SC 625 - Systems Theory

Assignment 3

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Problem 1.

Solution. ☐

$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

a)

$$W_c(0, 2\pi) = \int_0^{2\pi} \Phi(0,t) B B^T \Phi^T(0,t) dt$$

We have $\Phi(0,t) = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix}$ and $BB^T = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

Hence, the controllability Grammian is given by

$$W_c(0, 2\pi) = \int_0^{2\pi} \begin{bmatrix} \frac{1}{2}(\cos 2t - 1) & -\frac{1}{2}\sin 2t \\ \frac{1}{2}\sin 2t & \frac{1}{2}(\cos 2t + 1) \end{bmatrix} dt$$

$$\therefore W_c(0, 2\pi) = \begin{bmatrix} \pi & 0 \\ 0 & \pi \end{bmatrix}$$

b)

$$W_c(0, 2\pi)\eta_0 = x_0 - \Phi(0, 2\pi) x_1$$

$$\begin{bmatrix} \pi & 0 \\ 0 & \pi \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\therefore \eta_1 = \frac{1}{\pi}, \eta_2 = 0$$

Now, we know that the control law is given by $u(t) = -B^T \Phi^T(0,t)\eta_0$. Therefore,

$$u(t) = -\begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix} \begin{bmatrix} \frac{1}{\pi} \\ 0 \end{bmatrix}$$

$$\therefore u(t) = \frac{1}{\pi}\sin t$$

Thus, there exists a control u which transfers the state from $x_0 = \begin{bmatrix} 1 & 0 \end{bmatrix}^T$ to $x_1 = \begin{bmatrix} 0 & 0 \end{bmatrix}^T$ in time $t = 2\pi$.

c)

Problem 2.

Solution. □

$A = \begin{bmatrix} 6 & 4 & 1 \\ -5 & -4 & 0 \\ 0 & -3 & -1 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

First, let us compute the controllability matrix $C = \begin{bmatrix} B & AB & A^2B \end{bmatrix}$. Thus, we obtain:

$$C = \begin{bmatrix} 1 & 6 & 16 \\ 0 & -5 & -10 \\ 0 & 0 & 15 \end{bmatrix}$$

Now, we know that the controllability matrix is full rank if and only if the system is controllable. Thus, we can see that the system is controllable. To perform a similarity transformation to the controllable canonical form, we need to find the transformation matrix T such that $TAT^{-1} = A_c$ and $TB = B_c$.

Let us find C^{-1}. $C^{-1} = \dfrac{1}{75} \begin{bmatrix} 75 & 90 & -20 \\ 0 & -15 & -10 \\ 0 & 0 & 5 \end{bmatrix}$

Consider the last row of C^{-1} as $q = \begin{bmatrix} 0 & 0 & \frac{1}{15} \end{bmatrix}$. We can construct the transformation matrix T as follows:

$$T = \begin{bmatrix} q^T \\ q^T A \\ q^T A^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & \frac{1}{15} \\ 0 & -\frac{1}{5} & -\frac{1}{15} \\ 1 & 1 & \frac{1}{15} \end{bmatrix}$$

Thus,

$$A_c = TAT^{-1} = \begin{bmatrix} 6 & 4 & 1 \\ -5 & -4 & 0 \\ 0 & -3 & -1 \end{bmatrix} \begin{bmatrix} 4 & 5 & 1 \\ -5 & -5 & 0 \\ 15 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & \frac{1}{15} \\ 0 & -\frac{1}{5} & -\frac{1}{15} \\ 1 & 1 & \frac{1}{15} \end{bmatrix}$$

$$\implies A_c = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 19 & 6 & 1 \end{bmatrix}$$

Similarly we have $B_c = TB$

$$\implies B_c = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Thus, (A_c, B_c) is the controllable canonical form of the system.

Now, we need to find K such that the eigenvalues of $A + BK$ are $-1, -1, -1$. Since we have a similarity transformation from (A, B) to (A_c, B_c), we can find K_c such that the eigenvalues of $A_c + B_c K_c$ are $-1, -1, -1$. Here $K_c = KT^{-1}$. Let $K = \begin{bmatrix} k_1 & k_2 & k_3 \end{bmatrix}^T$. Thus, we have

$$K_c = \begin{bmatrix} k_1 & k_2 & k_3 \end{bmatrix} \begin{bmatrix} 4 & 5 & 1 \\ -5 & -5 & 0 \\ 15 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 4k_1 - 5k_2 + 15k_3 & 5k_1 - 5k_2 & k_1 \end{bmatrix}$$

$$\implies A_c + B_c K_c = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 19 & 6 & 10 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 4k_1 - 5k_2 + 15k_3 & 5k_1 - 5k_2 & k_1 \end{bmatrix}$$

$$\implies A_c + B_c K_c = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 19 + 4k_1 - 5k_2 + 15k_3 & 6 + 5k_1 - 5k_2 & 1 + k_1 \end{bmatrix}$$

We require that the characterstic polynomial of this matrix be $(\lambda + 1)^3$. Thus, we have

$$\lambda^3 + 3\lambda^2 + 3\lambda + 1 = 0$$

We also have the actual characterstic polynomial of the matrix $A_c + B_c K_c$ as

$$\lambda^3 - (1 + k_1)\lambda^2 + (6 + 5k_1 - 5k_2)\lambda + (19 + 4k_1 - 5k_2 + 15k_3) = 0$$

Comparing the coefficients of the two polynomials, we obtain the following equations:

$$1 + k_1 = -3$$

$$6 + 5k_1 - 5k_2 = -3$$

$$19 + 4k_1 - 5k_2 + 15k_3 = -1$$

Solving these equations, we obtain $k_1 = -4, k_2 = \frac{11}{5}, k_3 = \frac{7}{15}$.

Thus, we have

$$K = \begin{bmatrix} -4 & \frac{11}{5} & \frac{7}{15} \end{bmatrix}$$

, which is the required feedback gain matrix for the closed loop system to have eigenvalues $-1, -1, -1$.

Problem 3.

Solution. □

$$A = \begin{bmatrix} -1 & -1 & -4 \\ 3 & 3 & 4 \\ -3 & 1 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

a) The controllability matrix is given by $C = \begin{bmatrix} B & AB & A^2B \end{bmatrix}$. Thus, we obtain

$$C = \begin{bmatrix} 0 & -1 & -6 \\ 1 & 3 & 10 \\ 0 & 1 & 6 \end{bmatrix}$$

We can see that the rank of C is 2 (first and last row are linearly dependent). Thus, the system is not controllable.

b) We can say that the system is stabilizable iff rank $\begin{bmatrix} A - \lambda I & B \end{bmatrix} = 3$ for all $\lambda \in \mathbb{C}, Re(\lambda) \geq 0$.

$$\therefore \begin{bmatrix} A - \lambda I & B \end{bmatrix} = \begin{bmatrix} -1-\lambda & -1 & -4 & 0 \\ 3 & 3-\lambda & 4 & 1 \\ -3 & 1 & -\lambda & 0 \end{bmatrix}$$

The eigenvalues of A are $-1, 1 \pm 2i$. So, if we satisfy the condition at $\lambda = 1 \pm 2i$, the system will be stabilizable.

$$\therefore \begin{bmatrix} A - (1+2i)I & B \end{bmatrix} = \begin{bmatrix} -2-2i & -1 & -4 & 0 \\ 3 & 2-2i & 4 & 1 \\ -3 & 1 & -1-2i & 0 \end{bmatrix}$$

We can see that the rank of this matrix is 3. Similarly, we get that the rank of the matrix when we use $\lambda = 1 - 2i$ is also 3.

Thus, the system is stabilizable from the PBH test.

c) If $A + BK$ should be Hurwitz, then the eigenvalues of $A + BK$ should have negative real parts. Thus, we need to find K such that the eigenvalues of $A + BK$ have negative real parts.

Let $K = \begin{bmatrix} k_1 & k_2 & k_3 \end{bmatrix}$. Thus, we have

$$A + BK = \begin{bmatrix} -1 & -1 & -4 \\ 3 & 3 & 4 \\ -3 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} k_1 & k_2 & k_3 \end{bmatrix}$$

$$\implies A + BK = \begin{bmatrix} -1 & -1 & -4 \\ 3+k_2 & 3+k_2 & 4+k_3 \\ -3 & 1 & 0 \end{bmatrix}$$

We need to find K such that the eigenvalues of this matrix have negative real parts. The eigenvalues of the matrix are given by:

$$\lambda = -4, 3 + \frac{k_2 \pm \sqrt{k_2^2 - 4k_1 + 4k_3 + 4}}{2}$$

We need to find K such that the real parts of the eigenvalues are negative.

If $k_2^2 - 4k_1 + 4k_3 + 4 < 0$, then the real part of the eigenvalues becomes $3 + \frac{k_2}{2}$. For this to be negative, we need $k_2 < -6$. Therefore, let us choose $k_2 = -7$. In order to satisfy the first constraint, we have $49 - 4k_1 + 4k_3 + 4 < 0$. Thus, we have $k_1 - k_3 > 13.25$.

Hence, one choice of K is

$$K = \begin{bmatrix} 14 & -7 & 0 \end{bmatrix}$$

.

This choice of K makes the real part of the eigenvalues of $A + BK$ negative. Thus, $A + BK$ is Hurwitz.

Problem 4.

Solution. □

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & g & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 2g & 0 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

a) The controllability matrix is given by $C = \begin{bmatrix} B & AB & A^2B & A^3B \end{bmatrix}$. Thus, we obtain

$$C = \begin{bmatrix} 0 & 1 & 0 & g \\ 1 & 0 & g & 0 \\ 0 & 1 & 0 & 2g \\ 1 & 0 & 2g & 0 \end{bmatrix}$$

We can see that the rank of C is 4. Thus, the system (A, B) is controllable.

b) Using the Lyapunov test, we need to find K such that all the eigenvalues of $A + BK$ have real part less than -1.

We have $AW + WA^T = -BB^T$, where W is some positive definite matrix. We need to find K such that $A + BK$ has eigenvalues with real part less than -1.

Let us initially consider $W = \int_0^\infty e^{A^T t} BB^T e^{At} dt$. But in the term $e^{A^T t} BB^T e^{At}$ we will have diverging terms as $t \to \infty$. Thus, we need to choose a different W.

Let us consider $\bar{A} = -\mu I - A$, where $\mu > 0$. On considering $W = \int_0^\infty e^{\bar{A}^T t} BB^T e^{\bar{A} t} dt$, we observe that in order to get a positive definite matrix W, we would need to choose $\mu > \frac{7\sqrt{10}}{5}$.

Thus, let us choose $\mu = 5$. This naturally ensures that the eigenvalues for the closed loop system are less than -1 (since they are less than -5).

Thus, we get a positive definite matrix W which satisfies $AW + WA^T = -BB^T$.

Correspondingly, we choose $K = -\frac{1}{2} B^T W^{-1}$. Thus, we have

$$K = \begin{bmatrix} 4020g & 1804g & -6960g & -2000g \end{bmatrix}$$

c) The simulations for the system for $F = K \begin{bmatrix} x & \dot{x} & \theta & \dot{\theta} \end{bmatrix}^T$ are shown below:

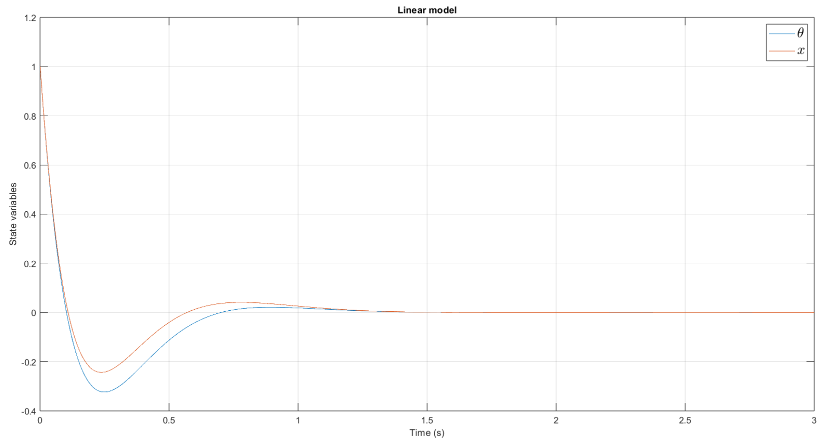

Figure 1: Initial conditions $x(0) = \dot{x}(0) = \theta(0) = \dot{\theta}(0) = 1$

Hence, it is verified from the simulations that the state trajectory of the LTI model for the given initial conditions converge to 0

d) The simulations for the nonlinear model are shown below:

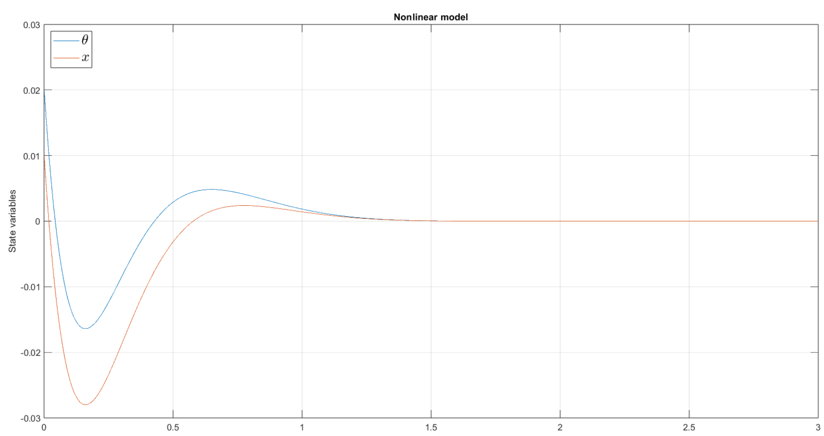

Figure 2: Initial conditions $x(0) = 0.01, \dot{x}(0) = 0.01, \theta(0) = 0.02, \dot{\theta}(0) = 0.02$

Thus, the nonlinear model also converges to zero for small initial states for the same state-feedback control law.

For the same nonlinear model, if we give initial states $x(0) = \dot{x} = \theta(0) = \dot{\theta}(0) = 1$, we can observe that the system does not converge to zero.