PAPER SOLUTION
WINTER 2023

Subject code : 3110014 **Subject Name : Mathematics - I**

Q-1 (a) Evaluate $\lim_{x\to 0^+} xlnx$

$$l = \lim_{x\to 0} xlnx \qquad [\,0\times\infty\ from\,]$$

$$= \lim_{x\to 0}\frac{lnx}{\frac{1}{x}}$$

$$= \lim_{x\to 0}\frac{\frac{1}{x}}{\frac{-1}{x^2}} \qquad [Applying\ L'sHospital\ rule]$$

$$= \lim_{x\to 0}(-x)$$

$$= 0$$

(b) Define beta and gamma functions. What is the relationship between beta and gamma functions?

→ Gamma Function is defined by the improper integral $\int_0^\infty e^{-x}.x^{n-1}\,.dx \quad n>0$ and is denoted by Γn
Γn = $\int_0^\infty e^{-x}.x^{n-1}\,.dx$

→ Beta function $\beta(m,n)$ is defined by
B(m,n) = $\int_0^1 x^{m-1}\,(1-x)^{n-1}\,.dx$ m>0 , n>0

→ The relationship between beta and gamma function can be mathematically expresses as

$$\beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m\ +\ n)}$$

→ Where $\beta(m,n)$ is the beta function with two variables $m\ and\ n$.
Γm is the gamma function with variable m
Γn is the gamma function with variable n

(c) Solve the following System of linear equation using Gauss – Jordan Elimination :

$$\boldsymbol{x_3 + x_4 + x_5 = 0}$$

$$\boldsymbol{-x_1 - x_2 + 2x_3 - 3x_4 + x_5 = 0}$$

$$\boldsymbol{x_1 + x_2 - 2x_3 - x_5 = 0}$$

$$\boldsymbol{2x_1 + 2x_2 - x_3 + x_5 = 0}$$

Ans: The matrix from of the system is

$$\begin{bmatrix} 0 & 0 & 1 & 1 & 1 \\ -1 & -1 & 2 & -3 & 1 \\ 1 & 1 & -2 & 0 & -1 \\ 2 & 2 & -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

→ The augmented matrix of the system is

$$\left[\begin{array}{ccccc|c} 0 & 0 & 1 & 1 & 1 & 0 \\ -1 & -1 & 2 & -3 & 1 & 0 \\ 1 & 1 & -2 & 0 & -1 & 0 \\ 2 & 2 & -1 & 0 & 1 & 0 \end{array}\right]$$

$\rightarrow$ Reducing the augmented matrix to reduced row echelon from

$R_1 \leftrightarrow R_3$

$$= \left[\begin{array}{ccccc|c} 1 & 1 & -2 & 0 & -1 & 0 \\ -1 & -1 & 2 & -3 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 2 & 2 & -1 & 0 & 1 & 0 \end{array}\right]$$

$R_2 \rightarrow R_2 + R_1$

$$= \left[\begin{array}{ccccc|c} 1 & 1 & -2 & 0 & -1 & 0 \\ 0 & 0 & 0 & -3 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 2 & 2 & -1 & 0 & 1 & 0 \end{array}\right]$$

$R_2 \leftrightarrow R_4$

$$= \left[\begin{array}{ccccc|c} 1 & 1 & -2 & 0 & -1 & 0 \\ 2 & 2 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & -3 & 0 & 0 \end{array}\right]$$

$R_2 \rightarrow \frac{1}{2} R_2$

$$= \left[\begin{array}{ccccc|c} 1 & 1 & -2 & 0 & -1 & 0 \\ 1 & 1 & -\frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & -3 & 0 & 0 \end{array}\right]$$

$R_2 \rightarrow R_2 - R_1$

$$= \left[\begin{array}{ccccc|c} 1 & 1 & -2 & 0 & -1 & 0 \\ 0 & 0 & \frac{3}{2} & 0 & \frac{3}{2} & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & -3 & 0 & 0 \end{array}\right]$$

$\rightarrow$ The Corresponding System of equation is

$x_1 + x_2 - x_3 - x_5 = 0$ -------------------------- 1

$\frac{3}{2} x_3 + \frac{3}{2} x_5 = 0$ ---------------------------------- 2

$x_3 + x_4 + x_5 = 0$ ------------------------------- 3

$-3x_4 = 0$

$\therefore x_4 = 0$

The variable x_5 Let's take t_1 and $x_2 \longrightarrow t_2$

From ③

$$x_3 + 0 + t_1 = 0$$

$$x_3 = -t_1$$

From ①

$$x_1 + t_2 + t_1 - t_1 = 0$$

$$x_4 = -t_2$$

→ Hence $x_1 = -t_2, x_2 = t_2, x_3 = -t_1, x_4 = 0$ and $x_5 = t_1$ is the solution of the system where t_1 and t_2 are the parameters.

Q 2 (a) Define rank of matrix. Find rank(A) if $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

Ans : The rank of matrix in echelon form is equal to the number of non-zero rows of the matrix

Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

→ Reducing the matrix A to echelon from

$R_2 \rightarrow R_2 - 2R_1$

$= \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$

→ The equivalent matrix is in echelon from

Number of the non zero rows = 1

$\rho(A) = 1$

(b) Test the Convergence of the series $\sum_{n=1}^{\infty} \left(\frac{1}{1+n}\right)^n$

Ans: Let $u_n = \left(\frac{1}{1+n}\right)^n$

$$(u_n)^{\frac{1}{n}} = \frac{1}{1+n}$$

$$= \frac{1}{n\left(1+\frac{1}{n}\right)}$$

$$\lim_{n\to\infty} (u_n)^{\frac{1}{n}} = \lim_{n\to\infty} \frac{1}{n\left(1+\frac{1}{n}\right)}$$

$$= 0 < 1$$

→ Hence, by cauchy's root test, the series is Convergent.

(C) Find eigenvalues and eigenvectors of the matrix $A = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix}$

Ans : Let A $= \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix}$

→ The characteristic equation is ,

$$\det(A - \lambda I) = 0$$

$$\begin{vmatrix} -5-\lambda & 2 \\ 2 & -2-\lambda \end{vmatrix} = 0$$

$\lambda^2 - S_1\lambda + S_2 = 0$

Where $S_1 = -5 - 2 = -7$

$S_2 = \det(A) = 10 - 4 = 6$

$\rightarrow$ Hence, the characteristic equation is

$\lambda^2 - 7\lambda + 6 = 0$

$\lambda = 6,1$

(a) For $\lambda = 6$,

$[A - \lambda I]x = 0$

$$\therefore \begin{bmatrix} -11 & 2 \\ 2 & 8 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\therefore\ -11x + 2y = 0$

Let $y = t$

$x = \frac{2}{11}t$

$$X = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{2}{11}t \\ t \end{bmatrix} = t\begin{bmatrix} \frac{2}{11} \\ 1 \end{bmatrix} = t \times 1$$

Where $x_1$is an eigenvector corresponding to $\lambda = 6$

(b) For $\lambda = 1$

$[A - \lambda I]x = 0$

$$\therefore \begin{bmatrix} 6 & 2 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\therefore\ -6x + 2y = 0$

Let $y = t$

$x = \frac{1}{3}t$

$$X = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{1}{3}t \\ t \end{bmatrix} = t\begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix} = t \times 2$$

Where x_2 is an eigenvector corresponding to $\lambda = 1$

(c) Find the fourier series of the function $f(x) = x + \pi\ if\ -\pi < x < \pi\ and\ f(x + 2\pi) = f(x)$

Ans : The fourier series of $f(x)$with period $2\pi\ is$ given by

$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$

$a_0 = \frac{1}{2\pi}\int_{\pi}^{-\pi} f(x).dx$

$= \frac{1}{2\pi}\int_{\pi}^{-\pi}(x + \pi).dx$

$= \frac{1}{2\pi}\left[\frac{x^2}{2} + \pi x\right]_{-\pi}^{\pi}$

$= \frac{1}{2\pi}\left[\frac{\pi^2}{2} + \pi^2 - \frac{\pi^2}{2} - \pi^2\right]$

$= 0$

$a_n = \frac{1}{\pi}\int_{\pi}^{-\pi} f(x)\cos nx\, dx$

$= \frac{1}{\pi}\left[\int_{\pi}^{-\pi}(x+\pi)\cos nx\, dx\right]$

$= \frac{1}{\pi}\left[(x+\pi)\left(\frac{\sin nx}{n}\right) - (1)\left(-\frac{\cos nx}{n^2}\right)\right]_{-\pi}^{\pi}$

$= \frac{1}{\pi}\left[\left(\frac{\cos nx}{n^2}\right) - \left(\frac{\cos(-n\pi)}{n^2}\right)\right]$

$= \frac{1}{\pi}\left[\frac{(-1)^n}{n^2} - \frac{(-1)^n}{n^2}\right]$

$= 0$

$b_n = \frac{1}{\pi}\int_{\pi}^{-\pi} f(x)\sin nx\, dx$

$= \frac{1}{\pi}\left[\int_{\pi}^{-\pi}(x+\pi)\sin nx\, dx\right]$

$= \frac{1}{\pi}\left[(x+\pi)\left(\frac{-\cos nx}{n}\right) - (1)\left(-\frac{\sin nx}{n^2}\right)\right]_{-\pi}^{\pi}$

$= \frac{1}{\pi}\left[2\pi\left(\frac{\cos(-n\pi)}{n}\right) - 0\right]$

$= \frac{-2}{n}(-1)^n$

$\rightarrow$ Hence, $f(x) = \sum_{n=1}^{\infty}\frac{-2}{n}(-1)^n \sin nx$

$f(x+2\pi) = \sum_{n=1}^{\infty}\frac{-2}{n}(-1)^n \sin n(x+2\pi)$

$= \sum_{n=1}^{\infty}\frac{-2}{n}(-1)^n \sin nx$

So, $f(x+2\pi) = f(x)$

Q 3 (a) If $w = x^2 + y^2, x = r - s, y = r + s$, using chain rule , prove that $\frac{\partial w}{\partial s} = 4s$.

Ans : $w = x^2 + y^2, x = r - s, y = r + s$

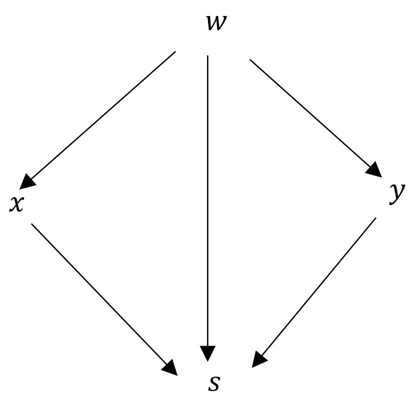
$\rightarrow \frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}.\frac{\partial y}{\partial s}$ -------------------------- 1

$\rightarrow \frac{\partial w}{\partial x} = \frac{\partial}{\partial x}(x^2 + y^2) = 2x$

$\frac{\partial w}{\partial y} = \frac{\partial}{\partial x}(x^2 + y^2) = 2y$

$\frac{\partial x}{\partial s} = \frac{\partial}{\partial s}(r - s) = -1$

$\frac{\partial y}{\partial s} = \frac{\partial}{\partial s}(r + s) = 1$



→ From ①

$$\frac{\partial w}{\partial s} = (2x)(-1) + (2y)(1)$$
$$= -2x + 2y$$

→ Substituting x_1 and y

$$\frac{\partial w}{\partial s} = -2(r-s) + 2(r+s)$$
$$= -2r + 2s + 2r + 2s$$
$$= 4s$$

(b) Find the directional derivative of the $f(x,y,z) = 2x^2 + 3y^2 + z^2$ at point (2,1,3) in the direction of the vector $i - 2k$.

Ans : $\nabla f = \hat{\imath}\frac{\partial f}{\partial x} + \hat{\jmath}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z}$

$$= \hat{\imath}(4x) + \hat{\jmath}(6y) + \hat{k}(2z)$$

→ AT the point (2,1,3)

$$\nabla f = 8\hat{\imath} + 6\hat{\jmath} + 6\hat{k}$$

→ Directional derivative in the direction of the vector

$$\vec{a} = \hat{\imath} + 0\hat{\jmath} - 2\hat{k}$$

→ $D_a t = \nabla f \cdot \frac{\vec{a}}{|\vec{a}|}$

$$= (8\hat{\imath} + 6\hat{\jmath} + 6\hat{k}) + \frac{(\hat{\imath}+\hat{\jmath}-2\hat{k})}{\sqrt{1+0+4}}$$
$$= \frac{8-12}{\sqrt{5}}$$
$$= -\frac{4}{\sqrt{5}}$$

(c) Find local extreme values of the function $f(x,y) = 4x^2 + 9y^2 + 8x - 36y + 24$

Ans: Let $f(x,y) = 4x^2 + 9y^2 + 8x - 36y + 24$

→ Step I for extreme values,

$$\frac{\partial f}{\partial x} = 0$$
$$8x + 8 = 0$$
$$x = 1$$

And $\frac{\partial f}{\partial y} = 0$

$$18y - 36 = 0$$
$$y = 2, x = -1$$

→ Stationary Point is $(-1,2)$

→ Step II

$$r = \frac{\partial^2 f}{\partial x^2} = 8$$
$$s = \frac{\partial^2 f}{\partial x \partial y} = 0$$
$$r = \frac{\partial^2 f}{\partial y^2} = 18$$

→ Step III

(x, y)	r	s	t	$rt - s^2$		Conclusion
$(-1,2)$	8	0	18	144>0	r>0	Minimum

→ Hence $f(x, y)$ is minimum at $(-1,2)$

$t_{min} = 4(-1)^2 + 9(2)^2 + 8(-1) - 36(2) + 24$

$= 4 + 36 - 8 - 72 + 24$

$= -16$

$Q3$(a) Determine weather $\lim_{(x,y)\to(0,0)} \frac{2x^2y}{x^3y^3}$ *exists and* find it if exists.

Ans : $\lim_{x\to 0}\left[\lim_{y\to 0} f(x, y)\right] = \lim_{x\to 0}\left[\lim_{y\to 0} \frac{2x^2y}{x^3y^3}\right]$

$= \lim_{x\to 0}(0) = 0$

$\lim_{y\to 0}\left[\lim_{x\to 0} f(x, y)\right] = \lim_{y\to 0}\left[\lim_{x\to 0} \frac{2x^2y}{x^3y^3}\right]$

$= \lim_{y\to 0}(0) = 0$

→ Putting $y = mx$ and taking limit $x \to 0$.

$\lim_{x\to 0} \frac{2x^2(mx)}{x^3 + (mx)^3} = \lim_{x\to 0} \frac{2mx^3}{x^3(1 + m^3)}$

$= \frac{2m}{1+m^3}$

→ Since, the last limit depends on m *and* n is not the limit does not exist Hence, $f(x)$ is discontinuous
origin.

(b) Find the equation of tangent pale to $z = 3x^2 - xy$ at the point $(1, 2, 1)$.

Ans : Let $f(x, y, z) = 3x^2 - xy - z$

$f_x(x, y, z) = 6x - y$ $\quad$ $f_x(1,2,1) = 4$

$f_y(x, y, z) = -x$ $\quad$ $f_x(1,2,1) = -1$

$f_z(x, y, z) = 6x - y$ $\quad$ $f_x(1,2,1) = 4$

→ Hence, the equation of the tangent plane at (1,2,1) is

$\therefore 4(x - 1) - 1(y - 2) - 1(z - 1) = 0$

$\therefore 4x - 4 - y + 2 - z + 1 = 0$

$\therefore 4x - y - z - 1 = 0$

$\therefore 4x - y - z = 1$

(c) Find the maximum and minimum value of the function $F(x, y) = 3x + 4y$ on the circle $x^2 + y^2 = 1$

Ans : To find stationary value of $u = f(x, y) = 3x + 4y$ ----------------------- 1

Subject to the condition $\phi\ (x, y) = x^2 + y^2 - 1 = 0$ --------------------- 2

→ Lagrangel's Function

$$F(x,y) = f(x,y) + \lambda\, \phi(x,y) = 0$$

$$= (3x + 4y) + \lambda\,(x^2 + y^2 - 1) = 0 \text{ ------------------------------- } ③$$

→ Differentiating Eq.(3) partially w.r.t x

$$3 + \lambda(2x) = 0$$

$$\lambda = -\frac{3}{2x} \text{ -- } ④$$

→ Differentiating Eq.(3) partially w.r.t x

$$4 + \lambda(2y) = 0$$

$$\lambda = -\frac{4}{2y} \text{ -- } ⑤$$

→ From ⑤ & ④

$$\therefore\ -\frac{3}{2x} = -\frac{2}{y}$$

$$\because\ y = \frac{4}{3}x$$

→ From Equation ②

$$x^2 + y^2 = 1$$

$$x^2 + \frac{16}{9}x^2 = 1$$

$$\because\ \frac{25x^2}{9} = 1$$

$$\because\ x = +\frac{3}{5}$$

$$\because\ y = \frac{+4}{5}$$

→ So , $+\left(\frac{3}{5},\frac{4}{5}\right) = 3\left(\frac{3}{5}\right) + 4\left(\frac{4}{5}\right) = \frac{9}{5} + \frac{16}{5} = 5 \longrightarrow Maxima$

→ $f\left(-\frac{3}{5},-\frac{4}{5}\right) = 3\left(-\frac{3}{5}\right) + 4\left(-\frac{4}{5}\right) = -5 \rightarrow Minima$

Q 4: (a) Calculate $\iint f(x,y)\ .dA\ for\ f(x,y) = 100 - 6x^2y\ and\ R{:}\ 0 \le x \le 2, -1 \le y \le 1$

Ans : $\iint f(x,y).\,dA = \int_{-1}^{1}\int_{0}^{2}(100 - 6x^2y)\,dx.\,dy$

$$= \int_{-1}^{1}[100x - 2x^3y]_0^2.\,dy$$

$$= \int_{-1}^{1}[200 - 16y].\,dy$$

$$= [200y - 8y\text{^}2]_{-1}^{1}$$

$$= 400$$

(b) Sketch the Region of the integral $\int_0^2\int_{x^2}^{2x}(4x+2)\,dy.\,dx$ and write an equivalent integral with the order of integration reversed.

Ans : I= $\int_0^4\int_{\frac{y}{2}}^{\sqrt{y}}(4x+2)\,dx.\,dy$

$= \int_0^4 \left[\frac{4x^2}{2} + 2x\right]_{\frac{y}{2}}^{\sqrt{y}} .dy$

$= \int_0^4 \left[2y + 2\sqrt{y} - \frac{y^2}{4} - y\right].dy$

$= \int_0^4 \left(y + 2\sqrt{y} - \frac{y^2}{4}\right).dy$

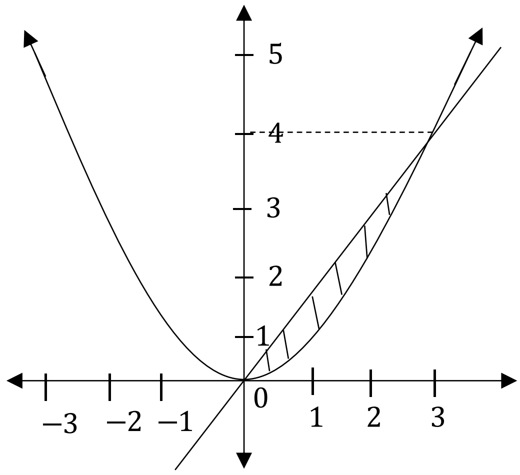
$= \left[\frac{y^2}{2} + \frac{2y^{\frac{3}{2}}}{\frac{3}{2}} - \frac{y^3}{12}\right]_0^4$

$= \left[\frac{16}{2} + \frac{4}{3\left(\frac{3}{2}\right)} - \frac{(4)^3}{12}\right]$

$= 8 + \frac{32}{3} - \frac{16}{3}$

$= 8 + \frac{16}{3}$

$x = \frac{40}{3}$



(c) Calculate $\iint \frac{\sin x}{x} . dA$ where R is the triangle in the xy plane bounded by the xaxis the line y=x and the line x=1

Ans : I $= \int_0^1 \int_0^x \frac{\sin x}{x}\, dy.dx$

$= \int_0^1 \frac{\sin x}{x} [y]_0^x .dx$

$= \int_0^1 \sin x\, dx$

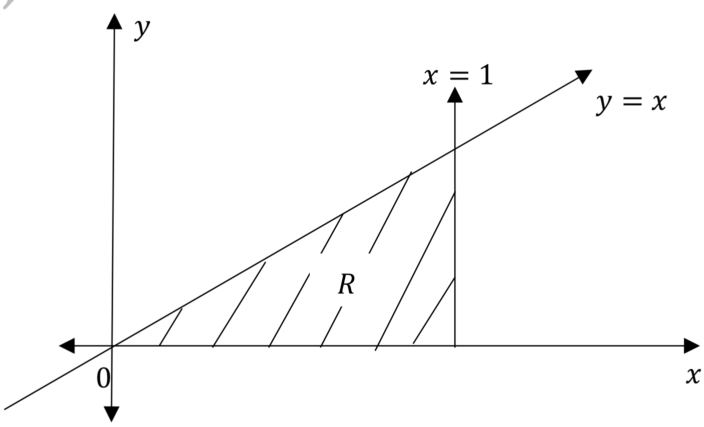
$= \left| -\cos x \right|_0^1$

$= -\cos 1 - (-\cos 0)$

$= -\cos 1 + 1$

$= 1 - \cos 1$

$\cong 0.46$



Q 4 : (a) Evaluate $\int_1^{\ln 8} \int_0^{\ln y} e^{x+y}\, dx.dy$

Ans : I $= \int_1^{\ln 8} \int_0^{\ln y} e^{x+y}\, dx.dy$

$= \int_1^{\ln 8} \int_0^{\ln y} e^x . e^y\, dx.dy$

$= \int_1^{\ln 8} e^y \left(\int_0^{\ln y} e^x . dx\right).dy$

$= \int_1^{\ln 8} e^y\, [e^x]_0^{\ln y} .dy$

$= \int_1^{\ln 8} e^y \left[e^{\ln y} - e^0\right].dy$

$= \int_1^{\ln 8} e^y\,[y-1].\,dy$

$= \int (y-1)e^y.\,dy$ --------------------- ①

$u = y-1 \qquad\qquad dv = e^y dy$

$du = dy \qquad\qquad v = e^y$

$\int (y-1)e^y\,dy = uv - \int v.\,du$

$\int (y-1)e^y dy = (y-1)e^y - \int e^y.\,dy$

$= (y-1)e^y - e^y$

$= ye^y - 2e^y$

→ From ①

$= \int_1^{\ln 8}(y-1)e^y\,dy$

$= [ye^y - 2e^y]_1^{\ln 8}$

$= [ye^y]_1^{\ln 8} - 2[e^y]_1^{\ln 8}$

$= \left[(\ln 8)e^{\ln 8} - 1e^1\right] - 2\,[e^{\ln 8} - e^1]$

$= (\ln 8)\,(8) - e - 2(8-e)$

$= 8ln8 - e - 16 + 2e$

$= 8ln8 - 16 + e$

(b) Find the area of the origin R bounded by the $y = x$ and $y = x^2$ in the first quadrant

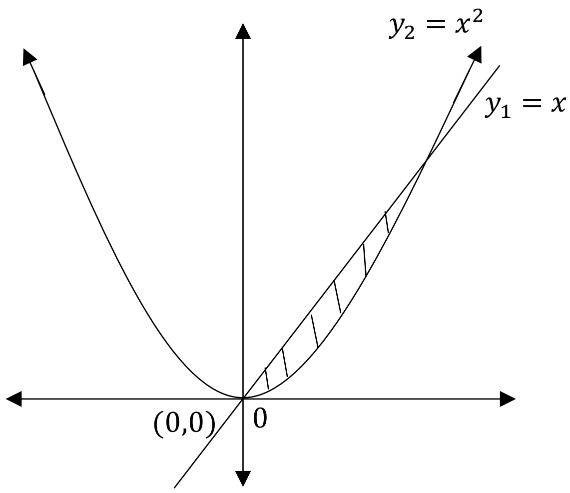
Ans : $y = x\ and\ y = x^2$

$x = x^2$

$x = 0,1$

The value $x = 0$ correspond to $0(0,0)$

For $x = 1, y = 1$



$A = \int_0^1 (y_1 - y_0)dx$

$= \int_0^1 (x - x^2)dx$

$= \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1$

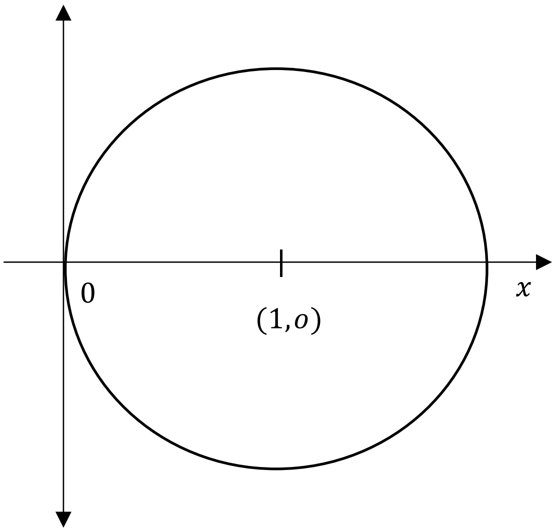
$= \frac{1}{2} - \frac{1}{6}$

$A = \frac{1}{6}$

(c) Evaluate $\int_0^1\int_0^{\sqrt{1-x^2}}(x^2+y^2)\,dydx\ \ by\ changing\ into\ polar\ coordinates$

Ans :

→ Limits of $y : y = 0\ to\ y = \sqrt{1-x^2}$



→ Limits of $x : x = 0\ to\ x = 1$

$y = 0$ and the circle $x^2 + y^2 = 1$

→ Putting $x = r\cos\theta\,, y = r\sin\theta$
$(i)\ y = 0$
$(ii)\ x^2 + y^2 = 0$

$r^2\cos^2\theta + r^2\sin^2\theta = 0$
$\therefore r = 0$

→ Therefore, in polar coordinates integral will be

$$\int_0^{\frac{\pi}{2}}\int_0^1 (r^2)\,r\,dr.\,d\theta$$

$$= \int_0^{\frac{\pi}{2}}\int_0^1 r^3\,dr.\,d\theta$$

$$= \int_0^{\frac{\pi}{2}}\left[\frac{r^4}{4}\right]_0^1 d\theta$$

$$= \int_0^{\frac{\pi}{2}}\left[\frac{1}{4}\right].\,d\theta$$

$$= \frac{1}{4}[\theta]_0^{\frac{\pi}{2}}$$

$$= \frac{\pi}{8}$$

Q 5 (a) Find the Maclaurin Series for the Cos x

Ans : Let $y = cos\ x$

$y(0) = \cos 0 = 1$

→ Now , $y_n = \frac{d^n}{dx^n}(cosx) = \cos\left(x + \frac{n\pi}{2}\right)$ $\qquad y_n(0) = \cos\left(\frac{n\pi}{2}\right)$

→ Putting n=1,2,3,4,…..

$y_1(0) = 0$

$y_2(0) = -1$

$y_3(0) = 0$

$y_4(0) = 1$

→ Substituting in Maclaurin's Series

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} \dots \dots \dots$$

(b) Determine the convergence or divergence of the series $\sum_{n=1}^{\infty} ne^{-n^2}$

Ans: Let $u_n = ne^{-n^2} = f(n)$

$f(x) = x.e^{-x^2}$

$\int_1^{\infty} f(x).dx \quad = \quad \int_1^{\infty} x > e^{-x^2}.dx$

$$= \lim_{m\to\infty}[-\frac{1}{2}\int_1^m e^{-x^2}(-2x).dx]$$

$$= \lim_{m\to\infty}\left[-\frac{1}{2}\left|e^{-x^2}\right|_1^m\right] \qquad [\because e^{f(x)}f'(x).dx = e^{f(x)}]$$

$$= \lim_{m\to\infty}[-\frac{1}{2}(e^{-m^2}-e^{-1})]$$

$$= -\frac{1}{2}(e^{-\infty}-e^{-1})$$

$$= -\frac{1}{2}\left(0+\frac{1}{e}\right)$$

$$= \frac{1}{2e}$$

→ Hence, by Cauchy's integral test ,the series is Convergent .

(c) Test the Convergence of the series $\sum_{n-1}^{\infty}\frac{(-1)^{n-1}}{n^2}$

Ans : Let $u_n = (-1)^{n-1}.\frac{1}{n^2}$

$|u_n| = \frac{1}{n^2}$

→ The given series is an alternating series

$(i)\ |u_n| - |u_{n+1}| = \frac{1}{n^2} - \frac{1}{(n+1)^2}$

$= \frac{2n+1}{n^2(n+1)^2} > 0\ for\ all\ n \in N$

$\therefore |u_n| > |u_{n+1}|$

$(ii)\ \lim_{n\to\infty}|u_n| = \lim_{n\to\infty}\frac{1}{n^2}$

$= 0$

→ Hence, by Leibnitz's test, the series is convergent

Q 5(a) Define monotonic sequence Is the Sequence $\left\{\frac{1}{n^2}\right\}$ monotonic ?

Ans : A sequence is said to be monotonically increasing if $u_{n+1} \geq u_n$ for each value of n and is monotonically decreasing if $u_{n+1} \leq u_n$ for each value of the sequence is called alternating sequence if the terms are alternate positive and negative.

$u_n = \frac{1}{n^2}$

$u_{n+1} = \frac{1}{(n+1)^2}$

→ $u_{n+1} - u_n = \frac{1}{(n+1)^2} - \frac{1}{n^2}$

$= \frac{n^2-(n+1)^2}{n^2(n+1)^2}$

$= \frac{n^2-n^2-2n-1}{n^2(n+1)^2} < 0$

$= \frac{-2n-1}{n^2(n^2+2n+1)} < 0$

$|u_{n+1}| < |u_n|$

→ Hence , $\{u_n\}$ is monotonic decreasing sequence

$$\therefore u_n = \frac{1}{n^2} > 0$$

(b) Investing the convergence of the series $\sum_{n=0}^{\infty} \frac{2^n+5}{3^n}$

Ans : $\sum_{n=1}^{\infty} \frac{2^n+5}{3^n} = \sum_{n=1}^{\infty} \left[\left(\frac{2}{n}\right)^n + 5\left(\frac{1}{3}\right)^n \right]$

$$= \sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n + \sum_{n=1}^{\infty} 5\left(\frac{1}{3}\right)^n$$

$$= \sum_{n=1}^{\infty} \left(\frac{2}{3}\right)\left(\frac{2}{3}\right)^{n-1} + \sum_{n=1}^{\infty} \left(\frac{5}{3}\right)\left(\frac{1}{3}\right)^{n-1}$$

→ Both the series are geometric series with $r_1 = \frac{2}{3}$ *and* $r_2 = \frac{1}{3}$ respectively

$$|r_1| < 1 \text{ and } |r_2| < 1$$

→ Hence, both the series are convergent

$$S_1 = \frac{a_1}{1-n_1} = \frac{\frac{2}{3}}{1-\frac{2}{3}} = \frac{\frac{2}{3}}{\frac{1}{3}} = 2$$

$$S_2 = \frac{a_2}{1-r_2} = \frac{\frac{5}{3}}{1-\frac{1}{3}} = \frac{\frac{5}{3}}{\frac{2}{3}} = \frac{5}{2}$$

→ Hence, $\sum_{n=1}^{\infty} \frac{2^n+5}{3^n} = 2 + \frac{5}{2} = \frac{9}{2}$

(c) Find the interval of Convergence of the series $x - \frac{x^2}{2} + \frac{x^3}{3}$ …….

Ans : Let , $u_n = (-1)^{n-1} . \frac{x^n}{n}$

$$|u_n| = \frac{x^n}{n}$$

$$|u_{n+1}| = \frac{x^{n+1}}{n+1}$$

→ $\frac{|u_n|}{|u_{n+1}|} = \frac{x^n}{n} . \frac{n+1}{x^{n+1}}$

$$= 1 + \frac{1}{n} . \frac{1}{x}$$

$$\lim_{n\to\infty} \frac{|u_n|}{|u_{n+1}|} = \lim_{n\to\infty} \frac{\left(1+\frac{1}{n}\right)}{x}$$

$$= \frac{1}{x}$$

→ By D'Alemblert's ratio test $\sum |u_n|$is convergent if $\frac{1}{x} > 1$ *or* $x < 1$ $[\therefore x > 0]$

→ Thus, the given series is absolutely convergent and hence, is convergent for $x < 1$

If $x = 1$ $[x > 0]$

$$u_n = \frac{(-1)^{n-1}}{n}$$

$$|u_n| = \frac{1}{n}$$

→ The given Series is an alternating series

(i) $|u_n| - |u_{n+1}| = \frac{1}{n} - \frac{1}{n+1}$

$$= \frac{1}{n(n+1)} > 0 \; for \; all \; n \; \in N$$

$(ii) \lim_{n\to\infty} |u_n| = \lim_{n\to\infty} \frac{1}{n}$

$$= 0$$

→ By leibnitz's test , the series is convergent for $x = 1$

→ Hence , the series is convergent for $x \leq 1$