BUSI4496
Supply Chain Planning & Management

Lecture 5: Inventory Management

Agenda

- Inventory modelling
  - Key Characteristics
  - Inventory costs
  - Inventory policies

- Deterministic models:

  - EOQ: Economic order quantity
  - POQ: Production order quantity

- Stochastic models:

  - (R,Q): Reorder point – order quantity model

Inventory System

- Inventory is the stock of any item or resource used in an organisation and can include: raw materials, component parts, supplies, work-in-process and finished goods.

- An inventory system is the set of policies and controls that monitors levels of inventory and determines what levels should be maintained, when stock should be replenished, and how large orders should be.

This distinction is important:
Companies don’t just need to hold **inventory**, they need a **structured way** of deciding how much to hold, when to reorder, and in what quantities, in order to minimise costs and ensure smooth operations.

Inventory modelling: Key characteristics

- **Demand**
  - Constant ↔ variable
  - Known (deterministic) ↔ random (stochastic)
- **Lead time (τ)**
  - External order: time between placement of an order until arrival of goods
  - Internal production: amount of time required to produce a batch of items
  - Lead time can be deterministic or random
- **Replenishment**
  - How does the order arrive? uniform over time, instantaneous, batches
- **Review time**
  - Continuous review: inventory level is known at all times
  - Periodic review: inventory level is known only at discrete points in time

Inventory modelling: Key characteristics

- **Excess demand**
  - Back ordering: excess is satisfied in the future
  - Lost sales: excess demand is lost
  - Partial backordering
- **Changing inventory**
  - Inventory may change over time: limited shelf life (perishable goods - food), obsolescence (e.g. automotive parts)
- **Single vs. multi period**
  - Short selling period (news vendor problem)
- **Single vs. multiple stocking points**
  - Multi-echelon inventory systems
- **Single vs. multiple items**
  - Overall constraints on budget or space
  - Coordinated control, joint replenishment

Inventory Modelling: Costs

Optimisation criterion in models: **cost minimisation**

- **Holding/ carrying costs (h)**
  - Cost for storage, handling, tax / insurance, breakage deterioration, obsolescence, opportunity cost of alternative investment (cost of capital), etc.
  - **h = Ic** (with I = annual interest rate e.g. 20-35%)
  - Dimension of h: £ per unit per year
  - Total holding cost is proportional to the amount of inventory on hand
- **Production (set up or production change over)/ Ordering costs (K, c)**
  - Cost of arranging specific equipment setups or someone placing an order
  - Fixed and variable part (x units): **C(x) = K + cx**
- **Penalty (shortage or stock-out) costs (p)**
  - Lost sales, loss of goodwill
  - Assume p is charged on a per unit basis

Inventory Modelling: Service level

Penalty cost vs. service levels

Cost of stock-out p is difficult to estimate (intangible components: loss of goodwill, delays …)

So, we use Service level type, instead:

- Type 1 Service: Probability of not stocking-out during lead time (α)
- Type 2 Service (Fill rate): Proportion of demands immediately met from stock (β)

If you want to ensure customers rarely experience a stockout, you focus on **Type I**.
If your goal is to meet most of the demand even if some small shortages happen, you focus on **Type II**.

Inventory modelling: Service level

Order cycle*	Demand	Stock-outs
1	180	0
2	75	0
3	235	45
4	140	0
5	180	0
6	200	10
7	150	0
8	90	0
9	160	0
10	40	0

*Order cycle = time between two replenishments

Type 1 service:
- Probability of not stocking-out during lead time
  - $No° \ of \ cycles \ with \ no \ stock \ out \ = \ 8$
  - $Fraction \ of \ cycles \ with \ no \ stock \ out \ = \frac{8}{10}$
  - $\alpha = 80\%$

Type 2 service:
- Proportion of demands immediately met from stock
  - $total\_demand = 1450$
  - $stock\_outs = 55$
  - $Immediately\_satisfied = 1450 - 55 = 1395$
  - $Fraction \ met \ from \ stock \ = \frac{1395}{1450} = 0.962$
  - $B \ = 96\%$

Inventory policies

- Two basic questions:
  - Q1: When to order (or produce)?
  - Q2: How much to order (or produce)?
- Typical answers to Q1:
  - When inventory position is equal (or below) a level **R (or s)**
  - Every **T** time units
- Typical answers to Q2:
  - Order or produce **Q** units
  - Order or produce such that the inventory position becomes **S**
- Models
  - Continuous review: (R, Q) and (s, S)
  - Periodic review: (T, Q) and (T, S)
  - Values of parameters R (or s), T, Q, S?

Inventory Policy: (When, How much)

So, Key Parameters to know?	• Reorder point: **R (or s)** • Review interval: **T** • Order quantity: **Q** • Target level: **S**

Inventory Policy: (When, How much)

		WHEN to order/ produce?	
		Continuous Review ***R*** *(or* ***s****) level triggers when*	**Periodic Review** ***T*** *time unite triggers when*
HOW MUCH to order/ produce?	**Fixed** fixed amount of **Q** each time	**(R, Q)** When Inventory position ≤ R → Order/ Produce fixed Q	**(T, Q)** Every T → Order/ Produce fixed Q
	Variable to reach target level of **S** each time	**(s, S)** When Inventory position ≤ s → Order/ Produce up to S	**(T, S)** Every T → Order/ Produce up to S

So, Key Parameters to know?	• Reorder point: **R (or s)** • Review interval: **T** • Order quantity: **Q** • Target level: **S**

Inventory policies: Inventory position

We reorder based on **Inventory Position**, not just what's on the shelf.

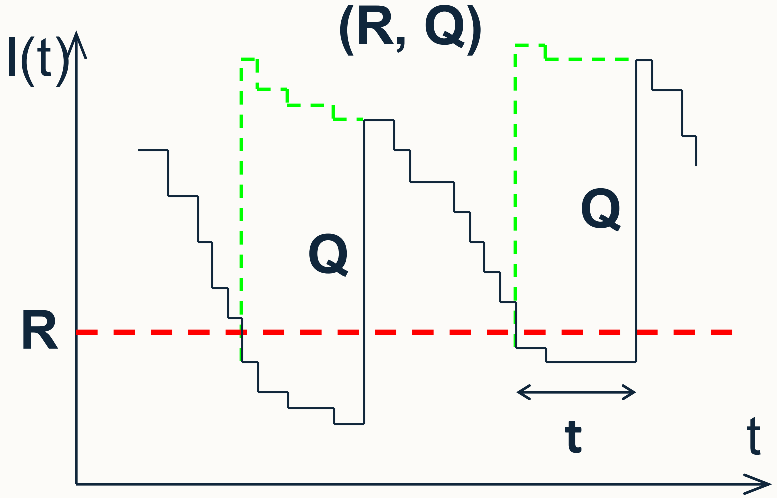
Why? It accounts for:

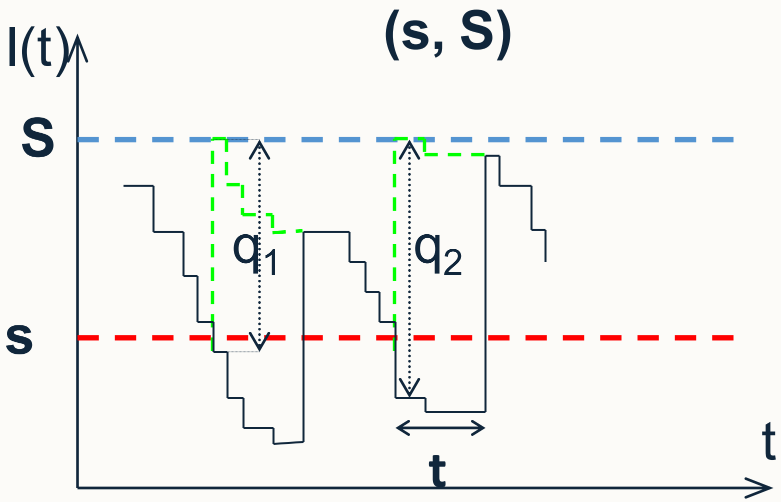
- *Stock-on-hand:* Inventory physically available at warehouse. (always ≥ 0)
- *Stock-on-order:* on the way.
- *Backorders:* owed to customers.

$$\text{Inventory } Position = Stock{-}on{-}hand + Stock{-}on{-}order - Backorders$$

*Inventory Position is the true measure that decides **when to reorder.***

Inventory policies





(R, Q): continuous review

Reorder point – reorder quantity model: order Q units when the *inventory position* reaches level R. The Q units arrive in stock after lead time t.

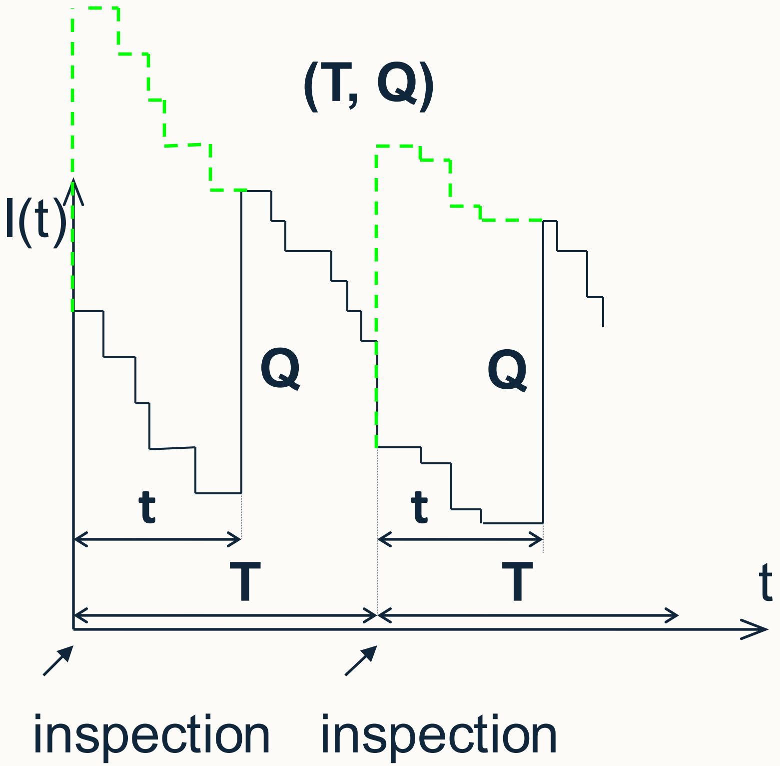
'black line' denotes inventory level = stock on hand

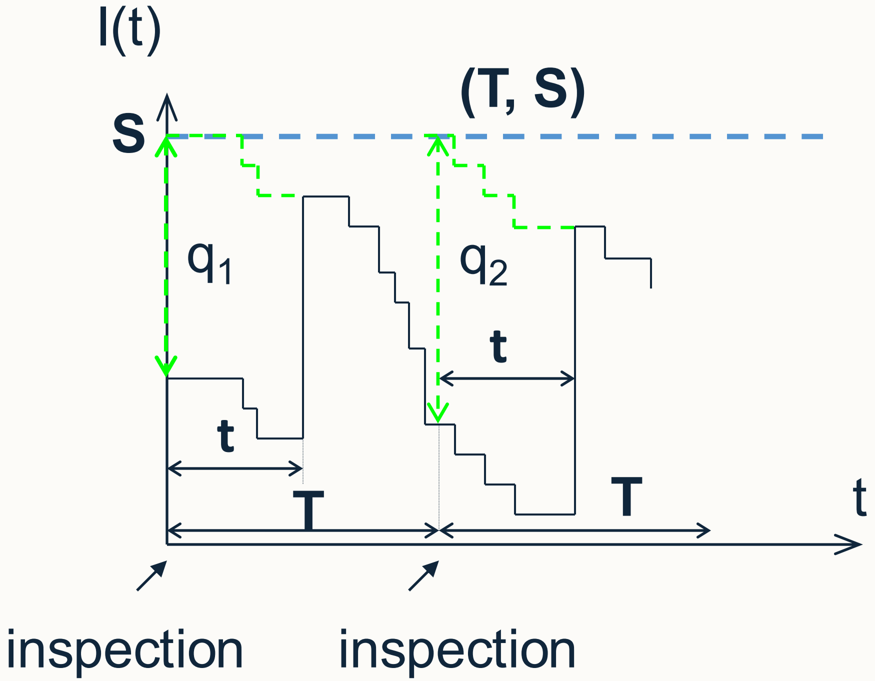
'green dotted line' denotes the inventory position = stock on hand plus stock on order (in-transit) – backorders

(s, S): continuous review

Reorder point – order up to level model: when the *inventory position* reaches level s, order an amount to bring it back to target level S.

Inventory policies





(T, Q): periodic review - reorder quantity model: at the beginning of every review period T, order Q units.
The Q units arrive in stock after lead time t.

(T, S): periodic review – order up to model: at the beginning of every review period T, order an amount to bring the inventory position back to the target level S.
The units arrive in stock after lead time t.

Inventory Models: Deterministic Models

- EOQ
- POQ

Deterministic Inventory Models: Overview

Deterministic demand assumption:

- Demand is known and constant over time
- Lead time is fixed
- No uncertainty

Key models:

- **EOQ (Economic Order Quantity):** Optimal fixed order size
- **POQ / EPQ (Production Order Quantity):** EOQ adjusted for gradual replenishment

Deterministic models: EOQ

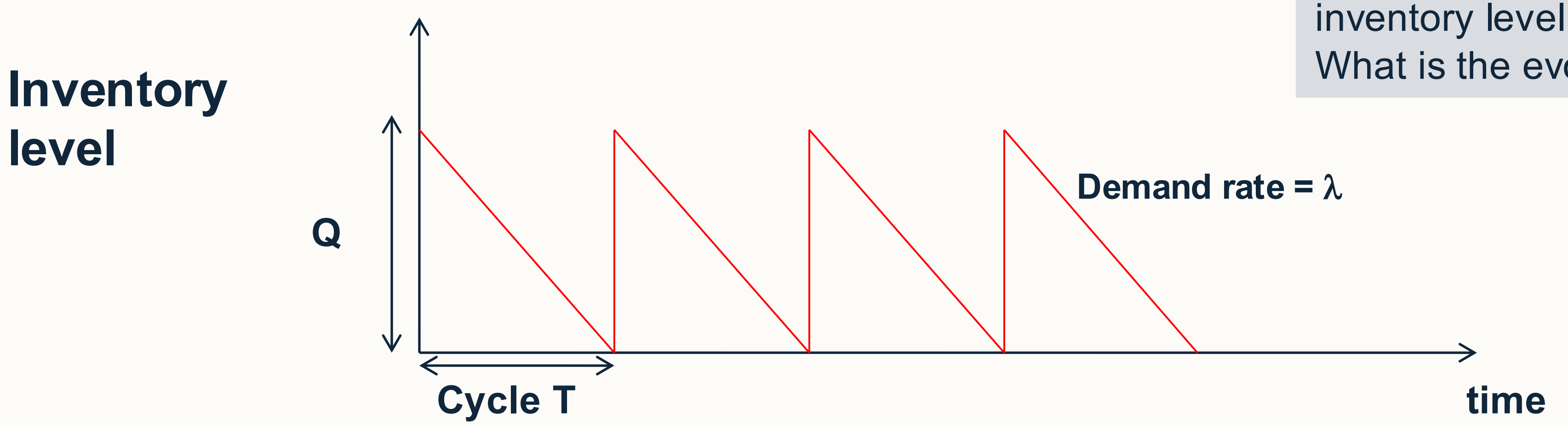
EOQ: Economic Order Quantity

- Simplest, most fundamental model, basis for more complex models;
- Trade-off: **fixed order costs** and **holding costs**
- Assumptions
  - Demand rate is **high**, **known** and **constant**: λ units / time
  - **Demand transactions** are **small** compared with the overall rate
  - **No shortages** permitted
  - **Zero lead time** (*will be relaxed later*)
  - Costs
    - Fixed setup (order) cost per order placed: K
    - Variable order cost per unit ordered: c
    - Holding cost per unit held per unit time: h

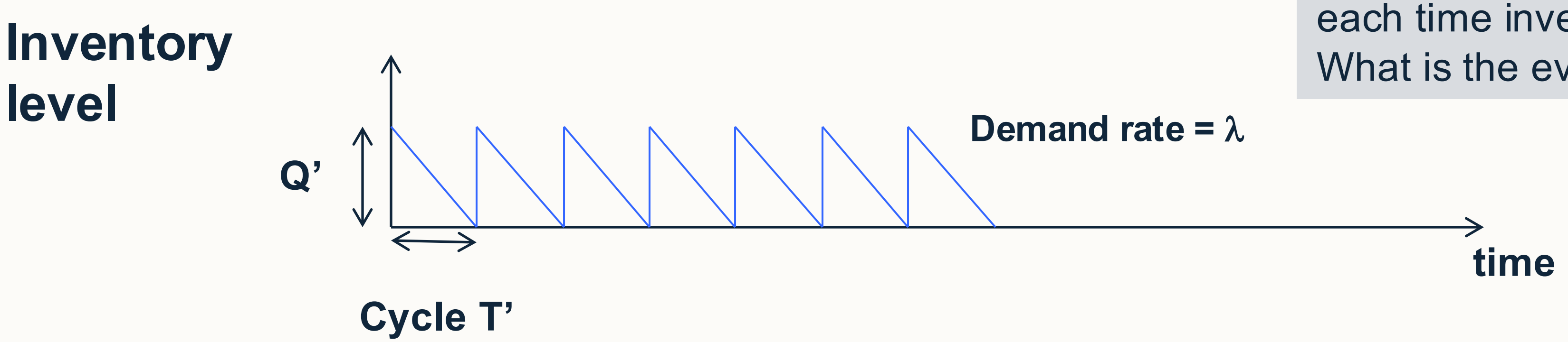
EOQ

Constant quantity to be ordered (when on-hand inventory is zero) = Q

Suppose I order an amount Q each time inventory level becomes zero.
What is the evolution of inventory over time?



Suppose I order an amount Q’ (with Q’ < Q) each time inventory level becomes zero.
What is the evolution of inventory over time?



- What is the optimal Q? → Minimise average cost per unit time G(Q)

EOQ

Average cost per unit time $G(Q)$=
Ordering cost per unit time + Holding cost per unit time.

- Ordering cost for each cycle T:

$$\boldsymbol{K + cQ}$$

- Ordering cost per unit time:

$$\frac{\boldsymbol{K + cQ}}{\boldsymbol{T}} \text{ and } \boldsymbol{T = \frac{Q}{\lambda}}$$

- Average inventory holding cost:

$$\boldsymbol{½hQ}$$

$$\boldsymbol{G(Q) = \frac{K + cQ}{T} + ½hQ}$$

$$= \frac{K+cQ}{\frac{Q}{\lambda}} + ½hQ$$

$$= \frac{K\lambda}{Q} + \lambda c + ½hQ$$

To minimise $G(Q)$ set 1st derivative = 0

$$G'(Q) = -\frac{K\lambda}{Q^2} + \frac{1}{2}\mathrm{h} = 0$$

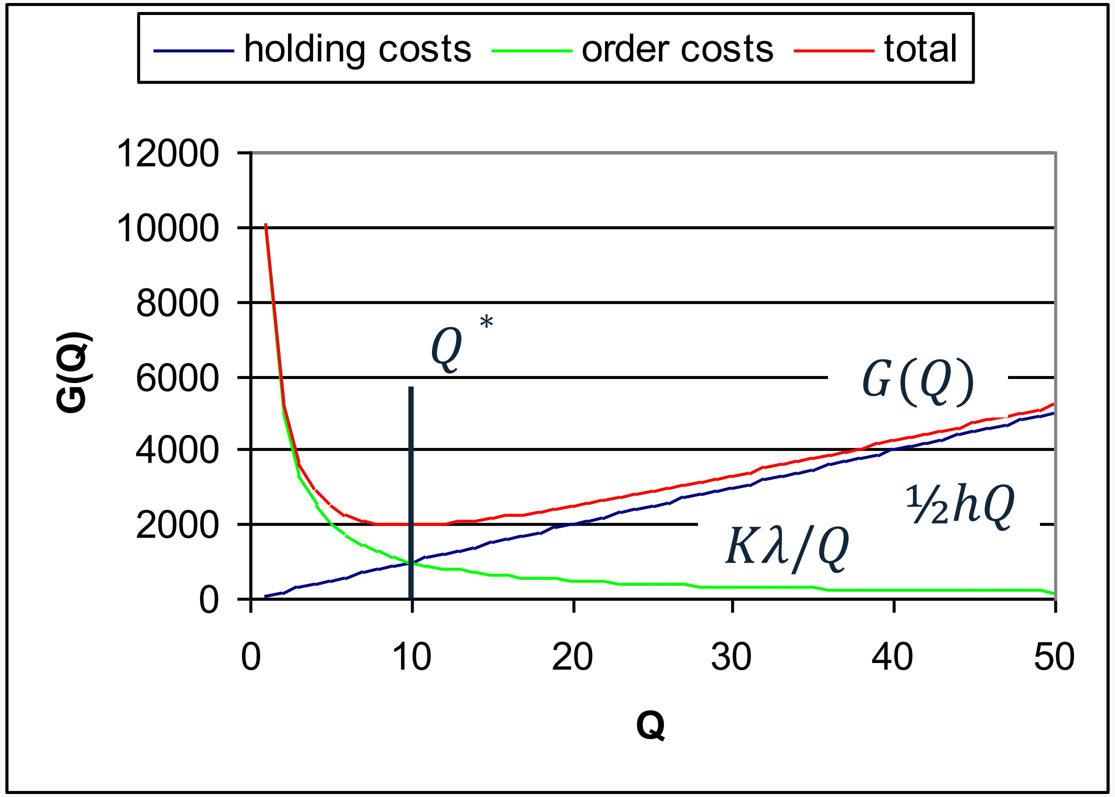
$$\boldsymbol{Q^* = EOQ = \left(\frac{2K\lambda}{h}\right)^{\frac{1}{2}}}$$

(Check $G''(Q) = \frac{2K\lambda}{Q^3} > 0$ for $Q > 0$)

EOQ

K = £100 per order; λ = 100 units per year; h = £200 per unit per year;

c = £5000 per unit



Note: c not in formula

at Q^*: $\frac{1}{2}hQ = K\lambda/Q$

Optimal cost G(Q*)

$$= \sqrt{(2K\lambda/h)}$$

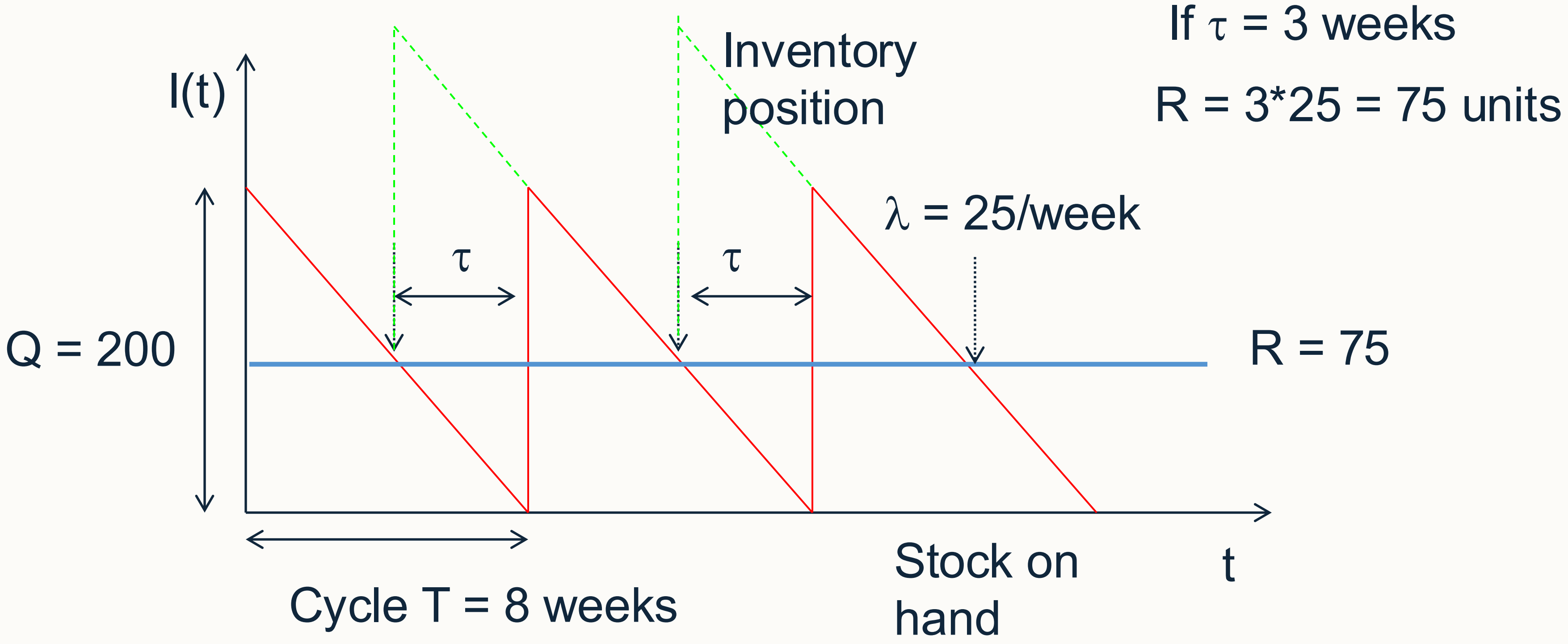
$$= \sqrt{(2 * 100 * 100/200)}$$

$$= \sqrt{100} = 10$$

G(Q) not very sensitive around Q^*

Inclusion of order lead time (τ)

- Order lead time τ
- Reorder point R = $\lambda\tau$ = demand during lead time = inventory position at order instant
- Inventory position I(t) = stock on hand + stock on order - backorders



EOQ

Earlier we found the economic order quantity, now let's find the reorder point,

- Demand $= 1000$ units per year
- Ordering costs K = £5 per order
- Holding costs h = £1.25 per unit per year
- Order lead time τ = 5 days

Recall that we need to find R i.e., the demand during lead time:

$$R = \lambda\,\tau, \quad \lambda = \frac{1000}{365}, \ \tau = 5$$

$$\Rightarrow R = \frac{1000}{365} * 5$$

$$= 13.69 \text{ units}$$

EOQ Summary

Goal is to find the order size Q^* that balances **ordering cost** and **holding cost**.

How much to order (Q*)?

EOQ formula: $Q^* = \sqrt{\dfrac{2K\lambda}{h}}$

- K =setup/order cost per order
- λ= demand rate (units/time)
- h= holding cost per unit per time

At Q*:

- Ordering cost = Holding cost
- Total cost minimized
- Purchase cost (c) drops out

Total cost curve is **flat near Q*** → rounding is OK.

When to order (R)?

Reorder Point: $R = \lambda\tau$

λ= demand rate

τ= lead time

Meaning:

- Place order when inventory position hits R
- So, replenishment arrives just in time

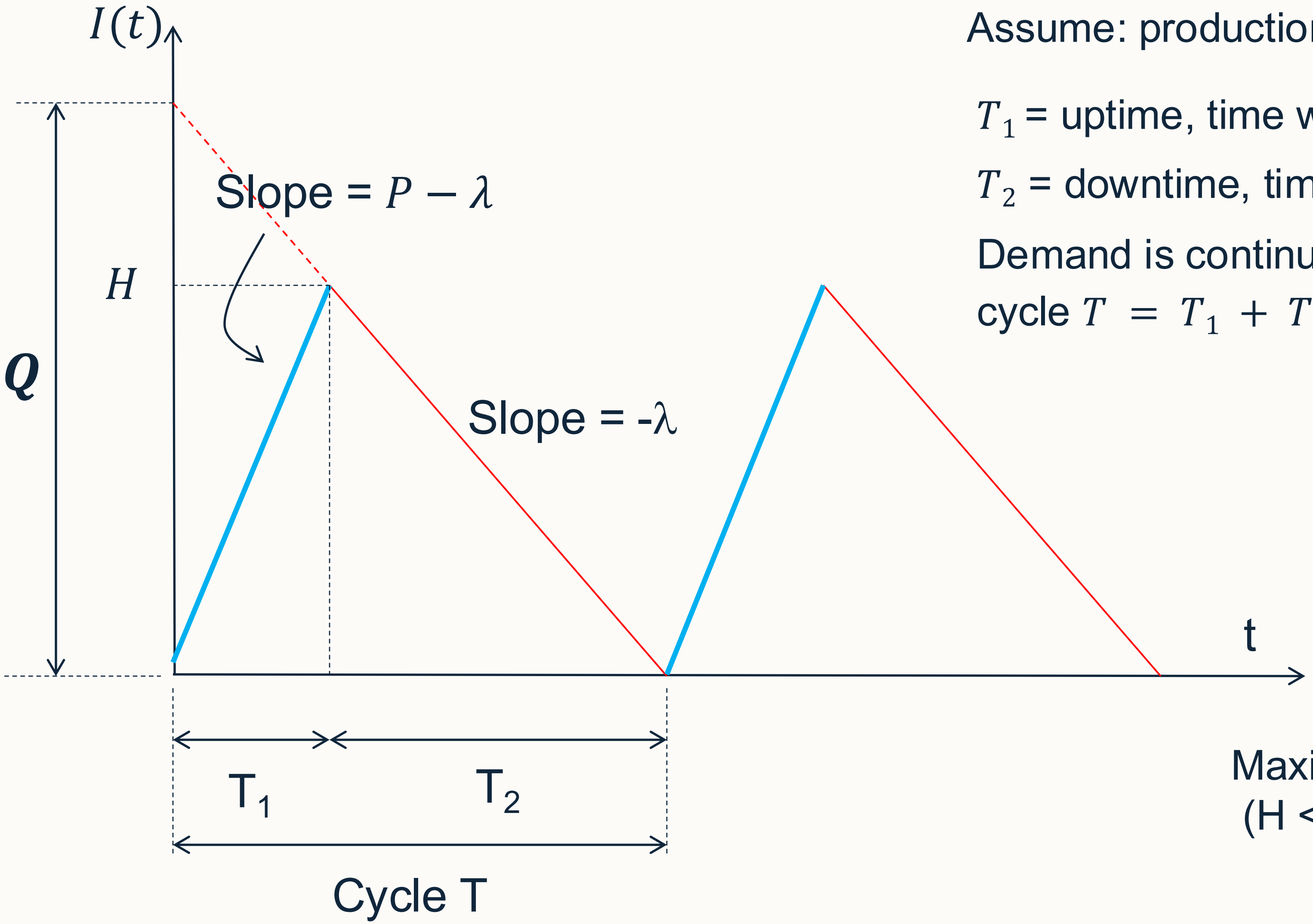
Deterministic models: POQ

POQ : Production Order Quantity

→ extension of EOQ to finite production rate

- **Assumptions**
  - Finite and **constant production rate**: P units / time
  - **Demand** rate is **known and constant**: λ units / time
  - P > λ for feasibility
  - **No shortages**
  - Costs
    - Fixed setup cost: K
    - (Ignore per unit production cost: c)
    - Holding cost per unit held per unit time: h

POQ



Assume: production batch Q

T_1 = uptime, time when machine is busy

T_2 = downtime, time when machine is idle

Demand is continuous, i.e. over entire cycle $T = T_1 + T_2$

Maximal inventory level = H
(H < Q production batch)

POQ

- Per cycle:

1. Number ordered (Q) = Number produced (PT_1) = Number consumed (λT)
   T_1 : Production run time; T: Cycle length

   - $T_1 = Q / P$
   - $H / T1 = P - \lambda$
   - $H = Q(1 - \lambda / P)$

- Average cost per unit time
  - $G(Q) = K/T + ½hH$
  - $G(Q) = K\lambda/Q + ½hQ(1 - \lambda / P)$
  - Define **$h' = h(1 - \lambda / P)$**
  - $G(Q) = K\lambda/Q + ½h'Q$

  $POQ = (2K\lambda/h')^{1/2}$

Other extensions

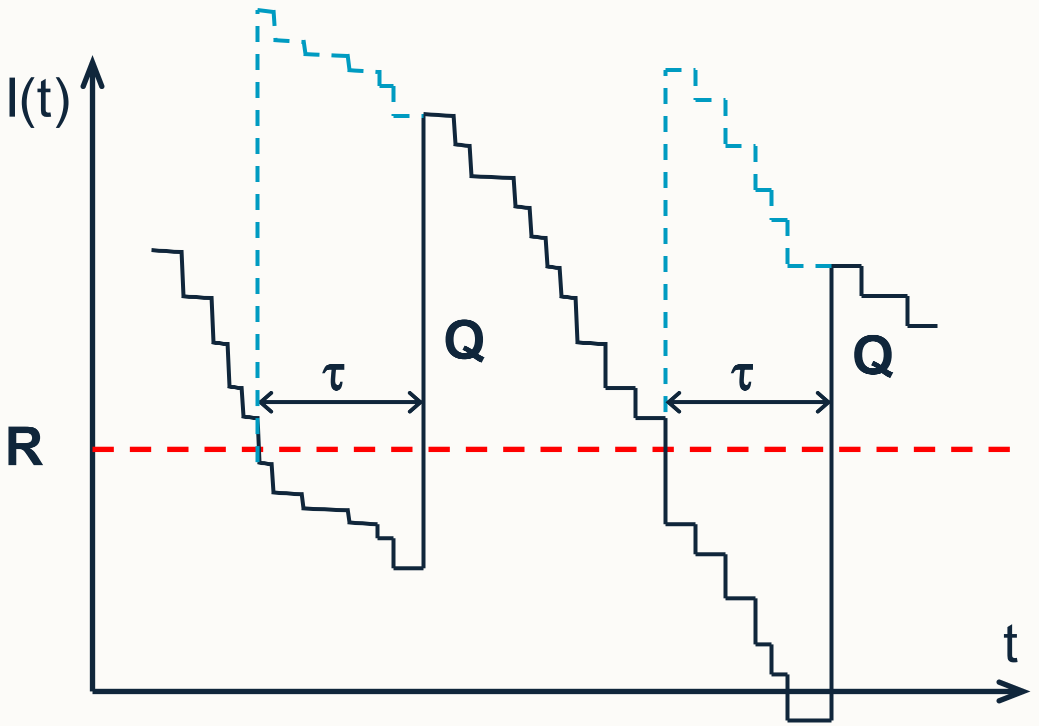
- There exist many other extensions of the basic EOQ, POQ model
  - Joint replenishment EOQ
  - Multi-product POQ
  - … $\rightarrow$ Advanced Operations Analysis

Inventory Models: Stochastic Model

- (R, Q)

Stochastic models

(R, Q) models: reorder point / reorder quantity systems – fast moving items



R = reorder point

Q = reorder quantity

τ = lead time

Inventory position = stock-on-hand + on order – backorders

Net stock = stock-on-hand – backorders

Rule: if inventory position ≤ R, order Q

Key assumptions (R, Q)-model

- Continuous review
- Stochastic but stationary demand; expected demand rate λ (units per year)
- Constant lead time τ (in years)
- **Stock-outs** can only **occur during the replenishment lead time**

  → in (R, Q) model, **we are interested in the demand D during the lead time** (probability distribution)

  - Expected demand during lead time τ : $E(D) = \mu = \lambda\tau$
  - Standard deviation of demand during lead time $\sigma = var(D)^{½}$
- Often it is assumed that demand during lead time is normally distributed (this is a good approximation for fast moving items)
- Decision variables: Q, R ?

Calculating R and Q: managerial approach (service level)

- Key idea: **decouple the calculation** of R and Q

(1) Order quantity Q:

use the EOQ formula

(2) Reorder point R in stochastic models:

R = **expected** demand during lead time **+ safety stock**

$R = \lambda\tau + SS$

How to calculate SS?

- Service level models: type 1 / type 2
- Cost model: specify penalty cost instead of service level
- (other assumptions such as backordering vs. lost sales etc)

Example service levels

- Demand rate: λ = 2400 per year; h → annual interest rate: 30%; c = £40 per unit; K = £500 per order; τ = 1 month; demand during lead time ~ N($\mu = \lambda\tau = 200$, $\sigma = 50$);

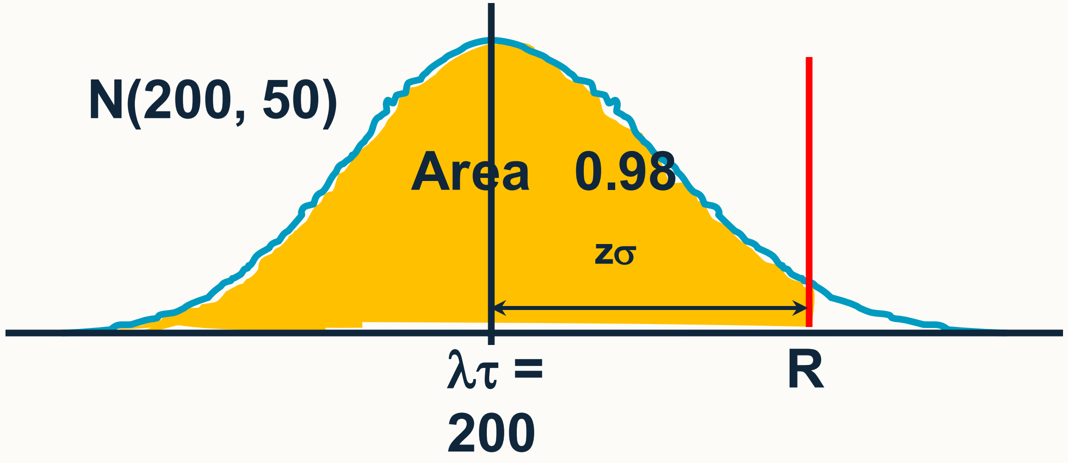
Calculate Q and R for type 1 and type 2 service of 98% ($\alpha = 0.98$ and $\beta = 0.98$)

- Q = $\text{EOQ} = \sqrt{\dfrac{2K\lambda}{h}} = \sqrt{\dfrac{2(500)(2400)}{(0.3)(40)}} \approx 447$

- R = $\lambda\tau$ + SS = 2400(1/12) + SS = 200 + SS

Example service levels (cont'd)

- **Type 1 service level:** probability of no stock-out during lead time (α = 0.98)



Reorder point R = $\lambda\tau$ + SS = $\lambda\tau + z\sigma$ with **z the safety stock factor** (higher z results in higher SS and higher service)

We want R such that **F(R) = Prob(demand during lead time ≤ R) = 0.98**; this corresponds to finding z from the standard normal distribution N(0, 1) **table (z, F(z))** such that F(z) = 0.98.

Looking at the table, we find z = 2.05

SS = $z\sigma$ = (2.05)(50) = 102.5 ≈ 103, thus R = 200 + 103 = 303

This means:
If we **reorder when stock reaches 303 units**,
We'll avoid a stockout in 98% of lead times.

Example service levels (cont'd)

- **Type 2 service level:** fill rate: fraction of demand immediately met from stock $\beta = 0.98$
  - Reorder point $R = \lambda\tau + SS = \lambda\tau + z\sigma$ with z the safety stock factor, but **now we calculate z in a different way !!**
  - Let n(R) be the expected number of units stock-out per order cycle
  - Q (= EOQ) is the expected demand per order cycle
  - We want R such that **$n(R)/Q = 1-\beta$**
  - For the normal distribution, we can relate n(R) to the standard loss function L(z) via **$n(R) = \sigma L(z)$**, with σ the standard deviation of demand during lead time
  - We can find **(z, L(z))** in the standard loss function table

Example service levels (type 2 cont'd)

- Type 2 service: fraction of demand immediately met from stock $\beta = 0.98$

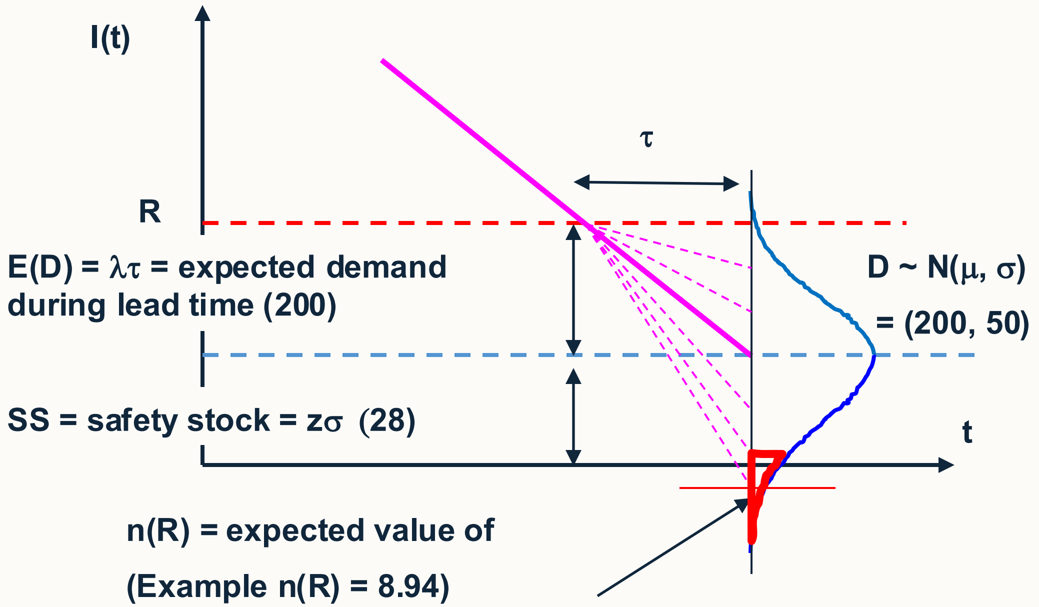
  $n(R)/Q = (1 - \beta)$ take $Q = EOQ$

  $n(R) = (1 - \beta)EOQ = (1- 0.98)447 = 8.94$

  $L(z) = n(R) / \sigma = 8.94 / 50 = 0.1788$

  Table $(z, L(z))$: for $L(z) = 0.1788$ we find $z = 0.563$

  $R = \lambda\tau + z\sigma = 200 + (0.563)(50) \approx$ **228**



Remarks (cont’d)

- There are many other stochastic inventory models
  - Cost optimisation vs service level models
  - Periodic review models
  - Models for slow moving items (S-1, S)
  - Models for items with a short selling period (newsboy models)
  - Multi-echelon models

  → Advanced Operations Analysis

References and recommended reading

- Nahmias, S. (2009). Production and Operations Analysis (6th edition), Singapore: McGraw Hill. (Chapters 4 & 5)

- Taylor III, B.W. (2016). Introduction to Management Science (12th edition), Harlow: Pearson (Chapter 17).

Thank you

Any Question?