Digital Signal Processing

Shreyas Wankhede

Contents

***Abstract*—This manual provides a simple introduction to digital signal processing.**

1 Software Installation

Run the following commands

```
sudo apt−get update
sudo apt−get install libffi−dev libsndfile1 python3
    −scipy python3−numpy python3−matplotlib
sudo pip install cffi pysoundfile
```

2 Digital Filter

2.1 Download the sound file from

```
https://github.com/
    shreyaswankhede12/EE3900/blob
    /master/Assignment%201/sound/
    Sound_Noise.wav
```

2.2 You will find a spectrogram at https://academo.org/demos/spectrum-analyzer. Upload the sound file that you downloaded in Problem **??** in the spectrogram and play. Observe the spectrogram. What do you find?
Solution: There are a lot of yellow lines between 440 Hz to 5.1 KHz. These represent the synthesizer key tones. Also, the key strokes are audible along with background noise.

2.3 Write the python code for removal of out of band noise and execute the code.
Solution:

```
https://github.com/
    shreyaswankhede12/EE3900/blob
    /master/Assignment%201/codes/
    qs%202/Cancel_noise.py
```

2.4 The output of the python script in Problem **??** is the audio file Sound_With_ReducedNoise.wav. Play the file in the spectrogram in Problem **??**. What do you observe?
Solution: The key strokes as well as background noise is subdued in the audio. Also, the signal is blank for frequencies above 5.1 kHz.

3 Difference Equation

3.1 Let

$$x(n) = \left\{ \underset{\uparrow}{1}, 2, 3, 4, 2, 1 \right\} \tag{3.1}$$

Sketch $x(n)$.

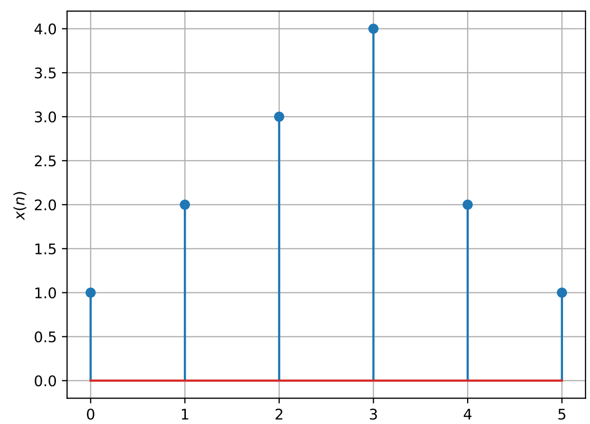

Fig. 3.1

3.2 Let

$$y(n) + \frac{1}{2}y(n-1) = x(n) + x(n-2),$$
$$y(n) = 0, n < 0 \tag{3.2}$$

Sketch $y(n)$.
Solution: The following code yields Fig. **??**.

```
https://github.com/
    shreyaswankhede12/EE3900/blob
    /master/Assignment%201/codes/
    qs%203/xn.py\\
https://github.com/
    shreyaswankhede12/EE3900/blob
```

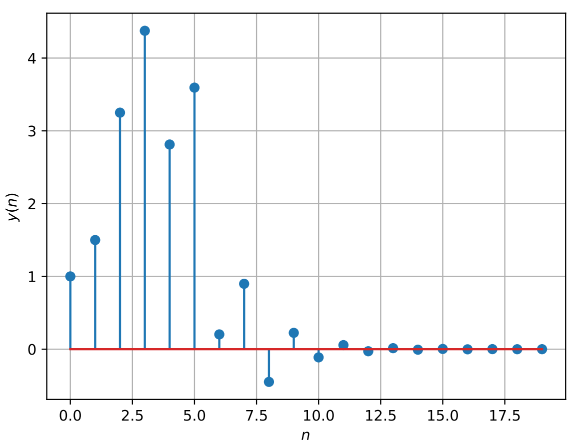

Fig. 3.2

/master/Assignment%201/codes/
qs%203/yn.py

3.3 Repeat the above exercise using a C code.
Solution: Run the following C code.

https://github.com/
shreyaswankhede12/EE3900/blob
/master/Assignment%201/codes/
qs%203/xn_yn.C

4 Z-TRANSFORM

4.1 The Z-transform of $x(n)$ is defined as

$$X(z) = \mathcal{Z}\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n)z^{-n} \tag{4.1}$$

Show that

$$\mathcal{Z}\{x(n-1)\} = z^{-1}X(z) \tag{4.2}$$

and find

$$\mathcal{Z}\{x(n-k)\} \tag{4.3}$$

Solution: From (**??**),

$$\mathcal{Z}\{x(n-k)\} = \sum_{n=-\infty}^{\infty} x(n-1)z^{-n} \tag{4.4}$$

$$= \sum_{n=-\infty}^{\infty} x(n)z^{-n-1} = z^{-1}\sum_{n=-\infty}^{\infty} x(n)z^{-n} \tag{4.5}$$

resulting in (**??**). Similarly, it can be shown that

$$\mathcal{Z}\{x(n-k)\} = z^{-k}X(z) \tag{4.6}$$

4.2 Obtain $X(z)$ for $x(n)$ defined in problem 3.1
Solution: Finding Z transform of $x(n)$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n} \tag{4.7}$$

$$= \sum_{n=0}^{5} x(n)z^{-n} \tag{4.8}$$

$$= 1 + 2z^{-1} + 3z^{-2} + 4z^{-3} + 2z^{-4} + z^{-5} \tag{4.9}$$

4.3 Find

$$H(z) = \frac{Y(z)}{X(z)} \tag{4.10}$$

from (**??**) assuming that the Z-transform is a linear operation.
Solution: Applying (**??**) in (**??**),

$$Y(z) + \frac{1}{2}z^{-1}Y(z) = X(z) + z^{-2}X(z) \tag{4.11}$$

$$\implies \frac{Y(z)}{X(z)} = \frac{1 + z^{-2}}{1 + \frac{1}{2}z^{-1}} \tag{4.12}$$

4.4 Find the Z transform of

$$\delta(n) = \begin{cases} 1 & n = 0 \\ 0 & \text{otherwise} \end{cases} \tag{4.13}$$

and show that the Z-transform of

$$u(n) = \begin{cases} 1 & n \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{4.14}$$

is

$$U(z) = \frac{1}{1 - z^{-1}}, \quad |z| > 1 \tag{4.15}$$

Solution: It is easy to show that

$$\delta(n) \overset{\mathcal{Z}}{\rightleftharpoons} 1 \tag{4.16}$$

and from (**??**),

$$U(z) = \sum_{n=0}^{\infty} z^{-n} \tag{4.17}$$

$$= \frac{1}{1 - z^{-1}}, \quad |z| > 1 \tag{4.18}$$

using the fomula for the sum of an infinite geometric progression.

4.5 Show that

$$a^n u(n) \overset{\mathcal{Z}}{\rightleftharpoons} \frac{1}{1 - az^{-1}} \quad |z| > |a| \tag{4.19}$$

Solution:

$$\mathcal{Z}\{a^n u(n)\} = \sum_{n=-\infty}^{\infty} a^n u(n) z^{-n} \tag{4.20}$$
$$= \sum_{n=0}^{\infty} a^n z^{-n} \tag{4.21}$$
$$= \sum_{n=0}^{\infty} \left(az^{-1}\right)^n \tag{4.22}$$
$$= \frac{1}{1-az^{-1}}, \quad \left|az^{-1}\right| < 1 \tag{4.23}$$

using the fomula for the sum of an infinite geometric progression.

4.6 Let

$$H\left(e^{\jmath\omega}\right) = H\left(z = e^{\jmath\omega}\right). \tag{4.24}$$

Plot $\left|H\left(e^{\jmath\omega}\right)\right|$. Comment. $H(e^{\jmath\omega})$ is known as the *Discret Time Fourier Transform* (DTFT) of $x(n)$. **Solution:** The following code plots Fig. **??**.

https://github.com/shreyaswankhede12/EE3900/blob/master/Assignment%201/codes/qs%204/%20dtft.py

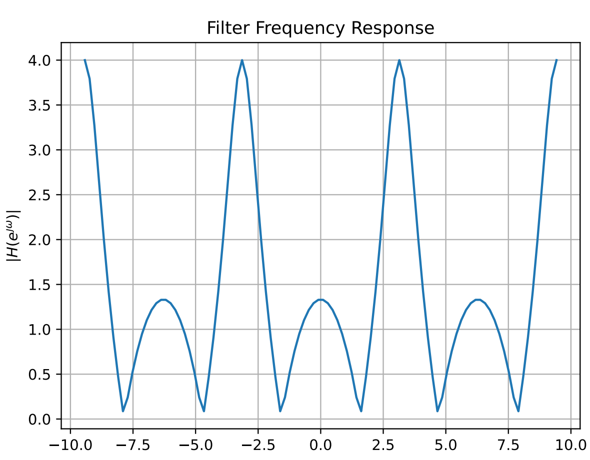

Fig. 4.6: $\left|H\left(e^{\jmath\omega}\right)\right|$

$$\left|H\left(e^{\jmath\omega}\right)\right| = \left|\frac{1+e^{-2\jmath\omega}}{1+\frac{1}{2}e^{-\jmath\omega}}\right| \tag{4.25}$$
$$= \sqrt{\frac{(1+\cos 2\omega)^2 + (\sin 2\omega)^2}{\left(1+\frac{1}{2}\cos\omega\right)^2 + \left(\frac{1}{2}\sin\omega\right)^2}} \tag{4.26}$$
$$0 = \sqrt{\frac{2\left(1+\cos 2\omega\right)}{\frac{5}{4}+\cos\omega}} \tag{4.27}$$
$$= \sqrt{\frac{2\left(2\cos^2\omega\right)}{\frac{5}{4}+\cos\omega}} \tag{4.28}$$
$$= \frac{4|\cos\omega|}{\sqrt{5+4\cos\omega}} \tag{4.29}$$

and so its fundamental period is 2π.

4.7 Express $x(n)$ in terms of $H\left(e^{j\omega}\right)$.
Solution:

$$h(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H\left(e^{\jmath\omega}\right) e^{j\omega n}\, d\omega \tag{4.30}$$
$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1+e^{-2\jmath\omega}}{1+\frac{1}{2}e^{-\jmath\omega}} e^{j\omega n}\, d\omega \tag{4.31}$$

5 Impulse Response

5.1 Using long division, find

$$h(n), \quad n < 5 \tag{5.1}$$

for H(z) in (**??**). **Solution:**

$$H(z) = \frac{1+z^{-2}}{1+\frac{1}{2}z^{-1}} \tag{5.2}$$
$$1+z^{-2} = \left(1+\frac{1}{2}z^{-1}\right) * \left(2z^{-1}-4\right) + 5 \tag{5.3}$$
$$H(z) = \frac{\left(1+\frac{1}{2}z^{-1}\right) * \left(2z^{-1}-4\right) + 5}{1+\frac{1}{2}z^{-1}} \tag{5.4}$$
$$= 2z^{-1} - 4 + \frac{5}{1+\frac{1}{2}z^{-1}} \tag{5.5}$$

Now,

$$\frac{5}{1+\frac{1}{2}z^{-1}} = 5\left(1 - \frac{z^{-1}}{2} + \frac{z^{-2}}{4} - \frac{z^{-3}}{8} + \ldots\right) \tag{5.6}$$

$$= 5 - \frac{5}{2}z^{-1} + \frac{5}{4}z^{-2} - \frac{5}{8}z^{-3} + \ldots \tag{5.7}$$

$$= \sum_{n=0}^{\infty} 5\left(\frac{-z^{-1}}{2}\right)^n \tag{5.8}$$

$$H(z) = 2z^{-1} - 4 + \frac{5}{1+\frac{1}{2}z^{-1}} \tag{5.9}$$

$$= 2z^{-1} - 4 + \sum_{n=0}^{\infty} 5\left(\frac{-z^{-1}}{2}\right)^n \tag{5.10}$$

As $n < 5$,

$$H(z) = 2z^{-1} - 4 + \sum_{n=0}^{4} 5\left(\frac{-z^{-1}}{2}\right)^n \tag{5.11}$$

$$H(z) = 1 - \frac{1}{2}z^{-1} + \frac{5}{4}z^{-2} - \frac{5}{8}z^{-3} + \frac{5}{16}z^{-4} \tag{5.12}$$

$$\implies h(n) = \left(1, \frac{-1}{2}, \frac{5}{4}, \frac{-5}{8}, \frac{5}{16}\right) \tag{5.13}$$

for general n,

$$h(n) = \begin{cases} 1 & n = 0 \\ -\frac{1}{2} & n = 1 \\ \frac{3}{2}\left(-\frac{1}{2}\right)^{n-2} & n \geq 2 \end{cases} \tag{5.14}$$

5.2 Find an expression for $h(n)$ using $H(z)$, given that

$$h(n) \overset{\mathcal{Z}}{\rightleftharpoons} H(z) \tag{5.15}$$

and there is a one to one relationship between $h(n)$ and $H(z)$. $h(n)$ is known as the *impulse response* of the system defined by (**??**).
Solution: From (**??**),

$$H(z) = \frac{1}{1+\frac{1}{2}z^{-1}} + \frac{z^{-2}}{1+\frac{1}{2}z^{-1}} \tag{5.16}$$

$$\implies h(n) = \left(-\frac{1}{2}\right)^n u(n) + \left(-\frac{1}{2}\right)^{n-2} u(n-2) \tag{5.17}$$

using (**??**) and (**??**).

5.3 Sketch $h(n)$. Is it bounded? Convergent?
Solution: The following code plots Fig. **??**.

```
https://github.com/
    shreyaswankhede12/EE3900/blob
    /master/Assignment%201/codes/
    qs%205/hn.py
```

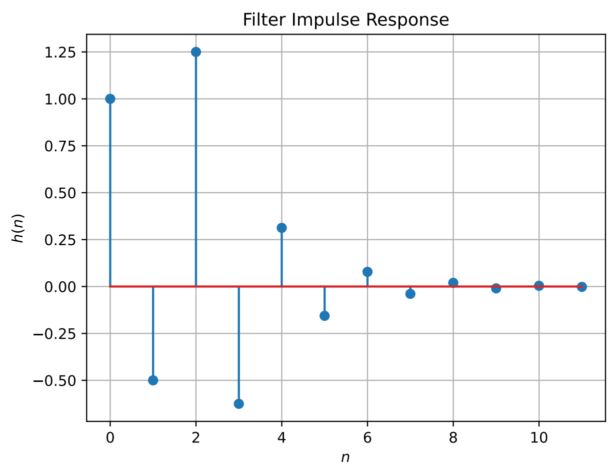

Fig. 5.3: $h(n)$ as the inverse of $H(z)$

The sequence is convergnet to 0 and hence bounded as well.

5.4 Convergent? Justify using the ratio test.
Solution: Using the ratio test for convergence

$$\lim_{n\to\infty}\left|\frac{h(n+1)}{h(n)}\right| = \lim_{n\to\infty}\left|\frac{\left(-\frac{1}{2}\right)^{n-1}\left(\frac{1}{4}+1\right)}{\left(-\frac{1}{2}\right)^{n-2}\left(\frac{1}{4}+1\right)}\right| \tag{5.18}$$

$$= \lim_{n\to\infty}\left|-\frac{1}{2}\right| \tag{5.19}$$

$$= \frac{1}{2} < 1 \tag{5.20}$$

Therefore, $h(n)$ is convergent.

5.5 The system with $h(n)$ is defined to be stable if

$$\sum_{n=-\infty}^{\infty} h(n) < \infty \tag{5.21}$$

Is the system defined by (**??**) stable for the impulse response in (**??**)?

Solution:

$$\sum_{n=-\infty}^{\infty} h(n) = \sum_{n=-\infty}^{\infty} \left(-\frac{1}{2}\right)^n u(n) + \sum_{n=-\infty}^{\infty} \left(-\frac{1}{2}\right)^{n-2} u(n-2) \quad (5.22)$$

$$\sum_{n=-\infty}^{\infty} h(n) = \sum_{n=0}^{\infty} \left(-\frac{1}{2}\right)^n + \sum_{n=2}^{\infty} \left(-\frac{1}{2}\right)^{n-2} \quad (5.23)$$

$$\sum_{n=-\infty}^{\infty} h(n) = \frac{1}{1-\left(-\frac{1}{2}\right)} + \frac{1}{1-\left(-\frac{1}{2}\right)} \quad (5.24)$$

$$= \frac{4}{3} < \infty \quad (5.25)$$

Therefore, the system is stable.

5.6 Verify the above result using a python code. **Solution:** :

```
https://github.com/
        shreyaswankhede12/EE3900/blob
        /master/Assignment%201/codes/
        qs%205/5.6.py
```

5.7 Compute and sketch $h(n)$ using

$$h(n) + \frac{1}{2}h(n-1) = \delta(n) + \delta(n-2), \quad (5.26)$$

This is the definition of $h(n)$.

Solution:

$$h(n) + \frac{1}{2}h(n-1) = \delta(n) + \delta(n-2), \quad (5.27)$$

This is the definition of $h(n)$.

$$h(0) = 1 \quad (5.28)$$

Now, for $n = 1$,

$$h(1) + \frac{1}{2}h(0) = \delta(1) + \delta(-1) = 0 \quad (5.29)$$

$$\implies h(1) = -\frac{1}{2}h(0) = -\frac{1}{2} \quad (5.30)$$

For $n = 2$,

$$h(2) + \frac{1}{2}h(1) = \delta(2) + \delta(0) = 1 \quad (5.31)$$

$$\implies h(2) = 1 - \frac{1}{2}h(1) = \frac{5}{4} \quad (5.32)$$

For $n > 2$, the right hand side of the equation is always zero. Thus,

$$h(n) = -\frac{1}{2}h(n-1) \qquad n > 2 \quad (5.33)$$

$$h(3) = \frac{5}{4}\left(-\frac{1}{2}\right) \quad (5.34)$$

$$h(4) = \frac{5}{4}\left(-\frac{1}{2}\right)^2 \quad (5.35)$$

$$\vdots \quad (5.36)$$

$$h(n) = \frac{5}{4}\left(-\frac{1}{2}\right)^{n-2} \quad (5.37)$$

Therefore,

$$h(n) = \begin{cases} 1 & n = 0 \\ -\frac{1}{2} & n = 1 \\ \frac{5}{4}\left(-\frac{1}{2}\right)^{n-2} & n \geq 2 \end{cases} \quad (5.38)$$

Thus, it is bounded and convergent to 0

$$\lim_{n\to\infty} h(n) = 0 \quad (5.39)$$

The following code plots Fig. **??**. Note that this is the same as Fig. **??**.

```
https://github.com/
        shreyaswankhede12/EE3900/blob
        /master/Assignment%201/codes/
        qs%205/hndef.py
```

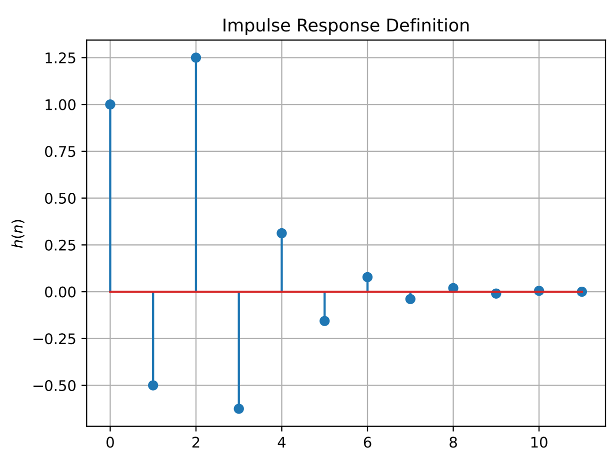

Fig. 5.7: $h(n)$ from the definition

5.8 Compute

$$y(n) = x(n) * h(n) = \sum_{n=-\infty}^{\infty} x(k)h(n-k) \quad (5.40)$$

Comment. The operation in (**??**) is known as *convolution*.

Solution:

$$x(n) * h(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k) \quad (5.41)$$

$$= \sum_{k=0}^{5} x(k)h(n-k) \quad (5.42)$$

Solution: The following code plots Fig. **??**. Note that this is the same as $y(n)$ in Fig. **??**.

```
https://github.com/
    shreyaswankhede12/EE3900/blob
    /master/Assignment%201/codes/
    qs%205/ynconv.py
```

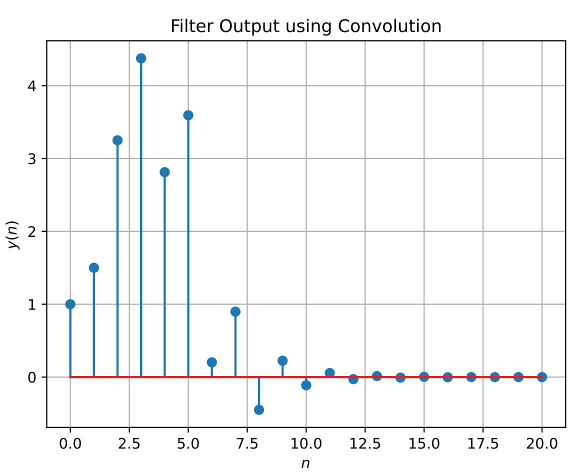

Fig. 5.8: $y(n)$ from the definition of convolution

5.9 Express the above convolution using a Teoplitz matrix.

Solution:

$$\mathbf{x} = \begin{pmatrix} 1 & 2 & 3 & 4 & 2 & 1 \end{pmatrix}^\top \quad (5.43)$$

$$\mathbf{h} = \begin{pmatrix} h_0 & h_1 & \cdots & h_{N-1} \end{pmatrix}^\top \quad (5.44)$$

$$\mathbf{y} = \mathbf{x} \circledast \mathbf{h} \quad (5.45)$$

$$\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_{N+5} \end{pmatrix} = \begin{pmatrix} h_0 & 0 & 0 & \cdots & 0 \\ h_1 & h_0 & 0 & \cdots & 0 \\ h_2 & h_1 & h_0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ h_{N-1} & h_{N-2} & h_{N-3} & \cdots & h_{N-6} \\ 0 & h_{N-1} & h_{N-2} & \cdots & h_{N-5} \\ 0 & 0 & h_{N-1} & \cdots & h_{N-4} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & h_{N-1} \end{pmatrix} \begin{pmatrix} 1.0 \\ 2.0 \\ 3.0 \\ 4.0 \\ 2.0 \\ 1.0 \end{pmatrix} \quad (5.46)$$

5.10 Show that

$$y(n) = \sum_{n=-\infty}^{\infty} x(n-k)h(k) \quad (5.47)$$

Solution: We know that

$$y(n) = x(n) * h(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k) \quad (5.48)$$

Substitute $k = n - i$

$$\sum_{k=-\infty}^{\infty} x(k)h(n-k) = \sum_{n-i=-\infty}^{\infty} x(n-i)h(n-(n-i)) \quad (5.49)$$

$$= \sum_{i=\infty}^{-\infty} x(n-i)h(i) \quad (5.50)$$

$$= \sum_{i=-\infty}^{\infty} x(n-i)h(i) \quad (5.51)$$

since the order of limits does not matter for a summation. Thus,

$$\sum_{k=-\infty}^{\infty} x(k)h(n-k) = \sum_{k=-\infty}^{\infty} x(n-k)h(k) \quad (5.52)$$

$$\implies x(n) * h(n) = h(n) * x(n) \quad (5.53)$$

6 DFT and FFT

6.1 Compute

$$X(k) \triangleq \sum_{n=0}^{N-1} x(n)e^{-\jmath 2\pi kn/N}, \quad k = 0, 1, \ldots, N-1 \quad (6.1)$$

and $H(k)$ using $h(n)$.
Solution: Download the following Python code that plots Fig. ??.

```
https://github.com/
    shreyaswankhede12/
    EE3900/blob/master/
    Assignment%201/codes/
    qs%206/6.1.py
```

Run the code by executing

```
python 6.1.py
```

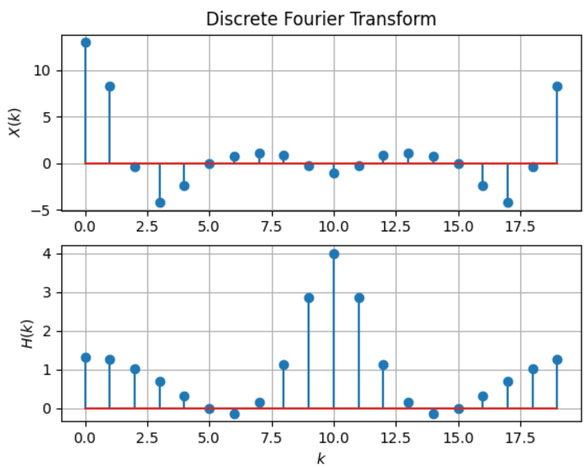

Fig. 6.1: Plots of the real parts of the discrete Fourier transforms of $x(n)$ and $h(n)$

6.2 Compute

$$Y(k) = X(k)H(k) \tag{6.2}$$

Solution: Download the following Python code that plots Fig. ??.

```
https://github.com/
    shreyaswankhede12/
    EE3900/blob/master/
    Assignment%201/codes/
    qs%206/6.2.py
```

Run the code by executing

```
python 6.2.py
```

6.3 Compute

$$y(n) = \frac{1}{N} \sum_{k=0}^{N-1} Y(k) \cdot e^{j2\pi kn/N}, \quad n = 0, 1, \dots, N-1 \tag{6.3}$$

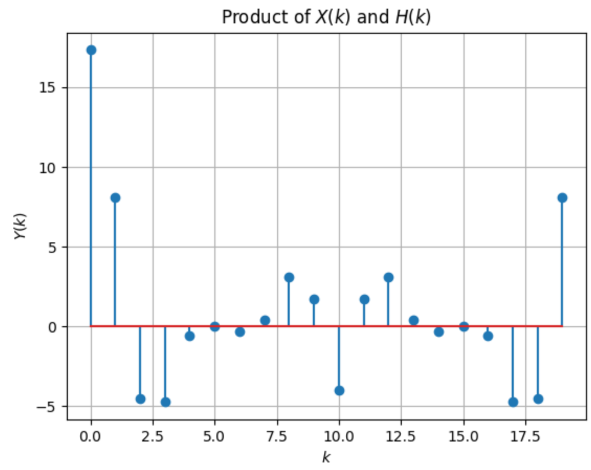

Fig. 6.2: Plot of $Y(k)$

Solution: Download the following Python code that plots Fig. ??.

```
https://github.com/
    shreyaswankhede12/
    EE3900/blob/master/
    Assignment%201/codes/
    qs%206/6.3.py
```

Run the code by executing

```
python 6.3.py
```

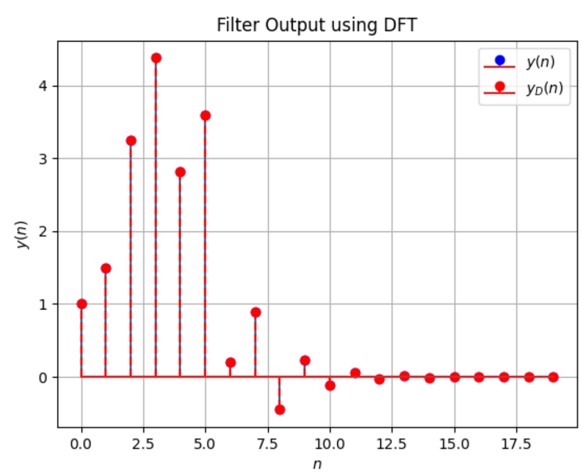

Fig. 6.3: Plot of the inverse discrete Fourier transform of $Y(k)$

The plot is exactly the same as that obtained in Fig. 3.2. Therefore, we conclude that

$$y(n) = x(n) * h(n) \tag{6.4}$$
$$\iff Y(k) = X(k)H(k) \tag{6.5}$$

6.4 Repeat the previous exercise by computing $X(k), H(k)$ and $y(n)$ through FFT and IFFT.
Solution: Download the following Python code that plots Fig. **??**.

```
https://github.com/
    shreyaswankhede12/
    EE3900/blob/master/
    Assignment%201/codes/
    qs%206/6.4.py
```

Run the code by executing

```
python 6.4.py
```

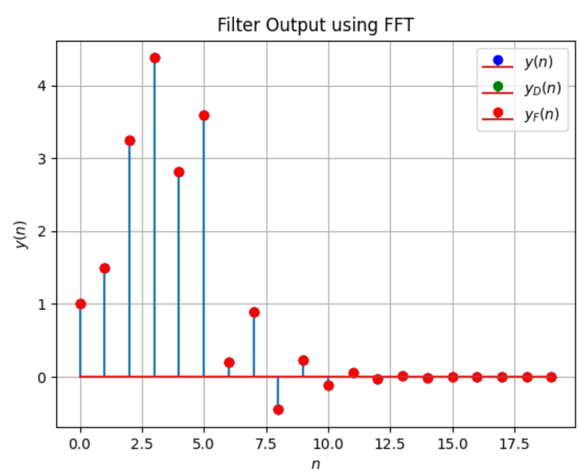

Fig. 6.4: Plot of $y(n)$ by fast Fourier transform

The plot is exactly the same as that obtained in Fig. 3.2.

6.5 Wherever possible, express all the above equations as matrix equations.

Solution:

$$\mathbf{x} = \begin{pmatrix} x_0 & x_1 & \cdots & x_{N-1} \end{pmatrix}^\top \tag{6.6}$$
$$\mathbf{h} = \begin{pmatrix} x_0 & x_1 & \cdots & x_{N-1} \end{pmatrix}^\top \tag{6.7}$$
$$\mathbf{y} = \mathbf{x} \circledast \mathbf{h} \tag{6.8}$$

$$\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_{2N-1} \end{pmatrix} = \begin{pmatrix} h_0 & 0 & 0 & \cdots & 0 \\ h_1 & h_0 & 0 & \cdots & 0 \\ h_2 & h_1 & h_0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ h_{N-1} & h_{N-2} & h_{N-3} & \cdots & h_0 \\ 0 & h_{N-1} & h_{N-2} & \cdots & h_1 \\ 0 & 0 & h_{N-1} & \cdots & h_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & h_{N-1} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{pmatrix} \tag{6.9}$$

The convolution can be written using a Toeplitz matrix.
Consider the DFT matrix

$$\mathbf{W} = \begin{pmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega & \omega^2 & \omega^3 & \cdots & \omega^{N-1} \\ 1 & \omega^2 & \omega^4 & \omega^6 & \cdots & \omega^{2(N-1)} \\ 1 & \omega^3 & \omega^6 & \omega^9 & \cdots & \omega^{3(N-1)} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{N-1} & \omega^{2(N-1)} & \omega^{3(N-1)} & \cdots & \omega^{(N-1)(N-1)} \end{pmatrix} \tag{6.10}$$

where $\omega = e^{-\jmath 2\pi/N}$ is the N^{th} root of unity
Then the discrete Fourier transforms of $\mathbf{x}$ and $\mathbf{h}$ are given by

$$\mathbf{X} = \mathbf{W}\mathbf{x} \tag{6.11}$$
$$\mathbf{H} = \mathbf{W}\mathbf{h} \tag{6.12}$$

$\mathbf{Y}$ is then given by

$$\mathbf{Y} = \mathbf{X} \circ \mathbf{H} \tag{6.13}$$

where $\circ$ denotes the Hadamard product (element-wise multiplication)
But $\mathbf{Y}$ is the discrete Fourier transform of the filter output $\mathbf{y}$

$$\mathbf{Y} = \mathbf{W}\mathbf{y} \tag{6.14}$$

Thus,

$$\mathbf{W}\mathbf{y} = \mathbf{X} \circ \mathbf{H} \tag{6.15}$$
$$\implies \mathbf{y} = \mathbf{W}^{-1}(\mathbf{X} \circ \mathbf{H}) \tag{6.16}$$
$$= \mathbf{W}^{-1}(\mathbf{W}\mathbf{x} \circ \mathbf{W}\mathbf{h}) \tag{6.17}$$

7 FFT

1. The DFT of $x(n)$ is given by
$$X(k) \triangleq \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N}, \quad k = 0, 1, \dots, N-1 \tag{7.1}$$
2. Let
$$W_N = e^{-j2\pi/N} \tag{7.2}$$
Then the N-point *DFT matrix* is defined as
$$\mathbf{F}_N = \left[W_N^{mn}\right], \quad 0 \le m, n \le N-1 \tag{7.3}$$
where W_N^{mn} are the elements of $\mathbf{F}_N$.
3. Let
$$\mathbf{I}_4 = \begin{pmatrix} \mathbf{e}_4^1 & \mathbf{e}_4^2 & \mathbf{e}_4^3 & \mathbf{e}_4^4 \end{pmatrix} \tag{7.4}$$
be the 4×4 identity matrix. Then the 4 point *DFT permutation matrix* is defined as
$$\mathbf{P}_4 = \begin{pmatrix} \mathbf{e}_4^1 & \mathbf{e}_4^3 & \mathbf{e}_4^2 & \mathbf{e}_4^4 \end{pmatrix} \tag{7.5}$$
4. The 4 point *DFT diagonal matrix* is defined as
$$\mathbf{D}_4 = diag\begin{pmatrix} W_8^0 & W_8^1 & W_8^2 & W_8^3 \end{pmatrix} \tag{7.6}$$
5. Show that
$$W_N^2 = W_{N/2} \tag{7.7}$$
Solution:
$$W_N^2 = \left(\exp\left(-j\frac{2\pi}{N}\right)\right)^2 \tag{7.8}$$
$$= \exp\left(-j\frac{2\pi}{N} \cdot 2\right) \tag{7.9}$$
$$= \exp\left(-j\frac{2\pi}{N/2}\right) \tag{7.10}$$
$$= W_{N/2} \tag{7.11}$$
6. Show that
$$\mathbf{F}_4 = \begin{bmatrix} \mathbf{I}_2 & \mathbf{D}_2 \\ \mathbf{I}_2 & -\mathbf{D}_2 \end{bmatrix} \begin{bmatrix} \mathbf{F}_2 & 0 \\ 0 & \mathbf{F}_2 \end{bmatrix} \mathbf{P}_4 \tag{7.12}$$
Solution:
$$\begin{bmatrix} \mathbf{I}_2 & \mathbf{D}_2 \\ \mathbf{I}_2 & -\mathbf{D}_2 \end{bmatrix} \begin{bmatrix} \mathbf{F}_2 & 0 \\ 0 & \mathbf{F}_2 \end{bmatrix} \tag{7.13}$$
$$= \begin{bmatrix} \mathbf{F}_2 & \mathbf{D}_2\mathbf{F}_2 \\ \mathbf{F}_2 & -\mathbf{D}_2\mathbf{F}_2 \end{bmatrix} \tag{7.14}$$
$$= \begin{bmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ 0 & -\jmath \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \\ \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} & -\begin{pmatrix} 1 & 0 \\ 0 & -\jmath \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \end{bmatrix} \tag{7.15}$$
$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -\jmath & \jmath \\ 1 & 1 & -1 & -1 \\ 1 & -1 & \jmath & -\jmath \end{bmatrix} \tag{7.16}$$
because $W_2^0 = 1$ and $W_2^1 = e^{-\jmath\pi} = -1$
Now
$$\begin{bmatrix} \mathbf{I}_2 & \mathbf{D}_2 \\ \mathbf{I}_2 & -\mathbf{D}_2 \end{bmatrix} \begin{bmatrix} \mathbf{F}_2 & 0 \\ 0 & \mathbf{F}_2 \end{bmatrix} \mathbf{P}_4 \tag{7.17}$$
$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -\jmath & \jmath \\ 1 & 1 & -1 & -1 \\ 1 & -1 & \jmath & -\jmath \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{7.18}$$
$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -\jmath & -1 & \jmath \\ 1 & -1 & 1 & -1 \\ 1 & \jmath & -1 & -\jmath \end{bmatrix} \tag{7.19}$$
$$= \begin{bmatrix} W_4^0 & W_4^0 & W_4^0 & W_4^0 \\ W_4^0 & W_4^1 & W_4^2 & W_4^3 \\ W_4^0 & W_4^2 & W_4^4 & W_4^6 \\ W_4^0 & W_4^3 & W_4^6 & W_4^9 \end{bmatrix} \tag{7.20}$$
$$= \mathbf{F}_4 \tag{7.21}$$
because
$$W_4^0 = 1 \tag{7.22}$$
$$W_4^1 = e^{-\jmath\frac{\pi}{2}} = -\jmath \tag{7.23}$$
$$W_4^2 = e^{-\jmath\pi} = -1 \tag{7.24}$$
$$W_4^3 = e^{-\jmath\frac{3\pi}{2}} = \jmath \tag{7.25}$$
$$W_4^n = W_4^{n-4} \qquad \forall n \ge 4 \tag{7.26}$$
7. Find
$$\mathbf{P}_4\mathbf{x} \tag{7.27}$$

Solution: Let $\mathbf{x} = \begin{pmatrix} x(0) & x(1) & x(2) & x(3) \end{pmatrix}^\top$

$$\mathbf{P}_4\mathbf{x} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \end{bmatrix} \tag{7.28}$$

$$= \begin{bmatrix} x(0) \\ x(2) \\ x(1) \\ x(3) \end{bmatrix} \tag{7.29}$$

8. Show that

$$\mathbf{X} = \mathbf{F}_N\mathbf{x} \tag{7.30}$$

where $\mathbf{x}, \mathbf{X}$ are the vector representations of $x(n), X(k)$ respectively.
Solution:

$$X(k) = \sum_{n=0}^{N-1} x(n)e^{-j2\pi kn/N}, \quad k = 0, 1, \dots, N-1 \tag{7.31}$$

$$\implies \mathbf{X} = \begin{bmatrix} \sum_{n=0}^{N-1} x(n)e^{-j2\pi n(0)/N} \\ \vdots \\ \sum_{n=0}^{N-1} x(n)e^{-j2\pi n(N-1)/N} \end{bmatrix} \tag{7.32}$$

$$= \begin{bmatrix} x(0) + \dots + x(N-1) \\ \vdots \\ x(0) + \dots + x(N-1)e^{-j2\pi(N-1)^2/N} \end{bmatrix} \tag{7.33}$$

$$\mathbf{X} = x(0)\begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} + \dots + x(N-1)\begin{bmatrix} 1 \\ \vdots \\ e^{-j2\pi(N-1)^2/N} \end{bmatrix} \tag{7.34}$$

$$= \begin{bmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & e^{-j2\pi(N-1)^2/N} \end{bmatrix} \begin{bmatrix} x(0) \\ \vdots \\ x(N-1) \end{bmatrix} \tag{7.35}$$

$$= \mathbf{F}_N\mathbf{x} \tag{7.36}$$

9. Derive the following step-by-step visualisation of 8-point FFTs into 4-point FFTs and so on

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} = \begin{bmatrix} X_1(0) \\ X_1(1) \\ X_1(2) \\ X_1(3) \end{bmatrix} + \begin{bmatrix} W_8^0 & 0 & 0 & 0 \\ 0 & W_8^1 & 0 & 0 \\ 0 & 0 & W_8^2 & 0 \\ 0 & 0 & 0 & W_8^3 \end{bmatrix} \begin{bmatrix} X_2(0) \\ X_2(1) \\ X_2(2) \\ X_2(3) \end{bmatrix} \tag{7.37}$$

$$\begin{bmatrix} X(4) \\ X(5) \\ X(6) \\ X(7) \end{bmatrix} = \begin{bmatrix} X_1(0) \\ X_1(1) \\ X_1(2) \\ X_1(3) \end{bmatrix} - \begin{bmatrix} W_8^0 & 0 & 0 & 0 \\ 0 & W_8^1 & 0 & 0 \\ 0 & 0 & W_8^2 & 0 \\ 0 & 0 & 0 & W_8^3 \end{bmatrix} \begin{bmatrix} X_2(0) \\ X_2(1) \\ X_2(2) \\ X_2(3) \end{bmatrix} \tag{7.38}$$

4-point FFTs into 2-point FFTs

$$\begin{bmatrix} X_1(0) \\ X_1(1) \end{bmatrix} = \begin{bmatrix} X_3(0) \\ X_3(1) \end{bmatrix} + \begin{bmatrix} W_4^0 & 0 \\ 0 & W_4^1 \end{bmatrix} \begin{bmatrix} X_4(0) \\ X_4(1) \end{bmatrix} \tag{7.39}$$

$$\begin{bmatrix} X_1(2) \\ X_1(3) \end{bmatrix} = \begin{bmatrix} X_3(0) \\ X_3(1) \end{bmatrix} - \begin{bmatrix} W_4^0 & 0 \\ 0 & W_4^1 \end{bmatrix} \begin{bmatrix} X_4(0) \\ X_4(1) \end{bmatrix} \tag{7.40}$$

$$\begin{bmatrix} X_2(0) \\ X_2(1) \end{bmatrix} = \begin{bmatrix} X_5(0) \\ X_5(1) \end{bmatrix} + \begin{bmatrix} W_4^0 & 0 \\ 0 & W_4^1 \end{bmatrix} \begin{bmatrix} X_6(0) \\ X_6(1) \end{bmatrix} \tag{7.41}$$

$$\begin{bmatrix} X_2(2) \\ X_2(3) \end{bmatrix} = \begin{bmatrix} X_5(0) \\ X_5(1) \end{bmatrix} - \begin{bmatrix} W_4^0 & 0 \\ 0 & W_4^1 \end{bmatrix} \begin{bmatrix} X_6(0) \\ X_6(1) \end{bmatrix} \tag{7.42}$$

$$\mathbf{P}_8 \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \\ x(4) \\ x(5) \\ x(6) \\ x(7) \end{bmatrix} = \begin{bmatrix} x(0) \\ x(2) \\ x(4) \\ x(6) \\ x(1) \\ x(3) \\ x(5) \\ x(7) \end{bmatrix} \tag{7.43}$$

$$\mathbf{P}_4 \begin{bmatrix} x(0) \\ x(2) \\ x(4) \\ x(6) \end{bmatrix} = \begin{bmatrix} x(0) \\ x(4) \\ x(2) \\ x(6) \end{bmatrix} \tag{7.44}$$

$$\mathbf{P}_4 \begin{bmatrix} x(1) \\ x(3) \\ x(5) \\ x(7) \end{bmatrix} = \begin{bmatrix} x(1) \\ x(5) \\ x(3) \\ x(7) \end{bmatrix} \tag{7.45}$$

Therefore,

$$\begin{bmatrix} X_3(0) \\ X_3(1) \end{bmatrix} = \mathbf{F}_2 \begin{bmatrix} x(0) \\ x(4) \end{bmatrix} \tag{7.46}$$

$$\begin{bmatrix} X_4(0) \\ X_4(1) \end{bmatrix} = \mathbf{F}_2 \begin{bmatrix} x(2) \\ x(6) \end{bmatrix} \tag{7.47}$$

$$\begin{bmatrix} X_5(0) \\ X_5(1) \end{bmatrix} = \mathbf{F}_2 \begin{bmatrix} x(1) \\ x(5) \end{bmatrix} \tag{7.48}$$

$$\begin{bmatrix} X_6(0) \\ X_6(1) \end{bmatrix} = \mathbf{F}_2 \begin{bmatrix} x(3) \\ x(7) \end{bmatrix} \tag{7.49}$$

Solution:

$$X(k) = \sum_{n=0}^{7} x(n)e^{-j2\pi kn/8}, \quad k = 0, \dots, 7 \tag{7.50}$$

$$= \sum_{n=0}^{7} x(n)W_8^{kn} \tag{7.51}$$

$$= \sum_{n \text{ is even}} x(n)W_8^{kn} + \sum_{n \text{ is odd}} x(n)W_8^{kn} \tag{7.52}$$

$$= \sum_{m=0}^{3} x(2m)W_8^{2km} + \sum_{m=0}^{3} x(2m+1)W_8^{2km+k} \tag{7.53}$$

Now substitute $W_8^2 = W_4$

$$X(k) = \sum_{m=0}^{3} x(2m)W_4^{km} + W_8^k \sum_{m=0}^{3} x(2m+1)W_4^{km} \tag{7.54}$$

Consider

$$x_1(n) = \{x(0), x(2), x(4), x(6)\} \tag{7.55}$$

$$x_2(n) = \{x(1), x(3), x(5), x(7)\} \tag{7.56}$$

Thus

$$X(k) = X_1(k) + W_8^k X_2(k) \qquad k = 0, \dots, 7 \tag{7.57}$$

Now, $X_1(k)$ and $X_2(k)$ are 4-point DFTs which means they are periodic with period 4

$$X(k+4) = X_1(k+4) + W_8^{k+4}X_2(k+4) \tag{7.58}$$

$$= X_1(k) + e^{-\jmath 2\pi(k+4)/8}X_2(k) \tag{7.59}$$

$$= X_1(k) + e^{-\jmath(2\pi k/8+\pi)}X_2(k) \tag{7.60}$$

$$= X_1(k) - e^{-\jmath 2\pi k/8}X_2(k) \tag{7.61}$$

$$= X_1(k) - W_8^k X_2(k) \tag{7.62}$$

Therefore, for $k = 0, 1, 2, 3$

$$X(k) = X_1(k) + W_8^k X_2(k) \tag{7.63}$$

$$X(k+4) = X_1(k) - W_8^k X_2(k) \tag{7.64}$$

which is the same as

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} = \begin{bmatrix} X_1(0) \\ X_1(1) \\ X_1(2) \\ X_1(3) \end{bmatrix} + \begin{bmatrix} W_8^0 & 0 & 0 & 0 \\ 0 & W_8^1 & 0 & 0 \\ 0 & 0 & W_8^2 & 0 \\ 0 & 0 & 0 & W_8^3 \end{bmatrix} \begin{bmatrix} X_2(0) \\ X_2(1) \\ X_2(2) \\ X_2(3) \end{bmatrix} \tag{7.65}$$

$$\begin{bmatrix} X(4) \\ X(5) \\ X(6) \\ X(7) \end{bmatrix} = \begin{bmatrix} X_1(0) \\ X_1(1) \\ X_1(2) \\ X_1(3) \end{bmatrix} - \begin{bmatrix} W_8^0 & 0 & 0 & 0 \\ 0 & W_8^1 & 0 & 0 \\ 0 & 0 & W_8^2 & 0 \\ 0 & 0 & 0 & W_8^3 \end{bmatrix} \begin{bmatrix} X_2(0) \\ X_2(1) \\ X_2(2) \\ X_2(3) \end{bmatrix} \tag{7.66}$$

Similarly, we can divide $x_1(n)$ into

$$x_3(n) = \{x(0), x(4)\} \tag{7.67}$$

$$x_4(n) = \{x(2), x(6)\} \tag{7.68}$$

i.e.,

$$\begin{bmatrix} X_3(0) \\ X_3(1) \end{bmatrix} = \mathbf{F}_2 \begin{bmatrix} x(0) \\ x(4) \end{bmatrix} \tag{7.69}$$

$$\begin{bmatrix} X_4(0) \\ X_4(1) \end{bmatrix} = \mathbf{F}_2 \begin{bmatrix} x(2) \\ x(6) \end{bmatrix} \tag{7.70}$$

to get

$$X_1(k) = X_3(k) + W_4^k X_4(k) \tag{7.71}$$

$$X_1(k+2) = X_3(k) - W_4^k X_4(k) \tag{7.72}$$

for $k = 0, 1$

$$\begin{bmatrix} X_1(0) \\ X_1(1) \end{bmatrix} = \begin{bmatrix} X_3(0) \\ X_3(1) \end{bmatrix} + \begin{bmatrix} W_4^0 & 0 \\ 0 & W_4^1 \end{bmatrix} \begin{bmatrix} X_4(0) \\ X_4(1) \end{bmatrix} \tag{7.73}$$

$$\begin{bmatrix} X_1(2) \\ X_1(3) \end{bmatrix} = \begin{bmatrix} X_3(0) \\ X_3(1) \end{bmatrix} - \begin{bmatrix} W_4^0 & 0 \\ 0 & W_4^1 \end{bmatrix} \begin{bmatrix} X_4(0) \\ X_4(1) \end{bmatrix} \tag{7.74}$$

And on dividing $x_2(n)$ into

$$x_5(n) = \{x(1), x(5)\} \tag{7.75}$$

$$x_6(n) = \{x(3), x(7)\} \tag{7.76}$$

i.e.,

$$\begin{bmatrix} X_5(0) \\ X_5(1) \end{bmatrix} = \mathbf{F}_2 \begin{bmatrix} x(1) \\ x(5) \end{bmatrix} \tag{7.77}$$

$$\begin{bmatrix} X_6(0) \\ X_6(1) \end{bmatrix} = \mathbf{F}_2 \begin{bmatrix} x(3) \\ x(7) \end{bmatrix} \tag{7.78}$$

to get

$$X_2(k) = X_5(k) + W_4^k X_6(k) \tag{7.79}$$

$$X_2(k+2) = X_5(k) - W_4^k X_6(k) \tag{7.80}$$

for $k = 0, 1$

$$\begin{bmatrix} X_2(0) \\ X_2(1) \end{bmatrix} = \begin{bmatrix} X_5(0) \\ X_5(1) \end{bmatrix} + \begin{bmatrix} W_4^0 & 0 \\ 0 & W_4^1 \end{bmatrix} \begin{bmatrix} X_6(0) \\ X_6(1) \end{bmatrix} \tag{7.81}$$

$$\begin{bmatrix} X_2(2) \\ X_2(3) \end{bmatrix} = \begin{bmatrix} X_5(0) \\ X_5(1) \end{bmatrix} - \begin{bmatrix} W_4^0 & 0 \\ 0 & W_4^1 \end{bmatrix} \begin{bmatrix} X_6(0) \\ X_6(1) \end{bmatrix} \tag{7.82}$$

10. For

$$\mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 2 \\ 1 \end{pmatrix} \tag{7.83}$$

compte the DFT using (??)
Solution: Download the following Python code that plots Fig. ??.

```
wget https://github.com/
    Ankit−Saha−2003/
    EE3900/raw/main/
    Assignment_1/codes
    /7.11.py
```

Run the code by executing

```
python 7.11.py
```

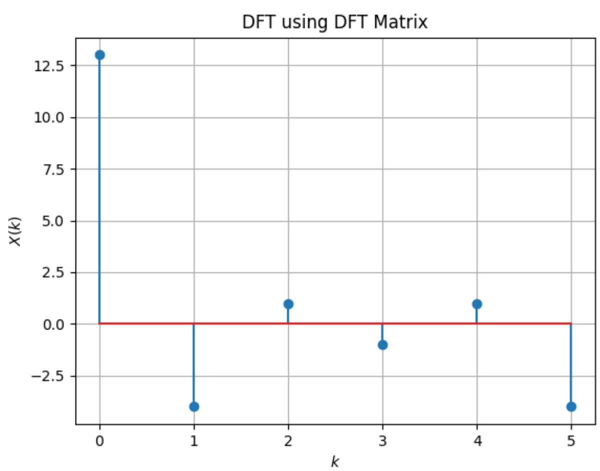

Fig. 7.10: Plot of the discrete fourier transform of **x** using the DFT matrix

11. Repeat the above exercise using the FFT after zero padding **x**.
Solution: Download the following Python code that plots Fig. ??.

```
wget https://github.com/
    Ankit−Saha−2003/
    EE3900/raw/main/
    Assignment_1/codes
    /7.12.py
```

Run the code by executing

```
python 7.12.py
```

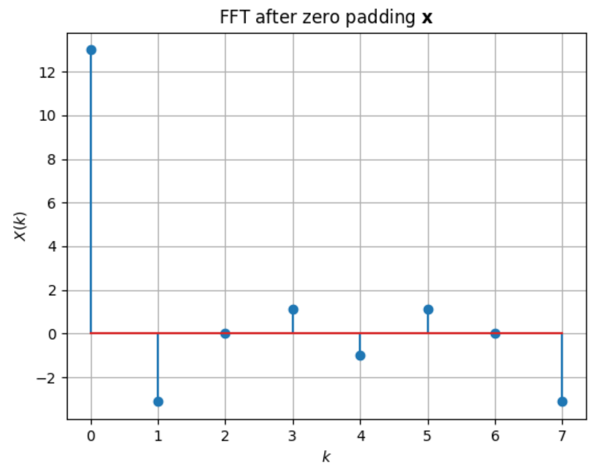

Fig. 7.11: Plot of the fast fourier transform of **x** after zero padding

12. Write a C program to compute the 8-point FFT.
Solution: Download the following C codes that generate the values of $X(k)$ using 8-point FFT

```
wget https://github.com/
    Ankit−Saha−2003/
    EE3900/raw/main/
    Assignment_1/codes/
    header.h
wget https://github.com/
    Ankit−Saha−2003/
    EE3900/raw/main/
    Assignment_1/codes
    /7.13.c
```

Compile and run the C program by executing the following

```
cc −lm 7.13.c
./a.out
```

13. Compare and determine the running time complexities of FFT/IFFT and convolution graphically
Solution: Download the following C codes that measure the running times of both the algorithms

```
wget https://github.com/
    Ankit−Saha−2003/
    EE3900/raw/main/
    Assignment_1/codes/
    header.h
wget https://github.com/
    Ankit−Saha−2003/
```

```
EE3900/raw/main/
Assignment_1/codes
/7.14.c
```

Compile and run the C program by executing the following

```
cc −lm 7.14.c
./a.out
```

Download the following Python code that plots Fig. **??** using the running times generated by the C code and fits them to appropriate functions of the input size

```
wget https://github.com/
    Ankit−Saha−2003/
    EE3900/raw/main/
    Assignment_1/codes
    /7.14.py
```

Run the code by executing

```
python 7.14.py
```

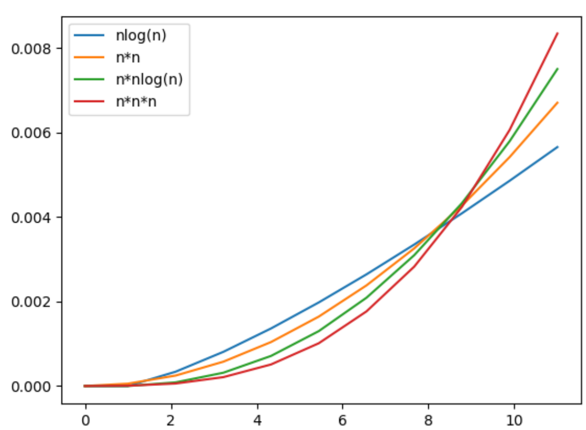

Fig. 7.13: Plot of the running times of FFT/IFFT and convolution

From the plot, it is evident that the time complexity of FFT/IFFT is $O(n \log n)$ and that of convolution is $O(n^2)$