# Indian Institute of Technology Bombay

Project Report

2023-2024

---

# AE 308
# Control Theory

---


*Name:* Shreyas N B

*Roll No.:* 210010061

*Course Instructor:* Prof. Arnab Maity

# 1 Introduction

$$G(s) = \frac{K}{s(s+5)^2}$$

Performance parameters of the system:

Finding the cross-over frequencies:

$$|G(j\omega_{gc})| = \frac{1}{\sqrt{100\omega_{gc}^4 + (25\omega_{gc} - \omega_{gc}^3)^2}} = 1$$

$$\omega_{gc}^6 + 50\omega_{gc}^4 + 625\omega_{gc}^2 - 1 = 0$$

$$\boxed{\omega_{gc} = 0.04 \text{ rad/s}}$$

$$\angle G(j\omega_{pc}) = -90^\circ - 2\tan^{-1}\frac{\omega_{pc}}{5} = -180^\circ$$

$$\boxed{\omega_{pc} = 5 \text{ rad/s}}$$

Finding the margins:

$$GM = 20\log_{10}|G(j\omega_{pc})| = -20.0\log_{10}(0.004) \text{ dB}$$

$$\boxed{GM = 47.9588 \text{ dB}}$$

$$PM = 180^\circ + \angle G(j\omega_{gc}) = 90^\circ - 2\tan^{-1}\left(\frac{0.04}{5}\right)$$

$$\boxed{PM = 89.083^\circ}$$

Following are the plots of the true model:

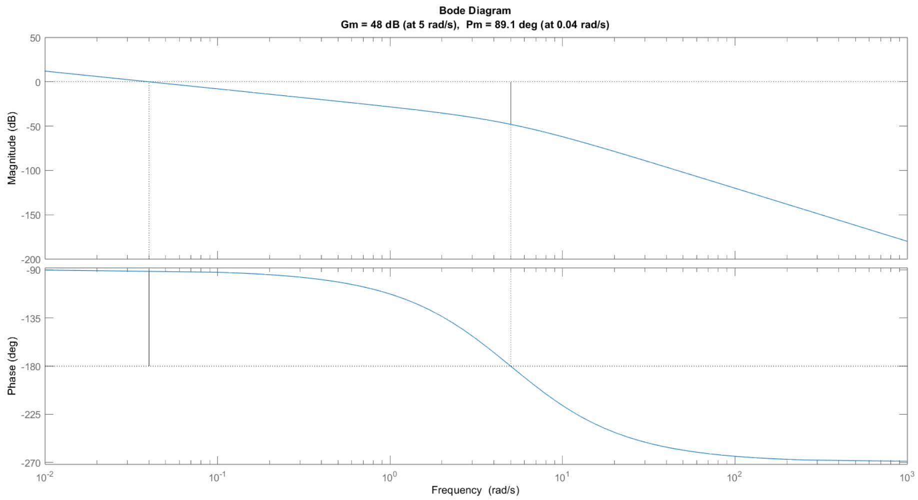


Figure 1: Bode Plot

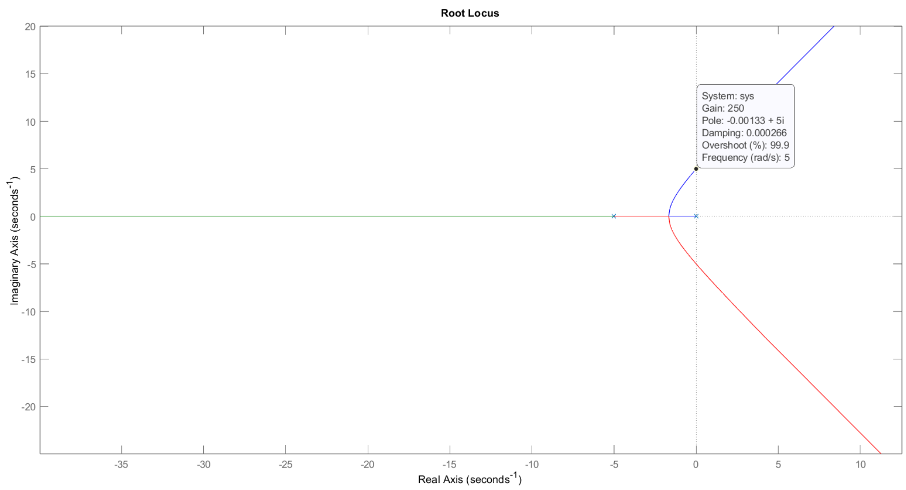


Figure 2: Root Locus Plot

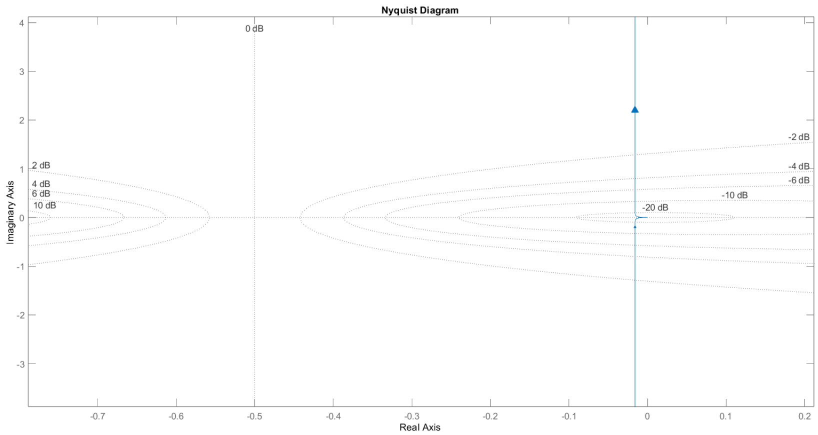


Figure 3: Nyquist Plot

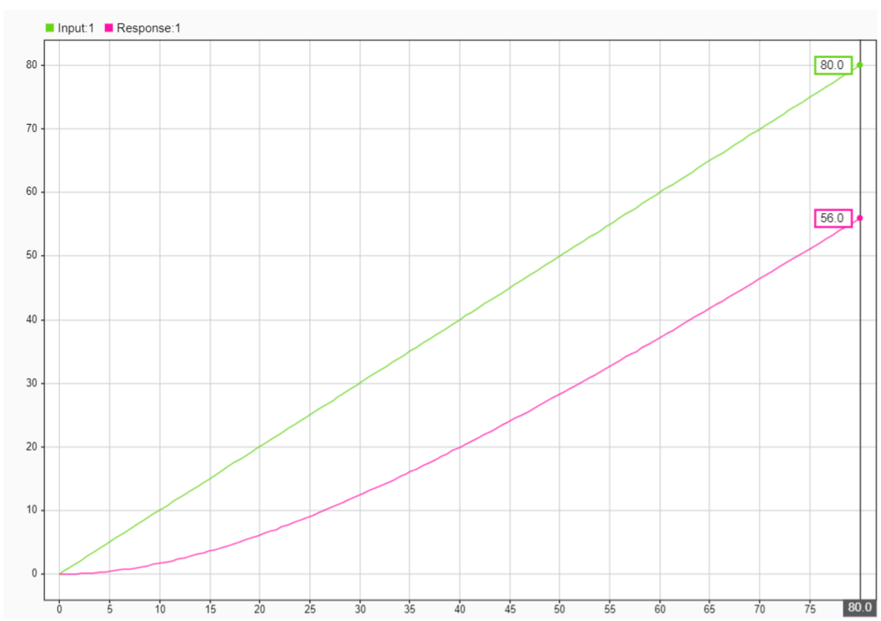


Figure 4: Ramp Response for $K = 1$

As we can see, we need to reduce the steady state error using a controller. The uncontrolled system has a $K_v = \frac{1}{25}$. This implies the steady state error is

$$e_{ss} = \lim_{s \to 0} \frac{R(s)}{1 + G(s)}$$

For ramp input, $R(s) = \frac{1}{s^2}$

$$\therefore e_{ss} = 25$$

This is seen in the ramp response plot too $(80 - 55 = 25)$.

# 2 Control Objectives

We need to design a PD controller such that the following specifications are met:

- The ramp-error constant $K_v = 10$.
- $PM \geq 70°$

# 3 Controller Design

To meet the first specification, we need to have the ramp error constant $K_v = 10$. The ramp error constant is given by:

$$K_v = \lim_{s \to 0} sG(s) = \lim_{s \to 0} s\frac{K}{s(s+5)^2} = \frac{K}{25} = 10$$

$$\implies \boxed{K = 250}$$

Calculating the gain and phase margins:

$$|G(j\omega_{gc})| = \frac{250}{\sqrt{100\omega_{gc}^4 + (25\omega_{gc} - \omega_{gc}^3)^2}} = 1$$

$$\omega_{gc}^6 + 50\omega_{gc}^4 + 625\omega_{gc}^2 - 250 = 0$$

$$\boxed{\omega_{gc} = 5 \text{ rad/s}}$$

$$\angle G(j\omega_{pc}) = -90° - 2\tan^{-1}\frac{\omega_{pc}}{5} = -180°$$

$$\boxed{\omega_{pc} = 5 \text{ rad/s}}$$

$$GM = -20\log_{10}|G(j\omega_{pc})| = 20.0\log_{10}(250) - 20.0\log_{10}(250) \text{ dB} = 0 \text{ dB}$$

$$\boxed{GM = 0 \text{ dB}}$$

$$PM = 180° + \angle G(j\omega_{gc}) = 90° - 2\tan^{-1}\left(\frac{5}{5}\right)$$

$$\boxed{PM = 0°}$$

Let us see the frequency response and margins of the system with $K = 250$:

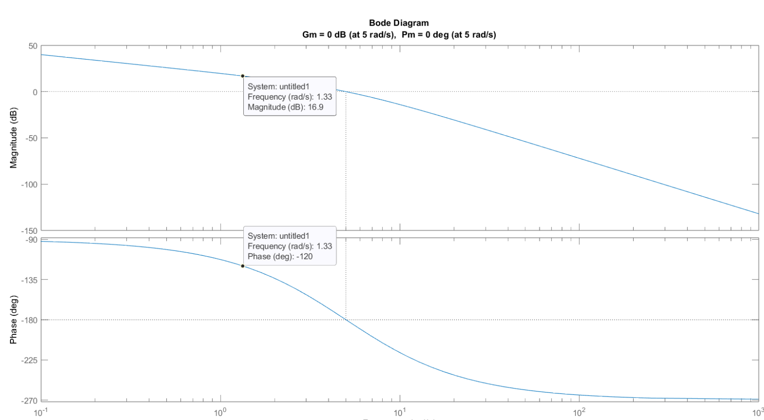


Figure 5: Bode Plot

Since our system is of Type 1 and we wish to have a finite value of $K_v$, we are restricted to using $P$ or $PD$ controller. In order to change the phase margins as required without affecting overall DC gain, it is optimal to try and implement a PD controller of the following form:

$$G_{PD}(s) = 1 + K_d s$$

Our current phase margin is $0°$. Since we require a phase margin of greater than or equal to $70°$, we can add a margin of $10°$ and add a total phase of $80°$.

The frequency at the required phase margin is $1.33 rad/s$ and the corresponding gain is $16.9 dB$.

Hence, we need to add $80°$ at $\omega = 5 rad/s$.

$$\implies \angle(1 + K_d j\omega)_{\omega=5} = \tan(80°)$$

$$5K_d = \tan(80°)$$

$$\boxed{K_d = 1.134}$$

Let us see if this satisfies our specifications by plotting the Bode plot and checking the margins.

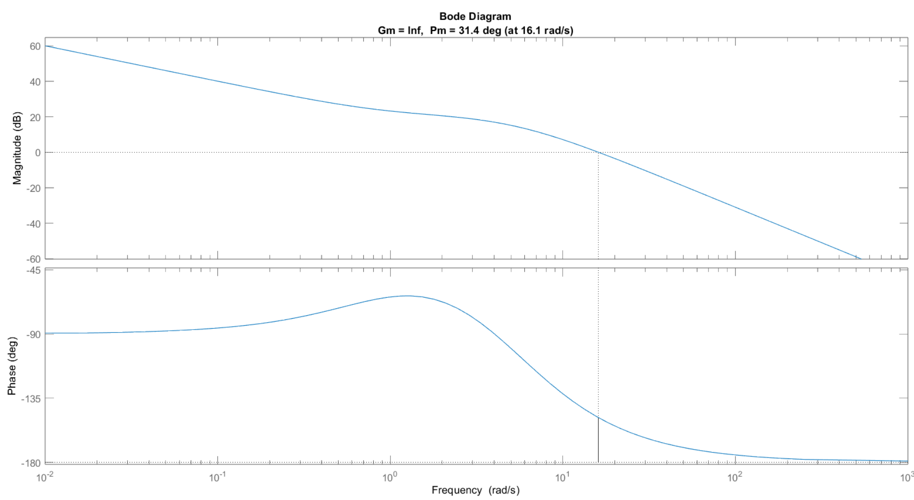


Figure 6: Bode Plot

As we can see, the phase margin we get is $31.4°$ which does not satisfy our specification. In fact, the maximum phase margin we can get with a PD controller is for when $T_d = 0.33$ and corresponding $PM = 43.5°$.

This is because the maximum phase margine which can be added to a system is typically around $55°$ using a single controller/compensator.

So what do we do? Does this mean we can never achieve the specification? Even changing the controller to a compensator would not help as the maximum phase margin we can add will still be $55°$.

Hence, we need to add a second controller in series with the first one. This is called a **cascade controller**.

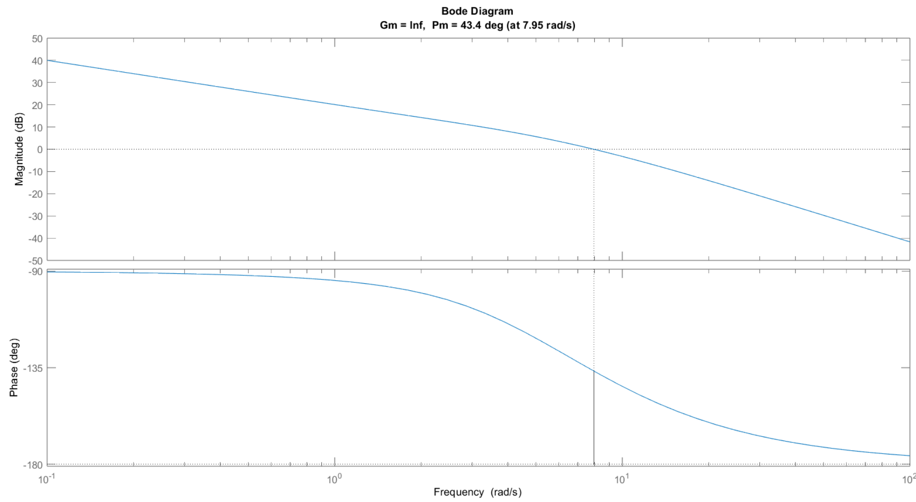


Figure 7: Bode Plot with maximum possible phase margin

Cascading the same PD controller with itself, we get the following transfer function:

$$G_{PD}^2(s) = (1 + T_d s)^2 = 1 + 2T_d s + T_d^2 s^2$$

Using $T_d = 1.134$ and closing the loop, we get the following closed loop system:

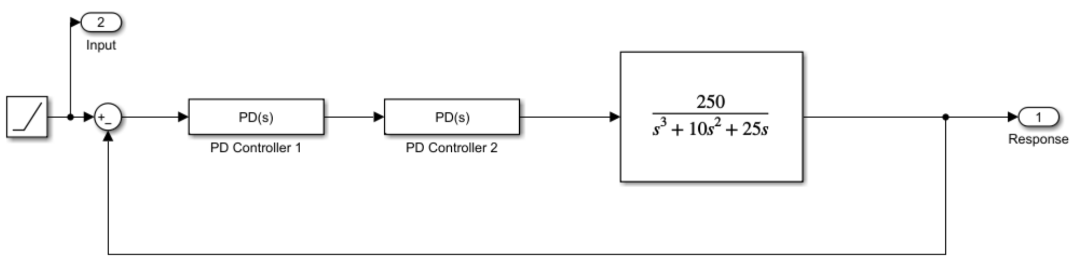


Figure 8: Closed loop system

# 4 Simulation Results

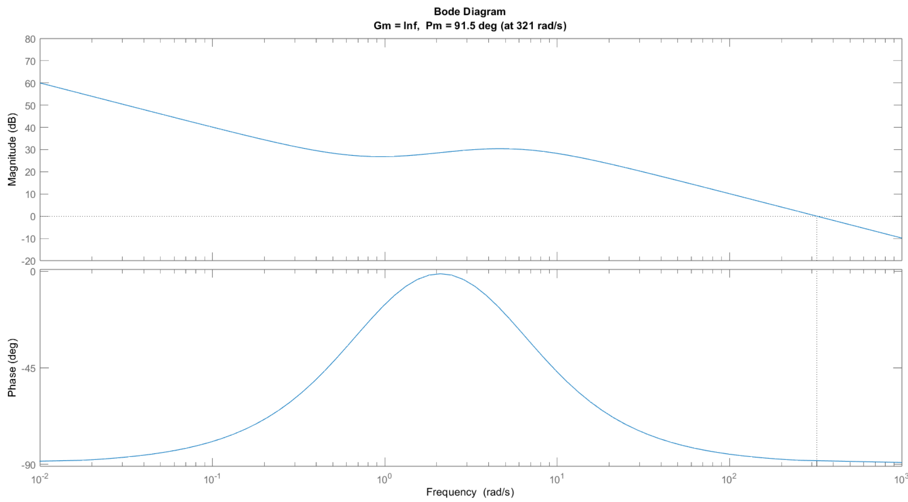


Figure 9: Bode Plot using cascaded controllers

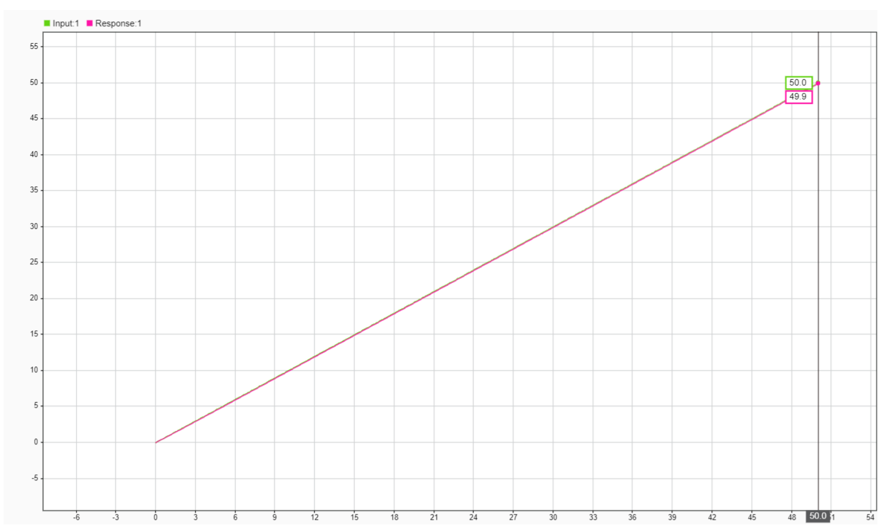


Figure 10: Ramp Response with cascaded controllers

As we can see, the ramp error constant is 10 as required, since the steady state error from the plot is seen to be 0.1. The phase margin is 90.5° which satisfies the specification.

# 5 Conclusion

Thus, we have designed a controller which satisfies the given specifications. The controller is a cascade of two PD controllers. The first PD controller adds a phase of 43.5° and the second PD controller adds a phase of 47°, giving us a total phase margin of 90.5°. This is an example of a **cascaded controller** which is used to add more phase margin to a system than what is possible with a single controller. Note that $PI$ or $PID$ controllers would not help in this case since we would not be able to get a finite value of $K_v$ with it as our system is of Type 1. We may also use lead compensators in cascade to achieve similar results.