2 SAMPLE SPACE AND EVENTS

2.1 Introduction

After having learnt the techniques of permutations and combinations in Chapter 1, we now set ourselves for understanding some concepts and terminologies needed further in the discrete probability theory. The concepts of sample space and algebra of events are solely based on the set theory, which you have learnt already in higher secondary classes.

Let us begin the discussion from what is meant by experiments; deterministic and non deterministic experiments.

2.2 Deterministic and Non deterministic (Random) Experiments

You are familiar with the word 'experiment'. You perform experiments in Physics, Chemistry or Biology. For example, in Chemistry, you estimate the exact amount of alkali required to neutralize acid using titration method. Or in Physics, velocity (v) of a particle can be determined using $v = u + at$, where u is the initial velocity and a is the acceleration. In Biological experiments, a type of diet is fed to animals and increase in their weights are recorded.

However, in Statistics, the word 'experiment' is used in a wider sense. It is not necessarily restricted to laboratory experiments,

Experiment : An experiment is virtually any operation that results in one or more outcomes.

For instance,

(i) Appearing for F.Y.B.Sc. examination is an experiment with possible outcomes as PASS or FAIL.

(ii) Casting a vote in the election is an experiment with outcomes; the party you voled for wins or looses.

(iii) Releasing a stone from hand is an experiment with the outcome that 'it will fall on the ground'.

(iv) Tossing a coin is an experiment with two possible outcomes, 'Head up' or 'Tail up'.

Head

Tail

Fig. 2.1

(v) Rolling a six faced die; outcomes are 1, 2, 3, 4, 5, 6.

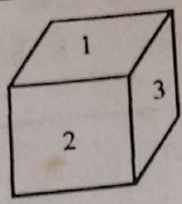


Fig. 2.2

Trial of an experiment : A trial of an experiment is nothing but performing the experiment once. So 'n' trials of an experiment means that the experiment is performed 'n' times either sequentially (or simultaneously). Experiments are classified into (i) Deterministic Experiments and (ii) Non-deterministic or Random Experiments.

(i) Deterministic Experiments : A deterministic experiment is an experiment for which the outcome is unique; hence certain. The outcome can therefore be predicted before performing the experiment. In other words, deterministic experiments are predictable phenomena. Following are some examples of deterministic experiments.

(i) Throwing a ball in the sky; Outcome: It falls down.

(ii) Cooling water below 0^oC; Outcome : It will freeze.

(iii) Determining the pressure of a gas using Boyle's law.

$$\text{Outcome : } P = \frac{\text{constant}}{\text{Volume of the gas}}$$

Observe that, in all the above experiments, the outcome is certain, unique and predictable.

Such experiments are known as deterministic experiments. All the deterministic experiments can be described by mathematical formulae. These mathematical formulae are called as deterministic models. For example, $PV = \text{constant}$, $S = \frac{1}{2} gt^2$ are deterministic models used in Physics, Since there is no uncertainty in the result of the experiment, probability theory does not play any role.

(ii) Non-deterministic (Random) Experiments : A non-deterministic experiment or a random experiment is an experiment, for which there are more than one possible outcomes and the result of the experiment cannot be predicted in advance.

For instance,

(i) Sex of a new born baby is recorded. Outcomes : Male or Female.

(ii) Rolling a die. Outcomes : 1, 2, 3, 4, 5, 6..

(iii) Tossing of a coin; Outcomes : Head or Tail.

(iv) Blood group of a person recorded in a blood donation camp. Outcomes : O, A, B, AB.

In random experiments, 'chance' element plays a vital role in determining the outcome. No mathematical formula can describe these experiments. These are the expermints we are interested in.

Though, we can not predict the outcome of a single trial of the experiment, we can get some knowledge about the pattern among the outcomes, when the experiment is performed repeatedly for a large number of times. For example, we can't say whether 'head' or 'tail' will come up when a fair coin is tossed. However, if we toss the coin say 1000 times, what will we expect ? *About* 500 times 'head' will turn up and *about* 500 times 'tail' will turn up. If we further increase the number of tossings, we expect that the proportion of getting 'head' should approach the value $\frac{1}{2}$.

Thus, though the outcome of any particular experiment may be uncertain, there exists a long term regularity. This is the basis for the development of random models. These models help in predicting the outcome of the non deterministic experiment in probabilistic terms. Therefore, these models are also known as probabilistic models. For example, using a non-deterministic i.e. random model, one can make the statement as 'Probability of death at the age of 80 years is 0.9.

These models are helpful in dealing with groups of cases. Random models are used in almost all branches of physical and social sciences. In fact, applicaiton of random models in different fields have given rise to several branches of Statistics such as Industrial Statistics, Econometrics, Statistical Ecology, Medical Statistics, Biometry, Acturial Science, Demography etc.

We cite here some situations where random models are applied.

(i) In industries, quality of the product manufactured is maintained.

(ii) Decision on whether to start an additional booking counter at railway station is taken by using random models in queueing theory.

(iii) A physicist studies the motion of particles emitted by a radio active substance.

(iv) An economist, using random models can study the changes in price levels and construct index numbers.

(v) In a sociological survey, these models may be used to investigate the relationship between women literacy and success of family planning programmes.

(vi) In ecology, changes in the population of endangered species can be studied with a view to provide remedial measures.

Probability theory deals with non deterministic or random models which describe and study the various phenomena happening in the world. We shall learn some of the simplest types of random moels in this course in later chapers.

We are now in the position of learning some terminologies and concepts used in the theory of probability.

2.3 Sample Space

In the previous section, we talked about random experiments which have more than one possible outcome. In order to develop the probability theory, naturally, the first step is to *group* all possible outcomes of the experiment in a set. This set is called as 'Sample Space'.

Definition : Sample Space : The set of all possible distinct outcomes of an experiment is called a sample space.

Sample space is denoted by Ω or S. Thus, a sample space is nothing but the universal set concerned with the experiment. For example, consider the experiment of tossing a coin. The corresponding sample space will be

$$\Omega = \{\text{Head, Tail}\} = \{H, T\}$$

The elements of sample space are different outcomes which are called as *sample points*. Thus, H and T are sample points of the above sample space Ω.

Depending upon the number of sample points, the sample spaces are categorized into two types (i) Discrete (ii) Continuous.

(i) Discrete Sample Space : A sample space containing a finite number of points or countably infinite points is called a discrete sample space. In other words a discrete sample space is either (a) a finite sample space or (b) a countably infinite sample space.

(a) Finite sample space : A sample space Ω is called a finite sample space if the number of elements contained in Ω is finite. Such a sample space can be denoted as

$$\Omega = \{\omega_1, \omega_2, \ldots\ldots \omega_n\}$$

where n = number of elements i.e. number of possible outcomes of the experiment. $\omega_1, \omega_2, \ldots\ldots \omega_n$ are the outcomes.

For instance,

(i) Suppose two coins are tossed. Then

$$\Omega = \{HH, HT, TH, TT\}, \quad n = 4$$

where H : Head, T : Tail.

(ii) A die is rolled, then $\Omega = \{1, 2, 3, 4, 5, 6\}$

$$n = 6$$

(iii) Suppose a pair of dice is rolled and the numbers on the uppermost faces are noted. Then,

$$\Omega = \left\{\begin{matrix} (1,1) & (1,2) & \ldots & (1,6) \\ (2,1) & (2,2) & \ldots & (2,6) \\ (3,1) & (3,2) & \ldots & (3,6) \\ (4,1) & (4,2) & \ldots & (4,6) \\ (5,1) & (5,2) & \ldots & (5,6) \\ (6,1) & (6,2) & \ldots & (6,6) \end{matrix}\right\}; \qquad n = 36$$

(iv) If a card is drawn from a well shuffled pack of playing cards and suit is recorded, then

$$\Omega = \{\text{Hearts, Spades, Diamonds, Clubs}\}; \; n = 4$$

(v) If a card is drawn from a pack of cards and denomination is noted, then

$$\Omega = \{\text{ace}, 2, 3, 4, 5, 6, 7, 8, 9, 10, \text{jack}, \text{queen}, \text{king}\}$$

$$n = 13$$

(vi) Three coins are tossed,

$$\Omega = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$$

$$n = 8$$

(b) Countably Infinite Sample Space : A sample space Ω is called a countably infinite sample space if the number of elements in Ω are countably infinite. This means that there is a one-one correspondence between Ω and the set of natural numbers. It may be represented as

$$\Omega = \{\omega_1, \omega_2, \omega_3, \ldots\ldots, \omega_n, \ldots\ldots\}$$

For example,

(i) A student appears for an examination till he passes. Then

$$\Omega = \{P, FP, FFP, FFFP, \ldots\ldots\}$$

where F : Fail P : Pass.

(ii) The number of accidents on the Bombay-Pune Road in a month.

$$\Omega = \{0, 1, 2, 3, \ldots\ldots\}$$

(iii) Number of customers arriving at a telephone booth during a day.

$$\Omega = \{0, 1, 2, 3, \ldots\ldots\}$$

Note that, in all the above three examples, although possibility of infinite number of elements may not seem practicable, theoretically, we have to consider all possibilities in Ω.

(ii) Continuous Sample Space : A sample space Ω, is called a continuous sample space, if the number of elements in Ω are uncountably infinite. For example, consider the experiment of measuring height of a person. We may take $\Omega = (0, \infty)$. If we know that the lowest height is say 130 cm and heighest is say 190 cm, we may as well consider $\Omega = (130, 190)$. Note that for both these sample spaces, the elements in the interval cannot be arranged in a sequence. Thus, there are countably infinite elements in Ω.

In other words, whenever, the observations on a characteristic can take any values in an interval, the concerned sample space is countably infinite.

The following diagram in fig. 2.3 represents the different categories of sample spaces.

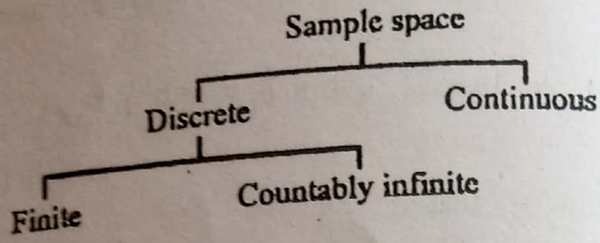


Fig. 2.3

2.4 Events

Consider the experiment of tossing two coins. Then the sample space is $\Omega = \{HH, HT, TH, TT\}$. We might be interested in getting a single head. i.e. in the set $\{HT, TH\}$. Thus we can associate the *event* of 'getting a single head' with the set $\{HH, TH\}$ which is the subset of Ω.

Definition : Event : An event is a subset of the sample space. It consists of some or all points of the sample space.

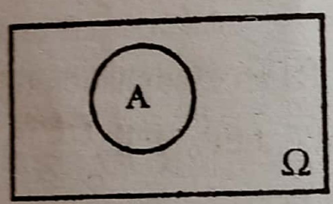


Fig. 2.4

Events are denoted by capital letters A, B, C, ...

Remark : For a sample space containing n elements, there are 2^n events (including ϕ and Ω).

Occurrence of an event : We say that an event A has occurred, if after performing the experiment, the outcome belongs to the set A.

For example, suppose the experiment is of rolling a die. Then $\Omega = \{1, 2, 3, 4, 5, 6\}$.

Let A be the event that even number appears on the upper face of the die.

$$\therefore \quad A = \{2, 4, 6\}.$$

Suppose we roll the die, and 4 comes; then we shall say that 'event A has occurred'. If any of 1, 3, 5 appears, then we shall say that 'event A has not occurred' as 1, 3 or 5 are not the elements of A. On the other hand, if we only know that the event A has occurred, then all we know is that the outcome is one of the points of A.

We give below some examples to clarify the meaning of event.

Examples :

1. Experiment : Tossing two coins.

Ω = {HH, HT, TH, TT}

event A = occurrence of single head.

= {HT, TH}

2. Experiment : Rolling two dice.

Ω contains 36 elements, see 2.3 (a) (iii).

Let the event A = sum on the two uppermost faces is 10. Then

A = {(6, 4), (5, 5), (4, 6)}

Let the event B = getting sum of 13 of the numbers on uppermost faces.

B = { } = ϕ.

2.5 Types of Events

According to the nature of the set, types of events are defined as follows.

1. Elementary event or simple event : An event containing only one element is called as *elementary event* or *simple event.* In other words, a singleton set is called as elementary event.

For example; A = getting a multiple of 5 on a die A = {5}.

2. Impossible event : An event corresponding to empty set is called as an impossible event. In other words, an event which does not contain any sample point is called as an impossible event. For example, if a single coin is tossed, then getting two heads is an impossible event.

3. Sure event or certain event : An event containing all the points of Ω is called a *sure event* or *certain event.* In other words, an event corresponding to the entire sample space is called a sure event. For example, getting a number either even or odd on a rolled die is a sure event.

We have seen in previous sections what is meant by sample space and events. As events are subsets of sample space, using set theory, we can generate new events from specified events. We shall use very often the following set identities.

Set Identities :

(i) $(A')' = A$ where A' denotes the complement of A.

$\Omega' = \phi, \ \phi' = \Omega$

$A \cup B = B \cup A.$

(ii) $A \cup B = B \cup A \quad A \cup \phi = A, \ A \cup \Omega = \Omega, \ A \cup A' = \Omega, \ A \cup A = A.$

(iii) $A \cap B = B \cap A, \ A \cap \phi = \phi, \ A \cap \Omega = A$

$A \cap A = \phi, \ A \cap A = A$

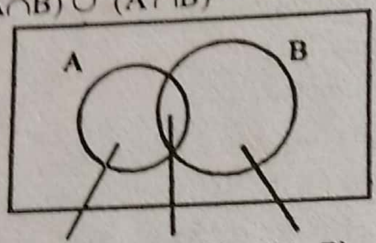
(iv) De Morgan's laws

$(A \cup B)' = A' \cap B'$ and $(A \cap B)' = A' \cup B'$

(iv) If $A \subset B$, then $A \cup B = B$ and $A \cap B = A$

(v) $A = (A \cap B) \cup (A \cap B')$

$B = (A \cap B) \cup (A' \cap B)$



$(A \cap B')$ $(A \cap B)$ $(A' \cap B)$

Fig. 2.5

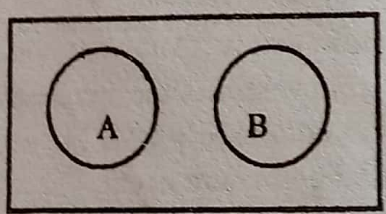
(vi) $(A \cup B) = (A \cap B') \cup (A \cap B) \cup (A' \cap B)$

4. Mutually Exclusive Events : (Disjoint events) : Events A and B are said to be mutually exclusive if there is no common element in A and B. That is,

$$A \cap B = \phi$$

For example : In an experiment of drawing a card from a well-shuffled pack of playing cards, if A = occurrence of red card and B = occurrence of a spade card, then

$A \cap B = \phi$. Hence A and B are mutually exclusive events. See figure fig. 2.6.

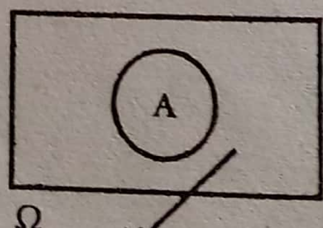


Ω

Fig. 2.6

5. Complement of an event : If A is an event on Ω then complement of A is the event corresponding to the set A'. In other words A' is the event containing all points in Ω which are not in A. (Fig. 2.7).

For example, if in the experiment of rolling a die, A = occurence of an even number. then complement of event A is A' = occurence of an odd number.



Ω A'

Fig. 2.7

Note that A and A' are mutually exclusive events.

6. Exhaustive events : Events A and B on Ω are said to be exhaustive events, if $A \cup B = \Omega$. In general, events $A_1, A_2, \ldots\ldots, A_n$ are said to be exhaustive if $A_1 \cup A_2 \ldots\ldots \cup A_n = \Omega$.

For example, if $\Omega = \{1, 2, 3, 4, 5, 6\}$

$$A_1 = \{1, 3\} \qquad A_2 = \{2, 3, 5\} \qquad A_3 = \{4, 5, 6\}$$

Then $A_1 \cup A_2 \cup A_3 = \Omega$. Therefore, A_1, A_2, A_3 are exhaustive events.

Remark : 1. A and A' are exhaustive.

2. By mutually exclusive and exhaustive events we mean a partition of Ω.

For example, with the above Ω, it

$$A_1 = \{1, 3\}, \quad A_2 = \{2, 5\} \quad A_3 = \{4, 6\}$$

then $A_1 \cap A_2 = \phi$, $A_1 \cap A_3 = \phi$, $A_2 \cap A_3 = \phi$,

$$A_1 \cup A_2 \cup A_3 = \Omega$$

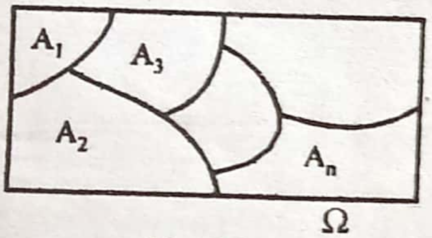


Fig. 2.8

It $A_1, A_2, \ldots\ldots, A_n$ are mutually exclusive and exhaustive events, then they can be represented as in the venn diagram given in Fig. 2.8.

Union and intersection of two or more events give rise to the following different concepts.

1. Occurrence of at least one of the given events : If A and B are two events on Ω, then occurrence of at least one of the events A and B is defined as $A \cup B$. (fig. 2.9)

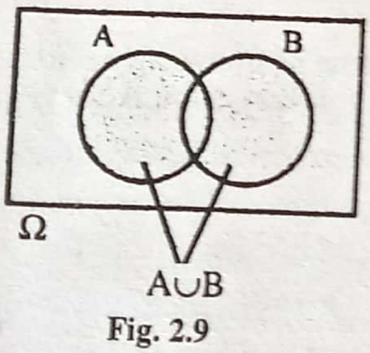


Fig. 2.9

For example, let $\Omega = \{1, 2, 3, 4, 5, 6, 7, 8\}$

A = occurrence of an even number.

= $\{2, 4, 6, 8\}$

B = occurrence of a multiple of 3 = $\{3, 6\}$

Here occurrence of number either even or multiple of 3 is given by

$$A \cup B = \{2, 3, 4, 6, 8\}$$

On similar lines, occurrence of *at least one* of the events of $A_1, A_2, \ldots A_n$ is given by $A_1 \cup A_2 \cup A_3 \ldots\ldots \cup A_n = \bigcup_{i=1}^{n} A_i$

2. **Occurrence of all of the given events** or **simultaneous occurence of the events :** If A and B are events on Ω, then $A \cap B$ is the occurrence of both the events A and B. It is also called as the simultaneous occurrence of events A and B (fig. 2.10.)

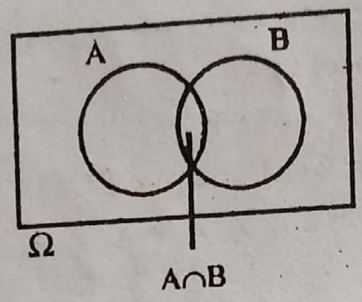


Fig. 2.10

For example, let $\Omega = \{1, 2, 3, 4, 5, 6\}$.

Event A : Occurrence of number multiple of 3.

B : Occurrence of even number.

Then $A = \{3, 6\}$ $B = \{2, 4, 6\}$

$A \cap B = \{6\}$.

We say that A and B occur simultaneously if '6' appears.

Similarly, occurrence of all the events $A_1, A_2, \ldots\ldots, A_n$ is given by

$$A_1 \cap A_2, \ldots\ldots, \cap A_n = \bigcap_{i=1}^{n} A_i$$

3. **Occurrence of none of the given events :** Let A and B be two events on Ω. Occurrence of none of the events A and B is given by

$(A \cup B)' = A' \cap B'$ by De Morgan's law (fig.2. 11).

Similarly, occurrence of none of the given events $A_1, A_2, \ldots\ldots, A_n$ is given by $\left(\bigcup_{i=1}^{n} A_i\right)' = \bigcap_{i=1}^{n} A_i$

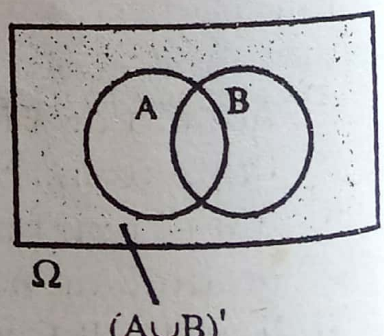


Fig. 2.11

For example,

let $\Omega = \{1, 2, 3, 4, 5, 6\}$

A = even number appears.

B = multiple of 3 appears.

Then none of the events A and B appear is

$(A \cup B)' = \{1, 5\} = A' \cap B'$

4. **Relative complementation :** Let A and B be two events on Ω. The relative complement of A with respect to be B is given by $A' \cap B$. That is, it is the set of all points which are *not in* A but *in* B. Similarly, the relative complement of B with respect to A is given by $A \cap B'$. (fig. 2.12).

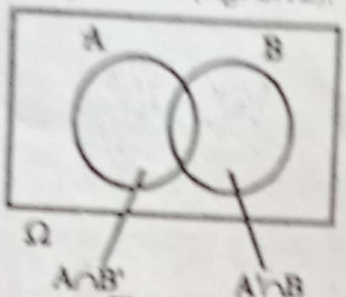


Fig. 2.12

Example 1 : Write down the sample spaces for the following experiments. Also, state the type of the sample space.

(i) Examination results (P : Pass, F : Fail) are noted for three students.

(ii) Ten radio sets are checked and number of defective sets are noted.

(iii) Items coming off a production line are marked defective (D) or non-defective (N). This is continued until two consecutive defectives are produced or four items have been checked, whichever occurs first.

(iv) A coin is tossed until 'head' appears for the first time.

(v) Life of an electric tube produced by a company is measured.

Solution : (i) Ω = {PPP, PPF, PFP, FPP, PFF, FPF, FFP, FFF}

Sample space is finite, hence discrete.

(ii) Ω = {0, 1, 2, ..., 10}

Sample space is finite, hence discrete.

(iii) Ω = {DD, NDD, NNDD, DNDD, NDND, DNDN, NDNN, NNDN, NNNN, NNND, DNNN}

Sample space is finite, hence discrete.

(iv) Ω = {H, TH, TTH, TTTH, ...}

Sample space is countably infinite, hence discrete.

(v) $\Omega = [0, \infty]$

Sample space is continuous or uncountably infinite.

Example 2 : Let A, B, C be any three events on a sample space Ω. Write expressions for the events.

(a) At least one of the events A, B, C occurs.

(b) Only A occurs.

(c) A and B occur but not C.

(d) All three events occur.

(e) None of A, B, C occurs.

(f) Exactly one occurs.

(g) Exactly two occurs.