Q1)

#6 1) (a) $(A \cap B) \cup (A \cap C)$

(b) A

(c) $x \in C$

(d) $x \in (A \cap B) \cup (A \cap C)$

2) (a) or

(b) and

(c) $x \in (A \cap (B \cup C))$

(d) subset

#11 For all sets A, B, and C, $A \cap (B - C) \subseteq (A \cap B) - (A \cap C)$

Proof: Suppose there exist sets A, B, C such that there is any element $x \in (A \cap (B - C))$

[To prove: $x \in ((A \cap B) - (A \cap C))$]

by definition of $\cap$,

$x \in A$ and $x \in (B - C)$

by definition of set difference,

$x \in A$ and $x \in B$ and $x \notin C$.

$(x \in A$ and $x \in B)$, in addition to $x \notin C$ by associativity,

$x \in (A \cap B)$, in addition to $x \notin C$ by def of intersection,

$x \in (A \cap B)$ and $(x \notin C$ or $x \notin A)$ by generalization

$x \in (A \cap B)$ and $(x \in C$ and $x \in A)^c$, by De Morgan's Law

$x \in (A \cap B)$ and $(x \in A$ and $x \in C)^c$, by commutativity

$x \in (A \cap B)$ and $(x \in (A \cap C))^c$, by def. of intersection

$x \in (A \cap B)$ and $(x \notin (A \cap C))$, by complement law

$\Rightarrow (A \cap B) - (A \cap C)$

Thus, x is in $(A \cap B) - (A \cap C)$, which proves that every element in $A \cap (B - C)$ is in $(A \cap B) - (A \cap C)$

$\therefore A \cap (B - C) \subseteq (A \cap B) - (A \cap C)$ $\square$

#15 For every set A, $A \cup \emptyset = A$

Proof: There are 2 cases: $A \cup \emptyset \subseteq A$, $A \subseteq A \cup \emptyset$

Case 1 $A \cup \emptyset \subseteq A$

Suppose there exists a set A, such that

there is any element $x \in (A \cup \emptyset)$

$x \in A$ or $x \in \emptyset$, by def. of union

since $\emptyset$ represents an empty set

$x \in \emptyset$ means that x will not represent

any element in an empty set.

Hence, $x \in A$. Thus, x is in A, which proves

that x is in $A \cup \emptyset$ and x is also in A. $\therefore A \cup \emptyset \subseteq A$

Case 2 $A \subseteq A \cup \emptyset$

Suppose there exists a set A, such that

there is any element $x \in A$

By generalization,

$x \in A$ or $x \in \emptyset$

Hence, by def. of union, $x \in A \cup \emptyset$

Thus, x is in $A \cup \emptyset$, which proves that x is

in A and x is also in $A \cup \emptyset$. $\therefore A \subseteq A \cup \emptyset$

In both cases 1 & 2, $A \subseteq A \cup \emptyset$ & $A \cup \emptyset \subseteq A$

then by definition of set equality,

$A \cup \emptyset = A$

$\square$

Q2)

#22 For all sets $A, B,$ and C,

$$A \times (B \cap C) = (A \times B) \cap (A \times C)$$

Proof: There are 2 Cases: $A \times (B \cap C) \subseteq (A \times B) \cap (A \times C)$ and $(A \times B) \cap (A \times C) \subseteq A \times (B \cap C)$

Case ①: Suppose there exists sets A, B, C such that $(x,y) \in A \times (B \cap C)$

By definition of Cartesian Product, $x \in A$ and $y \in (B \cap C)$

By definition of Intersection, $y \in B$ and $y \in C$.

Overall, $x \in A$ and $y \in B$ and $y \in C$ - ②

$(A \cap A) = A$, by idempotent Law

by def. of intersection, $x \in A = x \in A$ and $x \in A$ - ①

By substituting ① in ②

$x \in A$ and $x \in A$ $y \in B$ and $y \in C$

$x \in A$ and $y \in B$ and $x \in A$ and $y \in C$, by commutativity

$(x,y) \in (A \times B)$ and $(x,y) \in (A \times C)$, by def. of Cartesian Product

by definition of intersection,

$x,y \in ((A \times B) \cap (A \times C))$

Thus, (x,y) is in $(A \times B) \cap (A \times C)$, which proves that every element in $A \times (B \cap C)$ is in $(A \times B) \cap (A \times C)$

$\therefore A \times (B \cap C) \subseteq (A \times B) \cap (A \times C)$

Case ②: Suppose there exists sets A, B, C such that $(x,y) \in (A \times B) \cap (A \times C)$

by def. of intersection, $(x,y) \in (A \times B)$ and $(x,y) \in (A \times C)$

by def. of Cartesian Product, $x \in A$ and $y \in B$ and $x \in A$ and $y \in C$

by commutativity, $(x \in A$ and $x \in A)$ and $y \in B$ and $y \in C$.

by idempotent law, $x \in A$ and $y \in B$ and $y \in C$

by def. of intersection, $x \in A$ and $y \in (B \cap C)$

by def of Cartesian Product, $(x,y) \in (A \times (B \cap C))$.

Thus, (x,y) is in $A \times (B \cap C)$, which proves that every element in $(A \times B) \cap (A \times C)$ is also in $A \times (B \cap C)$. $\therefore (A \times B) \cap (A \times C) \subseteq A \times (B \cap C)$

Since in both cases, $A \times (B \cap C) \subseteq (A \times B) \cap (A \times C)$ and $(A \times B) \cap (A \times C) \subseteq A \times (B \cap C)$. By def of set equality, $A \times (B \cap C) = (A \times B) \cap (A \times C)$. $\square$

#35 For all sets A, B, and C, if $A \subseteq B$ and $B \cap C = \emptyset$ then $A \cap C = \emptyset$.

Proof: Suppose not. Suppose there exist sets A, B, C such that $A \subseteq B$, and $B \cap C = \emptyset$ and $A \cap C \neq \emptyset$.

Let any element $x \in A \cap C$

by def. of intersection, $x \in A$ and $x \in C$.

by def. of subset, if $x \in A$, then $x \in B$

Now, $x \in A$ and $x \in B$ and $x \in C$

From this, by def. of intersection, $x \in (B \cap C)$.

this means that $(B \cap C)$ has an element, x $\therefore B \cap C \neq \emptyset$

Which is a contradiction to the supposition, $B \cap C = \emptyset$

$B \cap C \neq \emptyset$ and $B \cap C = \emptyset$ is not possible. And,

$\therefore$ for all sets A, B, and C, if $A \subseteq B$ and $B \cap C = \emptyset$, then $A \cap C = \emptyset$.

$\square$

Q3)

#2 For all sets A and B, $(A \cup B)^c = A^c \cup B^c$

Let's suppose $A = \{x \mid x \text{ is prime}\}$

$B = \{x \mid x \text{ is composite}\}$

$A \cup B = \{x \mid x \in \mathbb{Z}^+ - \{1\}\}$

$(A \cup B)^c = \{x \mid x \in \mathbb{Z}^{\text{nonneg}} \cup \{1\}\}$

$A^c = \{x \mid x \text{ is composite}\}$

$B^c = \{x \mid x \text{ is prime}\}$

$A^c \cup B^c = \{x \mid x \in \mathbb{Z}^+ - \{1\}\}$

$(A \cup B)^c \neq A^c \cup B^c$

in fact, they're disjoint.

Hence, the statement is a false, as proved by the counterexample above.

#22 a. Negation: $\exists$ set S, such that $\forall$ sets T, $S \cap T \neq \emptyset$.

The statement is True. Not the Negation.

Given any set S, take $T = S^c$.

Then, $S \cap T = S \cap S^c = \emptyset$ by the complement law for $\cap$.

b. Negation: $\forall$ sets S, $\exists$ set T such that $S \cup T \neq \emptyset$.

The negation is True.

Given any set S, take $T = \{1, 2\}$

$S \cup T = S \cup \{1,2\} \neq \emptyset$ since even if $S = \emptyset$, $\{1, 2\} \neq \emptyset$.

#28 (a) set difference law (b) ~~de~~ set difference law (c) commutative law for $\cap$

(d) De Morgan's law (e) Double Complement Law (f) Distributivity law

(g) set difference law

Q4)

#33 For all sets A and B, $(A-B) \cap (A \cap B) = \emptyset$

Proof: $(A-B) \cap (A \cap B)$

$= (A \cap B^c) \cap (A \cap B)$ by set difference law

$= A \cap A \cap B^c \cap B$ by commutative law of $\cap$

$= (A \cap A) \cap (B \cap B^c)$ by commutative law of $\cap$

$= A \cap \emptyset$ by complement law

$= \emptyset$ by universal bound law

$\therefore (A-B) \cap (A \cap B) = \emptyset$

#38 For all sets A and B, $(A \cap B)^c \cap A = A - B$

Proof: $(A \cap B)^c \cap A$

$(A^c \cup B^c) \cap A$, by De Morgan's Law

$A \cap (A^c \cup B^c)$, by commutative law of $\cap$

$(A \cap A^c) \cup (A \cap B^c)$, by distributive law

$\emptyset \cup (A \cap B^c)$, by Complement Law

$(A \cap B^c) \cup \emptyset$, by commutative law of $\cap$

$A \cap B^c$, by Identity Law

$\Rightarrow A - B$, by set difference law

$\therefore (A \cap B)^c \cap A = A - B$ for all sets A and B

#43 $((A \cap (B \cup C)) \cap (A - B)) \cap (B \cup C^c)$

$(((A \cap B) \cup (A \cap C)) \cap (A - B)) \cap (B \cup C^c)$, by distributive law

$(((A \cap B) \cup (A \cap C)) \cap (A \cap B^c)) \cap (B \cup C^c)$, by set difference law

$[[((A \cap B) \cup A) \cap ((A \cap B) \cup C)] \cap (A \cap B^c)] \cap (B \cup C^c)$, by distributive law

$((A \cup (A \cap B)) \cap (C \cup (A \cap B)) \cap (A \cap B^c)) \cap (B \cup C^c)$, by commutative law of $\cup$

$((A \cup A) \cap (A \cup B)) \cap ((C \cup A) \cap (C \cup B)) \cap (A \cap B^c)) \cap (B \cup C^c)$, by distributive law

$(A \cap (A \cup B)) \cap ((C \cup A) \cap (C \cup B)) \cap (A \cap B^c)) \cap (B \cup C^c)$, by idempotent law

$(A) \cap ((C \cup A) \cap (C \cup B)) \cap (A \cap B^c)) \cap (B \cup C^c)$, by absorption law

$A \cap (A \cap B^c) \cap ((C \cup A) \cap C) \cup ((C \cup A) \cap B) \cap (B \cup C^c)$, by commutative law of $\cap$ and distributive law

$(A \cap A) \cap B^c \cap (((C \cap (C \cup A)) \cup (B \cap (C \cup A))) \cap (B \cup C^c)$, by associative law & commutative law of $\cap$

$A \cap B^c \cap ((C \cap (C \cup A)) \cup (B \cap (C \cup A))) \cap (B \cup C^c)$, by idempotent law

$A \cap B^c \cap (C \cup (B \cap (C \cup A))) \cap (B \cup C^c)$, by absorption law

$A \cap B^c \cap (C \cup ((B \cap C) \cup (B \cap A))) \cap (B \cup C^c)$, by distributive law

$A \cap B^c \cap ((C \cup (B \cap C)) \cup (B \cap A)) \cap (B \cup C^c)$, by associativity

$A \cap B^c \cap (((C \cup B) \cap (C \cup C)) \cup (B \cap A)) \cap (B \cup C^c)$, by distributive law

$A \cap B^c \cap (((C \cup B) \cap C) \cup (B \cap A)) \cap (B \cup C^c)$, by idempotent law

$A \cap B^c \cap ((C \cap (C \cup B)) \cup (B \cap A)) \cap (B \cup C^c)$, by commutative law of $\cap$

$A \cap B^c \cap ((C \cup (B \cap A)) \cap (B \cup C^c)$, by absorption law

$A \cap B^c \cap [(C \cup B] \cap (C \cup A)] \cap (B \cup C^c)$, by distributive law

$(A \cap (C \cup A)) \cap B^c \cap ((C \cup B] \cap (B \cup C^c)]$, by commutative law of $\cap$

$A \cap B^c \cap ((C \cup B] \cap (B \cup C^c))$, by absorption law

$A \cap B^c \cap (((C \cup B] \cap B) \cup ((C \cup B] \cap C^c)$, by distributive law

$A \cap B^c \cap (B \cap (C \cup B)) \cup (C^c \cap (C \cup B))$, by commutative law of $\cap$

$A \cap B^c \cap (B \cup (C^c \cap (C \cup B)))$, by absorption law

$A \cap B^c \cap (B \cup ((C^c \cap C) \cup (C^c \cap B)))$, by distributive law

$A \cap B^c \cap (B \cup (\emptyset \cup (C^c \cap B)))$, by complement law

$A \cap B^c \cap (B \cup (C^c \cap B))$, by identity law

$A \cap B^c \cap (B \cup (B \cap C^c)$, by commutative law of $\cap$

$A \cap B^c \cap B$, by absorption law

$A \cap (B \cap B^c)$, by associative law

$A \cap \emptyset$, by complement law

$\Rightarrow \emptyset$, by universal bound law

Q5)

#46

a.) $\{1, 2, 5, 6\}$

b.) $\{3, 4, 7, 8\}$

c.) $\{1, 2, 3, 4, 5, 6, 7, 8\}$

d.) $\{1, 2, 7, 8\}$

#52

$(A \Delta B) \Delta C = A \Delta (B \Delta C)$

Proof: Suppose for all sets A, B, C,

$LHS = (A \Delta B) \Delta C = (((A-B) \cup (B-A)) \Delta C)$ by symmetric difference def.

$= ((((A-B) \cup (B-A)) - C) \cup (C - ((A-B) \cup (B-A))))$

$= ((((A \cap B^c) \cup (B \cap A^c)) \cap C^c) \cup (C \cap ((A \cap B^c) \cup (B \cap A^c))^c))$

by set difference law and def. of symmetric diff.

$= ((C^c \cap ((A \cap B^c) \cup (B \cap A^c))) \cup (C \cap ((A^c \cup B) \cap (B^c \cup A)))$

by commutative law of $\cap$ and De Morgan's Law

$= ((C^c \cap A \cap B^c) \cup (C^c \cap B \cap A^c) \cup ((C \cap (A^c \cup B)) \cap (B^c \cup A)))$

by associative law & distributive law

$= (C^c \cap A \cap B^c) \cup (A^c \cap B \cap C^c) \cup (((C \cap A^c) \cup (C \cap B)) \cap (B^c \cup A))$

by distributive law

$= (C^c \cap A \cap B^c) \cup (A^c \cap B \cap C^c) \cup ((A \cup B^c) \cap (C \cap A^c)) \cup$
$(A \cup B^c) \cap (C \cap A^c))$

by distributive & commutative law

$= (C^c \cap A \cap B^c) \cup (A^c \cap B \cap C^c) \cup ((((C \cap A^c) \cap A) \cup (C \cap A^c \cap B^c)) \cup$
$((A \cap C \cap A^c) \cup (B^c \cap C \cap A))$

by distributive & commutative law

$= (C^c \cap A \cap B^c) \cup (A^c \cap B \cap C^c) \cup ((C \cap (A^c \cap A)) \cup (C \cap A^c \cap B^c)$
$\cup (C \cap (A \cap A^c))$ by associative law

$= (C^c \cap A \cap B^c) \cup (A^c \cap B \cap C^c) \cup (C \cap \emptyset) \cup (C \cap A^c \cap B^c) \cup (C \cap \emptyset)$

by Universal Bound Laws

$= (C^c \cap A \cap B^c) \cup (A^c \cap B \cap C^c) \cup \emptyset \cup (C \cap A^c \cap B^c) \cup \emptyset$ by Universal Bound Laws

$= (C^c \cap A \cap B^c) \cup (A^c \cap B \cap C^c) \cup (C \cap A^c \cap B^c)$ by Identity laws.

1

$$A \Delta (B \Delta C) = A \Delta ((B-C) \cup (C-B)), \text{ by def. of sym. difference}$$

$$= (A - ((B-C) \cup (C-B))) \cup (((B-C) \cup (C-B)) - A), \text{ by def. of sym. difference}$$

$$= (A - ((B \cap C^c) \cup (C \cap B^c))) \cup (((B \cap C^c) \cup (C \cap B^c)) - A)), \text{ by set difference law}$$

$$= (A \cap ((B \cap C^c) \cup (C \cap B^c))^c) \cup (((B \cap C^c) \cup (C \cap B^c)) \cap A^c)), \text{ by set difference law}$$

$$= (A \cap ((B^c \cup C) \cap (C^c \cup B))) \cup ((A^c \cap B \cap C^c) \cup (A^c \cap C \cap B^c)), \text{ by De Morgan's Law and Distributivity Law}$$

$$= ((A \cap (B^c \cup C)) \cap (C^c \cup B)) \cup (A^c \cap B \cap C^c) \cup (A^c \cap C \cap B^c), \text{ by Associative law}$$

$$= (((A \cap B^c) \cup (A \cap C)) \cap (C^c \cup B)) \cup (A^c \cap B \cap C^c) \cup (A^c \cap C \cap B^c), \text{ by Distributive Law}$$

$$= (((C^c \cup B) \cap (A \cap B^c)) \cup ((C^c \cup B) \cap (A \cap C))) \cup (A^c \cap B \cap C^c) \cup (A^c \cap C \cap B^c), \text{ by Distributive Law}$$

$$= (A \cap B^c \cap C^c) \cup (A \cap (B^c \cap B)) \cup (A \cap (C \cap C^c)) \cup (A \cap C \cap B) \cup (A^c \cap B \cap C^c) \cup (A^c \cap C \cap B^c),$$

by Associative Law & by Distributive Law

$$= (A \cap B^c \cap C^c) \cup (A \cap \emptyset) \cup (A \cap \emptyset) \cup (A \cap B \cap C) \cup (A^c \cap B \cap C^c) \cup (A^c \cap C \cap B^c)$$

by Universal Bound Laws

$$= (A \cap B^c \cap C^c) \cup (A \cap B \cap C) \cup (A^c \cap B \cap C^c) \cup (A^c \cap C \cap B^c), \text{ by Identity Laws}$$

From ①, $(C^c \cap A \cap B^c) \cup (A^c \cap B \cap C^c) \cup (C \cap A^c \cap B^c) \cup (A \cap B \cap C)$

by generalization.

Now, This shows that

$$\therefore (A \Delta B) \Delta C = A \Delta (B \Delta C)$$