In [1]:
```
version = "REPLACE_PACKAGE_VERSION"
```

Assignment 3 Part 2: Multiple Time Series Forecasting (50 pts) ¶

In this assignment, we're going to study forecasting and causality testing that involve multiple time series.

In [2]:
```
import numpy as np
import pandas as pd
import matplotlib.pyplot as plt
%matplotlib inline

from pandas.plotting import register_matplotlib_converters
register_matplotlib_converters()

# Suppress warnings
import warnings
from statsmodels.tools.sm_exceptions import ValueWarning
warnings.simplefilter("ignore", ValueWarning)
```

We will explore the same five time series about **daily new COVID-19 cases for the top 5 countries with the most cumulative cases as of August 21, 2020** as we had in **Assignment 2 Part 2**. In order not to reinvent the suspension either, let's import the other `load_data` function you wrote previously.

In [3]:
```
# Copy and paste the function you wrote in Assignment 2 Part 2 here and 
# We have tried a more elegant solution by using
# from ipynb.fs.defs.assignment2_part2 import load_data
# but it doesn't work with the autograder...

def load_data():
    df = pd.read_csv('assets/time_series_covid19_confirmed_global.csv')
    droppers = ['Province/State','Lat','Long']
    df.drop(droppers, axis=1, inplace=True)
    df = df.groupby('Country/Region').sum().reset_index()
    df = df.sort_values('8/21/20', ascending=False)[:5]
    df = df.T
    df, df.columns = df[1:], df.iloc[0]
    df = df.diff(periods=1).dropna()
    df.index = pd.to_datetime(df.index)

    columns = df.columns
    df[columns] = df[columns].astype(float)

    daily_new_cases = df

    return daily_new_cases
```

In [4]:
```
# Sanity checks to make sure you have imported the correct function - no

stu_ans = load_data()

assert isinstance(stu_ans, pd.DataFrame), "Q0: Your function should retu
assert stu_ans.shape == (212, 5), "Q0: The shape of your pd.DataFrame re
assert isinstance(stu_ans.index, pd.DatetimeIndex), "Q0: The index of yo
assert (("2020-01-23" <= stu_ans.index) & (stu_ans.index <= "2020-08-21"
assert not stu_ans.isna().any(axis=None), "Q0: Your pd.DataFrame contain
assert stu_ans.dtypes.apply(lambda x: np.issubdtype(x, np.floating)).all

del stu_ans
```

Question 1: Vector Autoregression (VAR) (35 pts)

There may be interesting relationships that exist among multiple time series. One way of uncovering such relationships is to perform a VAR, where we model a time series not only with its own observations but also observations from other possibly related time series. In this question, we'll explore how to apply VAR to the five time series about daily new COVID-19 cases.

Question 1a (25 pts)

Complete the function below that fits a $\mathrm{VAR}(p)$ model on the **first-order differences** of multiple input time series given as a `pd.DataFrame` and that makes forecasts in the original data space (i.e., the number of daily new cases). The function should return the trained VAR model as either a `VARResults` or `VARResultsWrapper` object, and its forecasts as a `pd.DataFrame` .

For example, when parameter `num_forecasts=20` , the forecasts should be a `pd.DataFrame` like the following:

	?	?	?	?	?
2020-08-22	34966.352289	27554.442883	66616.742795	4880.479802	1304.572448
2020-08-23	29196.140510	18778.000769	64704.583332	4644.870185	963.191434
2020-08-24	29317.565141	27333.717530	67000.950077	4454.998897	-305.011015
...	...	...	...	...	...
2020-09-08	30778.194852	54899.944448	77734.373256	4293.506363	-1871.868813
2020-09-09	32638.701231	55789.197274	80176.803330	4476.211786	-299.954762
2020-09-10	29185.068081	41457.646648	79565.796202	4678.429342	-348.964443

where

- the index of the DataFrame is a `pd.DatetimeIndex` ;
- the column names "?" are the top 5 countries with the most cumulative cases as of August 21, 2020, sorted in descending order from left to right;
- the values of the DataFrame are the forecasts; and
- the DataFrame doesn't contain any `NaN` values.

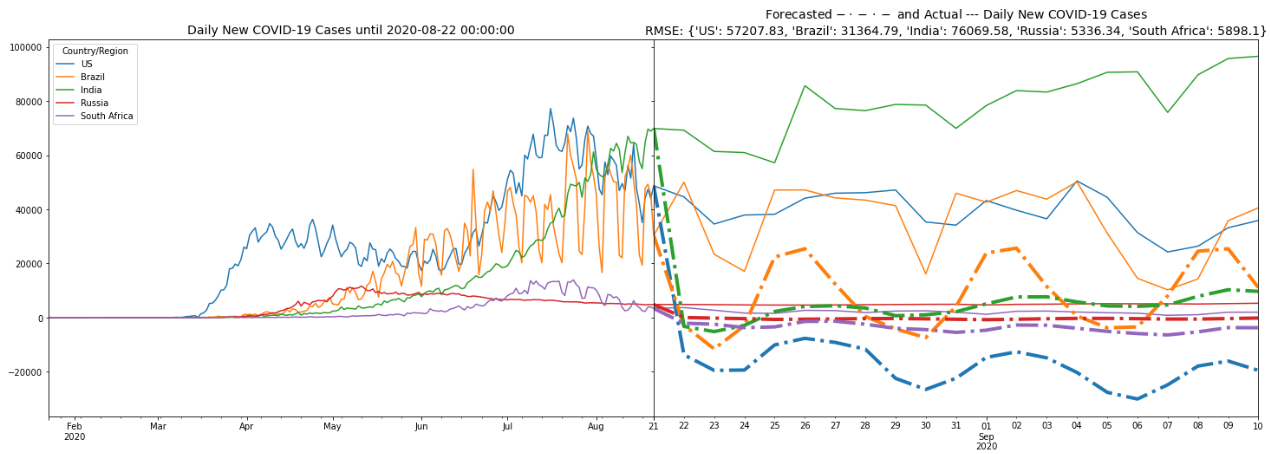
This function should return a `tuple` of length 2, whose first element is either a `VARResults` or `VARResultsWrapper` object representing a trained VAR model and whose last element is a `pd.DataFrame` of shape `(num_forecasts, 5)` representing the forecasts.

In [5]:
```
from statsmodels.tsa.api import VAR
from statsmodels.tsa.vector_ar.var_model import VARResults, VARResultsWra

def var_first_diff(df, p, num_forecasts):
    """
    Fits a VAR(p) model on the first-order diff on a df and makes num_fo
    """
    var_res, forecasts = None, None

    # YOUR CODE HERE
    #raise NotImplementedError()
    var_data = df.diff().dropna()
    var_res = VAR(var_data).fit(p)
    forecasts = pd.DataFrame(var_res.forecast(var_data.values[-var_res.k_

    forecasts.iloc[0] + df.iloc[1]
    for i in range(1, num_forecasts):
        forecasts.iloc[i] += forecasts.iloc[i-1]

    lag_order = var_res.k_ar
    forecast_input = df.values[-lag_order:]

    return var_res, forecasts
```

In [6]:
```
# Autograder tests

stu_df = load_data()
p, num_forecasts = 7, 20
stu_ans = var_first_diff(stu_df, p, num_forecasts)

assert isinstance(stu_ans, tuple), "Q1a: Your function should return a tu
assert len(stu_ans) == 2, "Q1a: The length of the tuple returned is inco

# Check the trained VAR model
assert isinstance(stu_ans[0], (VARResults, VARResultsWrapper)), "Q1a: You
assert stu_ans[0].nobs == stu_df.shape[0] - 1 - p, "Q1a: The VAR model wa
assert stu_ans[0].neqs == stu_df.shape[1], "Q1a: The VAR model was fit wi
assert stu_ans[0].k_ar == p, "Q1a: The VAR model was fit with an incorrec

# Check the forecasts
assert isinstance(stu_ans[1], pd.DataFrame), "Q1a: Your forecasts should
assert stu_ans[1].shape == (num_forecasts, stu_df.shape[-1]), "Q1a: The s
assert isinstance(stu_ans[1].index, pd.DatetimeIndex), "Q1a: The index of
assert (("2020-08-22" <= stu_ans[1].index) & (stu_ans[1].index <= "2020-0
assert not stu_ans[1].isna().any(axis=None), "Q1a: Your forecasts contain
assert stu_ans[1].dtypes.apply(lambda x: np.issubdtype(x, np.floating)).a

# Some hidden tests


del stu_ans, stu_df, p, num_forecasts
```

Let's plot and see your forecasts. Is your VAR model doing a good job? Why or why not?

In [7]:
```
p, num_forecasts = 7, 20

stu_df = load_data()
_, forecasts = var_first_diff(stu_df, p, num_forecasts)
actual = pd.read_pickle("assets/actual_multi.pkl")
rmse = np.sqrt(np.mean((actual - forecasts) ** 2, axis=0)).round(2)

fig, axes = plt.subplots(1, 2, figsize=(25, 8), sharey=True, gridspec_kw=

stu_df.plot(ax=axes[0])

stu_df.iloc[-1:].append(actual).plot(ax=axes[1], legend=False)
axes[1].set_prop_cycle(None)

stu_df.iloc[-1:].append(forecasts).plot(ax=axes[1], legend=False, style=

axes[0].set_title(f"Daily New COVID-19 Cases until {forecasts.index[0]}"
axes[1].set_title(r"Forecasted $-\cdot-\cdot-$ and Actual --- Daily New (

del fig, axes, stu_df, p, num_forecasts, forecasts, actual, rmse
```



Question 1b (10 pts)

Now, let's compare the forecasts made by the $\mathrm{VAR(p)}$ model you trained above with that made by five independent $\mathrm{AR(p)}$ models for each time series. This way, we will be able to see the effect of including observations from possibly related time series on modelling each invidual time series more clearly.

Complete the function below that uses the `arma_first_diff` function you wrote in **Assignment 3 Part 1** to fit five $\mathrm{AR}(p)$ models, one time series each, and make forecasts for each of the five time series. Return the forecasts as a `pd.DataFrame` .

For example, when parameter `num_forecasts=20` , the forecasts should be a `pd.DataFrame` like the following:

	?	?	?	?	?
2020-08-22	37606.337737	30054.801607	65293.198373	4799.898341	2613.269306
2020-08-23	34659.880538	21722.458140	60235.575280	4795.226196	2096.298645
2020-08-24	31822.923153	24670.815522	64022.272121	4773.476292	1556.778499
...	...	...	...	...	...
2020-09-08	36489.857891	41764.328105	73674.836876	4767.958973	2249.096891
2020-09-09	38427.089594	44335.452439	74071.644729	4767.761722	2587.726648
2020-09-10	39442.677738	39179.959665	71831.680217	4767.538096	2503.294019

where

- the index of the DataFrame is a `pd.DatetimeIndex` ;
- the column names "?" are the top 5 countries with the most cumulative cases as of August 21, 2020, sorted in descending order from left to right;
- the values of the DataFrame are the forecasts; and
- the DataFrame doesn't contain any `NaN` values.

This function should return a `pd.DataFrame` of shape `(num_forecasts, 5)` representing the forecasts.

In [8]:
```
# Copy and paste the function you wrote in Assignment 3 Part 1
from statsmodels.tsa.arima.model import ARIMA

def arma_first_diff(ser, p, q, num_forecasts):
    """
    Takes a series and fits an ARMA(p, q) model on first-order diff.
    Returns a number of forecasts as specified by num_forecasts.
    """
    model= ARIMA (ser, order = (p, 0, q))
    fit=model.fit()
    forecasts= fit.forecasts(steps=num_forecasts)

    # YOUR CODE HERE
    #raise NotImplementedError()

    return forecasts
```

In [9]:
```
from statsmodels.tsa.ar_model import AutoReg
def ar_first_diff(df, p, num_forecasts):
    """
    Fits an AR(p) model on the first-order diff on each time series in d
    """
    model= AutoReg (df, p)
    fit=model.fit()
    forecasts= fit.forecasts(steps=num_forecasts)

    # YOUR CODE HERE
    #raise NotImplementedError()

    return forecasts
```

In [10]:
```
# Autograder tests

stu_df = load_data()
p, num_forecasts = 7, 20
stu_ans = ar_first_diff(stu_df, p, num_forecasts)

assert isinstance(stu_ans, pd.DataFrame), "Q1b: Your forecasts should be
assert stu_ans.shape == (num_forecasts, stu_df.shape[-1]), "Q1b: The sha
assert isinstance(stu_ans.index, pd.DatetimeIndex), "Q1b: The index of yo
assert (("2020-08-22" <= stu_ans.index) & (stu_ans.index <= "2020-09-10"
assert not stu_ans.isna().any(axis=None), "Q1b: Your forecasts contain Na
assert stu_ans.dtypes.apply(lambda x: np.issubdtype(x, np.floating)).all

# Some hidden tests


del stu_ans, stu_df, p, num_forecasts
```

```
del stu_ans, stu_df, p, num_forecasts
```

```
---------------------------------------------------------------------------
TypeError                                 Traceback (most recent call last)
/tmp/ipykernel_75/4271707629.py in <module>
      3 stu_df = load_data()
      4 p, num_forecasts = 7, 20
----> 5 stu_ans = ar_first_diff(stu_df, p, num_forecasts)
      6 
      7 assert isinstance(stu_ans, pd.DataFrame), "Q1b: Your forecasts should be a pd.DataFrame. "

/tmp/ipykernel_75/3387642376.py in ar_first_diff(df, p, num_forecasts)
      4     Fits an AR(p) model on the first-order diff on each time series in df and makes num_forecasts forecasts
      5     """
----> 6     model= AutoReg (df, p)
      7     fit=model.fit()
      8     forecasts= fit.forecasts(steps=num_forecasts)

/opt/conda/lib/python3.8/site-packages/statsmodels/tsa/ar_model.py in __init__(self, endog, lags, trend, seasonal, exog, hold_back, period, missing, deterministic, old_names)
    254                 self._old_names = True
    255             self._check_lags()
--> 256             self._setup_regressors()
    257             self.nobs = self._y.shape[0]
    258             self.data.xnames = self.exog_names

/opt/conda/lib/python3.8/site-packages/statsmodels/tsa/ar_model.py in _setup_regressors(self)
    317             x, y = lagmat(self.endog, maxlag, original="sep")
    318             exog_names.extend(
--> 319                 [endog_names + ".L{0}".format(lag) for lag in self._lags]
    320             )
    321             if len(self._lags) < maxlag:

/opt/conda/lib/python3.8/site-packages/statsmodels/tsa/ar_model.py in <listcomp>(.0)
    317             x, y = lagmat(self.endog, maxlag, original="sep")
    318             exog_names.extend(
--> 319                 [endog_names + ".L{0}".format(lag) for lag in self._lags]
    320             )
    321             if len(self._lags) < maxlag:

TypeError: can only concatenate list (not "str") to list
```

We can of course plot the forecasts made by the five $\mathrm{AR}(p)$ models. How does the RMSE for each time series compare with that calculated from the forecasts made by a single $\mathrm{VAR}(p)$ model?

In [11]:
```
p, num_forecasts = 7, 20

stu_df = load_data()
forecasts = ar_first_diff(stu_df, p, num_forecasts)
actual = pd.read_pickle("assets/actual_multi.pkl")
rmse = np.sqrt(np.mean((actual - forecasts) ** 2, axis=0)).round(2)

fig, axes = plt.subplots(1, 2, figsize=(25, 8), sharey=True, gridspec_kw=

stu_df.plot(ax=axes[0])

stu_df.iloc[-1:].append(actual).plot(ax=axes[1], legend=False)
axes[1].set_prop_cycle(None)

stu_df.iloc[-1:].append(forecasts).plot(ax=axes[1], legend=False, style=

axes[0].set_title(f"Daily New COVID-19 Cases until {forecasts.index[0]}"
axes[1].set_title(r"Forecasted $-\cdot-\cdot-$ and Actual --- Daily New (

del fig, axes, stu_df, p, num_forecasts, forecasts, actual, rmse
```

```
---------------------------------------------------------------------------
TypeError                                 Traceback (most recent call
last)
/tmp/ipykernel_75/2310546019.py in <module>
      2
      3 stu_df = load_data()
----> 4 forecasts = ar_first_diff(stu_df, p, num_forecasts)
      5 actual = pd.read_pickle("assets/actual_multi.pkl")
      6 rmse = np.sqrt(np.mean((actual - forecasts) ** 2, axis=0)).rou
nd(2)

/tmp/ipykernel_75/3387642376.py in ar_first_diff(df, p, num_forecasts)
      4     Fits an AR(p) model on the first-order diff on each time s
eries in df and makes num_forecasts forecasts
      5     """
----> 6     model= AutoReg (df, p)
      7     fit=model.fit()
      8     forecasts= fit.forecasts(steps=num_forecasts)

/opt/conda/lib/python3.8/site-packages/statsmodels/tsa/ar_model.py in
__init__(self, endog, lags, trend, seasonal, exog, hold_back, period,
missing, deterministic, old_names)
```

```
    254                 self._old_names = True
    255             self._check_lags()
--> 256             self._setup_regressors()
    257             self.nobs = self._y.shape[0]
    258             self.data.xnames = self.exog_names

/opt/conda/lib/python3.8/site-packages/statsmodels/tsa/ar_model.py in
_setup_regressors(self)
    317             x, y = lagmat(self.endog, maxlag, original="sep")
    318             exog_names.extend(
--> 319                 [endog_names + ".L{0}".format(lag) for lag in self
._lags]
    320             )
    321             if len(self._lags) < maxlag:

/opt/conda/lib/python3.8/site-packages/statsmodels/tsa/ar_model.py in
<listcomp>(.0)
    317             x, y = lagmat(self.endog, maxlag, original="sep")
    318             exog_names.extend(
--> 319                 [endog_names + ".L{0}".format(lag) for lag in self
._lags]
    320             )
    321             if len(self._lags) < maxlag:

TypeError: can only concatenate list (not "str") to list
```

Question 2: Granger Causality (15 pts)

By comparing the forecasts made by a single $\mathrm{VAR}(p)$ model and by five independent $\mathrm{AR}(p)$ models above, you may notice that the RMSE of one country has improved after we "upgrade" an $\mathrm{AR}(p)$ model to include observations from all other time series. It alerts us about the possible *causality* that may exist among these time series, because the inclusion of some other time series enables us to make better forecasts on another one.

This is exactly the principle behind Granger Causality test, a statistical test for causality that works by determining whether the inclusion of one time series significantly improves the prediction of the other. Let's now perform a Granger Causality test on each pair of countries and see what we can conclude.

Complete the function below that first fits a $\mathrm{VAR}(p)$ model on the **first-order differences** of the input `pd.DataFrame` and that then performs a pairwise Granger Causality test **based on F-test** for all possible pairs of the five countries (excluding the pairs formed by a country and itself). The function should return the p-value of each pairwise test in a `pd.DataFrame` like the following:

	?	?	?	?	?
?	NaN	1.234348e-05			
?	6.323140e-01	NaN			
?			NaN		
?				NaN	
?					NaN

where

- the index and the column names "?" are the top 5 countries with the most cumulative cases as of August 21, 2020, sorted in descending order from top to bottom and from left to right; and
- **each row represents the *caused* variable and each column represents the *causing* variable**

For example, `1.234348e-05` is the p-value of the F-test performed to test the null hypothesis that the daily new cases in the Rank 1 country is not *caused by* that in the Rank 2 country. Notice that the "caused-by" relation is not symmetric, so the `pd.DataFrame` above is not symmetric either. You may use the `test_causality` function of either `VARResults` or `VARResultsWrapper` class to perform Granger Causality tests.

The object returned from the `test_causality` function possesses an attribute that gives you the p-value of the test as a single number. How do you identify that attribute? (Hint: Python's built-in `dir` (https://docs.python.org/3/library/functions.html#dir) function can be helpful.)

This function should return a `pd.DataFrame` of the shape `(5, 5)`, representing the p-value matrix for all pairwise Granger Causality tests.

In [12]:
```
from statsmodels.tsa.api import VAR
from statsmodels.tsa.vector_ar.var_model import VARResults, VARResultsWra

def test_granger(df, p):
    """
    Fits a VAR(p) model on the input df and performs pairwise Granger Cau
    """
    granger_df = None

    # YOUR CODE HERE
    #raise NotImplementedError()

    return granger_df
```

In [ ]:
```
# Autograder tests

stu_df, p = load_data(), 7
stu_ans = test_granger(stu_df, 7)

assert isinstance(stu_ans, pd.DataFrame), "Q2: Your function should retu
assert stu_ans.shape == (5, 5), "Q2: The shape of your pd.DataFrame is no
assert (stu_ans.index == stu_ans.columns).all(), "Q2: Your pd.DataFrame s
assert stu_ans.dtypes.apply(lambda x: np.issubdtype(x, np.floating)).all

# Some hidden tests


del stu_df, stu_ans, p
```

If we believe in the magic threshold of 0.01 (or 0.05) for rejecting null hypotheses, we will obtain the following "causality matrix".

In [ ]:
```
# Show the causality matrix

stu_df, p = load_data(), 7
stu_ans = test_granger(stu_df, 7)
caul_mtrx = stu_ans.rename(index={item: f"{item} caused by" for item in s
caul_mtrx.where(caul_mtrx.isna(), caul_mtrx <= 0.01)
```

What do you think about the causality matrix above? Are there any surprising conclusions? Do you believe in the Granger Causality test we just performed? Why or why not?