Consider the rigid body dynamics

$$\dot{R}(t) = R(t)\,\Omega^{\wedge}(t) \longrightarrow (1)$$

where $\Omega \in \mathbb{R}^3$, $\Omega^{\wedge} \in so(3)$

$\hat{(\cdot)}$ — skew symmetric operator
$\check{(\cdot)}$ — "vectorize" operator (inverse).

Let $J \in \mathbb{R}^{3\times 3}$ be the inertia matrix and $\tau(t) \in \mathbb{R}^3$ be the torque inputs in the body axis.

Then we have

$$J\dot{\Omega}(t) = -\Omega(t) \times J\Omega(t) + \tau(t)$$

$$\Rightarrow \quad \dot{\Omega}(t) = J^{-1}\left(\tau(t) - \Omega \times J\Omega(t)\right) \longrightarrow (2)$$

We may simplify this using $\tau(t) = \Omega(t) \times J\Omega(t) + Ju(t) \longrightarrow (3)$

such that $\quad \dot{\Omega}(t) = u(t) \longrightarrow (4)$

Thus we have the rigid body dynamics:

$$\boxed{\begin{aligned} \dot{R} &= R\,\Omega^{\wedge} \\ \dot{\Omega} &= u \end{aligned}}$$

(fully actuated)

set of two 1st order eqs.

Step 1: Obtaining the control affine form

In order to represent this in the form of a control affine nonlinear system, we vectorize the matrix R using a diffeomorphism $\mathcal{V} : \mathbb{R}^{3\times 3} \to \mathbb{R}^9$

$$\begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \longmapsto \begin{bmatrix} r_{11} \\ r_{21} \\ r_{31} \\ \vdots \\ r_{13} \\ r_{23} \\ r_{33} \end{bmatrix}$$

Let $\Omega = [p, q, r]^T$ be the angular velocity.

Now, $\mathcal{V}(\dot{R}) = \mathcal{V}(R\Omega^\wedge)$

$$\Rightarrow \left[\begin{array}{lcl} \dot{r}_{i1} = r_{i2}r - r_{i3}q &,& \dot{p} = u_1 \\ \dot{r}_{i2} = r_{i3}p - r_{i1}r &,& \dot{q} = u_2 \\ \dot{r}_{i3} = r_{i1}q - r_{i2}p &,& \dot{r} = u_3 \end{array}\right]$$

$\forall\, i = \{1, 2, 3\}$

$\therefore$ If we select our state

$x = [\mathcal{V}(R), \Omega]^T \in \mathbb{R}^{12}$, then

$$f(x) = \begin{bmatrix} r_{i2}r - r_{i3}q \\ r_{i3}p - r_{i1}r \\ r_{i1}q - r_{i2}p \\ 0_{3,1} \end{bmatrix}, \quad g(x) = \begin{bmatrix} 0_{9,3} \\ I_3 \end{bmatrix}$$

$$\Rightarrow \quad \underline{\dot{x} = f(x) + g(x)u}$$

Step 2: The feedback linearizing diffeomorphism

Let $z(t) = [\xi, \eta]^T$ such that $\quad z(t) \in \mathbb{R}^6$

$$\boxed{\xi = (\mathrm{Log}(R))^\vee, \quad \eta = \Omega} \qquad \xi, \eta \in \mathbb{R}^3$$

$$\Rightarrow \quad \dot{\xi} = \Omega, \quad \dot{\eta} = v$$

$$\therefore \quad \dot{z}(t) = \underbrace{\begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix}}_{A} z(t) + \underbrace{\begin{bmatrix} 0 \\ I \end{bmatrix}}_{B} v \qquad \boxed{FL}$$

Note that here, the inverse diffeomorphism yields

$$\boxed{R = \exp(\xi^\wedge) \quad , \quad \Omega = \eta}$$

We may also simply design a stabilizer by choosing $v = -Kz$

$$\Rightarrow \quad \dot{z}(t) = (A - BK) z(t)$$

$$(\xi \to 0 \ , \ \eta \to 0) \Leftrightarrow (R \to I, \Omega \to 0)$$

If $K = [k_1 \quad k_2]$,

$$u(t) = -k_1 (\mathrm{Log}(R))^\vee - k_2 \Omega(t)$$

$$\Rightarrow \boxed{\tau(t) = \Omega(t) \times J\Omega(t) - k_1 J (\mathrm{Log}(R(t)))^\vee - k_2 J\Omega(t)}$$

stabilizing torque input

for some k_1, k_2.

Step 3: Choose the discretization map on $\mathcal{N}$

Let us choose Forward Euler:

$$z_{k+1} = (I + h(A - BK)) z_k \qquad h - \text{step size}$$

$$z_k = \begin{bmatrix} \xi_k \\ \eta_k \end{bmatrix} \quad , \quad R_k = \exp(\xi_k^\wedge) \ , \ \Omega_k = \eta_k$$

$$\xi_{k+1} = \xi_k + h\eta_k$$

$$\eta_{k+1} = \eta_k - hk_1\xi_k - hk_2\eta_k$$

Step 4: Lift it back to obtain the FL discretization

$$\mathrm{Log}(R_{k+1})^\vee = \mathrm{Log}(R_k)^\vee + h\Omega_k$$

$$\Rightarrow \mathrm{Log}(R_{k+1}) = \mathrm{Log}(R_k) + h\hat{\Omega}_k$$

$$\therefore \boxed{\begin{aligned} R_{k+1} &= R_k \exp(h\hat{\Omega}_k) \quad ** \\ \Omega_{k+1} &= \Omega_k - hk_1 \mathrm{Log}(R_k)^\vee - hk_2\Omega_k \end{aligned}}$$

Remarks

* Must be noted that the problem is simplified to make sure that the second-order differential eqn. satisfies the Mechanical Feedback Linearizability conditions (in paper):

If $\alpha(R) = (\mathrm{Log}(R))^\vee$, $\alpha : SO(3) \to \mathbb{R}^3$, $\alpha(0) = I$,

1) $\xi = \alpha(R)$, $\eta = \mathcal{L}_f \alpha(R) = \Omega$
2) $\ker(\mathcal{L}_g \mathcal{L}_f \alpha) = 0$

* Strictly $R(t) \in SO(3) \setminus \{\mathrm{diag}(1,-1,-1), \mathrm{diag}(-1,1,-1), \mathrm{diag}(-1,-1,1)\}$

since these are Euler angle singularities. [body is completely flipped 180°]

Trajectory tracking in dynamic FL for $SO(3)$?

Inclusion of state estimation in the above?