Feedback Linearizable Discretizations of Mechanical Systems using Retraction Maps

BTP Stage I Presentation

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Outline

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Motivation

Consider a continous-time nonlinear system of the form:

$$\dot{x}(t) = f(x(t), u(t))$$

Assuming the following:

1. There exists a coordinate transformation $z := \varphi(x)$ and an auxiliary control $v := \psi(x, u)$ such that $\dot{z}(t) = Az(t) + Bv(t)$ where A, B are constant matrices.

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1. There exists a coordinate transformation $z := \varphi(x)$ and an auxiliary control $v := \psi(x, u)$ such that $\dot{z}(t) = Az(t) + Bv(t)$ where A, B are constant matrices.
2. The discretization scheme is arbitrary.

Motivating Example

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} (1+2u(t))x_2(t) \\ u(t) \end{pmatrix} \tag{2.1}$$

Taking $\tilde{x}_1 = x_1 - x_2^2$ and $\tilde{x}_2 = x_2$, we get the transformation $\varphi(x_1, x_2) = \left(x_1 - x_2^2, x_2\right)$, we get the feedback linearized system:

$$\begin{pmatrix} \dot{z}_1(t) \\ \dot{z}_2(t) \end{pmatrix} = \begin{pmatrix} z_2(t) \\ u(t) \end{pmatrix} \tag{2.2}$$

Definitions

Continuous Feedback Linearization

A continous-time nonlinear system $\dot{x}(t) = f(x(t), u(t))$ is said to be feedback linearizable if there exists a coordinate transformation $z = \varphi(x)$ and a feedback control law $v = \psi(x, u)$ such that the transformed system is linear $\dot{z}(t) = Az(t) + Bv(t)$.

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Continuous Feedback Linearization

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Discrete Feedback Linearization

A discrete-time nonlinear system $x_{k+1} = F(x_k, u_k)$ is said to be feedback linearizable if there exists a coordinate transformation $z_k = \varphi(x_k)$ and a feedback control law $v_k = \psi(x_k, u_k)$ such that the transformed system is linear $z_{k+1} = Az_k + Bv_k$, where $x_k = x(t_k)$.

Motivating Example

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} (1+2u(t))x_2(t) \\ u(t) \end{pmatrix} \tag{2.3}$$

Taking $\tilde{x}_1 = x_1 - x_2^2$ and $\tilde{x}_2 = x_2$, we get the diffeomorphism $\varphi(x_1, x_2) = \left(x_1 - x_2^2, x_2\right)$, we get the feedback linearized system:

$$\begin{pmatrix} \dot{z}_1(t) \\ \dot{z}_2(t) \end{pmatrix} = \begin{pmatrix} z_2(t) \\ u(t) \end{pmatrix} \tag{2.4}$$

Type 1 Discretization

First, we consider the forward Euler discretization scheme. This gives a discrete-time system:

$$\begin{pmatrix} x_{1,k+1} \\ x_{2,k+1} \end{pmatrix} = \begin{pmatrix} x_{1,k} \\ x_{2,k} \end{pmatrix} + h \begin{pmatrix} (1+2u_k)x_{2,k} \\ u_k \end{pmatrix} \tag{2.5}$$

Taking the same diffeomorphism φ as before, we get:

$$\begin{pmatrix} z_{1,k+1} \\ z_{2,k+1} \end{pmatrix} = \begin{pmatrix} z_{1,k} + hz_{2,k} - h^2 u_k^2 \\ z_{2,k} + hu_k \end{pmatrix} \tag{2.6}$$

Thus, we see that it is NOT feedback linearizable.

Type 2 Discretization

Now, let's consider an alternate discretization which yields the discrete system:

$$\begin{pmatrix} x_{1,k+1} \\ x_{2,k+1} \end{pmatrix} = \begin{pmatrix} x_{1,k} \\ x_{2,k} \end{pmatrix} + h \begin{pmatrix} (1+2u_k)x_{2,k} \\ u_k \end{pmatrix} + h^2 \begin{pmatrix} u_k^2 \\ 0 \end{pmatrix} \tag{2.7}$$

Again taking the same diffeomorphism φ, we get:

$$\begin{pmatrix} z_{1,k+1} \\ z_{2,k+1} \end{pmatrix} = \begin{pmatrix} z_{1,k} + hz_{2,k} \\ z_{2,k} + hu_k \end{pmatrix} \tag{2.8}$$

which is indeed feedback linearizable.

Motivation

Question 1

Can we construct a discretization scheme such that the discrete system can also be linearized using $\varphi(x)$ and $\psi(x, u)$ similarly?

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Question 2

Can we extend this scheme (geometrically) to second-order nonlinear mechanical systems?

Motivation

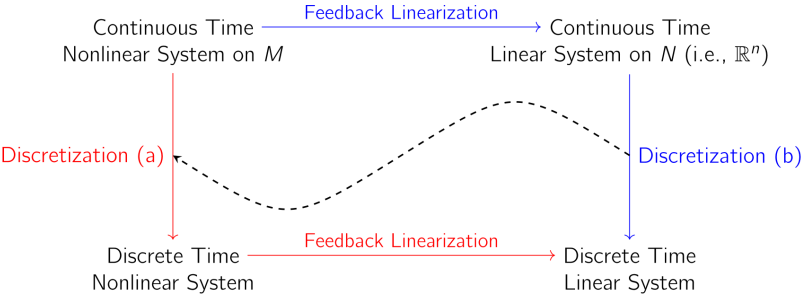

Figure: *Feedback Linearizable* Discretization?

Observations

Problem

Feedback linearizability of discrete-time systems depends on the choice of the discretization scheme.

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Given a (locally) feedback linearizable continuous-time nonlinear system, construct a discretization scheme such that the discrete-time system is also (locally) feedback linearizable.

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Feedback linearizability of discrete-time systems depends on the choice of the discretization scheme.

Objective

Given a (locally) feedback linearizable continuous-time nonlinear system, construct a discretization scheme such that the discrete-time system is also (locally) feedback linearizable.

Strategy

We utilize the concept of **retraction maps** to construct such a discretization scheme.

Definition

We define a **retraction map** on a manifold M as a smooth map $\mathcal{R} : TM \to M$, such that if $\mathcal{R}_x$ be the restriction of $\mathcal{R}$ to T_xM, then the following properties are satisfied:

1. $\mathcal{R}_x(0_x) = x$ where 0_x is the zero element of T_xM.

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1. $\mathcal{R}_x(0_x) = x$ where 0_x is the zero element of T_xM.
2. $\mathrm{D}\mathcal{R}_x(0_x) = T_{0_x}\mathcal{R}_x = \mathbb{I}_{T_xM}$, where $\mathbb{I}_{T_xM}$ is the identity mapping on T_xM.

Retraction Map

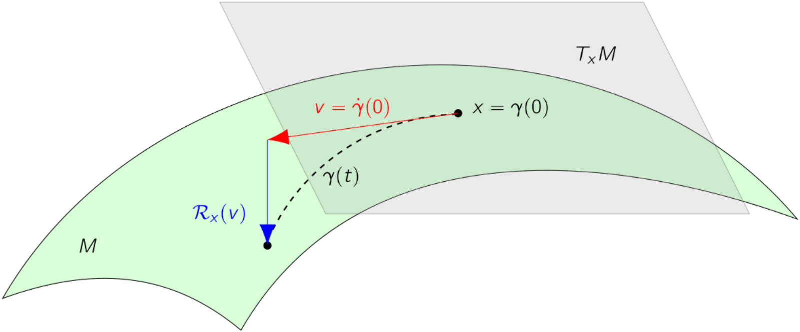

Figure: A visualization

Discretization Map

A map $\mathcal{D} : U \subset TM \longrightarrow M \times M$ given by

$$\mathcal{D}(x, v) \equiv \mathcal{D}_x(v) = \left(R^1_x(v), R^2_x(v)\right)$$

where U is the open neighborhood of the zero section $0_x \in TM$, is called a **discretization map** on M, if the following properties are satisfied:

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1. $\mathcal{D}(x, 0_x) = (x, x)$
2. $T_{0_x} R^2_x - T_{0_x} R^1_x = \mathbb{I}_{T_x M}$, which is the identity map on $T_x M$ for any $x \in M$.

Discretization Map

Some examples of discretization maps on $\mathbb{R}^n$ for $\dot{x}(t) = X(x(t))$:

Discretization map $\mathcal{D}$	Scheme	Order
$\mathcal{D}(x, v) = (x, x + v)$	Forward Euler $x_{k+1} = x_k + hX(x_k)$	$\mathcal{O}(h)$
$\mathcal{D}(x, v) = (x - v, x)$	Backward Euler $x_k = x_{k+1} - hX(x_{k+1})$	$\mathcal{O}(h)$
$\mathcal{D}(x, v) = \left(x - \frac{v}{2}, x + \frac{v}{2}\right)$	Symmetric Euler $x_{k+1} = x_k + hX\left(\frac{x_k + x_{k+1}}{2}\right)$	$\mathcal{O}(h^2)$

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- $\tau_M : TM \longrightarrow M$: canonical projection M s.t. $\tau_M(x, v) = x$.
- $h = t_{k+1} - t_k$: time step of discretization.
- $\mathcal{D}^{TM}$ is a discretization map on M.

Discretization of Vector Fields

Proposition

Let $X(\cdot, u_k) \in \mathfrak{X}(M)$ be a controlled vector field on M. Then, for a given discretization scheme $\mathcal{D}$,

$$\mathcal{D}^{-1}(x_k, x_{k+1}) = hX(\tau_M(\mathcal{D}^{-1}(x_k, x_{k+1})), u_k)$$

is an implicit numerical discretization of $\dot{x}(t) = X(x(t), u(t))$.

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Example

The forward Euler discretization scheme $\mathcal{D}(x, v) = (x, x + v)$ yields the explicit Euler form $x_{k+1} = x_k + hX(x_k, u_k)$.

Tangent Lift

Proposition

Let $\varphi : M \longrightarrow N$ be a smooth map (diffeomorphism). For a given discretization map $\mathcal{D}^{TM} : TM \longrightarrow M \times M$ on M, the map $\mathcal{D}^{TN} := (\varphi \times \varphi) \circ \mathcal{D}^{TM} \circ T\varphi^{-1}$ is a discretization map on N i.e., $\mathcal{D}^{TN} : TN \longrightarrow N \times N$.

$$\begin{tikzcd} TM \arrow[r, "T\varphi"] \arrow[d, "\mathcal{D}^{TM}"'] & TN \arrow[d, "\mathcal{D}^{TN}"] \\ M \times M \arrow[r, "\varphi \times \varphi"'] & N \times N \end{tikzcd}$$

Feedback Linearizable Discretization

Proposition

Let φ be the linearizing coordinate transformation and ψ be the linearizing feedback. Let $\mathcal{D}^{TN}$ be a discretization map that **discretizes the continuous-time linear system to a discrete-time linear system**. Then,

$$\mathcal{D}^{TM} = (\varphi \times \varphi)^{-1} \circ \mathcal{D}^{TN} \circ T\varphi$$

is a discretization on M which discretizes the continuous-time system to a discrete-time nonlinear system such that the discrete-time system is feedback linearizable using $z_k := \varphi(x_k)$ and $v_k := \psi(x_k, u_k)$.

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Motivation

1. Mechanical systems are usually described by nonlinear second-order differential equations (SODEs).

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2. These systems also have underlying mechanical structures (symmetry, conservation laws, etc.) which needs to be preserved while linearizing and discretizing.
3. Since the notion of mechanical feedback linearization has been well-studied, it is natural to extend this to the discrete-time setting.

Second Order Differential Equations

A second-order differential equation (SODE) is a vector field X such that locally,

$$X = \dot{x}^i \frac{\partial}{\partial x^i} + X^i(x^i, \dot{x}^i) \frac{\partial}{\partial \dot{x}^i} \tag{4.1}$$

To find the integral curves of X is equivalent to solving the SODE:

$$\frac{d^2}{dt^2} x(t) = X\left(x(t), \frac{d}{dt} x(t)\right) \tag{4.2}$$

Discretization of SODEs

Now, we wish to discretize this using the notion of the discretization map on TM. We would like to tangently lift a discretization on M to obtain $\mathcal{D}^{TTM} : TTM \longrightarrow TM \times TM$. This yields the following numerical scheme:

$$\begin{aligned} &\left(\mathcal{D}^{TTM}\right)^{-1}(x_k, y_k; x_{k+1}, y_{k+1}) \\ &\quad = hX\left(\tau_{TM}\left(\left(\mathcal{D}^{TTM}\right)^{-1}(x_k, y_k; x_{k+1}, y_{k+1})\right)\right) \end{aligned} \tag{4.3}$$

What is different here?

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Denote the canonical involution map $\kappa_M : TTM \longrightarrow TTM$ which is a vector bundle isomorphism, over the identity of TM.

$$\kappa_M(x, v, \dot{x}, \dot{v}) = (x, \dot{x}, v, \dot{v})$$

Why is this important?

The tangent lift of a vector field X on M does not define a vector field on TM. It is necessary to consider the composition $\kappa_M \circ TX$ to obtain a vector field on TM, and this is called the **complete lift** X^c of the vector field X. Hence, a similar technique must be used to lift a discretization map from TM to TTM.

Proposition

If $\mathcal{D}^{TM} : TM \longrightarrow M \times M$ is a discretization map on M, then $\mathcal{D}^{TTM} = T\mathcal{D}^{TM} \circ \kappa_M$ is a discretization map on TM.

Tangent Lift of Discretization Map

$$
\begin{tikzcd}
TTM \arrow[r, "\mathcal{D}^{TTM}"] \arrow[d, leftrightarrow, "\kappa_M"', "\kappa_M"] & TM \times TM \arrow[d, equal] \\
TTM \arrow[r, "T\mathcal{D}^{TM}"] \arrow[d, "\tau_{TM}"'] & T(M \times M) \arrow[d, "\tau_{M \times M}"] \\
TM \arrow[r, "\mathcal{D}^{TM}"] & M \times M
\end{tikzcd}
$$

Figure: Commutation of maps around TTM

The whole (slightly intimidating) picture

$$
\begin{tikzcd}
TM \arrow[r, equal] \arrow[dddr, bend right, "\mathcal{D}^{TM}"'] & TM \arrow[r, "T\varphi"] & TN \arrow[r, equal] & TN \arrow[dddl, bend left, "\mathcal{D}^{TN}"] \\
TTM \arrow[u, "T\tau_M"] \arrow[r, leftrightarrow, "\kappa_M"] \arrow[d, "T\mathcal{D}^{TM}"'] & TTM \arrow[u, "\tau_{TM}"'] \arrow[r, "TT\varphi"] \arrow[d, "\mathcal{D}^{TTM}"'] & TTN \arrow[u, "\tau_{TN}"] \arrow[r, leftrightarrow, "\kappa_N"] \arrow[d, "\mathcal{D}^{TTN}"] & TTN \arrow[u, "T\tau_N"] \arrow[d, "T\mathcal{D}^{TN}"] \\
T(M \times M) \arrow[r, equal] & TM \times TM \arrow[r, "T\varphi \times T\varphi"] \arrow[d, "\tau_{M \times M}"'] & TN \times TN \arrow[r, equal] \arrow[d, "\tau_{N \times N}"] & T(N \times N) \\
 & M \times M \arrow[r, "\varphi \times \varphi"] & N \times N &
\end{tikzcd}
$$

Figure: The Commutator

More Notation

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- $\mathfrak{g} = \{g_1, \ldots, g^r, \ldots, g_m\}$: control vector fields.
- e : uncontrolled vector field.

Definition

Mechanical Systems

A mechanical control system $(\mathcal{MS})_{(n,m)}$ is defined by a 4-tuple $(M, \nabla, \mathfrak{g}, e)$ where:

$$m\nabla_{\dot{x}}\dot{x} = e(x) + \sum_{r=1}^{m} g_r(x)u_r \tag{4.4}$$

Or equivalently in local coordinates $x = (x^1, \ldots, x^n)$ on M,

$$m\ddot{x}^i = -\Gamma^i_{jk}(x)\dot{x}^j\dot{x}^k + e^i(x) + \sum_{r=1}^{m} g^i_r(x)u_r \tag{4.5}$$

Definition

We can write this as two first-order differential equations:

$$\dot{x}^i = y^i;$$
$$\dot{y}^i = -\Gamma^i_{jk}(x) y^j y^k + e^i(x) + \sum_{r=1}^{m} g^i_r(x) u_r \qquad (\mathcal{MS})$$

Conclusion

Given a mechanical control system $(\mathcal{MS})_{(n,m)}$, we wish to construct a discretization scheme such that the discrete-time system is **mechanical feedback linearizable**.

Outline

Inertia Wheel Pendulum

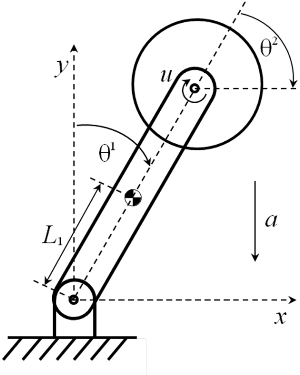

Figure: Mechanism

Inertia Wheel Pendulum - The Dynamical Equations

The equations of motion ($M = S^1 \times S^1$) are given by:

$$\begin{aligned} (m_d + J_2)\ddot{\theta}^1 + J\ddot{\theta}^2 - m_0 \sin\theta^1 &= 0 \\ J\ddot{\theta}^1 + J\ddot{\theta}^2 &= u \end{aligned} \tag{5.1}$$

where

$$m_d = L_1^2(m_1 + 4m_2) + J_1, \;\; m_0 = aL_1(m_1 + 2m_2)$$

m_1 - mass of the pendulum, m_2 - mass of the wheel, J - moment of inertia of the wheel

Inertia Wheel Pendulum - General form

Taking $(\theta^1, \theta^2) = (x_1, x_2)$ and correspondingly $(\dot{\theta}^1, \dot{\theta}^2) = (y_1, y_2)$, we get the following equations:

$$\begin{aligned} \dot{x}_1 &= y_1, \;\; \dot{x}_2 = y_2 \\ \dot{y}_1 &= e_1 + g_1 u, \;\; \dot{y}_2 = e_2 + g_2 u \end{aligned} \tag{5.2}$$

where

$$\begin{aligned} e_1 &= \frac{m_0}{m_d} \sin x_1, \quad g_1 = -\frac{1}{m_d} \\ e_2 &= -\frac{m_0}{m_d} \sin x_1, \quad g_2 = \frac{m_d + J_2}{m_d J_2} \end{aligned}$$

Note that this system has all the Christoffel symbols $\Gamma^i_{jk} = 0$

Inertia Wheel Pendulum - Feedback Linearization

Choosing $\Phi(x, y) = (\varphi(x), D\varphi(x)y)$, which is given by (where $\tilde{x} = \varphi(x)$):

$$\begin{aligned} \tilde{x}_1 &= \frac{m_d + J_2}{J_2} x_1 + x_2, \ \tilde{x}_2 = \frac{m_0}{J_2} \sin x_1 \\ \tilde{y}_1 &= \frac{m_d + J_2}{J_2} y_1 + y_2, \ \tilde{y}_2 = \frac{m_0}{J_2} \cos x_1 y_1 \end{aligned} \tag{5.3}$$

Taking $\tilde{\mathbf{x}} = \begin{pmatrix} \tilde{x}_1 & \tilde{x}_2 & \tilde{y}_1 & \tilde{y}_2 \end{pmatrix}^T$, such that the linearized equations become $\dfrac{d}{dt}\tilde{\mathbf{x}} = A\tilde{\mathbf{x}} + B\tilde{u}$

Inertia Wheel Pendulum - Feedback Linearization

Here, the matrices $A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$ and

$\tilde{u} = \psi(x, y, u)$ is the auxiliary control, such that:

$$\tilde{u} = -\frac{m_0}{J_2}\sin x_1 y_1^2 + \frac{m_0^2}{2m_d J_2}\sin 2x_1 - \frac{m_0}{m_d J_2}\cos x_1 u \qquad (5.4)$$

Taking $\tilde{u} = -K\tilde{\mathbf{x}}$, such that the closed-loop system is stable, we get $\dot{\tilde{\mathbf{x}}} = F\tilde{\mathbf{x}}$.

Inertia Wheel Pendulum - Discretization

Let h denote a (fixed) sampling time and $h' = \dfrac{h}{2}$. We utilize the symmetric discretization:

$$F(\mathrm{x}_k; h/2) = F(\mathrm{x}_{k+1}; -h/2)$$

$$\mathrm{x}_k + h'(A - BK)\mathrm{x}_k = \mathrm{x}_{k+1} - h'(A - BK)\mathrm{x}_{k+1}$$

$$\therefore \mathrm{x}_{k+1} = (I - h'(A - BK))^{-1}(I + h'(A - BK))\mathrm{x}_k \tag{5.5}$$

Results

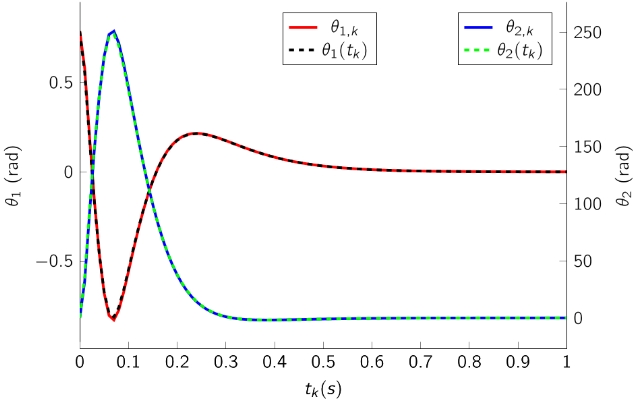

Figure 4.1: System states x_k for symmetric discretization plotted against exact discretization (ODE45) $x(t_k)$ for $t_k \in [0, 1]$

Results

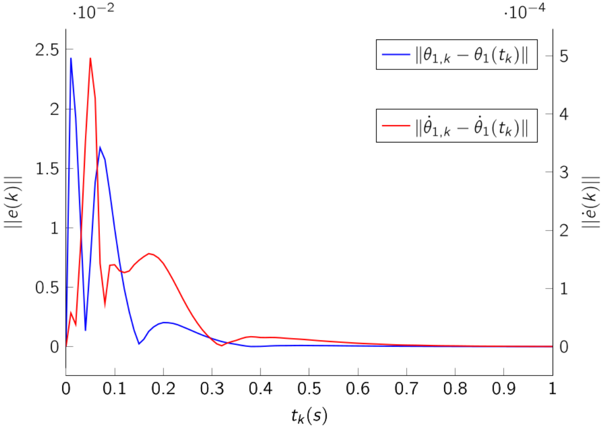

Figure 4.2: Magnitude of error norm for θ_1 and $\dot{\theta}_1$

Results

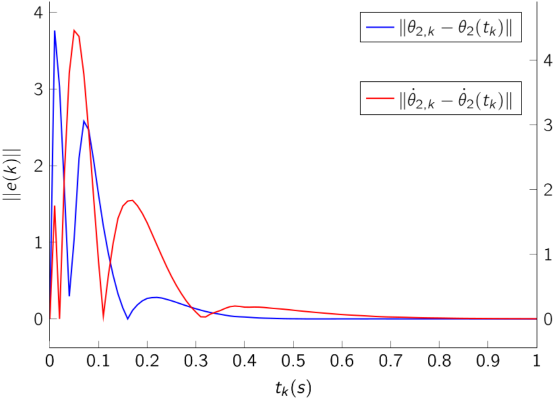

Figure 4.3: Magnitude of error norm for θ_2 and $\dot{\theta}_2$

TORA System

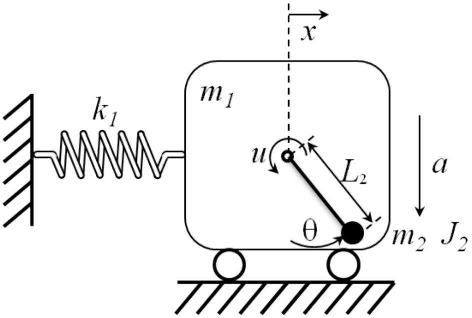

Figure: Translational Oscillator with Rotational Actuator

TORA System - The Dynamical Equations

The equations of motion ($M = S^1 \times \mathbb{R}$) are given by:

$$\begin{aligned} (m_1 + m_2)\ddot{x} + m_2 L_2 \cos\theta\ddot{\theta} - m_2 L_2 \sin\theta\dot{\theta}^2 + k_1 x &= 0 \\ m_2 L_2 \cos\theta\ddot{x} + (m_2 L_2^2 + J_2)\ddot{\theta} + m_2 L_2 a \sin\theta &= u \end{aligned} \tag{5.6}$$

TORA System - General form

Taking $(x, \theta) = (x_1, x_2)$ and correspondingly $(\dot{x}, \dot{\theta}) = (y_1, y_2)$, we get the following equations:

$$\begin{aligned} \dot{x}_1 &= y_1 \\ \dot{x}_2 &= y_2 \\ \dot{y}_1 &= -\Gamma^1_{22} y_2 y_2 + e_1 + g_1 u \\ \dot{y}_2 &= -\Gamma^2_{22} y^2 y^2 + e_2 + g_2 u \end{aligned} \tag{5.7}$$

Here, the Christoffel symbols are non-zero:

$$\Gamma^1_{22} = \frac{-m_2 L_2 (m_2 L_2^2 + J_2) \sin x_2}{(m_1 + m_2)(m_2 L_2^2 + J_2) - m_2^2 L_2^2 \cos^2 x_2}$$

$$\Gamma^2_{22} = \frac{m_2^2 L_2^2 \sin x_2 \cos x_2}{(m_1 + m_2)(m_2 L_2^2 + J_2) - m_2^2 L_2^2 \cos^2 x_2}$$

TORA System - General form

$$e_1 = \frac{m_2^2 L_2^2 a \sin x_2 \cos x_2 - k_1(m_2 L_2^2 + J_2)x_1}{(m_1 + m_2)(m_2 L_2^2 + J_2) - m_2^2 L_2^2 \cos^2 x_2}$$

$$e_2 = \frac{-m_2 L_2 (m_1 + m_2) a \sin x_2 + k_1 m_2 L_2 x_1 \cos x_2}{(m_1 + m_2)(m_2 L_2^2 + J_2) - m_2^2 L_2^2 \cos^2 x_2}$$

$$g_1 = \frac{-m_2 L_2 \cos x_2}{(m_1 + m_2)(m_2 L_2^2 + J_2) - m_2^2 L_2^2 \cos^2 x_2}$$

$$g_2 = \frac{m_1 + m_2}{(m_1 + m_2)(m_2 L_2^2 + J_2) - m_2^2 L_2^2 \cos^2 x_2}$$

TORA System - Feedback Linearization

Choosing $\Phi(x, y) = (\varphi(x), D\varphi(x)y)$, which is given by:

$$\tilde{x}_1 = m_{11}x_1 + m_{12}\sin x_2, \;\; \tilde{x}_2 = -k_1 x_1$$
$$\tilde{y}_1 = m_{11}y_1 + m_{12}\cos x_2 y_2, \;\; \tilde{y}_2 = -k_1 y_1$$

Again taking $\tilde{\mathbf{x}} = \begin{pmatrix} \tilde{x}_1 & \tilde{x}_2 & \tilde{y}_1 & \tilde{y}_2 \end{pmatrix}^T$, such that the linearized equations become $\dfrac{d}{dt}\tilde{\mathbf{x}} = A\tilde{\mathbf{x}} + B\tilde{u}$

TORA System

Note that $\tilde{u} = \psi(x, y, u)$ is the auxiliary control, such that:

$$\tilde{u} = -k_1(-\Gamma^1_{22} y_2^2 + e_1 + g_1 u) \tag{5.8}$$

Taking $\tilde{u} = -K\tilde{\mathbf{x}}$, such that the closed-loop system is stable, we get $\dot{\tilde{\mathbf{x}}} = F\tilde{\mathbf{x}}$.
We again utilize the symmetric discretization scheme to obtain the discrete-time system.

Results

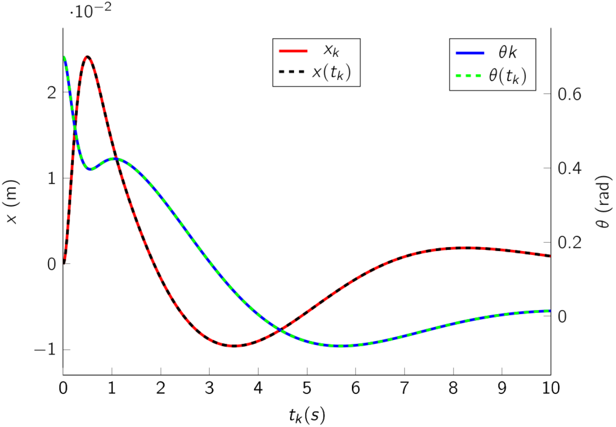

Figure 4.4: System states x_k for symmetric discretization plotted against exact discretization (ODE45) $x(t_k)$ for $t_k \in [0, 10]$

Results

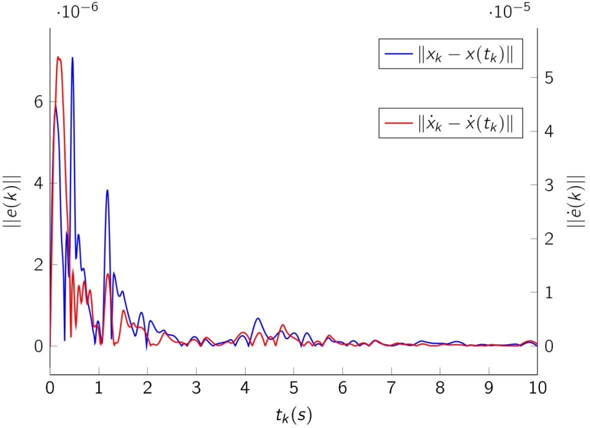

Figure 4.5: Magnitude of error norm for x and $\dot{x}$

Results

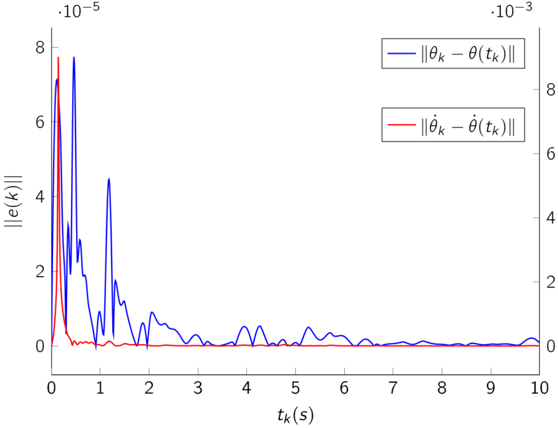

Figure 4.6: Magnitude of error norm for θ and $\dot{\theta}$

Thank You!