

$$\begin{aligned}
&(T_{(0_{\tilde{x}}, 0_{\dot{\tilde{x}}})}(T\mathcal{R}^2_\varphi)_{(\tilde{x}, \dot{\tilde{x}})} - T_{(0_{\tilde{x}}, 0_{\dot{\tilde{x}}})}(T\mathcal{R}^1_\varphi)_{(\tilde{x}, \dot{\tilde{x}})})(\tilde{x}, \dot{\tilde{x}}, \tilde{y}, \dot{\tilde{y}}) \\
&= \left.\frac{d}{ds}\right|_{s=0} [(T\varphi \circ (T\mathcal{R}^1) \circ TT\varphi^{-1})(\tilde{x}, \dot{\tilde{x}}, s\tilde{y}, s\dot{\tilde{y}}) \\
&- (T\varphi \circ (T\mathcal{R}^2) \circ TT\varphi^{-1})(\tilde{x}, \dot{\tilde{x}}, s\tilde{y}, s\dot{\tilde{y}})] \\
&= T_{(\tilde{x}, \dot{\tilde{x}})} T\varphi \left(\left.\frac{d}{ds}\right|_{s=0} \begin{array}{l} [(T\mathcal{R}^1)(s(TT\varphi^{-1})(\tilde{x}, \dot{\tilde{x}}, \tilde{y}, \dot{\tilde{y}})) \\ -(T\mathcal{R}^2)(s(TT\varphi^{-1})(\tilde{x}, \dot{\tilde{x}}, \tilde{y}, \dot{\tilde{y}}))] \end{array} \right) \\
&= T_{(\tilde{x}, \dot{\tilde{x}})} T\varphi((TT\varphi^{-1})(\tilde{x}, \dot{\tilde{x}}, \tilde{y}, \dot{\tilde{y}})) = (\tilde{x}, \dot{\tilde{x}}, \tilde{y}, \dot{\tilde{y}})
\end{aligned}$$

which proves the second condition in (1.1.1).

Note that $\mathcal{R} : TM \longrightarrow M \times M$ and $\mathcal{R}_\varphi : TN \longrightarrow N \times N$ are retraction maps on M and N respectively. Thus, using the linearity of the map $TT\varphi$, we prove that $\mathcal{D}^{TTN}$ is indeed a discretization map on TN. □

Appendix A

Appendix Title Here

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