Feedback Linearizable Discretizations of Second Order Mechanical Systems using Retraction Maps

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***Abstract*— Mechanical systems, in nature, are often described by a set of continuous-time, nonlinear, second-order differential equations (SODEs). This has motivated designs of various control laws implemented on digital controllers, consequently requiring numerical discretization schemes. Feedback linearizability of such sampled systems depends on the discretization scheme or map choice. In this article, we utilize retraction maps and their lifts to construct feedback linearizable discretizations for SODEs, which can be applied to various mechanical systems.**

I. Introduction

The design of control laws for mechanical systems has evolved rapidly in the last century. Digital controllers facilitate the implementation of these algorithms in various ways, a popular way being discretization schemes that evolve numerically with the continuous-time dynamics. While dealing with systems on Euclidean spaces, we could use Runge-Kutta, Euler discretization, etc., these discretization schemes, when applied to general manifolds, do not assure that the system states stay on the manifold. In other words, we must know how to move on the manifold. For example, on the Riemannian manifold (M, g), this notion would be described by the exponential map $\exp_x : T_xM \longmapsto M$ at the point $x \in M$. Thus, there is a need for a universal tool that allows us to construct integrators that respect the underlying geometry of general manifolds. Retraction and discretization maps generalize Euclidean discretizations on general manifolds [1].

Another aspect related to the control of mechanical systems that has been under study for some time is transforming nonlinear systems to a locally linear system and an invertible control law through a coordinate transformation. Feedback linearization allows us to do this, enabling the utilization of standard powerful methods that can be applied to linear systems like pole placement to design control laws for nonlinear systems. This has led to the work on constructing feedback linearizable discretizations for first-order nonlinear systems (refer [2]). Although feedback linearization for continuous-time systems has been developed well in [3], [4], [5], when addressing mechanical systems of the second-order, we must ensure that the coordinate transformation yields a linear system which is also mechanically equivalent to the original mechanical system, i.e., we must design feedback which linearizes the mechanical system, while simultaneously preserving its mechanical structure (see [6], [7], [8], [9]).

Contribution: In this article, given a mechanical system that falls under a specific class, we utilize lifts of retraction maps to construct discretizations that are feedback linearizable. We propose a structured discretization scheme that can be applied to second-order mechanical systems.

Organization: The article is organized as follows: Section II introduces us to the field, with the definition of a mechanical control system, as defined in [6], [9]. In Section III, we look at the feedback linearizability of such mechanical systems in continuous time, where we define the existence of a specific class of systems that are feedback linearizable while simultaneously preserving its mechanical structure. In Section IV, we discuss discretization maps and their tangent lifts to construct integrators on the double-tangent bundle TTM for second-order systems. We present our main result in Section V, where we utilize the tangent lifts defined in Section IV to construct a feedback linearized discrete system, and we demonstrate these results on a simple mechanical system in Section VI.

A. Previous Work

1) Retraction and Discretization Maps:

Definition 1.1: Let M be an n dimensional manifold, and TM be its tangent bundle. Denote the canonical projection on the manifold to be $\tau_M(x, y) = x$, where $\tau_M : TM \to M$. We define a **retraction map** on a manifold M as a smooth map $R : TM \to M$, such that if R_x be the restriction of R to T_xM, then the following properties are satisfied:

1) $R_x(0_x) = x$ where 0_x is the zero element of T_xM.
2) $\mathrm{D}R_x(0_x) = T_{0_x}R_x = \mathrm{Id}_{T_xM}$, where Id_{T_xM} denotes the identity mapping on T_xM.

Define an open neighborhood $U \subset TM$ around the zero section of the tangent bundle. If $(x, y) \in U$, where $y \in T_xM$, then $R_d : U \to M \times M$ is called a **discretization map**, if it satisfies the following properties:

1) $R_d(x, 0_x) = (x, x)$
2) $T_{(x,0_x)}R^2 - T_{(x,0_x)}R^1 = \mathrm{Id}_{T_xM}$, which is the identity map on T_xM for any $x \in M$.

We note that here, we define $R_d(x, v) = (R^1_x(v), R^2_x(v))$, and $T_{(x,0_x)}R^a = \mathrm{D}R^a(x, 0_x)$ for $a = 1, 2$. Consequently, it can be shown that any discretization map R is a local diffeomorphism.

Given a vector field $X \in \mathfrak{X}(M)$ on M and a discretization map R_d and a fixed time discretization map $t \longmapsto (t-\alpha h, t+$

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$(1-\alpha)h), \alpha \in [0,1]$, the discretization of X is defined by:

$$R_d^{-1}(x_k, x_{k+1}) = hX(\tau_M(R_d^{-1}(x_k, x_{k+1}))) \qquad (1.1)$$

Let us consider an example where $R_d(x, v) = (x, x+v)$. If we have $\dot{x} = X(x(t))$, then this choice of discretization map yields the scheme $x_{k+1} = x_k + hX(x_k)$ where $x(t_k) = x_k$. This is the standard *forward Euler* scheme. Similarly $R_d(x, v) = (x - v, x)$ yields the *backward Euler* method.

Remark 1.1: More generally if we fix a discretization map on the space $\mathbb{R} \times M$ given by $\tilde{R}_d : T(\mathbb{R} \times M) \to (\mathbb{R} \times M) \times (\mathbb{R} \times M)$ and a time-dependent dynamics given by $\dot{x} = X(t, x)$, we define the corresponding discretization as

$$\begin{aligned}
&(\mathrm{pr}_{TM} \circ \tilde{R}_d^{-1})(t_k, x_k, t_{k+1}, x_{k+1}) \\
&= hX(\tau_{\mathbb{R}\times M}(\tilde{R}_d^{-1}(t_k, x_k, t_{k+1}, x_{k+1}))) \\
&h = (\mathrm{pr}_{2,\mathbb{R}} \circ \tilde{R}_d^{-1})(t_k, x_k, t_{k+1}, x_{k+1})
\end{aligned}$$

where $\mathrm{pr}_{TM} : T(\mathbb{R} \times M) \to TM$ and $\mathrm{pr}_{T\mathbb{R}} : T(\mathbb{R} \times M) \to T\mathbb{R}$ are the canonical projections and $\mathrm{pr}_{2,\mathbb{R}} = \mathrm{pr}_2 \circ \mathrm{pr}_{T\mathbb{R}}$.

Moreover, these extended retractions allow us to introduce discrete time reparametrizations (discrete Sundman transformations) adding to the dynamics given by $\dfrac{dx}{dt} = X(t, x)$ the time transformation $\dfrac{d\tau}{dt} = \dfrac{1}{f(x)}$ where $f : M \to \mathbb{R}$ verifies $f > 0$ (see [10] and references therein).

2) Feedback Linearization: Let M be an n-dimensional manifold and $U \in \mathbb{R}^n$ be open subset representing the control variables. For $u \in U$, let $X(\cdot, u) \in \mathfrak{X}(M)$ be a vector field on M. Then, for a fixed time $T > 0$, a continuous-time dynamical system on M is given by

$$\frac{d}{dt}x(t) = X(x(t), u(t)) \text{ for all } t \in [0, T] \qquad (1.2)$$

Let M and N be two n-dimensional manifolds and $\phi : M \longmapsto N$ be a diffeomorphism. If $X \in \mathfrak{X}(M)$ is a vector field on M, then X_ϕ is a vector field on N such that $X_\phi := T\phi \circ X \circ \phi^{-1}$, which yields the following mechanical system on N:

$$\frac{d}{dt}\tilde{x}(t) = X_\phi(\tilde{x}(t), \tilde{u}(t)) \qquad (1.3)$$

with $\tilde{x}(0) = \phi(x(0))$ and $\tilde{x}(t) = \phi(x(t))$ for all $t \in [0, T]$.

Definition 1.2: A given system 1.2 is said to be locally feedback linearizable around a specified point if there exist matrices $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$ such that $X_\phi(\tilde{x}, v) = A\tilde{x} + Bv$ where $v = \psi(\phi^{-1}(\tilde{x}), u)$, ψ being the linearizing feedback, and the corresponding feedback linearized dynamical system is given by:

$$\frac{d}{dt}\tilde{x}(t) = A\tilde{x}(t) + Bv(t) \text{ for all } t \in [0, T] \qquad (1.4)$$

For detailed proofs and background on feedback linearization, we refer the reader to [2] – [5].

$$\begin{tikzcd}
TM \arrow[r, "T\phi"] \arrow[d, "R_{d,\phi}"'] & TN \arrow[d, "R_d"] \\
M \times M \arrow[r, "\phi \times \phi"'] & N \times N
\end{tikzcd}$$

Fig. 1.1. R_d and $R_{d,\phi}$ commute as shown

3) Constructing feedback linearizable discretization maps: From [2], we thus formulate the construction of feedback linearizable discretization maps, by lifting the discretization map R_d on TN to $R_{d,\phi}$ on TM as shown in Figure 1.1. The key concept from [2], is the following:

Proposition 1.1: Let $\phi : M \longrightarrow N := \mathbb{R}^n$ be the linearizing coordinate change, and $\psi : M \times U \longrightarrow \mathbb{R}^m$ be the linearizing feedback, where $U \subset \mathbb{R}^m$ is the control space. Let R_d be a discretization map on N that discretizes the continuous-time linear system (CTLS) to a discrete-time linear system (DTLS). Then,

$$R_{d,\phi} = (\phi \times \phi)^{-1} \circ R_d \circ T\phi \qquad (1.5)$$

is a discretization map on M, which discretizes the continuous-time system (CTS) to a feedback linearizable discrete-time system (DTS). Here, the linearizing coordinate is given by $x_k = \phi^{-1}(\tilde{x}_k)$ and the auxiliary control u_k is given by $\tilde{u}_k = \psi(x_k, u_k)$ for the DTS.

II. Mechanical Control Systems

We define a mechanical control system and its feedback linearization as proposed in [6] and [8].

Definition 2.1: A mechanical control system $(\mathcal{MS})_{(n,m)}$ is defined by a 4-tuple $(M, \nabla, \mathfrak{g}, e)$ where:

- M is an n-dimensional manifold
- ∇ is a symmetric affine connection on M
- $\mathfrak{g} = \{g_1, \ldots, g_m\}$ is an m-tuple of control vector fields on M
- e is an uncontrolled vector field on M

$(\mathcal{MS})_{(n,m)}$ can be represented by the differential equation:

$$\nabla_{\dot{x}}\dot{x} = e(x) + \sum_{r=1}^{m} g_r(x)u_r \qquad (2.1)$$

Or equivalently in local coordinates $x = (x^1, \ldots, x^n)$ on M,

$$\ddot{x}^i = -\Gamma^i_{jk}(x)\dot{x}^j\dot{x}^k + e^i(x) + \sum_{r=1}^{m} g^i_r(x)u_r \qquad (2.2)$$

If we write this as two first-order differential equations:

$$\dot{x}^i = y^i; \;\; \dot{y}^i = -\Gamma^i_{jk}(x)y^jy^k + e^i(x) + \sum_{r=1}^{m} g^i_r(x)u_r \quad (2.3)$$

$$
\begin{tikzcd}
TM \arrow[r, equal] \arrow[rddd, bend right=60, "R_d"'] & TM \arrow[r, "T\phi"] & TN \arrow[r, equal] & TN \arrow[lddd, bend left=60, "R_{d,\phi}"] \\
TTM \arrow[u, "T\tau_M"] \arrow[r, leftrightarrow, "\kappa_M"] \arrow[d, "TR"] & TTM \arrow[u, "\tau_{TM}"'] \arrow[r, "TT\phi"] \arrow[d, "R_d^T"'] & TTN \arrow[u, "\tau_{TN}"] \arrow[r, leftrightarrow, "\kappa_N"] \arrow[d, "R^T_{d,\phi}"] & TTN \arrow[u, "T\tau_N"'] \arrow[d, "TR_{d,\phi}"'] \\
T(M \times M) \arrow[r, equal] & TM \times TM \arrow[r, "T\phi \times T\phi"] \arrow[d, "\tau_{M\times M}"'] & TN \times TN \arrow[r, equal] \arrow[d, "\tau_{N\times N}"] & T(N \times N) \\
& M \times M \arrow[r, "\phi \times \phi"] & N \times N &
\end{tikzcd}
$$

Fig. 2.1. R_d^T and $R_{d,\phi}^T$ commute as shown. Description in IV-A.

III. Mechanical Feedback Linearization

We consider the problem of bringing a mechanical system $(\mathcal{MS})_{(n,m)}$ into a linear mechanical form through a transformation and mechanical feedback. This mechanical feedback linearization (or MF-linearization [7]) must preserve the structure of the tangent bundle TM

Note: This section uses various definitions based on propositions from the works on mechanical feedback linearization in [7], [9] by W. Respondek, et al.

Definition 3.1: Let MF be a group of transformations such that:

1) coordinate transformations given by diffeomorphisms

$$\phi : M \longrightarrow N; \;\; x \longmapsto \tilde{x} = \phi(x) \tag{3.1}$$

2) mechanical feedback transformations, denoted by (α, β, γ) such that

$$u_r = \gamma^r_{jk} y^j y^k + \alpha_r(x) + \sum_{s=1}^{m} \beta^r_s(x) \tilde{u}_s \tag{3.2}$$

or more compactly,

$$u = y^T \gamma y + \alpha + \beta \tilde{u} \tag{3.3}$$

Now, using this group, we define the feedback linearization for mechanical systems:

Definition 3.2: Two mechanical systems $(\mathcal{MS})_{(n,m)} = (M, \nabla, \mathfrak{g}, e)$ and $(\widetilde{\mathcal{MS}})_{(n,m)} = (N, \widetilde{\nabla}, \tilde{\mathfrak{g}}, \tilde{e})$ are mechanical feedback equivalent if there exists $(\phi, \alpha, \beta, \gamma) \in MF$ such that

$$\phi : M \longrightarrow N \;\; \phi(x) = \tilde{x}, \;\; \phi_* \left(\nabla - \sum_{r=1}^{m} g_r \otimes \gamma^r \right) = \widetilde{\nabla}$$
$$\phi_* \left(\sum_{r=1}^{m} \beta^r_s g_r \right) = \tilde{g}_s, \;\; \phi_* \left(e + \sum_{r=1}^{m} g_r \alpha^r \right) = \tilde{e} \tag{3.4}$$

We can see that the diffeomorphism on M induces a mechanical diffeomorphism on TM given by $(\tilde{x}, \tilde{y}) = (\phi(x), D\phi(x)y)$. The proof of the action of MF preserving the mechanical structure of $(\mathcal{MS})_{(n,m)}$ can be referred to in more detail in [7].

According to the definition in 3.1, the MF-transformations act on the vector fields g_r and e through (ϕ, α, β). We define the following relevant distributions:

$$
\begin{aligned}
\mathcal{E}^0 &= \mathrm{span}\{g_r, 1 \leq r \leq m\} \\
\mathcal{E}^j &= \mathrm{span}\{\mathrm{ad}^i_e g_r, 1 \leq r \leq m, 0 \leq i \leq j\}
\end{aligned}
\tag{3.5}
$$

Thus, we state the following theorem:

Theorem 3.1: A mechanical system $(\mathcal{MS})_{(n,m)}$ is said to be mechanical feedback (MF) linearizable, locally around $x_0 \in M$ if and only if, in the neighborhood of x_0, it satisfies the following conditions:

1) $(ML1)$ $\mathcal{E}^0$ and $\mathcal{E}^1$ are of constant rank
2) $(ML2)$ $\mathcal{E}^0$ is involutive
3) $(ML3)$ ann $\mathcal{E}^0 \subset$ ann $\mathcal{R}$
4) $(ML4)$ ann $\mathcal{E}^0 \subset$ ann ∇g_r for all $r : 1 \leq r \leq m$
5) $(ML5)$ ann $\mathcal{E}^1 \subset$ ann $\nabla^2 e$

where $\mathcal{R}$ is the Riemannian curvature tensor.

Note that the above conditions are valid without the assumption of controllability of the linearized mechanical system. The following proposition is explicitly stated for planar mechanical systems where $n = 2$. (refer [7]).

Proposition 3.2: A planar mechanical system $(\mathcal{MS})_{(2,1)}$ is locally MF-linearizable at $x_0 \in M$ to a controllable $(\mathcal{LMS})_{(2,1)}$, if and only if it satisfies the following conditions:

1) $(MD1)$ g and $\mathrm{ad}_e g$ are independent
2) $(MD2)$ $\nabla_g g \in \mathcal{E}^0$ and $\nabla_{\mathrm{ad}_e g} g \in \mathcal{E}^0$
3) $(MD3)$ $\nabla^2_{g, \mathrm{ad}_e g} \mathrm{ad}_e g - \nabla^2_{\mathrm{ad}_e g, g} \mathrm{ad}_e g \in \mathcal{E}^0$

Definition 3.3: A mechanical control system $(\mathcal{MS})_{(n,m)} = (M, \nabla, \mathfrak{g}, e)$ is called MF-linearizable if it is MF-equivalent to a linear mechanical system $(\mathcal{LMS})_{(n,m)} = (\mathbb{R}^n, \bar{\nabla}, \mathfrak{b}, A\tilde{x})$, where $\bar{\nabla}$ is an affine connection with the Christoffel symbols zero ($\bar{\nabla}$ is a flat connection) and $\mathfrak{b} = \{b_1, \ldots, b_m\}$ are constant vector fields.

In other words, there exists $(\phi, \alpha, \beta, \gamma) \in MF$ such that

$$\begin{aligned} \phi : M \longrightarrow N \quad \phi(x) = \tilde{x} \\ \phi_* \left(\nabla - \sum_{r=1}^{m} g_r \otimes \gamma^r \right) = \bar{\nabla} \\ \phi_* \left(\sum_{r=1}^{m} \beta_s^r g_r \right) = b_s, \; 1 \le s \le m \\ \phi_* \left(e + \sum_{r=1}^{m} g_r \alpha^r \right) = A\tilde{x} \end{aligned} \tag{3.6}$$

Equivalently, we have the corresponding linear mechanical system $(\mathcal{LMS})_{(n,m)}$ as:

$$\dot{\tilde{x}} = \tilde{y}; \; \dot{\tilde{y}} = A\tilde{x} + \sum_{s=1}^{m} b_s \tilde{u}_s \tag{3.7}$$

IV. Tangent Lifts of Discretization Maps

We have already seen the definition of a discretization map on M given by $R_d : TM \longmapsto M \times M$. To perform discretization in second-order differential equations, we require the definition of a discretization map on TM, denoted by $R_d^T : TTM \longmapsto TM \times TM$, where TTM is the double tangent bundle of M [1], [11].

Let M be an n-dimensional manifold and $\tau_M : TM \longrightarrow M$ be the canonical projection map of the tangent bundle, and TTM be the double tangent bundle. The manifold TTM intrinsically admits two vector bundle structures such that the first vector bundle structure is canonical with the vector bundle projection $\tau_{TM} : TTM \longrightarrow TM$, whereas the second vector bundle structure, the map $T\tau_M : TTM \longrightarrow TM$ gives the projection.

Thus, to define R_d^T from a given discretization map R_d, we would have to use the canonical involution map $\kappa_M : TTM \longrightarrow TTM$, which shows the dual vector bundle structure of TTM. Let $(x, y) \in TM$ and $(x, y, \dot{x}, \dot{y}) \in TTM$ be the canonical coordinates. Then:

$$\kappa_M(x, y, \dot{x}, \dot{y}) = (x, \dot{x}, y, \dot{y}) \tag{4.1}$$

It can be shown that $\kappa_M^2 = \mathrm{Id}_{TTM}$ implying κ_M is an involution of TTM.

Proposition 4.1: If R_d is a discretization map on M, then $R_d^T = TR_d \circ \kappa_M$ is a discretization map on TM.

Proof: For $(x, y, \dot{x}, \dot{y}) \in TTM$, we have that $TR_d(x, y, \dot{x}, \dot{y}) = \left(R_d(x, y), D_{(x,y)} R_d(x, y)(\dot{x}, \dot{y})^T\right)$ and

$$R_d^T(x, \dot{x}, y, \dot{y}) = (R_d(x, y), D_{(x,y)} R_d(\dot{x}, \dot{y})^T)$$

Using the properties defined in Definition 1.1,

1) We know that $R_d(x, 0) = (x, x)$ for all $x \in M$. Thus,

$$\begin{aligned} R_d^T(x, \dot{x}, 0, 0) &= \left(R_d(x, 0), D_{(x,0)} R_d(\dot{x}, 0)\right) \\ &= (x, x, \dot{x}, \dot{x}) \equiv (x, \dot{x}, x, \dot{x}) \end{aligned}$$

where we trivially identify $T(M \times M) \equiv TM \times TM$.

2) For this property, we know that

$$R_d^T(x, \dot{x}, y, \dot{y}) = \left(TR_d^1(x, y; \dot{x}, \dot{y}), TR_d^2(x, y; \dot{x}, \dot{y})\right)$$

So, we need to compute

$$T_{(0,0)_{(x,\dot{x})}}(TR_d^a)_{(x,\dot{x})}(x, \dot{x}) : T_{(x,\dot{x})}TM \longrightarrow T_{(x,\dot{x})}TM$$

for $a = 1, 2$, to prove that the map $T(TR_d^2)_{(x,\dot{x})} - T(TR_d^1)_{(x,\dot{x})}$ is the identity map at the zero section $(0, 0)_{(x,\dot{x})}$, from $T_{(x,\dot{x})}TM$ to itself.
We can calculate

$$\left.\frac{d}{ds}\right|_{s=0} (R_d^a(x, sy), \partial_x R_d^a(x, sy)\dot{x} + \partial_y R_d^a(x, sy)s\dot{y})$$

At $(x, \dot{x}, 0, 0)$, the map $T_{(0,0)_{(x,\dot{x})}}(TR_d^a)_{(x,\dot{x})}$ is thus given by:

$$\begin{pmatrix} \partial_{y^j}(R_d^a)^i(x, 0) & 0 \\ \partial_{x^k}\partial_{y^j}(R_d^a)^i(x, 0)\dot{x}^k & \partial_{y^j}(R_d^a)^i(x, 0) \end{pmatrix}$$

Thus, using the properties of the discretization map R_d, we have the Jacobian matrix of $(TR_d^2)_{(x,\dot{x})} - (TR_d^1)_{(x,\dot{x})}$ at $(0, 0)_{(x,\dot{x})}$ as:

$$\begin{pmatrix} \partial_y(R_d^2 - R_d^1)(x, 0) & 0 \\ \partial_x(\partial_y(R_d^2 - R_d^1)(x, 0))\dot{x} & \partial_y(R_d^2 - R_d^1)(x, 0) \end{pmatrix}$$

which is indeed equal to the identity $\mathrm{Id}_{2n\times 2n}$, since $\partial_y(R_d^2 - R_d^1)(x, 0) = \mathrm{Id}_{n\times n}$ which also implies $\partial_x(\partial_y(R_d^2 - R_d^1))(x, 0) = 0$

■

Proposition 4.2: Let M and N be n dimensional manifolds and $\phi(x) = \tilde{x}$, where ϕ is a diffeomorphism and $x \in M, \tilde{x} \in N$. Let TM and TN be the tangent bundles of M and N, respectively. By definition, if $(x, \dot{x}) \in TM$ and $(\tilde{x}, \dot{\tilde{x}}) \in TN$, then $T\phi(x, \dot{x}) = (\tilde{x}, \dot{\tilde{x}})$ through the same diffeomorphism. For a given discretization map R_d^T on TM, $R_{d,\phi}^T := (T\phi \times T\phi) \circ R_d^T \circ TT\phi^{-1}$ is a discretization map on TN (refer Figure 2.1).

Proof: For any given $(\tilde{x}, \dot{\tilde{x}}) \in TN$, we have that:

$$\begin{aligned} R_{d,\phi}^T(\tilde{x}, \dot{\tilde{x}}, 0, 0) &= \left((T\phi \times T\phi) \circ R_d^T \circ TT\phi^{-1}\right)(\tilde{x}, \dot{\tilde{x}}, 0, 0) \\ &= (T\phi \times T\phi) \circ R_d^T(x, \dot{x}, 0, 0) \\ &= (T\phi \times T\phi)(x, \dot{x}, x, \dot{x}) = (\tilde{x}, \dot{\tilde{x}}, \tilde{x}, \dot{\tilde{x}}) \end{aligned}$$

which proves the first condition in 1.1.

Now, for coordinates $(\tilde{x}, \dot{\tilde{x}}, \tilde{y}, \dot{\tilde{y}}) \in TTN$,

$$\begin{aligned} &(T_{(\tilde{x},\dot{\tilde{x}},0,0)}(R_{d,\phi}^T)^2 - T_{(\tilde{x},\dot{\tilde{x}},0,0)}(R_{d,\phi}^T)^1)(\tilde{x}, \dot{\tilde{x}}, \tilde{y}, \dot{\tilde{y}}) \\ &= \left.\frac{d}{ds}\right|_{s=0} [(T\phi \circ (R_d^T)^1 \circ TT\phi^{-1})(\tilde{x}, \dot{\tilde{x}}, s\tilde{y}, s\dot{\tilde{y}}) \\ &- (T\phi \circ (R_d^T)^2 \circ TT\phi^{-1})(\tilde{x}, \dot{\tilde{x}}, s\tilde{y}, s\dot{\tilde{y}})] \\ &= T_{(\tilde{x},\dot{\tilde{x}})}T\phi \left(\left.\frac{d}{ds}\right|_{s=0} \begin{matrix} [(R_d^T)^1(s(TT\phi^{-1})(\tilde{x}, \dot{\tilde{x}}, \tilde{y}, \dot{\tilde{y}})) \\ -(R_d^T)^2(s(TT\phi^{-1})(\tilde{x}, \dot{\tilde{x}}, \tilde{y}, \dot{\tilde{y}}))] \end{matrix} \right) \\ &= T_{(\tilde{x},\dot{\tilde{x}})}T\phi((TT\phi^{-1})(\tilde{x}, \dot{\tilde{x}}, \tilde{y}, \dot{\tilde{y}})) = (\tilde{x}, \dot{\tilde{x}}, \tilde{y}, \dot{\tilde{y}}) \end{aligned}$$

which proves the second condition in 1.1.

Thus, using the linearity of the map $TT\phi$, we prove that $R_{d,\phi}^T$ is indeed a discretization map on TN. ■

A. Commutator diagram

Fig. 2.1 shows how one can move from one space to another using various maps. The commutator on the double tangent space (inner block) yields to us the relation between the tangent lifts of the discretization maps on M and N, which are R_d^T and $R_{d,\phi}^T$, respectively. We also see the relation between the canonical isomorphism on the double tangent space of M and N, which can be related through the canonical involution maps κ_M and κ_N. Due to this, we have different projection maps τ_{TM} and $T\tau_M$ acting on corresponding double tangent spaces onto the tangent space TM. The commutator on the tangent space (outer block) yields us the result from [2] (see Fig. 1.1), which is the relation between the discretization maps on M and N, which are R_d and $R_{d,\phi}$ respectively.

V. Feedback Linearizable Discretizations for SODEs

Let $x \in M$ and $(x, \dot{x}) \in TM$ be the coordinates on the manifold M and the induced coordinates on the tangent bundle of M, respectively. We know that a second-order differential equation is a vector field X such that $\tau_{TM}(X) = T\tau_M(X)$. This implies that the vector field X on TM is a section of the second-order tangent bundle TTM. Locally, if we take coordinates (x^i) on M and induced coordinates $(x^i, \dot{x}^i)$ on TM, then:

$$X = \dot{x}^i \frac{\partial}{\partial x^i} + X^i(x^i, \dot{x}^i) \frac{\partial}{\partial \dot{x}^i} \tag{5.1}$$

To find the integral curves of X is equivalent to solving the SODE:

$$\frac{d^2}{dt^2} x(t) = X\left(x(t), \frac{d}{dt} x(t)\right) \tag{5.2}$$

Now, we wish to discretize this using the notion of the discretization map on TM. We would like to tangently lift a discretization on M to obtain $R_d^T : TTM \longrightarrow TM \times TM$ as defined in Proposition 4.1. This yields the following numerical scheme [1]:

$$\begin{aligned} hX\left(\left(\tau_{TM} \circ \left(R_d^T\right)^{-1}\right)(x_k, y_k; x_{k+1}, y_{k+1})\right) \\ = \left(R_d^T\right)^{-1}(x_k, y_k; x_{k+1}, y_{k+1}) \end{aligned} \tag{5.3}$$

A. Example: Symmetric Discretization

Let us say we choose the midpoint discretization on $N = \mathbb{R}^n$, denoted by R_d of the following form:

$$R_d(\tilde{x}, \tilde{y}) = \left(\tilde{x} - \frac{\tilde{y}}{2}, \tilde{x} + \frac{\tilde{y}}{2}\right) \tag{5.4}$$

for some $(\tilde{x}, \tilde{y}) \in TN$. From Proposition 4.1, we can find the tangent lift of R_d as follows:

$$D_{(\tilde{x},\tilde{y})} R_d(\tilde{x}, \tilde{y}) = \begin{pmatrix} \mathrm{Id} & -\frac{\mathrm{Id}}{2} \\ \mathrm{Id} & \frac{\mathrm{Id}}{2} \end{pmatrix}$$

$$D_{(\tilde{x},\tilde{y})} R_d(\tilde{x}, \tilde{y})(\dot{\tilde{x}}, \dot{\tilde{y}})^T = \left(\dot{\tilde{x}} - \frac{\dot{\tilde{y}}}{2}, \dot{\tilde{x}} + \frac{\dot{\tilde{y}}}{2}\right)$$

$$R_d^T(\tilde{x}, \dot{\tilde{x}}, \tilde{y}, \dot{\tilde{y}}) = \left(\tilde{x} - \frac{\tilde{y}}{2}, \tilde{x} + \frac{\tilde{y}}{2}, \dot{\tilde{x}} - \frac{\dot{\tilde{y}}}{2}, \dot{\tilde{x}} + \frac{\dot{\tilde{y}}}{2}\right) \tag{5.5}$$

which is a discretization on TN.

Now, to lift R_d^T to obtain $R_{d,\phi}^T$, we use Proposition 4.2, which gives:

$$R_{d,\phi}^T = (T\phi \times T\phi)^{-1} \circ R_d^T \circ TT\phi \tag{5.6}$$

which is also a discretization map on TM.

Using the numerical scheme from Equation 5.3, we obtain:

$$\begin{aligned} \frac{x_{k+1} - x_k}{h} &= \frac{y_{k+1} + y_k}{2}, \\ \frac{y_{k+1} - y_k}{h} &= X\left(\frac{x_k + x_{k+1}}{2}, \frac{y_k + y_{k+1}}{2}\right) \end{aligned} \tag{5.7}$$

which is the numerical scheme for a symmetric discretization of the SODE 5.2.

B. MF-Linearizable discretizations

We can also apply the discretization of second-order differential equations for controlled mechanical systems.

Theorem 5.1: Let R_d be a discretization map for the linear mechanical system $((\mathcal{LMS})_{(n,m)})$ given by (3.7) preserving linearity. Then the mechanical control system (2.3) $((\mathcal{MS})_{(n,m)})$ admits, using $R_{d,\phi}$, a discretization that is feedback linearizable.

Proof: Now if we transform the linearized system (3.7)

$$\frac{d^2 \tilde{x}}{dt^2} = A\tilde{x} + \sum_{s=1}^{m} b_s \tilde{u}_s$$

using a discretization map R_d that preserves linearity and defining $\tilde{u}_k = \tilde{u}(t_k)$ we obtain as a discretization the controlled second–order linear difference equation

$$\tilde{x}_{k+2} = \tilde{A}\tilde{x}_k + \tilde{B}\tilde{x}_{k+1} + \sum_{s=1}^{m} b_s(\tilde{u_k})_s \tag{5.8}$$

Now since

$$R_{d,\phi}^T = (T\phi \times T\phi)^{-1} \circ R_d^T \circ TT\phi$$

then the discrete system

$$\begin{aligned} &hX\left(\left(\tau_{TM} \circ \left(R_{d,\phi}^T\right)^{-1}\right)(x_k, y_k; x_{k+1}, y_{k+1}), u_k\right) \\ &= \left(R_{d,\phi}^T\right)^{-1}(x_k, y_k; x_{k+1}, y_{k+1}) \end{aligned}$$

is feedback linearizable to (5.8) where X is the second order system defined in (2.3) and

$$(\tilde{x}_k, \tilde{y}_k) = T\phi(x_k, y_k)$$

$$(u_k)_r = \gamma_{ij}^l(\bar{x}_k)\bar{y}_k^i \bar{y}_k^j + \alpha_l(\bar{x}_k) + \sum_{s=1}^{m} \beta_s^r(\bar{x}_k)(\tilde{u}_k)_s$$

where $(\bar{x}_k, \bar{y}_k) = \tau_M(R_{d,\phi}^{-1}(x_k, y_k; x_{k+1}, y_{k+1}))$ ∎

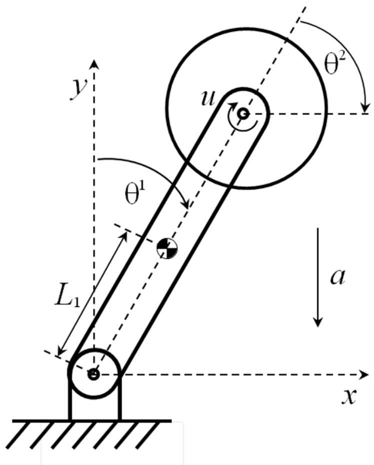

Fig. 6.1. The inertia wheel pendulum

VI. Example

Here, we consider an example: a simple mechanical system - the inertia wheel pendulum. The equations of motion are given by

$$\begin{aligned} \mathfrak{m}_{11}\ddot{\theta}^1 + \mathfrak{m}_{12}\ddot{\theta}^2 + c^1 &= 0 \\ \mathfrak{m}_{21}\ddot{\theta}^1 + \mathfrak{m}_{22}\ddot{\theta}^2 &= u \end{aligned} \tag{6.1}$$

where

$$\begin{aligned} \mathfrak{m}_{11} &= m_d + J_2,\ \mathfrak{m}_{12} = \mathfrak{m}_{21} = \mathfrak{m}_{22} = J_2 \\ m_d &= L_1^2(m_1 + 4m_2) + J_1,\ m_0 = aL_1(m_1 + 2m_2) \\ c^1 &= -m_0 \sin\theta^1 \end{aligned}$$

Taking $(\theta^1, \theta^2) = (x_1, x_2)$ and correspondingly $(\dot{\theta}^1, \dot{\theta}^2) = (y_1, y_2)$, we get the following equations:

$$\begin{aligned} \dot{x}_1 &= y_1,\ \dot{x}_2 = y_2 \\ \dot{y}_1 &= e_1 + g_1 u,\ \dot{y}_2 = e_2 + g_2 u \end{aligned} \tag{6.2}$$

where

$$\begin{aligned} e_1 &= \frac{m_0}{m_d}\sin x_1,\quad g_1 = -\frac{1}{m_d} \\ e_2 &= -\frac{m_0}{m_d}\sin x_1,\quad g_2 = \frac{m_d + J_2}{m_d J_2} \end{aligned}$$

A. MF-Linearization

We will verify $(MD1 - MD3)$ from Proposition 3.2, since the mechanical system here is a planar mechanical system.

First, we calculate:

$$\begin{aligned} \mathrm{ad}_e g &= 0 - \begin{pmatrix} \frac{m_0}{m_d}\cos x^1 & 0 \\ -\frac{m_0}{m_d}\cos x^1 & 0 \end{pmatrix} \begin{pmatrix} -\frac{1}{m_d} \\ \frac{m_d + J_2}{m_d J_2} \end{pmatrix} \\ &= \begin{pmatrix} \frac{m_0}{m_d^2}\cos x^1 \\ -\frac{m_0}{m_d^2}\cos x^1 \end{pmatrix} \end{aligned} \tag{6.3}$$

It can be seen that g and $\mathrm{ad}_e g$ are independent (except at $x_1 = \pm\frac{\pi}{2}$). Thus, $MD1$ is satisfied. To verify $MD2$,

$$\begin{aligned} \nabla_g g &= \left(\frac{\partial g_i}{\partial x_j} g_j + \Gamma^i_{jk} g_j g_k\right)\frac{\partial}{\partial x_i} = 0 \in \mathcal{E}^0 \\ \nabla_{\mathrm{ad}_e g} g &= 0 \in \mathcal{E}^0 \end{aligned} \tag{6.4}$$

which is also verified. Lastly, for $MD3$,

$$\begin{aligned} \nabla^2_{g,\mathrm{ad}_e g}\mathrm{ad}_e g &= \nabla_g \nabla_{\mathrm{ad}_e g}\mathrm{ad}_e g - \nabla_{\nabla_g \mathrm{ad}_e g}\mathrm{ad}_e g \\ &= \begin{pmatrix} \frac{m_0^2}{m_d^5}\cos^2 x_1 \\ -\frac{m_0^2}{m_d^5}\cos^2 x_1 \end{pmatrix} \end{aligned} \tag{6.5}$$

and,

$$\begin{aligned} \nabla^2_{\mathrm{ad}_e g, g}\mathrm{ad}_e g &= \nabla_{\mathrm{ad}_e g}\nabla_g \mathrm{ad}_e g - \nabla_{\nabla_{\mathrm{ad}_e g} g}\mathrm{ad}_e g \\ &= \begin{pmatrix} \frac{m_0^2}{m_d^5}\cos^2 x_1 \\ -\frac{m_0^2}{m_d^5}\cos^2 x_1 \end{pmatrix} \end{aligned} \tag{6.6}$$

Hence, from Equations 6.5, 6.6, we have:

$$\nabla^2_{g,\mathrm{ad}_e g}\mathrm{ad}_e g - \nabla^2_{\mathrm{ad}_e g, g}\mathrm{ad}_e g = 0 \in \mathcal{E}^0 \tag{6.7}$$

Therefore, all the conditions $(MD1 - MD3)$ are satisfied and the given system is MF-Linearizable.

We have the diffeomorphism $\Phi(x, y) = (\phi(x), D\phi(x)y)$, which is given by:

$$\begin{aligned} \tilde{x}_1 &= \frac{m_d + J_2}{J_2}x_1 + x_2,\ \tilde{x}_2 = \frac{m_0}{J_2}\sin x_1 \\ \tilde{y}_1 &= \frac{m_d + J_2}{J_2}y_1 + y_2,\ \tilde{y}_2 = \frac{m_0}{J_2}\cos x_1 y_1 \end{aligned} \tag{6.8}$$

Taking $\tilde{\mathrm{x}} = \begin{pmatrix}\tilde{x}_1 & \tilde{x}_2 & \tilde{y}_1 & \tilde{y}_2\end{pmatrix}^T$, such that the linearized equations become:

$$\frac{d}{dt}\tilde{\mathrm{x}} = A\tilde{\mathrm{x}} + C\tilde{u} \tag{6.9}$$

Here, the matrices $A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, $C = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$ and $\tilde{u} = \psi(x, y, u)$ is the auxiliary control, such that:

$$\tilde{u} = -\frac{m_0}{J_2}\sin x_1 y_1^2 + \frac{m_0^2}{2m_d J_2}\sin 2x_1 - \frac{m_0}{m_d J_2}\cos x_1 u \tag{6.10}$$

From Theorem 5.1 using for instance the symmetric map R_d, we obtain the discretization $R_{d,\phi}$ and the corresponding linearizable discretization of the initial system:

$$\begin{aligned} &\frac{x_{1,k+1} - x_{1,k}}{h} + \frac{J_2}{m_d + J_2}\left(\frac{x_{2,k+1} - x_{2,k}}{h} - \frac{y_{2,k+1} + y_{2,k}}{2}\right) \\ &= y_{1,k+1/2} \\ &\frac{\sin x_{2,k+1} - \sin x_{2,k}}{h} = \frac{y_{1,k}\cos x_{1,k} + \sin x_{2,k} y_{1,k}\cos x_{1,k}}{2} \\ &\frac{m_d + J_2}{J_2}\frac{y_{1,k+1} - y_{1,k}}{h} + \frac{y_{2,k+1} - y_{2,k}}{h} = \frac{m_0}{J_2}\sin x_{k+1/2} \\ &\frac{m_0}{hJ_2}\left(y_{1,k+1}\cos x_{1,k+1} - y_{1,k}\cos x_{1,k}\right) = \tilde{u}_k \end{aligned}$$

where

$$\begin{aligned} \tilde{u}_k = &-\frac{m_0}{J_2}\sin x_{1,k+1/2} y_{1,k+1/2}^2 + \frac{m_0^2}{2m_d J_2}\sin 2x_{1,k+1/2} \\ &- \frac{m_0}{m_d J_2}\cos x_{1,k+1/2} u_k \end{aligned}$$

B. Stabilization

We use the classical pole placement technique to obtain a control gain matrix K, such that $\tilde{u} = -K\tilde{x}$.

Let us choose the poles of the closed-loop system to be:

$$\lambda = -10, -20, -30, -40 \tag{6.11}$$

Correspondingly, we obtain

$$K = \begin{bmatrix} 240000 & 3500 & 50000 & 100 \end{bmatrix} \tag{6.12}$$

We denote $\mathrm{x} = \begin{bmatrix} x_1 & x_2 & y_1 & y_2 \end{bmatrix}^T$ to get:

$$\frac{d}{dt}\mathrm{x} = (A - CK)\,\mathrm{x} \tag{6.13}$$

C. Discretization

If we have the system as $\dot{\mathrm{x}} = (A - CK)\mathrm{x} = F(\mathrm{x})$. Let h denote a (fixed) sampling time and $h' = \frac{h}{2}$. We utilize the symmetric discretization formulated in Section V:

$$\begin{aligned} F(\mathrm{x}_k; h/2) &= F(\mathrm{x}_{k+1}; -h/2) \\ \mathrm{x}_k + h'(A-CK)\mathrm{x}_k &= \mathrm{x}_{k+1} - h'(A-CK)\mathrm{x}_{k+1} \\ \therefore \mathrm{x}_{k+1} &= (I - h'(A-CK))^{-1}(I + h'(A-CK))\mathrm{x}_k \end{aligned} \tag{6.14}$$

D. Results

We use the following parameters from [7] and [12]:

$$\begin{aligned} L_1 &= 0.063 \ [m] \\ m_1 &= 0.02 \ [kg] \\ m_2 &= 0.3 \ [kg] \\ J_1 &= 47 \cdot 10^{-6} \ [kg \cdot m^2] \\ J_2 &= 32 \cdot 10^{-6} \ [kg \cdot m^2] \\ a &= 9.81 \ [ms^{-2}] \\ m_0 &= 0.3832 \ [kg \cdot m^2 s^{-2}] \\ m_d &= 49 \cdot 10^{-4} \ [kg \cdot m^2] \end{aligned} \tag{6.15}$$

The comparison results between the proposed discretization scheme and ODE45 for the system, for a sampling time of $h = 0.01$, and initial conditions $\theta^1(0) = \frac{\pi}{4}, \theta^2(0) = \dot{\theta}^1(0) = \dot{\theta}^2(0) = 0$ are shown in Figure 6.2. The corresponding errors are plotted in Figures 6.5 and 6.6. In Figure 6.7, we compare the (percentage) relative error $100 \times \frac{\|e(t_k)\|}{\|x(t_k)\|}$.

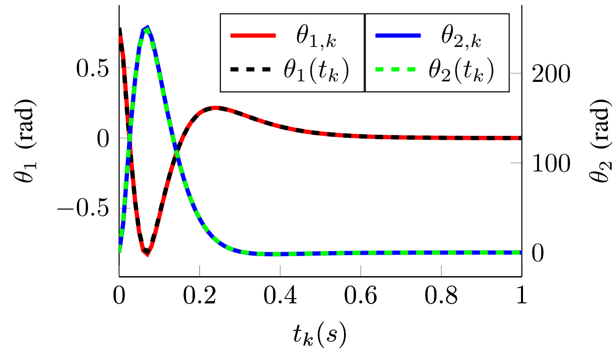

Fig. 6.2. System states x_k for symmetric discretization plotted against exact discretization (ODE45) $x(t_k)$ for $t_k \in [0, 1]$

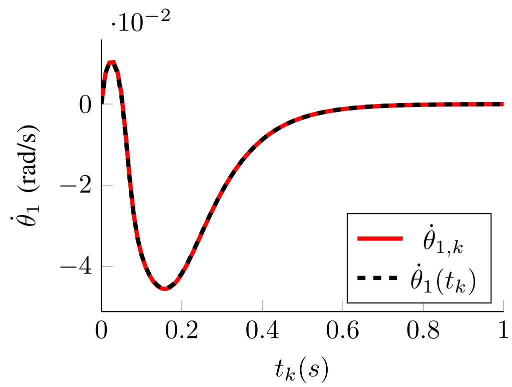

Fig. 6.3. $\dot{\theta}_{1,k}$ plotted against exact discretization (ODE45) $\dot{\theta}_1(t_k)$

VII. Conclusions

This paper provides a theoretical basis for further developments in feedback linearizable discretizations of second-order mechanical systems. In a forthcoming paper, we plan to propose a method to functionally compose discretizations to obtain higher-order integrators, using multi-rate sampling, that is feedback linearizable (see [2]). Moreover, we plan to analyze discrete Sundman transformations in this setting.

VIII. Acknowledgements

We acknowledge the work on feedback linearization by W. Respondek, Nowicki, et al., which has been significantly used in this article in Sections II, III to develop the work on retraction maps and discretizations in the context of MF-linearization.

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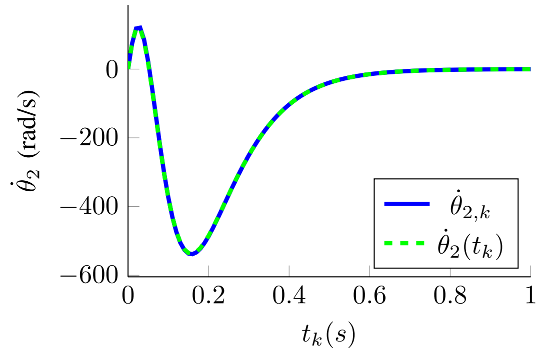

Fig. 6.4. $\dot{\theta}_{2,k}$ plotted against exact discretization (ODE45) $\dot{\theta}_2(t_k)$

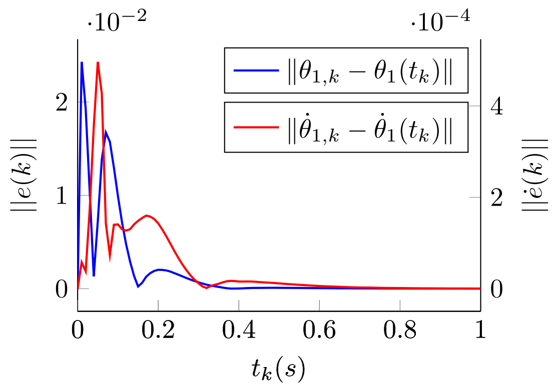

Fig. 6.5. Magnitude of error norm for θ_1 and $\dot{\theta}_1$

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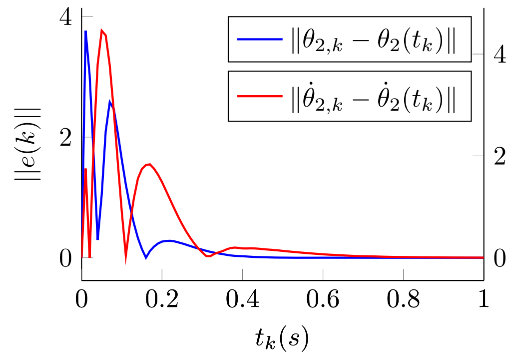

Fig. 6.6. Magnitude of error norm for θ_2 and $\dot{\theta}_2$

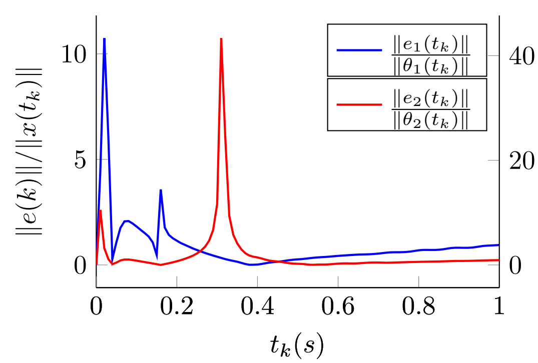

Fig. 6.7. Comparing percentage relative error in the proposed discretization, w.r.t ODE45 for $t_k \in [0, 1]$ and $h = 10^{-2}$