15.085 Midterm Cheat Sheet
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1 Prob. Models & Measures

Prob. Experiments

Probability space: $(\Omega, \mathcal{F}, \mathbb{P})$
–Ω sample space
–$\mathcal{F}$ σ-field
–$\mathbb{P}$ prob. measure

Disc. Prob. Space

Ω finite or countable,
$\mathcal{F}$ set of all subsets of Ω
$\mathbb{P} : \Omega \to [0,1]$ sums to 1

σ-fields

(a) $\emptyset \in \mathcal{F}$
(b) $A \in \mathcal{F} \implies A^c \in \mathcal{F}$
(c) $\{A_i\} \subseteq \mathcal{F} \implies \bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$
Set $A \in \mathcal{F}$ "event"/"measurable,"
$(\Omega, \mathcal{F})$ *measurable space*
$\mathcal{F} = \bigcap_{s \in S} \mathcal{F}_s$ also σ-field

Prob. Measures

Measure: $\mu : \mathcal{F} \to [0,\infty]$
(a) $\mu(\emptyset) = 0$;
(b) Countable additivity: $\{A_i\} \subseteq \mathcal{F}$ disjoint
$\implies \mu(\cup_i A_i) = \sum_{i=1}^{\infty} \mu(A_i)$
Prob. measure also has $\mathbb{P}(\Omega) = 1$
Field: like σ-field but finite

Continuity

Sequence of sets converge to union/intersection

2 Fundamental Models

Carathéodory's extension thm.

$\mathcal{F}_0$ field, $\mathcal{F}$ σ-field
$\mathbb{P}_0 : \mathcal{F}_0 \to [0,1]$, $\mathbb{P}_0(\Omega) = 1$
$\mathbb{P}_0$ yields $\mathbb{P}$ on $(\Omega, \mathcal{F})$

Lebesgue measure

Uniform measure on $[0,1]$

Borel σ-field

$\mathcal{B}$: smallest σ-field including every interval $[a,b] \subset [0,1]$
$A \subset [0,1]$, $A \in \mathcal{B}$ Borel set

3 Conditioning & Independence

Conditional probability

$\mathbb{P}(A \mid B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$
$\mathbb{P}_B(A) = \mathbb{P}(A \mid B)$
Bayes' rule: $\mathbb{P}_A(B_i) = \frac{\mathbb{P}(B_i)\mathbb{P}_{B_i}(A)}{\mathbb{P}(A)}$
$\mathbb{P}\left(\cap_{i=1}^{\infty} A_i\right) = \mathbb{P}(A_1) \prod_{i=2}^{\infty} \mathbb{P}\left(A_i \mid \cap_{j=1}^{i-1} A_j\right)$

Independence

Defn.: $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$
$\mathcal{F}_1, \mathcal{F}_2$ σ-fields indep. iff any $A_1 \in \mathcal{F}_1$ and $A_2 \in \mathcal{F}_2$ indep.

Borel-Cantelli lemma

Sequence of events $\{A_n\}$,
$A = \{A_n \text{ i.o.}\} = \bigcap_{n=1}^{\infty} \bigcup_{i=n}^{\infty} A_i$
(a) $\sum_{n=1}^{\infty} \mathbb{P}(A_n) < \infty \implies \mathbb{P}(A) = 0$
(b) $\sum_{n=1}^{\infty} \mathbb{P}(A_n) = \infty \implies \mathbb{P}(A) = 1$ if events A_n, $n \in \mathbb{N}$ indep.

Summation lemma

If $0 \le p_i \le 1$, $\forall i \in \mathbb{N}$, and $\sum_{i=1}^{\infty} p_i = \infty$, then $\prod_{i=1}^{\infty} (1 - p_i) = 0$

4 Combinatorial prob.

e

$\lim_{n\to\infty} \left(1 + \frac{r}{n}\right)^n = e^r$

Stirling's approx.

$n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$

5 Random Variables

Definition

(a) $X : \Omega \to \mathbb{R}$ s.t. $\{\omega \mid X(\omega) \le c\}$ $\mathcal{F}$-measurable $\forall c \in \mathbb{R}$
(b) Extended-valued r.v. if $\forall c \in \overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$
$X^{-1}(B) = \{X \in B\} = \{\omega \mid X(\omega) \in B\}$
Prob. law: $\mathbb{P}_X : \mathcal{B} \to [0,1]$, $B \mapsto \mathbb{P}(X \in B)$ (measure on $(\mathbb{R}, \mathcal{B})$)
$(\mathcal{F}_1, \mathcal{F}_2)$-measurable func.: $f : \Omega_1 \to \Omega_2$ s.t. $f^{-1}(B) \in \mathcal{F}_1, \forall B \in \mathcal{F}_2$
For $A \in \mathcal{F}$, I_A is $(\mathcal{F}, \mathcal{B})$-measurable

CDFs

Defn.: $F_X : \mathbb{R} \to [0,1]$, $x \mapsto \mathbb{P}(X \le x)$
(a) Monotonicity
(b) $\lim_{x\to-\infty} F_X(x) = 0$, $\lim_{x\to\infty} F_X(x) = 1$
(c) Right-continuity

Discrete RV's

Range $X(\Omega)$ finite/countable
$p_X : \mathbb{R} \to [0,1]$, $x \mapsto \mathbb{P}(X = x)$ PMF

Continuous RV's

$F_X(x) = \int_{-\infty}^{x} f(t)\,dt$, f PDF

6 Discrete RV's

Examples

Uniform: $p_X(k) = \frac{1}{b-a+1}$
Bernoulli: $p_X(1) = p$, $p_X(0) = 1 - p$
Binomial: $p_X(k) = \binom{n}{k} p^k (1-p)^{n-k}$
Geometric: $p_X(k) = (1-p)^{k-1} p$
Poisson: $p_X(k) = e^{-\lambda} \frac{\lambda^k}{k!}$
Power law: $p_X(k) = \frac{1}{k^\alpha} - \frac{1}{(k+1)^\alpha}$

7 More Discrete RV's

Expected Values

Bernoulli: $\mathbb{E}[X] = p$, $\mathrm{Var}(X) = p(1-p)$
Binomial: $\mathbb{E}[X] = np$, $\mathrm{Var}(X) = np(1-p)$
Geometric: $\mathbb{E}[X] = \frac{1}{p}$, $\mathrm{Var}(X) = \frac{1-p}{p^2}$
Poisson: $\mathbb{E}[X] = \lambda$, $\mathrm{Var}(X) = \lambda$
Power law: $\mathbb{E}[X] = \sum_{k=0}^{\infty} \frac{1}{(k+1)^\alpha}$

Cov. & Corr.

$\mathrm{Cov}(X,Y) = \mathbb{E}[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])]$
$\rho = \frac{\mathrm{Cov}(X,Y)}{\sigma_X \sigma_Y}$ Cauchy-Schwarz ineq: $(\mathbb{E}[XY])^2 \le \mathbb{E}\left[X^2\right]\mathbb{E}\left[Y^2\right]$

Conditional Expectations

$\mathbb{E}[\mathbb{E}[X \mid Y] g(Y)] = \mathbb{E}[X g(Y)]$

8 Continuous RV's

Examples

Uniform: $F_X(x) = \frac{x-a}{b-a}$
Exponential: $F_X(x) = 1 - e^{-\lambda x}$
Normal: $f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$
Cauchy: $f_X(x) = \frac{1}{\pi(1+x^2)}$
Power law: $f_X(t) = \frac{\alpha c^\alpha}{t^{\alpha+1}}$

Exp. Value

$\mathbb{E}[X] = \int_0^{\infty} (1 - F_X(t))\,dt$ for X non-negative

Joint Dist.'s

$F_{X,Y} = \mathbb{P}(X \le x, Y \le y)$
$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy$

Independence

$F_{X,Y} = F_X(x) F_Y(y)$

9 More Cont. RV's

Conditional PDFs

$f_{X|Y}(x \mid y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$

Bivariate Normal Dist.

f_X, f_Y Normal PDF's

Mixed Bayes' Rule

Sums for discrete RV, prods for cont. RV

10 Derived Dist.'s

Func. of Single RV

If $Y = g(X)$, calculate $F_Y(y) = \mathbb{P}(g(X) \le y) = \int_{\{x|g(x)\le y\}} f_X(x)\,dx$
Then $f_Y(y) = \frac{dF_Y}{dy}(y)$

Multivariate

$f_Y(y) = f_X\left(M^{-1} y\right) \cdot \left|M^{-1}\right|$

Max & Min of RV's

$\mathbb{P}\left(\max_j X_j \le x\right) = F_{X_1}(x) \cdots F_{X_n}(x)$
$\mathbb{P}\left(\min_j X_j \le x\right) = 1 - \left(1 - F_{X_1}(x)\right) \cdots \left(1 - F_{X_n}(x)\right)$

Convolution

$p_{X+Y}(z) = \sum_x p_X(x) p_Y(z - x)$
$f_{X+Y}(z) = \int_{-\infty}^{\infty} f_{X,Y}(x, z - x)\,dx$

11 Abstract Integration I

12 Abstract Integration II

Fatou's Lemma

Y s.t. $\mathbb{E}[|Y|] < \infty$: (a) If $Y \le X_n \forall n$, then $\mathbb{E}[\liminf_{n\to\infty} X_n] \le \liminf_{n\to\infty} \mathbb{E}[X_n]$
(b) If $X_n \le Y \forall n$, then $\mathbb{E}[\limsup_{n\to\infty} X_n] \ge \limsup_{n\to\infty} \mathbb{E}[X_n]$