SC 625 - Systems Theory

Assignment 5

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Problem 1.

Solution. □

$$A = \begin{bmatrix} 3 & 0 & 0 \\ 5 & 0 & -1 \\ 4 & -1 & 0 \end{bmatrix}, C = \begin{bmatrix} 1 & -1 & 1 \end{bmatrix}$$

a) To check if the system is observable, we check if the observability matrix is full rank. The observability matrix is given by:

$$\mathcal{O} = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & 1 \\ 5 & -1 & 1 \end{bmatrix}$$

The rank of this matrix is 2 and hence the system is not observable.

b) To check if the system is detectable, we can use the PBH test. The eigenvalues of the matrix A are $3, 1, -1$. The system (C, A) is detectable if $\begin{bmatrix} A - \lambda I \\ C \end{bmatrix}$ is full rank for all eigenvalues λ of A where $\text{Re}(\lambda) \geq 0$. We need to check the condition for $\lambda = 3, 1$. For $\lambda = 3$, we get the matrix:

$$\begin{bmatrix} 0 & 0 & 0 \\ 5 & -3 & -1 \\ 4 & -1 & -3 \\ 1 & -1 & 1 \end{bmatrix}$$

which has rank 3. For $\lambda = 1$, we get the matrix:

$$\begin{bmatrix} 2 & 0 & 0 \\ 5 & -1 & -1 \\ 4 & -1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$$

which also has a rank 3.

Hence, the system (C, A) is detectable.

c) To find an L such that $A + LC$ is Hurwitz, we can use the duality that (A^T, C^T) is detectable if (C, A) is observable. We can find a K such that $A^T + C^T K$ is Hurwitz by using the controllable canonical form. Let $A_c = A^T = \begin{bmatrix} 3 & 5 & 4 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix}$ and $B_c = C^T = \begin{bmatrix} 1 & -1 & 1 \end{bmatrix}$. The corresponding controllability matrix is given by:

$$\mathcal{C} = \begin{bmatrix} 1 & 2 & 5 \\ -1 & -1 & -1 \\ 1 & 1 & 1 \end{bmatrix}$$

We can see that the columns are not linearly dependent. In fact, $-3B_c + 4A_cB_c = A_c^2B_c$. Hence, the controllable canonical transformation matrix is given by

$$T = \begin{bmatrix} 1 & 2 & 0 \\ -1 & -1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

and the controllable canonical form is given by:

$$\bar{A}_c = T^{-1}A_cT = \begin{bmatrix} 0 & -3 & -2 \\ 1 & 4 & 3 \\ 0 & 0 & -1 \end{bmatrix}, \bar{B}_c = T^{-1}B_c = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Here, $A_o = \begin{bmatrix} 0 & -3 \\ 1 & 4 \end{bmatrix}$ and $B_o = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Let $K_o = \begin{bmatrix} k_1 & k_2 \end{bmatrix}$. Then, we get the following equations:

$$A_o + B_oK_o = \begin{bmatrix} k_1 & -3 + k_2 \\ 1 & 4 \end{bmatrix}$$

We can place the eigenvalues of this system at $\lambda = -2, -3$ to ensure that the system is Hurwitz. This gives us the following equations:

$$k_1 - 4 = 5, 3 + k_2 - 4k_1 = 6$$

Solving these equations, we get $k_1 = 9, k_2 = 39$. Hence, we get $K_o = \begin{bmatrix} 9 & 39 \end{bmatrix}$. Hence, $\bar{K} = \begin{bmatrix} 9 & 39 & 0 \end{bmatrix}$ Transforming back to the original coordinates, we get: $K = \bar{K}T^{-1}$

$$K = \begin{bmatrix} 30 & 21 & 0 \end{bmatrix}$$

Correspondingly, we get $L = -K^T$.

$$L = \begin{bmatrix} -30 \\ -21 \\ 0 \end{bmatrix}$$

Thus the eigenvalues of $A + LC$ are $-1, -2, -3$ and hence the system is Hurwitz.

Problem 2.

Solution. $\square$

$$\ddot{y} = u, V(u) = \int_0^{\infty} y(t)^2 + u(t)^2 dt$$

Let $y = x_1, \dot{x}_1 = x_2, \dot{x}_2 = u$. If $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ Then, we get the following equations:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

Also, $V(u) = \int_0^{\infty} x_1(t)^2 + u(t)^2 dt$. Comparing with $V(u) = \int_0^{\infty} \mathbf{x(t)}^T Q \mathbf{x(t)} + u(t)^2 dt$, we get $Q = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $R = 1$. The corresponding Riccati equation is given by:

$$A^T P + P^T A - PBR^{-1}B^T P + Q = 0$$

Using A, B, Q, R, solving for P, we get:

$$P = \begin{bmatrix} \sqrt{2} & 1 \\ 1 & \sqrt{2} \end{bmatrix}$$

Hence, the minimal cost is given by $V^* = \mathbf{x}^T(0) Q \mathbf{x}(0)$. Given that $\mathbf{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, we get $V^* = \sqrt{2}$.

Problem 3.

Solution. $\square$

$$\dot{x}_1 = x_2, \dot{x}_2 = -x_1 + u$$

. a) The state space model is given by:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

Hence, $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. The value function is given by:

$$V(u) = \int_0^\infty x_1(t)^2 + 2x_1(t)x_2(t) + x_2(t)^2 + 4u(t)^2 dt$$

Comparing with $V(u) = \int_0^\infty \mathbf{x(t)}^T Q\mathbf{x(t)} + u(t)^2 dt$, we get $Q = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, R = 4$. Using the Riccati equation,

$$A^T P + PA - PBR^{-1}B^T P + Q = 0$$

we get $P = \begin{bmatrix} \frac{\sqrt{80\sqrt{5}-140}}{2} - 1 & -4 + \sqrt{5} \\ -4 + \sqrt{5} & \sqrt{16\sqrt{5} - 28} \end{bmatrix}$. The optimal control law is given by $u^* = -R^{-1}B^T P\mathbf{x}$

Hence, we get $u^* = -\frac{-4+2\sqrt{5}}{4} x_1 - \frac{\sqrt{16\sqrt{5}-28}}{4} x_2$ or

$$u^*(\mathbf{x}) = \begin{bmatrix} -\frac{-4+2\sqrt{5}}{4} & -\frac{\sqrt{16\sqrt{5}-28}}{4} \end{bmatrix} \mathbf{x}$$

Here, $K = \begin{bmatrix} -\frac{-4+2\sqrt{5}}{4} & -\frac{\sqrt{16\sqrt{5}-28}}{4} \end{bmatrix}$ is the optimal gain matrix.

b) The closed loop system is given by $A + BK = \begin{bmatrix} 0 & 1 \\ -1 - \frac{-4+2\sqrt{5}}{4} & -\frac{\sqrt{16\sqrt{5}-28}}{4} \end{bmatrix}$ The eigenvalues of the closed loop system are given by:

$$\lambda = -\frac{\sqrt{16\sqrt{5} - 28}}{8} \pm \frac{\sqrt{16\sqrt{5} + 28}}{8} i$$

$$\lambda = -0.3486 \pm 0.9983i$$