

Probability and Random Variables Assignment

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1 UNIFORM RANDOM NUMBERS

Let U be a uniform random variable between 0 and 1.

- 1.1 Generate 10^6 samples of U using a C program and save into a file called uni.dat .

Solution: Download the following files and execute the C program.

```
wget https://github.com/shreyaswankhede12/
Random_numbers/blob/main/codes/1.1_exrand.
wget https://github.com/shreyaswankhede12/
Random_numbers/blob/main/codes/coeffs.h
```

Compile and run the C program by executing the following

```
cc -lm 1.1_exrand.c
./a.out
```

- 1.2 Load the uni.dat file into python and plot the empirical CDF of U using the samples in uni.dat. The CDF is defined as

$$F_U(x) = \Pr(U \leq x) \quad (1.1)$$

Solution: Download the following Python code that plots

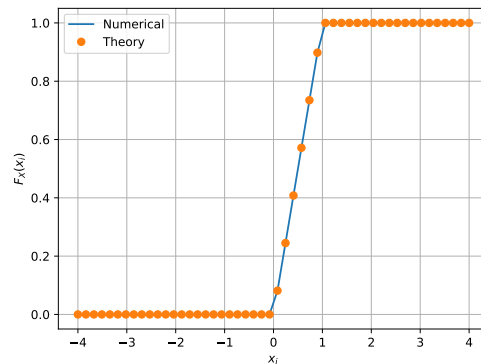


Fig:1.2 CDF of U

```
wget https://github.com/shreyaswankhede12/
Random_numbers/blob/main/codes/1.2_cdf.py
```

Run the code by executing

```
python 1.2_cdf.py
```

- 1.3 Find a theoretical expression for $F_U(x)$. **Solution:** The PDF of U is given by

$$p_U(x) = \begin{cases} 1 & x \in [0, 1] \\ 0 & \text{otherwise} \end{cases} \quad (1.2)$$

The CDF of U is given by

$$F_U(x) = \Pr(U \leq x) = \int_{-\infty}^x p_U(x) dx \quad (1.3)$$

If $x < 0$,

$$\int_{-\infty}^x p_U(x) dx = \int_{-\infty}^x 0 dx = 0 \quad (1.4)$$

If $0 \leq x < 1$,

$$\int_{-\infty}^x p_U(x) dx = \int_{-\infty}^0 0 dx + \int_0^x 1 dx \quad (1.5)$$

$$= 0 + x \quad (1.6)$$

$$= x \quad (1.7)$$

If $x \geq 1$,

$$\begin{aligned} \int_{-\infty}^x p_U(x) dx &= \int_{-\infty}^0 0 dx + \int_0^1 1 dx + \int_1^x 0 dx \\ &= 0 + 1 + 0 \end{aligned} \quad (1.8)$$

$$\int_{-\infty}^x p_U(x) dx = 0 + 1 + 0 \quad (1.9)$$

$$= 1 \quad (1.10)$$

Therefore, we obtain the CDF of U as

$$F_U(x) = \begin{cases} 0 & x < 0 \\ x & 0 \leq x \leq 1 \\ 1 & x > 1 \end{cases} \quad (1.11)$$

1.4 The mean of U is defined as

$$E[U] = \frac{1}{N} \sum_{i=1}^N U_i \quad (1.12)$$

and its variance as

$$\text{var}[U] = E[U - E[U]]^2 \quad (1.13)$$

Write a C program to find the mean and variance of U .

Solution: execute the following

```
wget https://github.com/
shreyaswankhede12/
Random_numbers/blob/
main/codes/1.4.c
```

1.5 Verify your result theoretically given that

$$E[U^k] = \int_{-\infty}^{\infty} x^k dF_U(x) \quad (1.14)$$

solution: From 1.11,

$$\begin{aligned} F_U(x) &= x \quad \forall x, 0 \leq x \leq 1 \\ \therefore dF_U(x) &= dx \end{aligned} \quad (1.15)$$

$$E[U] = \int_0^1 x dx = \frac{1}{2} = 0.5 \quad (1.16)$$

similarly,

$$E[U^2] = \int_0^1 x^2 dx = \frac{1}{3} \quad (1.17)$$

$$\begin{aligned} \text{Var}[U] &= E[U^2] - (E[U])^2 \\ &= \frac{1}{3} - \left(\frac{1}{2}\right)^2 \\ &= \frac{1}{3} - \frac{1}{4} \\ &= \frac{1}{12} \end{aligned} \quad (1.18)$$

2 CENTRAL LIMIT THEOREM

2.1 Generate 10^6 samples of the random variable

$$X = \sum_{i=1}^{12} U_i - 6 \quad (2.1)$$

using a C program, where $U_i, i = 1, 2, \dots, 12$ are a set of independent uniform random variables between 0 and 1 and save in a file called gau.dat

Solution: link for the code:

```
wget https://github.com/
shreyaswankhede12/
Random_numbers/blob/
main/codes/2.1.c\
wget https://github.com/
shreyaswankhede12/
Random_numbers/blob/
main/codes/coeffs.h
```

execute the following by

```
gcc 2.1.c -lm\
./a.out
```

- 2.2 Load gau.dat in python and plot the empirical CDF of X using the samples in gau.dat. What properties does a CDF have?

Solution: The CDF of X is plotted link to the python code

```
wget https://github.com/
shreyaswankhede12/
Random_numbers/blob/
main/codes/2.2_emp_cdf
.py
```

execute the following by

```
python 2.2_emp_cdf.py
```

- 2.3 Load gau.dat in python and plot the empirical PDF of X using the samples in gau.dat. The PDF of X is defined as

$$p_X(x) = \frac{d}{dx} F_X(x) \quad (2.2)$$

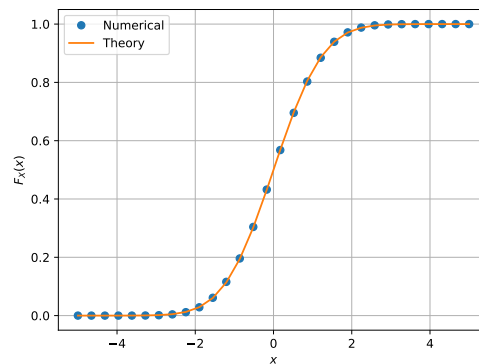


Fig:2.2 The CDF of X

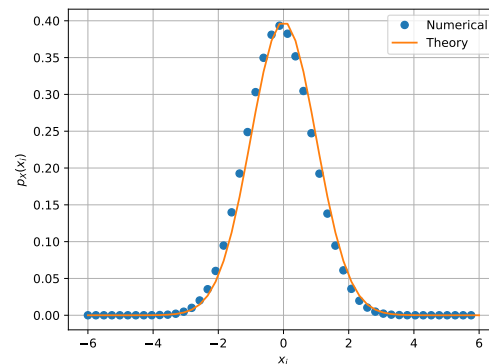


Fig:2.3 The PDF of X

What properties does the PDF have?

Solution: The PDF of X is plotted in Fig. ?? using the code below

```
wget https://github.com/
shreyaswankhede12/
Random_numbers/blob/
main/codes/2.3_emp_pdf
.py
```

- 2.4 Find the mean and variance of X by writing a C program.

Solution: link for c program

```
wget https://github.com/
shreyaswankhede12/
Random_numbers/blob/
main/codes/2.4.c
```

execute the following by

```
gcc 2.4.c -lm
```

2.5 Given that

$$p_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), -\infty < x < \infty, \quad (2.3)$$

repeat the above exercise theoretically.

solution:

$$E[X] = \int_{-\infty}^{\infty} x p_X(x) dx \quad (2.4)$$

$$= \int_{-\infty}^{\infty} \frac{x}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx \quad (2.5)$$

$$\text{Let } h(x) = \frac{x}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right)$$

Now,

$$h(-x) = -h(x)$$

Thus $h(x)$ is an odd function

$$\therefore E[X] = \int_{-\infty}^{\infty} h(x) dx = 0 \quad (2.6)$$

$$E[X^2] = \int_{-\infty}^{\infty} x^2 p_X(x) dx \quad (2.7)$$

$$= \int_{-\infty}^{\infty} \frac{x^2}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx \quad (2.8)$$

Using integration by parts:

$$= x \int x e^{-\frac{x^2}{2}} dx - \int \frac{d(x)}{dx} \int x e^{-\frac{x^2}{2}} dx \quad (2.9)$$

$$I = \int x e^{-\frac{x^2}{2}} \quad (2.10)$$

$$\text{Let } \frac{x^2}{2} = t \quad (2.11)$$

$$\Rightarrow x dx = dt \quad (2.12)$$

$$\Rightarrow \int e^{-t} dt = -e^{-t} + c \quad (2.13)$$

$$\therefore \int x e^{-\frac{x^2}{2}} = -e^{-\frac{x^2}{2}} + c \quad (2.14)$$

Using (2.14) in (2.9)

$$= -x e^{-\frac{x^2}{2}} + \int e^{-\frac{x^2}{2}} dx \quad (2.15)$$

$$\text{Also, } \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} dx = \sqrt{2\pi} \quad (2.16)$$

\therefore substituting limits we get

$$E[x^2] = 1 \quad (2.17)$$

$$\text{var}(X) = E[x^2] - (E[x])^2 = 1 - 0 \quad (2.18)$$

3 FROM UNIFORM TO OTHER

3.1 Generate samples of

$$V = -2 \ln(1 - U) \quad (3.1)$$

and plot its CDF.

Solution: : Run the following codes

```
wget https://github.com/
shreyaswankhede12/
Random_numbers/blob/
main/codes/3.1_code.c
```

execute by commands:

```
gcc 3.1_code.c -lm
```

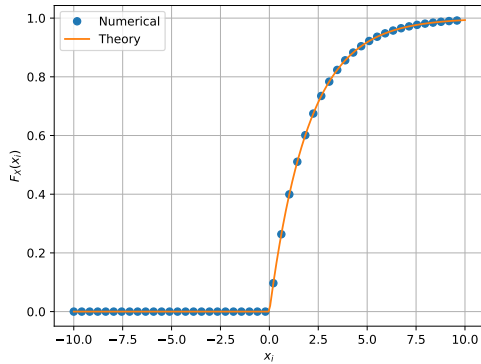


Fig:3.1 The CDF of V

```
./ a . out
```

```
wget https://github.com/
shreyaswankhede12/
Random_numbers/blob/
main/codes/3.1_plot.py
```

3.2 Find a theoretical expression for $F_V(x)$. **Solution:** We have

$$F_V(x) = \Pr(V \leq x) \quad (3.2)$$

$$= \Pr(-2 \ln(1 - U) \leq x) \quad (3.3)$$

$$= \Pr\left(\ln(1 - U) \geq -\frac{x}{2}\right) \quad (3.4)$$

$$= \Pr\left(1 - U \geq \exp\left(-\frac{x}{2}\right)\right) \quad (3.5)$$

$$= \Pr\left(U \leq 1 - \exp\left(-\frac{x}{2}\right)\right) \quad (3.6)$$

$$= F_U\left(1 - \exp\left(-\frac{x}{2}\right)\right) \quad (3.7)$$

Now,

$$0 \leq 1 - \exp\left(-\frac{x}{2}\right) < 1 \quad \text{if } x \geq 0 \quad (3.8)$$

$$1 - \exp\left(-\frac{x}{2}\right) < 0 \quad \text{if } x < 0 \quad (3.9)$$

Therefore, from (1.11)

$$F_V(x) = \begin{cases} 1 - \exp\left(-\frac{x}{2}\right) & x \geq 0 \\ 0 & x < 0 \end{cases} \quad (3.10)$$

4 TRIANGULAR DISTRIBUTION

4.1 Generate

$$T = U_1 + U_2 \quad (4.1)$$

Solution: this generates U_1 and U_2

```
wget https://github.com/
shreyaswankhede12/
Random_numbers/blob/
main/codes/4.1_gen_u1_
u2.c
```

this code generates $U_1 + U_2$

```
wget https://github.com/
shreyaswankhede12/
Random_numbers/blob/
main/codes/4.1_gen_u1+
u2.c
```

4.2 Find the CDF of T .

Solution: python code for plotting cdf:

```
wget https://github.com/
shreyaswankhede12/
Random_numbers/blob/
main/codes/4.2.py
```

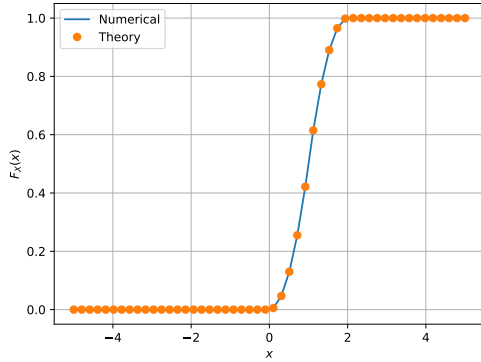


Fig:4.2 The CDF of T

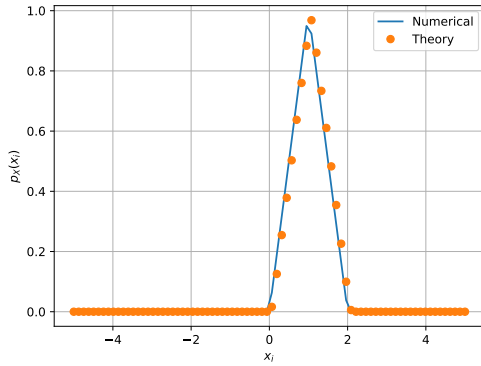


Fig:4.3 The PDF of T

4.3 Find the PDF of T .

Solution: python code for plotting pdf:

```
wget https://github.com/
shreyaswankhede12/
Random_numbers/blob/
main/codes/4.3.py
```

4.4 Find the theoretical expressions for the PDF and CDF of T .

Solution: The CDF of T is given by

$$F_T(t) = \Pr(T \leq t) = \Pr(U_1 + U_2 \leq t) \quad (4.2)$$

$$U_1 + U_2 \in [0, 2]$$

Thus, for $t \geq 2$, then $U_1 + U_2 \leq t$ is always true

for $t < 0$, then $U_1 + U_2 \leq t$ is always false.

Now, fix the value of U_1 to be some x

$$x + U_2 \leq t \implies U_2 \leq t - x \quad (4.3)$$

If $0 \leq t \leq 1$, then x can take all values in $[0, t]$

$$F_T(t) = \int_0^t F_{U_2}(t-x)p_{U_1}(x)dx \quad (4.4)$$

since, $t-x \in [0, 1]$

$$F_{U_2}(t-x) = t-x \quad (4.5)$$

$$F_T(t) = \int_0^t (t-x).1.dx \quad (4.6)$$

$$= \frac{t^2}{2} \quad (4.7)$$

If $1 < t < 2$, x can only take values in $[0, 1]$ as $U_1 \leq 1$

$$F_T(t) = \int_0^1 F_{U_2}(t-x).1.dx \quad (4.8)$$

since, $t-x \in [0, 1]$,

$$F_T(t) = \int_0^{t-1} 1dx + \int_{t-1}^1 (t-x)dx \quad (4.9)$$

$$= -\frac{t^2}{2} + 2t - 1 \quad (4.10)$$

Therefore,

$$F_T(t) = \begin{cases} 0 & t < 0 \\ \frac{t^2}{2} & 0 \leq t \leq 1 \\ 2t - \frac{t^2}{2} - 1 & 1 < t < 2 \\ 1 & t \geq 2 \end{cases} \quad (4.11)$$

On differentiating, we get,

$$P_T(t) = \begin{cases} 0 & t < 0 \\ t & 0 \leq t \leq 1 \\ 2 - t & 1 < t < 2 \\ 1 & t \geq 2 \end{cases} \quad (4.12)$$

4.5 Verify your results through a plot.

Solution: Refer to solution of qs 4.2 and 4.3

5 MAXIMUM LIKELIHOOD

5.1 Generate

$$Y = AX + N, \quad (5.1)$$

where $A = 5$ dB, $X \in \{1, -1\}$, is Bernoulli and $N \sim \mathcal{N}(0, 1)$.

5.2 Plot Y .

5.3 Guess how to estimate X from Y .

5.4 Find

$$P_{e|0} = \Pr(\hat{X} = -1 | X = 1) \quad (5.2)$$

and

$$P_{e|1} = \Pr(\hat{X} = 1 | X = -1) \quad (5.3)$$

5.5 Find P_e .

5.6 Verify by plotting the theoretical P_e .

6 GAUSSIAN TO OTHER

6.1 Let $X_1 \sim \mathcal{N}(0, 1)$ and $X_2 \sim \mathcal{N}(0, 1)$. Plot the CDF and PDF of

$$V = X_1^2 + X_2^2 \quad (6.1)$$

6.2 If

$$F_V(x) = \begin{cases} 1 - e^{-\alpha x} & x \geq 0 \\ 0 & x < 0, \end{cases} \quad (6.2)$$

find α .

6.3 Plot the CDF and PDF of

$$A = \sqrt{V} \quad (6.3)$$

7 CONDITIONAL PROBABILITY

7.1

7.2 Plot

$$P_e = \Pr(\hat{X} = -1 | X = 1) \quad (7.1)$$

for

$$Y = AX + N, \quad (7.2)$$

where A is Rayleigh with $E[A^2] = \gamma$, $N \sim \mathcal{N}(0, 1)$, $X \in (-1, 1)$ for $0 \leq \gamma \leq 10$ dB.

7.3 Assuming that N is a constant, find an expression for P_e . Call this $P_e(N)$

7.4 For a function g ,

$$E[g(X)] = \int_{-\infty}^{\infty} g(x)p_X(x)dx \quad (7.3)$$

Find $P_e = E[P_e(N)]$.

7.5 Plot P_e in problems 7.2 and 7.4 on the same graph w.r.t γ . Comment.

8 TWO DIMENSIONS

Let

$$\mathbf{y} = \mathbf{A}\mathbf{x} + \mathbf{n}, \quad (8.1)$$

where

$$x \in (\mathbf{s}_0, \mathbf{s}_1), \mathbf{s}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{s}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (8.2)$$

$$\mathbf{n} = \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}, n_1, n_2 \sim \mathcal{N}(0, 1). \quad (8.3)$$

8.1 Plot

$$\mathbf{y}|\mathbf{s}_0 \text{ and } \mathbf{y}|\mathbf{s}_1 \quad (8.4)$$

on the same graph using a scatter plot.

8.2 For the above problem, find a decision rule for detecting the symbols \mathbf{s}_0 and \mathbf{s}_1 .

8.3 Plot

$$P_e = \Pr(\hat{\mathbf{x}} = \mathbf{s}_1 | \mathbf{x} = \mathbf{s}_0) \quad (8.5)$$

with respect to the SNR from 0 to 10 dB.

8.4 Obtain an expression for P_e . Verify this by comparing the theory and simulation plots on the same graph.