Probability and Random Variables Assignment

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1 Uniform Random Numbers

Let U be a uniform random variable between 0 and 1.

1.1 Generate 10^6 samples of U using a C program and save into a file called uni.dat .
Solution: Download the following files and execute the C program.

```
wget https://github.com/
    shreyaswankhede12/
    Random_numbers/blob/
    main/codes/1.1_exrand.
wget https://github.com/
    shreyaswankhede12/
    Random_numbers/blob/
    main/codes/coeffs.h
```

Compile and run the C program by executing the following

```
cc -lm 1.1_exrand.c
./a.out
```

1.2 Load the uni.dat file into python and plot the empirical CDF of U using the samples in uni.dat. The CDF is defined as

$$F_U(x) = \Pr(U \le x) \tag{1.1}$$

Solution: Download the following Python code that plots

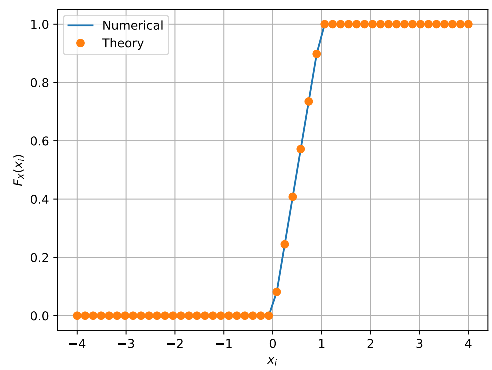

Fig:1.2 CDF of U

```
wget https://github.com/
    shreyaswankhede12/
    Random_numbers/blob/
    main/codes/1.2_cdf.py
```

Run the code by executing

```
python 1.2_cdf.py
```

1.3 Find a theoretical expression for $F_U(x)$. **Solution:** The PDF of U is given by

$$p_U(x) = \begin{cases} 1 & x \in [0,1] \\ 0 & \text{otherwise} \end{cases} \tag{1.2}$$

The CDF of U is given by

$$F_U(x) = \Pr(U \le x) = \int_{-\infty}^{x} p_U(x)\ \mathrm{d}x \tag{1.3}$$

If $x < 0$,

$$\int_{-\infty}^{x} p_U(x)\ \mathrm{d}x = \int_{-\infty}^{x} 0\ \mathrm{d}x = 0 \tag{1.4}$$

If c,

$$\int_{-\infty}^{x} p_U(x)\ \mathrm{d}x = \int_{-\infty}^{0} 0\ \mathrm{d}x + \int_{0}^{x} 1\ \mathrm{d}x \tag{1.5}$$

$$= 0 + x \tag{1.6}$$

$$= x \tag{1.7}$$

If $x > 1$,

$$\int_{-\infty}^{x} p_U(x)\ \mathrm{d}x$$

$$= \int_{-\infty}^{0} 0\ \mathrm{d}x + \int_{0}^{1} 1\ \mathrm{d}x + \int_{1}^{x} 0\ \mathrm{d}x \tag{1.8}$$

$$\int_{-\infty}^{x} p_U(x)\ \mathrm{d}x = 0 + 1 + 0 \tag{1.9}$$

$$= 1 \tag{1.10}$$

Therefore, we obtain the CDF of U as

$$F_U(x) = \begin{cases} 0 & x < 0 \\ x & 0 \le x \le 1 \\ 1 & x > 1 \end{cases} \tag{1.11}$$

1.4 The mean of U is defined as

$$E[U] = \frac{1}{N}\sum_{i=1}^{N} U_i \tag{1.12}$$

and its variance as

$$\mathrm{var}[U] = E[U - E[U]]^2 \tag{1.13}$$

Write a C program to find the mean and variance of U.
Solution: execute the following

```
wget https://github.com/
    shreyaswankhede12/
    Random_numbers/blob/
    main/codes/1.4.c
```

1.5 Verify your result theoretically given that

$$E\left[U^k\right] = \int_{-\infty}^{\infty} x^k dF_U(x) \tag{1.14}$$

solution: From 1.11,

$$F_U(X) = x \quad \forall x, 0 \le x \le 1$$

$$\therefore dF_U(X) = dx \tag{1.15}$$

$$E[U] = \int_0^1 x dx = \frac{1}{2} = 0.5 \tag{1.16}$$

similarly,

$$E[U^2] = \int_0^1 x^2 dx = \frac{1}{3} \tag{1.17}$$

$$Var[U] = E[U^2] - (E[U])^2 \tag{1.18}$$

$$= \frac{1}{3} - \left(\frac{1}{2}\right)^2$$

$$= \frac{1}{3} - \frac{1}{4}$$

$$= \frac{1}{12}$$

2 Central Limit Theorem

2.1 Generate 10^6 samples of the random variable

$$X = \sum_{i=1}^{12} U_i - 6 \tag{2.1}$$

using a C program, where $U_i, i = 1, 2, \ldots, 12$ are a set of independent uniform random variables between 0 and 1 and save in a file called gau.dat
Solution: link for the code:

```
wget https://github.com/
    shreyaswankhede12/
    Random_numbers/blob/
    main/codes/2.1.c\\
wget https://github.com/
    shreyaswankhede12/
    Random_numbers/blob/
    main/codes/coeffs.h
```

execute the following by

```
gcc 2.1.c -lm\\
./a.out
```

2.2 Load gau.dat in python and plot the empirical CDF of X using the samples in gau.dat. What properties does a CDF have?
Solution: The CDF of X is plotted link to the python code

```
wget https://github.com/
    shreyaswankhede12/
    Random_numbers/blob/
    main/codes/2.2_emp_cdf
    .py
```

execute the following by

```
python 2.2_emp_cdf.py
```

2.3 Load gau.dat in python and plot the empirical PDF of X using the samples in gau.dat. The PDF of X is defined as

$$p_X(x) = \frac{d}{dx} F_X(x) \tag{2.2}$$

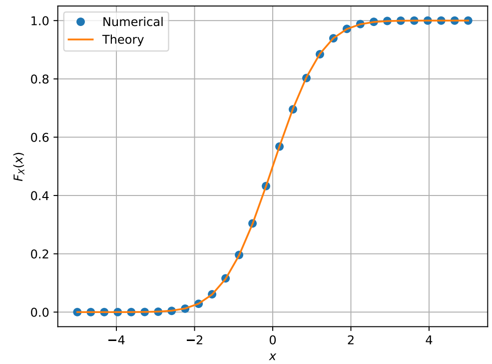

Fig:2.2 The CDF of X

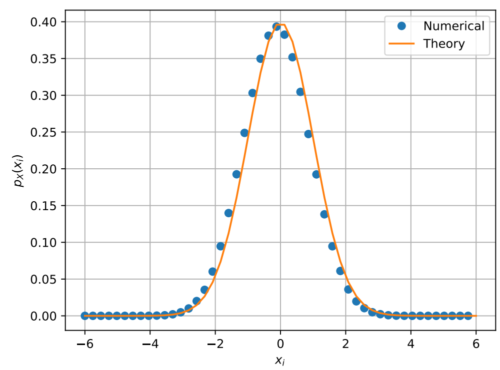

Fig:2.3 The PDF of X

What properties does the PDF have?
Solution: The PDF of X is plotted in Fig. **??** using the code below

```
wget https://github.com/
    shreyaswankhede12/
    Random_numbers/blob/
    main/codes/2.3_emp_pdf
    .py
```

2.4 Find the mean and variance of X by writing a C program.
Solution: link for c program

```
wget https://github.com/
    shreyaswankhede12/
    Random_numbers/blob/
    main/codes/2.4.c
```

execute the following by

```
gcc 2.4.c -lm
```

2.5 Given that

$$p_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), -\infty < x < \infty, \tag{2.3}$$

repeat the above exercise theoretically.

solution:

$$E[X] = \int_{-\infty}^{\infty} x P_X(x) dx \tag{2.4}$$

$$= \int_{-\infty}^{\infty} \frac{x}{\sqrt{2\pi}} \exp\left(\frac{-x^2}{2}\right) dx \tag{2.5}$$

$$\text{Let } h(x) = \frac{x}{\sqrt{2\pi}} \exp\left(\frac{-x^2}{2}\right)$$

Now,

$$h(-x) = -h(x)$$

Thus h(x) is an odd function

$$\therefore E[X] = \int_{-\infty}^{\infty} h(x) dx = 0 \tag{2.6}$$

$$E[X^2] = \int_{-\infty}^{\infty} x^2 P_X(x) dx \tag{2.7}$$

$$= \int_{-\infty}^{\infty} \frac{x^2}{\sqrt{2\pi}} \exp\left(\frac{-x^2}{2}\right) dx \tag{2.8}$$

$$\text{Let } g(x) = \frac{x^2}{\sqrt{2\pi}} \exp\left(\frac{-x^2}{2}\right)$$

Now,

$$g(-x) = g(x)$$

Thus g(x) is an even function
Using integration by parts:

$$= x \int x e^{-\frac{-x^2}{2}} dx - \int \frac{d(x)}{dx} \int x e^{-\frac{-x^2}{2}} dx \tag{2.9}$$

$$I = \int x e^{-\frac{-x^2}{2}} \tag{2.10}$$

$$\text{Let} \frac{x^2}{2} = t \tag{2.11}$$

$$\implies x dx = dt \tag{2.12}$$

$$\implies = \int e^{-t} dt = -e^{-t} + c \tag{2.13}$$

$$\therefore \int x e^{-\frac{-x^2}{2}} = -e^{-\frac{-x^2}{2}} + c \tag{2.14}$$

Using (2.14) in (2.9)

$$= -x e^{-\frac{-x^2}{2}} + \int e^{-\frac{-x^2}{2}} dx \tag{2.15}$$

$$\text{Also, } \int_{-\infty}^{\infty} e^{-\frac{-x^2}{2}} dx = \sqrt{2\pi} \tag{2.16}$$

$\therefore$ substituting limits we get

$$E[x^2] = 1 \tag{2.17}$$

$$\text{var}(X) = E[x^2] - (E[x])^2 = 1 - 0 \tag{2.18}$$

3 From Uniform to Other

3.1 Generate samples of

$$V = -2 \ln (1 - U) \tag{3.1}$$

and plot its CDF.
Solution: : Run the following codes

```
wget https://github.com/
    shreyaswankhede12/
    Random_numbers/blob/
    main/codes/3.1_code.c
```

execute by commands:

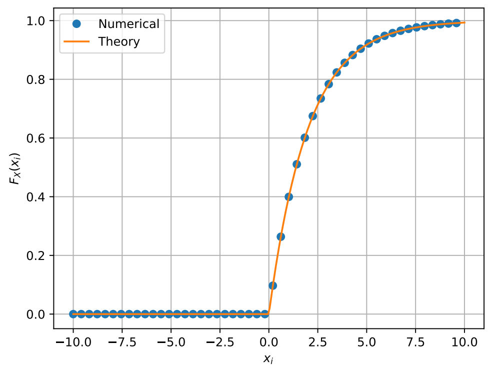

Fig:3.1 The CDF of V

```
gcc 3.1_code.c -lm
./a.out
```

```
wget https://github.com/
    shreyaswankhede12/
    Random_numbers/blob/
    main/codes/3.1_plot.py
```

3.2 Find a theoretical expression for $F_V(x)$. **Solution:** We have

$$F_V(x) = \Pr(V \le x) \tag{3.2}$$
$$= \Pr(-2\ln(1-U) \le x) \tag{3.3}$$
$$= \Pr\left(\ln(1-U) \ge -\frac{x}{2}\right) \tag{3.4}$$
$$= \Pr\left(1-U \ge \exp\left(-\frac{x}{2}\right)\right) \tag{3.5}$$
$$= \Pr\left(U \le 1-\exp\left(-\frac{x}{2}\right)\right) \tag{3.6}$$
$$= F_U\left(1-\exp\left(-\frac{x}{2}\right)\right) \tag{3.7}$$

Now,

$$0 \le 1-\exp\left(-\frac{x}{2}\right) < 1 \qquad \text{if } x \ge 0 \tag{3.8}$$
$$1-\exp\left(-\frac{x}{2}\right) < 0 \qquad \text{if } x < 0 \tag{3.9}$$

Therefore, from (1.11)

$$F_V(x) = \begin{cases} 1-\exp\left(-\frac{x}{2}\right) & x \ge 0 \\ 0 & x < 0 \end{cases} \tag{3.10}$$

4 Triangular Distribution

4.1 Generate

$$T = U_1 + U_2 \tag{4.1}$$

Solution: this generates U_1 and U_2

```
wget https://github.com/
    shreyaswankhede12/
    Random_numbers/blob/
    main/codes/4.1_gen_u1_
    u2.c
```

this code generates $U_1 + U_2$

```
wget https://github.com/
    shreyaswankhede12/
    Random_numbers/blob/
    main/codes/4.1_gen_u1+
    u2.c
```

4.2 Find the CDF of T.
Solution: python code for plotting cdf:

```
wgethttps://github.com/
    shreyaswankhede12/
    Random_numbers/blob/
    main/codes/4.2.py
```

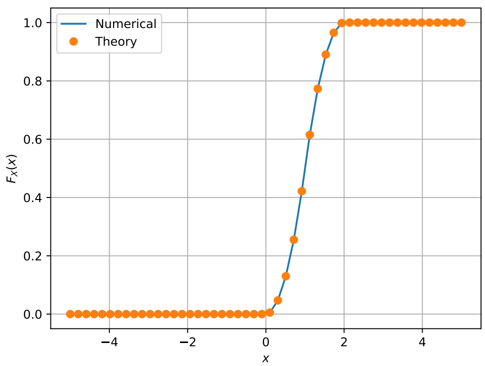

Fig:4.2 The CDF of T

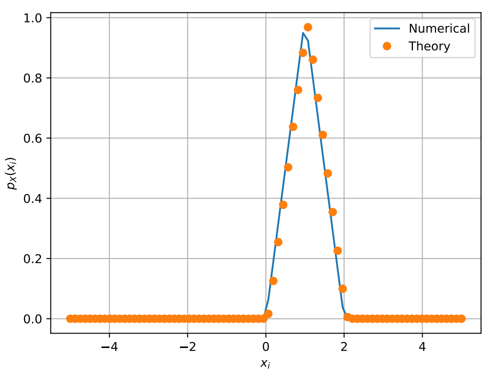

Fig:4.3 The PDF of T

4.3 Find the PDF of T.
Solution: python code for plotting pdf:

```
wget https://github.com/
    shreyaswankhede12/
    Random_numbers/blob/
    main/codes/4.3.py
```

4.4 Find the theoretical expressions for the PDF and CDF of T.

Solution: The CDF of T is given by

$$F_T(t) = \Pr(T \le t) = \Pr(U_1 + U_2 \le t) \tag{4.2}$$

$U_1 + U_2 \in [0, 2]$
Thus, for $t \ge 2$, then $U_1 + U_2 \le t$ is always true
for $t < 0$, then $U_1 + U_2 \le t$ is always false.
Now, fix the value of U_1 to be some x

$$x + U_2 \le t \implies U_2 \le t - x \tag{4.3}$$

If $0 \le t \le 1$, then x can take all values in $[0, t]$

$$F_T(t) = \int_0^t F_{U_2}(t-x) p_{U_1}(x) dx \tag{4.4}$$

since, $t - x \in [0, 1]$

$$F_{U_2}(t-x) = t - x \tag{4.5}$$

$$F_T(t) = \int_0^t (t-x).1.dx \tag{4.6}$$

$$= \frac{t^2}{2} \tag{4.7}$$

If $1 < t < 2$, x can only take values in $[0, 1]$ as $U_1 \le 1$

$$F_T(t) = \int_0^1 F_{U_2}(t-x).1.dx \tag{4.8}$$

since, $t - x \in [0, 1]$,

$$F_T(t) = \int_0^{t-1} 1dx + \int_{t-1}^1 (t-x)dx \tag{4.9}$$

$$= -\frac{t^2}{2} + 2t - 1 \tag{4.10}$$

Therefore,

$$F_T(t) = \begin{cases} 0 & t < 0 \\ \frac{t^2}{2} & 0 \le t \le 1 \\ 2t - \frac{t^2}{2} - 1 & 1 < t < 2 \\ 1 & t \ge 2 \end{cases} \tag{4.11}$$

On differentiating, we get,

$$P_T(t) = \begin{cases} 0 & t < 0 \\ t & 0 \le t \le 1 \\ 2 - t & 1 < t < 2 \\ 1 & t \ge 2 \end{cases} \tag{4.12}$$

4.5 Verify your results through a plot.
Solution: Refer to solution of qs 4.2 and 4.3

5 Maximul Likelihood

5.1 Generate

$$Y = AX + N, \tag{5.1}$$

where $A = 5$ dB, $X1\{1, -1\}$, is Bernoulli and $N \sim \mathcal{N}(0, 1)$.
5.2 Plot Y.
5.3 Guess how to estimate X from Y.
5.4 Find

$$P_{e|0} = \Pr\left(\hat{X} = -1 | X = 1\right) \tag{5.2}$$

and

$$P_{e|1} = \Pr\left(\hat{X} = 1 | X = -1\right) \tag{5.3}$$

5.5 Find P_e.
5.6 Verify by plotting the theoretical P_e.

6 Gaussian to Other

6.1 Let $X_1 \sim \mathcal{N}(0, 1)$ and $X_2 \sim \mathcal{N}(0, 1)$. Plot the CDF and PDF of

$$V = X_1^2 + X_2^2 \tag{6.1}$$

6.2 If

$$F_V(x) = \begin{cases} 1 - e^{-\alpha x} & x \ge 0 \\ 0 & x < 0, \end{cases} \tag{6.2}$$

find α.
6.3 Plot the CDF and PDf of

$$A = \sqrt{V} \tag{6.3}$$

7 Conditional Probability

7.1
7.2 Plot

$$P_e = \Pr\left(\hat{X} = -1 | X = 1\right) \tag{7.1}$$

for

$$Y = AX + N, \tag{7.2}$$

where A is Raleigh with $E\left[A^2\right] = \gamma, N \sim \mathcal{N}(0, 1), X \in (-1, 1)$ for $0 \le \gamma \le 10$ dB.
7.3 Assuming that N is a constant, find an expression for P_e. Call this $P_e(N)$
7.4 For a function g,

$$E[g(X)] = \int_{-\infty}^{\infty} g(x) p_X(x)\, dx \tag{7.3}$$

Find $P_e = E[P_e(N)]$.
7.5 Plot P_e in problems 7.2 and 7.4 on the same graph w.r.t γ. Comment.

8 Two Dimensions

Let

$$\mathbf{y} = A\mathbf{x} + \mathbf{n}, \tag{8.1}$$

where

$$x \in (\mathbf{s}_0, \mathbf{s}_1), \mathbf{s}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{s}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{8.2}$$

$$\mathbf{n} = \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}, n_1, n_2 \sim \mathcal{N}(0, 1). \tag{8.3}$$

8.1 Plot

$$\mathbf{y}|\mathbf{s}_0 \text{ and } \mathbf{y}|\mathbf{s}_1 \tag{8.4}$$

on the same graph using a scatter plot.

8.2 For the above problem, find a decision rule for detecting the symbols $\mathbf{s}_0$ and $\mathbf{s}_1$.

8.3 Plot

$$P_e = \Pr(\hat{\mathbf{x}} = \mathbf{s}_1 | \mathbf{x} = \mathbf{s}_0) \tag{8.5}$$

with respect to the SNR from 0 to 10 dB.

8.4 Obtain an expression for P_e. Verify this by comparing the theory and simulation plots on the same graph.