15.470 Midterm Cheat Sheet
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## 1 Basic Framework

**Environment**

State space $\Omega = \{\omega_i\}_{i=1}^M$
$\mathbb{P}(\omega_m) = p_m > 0$

**Agents**

Agents $k = 1, \dots, K$
Rational expectations, homogeneous beliefs
Prior $\mathbb{P}$ beliefs
Endowment $e_0^k$ and $e_{1\omega}^k$
Production $y_1(I) = [f_w(I)]^\top$
Consumption $c_0, c_{1\omega}$

**Securities Market**

Securities $n = 1, \dots, N$
Payoffs $D_{1n} = [D_{1\omega n}]$
Prices $P = [P_n]$
Portfolio $\theta = [\theta_n]$

**Mkt. Equilibrium**

Endowment $e^k$, constraints
$c_0^k = e_0^k - P^\top \theta$,
$c_1^k = e_1^k + D_1 \theta$,
$c^k \geq 0$
Optimize $u_k(c)$
Solution $\theta^k(P, e)$
Mkt. clearing $\sum_{k=1}^K \theta^k\left(P, e^k\right) = 0$

## 2 Mathematics

**Expected Value**

$\mathbb{E}[X] = \sum_{m=1}^M p_m X(\omega_m) = \int_{-\infty}^{\infty} x f_X(x)\, dx$
Linearity: $\mathbb{E}[aX + bY] = a\mathbb{E}[X] + b\mathbb{E}[Y]$
Jensen's: $g$ convex $\mathbb{E}[g(X)] \geq g(\mathbb{E}[X])$

**Variance**

$\mathrm{Cov}(X, Y) = \mathbb{E}[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])]$
$\mathrm{Var}(X) = \mathrm{Cov}(X, X)$
$\rho_{X,Y} = \frac{\mathrm{Cov}(X,Y)}{\sigma_X \sigma_Y}$
Billinearity, symmetry
$\mathrm{Var}(X + Y) = \mathrm{Var}(X) + \mathrm{Var}(Y) + 2\,\mathrm{Cov}(X, Y)$

**Independence**

$\mathbb{P}(X \in A, Y \in B) = \mathbb{P}(X \in A)\,\mathbb{P}(Y \in B)$
$\mathbb{E}[XY] = \mathbb{E}[X]\,\mathbb{E}[Y]$
$\mathrm{Cov}(X, Y) = 0$

**Log-normal RV**

$X \sim \mathcal{N}\left(\mu, \sigma^2\right) \implies$
$\mathbb{E}[\exp(X)] = \exp\left(\mu + \frac{\sigma^2}{2}\right)$

**Returns**

$1 + \underbrace{r_t}_{\text{Net Ret}} = \underbrace{R_t}_{\text{Gross Ret}}$
$I_t = I_0 \prod_{t=1}^T R_t$
$\overline{R}_T^A = \frac{1}{T}\sum_{t=1}^T R_t$
$\overline{R}_T^G = \left(\prod_{t=1}^T R_t\right)^{\frac{1}{T}}$

## 3 Utility Functions

**CARA (Exponential)**

$u(c) = \frac{1 - \exp(-\alpha c)}{\alpha}$, $\alpha > 0$
Abs. risk aversion $\alpha_u(w_0) = -\frac{u''(w_0)}{u'(w_0)}$

**CRRA**

$u(c) = \frac{c^{1-\gamma} - 1}{1-\gamma}$, $\gamma > 0$
Rel. risk aversion $\gamma_u(w_0) \equiv -\frac{w_0 u''(w_0)}{u'(w_0)} = w_0 \alpha_u(w_0)$

## 4 Canonical Model

**Redundancy**

$\exists \theta_{\setminus n} : D_{\setminus n}\theta_{\setminus n} = D_n$
Complete: $\forall c_1 \in \mathbb{R}^m \exists \theta \in \mathbb{R}^n : D\theta = c_1$
Complete $\iff \mathrm{rank}(D) = M$

**Arrow-Debreu Mkt.**

$\phi_\omega$ state-contingent claim
Arrow-Debreu mkt complete

**Risk Aversion**

Fair gamble: $\mathbb{E}[x] = 0$
Risk averse: $\mathbb{E}[u(w + x)] \leq \mathbb{E}[u(w)]$
Risk prem $\pi$: $u(w - \pi) = \mathbb{E}[u(w + x)]$

**Portfolio choice**

SDF $\eta = \frac{u_1'(c_{1\omega})}{u_0'(c_0)} = \frac{\phi_\omega}{p_\omega}$
$\eta_\omega c_{1\omega} \propto 1$

**Euler Equation**

$P_n = \mathbb{E}\left[\frac{u_1'(c_1)}{u_0'(c_0)} D_n\right]$

**Valuation Formulas**

DCF/PV: $P_n = \phi^\top D_n$
Risk-Neutral $q_\omega \equiv \frac{\phi_\omega}{\sum_{\omega'} \phi_{\omega'}}$
$P_n = \frac{\mathbb{E}^{\mathbb{Q}}[D_n]}{1 + r_f}$
SPD/SDF: $P_n = \mathbb{E}^{\mathbb{P}}[\eta D^n]$
$R_f = \mathbb{E}^{\mathbb{P}}[\eta]^{-1}$

**Risk Premia**

$\mathbb{E}\left[R_n - R_f\right] = -R_f\,\mathrm{Cov}\left[\eta, R_n - R_f\right]$
Lucas: $\mathbb{E}\left[R_n - R_f\right] \approx -\gamma\,\mathrm{Cov}\left[R_n, \frac{c_1}{c_0}\right]$

**SPD/SDF Bounds**

$$\frac{\sqrt{\mathrm{Var}^{\mathbb{P}}[\eta]}}{\mathbb{E}^{\mathbb{P}}[\eta]} \geq \frac{\mathbb{E}^{\mathbb{P}}[r_n - r_f]}{\sqrt{\mathrm{Var}^{\mathbb{P}}[r_n - r_f]}}$$

## 5 Arbitrage

**Definition**

1) $P^T \theta \leq 0$,
2) $D\theta \geq 0$,
3) One or both of above strict;
No Arbitrage in frictionless mkt.

**No Arbitrage**

$\exists V(\cdot)$ pricing operator
$P = V(d)$; Law of One Price:
$V(d_1) = V(d_2) \iff d_1 = d_2$;
$V$ monotonic & linear
FTAP: $\exists \phi : P^\top = \phi^\top D$
NA $\iff \exists \phi > 0$

## 6 Corporate Finance

**Opt Prod/Investment**

Production: $y_{1\omega} = y_\omega(y_0)$
NPV: $v = \sum_{\omega \in \Omega} \phi_\omega y_\omega(y_0) - y_0$
Max NPV *or* utility

**Capital Structure**

$y_0 = d_0 + e_0$
$D = \phi^\top d_1$
$E = \phi^\top e_1$

**$q$ Theory**

$q_j \equiv \frac{\mathbb{E}^{\mathbb{Q}}[A_j]}{1 + r_f}$

## 7 Fixed Income

**Term Structure**

$P_{N,t} = \mathbb{E}_t\left[\prod_{i=t+1}^{t+N} \eta_i\right]$
$P_{N,t} = Y_{N,t}^{-N}$
Nominal: $P_{N,t}^{\$} = P_{N,t} \times \mathrm{CPI}_t$
$\Pi_t = \mathrm{CPI}_t / \mathrm{CPI}_{t-1}$
$1 = \mathbb{E}_t\left[\frac{\eta_{t+1}}{\Pi_{t+1}} \frac{P_{N-1,t+1}^{\$}}{P_{N,t}^{\$}}\right]$
$\eta_{t+1}^{\$} = \frac{\eta_{t+1}}{\Pi_{t+1}}$ nominal SDF
Real: $1 = \mathbb{E}_t\left[\eta_{t+1} \frac{P_{N-1,t+1}}{P_{N,t}}\right]$

## 8 Option Pricing

**Derivative Defn.**

Payoff $f(X_T)$ at $t = T$
$P = \mathbb{E}_t\left[f(X_T) \prod_{i=t+1}^T \eta_i\right]$
$P = \sum_{\omega \in \Omega} \phi_\omega f(X_\omega)$
$P = \int \phi(\omega) f(X(\omega))\, d\omega$

**Options**

$[x]_+ \equiv \max[0, x]$
$c_1 = [S_1 - K]_+, p_1 = [K - S_1]_+$
$P_X = \mathbb{E}[\eta X] = \mathrm{Cov}[\eta, X] + R_f^{-1}\mathbb{E}[X]$
Intrinsic value, ITM/ATM/OTM;
Prices convex in $K$
$S \geq c(S, K) \geq \left[S - \frac{K}{1 + r_f}\right]_+$
$c(S, K) - p(S, K) = S - \frac{K}{1 + r_f}$

**Backing Out $\mathbb{Q}$**

$q(K) = \frac{\partial^2 c(K)}{\partial K^2} \Big/ \int_0^\infty \frac{\partial^2 c(K)}{\partial K^2}\, dK$

## 9 Arbitrage Pricing

**Decomposition**

$r_{n,t} - r_f = \alpha_n + \beta_n\left(r_{-\eta,t} - r_f\right) + \varepsilon_{n,t}$
$\mathrm{Var}\left[r_n - r_f\right] = \beta_n^2\,\mathrm{Var}\left[r_{-\eta} - r_f\right] + \mathrm{Var}[\varepsilon_n]$

**Factors**

$F$ basis for $R - \overline{R}_n$
$R_n = \overline{R}_n + F\beta_n$
$\mathbb{E}^{\mathbb{Q}}[R_n] = R_f$ APT: $\overline{R}_n - R_f = \lambda^\top \beta_n$,
where $\lambda_k = -\mathbb{E}^{\mathbb{Q}}[F_k]$
$\lambda_k$ factor premium

**Diversification**

Portfolio $\theta$ *well diversified* if
$\theta_n = O\left(\frac{1}{n}\right)$
Sequence $\{\theta_i\}$, $|\theta_1| = 1$
well diversified if $\exists 0 < \kappa < \infty$ s.t. $\theta_{n,j}^2 < \frac{\kappa}{n^2}$.

**APT**

Asymptotic: $\mathbb{E}\left[r_{\theta_n}\right] \to \alpha > 0$,
but $\mathrm{Var}\left[r_{\theta_n}\right] \to 0$.
No asymptotic arb. $\implies \exists r_f, \lambda, A$:
$\sum_{i=1}^n \left[\overline{r}_i - \left(r_f + \lambda^\top \beta_i\right)\right]^2 = \sum_{i=1}^n \left[\overline{r}_i - \left(r_f + \sum_k \lambda_k \beta_{ik}\right)\right]^2 < A < \infty$.