Feedback Linearizable Discretizations of Mechanical Systems using Retraction Maps

BTP Stage I Presentation

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Outline

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Motivation

Consider a continous-time nonlinear system of the form:

$$\dot{x}(t) = f(x(t), u(t))$$

The corresponding discrete-time nonlinear system is given by:

$$x_{k+1} = F(x_k, u_k)$$

Assuming the following:

1. There exists a coordinate transformation $z := \varphi(x)$ and an auxiliary control $v := \psi(x, u)$ such that $\dot{z}(t) = Az(t) + Bv(t)$ where A, B are constant matrices.

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Assuming the following:

1. There exists a coordinate transformation $z := \varphi(x)$ and an auxiliary control $v := \psi(x, u)$ such that $\dot{z}(t) = Az(t) + Bv(t)$ where A, B are constant matrices.
2. The discretization scheme is arbitrary.

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Can we construct a discretization scheme such that the discrete system can also be linearized using $\varphi(x)$ and $\psi(x, u)$ similarly?

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Question 2

Can we extend this scheme (geometrically) to second-order nonlinear mechanical systems?

Definitions

Continuous Feedback Linearization

A continous-time nonlinear system $\dot{x}(t) = f(x(t), u(t))$ is said to be feedback linearizable if there exists a coordinate transformation $z = \varphi(x)$ and a feedback control law $v = \psi(x, u)$ such that the transformed system is linear $\dot{z}(t) = Az(t) + Bv(t)$.

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Discrete Feedback Linearization

A discrete-time nonlinear system $x_{k+1} = F(x_k, u_k)$ is said to be feedback linearizable if there exists a coordinate transformation $z_k = \varphi(x_k)$ and a feedback control law $v_k = \psi(x_k, u_k)$ such that the transformed system is linear $z_{k+1} = Az_k + Bv_k$, where $x_k = x(t_k)$.

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Problem

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Given a (locally) feedback linearizable continuous-time nonlinear system, construct a discretization scheme such that the discrete-time system is also (locally) feedback linearizable.

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Feedback linearizability of discrete-time systems depends on the choice of the discretization scheme.

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Given a (locally) feedback linearizable continuous-time nonlinear system, construct a discretization scheme such that the discrete-time system is also (locally) feedback linearizable.

Strategy

We utilize the concept of **retraction maps** to construct such a discretization scheme.

Definition

We define a **retraction map** on a manifold M as a smooth map $\mathcal{R} : TM \to M$, such that if $\mathcal{R}_x$ be the restriction of $\mathcal{R}$ to T_xM, then the following properties are satisfied:

1. $\mathcal{R}_x(0_x) = x$ where 0_x is the zero element of T_xM.

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2. $D\mathcal{R}_x(0_x) = T_{0_x}\mathcal{R}_x = \mathbb{I}_{T_xM}$, where $\mathbb{I}_{T_xM}$ is the identity mapping on T_xM.

Retraction Map

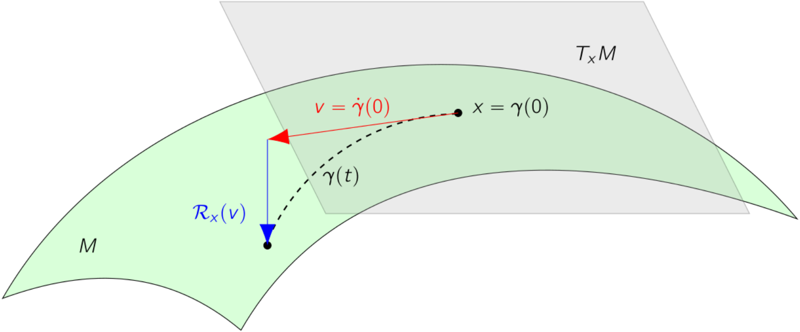

Figure: A visualization

Discretization Map

A map $\mathcal{D} : U \subset TM \longrightarrow M \times M$ given by

$$\mathcal{D}(x, v) \equiv \mathcal{D}_x(v) = \left(R_x^1(v), R_x^2(v)\right)$$

where U is the open neighborhood of the zero section $0_x \in TM$, is called a **discretization map** on M, if the following properties are satisfied:

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Example: The forward Euler discretization map is given by $\mathcal{D}(x, v) = (x, x + v)$.

Outline

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- $h = t_{k+1} - t_k$: time step of discretization.
- $\mathcal{D}^{TM}$ is a discretization map on M.

Discretization of Vector Fields

Proposition

Let $X(\cdot, u_k) \in \mathfrak{X}(M)$ be a controlled vector field on M. Then, for a given discretization scheme $\mathcal{D}$,

$$\mathcal{D}^{-1}(x_k, x_{k+1}) = hX(\tau_M(\mathcal{D}^{-1}(x_k, x_{k+1})), u_k)$$

is an implicit numerical discretization of $\dot{x}(t) = X(x(t), u(t))$.

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Example

The forward Euler discretization scheme $\mathcal{D}(x, v) = (x, x + v)$ yields the explicit Euler form $x_{k+1} = x_k + hX(x_k, u_k)$.

Tangent Lift

Proposition

Let $\varphi : M \longrightarrow N$ be a smooth map (diffeomorphism). For a given discretization map $\mathcal{D}^{TM} : TM \longrightarrow M \times M$ on M, the map $\mathcal{D}_\varphi := (\varphi \times \varphi) \circ \mathcal{D}^{TM} \circ T\varphi^{-1}$ is a discretization map on N i.e., $\mathcal{D}_\varphi \equiv \mathcal{D}^{TN} : TN \longrightarrow N \times N$.

$$\begin{tikzcd} TM \arrow[r, "T\varphi"] \arrow[d, "\mathcal{D}^{TM}"'] & TN \arrow[d, "\mathcal{D}^{TN}"] \\ M \times M \arrow[r, "\varphi \times \varphi"'] & N \times N \end{tikzcd}$$

Feedback Linearizable Discretization

Proposition

Let φ be the linearizing coordinate transformation and ψ be the linearizing feedback. Let $\mathcal{D}^{TN}$ be a discretization map that **discretizes the continuous-time linear system to a discrete-time linear system**. Then,

$$\mathcal{D}^{TM} = (\varphi \times \varphi)^{-1} \circ \mathcal{D}^{TN} \circ T\varphi$$

is a discretization on M which discretizes the continuous-time system to a discrete-time nonlinear system such that the discrete-time system is feedback linearizable using $z_k := \varphi(x_k)$ and $v_k := \psi(x_k, u_k)$.

Outline

Discretization of SODEs

A second-order differential equation (SODE) is a vector field X such that $\tau_{TM}(X) = T\tau_M(X)$. Locally,

$$X = \dot{x}^i \frac{\partial}{\partial x^i} + X^i(x^i, \dot{x}^i) \frac{\partial}{\partial \dot{x}^i} \tag{4.1}$$

To find the integral curves of X is equivalent to solving the SODE:

$$\frac{d^2}{dt^2} x(t) = X\left(x(t), \frac{d}{dt} x(t)\right) \tag{4.2}$$

Discretization of SODEs

Now, we wish to discretize this using the notion of the discretization map on TM. We would like to tangently lift a discretization on M to obtain $\mathcal{D}^{TTM} : TTM \longrightarrow TM \times TM$. This yields the following numerical scheme:

$$hX\left(\left(\tau_{TM} \circ \left(\mathcal{D}^{TTM}\right)^{-1}\right)(x_k, y_k; x_{k+1}, y_{k+1})\right) = \left(\mathcal{D}^{TTM}\right)^{-1}(x_k, y_k; x_{k+1}, y_{k+1}) \tag{4.3}$$

What is different here?

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Denote the canonical involution map $\kappa_M : TTM \longrightarrow TTM$ which is a vector bundle isomorphism, over the identity of TM.

$$\kappa_M(x, v, \dot{x}, \dot{v}) = (x, \dot{x}, v, \dot{v})$$

Why is this important?

The tangent lift of a vector field X on M does not define a vector field on TM. It is necessary to consider the composition $\kappa_M \circ TX$ to obtain a vector field on TM, and this is called the **complete lift** X^c of the vector field X. Hence, a similar technique must be used to lift a discretization map from TM to TTM.

Proposition

If $\mathcal{D}^{TM} : TM \longrightarrow M \times M$ is a discretization map on M, then $\mathcal{D}^{TTM} = T\mathcal{D}^{TM} \circ \kappa_M$ is a discretization map on TM.

Tangent Lift of Discretization Map

$$
\begin{tikzcd}
TTM \arrow[r, "\mathcal{D}^{TTM}"] \arrow[d, leftrightarrow, "\kappa_M"', "\kappa_M"] & TM \times TM \arrow[d, equal] \\
TTM \arrow[r, "T\mathcal{D}^{TM}"] \arrow[d, "\tau_{TM}"'] & T(M \times M) \arrow[d, "\tau_{M\times M}"] \\
TM \arrow[r, "\mathcal{D}^{TM}"] & M \times M
\end{tikzcd}
$$

Figure: Commutation of maps around TTM

The whole (slightly intimidating) picture

$$\begin{tikzcd}
TM \arrow[r, equal] \arrow[dddr, bend right, "\mathcal{D}^{TM}"'] & TM \arrow[r, "T\varphi"] & TN \arrow[r, equal] & TN \arrow[dddl, bend left, "\mathcal{D}^{TN}"] \\
TTM \arrow[u, "T\tau_M"] \arrow[r, leftrightarrow, "\kappa_M"] \arrow[d, "T\mathcal{D}^{TM}"'] & TTM \arrow[u, "\tau_{TM}"] \arrow[r, "TT\varphi"] \arrow[d, "\mathcal{D}^{TTM}"'] & TTN \arrow[u, "\tau_{TN}"'] \arrow[r, leftrightarrow, "\kappa_N"] \arrow[d, "\mathcal{D}^{TTN}"] & TTN \arrow[u, "T\tau_N"'] \arrow[d, "T\mathcal{D}^{TN}"] \\
T(M \times M) \arrow[r, equal] & TM \times TM \arrow[r, "T\varphi \times T\varphi"] \arrow[d, "\tau_{M \times M}"'] & TN \times TN \arrow[r, equal] \arrow[d, "\tau_{N \times N}"] & T(N \times N) \\
 & M \times M \arrow[r, "\varphi \times \varphi"] & N \times N &
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Figure: The Commutator

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- e : uncontrolled vector field.
- $\mathfrak{R}$: Riemannian curvature tensor
- ann : annihilator.

Definition

Mechanical Systems

A mechanical control system $(\mathcal{MS})_{(n,m)}$ is defined by a 4-tuple $(M, \nabla, \mathfrak{g}, e)$ where:

$$\nabla_{\dot{x}}\dot{x} = e(x) + \sum_{r=1}^{m} g_r(x)u_r \tag{4.4}$$

Or equivalently in local coordinates $x = (x^1, \dots, x^n)$ on M,

$$\ddot{x}^i = -\Gamma^i_{jk}(x)\dot{x}^j\dot{x}^k + e^i(x) + \sum_{r=1}^{m} g^i_r(x)u_r \tag{4.5}$$

Definition

We can write this as two first-order differential equations:

$$\dot{x}^i = y^i;$$
$$\dot{y}^i = -\Gamma^i_{jk}(x) y^j y^k + e^i(x) + \sum_{r=1}^{m} g^i_r(x) u_r \qquad (\mathcal{MS})$$

Objective

Given a mechanical control system $(\mathcal{MS})_{(n,m)}$, we wish to construct a discretization scheme such that the discrete-time system is **mechanical feedback linearizable**.

Mechanical Feedback Linearizability

$$\mathcal{E}^0 = \text{span}\{g_r, 1 \leqslant r \leqslant m\} \; ; \; \mathcal{E}^j = \text{span}\{\text{ad}_e^i g_r, 1 \leqslant r \leqslant m, 0 \leqslant i \leqslant j\}$$

Theorem

A mechanical system $(\mathcal{MS})_{(n,m)}$ is mechanical feedback (MF) linearizable, locally around $x_0 \in M$ iff, in the neighborhood of x_0:

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- $(ML3)$ *ann $\mathcal{E}^0 \subset$ ann $\mathfrak{R}$*

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- *(ML5) ann $\mathcal{E}^1 \subset$ ann $\nabla^2 e$*

Mechanical Feedback Linearizability

For planar mechanical systems ($n = 2$):

Proposition

A planar mechanical system $(\mathcal{MS})_{(2,1)}$ is locally MF-linearizable at $x_0 \in M$ to a controllable $(\mathcal{LMS})_{(2,1)}$, if and only if it satisfies the following conditions:

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1. $(MD1)$ g *and* $ad_e g$ *are independent*
2. $(MD2)$ $\nabla_g g \in \mathcal{E}^0$ *and* $\nabla_{ad_e g} g \in \mathcal{E}^0$
3. $(MD3)$ $\nabla^2_{g, ad_e g} ad_e g - \nabla^2_{ad_e g, g} ad_e g \in \mathcal{E}^0$

Outline

Inertia Pendulum

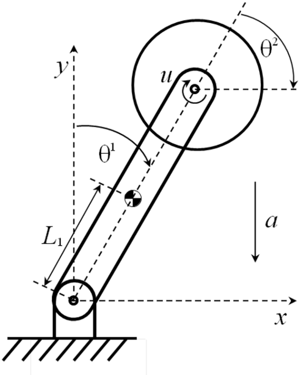

Figure: Inertia Wheel Pendulum

TORA System

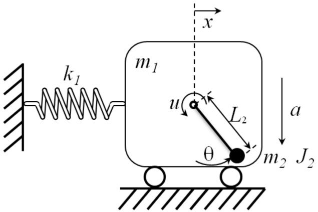

Figure: Translational Oscillator with Rotational Actuator