

1 Basic Framework

Environment

State space $\Omega = \{\omega_i\}_{i=1}^M$

$\mathbb{P}(\omega_m) = p_m > 0$

Agents

Agents $k = 1, \dots, K$

Rational expectations, homogeneous beliefs

Prior \mathbb{P} beliefs

Endowment e_0^k and $e_{1\omega}^k$

Production $y_1(I) = [f_w(I)]^\top$

Consumption $c_0, c_{1\omega}$

Securities Market

Securities $n = 1, \dots, N$

Payoffs $D_{1n} = [D_{1\omega n}]$

Prices $P = [P_n]$

Portfolio $\theta = [\theta_n]$

Mkt. Equilibrium

Endowment e^k , constraints

$c_0^k = e_0^k - P^\top \theta$,

$c_1^k = e_{1\omega}^k + D_1 \theta$,

$c^k \geq 0$

Optimize $u_k(c)$

Solution $\theta^k(P, e)$

Mkt. clearing $\sum_{k=1}^K \theta^k(P, e^k) = 0$

2 Mathematics

Expected Value

$\mathbb{E}[X] = \sum_{m=1}^M p_m X(\omega_m) = \int_{-\infty}^{\infty} x f_X(x) dx$

Linearity: $\mathbb{E}[aX + bY] = a\mathbb{E}[X] + b\mathbb{E}[Y]$

Jensen's: g convex $\mathbb{E}[g(X)] \geq g(\mathbb{E}[X])$

Variance

$\text{Cov}(X, Y) = \mathbb{E}[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])]$

$\text{Var}(X) = \text{Cov}(X, X)$

$\rho_{X,Y} = \frac{\text{Cov}(X,Y)}{\sigma_X \sigma_Y}$

Billinearity, symmetry

$\text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y) + 2\text{Cov}(X, Y)$

Independence

$\mathbb{P}(X \in A, Y \in B) = \mathbb{P}(X \in A) \mathbb{P}(Y \in B)$

$\mathbb{E}[XY] = \mathbb{E}[X] \mathbb{E}[Y]$

$\text{Cov}(X, Y) = 0$

Log-normal RV

$X \sim \mathcal{N}(\mu, \sigma^2) \Rightarrow$

$\mathbb{E}[\exp(X)] = \exp\left(\mu + \frac{\sigma^2}{2}\right)$

Returns

$1 + \underbrace{r_t}_{\text{Net Ret}} = \underbrace{R_t}_{\text{Gross Ret}}$

$I_t = I_0 \prod_{i=1}^T R_i$

$\bar{R}_T^A = \frac{1}{T} \sum_{t=1}^T R_t$

$\bar{R}_T^G = \left(\prod_{t=1}^T R_t\right)^{\frac{1}{T}}$

3 Utility Functions

CARA (Exponential)

$u(c) = \frac{1 - \exp(-\alpha c)}{\alpha}, \alpha > 0$

Abs. risk aversion $\alpha_u(w_0) = -\frac{u''(w_0)}{u'(w_0)}$

CRRA

$u(c) = \frac{c^{1-\gamma}-1}{1-\gamma}, \gamma > 0$

Rel. risk aversion $\gamma_u(w_0) \equiv$

$-\frac{w_0 u''(w_0)}{u'(w_0)} = w_0 \alpha_u(w_0)$

4 Canonical Model

Redundancy

$\exists \theta_{\setminus n} : D_{\setminus n} \theta_{\setminus n} = D_n$

Complete: $\forall c_1 \in \mathbb{R}^m \exists \theta \in \mathbb{R}^n : D\theta = c_1$

Complete $\iff \text{rank}(D) = m$

Arrow-Debreu Mkt.

ϕ_ω state-contingent claim

Arrow-Debreu mkt complete

Risk Aversion

Fair gamble: $\mathbb{E}[x] = 0$

Risk averse: $\mathbb{E}[u(w+x)] \leq \mathbb{E}[u(w)]$

Risk prem π : $u(w-\pi) = \mathbb{E}[u(w+x)]$

Portfolio choice

SDF $\eta = \frac{u'_1(c_{1\omega})}{u'_0(c_0)} = \frac{\phi_\omega}{P_\omega}$

$\eta_\omega c_{1\omega} \propto 1$

Euler Equation

$P_n = \mathbb{E}\left[\frac{u'_1(c_1)}{u'_0(c_0)} D_n\right]$

Valuation Formulas

DCF/PV: $P_n = \phi^\top D_n$

Risk-Neutral $q_\omega \equiv \frac{\phi_\omega}{\sum_{\omega'} \phi_{\omega'}}$

$P_n = \frac{\mathbb{E}^Q[D_n]}{1+r_f}$

SPD/SDF: $P_n = \mathbb{E}^P[\eta D^n]$

$R_f = \mathbb{E}^P[\eta]^{-1}$

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Risk Premia

$\mathbb{E}[R_n - R_f] = -R_f \text{Cov}[\eta, R_n - R_f]$

Lucas: $\mathbb{E}[R_n - R_f] \approx -\gamma \text{Cov}[R_n, \frac{c_1}{c_0}]$

SPD/SDF Bounds

$\frac{\sqrt{\text{Var}^P[\eta]}}{\mathbb{E}^P[\eta]} \geq \frac{\mathbb{E}^P[r_n - r_f]}{\sqrt{\text{Var}^P[r_n - r_f]}}$

$\frac{\sqrt{\text{Var}^P[\eta]}}{\mathbb{E}^P[\eta]} \geq \frac{\mathbb{E}^P[r_n - r_f]}{\sqrt{\text{Var}^P[r_n - r_f]}}$

5 Arbitrage

Definition

1) $P^T \theta \leq 0$,

2) $D\theta \geq 0$,

3) One or both of above strict;

No Arbitrage in frictionless mkt.

No Arbitrage

$\exists V(\cdot)$ pricing operator

$P = V(d)$; Law of One Price:

$V(d_1) = V(d_2) \iff d_1 = d_2$;

V monotonic & linear

FTAP: $\exists \phi : P^\top = \phi^\top D$

NA $\iff \exists \phi > 0$

6 Corporate Finance

Opt Prod/Investment

Production: $y_{1\omega} = y_\omega(y_0)$

NPV: $v = \sum_{\omega \in \Omega} \phi_\omega y_\omega(y_0) - y_0$

Max NPV or utility

Capital Structure

$y_0 = d_0 + e_0$

$D = \phi^\top d_1$

$E = \phi^\top e_1$

q Theory

$q_j \equiv \frac{\mathbb{E}^Q[A_j]}{1+r_f}$

7 Fixed Income

Term Structure

$P_{N,t} = \mathbb{E}_t \left[\prod_{i=t+1}^{t+N} \eta_i \right]$

$P_{N,t} = Y_{N,t}^{-N}$

Nominal: $P_{N,t}^\$ = P_{N,t} \times \text{CPI}_t$

$\Pi_t = \text{CPI}_t / \text{CPI}_{t-1}$

$1 = \mathbb{E}_t \left[\frac{\eta_{t+1}}{\Pi_{t+1}} \frac{P_{N-1,t+1}^\$}{P_{N,t}^\$} \right]$

$\eta_{t+1}^\$ = \frac{\eta_{t+1}}{\Pi_{t+1}}$ nominal SDF

Real: $1 = \mathbb{E}_t \left[\eta_{t+1} \frac{P_{N-1,t+1}}{P_{N,t}} \right]$

8 Option Pricing

Derivative Defn.

Payoff $f(X_T)$ at $t = T$

$P = \mathbb{E}_t[f(X_T) \prod_{i=t+1}^T \eta_i]$

$P = \sum_{\omega \in \Omega} \phi_\omega f(X_\omega)$

$P = \int \phi(\omega) f(X(\omega)) d\omega$

Options

$[x]_+ \equiv \max[0, x]$

$c_1 = [S_1 - K]_+, p_1 = [K - S_1]_+$

$P_X = \mathbb{E}[\eta X] = \text{Cov}[\eta, X] + R_f^{-1} \mathbb{E}[X]$

Intrinsic value, ITM/ATM/OTM;

Prices convex in K

$S \geq c(S, K) \geq \left[S - \frac{K}{1+r_f} \right]_+$

$c(S, K) - p(S, K) = S - \frac{K}{1+r_f}$

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Backing Out Q

$q(K) = \frac{\partial^2 c(K)}{\partial K^2} \Big/ \int_0^\infty \frac{\partial^2 c(K)}{\partial K^2} dK$

9 Arbitrage Pricing

Decomposition

$r_{n,t} - r_f = \alpha_n + \beta_n(r_{\setminus n,t} - r_f) + \varepsilon_{n,t}$

$\text{Var}[r_n - r_f] = \beta_n^2 \text{Var}[r_{\setminus n} - r_f] + \text{Var}[\varepsilon_n]$

$\text{Var}[\varepsilon_n]$

$\text{Var}[\varepsilon_n]$

$\text{Var}[\varepsilon_n]$

Factors

F basis for $R - \bar{R}_n$

$R_n = \bar{R}_n + F\beta_n$

$\mathbb{E}^Q[R_n] = R_f$ APT: $\bar{R}_n - R_f = \lambda^\top \beta_n$,

where $\lambda_k = -\mathbb{E}^Q[F_k]$

λ_k factor premium

Diversification

Portfolio θ well diversified if

$\theta_n = O\left(\frac{1}{n}\right)$

Sequence $\{\theta_i\}, |\theta_1| = 1$

well diversified if $\exists 0 < \kappa < \infty$ s.t.

$\theta_{n,j}^2 < \frac{\kappa}{n^2}$.

APT

Asymptotic: $\mathbb{E}[r_{\theta_n}] \rightarrow \alpha > 0$,

but $\text{Var}[r_{\theta_n}] \rightarrow 0$.

No asymptotic arb. $\implies \exists r_f, \lambda, A$:

$\sum_{i=1}^n [\bar{r}_i - (r_f + \lambda^\top \beta_i)]^2 =$

$\sum_{i=1}^n [\bar{r}_i - (r_f + \sum_k \lambda_k \beta_{ik})]^2 < A < \infty$.