Introduction to Model Predictive Control

Lecture 4: Nonlinear Models and S-Functions

Francesco Borrelli

University of California at Berkeley,
Mechanical Engineering Department

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Introduction

Today will discuss:

- Nonlinear Vehicle Model for Fuel Consumption
- S-Functions

Additional Material

- Nonlinear Vehicle 4-Wheels Model (for Vehicle Dynamics Control)

Nonlinear Vehicle Model for Fuel Consumption

Simplified vehicle dynamical model

Recall requirements from first lecture

- Speed needs to be a state. It will be constrained over the horizon.
- Input is “Engine Torque”
- Output is “Fuel”

Main issue

- Traffic speed sensors at fixed distance

Need a distance-based model

Simplified vehicle model

Use $s(k)$ as the independent variable, representing the k-th position, and a fixed step size of $\Delta s = s(k+1) - s(k)$ for all k.

Distance Based Model

$$
\begin{aligned}
\frac{1}{2}mv^2(k+1) - \frac{1}{2}mv^2(k) = \quad & F_w(k)\Delta s - \\
& -\frac{1}{2}\rho C_a v^2(k)\Delta s - \mu m g \Delta s - m g \theta(k)\Delta s
\end{aligned}
$$

where

- $v(k)$: the velocity of the vehicle at position $s(k)$
- $F_w(k)$: the wheel force at position $s(k)$
- ρC_a: the coefficient of drag multiplied by the car frontal area
- m: mass of the vehicle
- g: gravitational constant
- μ: friction coefficient associated with the rolling resistance
- $\theta(k)$: road grade at position $s(k)$

Simplified vehicle model

The "real" input is the engine torque $T(k)$:

$$F_w(k) = \frac{T(k)\eta R(k)}{r} \tag{1}$$

i.e.

$$T(k) = \frac{F_w(k)r}{\eta R(k)} \tag{2}$$

where r is the rolling radius of the tire, R is the (gear) ratio of wheel to engine speed and η is the driveline efficiency.

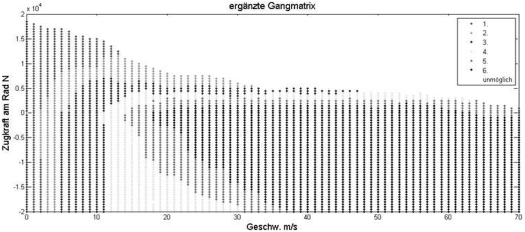
Simplified vehicle model

The gear ratio is given as a static nonlinear map of F_w and engine speed w_e:

$$R(k) = R(k, F_w(k), v(k))$$

so that

$$T(k) = \frac{F_w(k) r}{\eta R(k, F_w(k), v(k))}$$



Simplified vehicle model

Approach:

- The input for the model used in the control will be $F_w(k)$
- The engine torque will be an auxiliary variable function of state and input

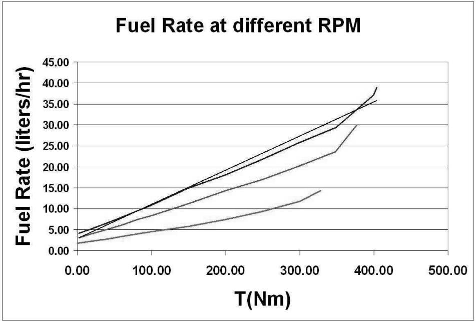
$$T(k) = \frac{F_w(k)r}{\eta R(k, F_w(k), v(k))}$$

$T(k)$ needed for

- Actual input to the vehicle
- Fuel model

Simplified fuel model

Denote the fuel rate as $\dot{f}$



Fist observation:

$$\dot{f} \simeq \alpha(\omega_e) + \beta(\omega_e)T(k)$$

where ω_e is the engine speed.

Simplified fuel model

Good empirical model:

$$\dot{f} \simeq c\omega_e^2 + d\omega_e T(k)$$

Note that

- at constant speed $w_e(k) = v(k)R/r$
- The fuel $f(k)$ used from position $s(k)$ till position $s(k+1)$ is $\dot{f}$ multiplied the time to complete the Δs segment: $(\Delta s/v(k))$.

In conclusion

$$\begin{aligned} f(k) &= c\left(\tfrac{R(k,F_w(k),v(k))}{r}\right)^2 \Delta s\ v(k) + d\left(\tfrac{R(k,F_w(k),v(k))}{r}\right)\Delta s\ T(k) \\ &= h(v(k), T(k), F_w(k)) \end{aligned}$$

Simplified fuel model

The resulting model is

Distance Based Model

$$
\begin{aligned}
&x(k+1) = ax(k) + bu(k) - b_2\theta(k) - \gamma \\
&T(k) = g(x(k), u(k)) \\
&y(k) = h(\sqrt{x(k)}, T(k), u(k))
\end{aligned}
$$

- distance based model
- $x(k) = v(k)^2$ is the speed squared, $u(k) = T_w(k)$ is the wheel torque, $y(k)$ is fuel used from $s(k)$ till $s(k+1)$
- nonlinear model, not in standard form.

Nonlinear Dynamical Systems in Matlab

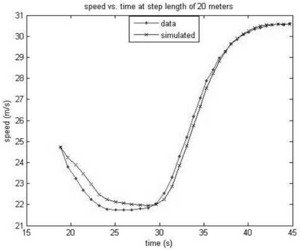
Discrete Time models

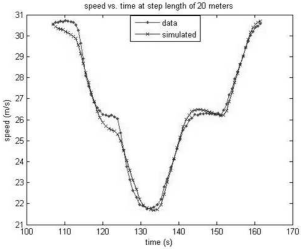
- Matlab: write a function $[y(k), x(k+1)]$=fun$(x(k), u(k))$
- Simulink: S-Functions

Continuous Time models

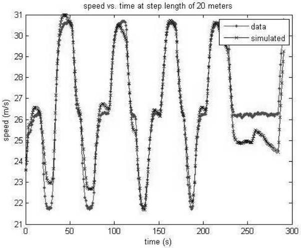
- Matlab: use ode45 (instead of lsim)
- Simulink: S-Functions

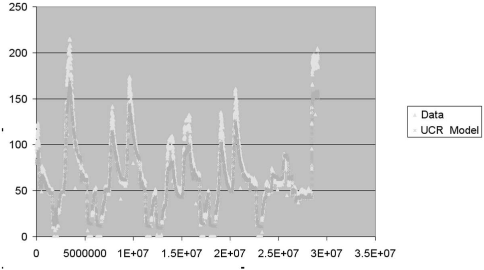
Is it a Good Model? - Identification





Is it a Good Model? - Validation





S-Functions

M-File S-Functions

MATLAB function of the following form:
[sys,x_0,str,ts]=f(t,x,u,flag,p1,p2,...)

where

- f is the S-function's name
- t is the current time
- x is the state vector of the corresponding S-function block
- u is the block's inputs
- flag indicates a task to be performed
- p1, p2, ... are the block's parameters.

During simulation of a model, Simulink repeatedly invokes **f**, using flag to indicate the task to be performed for a particular invocation.

M-File S-Functions

subfunctions, called S-function callback methods, perform the tasks required of the S-function during simulation.

Simulation Stage	S-Function Routine	Flag
Initialization	`mdlInitializeSizes`	`flag = 0`
Calculation of next sample hit (variable sample time block only)	`mdlGetTimeOfNextVarHit`	`flag = 4`
Calculation of outputs	`mdlOutputs`	`flag = 3`
Update of discrete states	`mdlUpdate`	`flag = 2`
Calculation of derivatives	`mdlDerivatives`	`flag = 1`
End of simulation tasks	`mdlTerminate`	`flag = 9`

M-File S-Functions

Consider the system with m inputs u, q outputs (denoted by y)

$$\left[\begin{array}{c} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \vdots \\ \dot{x}_n(t) \\ \hline y_1(t) \\ y_2(t) \\ \vdots \\ y_q(t) \end{array}\right] = \left[\begin{array}{c} f_1\left(x_1(t), x_2(t), \ldots, x_n(t), u_1(t), u_2(t), \ldots, u_m(t), t\right) \\ f_2\left(x_1(t), x_2(t), \ldots, x_n(t), u_1(t), u_2(t), \ldots, u_m(t), t\right) \\ \vdots \\ f_n\left(x_1(t), x_2(t), \ldots, x_n(t), u_1(t), u_2(t), \ldots, u_m(t), t\right) \\ \hline h_1\left(x_1(t), x_2(t), \ldots, x_n(t), u_1(t), u_2(t), \ldots, u_m(t), t\right) \\ h_2\left(x_1(t), x_2(t), \ldots, x_n(t), u_1(t), u_2(t), \ldots, u_m(t), t\right) \\ \vdots \\ h_q\left(x_1(t), x_2(t), \ldots, x_n(t), u_1(t), u_2(t), \ldots, u_m(t), t\right) \end{array}\right]$$

When flag 1 (mldDerivatives)

function sys=mdlDerivatives(t,x,u)

$sys(1) = f_1(\ldots)$, $\ldots$,$sys(n) = f_n(\ldots)$

When flag 3 (mldOutputs)

function sys=mdlOutputs(t,x,u)

$sys(1) = h_1(\ldots)$, $\ldots$,$sys(n) = h_q(\ldots)$

M-File S-Functions

Hints:

- in bSpace: a template implementation of an M-file S-function, sfuntmpl.m, (also in matlabroot/toolbox/simulink/blocks).
- in bSpace: a simple example for continuous time systems
- in bSpace: a simple example for discrete time systems
- In Matlab help search for “S-functions”

Homework: will be graded!

Due on Friday October 2nd, at beginning of lecture

Homework 2/1

Consider the vehicle model (for fuel consumption), assume zero grade, do exercise in Simulink.

- Use all data provided on bSpace and write an S-Function for simulating the system. Use as input the wheel force $F_w(k)$ and as outputs the vehicle speed and the fuel consumption
- Design a very simple controller (P, PI,..) which regulates the wheel force $F_w(k)$ to follow a desired speed profile.
- Take the average speed profile provided on b-Space (101 highway) as a reference (position,speed) (be careful that the signal is sampled every 10 meters while the suggested Δs for the model is 40 meters).
- Show with a plot that the designed controller allows the vehicle to track the given reference.
- Provide the total fuel consumption for the trip.
- Provide the total time to complete the trip.

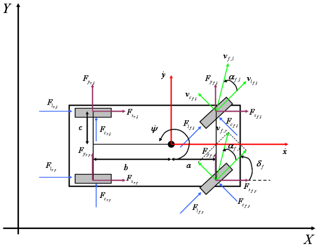
Homework 2/2

Consider the Ball and Plate model

- Consider the last part of Homework 1:
  - The control goal is to bring the ball from any admissible position to the origin of the state space (position and speed of the plate equal to zero and position and speed of the ball equal to zero). The state feedback discrete time controller $u(k) = Kx(k)$ with $K = [21.1 \quad 4.0 \quad -266.3 \quad -5.8]$ and sampling $\Delta T = 30ms$ should do the job.
  - Simulate for 3 seconds the closed loop system (DT controller and CT LTI system in state space form) when the system starts from the initial condition $x(0) = [20, 0, 0, 0]$ (plate and ball at rest and ball at distance 20cm from the origin)
- Redo the exercise by substituting the state-space simulation block with an S-Function.

Additional Material: 4-Wheels Model

Simplified vehicle dynamical model



$$
\begin{aligned}
m\ddot{y} &= -m\dot{x}\dot{\psi} + F_{y_{f,l}} + F_{y_{f,r}} + F_{y_{r,l}} + F_{y_{r,r}}, && \text{(3a)}\\
m\ddot{x} &= m\dot{y}\dot{\psi} + F_{x_{f,l}} + F_{x_{f,r}} + F_{x_{r,l}} + F_{x_{r,r}} && \text{(3b)}\\
I\ddot{\psi} &= a\left(F_{y_{f,l}} + F_{y_{f,r}}\right) - b\left(F_{y_{r,l}} + F_{y_{r,r}}\right) && \\
&\quad + c\left(-F_{x_{f,l}} + F_{x_{f,r}} - F_{x_{r,l}} + F_{x_{r,r}}\right), && \text{(3c)}\\
\dot{Y} &= \dot{x}\sin\psi + \dot{y}\cos\psi, && \text{(3d)}\\
\dot{X} &= \dot{x}\cos\psi - \dot{y}\sin\psi. && \text{(3e)}
\end{aligned}
$$

Simplified vehicle dynamical model

- $\dot{x}$ and $\dot{y}$: velocities along the longitudinal and lateral vehicle axes
- ψ: yaw angle, $\dot{\psi}$: yaw rate
- Y and X are the lateral and longitudinal positions of the vehicle Center of Gravity (CoG) in the inertial frame
- m is the mass of the car, I is the inertia of the vehicle along the vertical axis
- Constants a and b are the distances from the CoG of the front and rear axles, respectively, c is the distance of the left and right wheels from the longitudinal vehicle axis.

Simplified vehicle dynamical model

The lateral and longitudinal tire forces $F_{c_{\star,\bullet}}$ and $F_{l_{\star,\bullet}}$ generate $F_{y_{\star,\bullet}}$ and $F_{x_{\star,\bullet}}$, along the lateral and longitudinal vehicle axes:

$$F_{y_{\star,\bullet}} = F_{l_{\star,\bullet}} \sin\delta_\star + F_{c_{\star,\bullet}} \cos\delta_\star, \tag{4a}$$

$$F_{x_{\star,\bullet}} = F_{l_{\star,\bullet}} \cos\delta_\star - F_{c_{\star,\bullet}} \sin\delta_\star. \tag{4b}$$

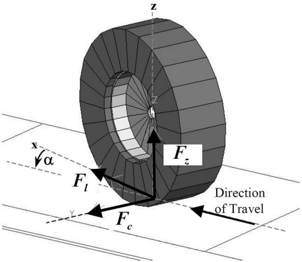


Figure: Illustration of tire model nomenclature

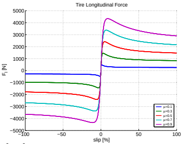
Tyre Model

The lateral and longitudinal tire forces $F_{c_{\star,\bullet}}$ and $F_{l_{\star,\bullet}}$ are directed as in previous Figure and computed as

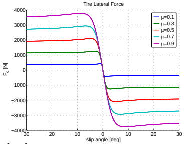
$$F_{c_{\star,\bullet}} = f_c(\alpha_{\star,\bullet}, s_{\star,\bullet}, \mu_{\star,\bullet}, F_{z_{\star,\bullet}}), \tag{5a}$$
$$F_{l_{\star,\bullet}} = f_l(\alpha_{\star,\bullet}, s_{\star,\bullet}, \mu_{\star,\bullet}, F_{z_{\star,\bullet}}), \tag{5b}$$

- $\alpha_{\star,\bullet}$: the tire slip angles,
- $s_{\star,\bullet}$: the slip ratios
- $\mu_{\star,\bullet}$ are the road friction coefficients
- $F_{z_{\star,\bullet}}$ are the tires normal forces.

Tyre Model

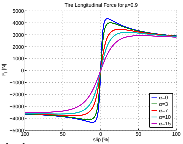


(a) Long. pure braking/driving.

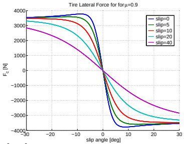


(b) Lat. pure cornering.

Figure: Longitudinal and lateral tire forces with different μ coefficient values.



(a) Long. function for different slip angle α.



(b) Lat. for different slip ratio s.

Figure: Longitudinal and lateral tire forces in combined braking/driving and

Tyre Model

The slip angle $\alpha_{\star,\bullet}$ in (5) represents the angle between the wheel velocity vector $v_{\star,\bullet}$ and the direction of the wheel itself, and can be compactly expressed as:

$$\alpha_{\star,\bullet} = \arctan \frac{v_{c_{\star,\bullet}}}{v_{l_{\star,\bullet}}}, \tag{6}$$

where $v_{c_{\star,\bullet}}$ and $v_{l_{\star,\bullet}}$ are the lateral and longitudinal wheels velocities, respectively.

$$v_{c_{\star,\bullet}} = v_{y_{\star,\bullet}} \cos \delta_\star - v_{x_{\star,\bullet}} \sin \delta_\star, \tag{7a}$$

$$v_{l_{\star,\bullet}} = v_{y_{\star,\bullet}} \sin \delta_\star + v_{x_{\star,\bullet}} \cos \delta_\star, \tag{7b}$$

Tyre Model

The velocities $v_{x_{\star,\bullet}}$ and $v_{y_{\star,\bullet}}$ for the four wheels are computed as follows:

$$v_{y_{f,l}} = \dot{y} + a\dot{\psi} \quad v_{x_{f,l}} = \dot{x} - c\dot{\psi}, \tag{8a}$$
$$v_{y_{f,r}} = \dot{y} + a\dot{\psi} \quad v_{x_{f,r}} = \dot{x} + c\dot{\psi}, \tag{8b}$$
$$v_{y_{r,l}} = \dot{y} - b\dot{\psi} \quad v_{x_{r,l}} = \dot{x} - c\dot{\psi}, \tag{8c}$$
$$v_{y_{r,r}} = \dot{y} - b\dot{\psi} \quad v_{x_{r,r}} = \dot{x} + c\dot{\psi}. \tag{8d}$$

The slip ratio $s_{\star,\bullet}$ in (5) is the defined as

$$s_{\star,\bullet} = \begin{cases} \dfrac{r_w \omega_{\star,\bullet}}{v_{l_{\star,\bullet}}} - 1 \text{ if } v_{l_{\star,\bullet}} > r_w \omega_{\star,\bullet},\ v_{l_{\star,\bullet}} \neq 0 \text{ for braking} \\ 1 - \dfrac{v_{l_{\star,\bullet}}}{r_w \omega_{\star,\bullet}} \text{ if } v_{l_{\star,\bullet}} < r_w \omega_{\star,\bullet},\ \omega_{\star,\bullet} \neq 0 \text{ for driving,} \end{cases} \tag{9}$$

where r_w and $\omega_{\star,\bullet}$ are the radius and the angular speed of the wheels, respectively, and $v_{l_{\star,\bullet}}$ are the wheel longitudinal velocities computed in (7).

Tyre Model

The wheel angular speeds $\omega_{\star,\bullet}$ in (9) are obtained by integrating the following set of differential equations:

$$J_{w_{\star,\bullet}}\dot{\omega}_{\star,\bullet} = -F_{l_{\star,\bullet}} r_w - T_{b_{\star,\bullet}} - b \cdot \omega_{\star,\bullet}, \tag{10}$$

where $J_{w_{\star,\bullet}}$ include the wheel and driveline inertias, b is the damping coefficient, $T_{b_{\star,\bullet}}$ are the braking torques at the braking pads.$F_{z_{\star,\bullet}}$ in (5) are the normal forces Assumption: the normal forces $F_{z_{\star,\bullet}}$ are constant and distributed between the front and rear axles as

$$F_{z_{f,\bullet}} = \frac{bmg}{2(a+b)}, \quad F_{z_{r,\bullet}} = \frac{amg}{2(a+b)}.$$

Final Model

4-Wheel Model

$$\dot{\xi}(t) = f_{\mu(t)}(\xi(t), u(t)),$$

where the state and input vectors are

- $\xi = [\dot{y},\ \dot{x},\ \psi,\ \dot{\psi},\ Y,\ X,\ \omega_{f,l},\ \omega_{f,r},\ \omega_{r,l},\ \omega_{r,r}]$
- $u = [\delta_f,\ T_{b_{f,l}},\ T_{b_{f,r}},\ T_{b_{r,l}},\ T_{b_{r,r}}]$,

and $\mu(t) = [\mu_{f,l}(t),\ \mu_{f,r}(t),\ \mu_{r,l}(t),\ \mu_{r,r}(t)]$.