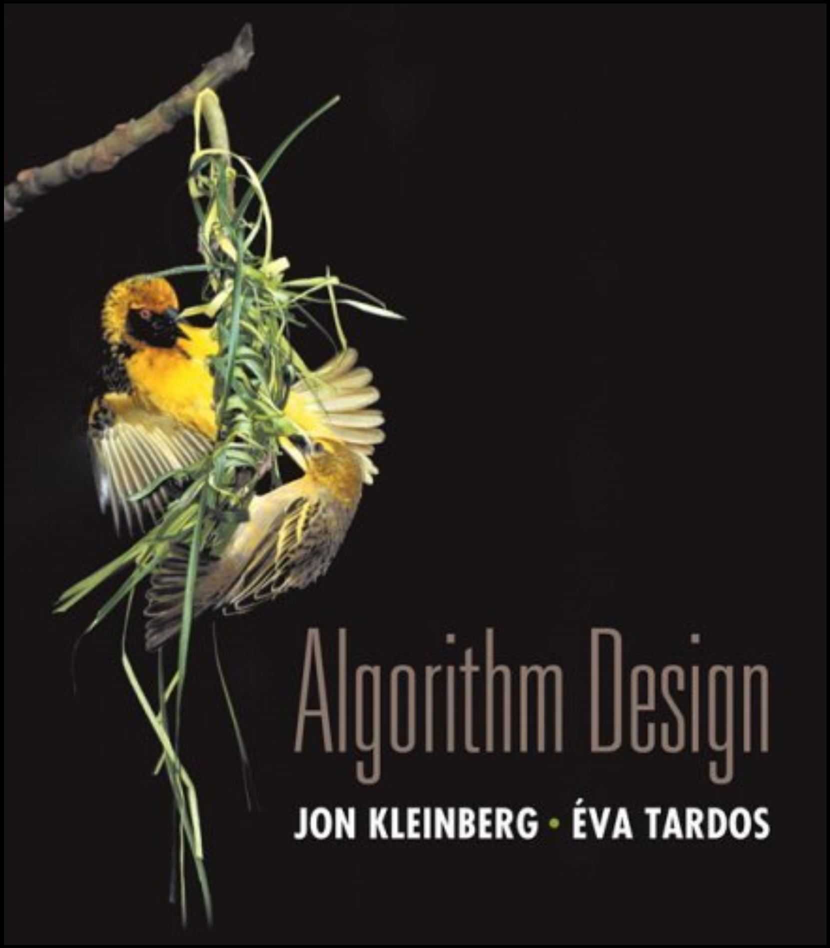

Chapter 8

NP and Computational Intractability

Slides by Kevin Wayne.
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Algorithm Design Patterns and Anti-Patterns

Algorithm design patterns.

- Greed.
- Divide-and-conquer.
- Dynamic programming.
- Duality.
- Reductions.
- Local search.
- Randomization.

Ex.

$O(n \log n)$ interval scheduling.
$O(n \log n)$ FFT.
$O(n^2)$ edit distance.
$O(n^3)$ bipartite matching.

Algorithm design anti-patterns.

- NP-completeness. — $O(n^k)$ algorithm unlikely.
- PSPACE-completeness. — $O(n^k)$ certification algorithm unlikely.
- Undecidability. — No algorithm possible.

8.1 Polynomial-Time Reductions

Bipartite Matching $\xrightarrow[\text{Polytime}]{\text{construct}}$ Flow Network $\underset{f^*}{\overset{G'}{\rightleftarrows}}$ $\boxed{\text{Max Flow}}$

$\xleftarrow{\quad}$ f^*

output : M

$X \underset{\text{convert}}{\overset{\text{construct}}{\rightleftarrows}} Y$

Classify Problems According to Computational Requirements

Q. Which problems will we be able to solve in practice?

A working definition. [Cobham 1964, Edmonds 1965, Rabin 1966]
Those with polynomial-time algorithms.

Yes	Probably no
Shortest path	Longest path
Matching	3D-matching
Min cut	Max cut
2-SAT	3-SAT
Planar 4-color	Planar 3-color
Bipartite vertex cover	Vertex cover

Primality testing	Factoring

Classify Problems

Desiderata. Classify problems according to those that can be solved in polynomial-time and those that cannot.

Provably requires exponential-time.

- Given a Turing machine, does it halt in at most k steps?
- Given a board position in an n-by-n generalization of chess, can black guarantee a win?

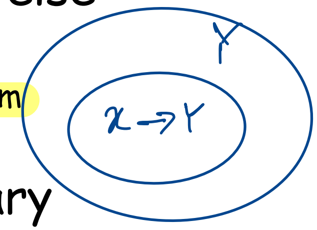
Frustrating news. Huge number of fundamental problems have defied classification for decades.

This chapter. Show that these fundamental problems are "computationally equivalent" and appear to be different manifestations of one really hard problem.

Polynomial-Time Reduction

Desiderata'. Suppose we could solve X in polynomial-time. What else could we solve in polynomial time?

don't confuse with reduces from



Reduction. Problem X polynomial reduces to problem Y if arbitrary instances of problem X can be solved using:

- Polynomial number of standard computational steps, plus
- Polynomial number of calls to oracle that solves problem Y.

computational model supplemented by special piece of hardware that solves instances of Y in a single step

- solving Y can solve 'x', but vice-versa not true.
- Thus solving 'x' is equal/easier to solve then 'Y'.

Notation. $X \leq_{P} Y$.

↳ polynomial time, e.g. 1st slide

Remarks.

- We pay for time to write down instances sent to black box $\Rightarrow$ instances of Y must be of polynomial size.
- Note: Cook reducibility.

in contrast to Karp reductions

Polynomial-Time Reduction

Purpose. Classify problems according to relative difficulty.

Design algorithms. If $\boxed{X \leq_P Y}$ and Y can be solved in polynomial-time, then X can also be solved in polynomial time.

Establish intractability. If $\boxed{X \leq_P Y}$ and X cannot be solved in polynomial-time, then Y cannot be solved in polynomial time.

Establish equivalence. If $X \leq_P Y$ and $Y \leq_P X$, we use notation $X \equiv_P Y$.

up to cost of reduction

Reduction By Simple Equivalence

Basic reduction strategies.

- Reduction by simple equivalence.
- Reduction from special case to general case.
- Reduction by encoding with gadgets.

Independent Set

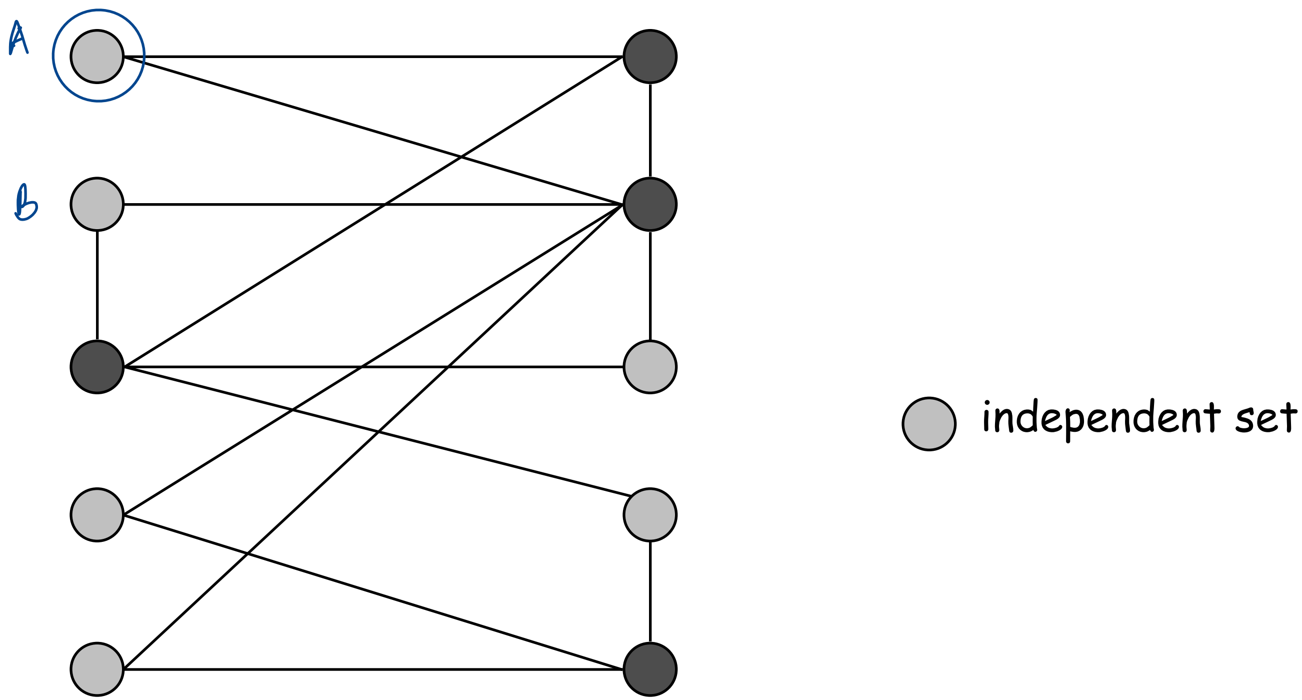
'Refer KT 8.1'

P1. **INDEPENDENT SET:** Given a graph $G = (V, E)$ and an integer k, is there a subset of vertices $S \subseteq V$ such that $|S| \geq k$, and for each edge at most one of its endpoints is in S?

- Influencer in SN, not connected

Ex. Is there an independent set of size ≥ 6? Yes.
Ex. Is there an independent set of size ≥ 7? No.

A & B has no exposure



Vertex Cover

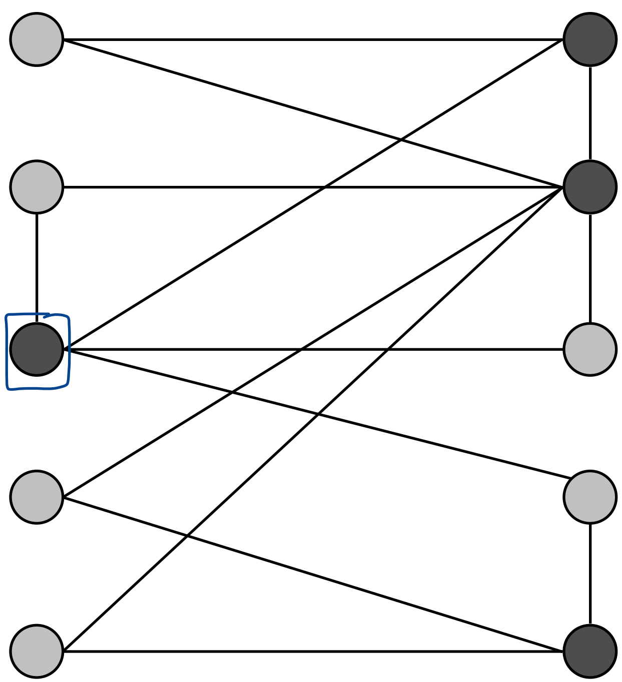
k = 2 (atmost?)
↳ but k ≥ n

VERTEX COVER: Given a graph $G = (V, E)$ and an integer k, is there a subset of vertices $S \subseteq V$ such that $|S| \leq k$, and for each edge, at least one of its endpoints is in S?

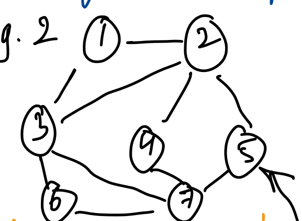
Ex. Is there a vertex cover of size ≤ 4? Yes.
Ex. Is there a vertex cover of size ≤ 3? No.

Influencer push 1 pdt!

Eg. 1



KT Eg. 2



~~1/2~~

$V = \{3, 4, 5\}$

1 – 3
1 – 2
3 – 6
3 – 7
3 – 2
2 – 4
2 – 5
4 – 7
5 – 7
6 – 7

$\boxed{\{1, 2, 6, 7\}}$ size=4

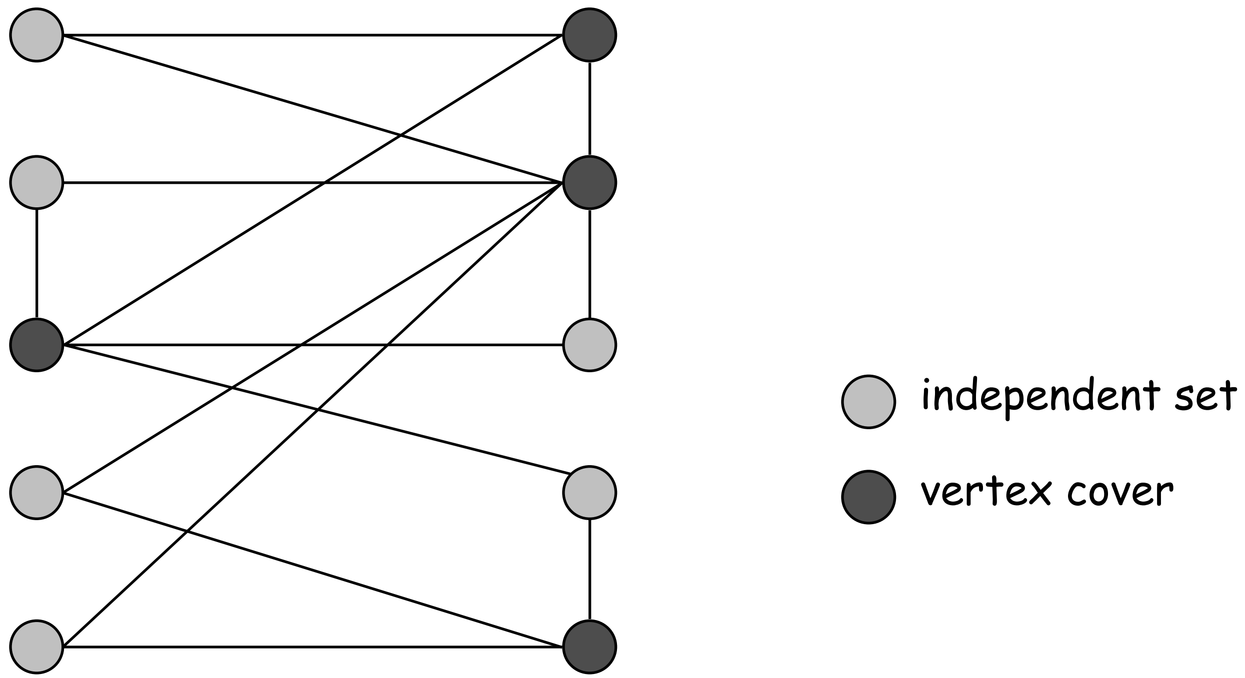
$\boxed{\{2, 3, 7\}}$ size=3

● vertex cover

Vertex Cover and Independent Set

Claim. VERTEX-COVER $\equiv_P$ INDEPENDENT-SET.

Pf. We show S is an independent set iff V – S is a vertex cover.



Vertex Cover and Independent Set

Claim. VERTEX-COVER $\equiv_P$ INDEPENDENT-SET.

Pf. We show S is an independent set iff V – S is a vertex cover.

$\Rightarrow$

- Let S be any independent set.
- Consider an arbitrary edge (u, v).
- S independent $\Rightarrow$ $\underline{u \notin S}$ or $\underline{v \notin S}$ $\Rightarrow$ $u \in V - S$ or $v \in V - S$.
- Thus, V – S covers (u, v). $\Rightarrow$ Vertex cover $\leq_p$ Independent set

$\Leftarrow$

- Let V – S be any vertex cover.
- Consider two nodes $u \in S$ and $v \in S$., if there is a $e = (u, v)$
- There is no edge (u, v) in E since V – S is a vertex cover (otherwise u or v has to be in V-S, the vertex cover)
- Thus, no two nodes in S are joined by an edge $\Rightarrow$ S independent set. ▪

$\Rightarrow$ Independent set $\leq_p$ Vertex cover

Reduction from Special Case to General Case

Basic reduction strategies.

- Reduction by simple equivalence.
- Reduction from special case to general case.
- Reduction by encoding with gadgets.

Set Cover

SET COVER: Given a set U of elements, a collection $S_1, S_2, \ldots, S_m$ of subsets of U, and an integer k, does there exist a collection of $\leq k$ of these sets whose union is equal to U?

Sample application.
- m available pieces of software.
- Set U of n capabilities that we would like our system to have.
- The ith piece of software provides the set $S_i \subseteq U$ of capabilities.
- Goal: achieve all n capabilities using fewest pieces of software.

Ex:

$U = \{ 1, 2, 3, 4, 5, 6, 7 \}$
$k = 2$

$S_1 = \{3, 7\}$	$S_4 = \{2, 4\}$
$S_2 = \{3, 4, 5, 6\}$	$S_5 = \{5\}$
$S_3 = \{1\}$	$S_6 = \{1, 2, 6, 7\}$

n elements · 2^n subsets

using 'k' subsets we should get 'U' back.

Vertex Cover Reduces to Set Cover

Claim. VERTEX-COVER $\le_P$ SET-COVER.

Pf. Given a VERTEX-COVER instance $G = (V, E)$, k, we construct a set cover instance whose size equals the size of the vertex cover instance.

Construction.

$U = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$

- Create SET-COVER instance:
  - $k = k$, $U = E$, $S_v = \{e \in E : e \text{ incident to } v\}$

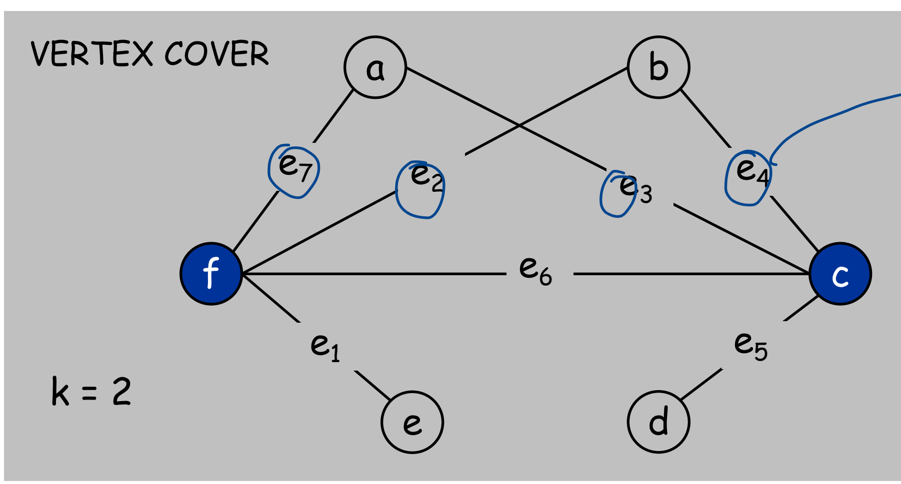
$S_a = \{e_7, e_3\}$ $S_b = \{e_2, e_4\}$

- Set-cover of size $\le k$ iff vertex cover of size $\le k$. ▪

$S_c = \{e_3, e_4, e_6, e_5\}$

Thus, S_c & S_f form 'U' & Vertex 'c' & 'f' are vertex cover.

$S_f = \{e_7, e_2, e_6, e_1\}$



SET COVER

$U = \{1, 2, 3, 4, 5, 6, 7\}$

$k = 2$

$S_a = \{3, 7\}$ $S_b = \{2, 4\}$

$S_c = \{3, 4, 5, 6\}$ $S_d = \{5\}$

$S_e = \{1\}$ $S_f = \{1, 2, 6, 7\}$

Node 'a' has edge '3' & '7'.

Polynomial-Time Reduction

Basic strategies.

- Reduction by simple equivalence.
- Reduction from special case to general case.
- Reduction by encoding with gadgets.

8.2 Reductions via "Gadgets"

Basic reduction strategies.

- Reduction by simple equivalence.
- Reduction from special case to general case.
- Reduction via "gadgets."

Satisfiability

Literal: A Boolean variable or its negation. x_i or $\overline{x_i}$

Clause: A disjunction of literals. $C_j = x_1 \underset{\text{OR}}{\vee} \overline{x_2} \underset{\text{OR}}{\vee} x_3$

Conjunctive normal form: A propositional formula Φ that is the conjunction of clauses. $\Phi = C_1 \underset{\text{AND}}{\wedge} C_2 \underset{\text{AND}}{\wedge} C_3 \underset{\text{AND}}{\wedge} C_4$

assignment is satisfying if cnf = true

SAT: Given CNF formula Φ, does it have a satisfying truth assignment?

3-SAT: SAT where each clause contains exactly 3 literals.

each corresponds to a different variable

Ex: $(\overline{x_1} \vee x_2 \vee x_3) \wedge (x_1 \vee \overline{x_2} \vee x_3) \wedge (x_2 \vee x_3) \wedge (\overline{x_1} \vee \overline{x_2} \vee \overline{x_3})$

Yes: $\boxed{x_1 = \text{true}, x_2 = \text{true } x_3 = \text{false.}}$ → assignment.

3 Satisfiability Reduces to Independent Set

Why 3? – It can be easily reduced to other NP-complete problem!

Claim. 3-SAT $\leq_P$ INDEPENDENT-SET.

Pf. Given an instance Φ of 3-SAT, we construct an instance (G, k) of INDEPENDENT-SET that has an independent set of size k iff Φ is satisfiable.

Construction.

- G contains 3 vertices for each clause, one for each literal.
- Connect 3 literals in a clause in a triangle.
- Connect literal to each of its negations.

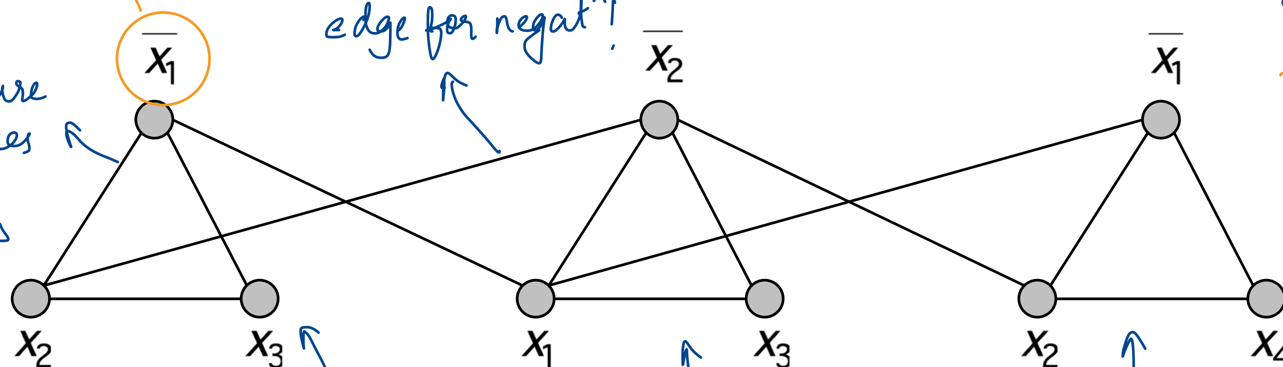
m clauses → · m Δ's
· 3m nodes
· 3m edges + edge for negat^n
– 3m·(m-1) worst case

can have 'm-1' edges

edge for negat^n!

Makes sure other vertices r not in ind-set, as any1 of it makes it true!

G



k = 3

$$\Phi = (\bar{x}_1 \vee x_2 \vee x_3) \wedge (x_1 \vee \bar{x}_2 \vee x_3) \wedge (\bar{x}_1 \vee x_2 \vee x_4)$$

3 Satisfiability Reduces to Independent Set

Claim. G contains independent set of size $k = |\Phi|$ iff Φ is satisfiable.

Pf. $\Rightarrow$ Let S be independent set of size k.

- S must contain exactly one vertex in each triangle.
- Set these literals to true. ⟵ and any other variables in a consistent way
- Truth assignment is consistent and all clauses are satisfied.

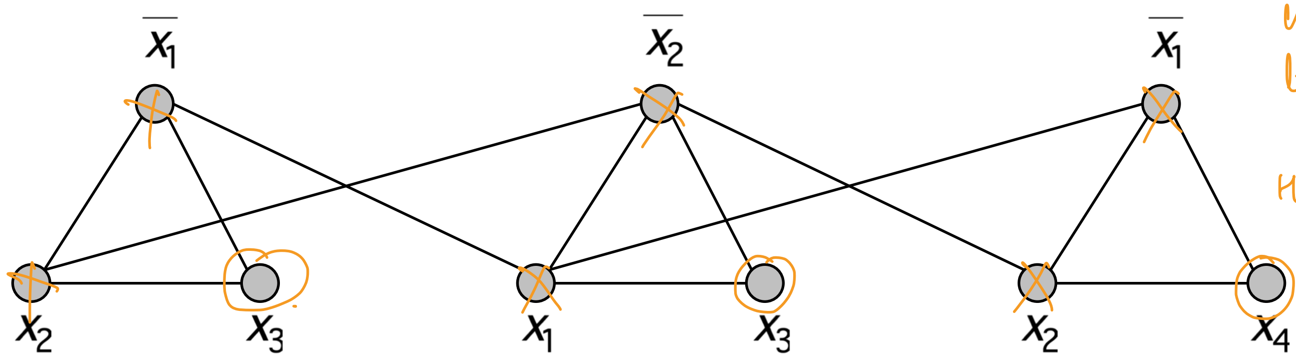
Time :- size of ind. set!

Pf $\Leftarrow$ Given satisfying assignment, select one true literal from each triangle. This is an independent set of size k. ▪

2 x_3 are considered as seperate nodes!

Independent set may not give us all sol^n but can tell us yes or no! Hence $\{x_3, x_4\}$ also $\{x_2, x_3, x_4\}$

G



$k = 3$

$$\Phi = (\bar{x}_1 \vee x_2 \vee x_3) \wedge (x_1 \vee \bar{x}_2 \vee x_3) \wedge (\bar{x}_1 \vee x_2 \vee x_4)$$

Review

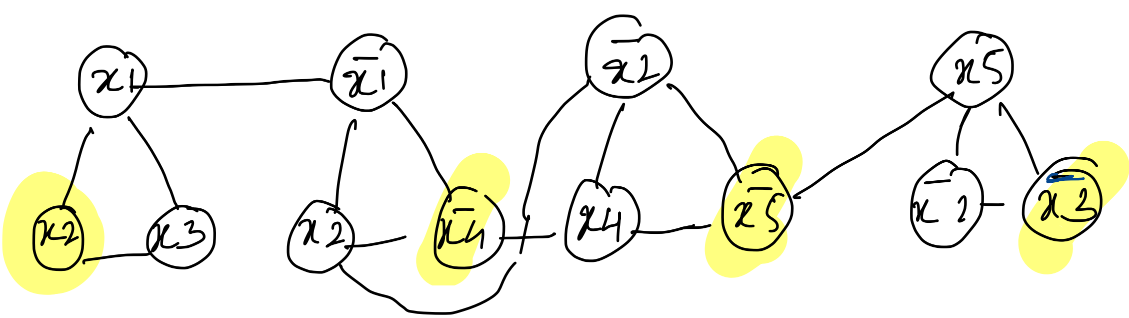
Basic reduction strategies.

- Simple equivalence: INDEPENDENT-SET $\equiv_P$ VERTEX-COVER.
- Special case to general case: VERTEX-COVER $\leq_P$ SET-COVER.
- Encoding with gadgets: 3-SAT $\leq_P$ INDEPENDENT-SET.

Transitivity. If X $\leq_P$ Y and Y $\leq_P$ Z, then X $\leq_P$ Z.

Pf idea. Compose the two algorithms.

Ex: 3-SAT $\leq_P$ INDEPENDENT-SET $\leq_P$ VERTEX-COVER $\leq_P$ SET-COVER.



$x2 \quad \bar{x}3 \quad \bar{x}4 \quad \bar{x}5$

T T T T

$\bar{x}2$ – F

$x5$

Self-Reducibility

Decision problem. Does there exist a vertex cover of size $\leq$ k?

Search problem. Find vertex cover of minimum cardinality.

Self-reducibility. Search problem $\leq_P$ decision version.

- Applies to all (NP-complete) problems in this chapter.
- Justifies our focus on decision problems.

Ex: to find min cardinality vertex cover.

- (Binary) search for cardinality k* of min vertex cover.
- Find a vertex v such that $G - \{ v \}$ has a vertex cover of size $\leq$ k* - 1.
  - any vertex in any min vertex cover will have this property
- Include v in the vertex cover.
- Recursively find a min vertex cover in $G - \{ v \}$.

delete v and all incident edges