CS 2810 Assignment 4

Shreyas Shukla

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Question 1

Given basis vectors $a = \begin{bmatrix} 7 \\ 1 \end{bmatrix}$ and $b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, and target vector $y = \begin{bmatrix} 1 \\ 8 \end{bmatrix}$:

1. **Identify if y is in the span of a and b:**

   To check if y is in the span, solve the equation:

   $$\begin{bmatrix} 7 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 1 \\ 8 \end{bmatrix}$$

   Solve the system:

   $$7\alpha = 1 \implies \alpha = \frac{1}{7}$$

   Substituting α in the second equation:

   $$\frac{1}{7} + \beta = 8 \implies \beta = \frac{55}{7}$$

   Therefore, y is in the span of a and b.

2. **Normalized version of a and b:**

   $$\text{Normalized } a = \frac{1}{\sqrt{7^2 + 1^2}} \begin{bmatrix} 7 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{50}} \begin{bmatrix} 7 \\ 1 \end{bmatrix}$$

   $$\text{Normalized } b = \frac{1}{\sqrt{0^2 + 1^2}} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

3. **Matrix A:**

   $$A = \begin{bmatrix} 7 & 0 \\ 1 & 1 \end{bmatrix}$$

4. **Matrix B:**

   $$B = \begin{bmatrix} 7 & 1 \\ 0 & 1 \end{bmatrix}$$

5. **Null space of A and B:**

   For A, solve $A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. The null space is $N(A) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ (trivial solution).

   For B, solve $B \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. The null space is $N(B) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ (trivial solution).

6. **Independence of A's columns:**

   The columns of A are independent since the determinant of A is non-zero:

   $$\det(A) = 7 \cdot 1 - 0 \cdot 1 = 7 \neq 0$$

7. **Independence of B's columns:**

   The columns of B are also independent since the determinant of B is non-zero:

   $$\det(B) = 7 \cdot 1 - 0 \cdot 1 = 7 \neq 0$$

8. **Dimension of the null space:**

   Both A and B have a null space of dimension 0.

9. **Rank of A and B:**

   The rank of both A and B is 2 (since both matrices have 2 independent columns).

Question 2

Given $v_1 = \begin{bmatrix} 0 \\ 8 \\ 0 \\ 3 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 0 \\ 8 \\ 1 \\ 0 \end{bmatrix}$, with target $y = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 2 \end{bmatrix}$:

1. **Identify if y is in the span of v_1 and v_2:**

   Solve the system:

   $$\begin{bmatrix} 0 & 0 \\ 8 & 8 \\ 0 & 1 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 2 \end{bmatrix}$$

   This system has no solution, hence y is not in the span.

2. **Normalized version of v_1 and v_2:**

   $$\text{Normalized } v_1 = \frac{1}{\sqrt{8^2 + 3^2}} \begin{bmatrix} 0 \\ 8 \\ 0 \\ 3 \end{bmatrix} = \frac{1}{\sqrt{73}} \begin{bmatrix} 0 \\ 8 \\ 0 \\ 3 \end{bmatrix}$$

$$\text{Normalized } v_2 = \frac{1}{\sqrt{8^2 + 1^2}} \begin{bmatrix} 0 \\ 8 \\ 1 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{65}} \begin{bmatrix} 0 \\ 8 \\ 1 \\ 0 \end{bmatrix}$$

3. **Matrix A:**

$$A = \begin{bmatrix} 0 & 0 \\ 8 & 8 \\ 0 & 1 \\ 3 & 0 \end{bmatrix}$$

4. **Matrix B:**

$$B = \begin{bmatrix} 0 & 8 \\ 8 & 1 \\ 0 & 0 \\ 3 & 0 \end{bmatrix}$$

5. **Null space of A and B:**

   The null space of A is non-trivial, so it has dimension greater than 0.

   The null space of B is also non-trivial.

6. **Rank of A and B:**

   The rank of both A and B is 2.

Question 3

Given $v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $v_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, and $v_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, with target $y = \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix}$:

1. **Identify if y is in the span of v_1, v_2, and v_3:**

   Solve the system:

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix}$$

   After solving, y is not in the span.

2. **Normalized version of v_1, v_2, and v_3:**

$$\text{Normalized } v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \text{Normalized } v_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \text{Normalized } v_3 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

3. **Matrix A:**

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

4. **Matrix B:**

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

5. **Null space of A and B:**

   The null space of both A and B is trivial.

6. **Rank of A and B:**

   The rank of A is 3, and the rank of B is 2.

Question 10pts:

Study the relationship between:

1. The number of basis vectors.
2. The dimension of the null space.
3. The rank.

The relationship between these concepts is captured by the rank-nullity theorem, which states that for a matrix A:

$$\text{rank}(A) + \text{nullity}(A) = n$$

where n is the number of columns of A.

Question 10pts:

Given vectors $v_1, v_2, v_3, \ldots$:

- **Definition of span:**

  The span of a set of vectors is the set of all linear combinations of those vectors.

- **Meaning of span:**

  Span represents the space that can be "reached" or generated by combining the vectors. In my life, the span could describe all the tasks I could accomplish if I combined all my different skills.

Question 10pts:

- **Definition of linear independence:**

  A set of vectors is linearly independent if none of the vectors can be written as a linear combination of the others.

- **Meaning of linear independence:**

  Linear independence means that each vector adds a new direction or dimension. In my life, this could describe how each of my friends brings a unique perspective, with no one repeating the role of the other.