15.470 Final Exam Cheat Sheet
by Shreyas V. Srinivasan, page 1 of 2

1 Basic Framework

Environment

State space $\Omega = \{\omega_i\}_{i=1}^M$
$\mathbb{P}(\omega_m) = p_m > 0$

Agents

Agents $k = 1, \dots, K$
Rational expectations, homogeneous beliefs
Prior $\mathbb{P}$ beliefs
Endowment e_0^k and $e_{1\omega}^k$
Production $y_1(I) = [f_\omega(I)]^\intercal$
Consumption c_0, $c_{1\omega}$

Securities Market

Securities $n = 1, \dots, N$
Payoffs $D_{1n} = [D_{1\omega n}]$
Prices $P = [P_n]$
Portfolio $\theta = [\theta_n]$

Mkt. Equilibrium

Endowment e^k, constraints
$c_0^k = e_0^k - P^\intercal \theta$,
$c_1^k = e_1^k + D_1 \theta$,
$c^k \geq 0$
Optimize $u_k(c)$
Solution $\theta^k(P, e)$
Mkt. clearing $\sum_{k=1}^K \theta^k\left(P, e^k\right) = 0$

2 Mathematics

Expected Value

$\mathbb{E}[X] = \sum_{m=1}^M p_m X(\omega_m) = \int_{-\infty}^{\infty} x f_X(x)\,\mathrm{d}x$
Linearity: $\mathbb{E}[aX + bY] = a\mathbb{E}[X] + b\mathbb{E}[Y]$
Jensen's: g convex $\mathbb{E}[g(X)] \geq g(\mathbb{E}[X])$

Variance

$\mathrm{Cov}(X,Y) = \mathbb{E}[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])]$
$\mathrm{Var}(X) = \mathrm{Cov}(X, X)$
$\rho_{X,Y} = \frac{\mathrm{Cov}(X,Y)}{\sigma_X \sigma_Y}$
Billinearity, symmetry
$\mathrm{Var}(X+Y) = \mathrm{Var}(X) + \mathrm{Var}(Y) + 2\mathrm{Cov}(X,Y)$

Independence

$\mathbb{P}(X \in A, Y \in B) = \mathbb{P}(X \in A)\mathbb{P}(Y \in B)$
$\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$
$\mathrm{Cov}(X,Y) = 0$

Log-normal RV

$X \sim \mathcal{N}\left(\mu, \sigma^2\right) \implies$
$\mathbb{E}[\exp(X)] = \exp\left(\mu + \frac{\sigma^2}{2}\right)$

Returns

$\underbrace{1 + r_t}_{\text{Net Ret}} = \underbrace{R_t}_{\text{Gross Ret}}$
$I_t = I_0 \prod_{t=1}^T R_t$
$\overline{R}_T^A = \frac{1}{T}\sum_{t=1}^T R_t$
$\overline{R}_T^G = \left(\prod_{t=1}^T R_t\right)^{\frac{1}{T}}$

3 Utility Functions

CARA (Exponential)

$u(c) = \frac{1 - \exp(-\alpha c)}{\alpha}$, $\alpha > 0$
Abs. risk aversion $\alpha_u(w_0) = -\frac{u''(w_0)}{u'(w_0)}$

CRRA

$u(c) = \frac{c^{1-\gamma} - 1}{1-\gamma}$, $\gamma > 0$
Rel. risk aversion $\gamma_u(w_0) \equiv -\frac{w_0 u''(w_0)}{u'(w_0)} = w_0 \alpha_u(w_0)$

4 Canonical Model

Redundancy

$\exists \theta_{\setminus n} : D_{\setminus n}\theta_{\setminus n} = D_n$
Complete: $\forall c_1 \in \mathbb{R}^m \exists \theta \in \mathbb{R}^n : D\theta = c_1$
Complete $\iff$ rank$(D) = M$

Arrow-Debreu Mkt.

ϕ_ω state-contingent claim
Arrow-Debreu mkt complete

Risk Aversion

Fair gamble: $\mathbb{E}[x] = 0$
Risk averse: $\mathbb{E}[u(w + x)] \leq \mathbb{E}[u(w)]$
Risk prem π: $u(w - \pi) = \mathbb{E}[u(w + x)]$

Portfolio choice

SDF $\eta = \frac{u_1'(c_{1\omega})}{u_0'(c_0)} = \frac{\phi_\omega}{p_\omega}$
$\eta_\omega c_{1\omega} \propto 1$

Euler Equation

$P_n = \mathbb{E}\left[\frac{u_1'(c_1)}{u_0'(c_0)} D_n\right]$

Valuation Formulas

DCF/PV: $P_n = \phi^\intercal D_n$
Risk-Neutral $q_\omega \equiv \frac{\phi_\omega}{\sum_{\omega'} \phi_{\omega'}}$
$P_n = \frac{\mathbb{E}^{\mathbb{Q}}[D_n]}{1 + r_f}$
SPD/SDF: $P_n = \mathbb{E}^{\mathbb{P}}[\eta D_n]$
$R_f = \mathbb{E}^{\mathbb{P}}[\eta]^{-1}$

Risk Premia

$\mathbb{E}\left[R_n - R_f\right] = -R_f \mathrm{Cov}\left[\eta, R_n - R_f\right]$
Lucas: $\mathbb{E}\left[R_n - R_f\right] \approx -\gamma \mathrm{Cov}\left[R_n, \frac{c_1}{c_0}\right]$

SPD/SDF Bounds

$\frac{\sqrt{\mathrm{Var}^{\mathbb{P}}[\eta]}}{\mathbb{E}^{\mathbb{P}}[\eta]} \geq \frac{\mathbb{E}^{\mathbb{P}}\left[r_n - r_f\right]}{\sqrt{\mathrm{Var}^{\mathbb{P}}\left[r_n - r_f\right]}}$

5 Arbitrage

Definition

1) $P^T \theta \leq 0$,
2) $D\theta \geq 0$,
3) One or both of above strict;
No Arbitrage in frictionless mkt.

No Arbitrage

$\exists V(\cdot)$ pricing operator
$P = V(d)$; Law of One Price:
$V(d_1) = V(d_2) \iff d_1 = d_2$;
V monotonic & linear
FTAP: $\exists \phi : P^\intercal = \phi^\intercal D$
NA $\iff \exists \phi > 0$

6 Corporate Finance

Opt Prod/Investment

Production: $y_{1\omega} = y_\omega(y_0)$
NPV: $v = \sum_{\omega \in \Omega} \phi_\omega y_\omega(y_0) - y_0$
Max NPV *or* utility

Capital Structure

$y_0 = d_0 + e_0$
$D = \phi^\intercal d_1$
$E = \phi^\intercal e_1$

q Theory

$q_j \equiv \frac{\mathbb{E}^{\mathbb{Q}}\left[A_j\right]}{1 + r_f}$

7 Fixed Income

Term Structure

$P_{N,t} = \mathbb{E}_t\left[\prod_{i=t+1}^{t+N} \eta_i\right]$
$P_{N,t} = Y_{N,t}^{-N}$
Nominal: $P_{N,t}^{\$} = P_{N,t} \times \mathrm{CPI}_t$
$\Pi_t = \mathrm{CPI}_t / \mathrm{CPI}_{t-1}$
$1 = \mathbb{E}_t\left[\frac{\eta_{t+1}}{\Pi_{t+1}} \frac{P_{N-1,t+1}^{\$}}{P_{N,t}^{\$}}\right]$
$\eta_{t+1}^{\$} = \frac{\eta_{t+1}}{\Pi_{t+1}}$ nominal SDF
Real: $1 = \mathbb{E}_t\left[\eta_{t+1} \frac{P_{N-1,t+1}}{P_{N,t}}\right]$

8 Option Pricing

Derivative Defn.

Payoff $f(X_T)$ at $t = T$
$P = \mathbb{E}_t\left[f(X_T) \prod_{i=t+1}^T \eta_i\right]$
$P = \sum_{\omega \in \Omega} \phi_\omega f(X_\omega)$
$P = \int \phi(\omega) f(X(\omega))\,\mathrm{d}\omega$

Options

$[x]_+ \equiv \max[0, x]$
$c_1 = [S_1 - K]_+, p_1 = [K - S_1]_+$
$P_X = \mathbb{E}[\eta X] = \mathrm{Cov}[\eta, X] + R_f^{-1}\mathbb{E}[X]$
Intrinsic value, ITM/ATM/OTM;
Prices convex in K
$S \geq c(S,K) \geq \left[S - \frac{K}{1 + r_f}\right]_+$
$c(S,K) - p(S,K) = S - \frac{K}{1 + r_f}$

Backing Out $\mathbb{Q}$

$q(K) = \frac{\partial^2 c(K)}{\partial K^2} \Big/ \int_0^\infty \frac{\partial^2 c(K)}{\partial K^2}\,\mathrm{d}K$

9 Arbitrage Pricing

Decomposition

$r_{n,t} - r_f = \alpha_n + \beta_n\left(r_{-\eta,t} - r_f\right) + \varepsilon_{n,t}$
$\mathrm{Var}\left[r_n - r_f\right] = \beta_n^2 \mathrm{Var}\left[r_{-\eta} - r_f\right] + \mathrm{Var}\left[\varepsilon_n\right]$

Factors

F basis for $R - \overline{R}_n$
$R_n = \overline{R}_n + F\beta_n$
$\mathbb{E}^{\mathbb{Q}}\left[R_n\right] = R_f$ APT: $\overline{R}_n - R_f = \lambda^\intercal \beta_n$,
where $\lambda_k = -\mathbb{E}^{\mathbb{Q}}\left[F_k\right]$
λ_k factor premium

Diversification

Portfolio θ *well diversified* if
$\theta_n = O\left(\frac{1}{n}\right)$
Sequence $\{\theta_i\}$, $|\theta_1| = 1$
well diversified if $\exists 0 < \kappa < \infty$ s.t.
$\theta_{n,j}^2 < \frac{\kappa}{n^2}$.

APT

Asymptotic: $\mathbb{E}\left[r_{\theta_n}\right] \to \alpha > 0$,
but $\mathrm{Var}\left[r_{\theta_n}\right] \to 0$.
No asymptotic arb. $\implies \exists r_f, \lambda, A$:
$\sum_{i=1}^n \left[\overline{r}_i - \left(r_f + \lambda^\intercal \beta_i\right)\right]^2 = \sum_{i=1}^n \left[\overline{r}_i - \left(r_f + \sum_k \lambda_k \beta_{ik}\right)\right]^2 < A < \infty$.

10 Mean-Variance & CAPM

Mean-Variance Preference

Def: $\mathbb{E}[u(W)] = v(\overline{w}, \sigma_w)$
Quadratic utility $\implies$ MV preference

Mean-Variance Frontier

$\min_x \frac{1}{2} x^\intercal \Sigma x$ s.t. $\mu^\intercal x = \mu_p$, $\iota^\intercal x = 1$
Soln: $x = \lambda_1 \Sigma^{-1}\iota + \lambda_2 \Sigma^{-2}\mu$

MVF Properties

Linear comb. of MVF portfolios is on MVF
Set of MVF portfolios lies on a line in R^n
MVP: Minimum Variance Portfolio among all;
has $\frac{\partial \sigma_p^2}{\partial \mu_p} = 0$ necessary and sufficient
For any p, $\mathrm{Cov}\left[r_p, r_{mvp}\right] = \sigma_{mvp}^2$
For any MVF p, $\exists zcp$ on MVF s.t. $\mathrm{Cov}\left[r_p, r_{zcp}\right] = 0$
For p MVF, q arbitrary: $\mu_q - \mu_{zcp} = \beta_{qp}\left(\mu_p - \mu_{zcp}\right)$
η^* projection of η on linear payoff space: $r^* = \frac{\eta^*}{\mathbb{E}\left[(\eta^*)^2\right]}$

Riskless Asset

Sol: $x_p = \frac{\hat{\mu}_p}{\hat{\mu}^\intercal \Sigma^{-1} \hat{\mu}} \Sigma^{-1} \hat{\mu}$
Tangent portfolio: $x_T = \frac{\Sigma^{-1}\hat{\mu}}{\hat{\mu}^\intercal \Sigma^{-1} \iota}$
Any MVF port is mix of riskless asset and tangent port Sharpe Ratio:
$SR = \frac{\hat{\mu}^\intercal x}{\sqrt{x^\intercal \Sigma x}}$
Tangent port has highest SR of all risky portfolios

CAPM Derivation

CAPM: $\mu_i - r_F = \beta_{i,m}(\mu_m - r_F)$, $\beta_{i,m} = \frac{\mathrm{Cov}[r_i, r_m]}{\mathrm{Var}[r_m]}$

11 Imperfect Information

Walrasian Equilibrium

Informed and uninformed agents trading
Issue: price reveals signal of informed agents, but uninformed do not take into account

Rational Expectations Equilibrium

N agents, utility U^i, wealth W_0^i, signal s^i
REE: price function $p\left(s^1, \dots, s^N\right)$ and demands $\left(x^1(p), \dots, x^N(p)\right)$ s.t.:
1) $x^i(p)$ solves $\max_x \mathbb{E}\left\{U^i\left[\left(W_0^i - xp\right)(1 + r) + xd\right] \mid s^i, p\right\}$
2) $\sum_{i=1}^N x^i(p) = S$
Equilibrium: $p = \frac{\overline{d}}{1+r} + \frac{\beta_s\left(s - \overline{d}\right)}{1+r} - \frac{S\alpha\sigma_{d|s}^2}{(1+r)N}$
$x_I(p) = \frac{\overline{d} + \beta_s\left(s - \overline{d}\right) - (1+r)p}{\alpha\sigma_{d|s}^2}$
$x_U(p) = \frac{S}{N}$

Noisy REE

REE with supply $S + u$, $u \sim \mathcal{N}\left(0, \sigma_u^2\right)$
Noise u is liquidity demand
Constant $C = \frac{\alpha\sigma_\varepsilon^2}{N_I}$
$C \propto N_I^{-1}$; $C \propto \alpha, \sigma_\varepsilon^2$
Price informativeness $\tau = \sigma_{d|p}^{-2}$
$\tau \propto \sigma_u^{-2}, \alpha^{-1}, \sigma_\varepsilon^{-2}$; $\tau \propto N_I$
Price sensitivity $\frac{\mathrm{d}x_U(p)}{\mathrm{d}p} = \frac{\beta_p - (1+r)}{\alpha\sigma_{d|p}^2}$

Differential Information

Multiple agents with signal $s^i = d + \varepsilon^i$, $\varepsilon^i \sim \mathcal{N}\left(0, \sigma^2\varepsilon\right)$
REE: $p = A + B\frac{\sum_{i=1}^N\left(s^i - \overline{d}\right)}{N}$
$\mathbb{E}[d \mid p] = \overline{d} + \beta_p(p - A)$

15.470 Final Exam Cheat Sheet
by Shreyas V. Srinivasan, page 2 of 2

$\beta_p = \frac{\sigma^2}{B\left(\sigma^2+\frac{\sigma_\epsilon^2}{N}\right)}$

$\sigma_{d|p}^2 = \frac{\sigma^2\frac{\sigma_\epsilon^2}{N}}{\sigma^2+\frac{\sigma_\epsilon^2}{N}}$

$A = \frac{\overline{d}}{1+r} - \frac{S\alpha\sigma_{d|p}^2}{(1+r)N}$, $B\beta_p = 1+r$

Kyle Model

Two periods, one risky pays d at $t=1$: $d \sim \mathcal{N}\left(\overline{d}, \sigma^2\right)$
Price p at $t=0$, riskless rate $r=0$
Insider (know d), market maker, noise traders (need quantity $u \sim \mathcal{N}\left(0, \sigma_u^2\right)$)
Insiders solve $\max_x \mathbb{E}\left[\left(d - p\left(x+u\right)\right)x \mid d\right]$
MM's solve $p\left(x+u\right) = \mathbb{E}\left[d \mid x+u\right]$
Sol: $x(d) = \beta\left(d-\overline{d}\right)$, $p(x+u) = \overline{d} + \lambda(x+u)$,
$\beta = \frac{\sigma_u}{\sigma}$, $\lambda = \frac{\sigma}{2\sigma_u}$

Glosten-Milgrom Model

T periods, asset payoff $\text{Bin}(p)$ at end
Informed traders (prob π), market makers, noise traders (prob $1-\pi$)
Order size fixed at 1, History of trades $\mathcal{H}_t$, bid and ask prices

12 Dynamic Asset Pricing

Model

Trading dates $\mathcal{T} = \{0, 1, \ldots, T\}$
Unique history's up to time t: ω_t
$N+1$ securities indexed $n = 0, \ldots, N$
Security 0 riskless, r short rate process
Risky: δ_n dividend and ex-dividend price S_n
Trading strategies can *finance* cash flows,
cash flow *marketable* iff financed by a strategy θ
Self-financing: $c_1 = c_2 = \cdots = c_{T-1} = 0$

FTAP

Positive: $X_t \geq 0, \forall t$
Strictly positive: $X_t > 0, \forall t$
Arbitrage: positive marketable cash flow, not identically 0
Strictly increasing: $\Psi : \mathcal{L} \to \mathbb{R} \iff X_t \geq Y_t, \forall t \implies \Psi(X) > \Psi(Y)$
FTAP: no arbitrage $\iff \exists \Psi : \mathcal{L} \to \mathbb{R}$ linear strictly increasing s.t. $\Psi(c) = 0, \forall c \in M$
State Prices ψ_t time-0 price of unit of consumption at ω_t:
$-c_0 = \mathbf{K}_0 \sum_{t=1}^T \psi_t c_t$

Security & State Prices

$S_t = \mathbf{K}_t \left(\sum_{s=t+1}^T \frac{\psi_s}{\psi_t}\delta_s + \frac{\psi_T}{\psi_t}S_T\right)$
$S_t = \mathbf{K}_t \frac{\psi_{t+1}}{\psi_t}\left(\delta_{t+1} + S_{t+1}\right)$

State-Price Density

SPD: $\pi_t = \frac{\psi_t}{p_t} > 0$, $\pi_0 = \psi_0 = 1$
$S_t = \mathbb{E}_t\left[\sum_{s=t+1}^T \frac{\pi_s}{\pi_t}\delta_s + \frac{\pi_T}{\pi_t}S_T\right] = \mathbb{E}\left[\frac{\pi_{t+1}}{\pi_t}\left(\delta_{t+1} + S_{t+1}\right)\right]$
Martingale $G_t^\pi = \pi_t S_t + \sum_{s=1}^t \pi_s \delta_s$

Equivalent Martingale Measure

$\mathbb{Q}$ s.t. $\Psi(c) = \mathbb{E}_0^{\mathbb{Q}}\left[\sum_{t=0}^T \frac{c_t}{S_{0,t}}\right]$
EMM: $\mathbb{Q}$ s.t. $q_t = p_t \pi_t S_{0,t}$

Redundant Securities

Cash flows can be replicated by trading strategy of other securities
Markets *complete* if all cash flows replicable
$d(\omega_t)$: # of time $t+1$ nodes subsequent to ω_t
$r(\omega_t)$: rank of $(N+1) \times d(\omega_t)$ payoff matrix
Markets *dynamically complete* if $r(\omega_t) = d(\omega_t)$
Markets dynamically complete iff state prices unique

Portfolio Choice

Gradient of U at c: $\nabla U(c)[\hat{c}] \equiv \lim_{\alpha\to\infty} \frac{U(c+\alpha\hat{c}) - U(c)}{\alpha}$
Riesz representation: $\nabla U(c)[\hat{c}] = \mathbb{E}_0\left[\sum_{t=0}^T R_t \hat{c}_t\right]$
Equilibrium: $\nabla U(c^*) = \lambda\pi$
U time-additive means $R_t = u_t'(c_t)$

Equilibrium Asset Pricing

Contingent Claims (CC) Equilibrium: c_i optimal for agent i,
(market-clearing) $\sum_{i=1}^I c_i = \sum_{i=1}^I e_i$