15.085 Midterm Cheat Sheet
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1 Prob. Models & Measures

Prob. Experiments

Probability space: $(\Omega, \mathcal{F}, \mathbb{P})$
–Ω sample space
–$\mathcal{F}$ σ-field
–$\mathbb{P}$ prob. measure

Disc. Prob. Space

Ω finite or countable,
$\mathcal{F}$ set of all subsets of Ω
$\mathbb{P} : \Omega \to [0,1]$ sums to 1

σ-fields

(a) $\emptyset \in \mathcal{F}$
(b) $A \in \mathcal{F} \implies A^c \in \mathcal{F}$
(c) $\{A_i\} \subseteq \mathcal{F} \implies \bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$
Set $A \in \mathcal{F}$ "event"/"measurable,"
$(\Omega, \mathcal{F})$ *measurable space*
$\mathcal{F} = \bigcap_{s \in S} \mathcal{F}_s$ also σ-field

Prob. Measures

Measure: $\mu : \mathcal{F} \to [0, \infty]$
(a) $\mu(\emptyset) = 0$;
(b) Countable additivity: $\{A_i\} \subseteq \mathcal{F}$ disjoint
$\implies \mu(\cup_i A_i) = \sum_{i=1}^{\infty} \mu(A_i)$
Prob. measure also has $\mathbb{P}(\Omega) = 1$
Field: like σ-field but finite

Continuity

Sequence of sets converge to union/intersection

2 Fundamental Models

Carathéodory's extension thm.

$\mathcal{F}_0$ field, $\mathcal{F}$ σ-field
$\mathbb{P}_0 : \mathcal{F}_0 \to [0,1]$, $\mathbb{P}_0(\Omega) = 1$
$\mathbb{P}_0$ yields $\mathbb{P}$ on $(\Omega, \mathcal{F})$

Lebesgue measure

Uniform measure on $[0,1]$

Borel σ-field

$\mathcal{B}$: smallest σ-field including every interval $[a,b] \subset [0,1]$
$A \subset [0,1]$, $A \in \mathcal{B}$ Borel set

3 Conditioning & Independence

Conditional probability

$\mathbb{P}(A \mid B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$
$\mathbb{P}_B(A) = \mathbb{P}(A \mid B)$
Bayes' rule: $\mathbb{P}_A(B_i) = \frac{\mathbb{P}(B_i)\mathbb{P}_{B_i}(A)}{\mathbb{P}(A)}$
$\mathbb{P}\left(\cap_{i=1}^{\infty} A_i\right) = \mathbb{P}(A_1) \prod_{i=2}^{\infty} \mathbb{P}\left(A_i \mid \cap_{j=1}^{i-1} A_j\right)$

Independence

Defn.: $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$
$\mathcal{F}_1, \mathcal{F}_2$ σ-fields indep. iff any $A_1 \in \mathcal{F}_1$ and $A_2 \in \mathcal{F}_2$ indep.

Borel-Cantelli lemma

Sequence of events $\{A_n\}$,
$A = \{A_n \text{ i.o.}\} = \bigcap_{n=1}^{\infty} \bigcup_{i=n}^{\infty} A_i$
(a) $\sum_{n=1}^{\infty} \mathbb{P}(A_n) < \infty \implies \mathbb{P}(A) = 0$
(b) $\sum_{n=1}^{\infty} \mathbb{P}(A_n) = \infty \implies \mathbb{P}(A) = 1$ if events A_n, $n \in \mathbb{N}$ indep.

Summation lemma

If $0 \le p_i \le 1$, $\forall i \in \mathbb{N}$, and $\sum_{i=1}^{\infty} p_i = \infty$, then $\prod_{i=1}^{\infty} (1 - p_i) = 0$

4 Combinatorial prob.

e

$\lim_{n \to \infty} \left(1 + \frac{r}{n}\right)^n = e^r$

Stirling's approx.

$n! \approx \sqrt{2\pi n} \left(\frac{n}{e}\right)^n$

5 Random Variables

Definition

(a) $X : \Omega \to \mathbb{R}$ s.t. $\{\omega \mid X(\omega) \le c\}$ $\mathcal{F}$-measurable $\forall c \in \mathbb{R}$
(b) Extended-valued r.v. if $\forall c \in \overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$
$X^{-1}(B) = \{X \in B\} = \{\omega \mid X(\omega) \in B\}$
Prob. law: $\mathbb{P}_X : \mathcal{B} \to [0,1]$, $B \mapsto \mathbb{P}(X \in B)$ (measure on $(\mathbb{R}, \mathcal{B})$)