ML EC3

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October 4, 2024

Question 1

Given basis vectors $a = \begin{bmatrix} 7 \\ 1 \end{bmatrix}$ and $b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, and target vector $y = \begin{bmatrix} 1 \\ 8 \end{bmatrix}$:

$$\begin{bmatrix} 7 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 1 \\ 8 \end{bmatrix}$$

$$7\alpha = 1 \implies \alpha = \frac{1}{7}$$

$$\frac{1}{7} + \beta = 8 \implies \beta = \frac{55}{7}$$

Therefore, y is in the span of a and b.

$$\|a\| = \sqrt{7^2 + 1^2} = \sqrt{50}$$

$$\text{Normalized } a = \frac{1}{\sqrt{50}} \begin{bmatrix} 7 \\ 1 \end{bmatrix}$$

$$\|b\| = \sqrt{0^2 + 1^2} = 1$$

$$\text{Normalized } b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 7 & 0 \\ 1 & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} 7 & 1 \\ 0 & 1 \end{bmatrix}$$

$$A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies N(A) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$B \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies N(B) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\det(A) = 7 \times 1 - 0 \times 1 = 7 \neq 0$$

$$\det(B) = 7 \times 1 - 0 \times 1 = 7 \neq 0$$

Both A and B have a null space of dimension 0.
The rank of both A and B is 2.

Question 2

Given $v_1 = \begin{bmatrix} 0 \\ 8 \\ 0 \\ 3 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 0 \\ 8 \\ 1 \\ 0 \end{bmatrix}$, with target $y = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 2 \end{bmatrix}$:

$$\begin{bmatrix} 0 & 0 \\ 8 & 8 \\ 0 & 1 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 2 \end{bmatrix}$$

The system has no solution, so y is not in the span.

$$\|v_1\| = \sqrt{8^2 + 3^2} = \sqrt{73}$$

$$\text{Normalized } v_1 = \frac{1}{\sqrt{73}} \begin{bmatrix} 0 \\ 8 \\ 0 \\ 3 \end{bmatrix}$$

$$\|v_2\| = \sqrt{8^2 + 1^2} = \sqrt{65}$$

$$\text{Normalized } v_2 = \frac{1}{\sqrt{65}} \begin{bmatrix} 0 \\ 8 \\ 1 \\ 0 \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & 0 \\ 8 & 8 \\ 0 & 1 \\ 3 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 8 \\ 8 & 1 \\ 0 & 0 \\ 3 & 0 \end{bmatrix}$$

The null space of A and B are non-trivial.
The rank of both A and B is 2.

Question 3

Given $v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $v_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, and $v_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, with target $y = \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix}$:

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix}$$

The system has no solution, so y is not in the span.

$$\text{Normalized } v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\|v_2\| = \sqrt{1^2 + 1^2} = \sqrt{2}$$

$$\text{Normalized } v_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$\|v_3\| = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$$

$$\text{Normalized } v_3 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

The null space of both A and B is trivial.
The rank of A is 3, and the rank of B is 2.

Relationship between Basis, Null Space, and Rank (10 pts)

The relationship between these concepts is captured by the rank-nullity theorem:

$$\text{rank}(A) + \text{nullity}(A) = n$$

where n is the number of columns of A.

Definition of Span and its Meaning (10 pts)

The span of a set of vectors is the set of all linear combinations of those vectors. It represents the space that can be "reached" or generated by combining the vectors.

Definition of Linear Independence and its Meaning (10 pts)

A set of vectors is linearly independent if none of the vectors can be written as a linear combination of the others. Linear independence means each vector adds a new direction or dimension.