SC 625 - Systems Theory

Assignment 1

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Problem 1.

Solution. **(a)**
v and w are orthogonal vectors in $\mathbb{R}^n$.

$$\implies v^T w = w^T v = 0 \tag{1}$$

Given that the norm on $\mathbb{R}^n$ is defined as $\|u\| = \sqrt{u^T u}$ for some $u \in \mathbb{R}^n$. Using this,

$$\|v + w\| = \sqrt{(v + w)^T (v + w)}$$

$$= \sqrt{v^T v + w^T w + v^T w + w^T v}$$

The last two terms are zero from (1). Squaring on both sides,

$$\|v + w\|^2 = v^T v + w^T w$$

Using definition of norm,

$$\boxed{\implies \|v + w\|^2 = \|v\|^2 + \|w\|^2} \tag{2}$$

□

Solution. **(b)**
Similar to **(a)**, we can consider $\|v - w\|$

$$\|v - w\| = \sqrt{(v - w)^T (v - w)}$$

$$= \sqrt{v^T v + w^T w - v^T w - w^T v}$$

The last two terms are zero again, from (1). Squaring on both sides,

$$\|v - w\|^2 = v^T v + w^T w$$

Using definition of norm,

$$\implies \|v - w\|^2 = \|v\|^2 + \|w\|^2 \tag{3}$$

Adding (2) and (3), we finally get:

$$\boxed{\|v + w\|^2 + \|v - w\|^2 = 2\|v\|^2 + 2\|w\|^2} \tag{4}$$

Hence, proved. □

Problem 2.

Solution. $\square$

We have $T \in \mathbb{R}^n$. Let $T : W \to W$. Consider $w \in \mathrm{Ran}(T) \cap \ker(T)$. Then,

$$Tw = 0$$

and there is a $v \in W$ such that $Tv = w$. So,

$$T^2 v = T(Tv) = Tw = 0$$

Thus, $v \in \ker(T^2) = \ker(T) \implies v \in \ker(T)$, yields:

$$w = Tv = 0$$

$$\implies \mathrm{Ran}(T) \cap \ker(T) = \{0\}$$

As $\ker(T) \subset \ker(T^2)$ always holds, in order to prove equality, we also need to show that $\ker(T^2) \subset \ker(T)$. Let $v \in \ker(T^2)$

$$\implies T^2 v = 0$$

$$\implies Tv \in \mathrm{Ran}(T) \cap \ker(T^2)$$

Thus, $Tv = 0$ by assumption $\implies v \in \ker(T)$.

Hence,

$$\boxed{\ker(T) = \ker(T^2) \iff \mathrm{Ran}(T) \cap \ker T = \{0\}} \tag{5}$$

Problem 3.

Solution. We have $A = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 5 & 5 & 5 \end{bmatrix}$. The corresponding characteristic equation is given by:

$$det(A - \lambda I) = (\lambda - 1)^2(\lambda - 5) = 0$$

Solving for the eigenvalues, we get $\lambda = 1, 1, 5$. The algebraic multiplicity of $\lambda = 1$ is 2. To find the geometric multiplicity, we need to find the null space of $A - I$. Let $v_1 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3$.

$$\begin{aligned}
(A - I)v_1 &= 0 \\
\implies \begin{bmatrix} 0 & 0 & 0 \\ -2 & 0 & 0 \\ 5 & 5 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} &= 0 \\
\implies \begin{bmatrix} 0 \\ -2x_1 \\ 5x_1 + 5x_2 + 4x_3 \end{bmatrix} &= 0 \\
\implies x_1 = 0, x_2 = -\frac{4}{5}x_3, x_3 &\in \mathbb{R}^3 \\
v_1 &= \begin{bmatrix} 0 \\ -4 \\ 5 \end{bmatrix} \qquad (6)
\end{aligned}$$

Thus, the geometric multiplicity of $\lambda = 1$ is 1.
This means that the algebraic multiplicity $\neq$ geometric multiplicity for $\lambda = 1$. So we will have 1 Jordan block of size 2×2 and 1 Jordan block of size 1×1.

Now, we need to find the generalized eigenvector corresponding to $\lambda = 1$. Let $v_2 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3$.

$$\begin{aligned}
(A - I)v_2 &= v_1 \\
\implies \begin{bmatrix} 0 & 0 & 0 \\ -2 & 0 & 0 \\ 5 & 5 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ -4 \\ 5 \end{bmatrix} \\
\implies \begin{bmatrix} 0 \\ -2x_1 \\ 5x_1 + 5x_2 + 4x_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ -4 \\ 5 \end{bmatrix} \\
\implies x_1 = 2, 5x_2 + 4x_3 &= -5
\end{aligned}$$

Let us make sure we have the same constraints when we find perform $(A - I)^2 v_2 = 0$.

$$\begin{aligned}
(A - I)^2 v_2 &= 0 \\
\implies \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 10 & 20 & 16 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} &= 0 \\
\implies \begin{bmatrix} 0 \\ 0 \\ 10x_1 + 20x_2 + 16x_3 \end{bmatrix} &= 0
\end{aligned}$$

Taking $x_1 = 2$, we again get, $5x_2 + 4x_3 = -5$. So, we can take $x_2 = -1, x_3 = 0$. Thus, we get the following generalized eigenvector:

$$v_2 = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} \tag{7}$$

The algebraic multiplicity of $\lambda = 5$ is 1. To find the geometric multiplicity, we need to find the null space of $A - 5I$. Let $v_3 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3$.

$$\begin{aligned} (A - 5I)v_3 &= 0 \\ \implies \begin{bmatrix} -4 & 0 & 0 \\ -2 & -4 & 0 \\ 5 & 5 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} &= 0 \\ \implies \begin{bmatrix} -4x_1 \\ -2x_1 - 4x_2 \\ 5x_1 + 5x_2 \end{bmatrix} &= 0 \\ \implies x_1 = 0, x_2 = 0, x_3 &\in \mathbb{R}^3 \end{aligned}$$

$$v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{8}$$

Now that we have v_1, v_2, v_3, the required matrix M is given by:

$$M = \begin{bmatrix} 0 & 2 & 0 \\ -4 & -1 & 0 \\ 5 & 0 & 1 \end{bmatrix} \tag{9}$$

which yields the Jordan normal form of A as:

$$J = M^{-1}AM = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix} \tag{10}$$

$\square$

Problem 4.

Solution. We have $A \in \mathbb{R}^n$. Take $B = A + A^T$, which is a symmetric matrix ($B^T = B$). Now, if B is a positive semi-definite matrix, then by definition we have the following:

$$B \geq 0$$

$$v^T B v \geq 0$$

Since B is symmetric, we can write $B = Q\Lambda_B Q^T$ where Q is an orthogonal matrix and Λ_B is a diagonal matrix with eigenvalues of B on the diagonal. So,

$$v^T B v = v^T Q \Lambda_B Q^T v = (Q^T v)^T \Lambda_B (Q^T v)$$

Taking $y = Q^T v$, we get,

$$v^T B v = y^T \Lambda_B y \geq 0$$

If $\Lambda_B = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$, then we have,

$$y^T \Lambda_B y = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \cdots + \lambda_n y_n^2 \geq 0 \tag{11}$$

Since $y_i^2 \geq 0$ for all i, we have $\lambda_i \geq 0$ for all i. Hence, B is positive semi-definite $\implies$ $\Lambda_B \geq 0$ i.e., all eigenvalues of B are non-negative.
Now, we have $B = A + A^T$. If we denote Λ_A as the diagonal matrix with eigenvalues of A on the diagonal, we can write:

$$\Lambda_B = \Lambda_A + \Lambda_{A^T}$$

Since A is a real matrix, we have $\Lambda_A = \Lambda_{A^T}$. Hence,

$$\Lambda_B = 2\Lambda_A$$

Since $\Lambda_B \geq 0$, we have $\Lambda_A \geq 0$.

Thus, all eigenvalues of A are non-negative from the same argument made in (11).
Hence, A is positive semi-definite.

$$\boxed{\therefore v^T A v \geq 0 \quad \forall \ v \in \mathbb{R}^n \iff A + A^T \text{ is positive semi-definite}} \tag{12}$$

□

Problem 5.

Solution. The nonlinear model for the inverted pendulum mounted on a cart is given by:

$$(M+m)\ddot{x} - ml\ddot{\theta}\cos\theta + ml\dot{\theta}^2\sin\theta = F$$

$$l\ddot{\theta} - g\sin\theta = \ddot{x}\cos\theta$$

where x is the position of the cart, θ is the angle of the pendulum from the vertical, M is the mass of the cart, m is the mass of the pendulum, l is the length of the pendulum, g is the acceleration due to gravity and F is the force applied on the cart.
Linearizing the above equations around the equilibrium point $(x_e, \theta_e) = (0, 0)$, we get:

$$(M+m)\ddot{x} - ml\ddot{\theta} = F$$

$$l\ddot{\theta} - g\theta = \ddot{x}$$

Taking $F = 0, M = 1, m = 1, l = 1$ and $g = 9.81$, we can solve the above equations.

Below are the plots which compares the linearized model with the nonlinear model for the respective parameters:

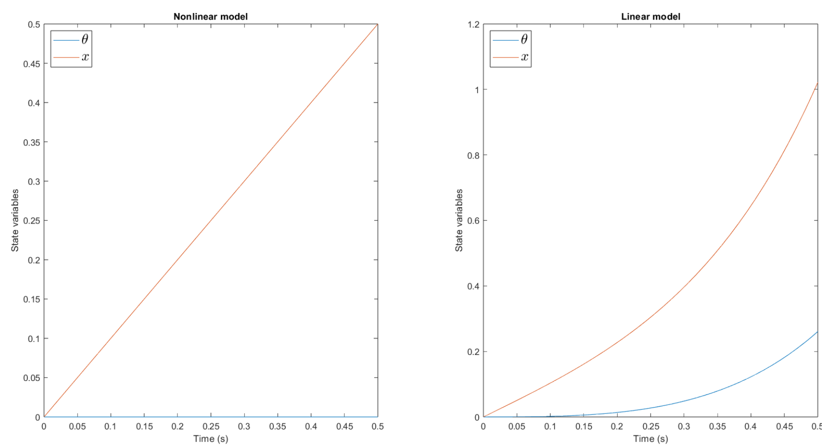

Figure 1: Initial conditions $x_0 = 0, \theta_0 = 0, \dot{x}_0 = 0.1, \dot{\theta}_0 = 0$

The initial conditions suggest that the cart is moving at a velocity of 0.1 m/s and the pendulum is at rest. So we expect $\theta(t) = 0$ for all t since $F = 0$. This is seen in the Nonlinear model plot clearly since we expect the cart to keep moving at constant velocity.

The plot shows that the linearized model is a good approximation of the nonlinear model for this set of initial conditions, but only up until $t = 0.2s$. This is because the linearized model is valid only for small deviations from the equilibrium point.

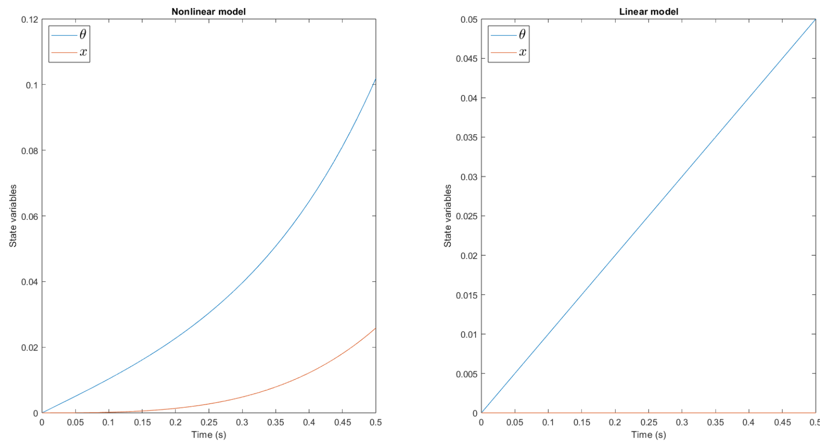

Figure 2: Initial conditions $x_0 = 0, \theta_0 = 0, \dot{x}_0 = 0, \dot{\theta}_0 = 0.1$

The initial conditions suggest that the cart is at rest and the pendulum is moving at a velocity of 0.1 rad/s. So we expect that the cart will move due to the motion of the pendulum. This is seen in the Nonlinear model plot clearly since we expect the cart to move in the opposite direction of the pendulum.

The plot shows that the linearized model does not show any response regarding the motion of the cart i.e., it considers the cart to be at rest (due to this particular I.C.) and the pendulum to be rotating at a constant angular speed.

Let us see the same plot after it has evolved for a longer time: Again, it can be seen that the nonlinear

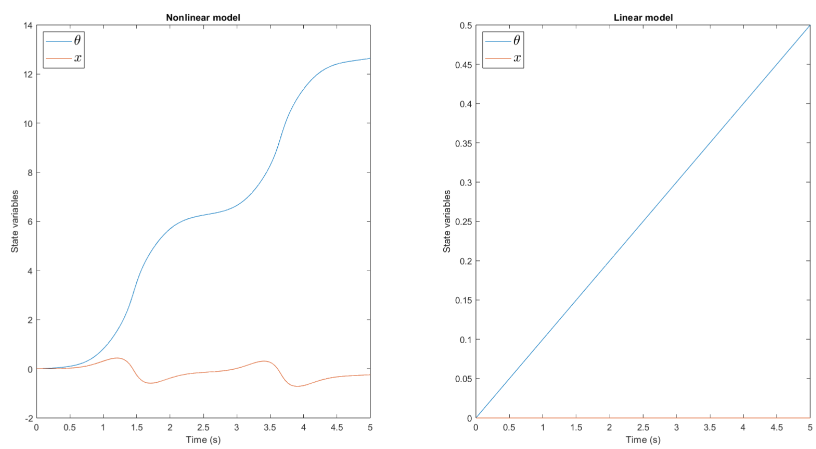

Figure 3: Initial conditions $x_0 = 0, \theta_0 = 0, \dot{x}_0 = 0, \dot{\theta}_0 = 0.1$

model is responding as we expect but the linear model is not, since it is valid only for small deviations from the equilibrium point.

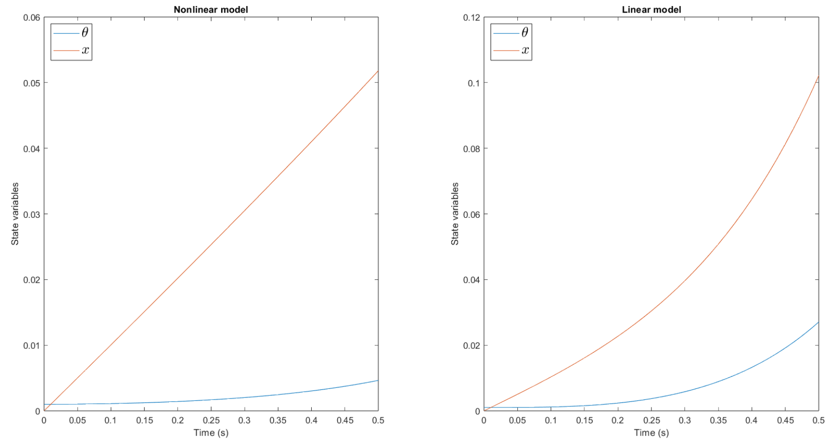

Figure 4: Initial conditions $x_0 = 0, \theta_0 = 0.001, \dot{x}_0 = 0.1, \dot{\theta}_0 = 0$

So, here we have given a small initial angular displacement to the pendulum and a small initial velocity to the cart. Here, we can see that the linear model is a good approximation as we have chosen a small enough perturbation from the equilibrium point and also chosen a small enough time interval.

Hence, so far we have been able to deduce that the linearized model is a good approximation of the nonlinear model only for small deviations from the equilibrium point and for a reasonably small time interval. Let us finally check this by making the derivatives zero and simply perturbing both the pendulum and cart by very small displacements:

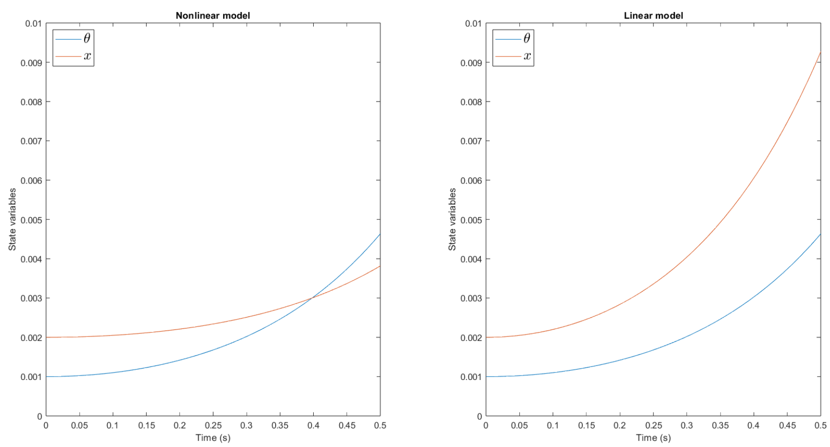

Figure 5: Initial conditions $x_0 = 0.002, \theta_0 = 0.001, \dot{x}_0 = 0, \dot{\theta}_0 = 0$

Here, we can confirm our findings, as the linear model plot is almost identical to the nonlinear model plot. Note that the time interval we have chosen here is $t = [0, 0.5]$. Choosing a smaller interval will make the linear model plot even more identical to the nonlinear model plot.

$\square$