1)
#79

1) If p is a prime number and r is an integer with $0 < r < p$, then $\binom{p}{r}$ is divisible by p.

Proof: Let p, r be any particular but arbitrarily chosen integers such that p is prime and $0 < r < p$.

[to show $\left[p \mid \binom{p}{r}\right]$]

By definition of a prime number, 1 and p are the only factors of p.

$$\binom{p}{r} = \frac{p!}{r!(p-r)!}$$

$$\binom{p}{r} = \frac{p\cdot(p-1)\cdot(p-2)\cdots(p-r+1)\cancel{(p-r)!}}{r!\cancel{(p-r)!}}$$

$r!$ is an integer because all integers are closed under factorial of integers, which is just repeated multiplication of integers, where integers are closed under multiplication and subtraction.

Let $t = \binom{p}{r} = \frac{a}{b}$ where $a = p(p-1)(p-2)\cdots(p-r+1)$ is an integer as integers are closed under multiplication.
for some $t \in \mathbb{Z}^+$
$b = r!$ is an integer as proved above

since $a = pm$
for some integer $m = (p-1)(p-2)\cdots(p-r+1)$ which is an integer as integers are closed under multiplication

By def of divisibility,

$p \mid pm$,

$p \mid a$.

$a = bt$

since $p \mid a$ and $p \nmid b$
we can conclude that $p \mid t$

$\therefore \quad p \mid \binom{p}{r}$ [or $\binom{p}{r}$ is divisible by p]

□

2)

2) $\sum_{a=1}^{b}\sum_{c=1}^{d}(a+c) = \frac{bd(b+d+2)}{2}$ for all $(b,d) \in \mathbb{Z}^+ \times \mathbb{Z}^+$.

Proof: Suppose a, b, c, d be particular but arbitrarily chosen positive integers

Base Case: For $(b,1)$ such that $b \geq 1$

$$\sum_{a=1}^{b}\sum_{c=1}^{1}(a+c) = \frac{bd(b+d+2)}{2}$$ by substitution

$$\sum_{a=1}^{b} a+1 = \frac{b(1)(b+1+2)}{2}$$ by

$$\sum_{a=1}^{b} a+1 = \frac{b(b+3)}{2}$$ algebra

$$(1+1)+(2+1)+(3+1)+\ldots+(b+1) = \frac{b(b+3)}{2}$$ by

$$(1+2+3+\ldots+b) + 1+1+1+1+\ldots+1) = \frac{b(b+3)}{2}$$ algebra and commutativity and associativity

$$\frac{b(b+1)}{2} + b \times 1 = \frac{b(b+3)}{2}$$

$$\frac{b(b+1)+2b}{2} = \frac{b(b+3)}{2}$$ using formula $1+2+3\ldots n = \frac{n(n+1)}{2}$ for first b numbers

$$\frac{b^2+b+2b}{2} = \frac{b(b+3)}{2}$$ using distributivity

$$\frac{b^2+3b}{2} = \frac{b(b+3)}{2}$$ using algebra

$$\frac{b(b+3)}{2} = \frac{b(b+3)}{2}$$ ✓ by factoring out b.

Base Case is True.

Induction Hypothesis: Suppose for any integer $b \geq 1$,

$$\sum_{a=1}^{b}\sum_{c=1}^{k}(a+c) = \frac{bk(b+k+2)}{2}$$ for all $(b,k) \in \mathbb{Z}^+ \times \mathbb{Z}^+$

(to show $\sum_{a=1}^{b}\sum_{c=1}^{k+1}(a+c) = \frac{b(k+1)(b+k+3)}{2}$ for all $(b,k+1) \in \mathbb{Z}^+ \times \mathbb{Z}^+$)

Proof : $\displaystyle\sum_{a=1}^{b}\sum_{c=1}^{k+1}(a+c) = \sum_{a=1}^{b}\sum_{c=1}^{k}(a+c) + \sum_{a=1}^{b}(a+k+1)$ by algebra

$$= \frac{bk(b+k+2)}{2} + \left(\sum_{a=1}^{b} a\right) + \left(\sum_{a=1}^{b} k+1\right)$$, using inductive hypothesis

$$= \frac{bk(b+k+2)}{2} + (1+2+3\cdots+b) + (b\times k+1)$$

$$= \frac{bk(b+k+2)}{2} + \frac{b(b+1)}{2} + bk+b$$ by algebra

$$= \frac{bk(b+k+2)}{2} + \frac{b(b+1)+2bk+2b}{2}$$ by commutativity

$$= \frac{b^2k+bk^2+2bk+b^2+b+2bk+2b}{2}$$ and associativity

$$= \frac{b^2k+bk^2+4bk+b^2+3b}{2}$$ by distributivity &

$$= \frac{b(bk+k^2+4k+b+3)}{2}$$ factoring out b

$$= \frac{b(k^2+k+3k+bk+b+3)}{2}$$ and

$$= \frac{b(k(k+1)+3(k+1)+b(k+1))}{2}$$ factoring out k+1

Hence, $\displaystyle\sum_{a=1}^{b}\sum_{c=1}^{k+1}(a+c) = \frac{b(k+1)(b+k+3)}{2}$, which was to be shown.

$$= \frac{b(k+1)(b+(k+1)+2)}{2}$$

Hence, $\displaystyle\sum_{a=1}^{b}\sum_{c=1}^{k+1}(a+c) = \frac{b(k+1)(b+k+3)}{2}$ for all $(b,k+1)\in\mathbb{Z}^+\times\mathbb{Z}^+$

for all $b\geq 1$

$\therefore$ By induction,

$$\sum_{a=1}^{b}\sum_{c=1}^{d}(a+c) = \frac{bd(b+d+2)}{2} \text{ for all } (b,d)\in\mathbb{Z}^+\times\mathbb{Z}^+.$$

$\square$

3)
#14

$$\sum_{i=1}^{n+1} i.2^i = n.2^{n+2} + 2, \text{ for every integer } n \geq 0$$

<u>Proof by Induction:</u>

Base Case: for $n=0$

$$\sum_{i=1}^{1} i.2^i = (0)2^{0+2} + 2$$

$$1.2^1 = 2$$

$$2 = 2$$

$$2 = 2 \checkmark$$

Base Case is True.

<u>Inductive Hypothesis</u>: Suppose $\sum_{i=1}^{k+1} i.2^i = k.2^{k+2} + 2$, for every integer $k \geq 0$

(to show $\sum_{i=1}^{k+1+1} i.2^i = (k+1)2^{k+1+2} + 2$).

Proof: $\sum_{i=1}^{k+2} i.2^i = \sum_{i=1}^{k+1} + (k+2).2^{k+2}$ using inductive hypothesis

$$= k.2^{k+2} + 2 + k.2^{k+2} + 2^{k+3}$$

$$= 2k.2^{k+2} + 2^{k+3} + 2$$

$$\sum_{i=1}^{k+2} i.2^i = k.2^{k+3} + 2^{k+3} + 2$$

$$= 2^{k+3}(k+1) + 2$$

$$\sum_{i=1}^{(k+1)+1} i.2^i = (k+1)\,2^{(k+1)+2} + 2, \text{ for every integer } k \geq 0$$

Hence, this is what we had to prove, and

by induction,

$$\therefore \sum_{i=1}^{n+1} i.2^i = n.2^{n+2} + 2, \text{ for every integer } n \geq 0$$

#18

$$\prod_{i=2}^{n}\left(1-\frac{1}{i}\right)=\frac{1}{n} \quad \text{for every integer } n \geq 2.$$

Proof by Induction: Suppose n is any particular but arbitrarily chosen such that $n \geq 2$

Base Step: For $n = 2$

$$\prod_{i=2}^{2}\left(1-\frac{1}{i}\right)=\frac{1}{2}$$

$$1-\frac{1}{2}=\frac{1}{2}$$

$$\frac{1}{2}=\frac{1}{2} \checkmark$$

Base Case is True.

Inductive Hypothesis: Suppose $\prod_{i=2}^{k}\left(1-\frac{1}{i}\right)=\frac{1}{k}$ for every integer $k \geq 2$.

[to show $\prod_{i=2}^{k+1}\left(1-\frac{1}{i}\right)=\frac{1}{k+1}$ for every integer $k \geq 2$.]

Proof: $$\prod_{i=2}^{k+1}\left(1-\frac{1}{i}\right)=\prod_{i=2}^{k}\left(1-\frac{1}{i}\right)\left(1-\frac{1}{k+1}\right)$$

$$=\frac{1}{k}\cdot\left(1-\frac{1}{k+1}\right) \text{, using Inductive Hypothesis}$$

$$=\frac{1}{k}\cdot\left(\frac{k+1-1}{k+1}\right) \text{, By Algebra}$$

$$=\left(\frac{1}{k}\cdot\frac{k}{k+1}\right)$$

$$\prod_{i=2}^{k+1}\left(1-\frac{1}{i}\right)=\frac{1}{k+1} \text{, for every } k \geq 2$$

$$\therefore \prod_{i=2}^{n}\left(1-\frac{1}{i}\right)=\frac{1}{n} \quad \text{for every integer } n \geq 2.$$

by induction.

4)
#40

If p is any prime number with $p \geq 5$, then the sum of the squares of any p consecutive integers is divisible by p.

Proof:

Suppose n, p are any particular but arbitrarily chosen integers such that $p \geq 5$.

$$n^2 + (n+1)^2 + (n+2)^2 + \dots + (n+(p-1))^2$$

$$= pn^2 + 2n(1+2+3+\dots+(p-1)) + (1+4+9+\dots+(p-1)^2)$$

$$= pn^2 + 2n\frac{(p(p+1))}{2} + (1+4+9+\dots+p^2)$$

Clearly, $p \mid pn^2$ & $p \mid p(n(p+1))$ because pn^2 & $p(n(p+1))$ both can be represented as pm, where m is some integer by def. of divisibility

(Here
- $1+2+3+4+\dots+n = \frac{n(n+1)}{2}$
- $1+2^2+3^2+4^2+\dots+n^2 = \frac{n(n+1)(2n+1)}{6}$)

Let $S = 1+4+9+\dots+p^2$

$$S = \frac{p(p+1)(2p+1)}{6}$$

$$6S = p(p+1)(2p+1)$$

Now, $p \mid p(p+1)(2p+1)$ By transitivity & substitution.

$\therefore p \mid 6S$

$\Rightarrow p \mid S$ by def. of divisibility.

Hence, p also divides the third term.

$\therefore p \mid$ All three terms above.

$\therefore$ If p is any prime number with $p \geq 5$, then the sum of the squares of any p consecutive integers is divisible by p.

5)

#3

#3

(a)

$5 = 5.1 + 8.0 = 5$

$8 = 5.0 + 8.1 = 8$

$10 = 5.2 + 8.0 = 10$

$13 = 5.1 + 8.1 = 13$

$15 = 5.3 + 8.0 = 15$

$16 = 5.0 + 8.2 = 16$

$20 = 5.4 + 8.0 = 20$

$21 = 5.1 + 8.2 = 21$

$24 = 5.0 + 8.3 = 24$

$25 = 5.5 + 8.0 = 25$

(b) Any qty. of at least 28 stamps can be obtained by buying a collection of 5-stamp packages and 8-stamp packages.

Let $P(n)$ = For any qty. $n \geq 28$ stamps, $n = 5m + 8n$ stamps. where m, n are some integers.

Base Case: $n = 28$

$28 = 5(4) + 8(1)$

$28 = 20 + 8$

$28 = 28$ ✓

Base Case is TRUE.

Inductive Hypothesis: Suppose $P(k)$ = For any qty. $k \geq 28$ stamps, $k = 5a + 8b$ stamps be true, $a, b \in \mathbb{Z}^+ - \{0\}$

to prove $P(k+1)$ = For any qty. $k \geq 28$ stamps, $k+1$ stamps can be made using 5 & 8 stamps is True.

Case 1 Suppose $0 \le b < 3$,

since $k \ge 28$,

$a \ge 3$

Define $y = b+2$ & $x = a-3$

$$
\begin{aligned}
5x + 8y &= 5(a-3) + 8(b+2) \\
&= 5a - 15 + 8b + 16 \\
&= 5a + 8b + 1 = k+1
\end{aligned}
$$

using induc. hypothesis

Case 2

Suppose $b \ge 3$, since $k \ge 28$, $a \ge 1$

Define $x = a+5$, $y = b-3$

$$
\begin{aligned}
5x + 8y &= 5(a+5) + 8(b-3) \\
&= 5a + 25 + 8b - 24 \\
&= 5a + 8b + 1 = k+1
\end{aligned}
$$

using induc. hypothesis

$\therefore$ By induction, for any qty. of stamps ≥ 28, you can use 3-stamp packages & 3-stamp packages to obtain that qty

$\square$

#12

#12 For any integer, $n \geq 0$, $7^n - 2^n$ is divisible by 5.

Let $P(n)$: For any integer, $n \geq 0$, $5 \mid 7^n - 2^n$

Base Case: $n = 0$, $7^0 - 2^0 = 5p$, for some integer p, by def of divisibility

$$1 - 1 = 5p$$
$$0 = 5p$$
$$0 = 5(0)$$, by zero product property

Inductive Hypothesis: Suppose, for any integer, $k \geq 0$, $7^k - 2^k = 5p$, for some integer p by def. of divisibility

$$\begin{aligned}
7^{k+1} - 2^{k+1} &= 7^k \cdot 7 - 2^k \cdot 2 \\
&= 7^k \cdot (5+2) - 2^k \cdot 2 \\
&= 7^k \cdot 5 + 7^k \cdot 2 - 2^k \cdot 2 && \text{by distributivity} \\
&= 7^k \cdot 5 + 2(7^k - 2^k) && \text{by factoring out 2} \\
&= 7^k \cdot 5 + 2(5p) && \text{using induc. hypothesis} \\
&= 7^k \cdot 5 + 5(2p) && \text{by associativity} \\
&= 5(7^k + 2p) \\
7^{k+1} - 2^{k+1} &= 5m && \text{for } m = 7^k + 2p \text{, where } m \text{ is some integer}
\end{aligned}$$

by def. of divisibility

$$5 \mid 7^{k+1} - 2^{k+1}$$

$\therefore$ $P(k+1)$ is True.

$\therefore$ By induction, for any integer $n \geq 0$, $7^n - 2^n$ is divisible by 5.

$\square$

#21

#21 $\sqrt{n} < \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \dots + \frac{1}{\sqrt{n}}$, for every integer $n \geq 2$

For every integer $n \geq 2$, $\sqrt{n} < \sum_{i=1}^{n} \frac{1}{\sqrt{i}}$

Base Case: $n = 2$

$$\sqrt{2} < \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}}$$

$$\sqrt{2} < \frac{\sqrt{2}+1}{\sqrt{2}}$$

$$2 < \sqrt{2}+1$$

$$2 < (1.414\ldots)+1$$

$$2 < 2.414 \checkmark$$

Base Case is TRUE.

Inductive Hypothesis: Suppose for every integer $k \geq 2$,

$$\sqrt{k} < \sum_{i=1}^{k} \frac{1}{\sqrt{i}}.$$

to show for every integer $k \geq 2$, $\sqrt{k+1} < \sum_{i=1}^{k+1} \frac{1}{\sqrt{i}}$

From Inductive Hypothesis,

$$\sqrt{k} < \sum_{i=1}^{k} \frac{1}{\sqrt{i}}$$

$$\sqrt{k} + \frac{1}{\sqrt{k+1}} < \sum_{i=1}^{k} \frac{1}{\sqrt{i}} + \frac{1}{\sqrt{k+1}}$$, adding $\frac{1}{\sqrt{k+1}}$ to both sides

$$\frac{(\sqrt{k})(\sqrt{k+1}) + 1}{\sqrt{k+1}} < \sum_{i=1}^{k+1} \frac{1}{\sqrt{i}}$$

$$\frac{\sqrt{k+1} \times \sqrt{k+1}}{\sqrt{k+1}} < \sum_{i=1}^{k+1} \frac{1}{\sqrt{i}}$$, since $\frac{\sqrt{k+1} \times \sqrt{k+1}}{\sqrt{k+1}} \leq \frac{\sqrt{k^2+k} + 1}{\sqrt{k+1}}$

when

$k+1 \leq \sqrt{k^2+k} + 1$

$$\sqrt{k+1} < \sum_{i=1}^{k+1} \frac{1}{\sqrt{i}}$$

$\therefore$ by induction, $\sqrt{n+1} < \sum_{i=1}^{n} \frac{1}{\sqrt{i}}$, for any integer $n \geq 2$

$\square$

6)
#27

#27 $d_1 = 2$, $d_k = \frac{d_{k-1}}{k}$ for each integer $k \geq 2$.

To prove: For every $n \geq 1$, $d_n = \frac{2}{n!}$, $n \in \mathbb{Z}^+$.

Base Cases: $k = 2$ $\quad\quad$ $k = 1$

$d_2 = \frac{d_1}{2} = \frac{2}{2} = 1$ $\quad\quad$ $d_1 = \frac{d_1}{1} = \frac{2}{1} = 2$

$n = 2$, $d_2 = \frac{2}{2!} = \frac{2}{2} = 1$ $\quad\quad$ $n = 1$, $d_1 = \frac{2}{1!} = \frac{2}{1} = 2$

$d_2 = d_2$ ✓ $\quad\quad$ $d_1 = d_1$ ✓

Base Cases are TRUE.

Inductive Hypothesis: Suppose any integer $m \geq 2$, $d_m = \frac{2}{m!}$

to show $d_{m+1} = \frac{2}{(m+1)!}$

$d_{m+1} = \frac{d_{m+1-1}}{m+1}$ by recursive definition

$d_{m+1} = \frac{d_m}{m+1}$

$d_{m+1} = \frac{2}{m!(m+1)}$ by inductive hypothesis

$d_{m+1} = \frac{2}{(m+1)!}$

$\therefore$ by induction,

for any $n \in \mathbb{Z}^+$, $d_n = \frac{2}{n!}$ for the sequence given.

$\square$

#28

#28 For $n \in \mathbb{Z}^+$,

$$\frac{1}{3} = \frac{1+3+5+\dots+(2n-1)}{(2n+1)+(2n+3)+\dots+(2n+(2n-1))}$$

$$\frac{1}{3} = \frac{\sum_{i=1}^{n}(2i-1)}{\sum_{i=1}^{n} 2n+(2i-1)}$$

$$\frac{1}{3} = \frac{\sum_{i=1}^{n}(2i-1)}{2n(n)+\sum_{i=1}^{n}(2i-1)}$$

$$2n^2 + \sum_{i=1}^{n}(2i-1) = 3\sum_{i=1}^{n}(2i-1)$$

$$2\sum_{i=1}^{n}(2i-1) = 2n^2$$

Let $P(n) = \sum_{i=1}^{n}(2i-1) = n^2$

Base Case: $n=1$

$$P(1) = \sum_{i=1}^{1}(2i-1) = (1)^2$$

$$= 2(1)-(1) = (1)^2$$

$$= 2-1 = 1$$

$$1 = 1 \checkmark$$

Base Case is TRUE.

Inductive Hypothesis: Suppose $P(k)$ is True,

$$P(k) = \sum_{i=1}^{k}(2i-1) = k^2$$

$$\sum_{i=1}^{k+1}(2i-1) = \sum_{i=1}^{k}(2i-1) + (2(k+1)-1)$$

$$= k^2 + 2k + 2 - 1$$, using induc. hypothesis

$$= k^2 + 2k + 1$$

$$\sum_{i=1}^{k+1}(2i-1) = (k+1)^2$$

$\therefore$ By induction, for every $n \in \mathbb{Z}^+$

$$\sum_{i=1}^{n}(2i-1) = n^2$$

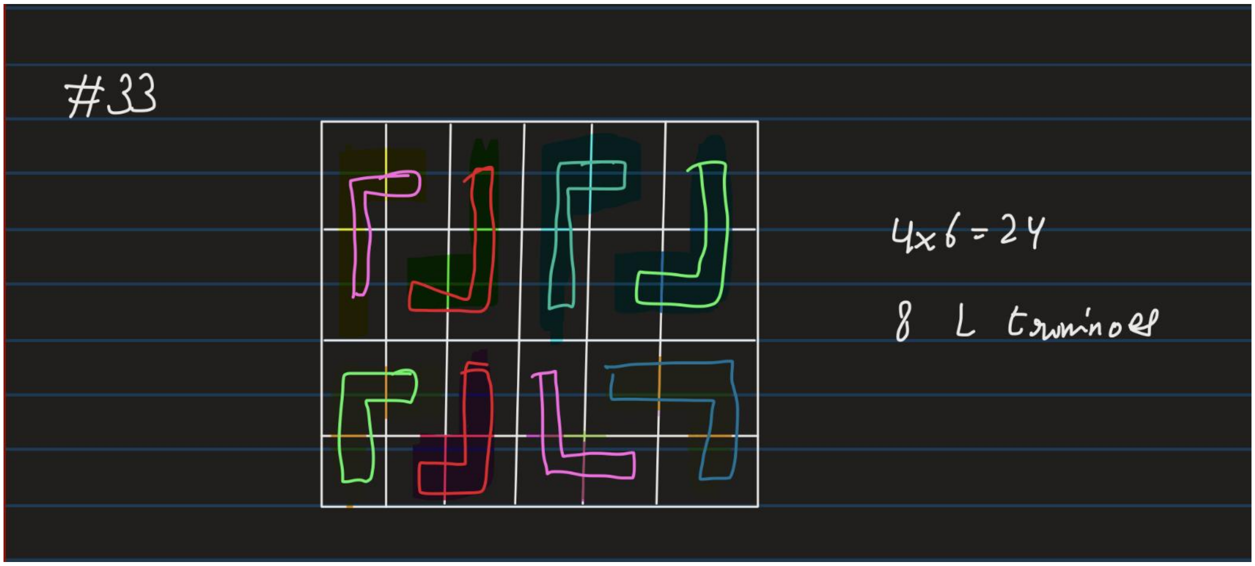
or.

$$\frac{1}{3} = \frac{1+3+5+\dots+(2n-1)}{(2n+1)+(2n+3)+\dots+(2n+(2n-1))}$$

$\square$

7)

#33



#45

The inductive step fails for going from n=1 to n=2, because when k=1 ,
A={a_1,a_2} and B = {a_1}
and no set C can be defined to have the properties claimed for the C in the proof. The reason is that C = {a_1} = B, and so an element of A, namely a_2, is not in either B or C.

Since the inductive step fails for going from n=1 to n=2, the truth of the following statement is never proved: “All the numbers in a set of two numbers are equal to each other.” This breaks the sequence of inductive steps, and so none of the statements for n>2 is proved true either.

#46

The basis step is False.
3^1 - 2 = 3 - 2 = 1, and 1 is not divisible by 2, since by def. of divisibility,
1 = 2p, for some integer p
p = ½ = 0.5 which is not an integer
Hence, 1 is not divisible by 2 and 1 is not even.
Base Case is False
The statement is False.