Moments

Q.1 Define (i) raw moments, (ii) central moments, for a frequency distribution of a variable x. Express the first four central moments in terms of raw moments.

Ans. Let the symbol x be used to represent the deviation of any item in a distribution from the arithmetic mean of that distribution. The arithmetic mean of the various powers of these deviations in any distribution is called the moment of the distribution. If we take the mean of the first power of the deviations we get the first moment about the mean ; the mean of the squares of the deviations gives us the second moment about the mean ; the mean of the cubes of the deviations gives us the third moment about the mean ; and so on. **The moments about mean are called as central moments.**

The central moment of order r is denoted by μ_r and is given by the formula

$$\mu_r = \frac{\sum (x_i - \bar{x})^r}{N} \quad \text{for individual observations.}$$

$$\text{and } \mu_r = \frac{\sum f_i (x_i - \bar{x})^r}{\sum f_i} \quad \text{for frequency distribution.}$$

Substituting r = 1, 2, 3, 4 we can get the first four central moments.

	Moments	Individual obs.	Freq. distribution
r=1	μ_1	$\frac{\sum (x_i - \bar{x})}{N} = 0$	$\frac{\sum f_i (x_i - \bar{x})}{\sum f_i} = 0$
r=2	μ_2	$\frac{\sum (x_i - \bar{x})^2}{N}$	$\frac{\sum f_i (x_i - \bar{x})^2}{\sum f_i}$
r=3	μ_3	$\frac{\sum (x_i - \bar{x})^3}{N}$	$\frac{\sum f_i (x_i - \bar{x})^3}{\sum f_i}$
r=4	μ_4	$\frac{\sum (x_i - \bar{x})^4}{N}$	$\frac{\sum f_i (x_i - \bar{x})^4}{\sum f_i}$

The moments taken about origin are called raw moments.

The raw moment of order r is denoted y μ'_r and is given by the formula

$$\mu'_r = \frac{\sum_{i=1}^{n} x_i^r}{N} \text{ for individual observations.}$$

$$\text{and } \mu'_r = \frac{\sum_{i=1}^{n} f_i x_i^r}{\sum_{i=1}^{n} f_i} \text{ for frequency distribution.}$$

Substituting r = 1, 2, 3, 4 we can get the first four raw moments.

	Moments	Individual obs.	Freq. distribution
r = 1	μ'_1	$\frac{\sum x_i}{N} = \bar{x}$	$\frac{\sum f_i x_i}{\sum f_i} = \bar{x}$
r = 2	μ'_2	$\frac{\sum x_i^2}{N}$	$\frac{\sum f_i x_i^2}{\sum f_i}$
r = 3	μ'_3	$\frac{\sum x_i^3}{N}$	$\frac{\sum f_i x_i^3}{\sum f_i}$
r = 4	μ'_4	$\frac{\sum x_i^4}{N}$	$\frac{\sum f_i x_i^4}{\sum f_i}$

Relation between raw and central moments

$$\mu_1 = \frac{\sum f_i (x_i - \bar{x})}{\sum f_i} = \frac{\sum f_i x_i}{\sum f_i} - \frac{\sum f_i \bar{x}}{\sum f_i}$$

$$\therefore \quad \mu_1 = \bar{x} - \bar{x} = 0 \qquad \therefore \quad \boxed{\mu_1 = 0}$$

$$\mu_2 = \frac{\sum f_i (x_i - \bar{x})^2}{\sum f_i} = \frac{1}{\sum f_i} \left[\sum f_i (x_i - \bar{x})^2\right]$$

$$= \frac{1}{\sum f_i} \sum f_i \left[x_i^2 - 2\bar{x}\, x_i + (\bar{x})^2\right]$$

$$= \frac{\sum f_i x_i^2}{\sum f_i} - 2\bar{x} \frac{\sum f_i x_i}{\sum f_i} + \frac{\sum f_i (\bar{x})^2}{\sum f_i}$$

$$= \frac{1}{\Sigma f_i}\left[\Sigma f_i (x_i^r - {}^r c_1 x_i^{r-1} \bar{x} + {}^r c_2 x_i^{r-2} (\bar{x})^2 - {}^r c_3 x_i^{r-3} (\bar{x})^3 + \ldots + (-1)^r (\bar{x})^r)\right]$$

$$= \frac{1}{\Sigma f_i}\left[\Sigma f_i x_i^r - {}^r c_1 \bar{x}\, \Sigma f_i x_i^{r-1} + {}^r c_2 (\bar{x})^2 \Sigma f_i x_i^{r-2} + \ldots (-1)^r (\bar{x})^r \Sigma f_i\right]$$

$$= \frac{\Sigma f_i\, x_i^r}{\Sigma f_i} - {}^r c_1 \mu_1' \frac{\Sigma f_i x_i^{r-1}}{\Sigma f_i} + {}^r c_2 (\mu_1')^2 \frac{\Sigma f_i\, x_i^{r-2}}{\Sigma f_i} + \ldots (-1)^r (\mu_1')^r \frac{\Sigma f_i}{\Sigma f_i}$$

$$= \mu_r' - {}^r c_1 \mu_1' \mu_{r-1}' + {}^r c_2 (\mu_1')^2\, \mu_{r-2}' - {}^r c_3 (\mu_1')^3 \mu_{r-3}' + \ldots (-1)^r (\mu_1')^r$$

Q. 3 Define moments about any arbitrary constant 'a'. Deduce the relation between moments about 'a' and central moments.

Ans. The r^{th} moment about any arbitrary constant 'a' is defined as,

$$\mu_r (a) = \frac{\Sigma (x_i - a)^r}{N} \text{ for individual observations}$$

$$\text{and } \mu_r (a) = \frac{\Sigma f_i (x_i - a^r)}{\Sigma f_i} \text{ for frequency distribution}$$

$$\text{or } \mu_r (a) = \frac{\Sigma f_i u_i^r}{\Sigma f_i} \text{ or } \frac{\Sigma f_i u_i^r}{\Sigma f_i} \times h^r \text{ where } u_i = \frac{x_i - A}{\Sigma f_i}$$ and h is class interval.

Relation between moments about any arbitrary constant 'a' and central moments :

$$\mu_1 = 0$$

$$\mu_2 = \frac{\Sigma f_i (x_i - \bar{x})^2}{\Sigma f_i} = \frac{\Sigma f_i \left[(x_i - a) - (\bar{x} - a)\right]^2}{\Sigma f_i}$$

$$= \frac{\Sigma f_i \left[(x_i - a)^2 - 2 (\bar{x} - a)(x_i - a) + (\bar{x} - a)^2\right]}{\Sigma f_i}$$

$$= \frac{\Sigma f_i (x_i - a)^2}{\Sigma f_i} - 2 (\bar{x} - a) \frac{\Sigma f_i (x_i - a)}{\Sigma f_i} + \frac{\Sigma f_i (\bar{x} - a)^2}{\Sigma f_i}$$

$$= \mu_2 (a) - 2 \mu_1^2 (a) + \mu_1^2 (a)$$

$$\therefore \quad \boxed{\mu_2 = \mu_2 (a) - \mu_1^2 (a)}$$

$$\mu_3 = \frac{\Sigma f_i (x_i - \bar{x})^3}{\Sigma f_i} = \frac{\Sigma f_i \left[(x_i - a) - (\bar{x} - a)\right]^3}{\Sigma f_i}$$

$$= \frac{\Sigma f_i (x_i - a)^3}{\Sigma f_i} - \frac{3 (\bar{x} - a) \Sigma f_i (x_i - a)^2}{\Sigma f_i}$$

$$= \mu_2' - 2(\bar{x})^2 + (\bar{x})^2$$

$$= \mu_2' - (\bar{x})^2 = \mu_2' - (\mu_1')^2$$

$$\therefore \quad \boxed{\mu_2 = \mu_2' - (\mu_1'^2)}$$

$$\mu_3 = \frac{\sum f_i (x_i - \bar{x})^3}{\sum f_i} = \frac{1}{\sum f_i}\left[\sum f_i (x_i - \bar{x})^3\right]$$

$$= \frac{1}{\sum f_i}\left[\sum f_i \{x_i^3 - 3x_i^2 \bar{x} + 3 x_i (\bar{x})^2 - (\bar{x})^3\}\right]$$

$$= \frac{\sum f_i x_i^3}{\sum f_i} - 3\bar{x}\frac{\sum f_i x_i^2}{\sum f_i} + 3(\bar{x})^2 \frac{\sum f_i x_i}{\sum f_i} - \frac{\sum f_i (\bar{x})^3}{\sum f_i}$$

$$= \mu_3' - 3\bar{x}\mu_2' + 3(\bar{x})^2 \mu_1' - (\bar{x})^3$$

$$= \mu_3' - 3\mu_2' \mu_1' + 3(\mu_1')^2 \mu_1' - (\mu_1')^3$$

$$= \mu_3' - 3\mu_2' \mu_1' + 3(\mu_1')^3 - (\mu_1')^3$$

$$\therefore \quad \boxed{\mu_3 = \mu_3' - 3\mu_2' \mu_1' + 2(\mu_1')^3}$$

$$\mu_4 = \frac{\sum f_i (x_i - \bar{x})^4}{\sum f_i} = \frac{1}{\sum f_i}\left[\sum f_i (x_i - \bar{x})^4\right]$$

$$= \frac{1}{\sum f_i}\left[\sum f_i \{x_i^4 - 4x_i^3 \bar{x} + 6x_i^2 (\bar{x})^2 - 4x_i (\bar{x})^3 - (\bar{x})^4\}\right]$$

$$= \frac{\sum f_i x_i^4}{\sum f_i} - 4\bar{x}\frac{\sum f_i x_i^3}{\sum f_i} + 6(\bar{x})^2 \frac{\sum f_i x_i^2}{\sum f_i} - 4(\bar{x})^3 \frac{\sum f_i x_i}{\sum f_i} + \frac{\sum f_i (\bar{x})^4}{\sum f_i}$$

$$= \mu_4' - 4\mu_1' \mu_3' + 6(\mu_1')^2 \mu_2' - 4(\mu_1')^3 \mu_1' + (\mu_1')^4$$

$$= \mu_4' - 4\mu_1' \mu_3' + 6(\mu_1')^2 \mu_2' - 3(\mu_1')^4$$

$$\therefore \quad \boxed{\mu_4 = \mu_4' - 4\mu_1' \mu_3' + 6(\mu_1')^2 \mu_2' - 3(\mu_1')^4}$$

Q.4 Define raw moments and central moments. Express central moments of r^{th} order in terms of raw moments of order 1 to r.

Ans. Raw and central moments : Please refer to the answer of Q.1 of the same chapter.

To express central moments of r^{th} order in terms of raw moments of order 1 to r.

Central moment of order r is given by,

$$\mu_r = \frac{\sum f_i (x_i - \bar{x})^r}{\sum f_i} = \frac{1}{\sum f_i}\left[\sum f_i (x_i - \bar{x})^r\right]$$

$$= \frac{1}{\Sigma f_i}\left[\Sigma f_i (x_i^r - {}^r c_1 x_i^{r-1} \bar{x} + {}^r c_2 x_i^{r-2} (\bar{x})^2 - {}^r c_3 x_i^{r-3} (\bar{x})^3 + \ldots + (-1)^r (\bar{x})^r)\right]$$

$$= \frac{1}{\Sigma f_i}\left[\Sigma f_i x_i^r - {}^r c_1 \bar{x}\, \Sigma f_i x_i^{r-1} + {}^r c_2 (\bar{x})^2 \Sigma f_i x_i^{r-2} + \ldots (-1)^r (\bar{x})^r \Sigma f_i\right]$$

$$= \frac{\Sigma f_i x_i^r}{\Sigma f_i} - {}^r c_1 \mu_1' \frac{\Sigma f_i x_i^{r-1}}{\Sigma f_i} + {}^r c_2 (\mu_1')^2 \frac{\Sigma f_i x_i^{r-2}}{\Sigma f_i} + \ldots (-1)^r (\mu_1')^r \frac{\Sigma f_i}{\Sigma f_i}$$

$$= \mu_r' - {}^r c_1 \mu_1' \mu_{r-1}' + {}^r c_2 (\mu_1')^2 \mu_{r-2}' - {}^r c_3 (\mu_1')^3 \mu_{r-3}' + \ldots (-1)^r (\mu_1')^r$$

Q. 3 Define moments about any arbitrary constant 'a'. Deduce the relation between moments about 'a' and central moments.

Ans. The r^{th} moment about any arbitrary constant 'a' is defined as,

$$\mu_r(a) = \frac{\Sigma (x_i - a)^r}{N} \text{ for individual observations}$$

$$\text{and } \mu_r(a) = \frac{\Sigma f_i (x_i - a^r)}{\Sigma f_i} \text{ for frequency distribution}$$

$$\text{or } \mu_r(a) = \frac{\Sigma f_i u_i^r}{\Sigma f_i} \quad \text{or} \quad \frac{\Sigma f_i u_i^r}{\Sigma f_i} \times h^r \text{ where } u_i = \frac{x_i - A}{\Sigma f_i}$$ and h is class interval.

Relation between moments about any arbitrary constant 'a' and central moments :

$$\mu_1 = 0$$

$$\mu_2 = \frac{\Sigma f_i (x_i - \bar{x})^2}{\Sigma f_i} = \frac{\Sigma f_i \left[(x_i - a) - (\bar{x} - a)\right]^2}{\Sigma f_i}$$

$$= \frac{\Sigma f_i \left[(x_i - a)^2 - 2(\bar{x} - a)(x_i - a) + (\bar{x} - a)^2\right]}{\Sigma f_i}$$

$$= \frac{\Sigma f_i (x_i - a)^2}{\Sigma f_i} - 2(\bar{x} - a)\frac{\Sigma f_i (x_i - a)}{\Sigma f_i} + \frac{\Sigma f_i (\bar{x} - a)^2}{\Sigma f_i}$$

$$= \mu_2(a) - 2\mu_1^2(a) + \mu_1^2(a)$$

$$\therefore \quad \boxed{\mu_2 = \mu_2(a) - \mu_1^2(a)}$$

$$\mu_3 = \frac{\Sigma f_i (x_i - \bar{x})^3}{\Sigma f_i} = \frac{\Sigma f_i \left[(x_i - a) - (\bar{x} - a)\right]^3}{\Sigma f_i}$$

$$= \frac{\Sigma f_i (x_i - a)^3}{\Sigma f_i} - \frac{3(\bar{x} - a)\Sigma f_i (x_i - a)^2}{\Sigma f_i}$$

Q. 5 What will happen to the central moments if there is change in origin and scale ?

Ans. Let $u = \dfrac{x - a}{h}$ $\therefore$ $\bar{u} = \dfrac{\bar{x} - a}{h}$

Be definition of moment,

$$\mu_r \text{ of } u = \frac{\Sigma (u_i - \bar{u})^r}{N}$$

$$= \frac{\Sigma \left[\dfrac{x_i - a}{h} - \dfrac{\bar{x} - a}{h}\right]^r}{N}$$

$$= \frac{1}{h^r} \cdot \frac{\Sigma (x_i - \bar{x})^r}{N}$$

$$\therefore \quad \mu_r \text{ of } u = \frac{1}{h^r} \mu_r \text{ of } x.$$

Q. 6 Describe in brief the significance of moments.

Ans. The concept of moments is of great significance in statistical work. With the help of moments we can measure the central tendency of a set of observations, their variability, their asymmetry and the height of the peak their curve would make. Because of the great convenience in obtaining measures of the various characteristics of a frequency distribution, the calculation of the first four moments about the mean may well be made the first step in the analysis of a frequency distribution.

Moments	What it measures ?
1. First moment about origin	Mean

$$+ \frac{3(\bar{x}-a)^2}{\Sigma f_i} \Sigma f_i (x_i - a) - \frac{(\bar{x}-a)^3}{\Sigma f_i} \Sigma f_i$$

$$= \mu_3(a) - 3\mu_2(a)\mu_1(a) + 3\mu_1^3(a) - \mu_1^3(a)$$

$$= \mu_3(a) - 3\mu_2(a)\mu_1(a) + 2\mu_1^3(a)$$

$$\therefore \quad \boxed{\mu_3 = \mu_3(a) - 3\mu_2(a)\mu_1(a) + 2\mu_1^3(a)}$$

$$\mu_4 = \frac{\Sigma f_i (x_i - \bar{x})^4}{\Sigma f_i} = \frac{\Sigma f_i \left[(x_i - a) - (\bar{x} - a)\right]^2}{\Sigma f_i}$$

$$= \frac{\Sigma f_i (x_i - a)^4}{\Sigma f_i} - \frac{4(\bar{x}-a)}{\Sigma f_i} \Sigma f_i (x_i - a)^3$$

$$+ \frac{6(\bar{x}-a)^2}{\Sigma f_i} \Sigma f_i (x_i - a)^2 - \frac{4(\bar{x}-a)^3}{\Sigma f_i} \Sigma f_i (x_i - a)$$

$$+ \frac{(\bar{x}-a)^4}{\Sigma f_i} \Sigma f_i$$

$$= \mu_4(a) - 4\mu_3(a)\mu_1(a) + 6\mu_2(a)\mu_1^2(a) - 4\mu_1^4(a) + \mu_1^4(a)$$

$$= \mu_4(a) - 4\mu_3(a)\mu_1(a) + 6\mu_2(a)\mu_1^2(a) - 3\mu_1^4(a)$$

$$\therefore \quad \boxed{\mu_4 = \mu_4(a) - 4\mu_3(a)\mu_1(a) + 6\mu_2(a)\mu_1^2(a) - 3\mu_1^4(a)}$$

Q. 4 Show that the central moments are invariant to the change of origin.

Ans. Let origin is changed so that $u = x - a$ where a is any arbitrary constant.

$$\therefore \quad \bar{u} = \bar{x} - a$$

By definition of moment,

$$\mu_r \text{ of } u = \frac{\Sigma (u_i - \bar{u})^r}{N}$$

$$= \frac{\Sigma \left[(x_i - a) - (\bar{x} - a)\right]^2}{N}$$

$$= \frac{\Sigma (x_i - \bar{x})^r}{N}$$

$$= \mu_r \text{ of } x.$$

Thus there is no change in the value of central moment though origin is changed. i.e. The central moments are invariant to the change of origin.

Q. 5 What will happen to the central moments if there is change in origin and scale ?

Ans. Let $u = \frac{x-a}{h}$ $\therefore$ $\bar{u} = \frac{\bar{x}-a}{h}$

Be definition of moment,

$$\mu_r \text{ of } u = \frac{\Sigma (u_i - \bar{u})^r}{N}$$

$$= \frac{\Sigma\left[\frac{x_i - a}{h} - \frac{\bar{x} - a}{h}\right]^r}{N}$$

$$= \frac{1}{h^r} \cdot \frac{\Sigma (x_i - \bar{x})^r}{N}$$

$$\therefore \quad \mu_r \text{ of } u = \frac{1}{h^r} \mu_r \text{ of } x.$$

Q. 6 Describe in brief the significance of moments.

Ans. The concept of moments is of great significance in statistical work. With the help of moments we can measure the central tendency of a set of observations, their variability, their asymmetry and the height of the peak their curve would make. Because of the great convenience in obtaining measures of the various characteristics of a frequency distribution, the calculation of the first four moments about the mean may well be made the first step in the analysis of a frequency distribution.

Moments	What it measures ?
1. First moment about origin	Mean
2. Second moment about the mean	Variance
3. Third moment about the mean	Skewness
4. Fourth moment about the mean	Kurtosis

SOLVED EXAMPLES

Ex. 1 : From the following data calculate the first four moments about (i) assumed mean 35, (ii) actual mean, (iii) zero.

Marks	0 - 10	10 - 20	20 - 30	30 - 40	40 - 50	50 - 60	60 - 70
No. of students	8	12	20	30	15	10	5

Solution : Let us denote the moments about zero by $\mu_1', \mu_2', \ldots\ldots$, moments about actual mean i.e. central moments by $\mu_1, \mu_2 \ldots\ldots$, and the moments about assumed mean by $\mu_1'', \mu_2'', \ldots\ldots$.

Marks	**Mid-pt. x_i**	**No. of students (f_i)**	$u_i = \dfrac{x_i - 35}{10}$	$f_i u_i$	$f_i u_i^2$	$f_i u_i^3$	$f_i u_i^4$
0 - 10	5	8	– 3	– 24	72	– 216	648
10 - 20	15	12	– 2	– 24	48	– 96	192
20 - 30	25	20	– 1	– 20	20	– 20	20
30 - 40	35	30	0	0	0	0	0
40 - 50	45	15	1	15	15	15	15
50 - 60	55	10	2	20	40	80	160
60 - 70	65	5	3	15	45	135	405
Total		100		– 18	240	– 102	1440

(i) Let us first calculate the moments about assumed mean 35.

$$\mu_1'' = \frac{\Sigma f_i u_i}{\Sigma f_i} \times h \text{ where } h = 10 \text{ i.e. class interval}$$

$$\mu_1'' = \frac{-18}{100} \times 10 = -1.8$$

$$\mu_2'' = \frac{\Sigma f_i u_i^2}{\Sigma f_i} \times h^2 = \frac{240}{100} \times 100 = 240.$$

$$\mu_3'' = \frac{\Sigma f_i u_i^3}{\Sigma f_i} \times h^3 = \frac{-102}{100} \times 1000 = -1020.$$

$$\mu_4'' = \frac{\Sigma f_i u_i^4}{\Sigma f_i} \times h^4 = \frac{1440}{100} \times 10000 = 144000.$$

(ii) Now let us convert the moments about assumed mean to moments about actual mean . i.e. central moments.

$\mu_1 = 0$

$\mu_2 = \mu_2'' - (\mu_1'')^2 = 240 - (-1.8)^2 = 240 - 3.24$

$\therefore \mu_2 = 236.76$

$\mu_3 = \mu_3'' - 3\,\mu_1''\,\mu_2'' + 2\,(\mu_1'')^3$

$= -1020 - 3\,(-1.8)\,(240) + 2\,(-1.8)^2$

$= -1020 + 1296 - 11.664$

$\therefore \mu_3 = 265.336$

$\mu_4 = \mu_4'' - 4\,\mu_3''\,\mu_1'' + 6\,\mu_2''\,(\mu_1'')^2 - 3\,(\mu_1'')^4$

$= 144000 - 4\,(-1.8)\,(-1020) + 6\,(-1.8)^2\,(240) - 3\,(-1.8)^4$

$= 144000 - 7354 + 4665.\,6 - 31.4928$

$\therefore \mu_4 = 141290.11$

(iii) Lastly, let us convert the moments about assumed mean to moments about zero.

$\mu_1' = A + \mu_1''$ where A is assumed mean i.e. 35.

$\therefore \mu_1' = 35 - 1.8 = 33.2$

$\mu_2' = \mu_2 + (\mu_1')^2 = 236.76 + (33.2)^2$

$\therefore \mu_2' = 1339$

$\mu_3' = \mu_3 + 3\,(\mu_1')^2\,(\mu_2') - 2\,(\mu_1')^3$

$= 265.336 + 3\,(33.2)^2\,(1339) - 2\,(33.2)^2$

$= 265.336 + 4427697.9 - 73188.736$

$= 4354774.5$

$\mu_4' = \mu_4 + 4\,(\mu_1'\,\mu_3') - 6\,(\mu_1')^2\,(\mu_2') + 3\,(\mu_1')^4$

$= 141290.11 + 4\,(33.2)\,(4354774.5) - 6\,(33.2)^2\,(1339) + 3\,(33.2)^2$

$\therefore \mu_4' = 573144746.9.$

Ex. 2 : The first three moments about the value 3 for a certain distribution are 1, 16 and – 40 respectively. Find the mean, variance and third central moment of the distribution.

Solution : We are given moments about arbitrary origin 3, i.e.

$\mu_1' = 1, \mu_2' = 16$, and $\mu_3' = -40$.

Mean $= \bar{x} = A + \mu_1'$ where A is arbitrary origin.

$\therefore \quad \bar{x} = 3 + 1 = 4$

$$\text{Variance} = \mu_2 = \mu_2' - (\mu_1')^2$$

$$= 16 - (1)^2 = 15$$

$$\text{Third central moment} = \mu_3 = \mu_3' - 3\,\mu_1'\,\mu_2' - 2\,(\mu_1')^3$$

$$= -40 - 3\,(1)\,(16) + 2\,(1)^3$$

$$= -40 - 48 + 2 = -86$$

Ex. 3 : If the first four moments of distribution about the value 4 are equal to – 2, 11, – 58 and 280, determine the corresponding moments (i) about mean and (ii) about zero.

Solution : We are given moments about an arbitrary origin 4.

Thus $\mu_1' = -2$, $\mu_2' = 11$, $\mu_3' = -58$ and $\mu_4' = 280$.

(i) Moments about mean :

$\mu_1 = 0$

$$\mu_2 = \mu_2' - (\mu_1')^2 = 11 - (-2)^2 = 7$$

$$\mu_3 = \mu_3' - 3\,\mu_1'\,\mu_2' + 2\,(\mu_1')^3$$

$$= -58 - 3\,(-2)\,(11) + 2\,(-2)^3$$

$$= -58 + 66 - 16 = -8$$

$$\mu_4 = \mu_4' - 4\,\mu_1'\,\mu_3' + 6\,\mu_2'\,(\mu_1')^2 - 3\,(\mu_1')^4$$

$$= 280 - 4\,(-2)\,(-58) + 6\,(11)\,(-2)^2 - 3\,(-2)^4$$

$$= 280 - 464 + 264 - 48$$

$$= 544 - 512 = 32$$

Thus the moments about mean are

$\mu_1 = 0$, $\mu_2 = 7$, $\mu_3 = -8$, and $\mu_4 = 32$.

(ii) Moments about zero :

$\mu_1' = A + \mu_1'$ where A is an arbitrary origin i.e. A = 4

$\therefore \quad \mu_1' = 4 + (-2) = 2$

$$\mu_2' = \mu_2 + (\mu_1')^2 = 7 + (2)^2 = 11$$

$$\mu_3' = \mu_3 + 3(\mu_1')^2(\mu_2') - 2(\mu_1')^3$$

$$= -8 + 3(2)^2(11) - 2(2)^3$$

$$= -8 + 132 - 16 = 108$$

$$\mu_4' = \mu_4 + 4\mu_1'\mu_3' - 6(\mu_1')^2(\mu_2') + 3(\mu_1')^4$$

$$= 32 + 4(2)(108) - 6(2)^2(11) + 3(2)^4$$

$$= 32 + 864 - 264 + 48 = 680.$$

Thus the moments about zero are,

$\mu_1' = 2$, $\mu_2' = 11$, $\mu_3' = 108$ and $\mu_4' = 680$.

Ex. 4 : For a group of 10 observations, $\Sigma f_i x_i = -100$, $\Sigma f_i x_i^2 = 400$, $\Sigma f_i x_i^3 = -1000$ and $\Sigma f_i x_i^4 = 5000$. Find the first four central moments.

Solutions : First we will find μ_1', μ_2', μ_3' and μ_4'

$$\mu_1' = \frac{\Sigma f_i x_i}{\Sigma f_i} = \frac{-10}{10} = -10$$

$$\mu_2' = \frac{\Sigma f_i x_i^2}{\Sigma f_i} = \frac{40}{10} = 40$$

$$\mu_3' = \frac{\Sigma f_i x_i^3}{\Sigma f_i} = \frac{-100}{10} = -100$$

$$\mu_4' = \frac{\Sigma f_i x_i^4}{\Sigma f_i} = \frac{500}{10} = 500$$

To find central moments :

$$\mu_1 = 0$$

$$\mu_2 = \mu_2' - (\mu_1')^2 = 40 - (-10)^2 = 40 - 100 = -60$$

$$\mu_3 = \mu_3' - 3\mu_1'\mu_2' + 2(\mu_1')^3$$

$$= -100 - 3(-10)(40) + 2(-10)^3$$

$$= -100 + 1200 - 2000 = -900$$

$$\mu_4 = \mu_4' - 4\mu_1'\mu_3' + 6(\mu_1')^2(\mu_2') - 3(\mu_1')^4$$

$$= 500 - 4(-10)(-100) + 6(-10)^2(40) - 3(-10)^4$$

$$= 500 - 4000 + 24000 - 30000 = -9500$$

Thus the central moments are

$\mu_1 = 0$, $\mu_2 = -60$, $\mu_3 = -900$, $\mu_4 = -9500$.

EXAMPLES FOR PRACTICE

1. Compute the first four central moments for the frequency distribution.

Age	0 - 10		
No. of Persons	6		26

(**Ans.** $\mu_1 = 0, \mu_2 = 87.79, \mu_3 = 181.38, \mu_4 = 23457.748$).

2. Find the first four central moments for the observations :

4, 0, 2, 6, 3, 1, –7, –5, 1, 5.

(**Ans.** $\mu_1 = 0, \mu_2 = 16.6, \mu_3 = -35.4, \mu_4 = 607.2$)

3. Find the first four moments for the following data (i) assumed mean 40.5 and (ii) actual mean.

Hours Worked	No. of Industries	Hours worked	No. of Industries
30 - 32.9	2	39 - 41.9	47
33 - 35.9	4	42 - 44.9	15
36 - 38.9	26	43 - 47.9	6

(**Ans.** (i) $\mu_1'' = -0.39, \mu_2'' = 8.91, \mu_3'' = -$

(ii) $\mu_1 = 0, \mu_2 = 8.76, \mu_3 = -2.928$ and $\mu_4 = 281.451$)

4. For a group of 100 observations, Σ –

$\Sigma f_i u_i^3 = 1948$ and $\Sigma f_i u_i^4 = 86752$ where $u_i = (x_i - 48)$. Find central moments.

(**Ans.** $\mu_1 = 0, \mu_2 = 19.45, \mu_3 = 0.18, \mu_4 = 837.9225$)

5. The first three moments of

10. Find its mean, standard deviation

(**Ans.** 3, $\sqrt{21}$, – 54).

6. The first four moments of a

are 2, 20,

(**Ans.** $\mu_1 = 0, \mu_2 = 16, \mu_3 = -64, \mu_4 = 322$).

the first four