1)
#36

In any round-robin tournament involving n teams, where $n \geq 2$ it is possible to label the teams $T_1, T_2, \ldots, T_n$ so the T_i beats T_{i+1} for all $i = 1, 2, \ldots n-1$

Proof: Proof by mathematical induction

Base Case: Let $n = 2$.

There was only one game played.
Label the winner as T_1 and the loser as T_2.
Hence, Base Case is TRUE.

Inductive Step: Let $k \geq 2$ for any $k \in \mathbb{Z}$.
Assume that for all tournaments with k teams, there is a labelling possible for teams $T_1, T_2, \ldots T_k$ so the T_{i+1} for all $i = 1, 2, \ldots, k-1$.

Proof: Consider a tournament on $k+1$ teams
When we remove one team, say team A, we have a tournament on k teams.
Thus, by our inductive hypothesis, there is a labelling $T_1, T_2, \ldots T_k$ as described above.
Either A beats team T_1, A loses to the first m teams (where $1 \leq m \leq k-1$) and beats the $(m+1)$st teams, or A loses to all the other teams.

In the first case, A, $T_1, T_2, \ldots, T_k$ is a desired ordering, so we relabel our teams so that A becomes T_1', T_1 becomes T_2', and so on.

In the second case, $T_1, T_2, \ldots, T_m$, A, $T_{m+1}, \ldots T_k$ is a desired ordering (since A lost to T_m but beat T_{m+1}), and so we relabel accordingly.

In the third case, $T_1, T_2, \ldots, T_k$, A is a desired ordering (since A lost to everyone, in particular they lost to T_k), and so we relabel accordingly.

Hence, in all cases, we have found the desired labelling.
$\therefore$ By mathematical induction,

In any round-robin tournament involving n teams, where $n \geq 2$ it is possible to label the teams $T_1, T_2, \ldots, T_n$ so the T_i beats T_{i+1} for all $i = 1, 2, \ldots n-1$

$\square$

2)
#13

Every integer greater then 1 is either a prime number or a product of prime numbers.

Proof: Proof by Strong Mathematical Induction

Let $P(n) = \forall n \in \mathbb{Z}^+ - \{1\}, n \in \mathbb{P}$ or $n = ab$ s.t. $a, b \in \mathbb{P}$.

Base Cases: $P(2) = 2$, which is a prime number ✓
$P(3) = 3$, which is a prime number ✓
$P(4) = 4 = 2 \times 2$, where $a = 2$, $b = 2$, a,b are prime numbers ✓

Base Cases are True.

Inductive Hypothesis: For every integer $k \geq 4$, assume $\forall i \in \mathbb{Z}^+ - \{1\}, i \in \mathbb{P}$ or $i = ab$ s.t. $a, b \in \mathbb{P}$ where $2 \leq i \leq k$.

To prove: $(k+1) \in \mathbb{P}$ or $k+1 = ab$ where $a, b \in \mathbb{P}$.

2 cases:

① Assume $k+1$ is prime,
then $k+1 \in \mathbb{P}$.
Hence, proved.

② Assume $k+1$ is composite,

$k+1 = mn$, where
$1 < m, n < k+1$
$2 \leq m, n \leq k$
By Inductive Hypothesis,
$k+1$ can be represented as a product of 2 numbers which can further be represented as products of 2 prime numbers.
As a result, ~~p~~
∴ by mathematical induction,

Every integer greater then 1 is either a prime number or a product of prime numbers.

□

#20

For $b_1, b_2, b_3, \ldots$ is a sequence: $b_1 = 0$, $b_2 = 3$, $b_k = 5 \cdot b_{k/2} + 6$ for every integer $k \geq 3$, b_n is divisible by 3 for $n \geq 1, n \in \mathbb{Z}$.

Proof: Proof by Strong Mathematical Induction,

Let $P(n) = 3 \mid b_n$ for $n \in \mathbb{Z}^+$.

Base Cases: $P(1) = b_1 = 0 = 3(0)$, $\therefore 3 \mid b_0$
$P(2) = b_2 = 3 = 3(1)$, $\therefore 3 \mid b_1$

$\therefore$ By def. of div.,
$P(1)$ & $P(2)$ are true.

Inductive Hypothesis: Suppose for any integer $k \geq 2$,
$P(i)$ is true for $1 \leq i \leq k$.

Proof: $b_{k+1} = 5\, b_{\left(\frac{k+1}{2}\right)} + 6$

since $1 \leq \frac{k+1}{2} \leq k$,

applying inductive hypothesis,

$b_{\left(\frac{k+1}{2}\right)}$ is divisible by 3.

by def. of divisibility,

$b_{\left(\frac{k+1}{2}\right)} = 3m$ for some integer m.

$b_{k+1} = 3m + 6$

$b_{k+1} = 3m + 3(2)$

$$b_{k+1} = 3(m+2)$$

since $m, 2$ are integers and integers are closed under addition.

hence, $m+2$ is an integer

$$b_{k+1} = 3q \quad \text{where } q = m+2$$

$\therefore$ $P(k+1)$ is true.

$\therefore$ By mathematical induction, $P(n)$ is true.

For $b_1, b_2, b_3, \ldots$ is a sequence: $b_1 = 0$, $b_2 = 3$, $b_k = 5 \cdot b_{k/2} + 6$ for every integer $k \geq 3$, b_n is divisible by 3 for $n \geq 1, n \in \mathbb{Z}$.

$\square$

3)

3) Consider the Fibonacci sequence $f_0 = f_1 = 1$ and $f_n = f_{n-1} + f_{n-2}$ for all $n \geq 2$. we need to prove that

$$f_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right]$$

Proof: Proof by Strong Mathematical Induction

Base Case:
$$f_0 = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{0+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{0+1}\right]$$
$$f_0 = \frac{1}{\sqrt{5}}\left[\frac{1+\sqrt{5}}{2} - \left(\frac{1-\sqrt{5}}{2}\right)\right]$$
$$f_0 = \frac{1}{\sqrt{5}}\left[\frac{\cancel{1}+\sqrt{5}-\cancel{1}+\sqrt{5}}{2}\right]$$
$$f_0 = \frac{1}{\sqrt{5}} \times \frac{2\sqrt{5}}{2}$$
$$f_0 = \frac{\sqrt{5}}{\sqrt{5}} = 1 = 1 \checkmark$$
$$f_1 = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{1+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{1+1}\right]$$
$$= \frac{1}{\sqrt{5}}\left[\frac{1+5+2\sqrt{5}}{2} - \left(\frac{1+5-2\sqrt{5}}{2}\right)\right]$$
$$= \frac{1}{\sqrt{5}}\left[\frac{6+2\sqrt{5}}{4} - \left(\frac{6-2\sqrt{5}}{4}\right)\right]$$
$$= \frac{1}{\sqrt{5}}\left[\frac{\cancel{6}+2\sqrt{5}\cancel{-6}+2\sqrt{5}}{4}\right]$$
$$= \frac{1}{\sqrt{5}}\left[\frac{4\sqrt{5}}{4}\right] = 1 = 1 \checkmark$$

Base Cases are TRUE.

Inductive Hypothesis: For any integer $k \geq 1$, assume

$$f_i = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{i+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{i+1}\right] \text{ is True.}$$

for $0 \leq i \leq k$

Proof: $f_{k+1} = f_{k+1-1} + f_{k+1-2}$

$f_{k+1} = f_k + f_{k-1}$

since $0 \leq k \leq k$ and $0 \leq k-1 \leq k$

Hence, by applying strong inductive hypothesis,

f_k and f_{k-1} are true.

$\therefore$ f_{k+1} is also true.

$\therefore$ By mathematical induction,

$$f_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right]$$

is True for any integer $n \geq 2$.

$\square$

4)
#25 and #32

#25

For $k=0$, $\frac{r^k \cdot r^k}{r^{k-1}} = \frac{r^0 r^0}{r^{0-1}} = r \neq 1$

Hence, we cannot apply the inductive hypothesis to this form of algebra and the proof has a mistake.

#32

No, it won't follow that $P(n)$ is true for every integer $n \geq 0$.

If $P(k)$ is true, then $P(3k)$ is also true.

But, if $P(3)$ is true, then it doesn't necessarily mean that $P(4)$ is true, since

$3k = 4$

$k = \frac{4}{3}$ which is not an integer.

Hence, the inductive hypothesis is not satisfied.

Let's consider a counter example where

$P(n) = n^2 + n + 1$ is prime for every natural number n.

When $n = 1, 2, 3$, $P(n) = 3, 7, 13$ which are primes & satisfy the property.

Hence, $P(k)$ is satisfied

Suppose $P(k)$ is prime

$$
\begin{aligned}
P(3k) &= (3k)^2 + 3k + 1 \\
&= 9k^2 + 3k + 1 \\
&= 8k^2 + 2k + k^2 + k + 1 \\
&= 2k(4k+1) + (k^2 + k + 1)
\end{aligned}
$$

is prime by properties of prime.

But, when $n = 4$,

$P(4) = 21$ which is not prime

$\therefore P(n)$ doesn't work & satisfy mathematical induction.

5)

(5)

(a) By well-ordering principle, since $n \in \mathbb{Z}$ and $n \geq a$, then the set S has a least element.

(b) As $n \in S \Rightarrow n \geq a$, so $x \in S \Rightarrow x \geq a$

If $P(a)$ is True, then $a \notin S$

So, $x > a$ (as $x \in S$ and $a \notin S$ and $x \geq a$)

(c) Since $x \in S$, so $P(x)$ is F

As x is the smallest element in S, so $x-1 \notin S$, otherwise we would get $x-1$ smaller then x s.t. $P(x-1)$ is F, which contradicts that x is the smallest element in S.

As $x-1 \notin S$, so either $(x-1) \leq a$ or $P(x-1)$ is T.

Now $x-1 \leq a \Rightarrow x \leq a+1$

However, in (b) we got $x > a$

$\Rightarrow x \geq a+1$

So, either $x = a+1$ or $P(x-1)$ is T.

If we consider $x = a+1$, then $P(x-1)$ $= P(a)$, which is true, which is what we assumed.

Hence, we always we get $P(x-1)$ is T.

(d) There are 2 cases:

Case 1: If we assume that $P(a)$ is T, then as from (c), we have proved that $P(x-1)$ is always T.

Case 2: Let us assume $P(a)$ is not T i.e $P(a)$ is F

Proof by Contradiction

so, $a \geq a$ and $P(a)$ is F

$\Rightarrow x = a$

$\Rightarrow x-1 = a-1 \notin S$ (as $x=a$ is the smallest element in S)

$\Rightarrow$ either $P(x-1)$ is T

or $x-1 < a$

as $x-1 = a-1 < a$ is always T

there is no guarenta that $P(x-1)$ is T in our contradiction proof.

$\therefore P(a) \Rightarrow P(x-1)$

$\neg P(a)$?

6)
#26

#26

Proof: Let n be any integer greater than 1. Consider the set S of all +ve integers other than 1 that divide n. Since $n \mid n$ and $n > 1$, there is at least one element in S. Hence, by well-ordering principle for integers, S has a smallest element call it p.

We claim p is prime.

Let's assume p is not prime.

then, $p = ab$, $1 < a < p$
$1 < b < p$.

By definition of divisibility, $a \mid p$. Also, $p \mid n$ because p is in S and every element in S divides n. $\therefore$ $a \mid p$ and $p \mid n$ and so, by transitivity of divisibility $a \mid n$.

Consequently, $a \in S$. But this contradicts the fact that $a < p$, and p is the smallest element of S.

Hence, p is prime, and we have shown the existence of a prime number that divides n.

#27

#27

Every integer greater than 1 is either prime or a product of prime numbers.

By well-ordering principle for integers,
let S be the set of integers containing one or more integers greater than a fixed number. Then S has a least element, call it p.

here, $p > 1$

Let $n > 1$ where $n \in S$

Here, $n > p$

We claim p is prime.

Proof by contradiction: Suppose p is not prime.

$\therefore p = ab, \; a, b \in \mathbb{Z}, \; 1 < a < p$ & $1 < b < p$.

Since $a < p$ and $p < n$,
by transitivity
$a < n$
& $a > 1 \; \therefore \; a \in S$ by the definition of S.

Also, $a < p$, this contradicts the fact p is the smallest element of S.

This contradiction shows that the supposition that p is not prime is false.

$\therefore p$ is prime

Hence, every integer greater than 1 is either prime or a product of prime numbers.

7)
#6

a.) $A \subseteq B$

For every integer $x \in A$, there is an integer $k \in Z$ such that $k = 10b-3$ for some integer b,
where $x = 5a+2$ for some integer a.

Proof: Let x be any integer $\in A$ such that
$x = 5a+2$ for some integer a.
Let k be any integer $\in B$.
$k = 10b-3$ where we assume
b to be some integer.
For $x = k$
$5a+2 = 10b-3$ by substitution

$$b = \frac{5a+5}{10}$$ by algebra

$$b = \frac{a+1}{2}$$

$\frac{a+1}{2}$ cannot always have an
integral value, i.e.,
b is not an integer,
which is a contradiction
to the assumption.

$\therefore A \subseteq B$ is False.

b) $B \subseteq A$

For every integer $y \in B$ of the form $y = 10b-3$ for some integer b, there exists an integer $m \in A$ such that $m = 5a+2$ for some integer a and $y = m$.

Proof: Suppose $y \in B$ & $y \in \mathbb{Z}$ of the form $y = 10b - 3$ for some integer b.

Suppose $m \in A$ & $m \in \mathbb{Z}$.

Assume a is some integer such that $m = 5a+2$

To prove the statement,

$y = m$ has to be True.

$10b-3 = 5a+2$ by substitution

$a = \frac{10b-5}{5}$ by algebra

$a = \frac{5(2b-1)}{5}$ by factoring out 5

$a = 2b-1$

since $2, b, 1$ are integers & integers are closed under multiplication and subtraction.

a is an integer.

which supports our assumption.

Putting $a = 2b-1$ in $m = 5a+2$

$m = 5(2b-1)+2 = 10b-5+2 = 10b-3$

which is equal to y.

Hence, proved!

$\therefore B \subseteq A$.

(c) $B = C$

we need to prove that $B \subseteq C$ and $C \subseteq B$.

(1) $B \subseteq C$

For every integer $y \in B$ of the form $y = 10b - 3$ for some integer b, there exists an integer $m \in C$ such that $m = 10c + 7$ for some integer c and $y = m$.

Proof: Suppose ~~for~~ for any $y \in B$ & $y \in \mathbb{Z}$ of the form $y = 10b - 3$ for some integer b.

Assume there exists an integer $m \in C$ of the form $m = 10c + 7$ ~~as~~ and assume c is some integer.

To prove that m is an element of B also,

$m = y$

$10c + 7 = 10b - 3$ by substitution

$c = \dfrac{10b - 10}{10}$ by algebra

$c = \dfrac{10(b-1)}{10}$ by factoring out 10

$c = b - 1$

since $b, 1$ are integers & integers are closed under subtraction.

$\therefore$ c is an integer

Hence, our assumption about c was correct

$m = 10c + 7 = 10(b-1) + 7$
$= 10b - 10 + 7 = 10b - 3 = y$

$\therefore$ $m = y$, & our assumption was also correct about $m = 10c + 7$.

Hence, $B \subseteq C$.

② $C \subseteq B$

For every integer $z \in C$ of the form $z = 10c+7$ for some integer c, there exists an integer $m \in B$ such that $m = 10b-3$ for some integer b and $z = m$.

Proof: Suppose for any $z \in C$ & $z \in \mathbb{Z}$ of the form $z = 10c+7$ for some integer c

Assume there exists an integer $m \in B$ of the form $m = 10b-3$ and assume b is some integer.

To prove that m is an element of C also,

$m = z$

$10b-3 = 10c+7$ by substitution

$b = \frac{10c+10}{10}$ by algebra

$b = \frac{10(c+1)}{10}$ by factoring out 10

$b = c+1$

since $c, 1$ are integers & integers are closed under subtraction.

$\therefore b$ is an integer

Hence, our assumption about b was correct

$m = 10b-3 = 10(c+1)-3$
$= 10c+10-3 = 10c+7 = z$

$\therefore m = z$, & our assumption was also correct about $m = 10b-3$.

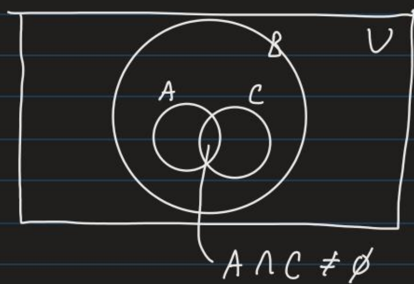
Hence, $C \subseteq B$.

In ① & ②, since $B \subseteq C$ & $C \subseteq B$.

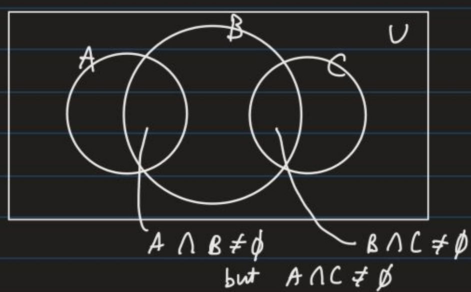
$B = C$.

#15(b),(c) and #18

15 (b)



15 (c)



$A \not\subseteq B$ &
$C \not\subseteq B$.

#18

a.) the number 0 is not in $\emptyset$, since $\emptyset$ represents a set of empty elements - False.

b) No, the left hand side is a set of empty elements & the RHS is a set containing one element False.

(c) Yes, $\emptyset \in \{\emptyset\}$, since

$\emptyset = \{\}$ & $\{\emptyset\} = \{\{\}\}$

Hence, $\emptyset$ or $\{\}$ is an element in $\{\emptyset\}$. True.

(d) No, since $\emptyset$ isn't in an empty set $\emptyset$. False.

8)
#25

#25

(a) $[1, 1+1] \cup [1, 1+\frac{1}{2}] \cup [1, 1+\frac{1}{3}] \cup [1, 1+\frac{1}{4}]$

$\therefore \quad [1, 2]$

(b) $[1, 1+1] \cap [1, 1+\frac{1}{2}] \cap [1, 1+\frac{1}{3}] \cap [1, 1+\frac{1}{4}]$

$\therefore \quad [1, \frac{5}{4}]$

(c) No, they all have some common part in each of them. One common part in all $[1, \frac{5}{4}]$. In fact, $R_{i+1} \subseteq R_i$

(d) $[1, 1+\frac{1}{1}] \cup [1, 1+\frac{1}{2}] \cup \dots \cup [1, 1+\frac{1}{n}]$

$\therefore [1, 2]$

(e) $[1, 1+\frac{1}{1}] \cap [1, 1+\frac{1}{2}] \cap \dots \cap [1, 1+\frac{1}{n}]$

$\therefore [1, 1+\frac{1}{n}]$

(f) Replacing $n = \infty$ for (d)

$\therefore \quad [1, 2]$

(g) Replacing $n = \infty$ for (e)

$\therefore [1, 1+\frac{1}{\infty}] = [1, 1+0] = [1, 1] = [1] = 1$

#27

#27

(a) No, d is in 2 of the sets

(b) Yes, since $\{w, x, v\} \cap \{u, y, q\} \cap \{p, z\} = \emptyset$

and $\{w, x, v\} \cup \{u, y, q\} \cup \{p, z\} = \{p, q, u, v, w, x, y, z\}$

(c) No, 4 is in 2 of the sets

(d) No, since there is no 6 in these sets.

(e) Yes, the intersection of all sets $= \emptyset$
& the union of all sets $= \{1, 2, 3, 4, 5, 6, 7, 8\}$

#32

(a) $P(A \times B) = \{\emptyset, \{(1,u)\}, \{(1,v)\}, \{(1,u), (1,v)\}\}$

(b) $P(X \times Y) = \{\emptyset, \{(a,x)\}, \{(a,y)\},$
$\{(b,x)\}, \{(b,y)\},$
$\{(a,x), (a,y)\}, \{(a,x), (b,x)\},$
$\{(a,x), (b,y)\}, \{(a,y), (b,x)\},$
$\{(a,y), (b,y)\}, \{(b,x), (b,y)\},$
$\{(a,x), (b,x), (a,y)\},$
$\{(a,x), (b,y), (a,y)\},$
$\{(b,x), (b,y), (a,y)\},$
$\{(b,x), (b,y), (a,x)\},$
$\{(a,x), (b,y), (a,y), (b,x)\}\}.$

9)

#32

#32

(a) $P(A\times B) = \{\emptyset, \{(1,u)\}, \{(1,v)\}, \{(1,u),(1,v)\}\}$

(b) $P(X\times Y) = \{\emptyset, \{(a,x)\}, \{(a,y)\}, \{(b,x)\}, \{(b,y)\}, \{(a,x),(a,y)\}, \{(a,x),(b,x)\}, \{(a,x),(b,y)\}, \{(a,y),(b,x)\}, \{(a,y),(b,y)\}, \{(b,x),(b,y)\}, \{(a,x),(b,x),(a,y)\}, \{(a,x),(b,y),(a,y)\}, \{(b,x),(b,y),(a,y)\}, \{(b,x),(b,y),(a,x)\}, \{(a,x),(b,y),(a,y),(b,x)\}\}.$

#33

(a) $P(\emptyset) = \{\emptyset\}$

(b) $P(P(\emptyset)) = P(\{\emptyset\}) = \{\emptyset, \{\emptyset\}\}$

(c) $P(P(P(\emptyset))) = P(\{\emptyset, \{\emptyset\}\}) = \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}$

10)

10)

$n=2$

$$\left[\sqrt{1+\tfrac{1}{2}},\ \sqrt{2-\tfrac{1}{2}}\right] = \left[\sqrt{\tfrac{3}{2}},\ \sqrt{\tfrac{3}{2}}\right]$$

$\cup$ $\qquad\qquad\qquad\qquad \cup$

$n=3$

$$\left[\sqrt{1+\tfrac{1}{3}},\ \sqrt{2-\tfrac{1}{3}}\right] = \left[\sqrt{\tfrac{4}{3}},\ \sqrt{\tfrac{5}{3}}\right]$$

$\cup$ $\qquad\qquad\qquad\qquad \cup$

$\vdots$ $\qquad\qquad\qquad\qquad \vdots$

$\cup$ $\qquad\qquad\qquad\qquad \cup$

$n=n$

$$\left[\sqrt{1+\tfrac{1}{n}},\ \sqrt{2-\tfrac{1}{n}}\right] \qquad \left[\sqrt{\tfrac{n+1}{n}},\ \sqrt{\tfrac{2n-1}{n}}\right]$$

$\sqrt{\tfrac{5}{4}},\ \sqrt{\tfrac{7}{4}} \ldots$

smaller $\qquad$ bigger

$\therefore$ Union upto $n=n$, $\left[\sqrt{\tfrac{3}{2}},\ \sqrt{\tfrac{2n-1}{n}}\right]$

$\sqrt{\tfrac{3}{2}} > \sqrt{\tfrac{4}{3}} > \sqrt{\tfrac{5}{4}} \ldots$ approaching to 1

and $\sqrt{\tfrac{3}{2}} < \sqrt{\tfrac{5}{3}} < \sqrt{\tfrac{7}{4}} \ldots$ approaching to 2

as $n \to \infty$ & $\frac{1}{n} \to 0$

union will approach 1 at left and 2 at right, but won't be equal to 1 or 2.

$\therefore$ Union $= (1, 2]$