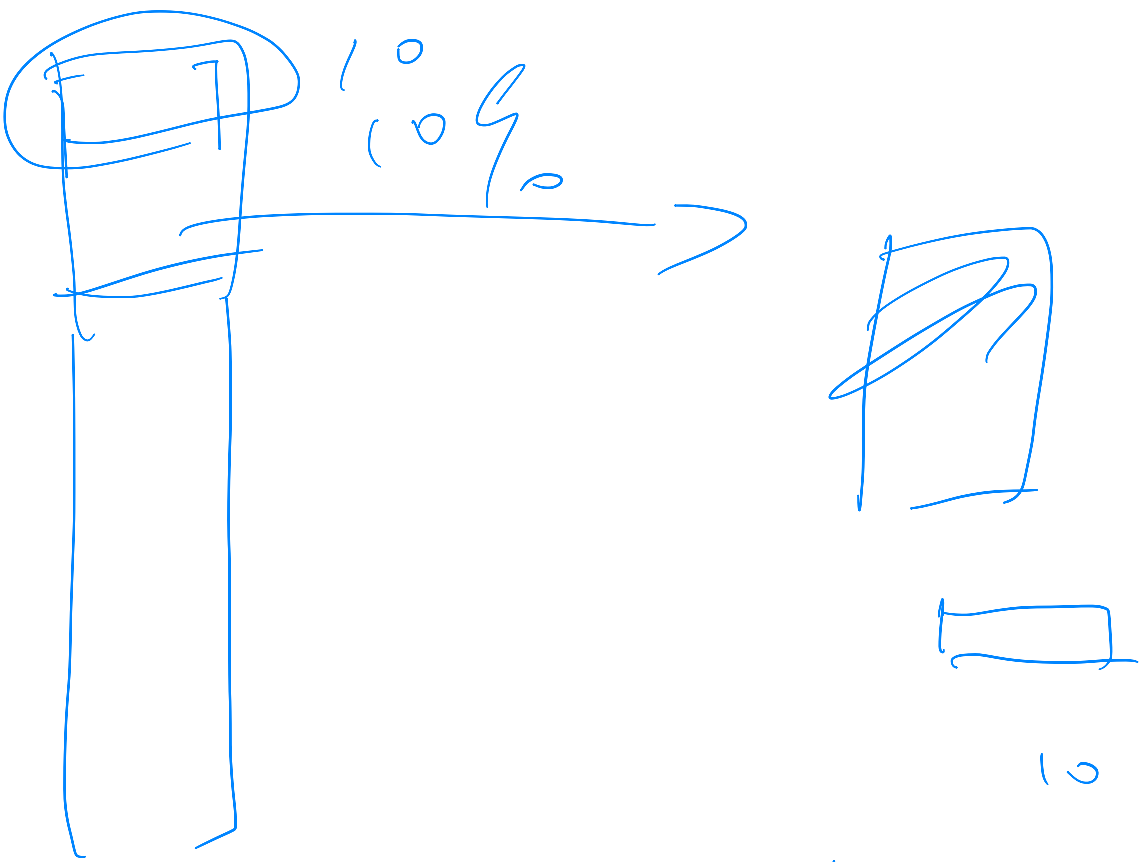
Dim Reduction



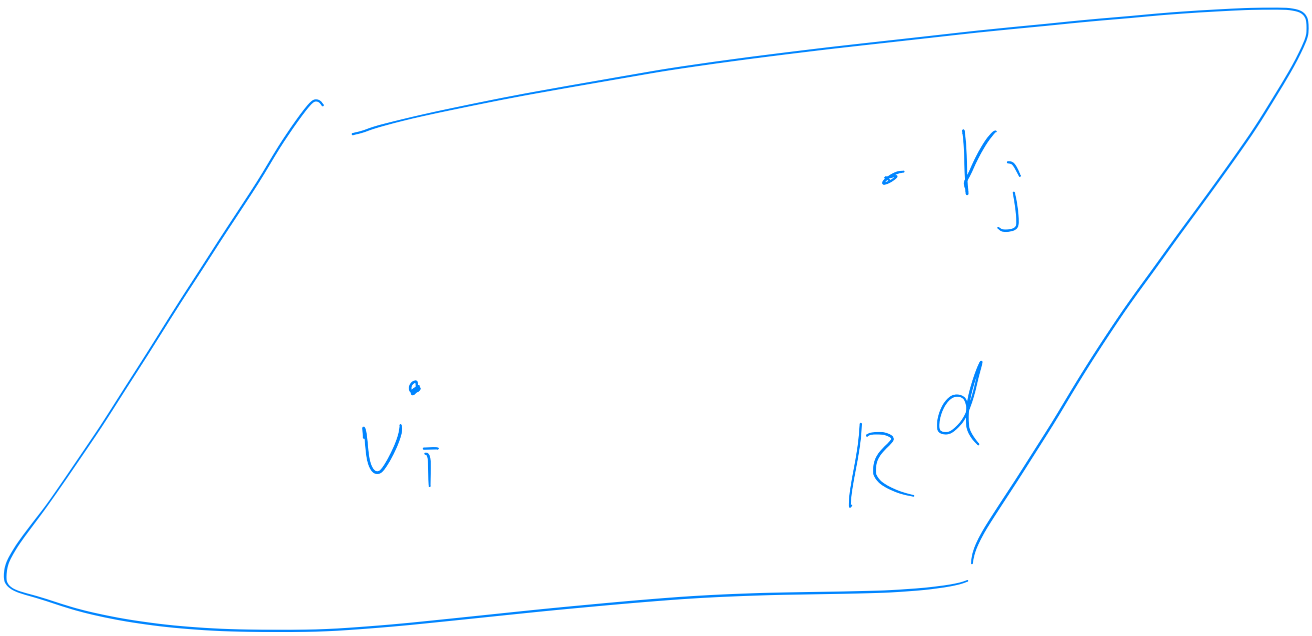
1M $\quad v \in R^d.$

Linear transformation

$M: \; R^{k \times d}. \qquad k = 10$

$Mv. \quad R^k.$

Property:



Far → Far

R^k

close → close.

$$\|v_i - v_j\| \le (1+\epsilon)\,\|v_i' - v_j'\|$$

$$\ge (1-\epsilon)\cdot\|v_i' - v_j'\|$$

$\boxed{\text{Goal.}}$

① $k \geq \square$

$k < d$

② $M = ?$

$M \neq 0$

$M = [I \; \ldots \;]$

$$\boxed{\begin{aligned} M &\sim N(0, I) \\ M_{ij} &\sim N(0, 1) \end{aligned}}$$

$k \times d$

HW

$$S = \{ v_1, \ldots\ldots v_n \}$$

$$\Big\downarrow M$$

$$S' = \{ v_1' \ldots\ldots v_n' \}$$

$$\boxed{v_i' = M \cdot v_i}$$

$$\forall\ i, j \quad \downarrow$$

$$(1-\epsilon)\underbrace{\|v_i - v_j\|}_{\checkmark} \leq \| v_i' - v_j' \| \leq (1+\epsilon) \cdot \underbrace{\|v_i - v_j\|}_{\checkmark}$$

$$\|$$

$$\| M v_i - M v_j \|$$

$$\overline{}$$

$$M (\underbrace{v_i - v_j}_{\checkmark} \|$$

b. $v \in \mathbb{R}^d$. w.h.p.

$$(1-\epsilon)\|v\|^2 \le \underline{\underline{\|Mv\|^2}} \le (1+\epsilon)\|v\|^2.$$

$$\|x\|^2 = x^T x$$

Pf:

$$E\left[\|Mv\|^2\right]$$

$$= E\left[(Mv)^T Mv\right]$$

$$= E\left[v^T M^T M v\right]$$

$$= v^T \cdot \underbrace{E\left[M^T M\right]}_{=I} \cdot v$$

$$= v^T v$$

$$= \|v\|^2$$

$$M = \begin{bmatrix} m_1 & \cdots & m_d \end{bmatrix}_{k \times d}$$

Here, $m_i \in \mathbb{R}^k$

$$M^T = \begin{bmatrix} m_1^T \\ m_2^T \\ \vdots \\ m_d^T \end{bmatrix}$$

$$E[M^T M] = k \cdot I$$

$$\Leftrightarrow \tilde{M} = M/\sqrt{k}$$

$$E[\tilde{M}^T \tilde{M}] = I$$

$$\underline{\underline{M^T M}} = \begin{bmatrix} \leftarrow m_1^T \rightarrow \\ \vdots \\ m_d^T \end{bmatrix} \begin{bmatrix} m_1 & \cdots & m_d \end{bmatrix}$$

$$E \begin{bmatrix} \overset{k}{m_1^T m_1} & m_1^T m_2 & \cdots & m_1^T m_d \\ m_2^T m_1 & \ddots & 0 & m_2^T m_d \\ \vdots & 0 & & k \\ m_d^T m_1 & \cdots & & m_d^T m_d \end{bmatrix}$$

$d \times d$

$$E[m_i^T m_j], \quad 1 \le i \ne j \le d.$$

1° $i \neq j$ $\quad N(0, 1)$

$$E[m_i^T m_j]$$

$$= E[m_i^T] \cdot E[m_j]$$

$$= 0^T \cdot 0$$

$$= 0$$

2° $i = j$

$$E[m_i^T m_i]$$

$$= E[\underbrace{\|m_i\|^2}]$$

$$= K.$$

$$m_i = \left(m_i^{(1)}, \; m_i^{(2)}, \ldots, m_i^{(k)} \right)$$

$$E\left[\| m_i \|^2 \right]$$

$$= E\left[\sum \left(m_i^{(j)} \right)^2 \right]$$

$$= \sum E\Big(\underbrace{m_i^{(j)}}_{N(0,1)} \Big)^2 = \sum_{j=1}^{k} 1 = k$$

$$\mathrm{Var}\left(m_i^{(j)} \right) = 1$$

$$\mathrm{Var} = \underbrace{E\left[\left(m_i^{(j)} \right)^2 \right]}_{=1} - \big(\underbrace{E}_{=0} \big)^2$$

① $k = \boxed{}$

② $\tilde{M} : k \times d.$

$\sim N(0, 1/k)$

$v_1, \ldots, v_n \quad \mathbb{R}^d.$

$\downarrow$

$\tilde{M}v_1, \ldots, \tilde{M}v_n \quad \mathbb{R}^k$

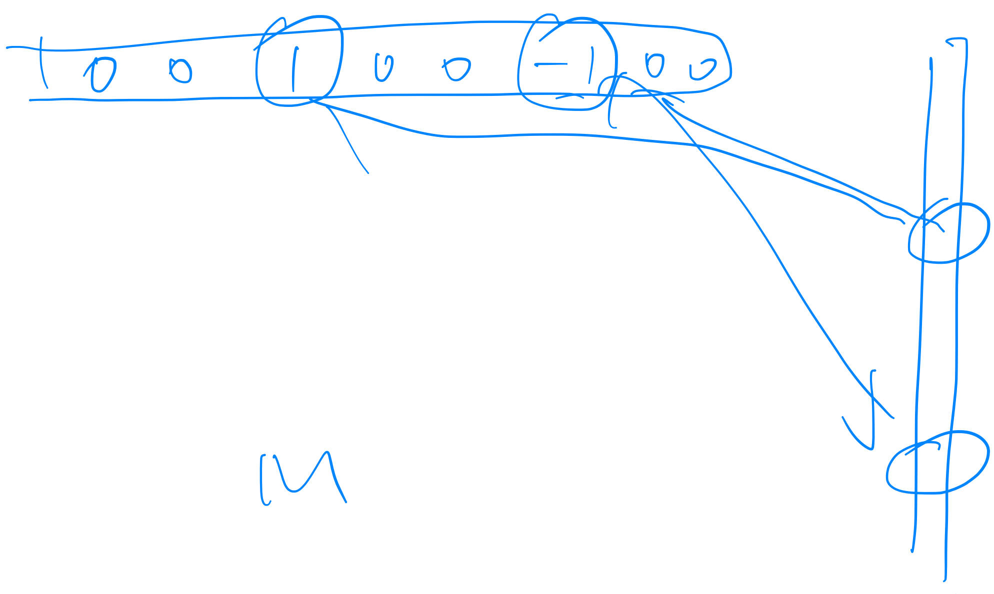
$$k \geq \frac{\log n}{\epsilon^2}$$

sparse matrix

M :

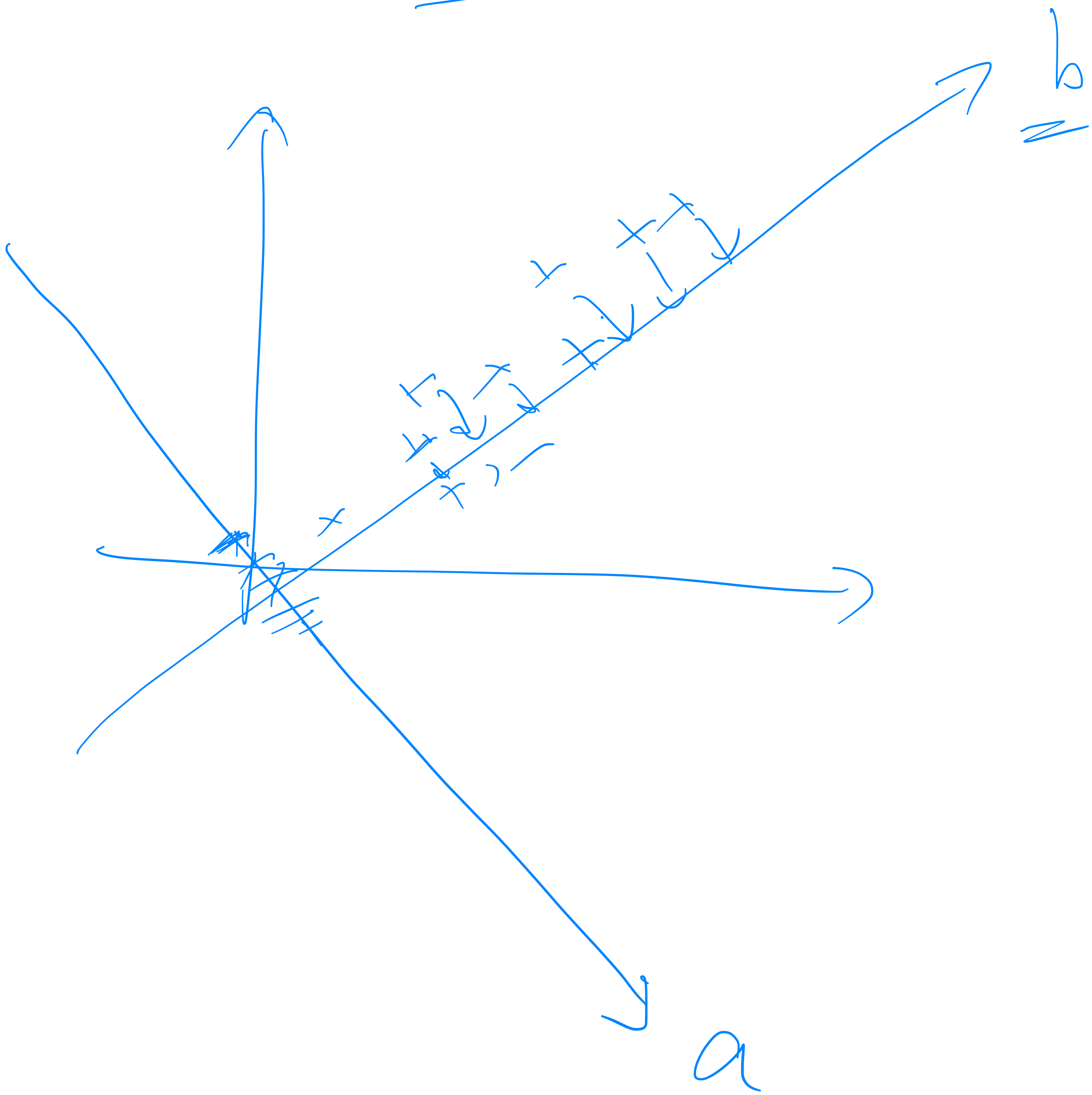
$$m_{ij} = \begin{cases} +1 & 1/6 \\ -1 & 1/6 \\ 0 & 2/3 \end{cases}$$

$2/3$



M

PCA



$$M = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 6 & 8 & 10 \end{bmatrix}$$

$$M' = \begin{bmatrix} 1.0001 & 2.001 & 3.01 & 3.99 & 4.99 \\ 2 & 4 & 6 & 8 & 10 \end{bmatrix}$$

① $M' \to \underline{M}$

② dim red.

find low-rank M

to fit M'

$$\left\{ \begin{array}{l} \min_{M} \| M' - M \|_F \\ \text{s.t.} \quad \text{rank}(M) \le r \end{array} \right.$$

non-convex

$M = ?$

SVD

1990's

$X \rightarrow \boxed{\text{SVD}} \rightarrow (U, S, V)$

① $X = U S V^T$

②