1)

#38

For every integer $m, m^2 = 5k,\ or\ m^2 = 5k + 1, or\ m^2 = 5k + 4$ for some integer k.
Proof: Suppose m is any particular but arbitrarily chosen integer,
By Quotient Remainder Theorem,
Let's consider the cases for m mod 5 and m div 5, for some integer a

1) m = 5a
2) m = 5a + 1
3) m = 5a + 2
4) m = 5a + 3
5) m = 5a + 4

Squaring all the cases to get a perfect square m^2 and then factoring out 5 from possible terms through commutativity

1) $m^2 = (5a)^2 = 25a^2 = 5(5a^2) = 5k,\ where\ k = 5a^2$
2) $m^2 = (5a + 1)^2 = 25a^2 + 1 + 10a = 5(5a^2 + 2a) + 1 = 5k + 1,\ where\ k = 5a^2 + 2a$
3) $m^2 = (5a + 2)^2 = 25a^2 + 4 + 20a = 5(5a^2 + 4a) + 4 = 5k + 4,\ where\ k = 5a^2 + 4a$
4) $m^2 = (5a + 3)^2 = 25a^2 + 9 + 30a = 25a^2 + 5 + 4 + 30a = 5(5a^2 + 6a + 1) + 4 = 5k + 4,$
   $where\ k = 5a^2 + 6a + 1$
5) $m^2 = (5a + 4)^2 = 25a^2 + 16 + 40a = 25a^2 + 1 + 15 + 40a = 5(5a^2 + 8a + 3) + 1 = 5k + 1,$
   $where\ k = 5a^2 + 8a + 3$

Hence, by Quotient Remainder Theorem
∴ for every integer m, $m, m^2 = 5k,\ or\ m^2 = 5k + 1, or\ m^2 = 5k + 4$ for some integer k.

□

#42

For all real numbers r and c with $c \geq 0, - c \leq r \leq c$ if, and only if, $|r| \leq c$.

1) For all real numbers r and c with $c \geq 0, - c \leq r \leq c$ if $|r| \leq c$.
2) For all real numbers r and c with $c \geq 0$, if $- c \leq r \leq c$, then $|r| \leq c$.

Proof:

1) Suppose r and c are any particular but arbitrarily chosen real numbers, s.t. $c \geq 0$ and $|r| \leq c$
   By definition of absolute value,
   $r \leq c$ and $- r \leq c$
   or, $r \leq c$ and $- c \leq r$
   $\therefore - c \leq r \leq c$

2) Suppose r and c are any particular but arbitrarily chosen real numbers, s.t. $c \geq 0$ and $-c \leq r \leq c$
Then, by algebra,
$r \leq c$ and $-c \leq r$
Hence, $r \leq c$ and $-r \leq c$
By definition of absolute value,
∴ $|r| \leq c$
□

#47

If m, n, and d are integers, d > 0, and $d|(m-n)$, then m mod d = n mod d

Proof: Suppose m, n, and d are any particular but arbitrarily chosen integers s.t. d > 0, and $d|(m-n)$.
Let p = m mod d and q = n mod d
By Quotient Remainder Theorem,
m = dr + p
n = dt + q
for some integers r and t
Subtracting n from m
m - n = dr + p - dt - q
m - n = dr - dt + p - q
m - n = d(r - t) + (p - q)
s = r - t is an integer, since integers are closed under subtraction
m - n = ds + (p - q)
However, it is given that $d|(m-n)$,
By definition of divisibility for $d|(m-n)$,
$(m-n)\ mod\ d = 0 = (p-q)$, by subsititution
(p-q) = 0, or p = q, by algebra
Since p = q and p = m mod d and q = n mod d
By transitivity,
∴ m mod d = n mod d
□

2)

Insert after the word “Proof:” -> Suppose negation is true for proof by contradiction

$\forall x \in R$, if $|x| < \varepsilon$ for any $\varepsilon > 0$, then $x = 0$

Contradiction: $\exists\, x \in R$, s.t. $|x| < \varepsilon$ for any $\varepsilon > 0$ and $x \neq 0$

Proof: Suppose x is any ~~to~~ particular but arbitrarily chosen integer such that $|x| < \varepsilon$ for any $\varepsilon > 0$ and $x \neq 0$

$x \neq 0$

$\therefore |x| > 0$ (By def of abs value) - (1)

$\frac{1}{4} > 0$ - (2)

(1) $\times$ (2), using Theorem T27 in Appendix A

$\frac{|x|}{4} > 0$

Let $\varepsilon = \frac{|x|}{4} > 0$

$\varepsilon = \frac{|x|}{4} < |x|$ - (3)

Given, $|x| < \varepsilon$ - (4)

By Transitivity in (3) & (4),

$\varepsilon < |x| < \varepsilon$

However, $\varepsilon \not< \varepsilon$, which is a contradiction

Hence, the assumed negation of the statement is false.

$\therefore \forall x \in R$, if $|x| < \varepsilon$ for any $\varepsilon > 0$, then $x = 0$.

□

3)

#9 Insert after the word “Proof:” -> Suppose negation is true for proof by contradiction

(a) Statement: The difference of any irrational number and any rational number is irrational.

Student's Statement: The difference of any irrational number and any rational number is rational.

The problems are marked in red. The negation, instead of being existential, it is universal. Instead of any, it should be some or atleast one for proper contradiction.

(b) The difference of any irrational number and any rational number is irrational.

Negation: The difference of some irrational number and some rational number is rational.

Proof: Let's assume that there exists x and y as particular but arbitrarily chosen rational & irrational numbers respectively such that

$$(y - x) \in \mathbb{Q}$$

By def. of rational number,

$x = \frac{a}{b}$ where a, b are some integers & $b \neq 0$

By. def. of rational number,

$(y - x) = \frac{p}{q}$, where p, q are some integers & $q \neq 0$

$$y - \frac{a}{b} = \frac{p}{q} \quad \text{, by substitution}$$

$$y = \frac{p}{q} + \frac{a}{b}$$

$$y = \frac{bp}{bq} + \frac{qa}{qb} \quad \text{, by multiplicative identity}$$

$$y = \frac{bp}{bq} + \frac{qa}{bq} \quad \text{, by commutativity}$$

$$y = \frac{bp+qa}{bq} \quad \text{, algebra}$$

since integers are closed under addition and subtraction and multiplication, $bp+qa$, bq are all integers and by zero product property and $b \neq 0$ & $q \neq 0$, $\therefore bq \neq 0$

$\therefore$ By def. of rational number, y is a rational number

which is a contradiction.

Hence, the negation assumed was false.

$\therefore$ The difference of any irrational number and any rational number is irrational.

□

#18

Insert after the word "Proof:" -> Suppose negation is true for proof by contradiction

If a and b are rational numbers, $b \neq 0$ and r is an irrational number, then $a + br$ is irrational.

Negation: a and b are rational numbers, $b \neq 0$ and r is an irrational number and $a + br$ is rational.

Proof: Suppose a, b are any particular but arbitrarily chosen rational numbers and r is any particular but arbitrarily chosen irrational number such that $b \neq 0$ and $a + br$ is rational.

By def of rational numbers, let p, q, m, n be some integers such that $q, n \neq 0$ for $a = \frac{p}{q}$ & $b = \frac{m}{n}$

then, $a + br = \frac{p}{q} + \frac{mr}{n}$ (By Substitution)

$\Rightarrow \frac{np}{nq} + \frac{qmr}{qn}$, By Multiplicative Identity

$= \frac{np}{nq} + \frac{qmr}{nq}$, By Commutativity

$= \frac{np + qmr}{nq}$, By Algebra — 1

since $a + br$ is rational, by def of rational

$a + br = \frac{i}{j}$ for some integers i, j & $j \neq 0$ — 2

By transitivity from 1 & 2,

$\frac{i}{j} = \frac{np + qmr}{nq}$

$$qmr = \frac{inq}{j} - np$$

$$qmr = \frac{inq - njp}{j}$$

$$r = \frac{inq - njp}{jqm}$$

Since integers are closed under multiplication, subtraction, and division.

$(inq - njp)$ and jqm are both integers.

$j \neq 0$, $q \neq 0$ and since $b \neq 0$, $\frac{m}{n} \neq 0$ $\Rightarrow m \neq 0$

By Zero Product Property,

$$jqm \neq 0$$

By def. of a rational number, r is rational. which is a contradiction.

The supposition is, thus, false

$\therefore$ If a, b are rational numbers, $b \neq 0$ and r is an irrational, then $a + br$ is irrational.

$\square$

4)

#22

For every real number r, if r^2 is irrational then r is irrational.

a) Proof by contradiction: Suppose not. That is, suppose there exists a real number r such that r^2 is irrational and r is rational. Show that this supposition leads logically to a contradiction, which will prove that r^2 is irrational, then r is irrational for every real number r.
b) Proof by contraposition: Suppose that for every real number r such that r is rational. Show that r^2 is rational.

Case 1): Suppose r = 0 then $|r| = 0$, by definition of absolute value
c = 0
r = c = 0
$|r|$ = r = 0
By transitivity,
r = c = 0

∴ r < c
Case 2): Suppose r > 0 then $|r| = r$, by definition of absolute value
$|r|$ = r > 0
c > 0
By transitivity,
c ≥ $|r|$ = r
∴ c ≥ r
□

#24

Insert after the word “Proof:” -> Suppose negation is true for proof by contradiction

The reciprocal of any irrational number is irrational.

Proof by Contradiction: Negation: There exists an irrational number whose reciprocal is rational.

Suppose m is any particular but arbitrarily chosen irrational number, such that $\frac{1}{m}$ is rational.

By def of rational number,

$$\frac{1}{m} = \frac{a}{b}, \text{ where } a, b \text{ are some integers such that } b \neq 0. \quad - (1)$$

since $\frac{1}{m}$ is rational,

1 and m must be integers.

By. def of rational numbers,

$\frac{m}{1} = m$ is rational (i.e. all integers are rational)

Hence, m is rational, which is a contradiction.

Hence, the supposition is false.

$\therefore$ The reciprocal of any irrational number is irrational.

□

5)

#28

For all integers a, b, and c, if $a \mid b$ and $a \nmid c$, then $a \nmid (b+c)$.

<u>Proof by Contraposition</u>: suppose a, b, c are any particular but arbitrarily chosen integers such that $a \mid (b+c)$.

To prove: $a \nmid b$ or $a \mid c$.

By def. of divisibility,

$b+c = ak$ for some integer k.

To prove: $a \mid (b+c) \rightarrow a \nmid b \vee a \mid c, \ \forall a,b,c \in \mathbb{Z}$

we can prove

$\equiv \forall a,b,c \in \mathbb{Z}, \ a \mid (b+c) \wedge \neg(a \nmid b) \Rightarrow a \mid c$

$\equiv \forall a,b,c \in \mathbb{Z}, \ a \mid (b+c) \wedge a \mid b \rightarrow a \mid c$

By def. of divisibility,

$b = am$ for some integer m

By Substitution

$am + c = ak$

$c = ak - am$

$c = a(k-m)$

since integers are closed under subtraction and k, m are integers, $(k-m)$ is an integer

by def. of divisibility,

$a \mid c$.

$\therefore \forall a,b,c \in \mathbb{Z}, \ a \mid (b+c) \Rightarrow a \nmid b \vee a \mid c$

$\therefore \forall a,b,c \in \mathbb{Z}, \ a \mid (b+c) \wedge \neg(a \nmid b) \rightarrow a \mid c$

$\therefore \forall a,b,c \in \mathbb{Z}, \ a \mid b \wedge a \nmid c \rightarrow a \nmid (b+c)$.

□

Insert after the word “Proof:” -> Suppose negation is true for proof by contradiction

<u>Proof by Contradiction:</u>

Negation: There exist some integers a, b, c such that $a \mid b$ and $a \nmid c$ and $a \mid (b+c)$.

Suppose a, b, c are any particular but arbitrarily chosen integers, such that $a \mid b$, $a \nmid c$, and $a \mid (b+c)$.

By def. of divisibility,

$b = ak$ for some integer k

$b + c = am$ for some integer m

By substitution,

$$ak + c = am$$

$$c = am - ak$$

$$c = a(m-k)$$

since integers are closed under subtraction and k, m are integers, $(m-k)$ is an integer

by def. of divisibility,

$a \mid c$.

which is a contradiction to $a \nmid c$.

Hence, the supposition is false.

$\therefore$ For all integers a, b, and c, if $a \mid b$ and $a \nmid c$, then $a \nmid (b+c)$.

□

6)

#31

(a) For all positive integers, n, r and s, if $rs \leq n$, then $r \leq \sqrt{n}$ or $s \leq \sqrt{n}$.

Proof by contraposition: Suppose n, r, s are any particular but arbitrarily chosen positive integers such that $r > \sqrt{n}$ and $s > \sqrt{n}$.

[to prove: $rs > n$]

Using Theorem T27 in Appendix A,

$$r.s > \sqrt{n}.\sqrt{n}$$

$$rs > (\sqrt{n})^2$$

$$rs > n$$

which was to be proved.

$\therefore$ For all positive integers, n, r and s, if $rs \leq n$, then $r \leq \sqrt{n}$ or $s \leq \sqrt{n}$. $\square$

(b) $\forall n \in \mathbb{Z}^+ - \{1\}$, n is not prime $\Rightarrow \exists$ prime no. p s.t. $p \leq \sqrt{n} \wedge p \mid n$.

Proof: Suppose n is any particular but arbitrarily chosen integer such that $n > 1$ and n is not prime.

since n is not prime,
by def. of composite numbers,
$n = ab$ for some integers a and b
s.t. $a \mid n$ and $b \mid n$ and
$a, b \neq n, 1$.

Using results from (a)

if $ab \leq n$, then $a \leq \sqrt{n}$ or $b \leq \sqrt{n}$.

Let's consider the case when
if $ab \leq n \Rightarrow a \leq \sqrt{n}$.

By substitution,

$$\frac{n}{b} \leq \sqrt{n}$$

$$b \geq \frac{n}{\sqrt{n}}$$

$$b \geq \sqrt{n}$$

case when if $ab \leq n \Rightarrow b \leq \sqrt{n}$

By substitution,

$$\frac{n}{a} \leq \sqrt{n}$$

$$a \geq \sqrt{n}$$

Hence, either of a or b $\geq n$ and then the other is $\leq n$.

Thus, n has a factor $p <= \sqrt{n}$.
(since all numbers can be expressed as a product of prime factors)
such that p is prime.

Since $a \mid n$ and $b \mid n$ and
p is a prime factor of
either a or b such that
either of them is $<= \sqrt{n}$
as proved above, i.e. $p \mid a$ or $p \mid b$.
then, $p \mid n$ by transitivity

$\therefore \forall n \in \mathbb{Z}^+ - \{1\}$, n is not prime $\Rightarrow \exists$ prime no. p s.t. $p \leq \sqrt{n} \wedge p \mid n$.

$\square$

(c) Contrapositive of (b): For each integer $n > 1$, if n is not divisible by any prime number p such that $p \leq \sqrt{n}$, then n is prime.