1)

#14

a) $\forall x \in \mathbb{Q},\ x^3 \in \mathbb{Q}$

b) True.

Suppose x is any particular but arbitrarily chosen rational number.

By definition of rational number, $x = \frac{a}{b}$, where a and b are some integers such that $b \neq 0$.

By substitution, $x^3 = (\frac{a}{b})^3 = \frac{a.a.a}{b.b.b} = \frac{a^3}{b^3}$, where both a^3 and b^3 are integers since integers are closed under multiplication of integers and since $b \neq 0$, then by zero product property, $b^3 \neq 0$

Since both a^3 and b^3 are integers and $b^3 \neq 0$,

$\therefore$ x^3 is a rational number.

□

#18

If r and s are any two rational numbers, then $\frac{r+s}{2}$ is rational.

TRUE

Suppose r and s are any two particular but arbitrarily chosen rational numbers.

By definition of rational number, r = $\frac{a}{b}$ and s = $\frac{c}{d}$, where a, b, c, and d are some integers such that b ≠ 0 and d ≠ 0.

$$\frac{r+s}{2} = \frac{\frac{a}{b}+\frac{c}{d}}{2},\ by\ substitution$$

$$= \frac{\frac{d}{d}\cdot\frac{a}{b}+\frac{b}{b}\cdot\frac{c}{d}}{2},\ by\ multiplicative\ identity$$

$$= \frac{\frac{ad}{db}+\frac{bc}{bd}}{2},\ by\ algebra$$

$$= \frac{\frac{ad}{bd}+\frac{bc}{bd}}{2},\ by\ commutativity$$

$$= \frac{\frac{ad+bc}{bd}}{2},\ by\ algebra$$

$$= \frac{ad+bc}{2bd},\ by\ algebra$$

By zero product property, bd ≠ 0.

Let q = 2bd, where q is an integer as integers are closed under multiplication of integers and p = ad+bc, where p is an integer as integers are closed under multiplication and addition of integers.

$\frac{r+s}{2} = \frac{p}{q}$, where p and q are integers

By definition of a rational number,

$\therefore \frac{r+s}{2}$ is a rational number.

□

2)

#32

Suppose c is any particular but arbitrarily chosen real number such that there exits a polynomial P with real number x, such that $P(x) = \frac{a_0}{b_0} + \left(\frac{a_1}{b_1}\right)x + \left(\frac{a_2}{b_2}\right)x^2 + \left(\frac{a_3}{b_3}\right)x^3 + \cdots \left(\frac{a_n}{b_n}\right)x^n$

where $a_0, b_0, a_1, b_1, a_2, b_2, a_3, b_3 \ldots, a_n, b_n$ are some integers and $b_0, b_1, b_2, b_3 \ldots b_n \neq 0$ and n is some positive Integer

Hence, coefficients in $P(x)$ are rational, by definition of rational

such that $P(c) = 0$.

$P(c) = \left(\frac{a_0}{b_0}\right) + \left(\frac{a_1}{b_1}\right)c + \left(\frac{a_2}{b_2}\right)c^2 + \left(\frac{a_3}{b_3}\right)c^3 + \cdots \left(\frac{a_n}{b_n}\right)c^n = 0$ By substitution,

$P(c) = (b_0 b_1 b_2 b_3 \cdots b_n)\left(\frac{a_0}{b_0} + \left(\frac{a_1}{b_1}\right)c + \left(\frac{a_2}{b_2}\right)c^2 + \left(\frac{a_3}{b_3}\right)c^3 + \cdots \left(\frac{a_n}{b_n}\right)c^n\right) = 0(b_0, b_1, b_2, b_3 \ldots b_n)$, By Multiplicating the LCM of all denominators of coefficients on both sides

$P(c) = a_0 b_1 b_2 b_3 \ldots b_n + (a_1 b_0 b_2 b_3 \cdots b_n)c + (a_2 b_0 b_1 b_3 \cdots b_n)c^2 + (a_3 b_0 b_1 b_2 \cdots b_n)c^3 + \cdots (a_n b_0 b_1 b_2 b_3 \cdots)c^n = 0$, By Algebra & Zero Product Property.

All coefficients of $P(c)$ are a product of integers and integers are closed under multiplication.

Hence, all coefficients of $P(c)$ are integers

$\therefore$ For ever real number c, if c is a root of polynomial with rational coefficients, then c is a root of a polynomial with integer coefficients.

□

#33

8 Suppose r and s are any particular but arbitrarily chosen odd integers for R a Polynomial $P(x)$ where x is any real number and $P(x) = (x-r)(x-s)$

By def of odd integers, $r = 2k+1$
$s = 2m+1$ for some integers k & m.

$P(x) = (x-r)(x-s)$

$= x^2 - (r+s)x + rs$, By Algebra

$= x^2 - (2k+1 + 2m+1)x + (2k+1)(2m+1)$, By substitution

$= x^2 - (2k+2m+2)x + 4km + 2k + 2m + 1$, By Algebra

$= x^2 - (2(k+m+1))x + 2(2km + k + m) + 1$, By Factoring out 2

Let $t = k+m+1$ & $n = 2km + k + m$

then since, $k, m, 1, 2$ are integers & integers are closed under multiplication & addition.

Hence, t & n are integers

$= (2(0)+1)x^2 - (2t)x + (2n+1)$, By substitution

∴ By definition of odd, coefficient of x^2 & constant coefficient are odd.

∴ By definition of even, coefficient of x is even.

8 Suppose r and s are any particular but arbitrarily chosen even integers for R a Polynomial $P(x)$ where x is any real number and $P(x) = (x-r)(x-s)$

By def of odd integers, $r = 2k$
$s = 2m$ for some integers k & m.

$P(x) = (x-r)(x-s)$

$= x^2 - (r+s)x + rs$, By Algebra

$= x^2 - (2k+2m)\ x + (2k)(2m)$, By substitution

$= x^2 - (2k+2m)x + 4km$, By Algebre

$= x^2 - (2(k+m))x + 2(2km)$, By Factoring out 2

Let $t = k+m$ & $n = 2km$
then since, k, m, 2 are integers &
integers are closed under multiplication & addition.
Hence, t & n are integers

$= (2(0)+1)x^2 - (2t)x + (2n)$, By substitution

∴ By definition of odd, coefficient of x^2 is odd

∴ By definition of even, coefficient of x and the coefficient of the constant are even.

~~a~~ Suppose r and s are any particular but arbitrarily chosen integers for a Polynomial $P(x)$ where x is any real number and $P(x) = (x-r)(x-s)$, such that r is odd & s is even

By def of odd integers, $r = 2k+1$
$s = 2m$ for some integers k & m.

$P(x) = (x-r)(x-s)$

$= x^2 - (r+s)x + rs$, By Algebra

$= x^2 - (2k+1+2m)x + (2k+1)(2m)$, By substitution

$= x^2 - (2k+2m+1)x + 4km + 2m$, By Algebra

$= x^2 - (2(k+m)+1)x + 2(2km+m)$, By Factoring out 2

Let $t = k+m$ & $n = 2km + m$
then since, $k, m, 2$ are integers &
integers are closed under multiplication & addition.
Hence, t & n are integers

$= (2(0)+1)x^2 - (2t+1)x + (2n)$, By substitution

$\therefore$ By definition of odd, coefficient of x^2 & coefficient of x are odd

$\therefore$ By definition of even, constant coefficient is even.

(b) 1253 is an odd integer and 255 is also an odd integer.
However, in all 3 possible cases of odd-even coefficients, discussed in (a) above does not include a conclusion where both constant coefficient & coefficient of linear variable x ~~a~~ are odd at the same time / in the same case.
Hence, ~~a~~ it is not possible to separate $x^2 - 1253x + 255$ as a product of 2 polynomials, since that is not possible in any case as proved above in (a).

#36

The given proof just randomly chooses 2 rational numbers that work perfectly to give a rational number on addition, but doesn’t generalize the case to all rational numbers which was asked in the statement initially. We are supposed to choose 2 particular but arbitrarily chosen rational numbers and not any two specific numbers. A proof should be universal, that is, it should hold for all possible values of r and s.