Model Predictive Control for Linear and Hybrid Systems
Multiparametric Programming

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Introductory example

Consider the following problem:

Example: Perturbed quadratic program

$$
\begin{array}{rl}
f^*(x) = \min\limits_{z} & \left\{ f(z,x) = \frac{1}{2}z^2 + 2xz \right\} \\
\text{subj. to} & z \leq 1 + x,
\end{array}
$$

$x \in \mathbb{R}$ is *an unknown* parameter.

The goals:

1. find $z^*(x) = argmin_z \; f(z,x)$,
2. find all x for which the problem has a solution
3. compute the *value function* $f^*(x)$

Define Lagrangian:

$$\mathcal{L}(z,x,\lambda) = f(z,x) + \lambda(z - x - 1)$$

KKT conditions:

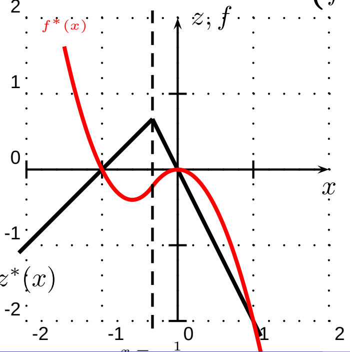
$$
\begin{aligned}
z + 2x + \lambda &= 0, && (1)\\
z - x - 1 &\leq 0, && (2)\\
\lambda(z - x - 1) &= 0, && (3)\\
\lambda &\geq 0. && (4)
\end{aligned}
$$

Introductory example

Consider the two complementary cases:

1. $$\begin{aligned} z - x - 1 &= 0 \\ \lambda &> 0 \end{aligned} \Rightarrow \begin{cases} z^*(x) = x + 1, \\ x \leq -\frac{1}{3} \\ f^*(x) = \frac{5}{2}x^2 + 3x + \frac{1}{2} \end{cases}$$

2. $$\begin{aligned} z - x - 1 &< 0 \\ \lambda &= 0 \end{aligned} \Rightarrow \begin{cases} z^*(x) = -2x, \\ x > -\frac{1}{3} \\ f^*(x) = -2x^2 \end{cases}$$



$$\boxed{z^*(x) = \begin{cases} x + 1, & \text{if } x \leq -\frac{1}{3}, \\ -2x, & \text{if } x \geq -\frac{1}{3} \end{cases}}$$

General Formulation

$$
\begin{aligned}
f^*(x) = \inf_{z} \quad & f(z,x), \\
\text{subj. to} \quad & g(z,x) \le 0,
\end{aligned}
$$

where $z \in \mathcal{Z} \subseteq \mathbb{R}^s$ is the optimization vector, $x \in \mathcal{X} \subseteq \mathbb{R}^n$ is the parameter vector, $f : \mathbb{R}^s \times \mathbb{R}^n \to \mathbb{R}$ is the cost function and $g : \mathbb{R}^s \times \mathbb{R}^n \to \mathbb{R}^{n_g}$ are the constraints.

- $x \in \mathbb{R}^n$, $n > 1 \Rightarrow$ multi-parametric program
- closely related to *sensitivity analysis*:
  - *sensitivity analysis*: local behavior of the solution for small perturbations
  - *parametric program*: solution for the full range of parameters x

General formulation. Notation

$$\begin{aligned} f^*(x) = \inf_z \quad & f(z,x), \\ \text{subj. to} \quad & g(z,x) \leq 0, \end{aligned}$$

- *Feasibility set (set of subsets). Point-to-set map* $R : \mathcal{X} \to 2^{\mathcal{Z}}$ defined as

$$R(x) = \{z \in \mathcal{Z} : \ g(z,x) \leq 0\}$$

- *Set of feasible parameters.*

$$\mathcal{K}^* = \{x \in \mathcal{X} : \ R(x) \neq \emptyset\}$$

- *Value function.* $f^* : \mathcal{K}^* \to \mathbb{R}$

$$f^*(x) = \inf_z \{f(z,x) : \ z \in R(x)\}$$

- *Optimizers set (set of subsets).* Point-to-set map $Z^* : \mathcal{K} \to 2^{\mathcal{Z}}$ defined as

$$\mathcal{Z}^*(x) = \{z \in R(x) : \ f(z,x) \leq f^*(x)\}$$

General formulation

$$\begin{aligned} f^*(x) = \inf_z \quad & f(z,x), \\ \text{subj. to} \quad & g(z,x) \leq 0, \end{aligned}$$

We aim at calculating

1. The *set of feasible parameters* $\mathcal{K}^* \subseteq \mathcal{K}$
2. The expression of *the value function* $f^*(x)$
3. The expression of *one of the* optimizers $z^*(x) \in Z^*(x)$.

- We consider two important classes:
  1. **multi-parametric LP (mpLP)**
  2. **multi-parametric QP (mpQP)**
- Results on general multiparametric programs require concepts of point-to-set function. Can be found in Chapter 5.

Multiparametric programs with linear constraints

Consider the multiparametric program

$$\begin{array}{rl} J^*(x) = \min\limits_{z} & J(z,x) \\ \text{subj. to} & Gz \le W + Sx, \end{array} \tag{5}$$

Given a closed and bounded polyhedral set $\mathcal{K} \subset \mathbb{R}^n$ of parameters, denote by $\mathcal{K}^* \subseteq \mathcal{K}$ the region of parameters $x \in \mathcal{K}$ such that (5) is feasible:

$$\mathcal{K}^* = \{x \in \mathcal{K} : \exists z \text{ satisfying } Gz \le W + Sx\} \tag{6}$$

Simple Result: If the domain of $J(z,x)$ is $\mathbb{R}^{s+n}$ then $\mathcal{K}^*$ is a polytope. (proof by projection)
We assume

1. $\mathcal{K}$ is fully dimensional.
2. S has full column rank.

Otherwise a smaller set of parameters can be used

Definition of Critical Region

Be $J \triangleq \{1, \ldots, m\}$ the set of constraint indices. For any $A \subseteq J$, G_A and S_A are the submatrices of G and S, with the rows indexed by A. G_j, S_j and W_j are the j-th row of G, S and W, respectively.

Definition (Critical Region)

We define CR_A as the set of parameters x for which the set A of constraints is active at the optimum.

At x define:

$$\begin{aligned} A(x) &\triangleq \{j \in J : G_j z^*(x) - S_j x = W_j \text{ for all } z^*(x) \in Z^*(x)\} \\ NA(x) &\triangleq \{j \in J : \exists z^*(x) \in Z^*(x),\ G_j z^*(x) - S_j x < W_j\}. \end{aligned}$$

$(A(x), NA(x))$ are disjoint and their union is J.

For a given $\bar{x} \in \mathcal{K}^*$ let $(A, NA) \triangleq (A(\bar{x}), NA(\bar{x}))$, and let

$$CR_A \triangleq \{x \in \mathcal{K}^* : A(x) = A\}. \tag{7}$$

mpLP. Problem formulation

Consider the Primal Problem (PP)

$$\begin{aligned} J^*(x) = \min_z \quad & J(z,x) = c^T z, \\ \text{subj. to} \quad & Gz \leq W + Sx \end{aligned}$$

with $c \in \mathbb{R}^s$, $G \in \mathbb{R}^{m\times s}$, $S \in \mathbb{R}^{m\times n}$, $W \in \mathbb{R}^m$ and

$$\mathcal{K} = \{x \in \mathcal{X} : \; Tx \leq Z\}$$

Given a parameter x, we will calculate CR_A, the optimizer $z^*(x)$ and the value function $J^*(x)$, for all $x \in CR_A$

- Assuming that an LP solver is available
- Assuming *no degeneracy*

Then we will iteratively partition the parameters space into critical regions

mpLP. Computation of the critical region, optimizer and value function

Consider the Dual Problem (DP)

$$\begin{array}{rc} \max\limits_{u} & (W+Sx)^T u, \\ \text{subj. to} & G^T u = c, \\ & u \leq 0 \end{array}$$

Optimality Conditions

- *primal feasibility*:

$$Gz \leq W + Sx$$

- *dual feasibility*:

$$G^T u = c, \quad u \leq 0$$

- *complementarity slackness*:

$$u_i(G_i z - S_i x - W_i) = 0, \quad \forall\, i \in \mathcal{I}$$

mpLP. Computation of the critical region, optimizer and value function

- Consider a point $x_0 \in \mathcal{K}^*$.
- Be z_0^* and u_0^* the solutions of the PP and DP, respectively, at x_0 (use a LP solver).
- Determine $(A, NA) \triangleq (A(x_0), NA(x_0))$ for the PP at the optimum.
- From the primal feasibility conditions:

$$\begin{array}{rcl} G_A z_0^*(x) & = & W_A + S_A x, \\ G_{NA} z_0^*(x) & < & W_{NA} + S_{NA} x, \end{array}$$

  for all $x \in CR_A$

mpLP. Computation of the critical region, optimizer and value function

- Assume *no degeneracy* $\Rightarrow \text{rank}(G_A) = s$. The general case $\text{rank}(G_A) = l$ in the textbook. The optimizer $z_0^*(x)$ is

$$z_0^*(x) = G_A^{-1} S_A x + G_A^{-1} W_A = F_0 x + g_0$$

- The critical region $\mathcal{CR}_A$ is

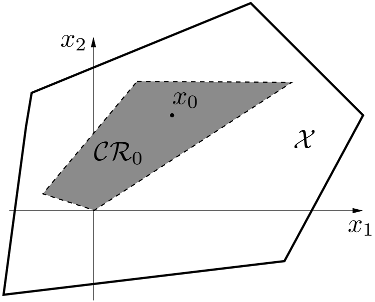
$$\mathcal{CR}_A = \{x \in \mathcal{K}^* \mid (G_{NA} F_0 - S_{NA}) x < W_{NA} - G_{NA} g_0\}$$

- Use the solution to the DP to compute the value function $J_0^*(x)$ (strong duality holds):

$$J_0^*(x) = (S_A x + W_A)^T u^*$$

mpLP. Solution Properties

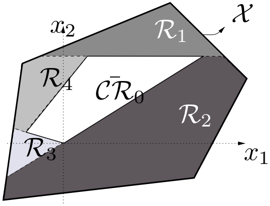
- Critical region CR_A is defined by *strict inequalities* $\Rightarrow$ an open polyhedral set



- Will consider the closure of CR_A (we will show that $z^*(x)$ is continuous)
- In CR_A, $J^*(x)$ is an affine function of the parameters vector
- In CR_A, $z^*(x)$ is an affine function of the parameters vector

mpLP. Partitioning the parameters space

I) Reversing inequalities

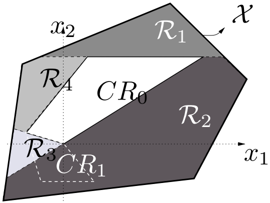


$$CR_0 = \{x \in \mathcal{X} \mid Hx \le K\}$$

$$\mathcal{R}_i = \{x \in \mathcal{X} \mid H_i x > K_i \\ H_j x \le K_j, \forall\, j < i\}$$

$$\boxed{\mathcal{X} \setminus CR_0 = \bigcup_i \mathcal{R}_i}$$

- **NOTE:** regions $\mathcal{R}_i$ are not critical regions
- proceed recursively: repeat the whole procedure for each $\mathcal{R}_i$
- **guaranteed:** set $\mathcal{X}$ explored in finite number of iterations
- **problem:** critical regions can be artificially divided among different $\mathcal{R}_i$



CR_1 is split between $\mathcal{R}_2$ and $\mathcal{R}_3$ $\Rightarrow$ keep track of the critical regions already "discovered"

mLP. Partitioning the parameters space

In each $\mathcal{R}_i$, select a point x_0 and calculate the critical region, optimizer and value function.

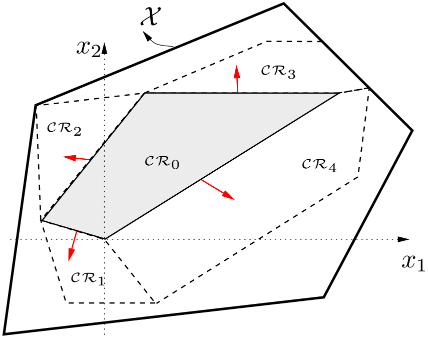
How do we chose x_0?
A possible choiche for x_0 is the Chebychev center of $\mathcal{R}_i$ with

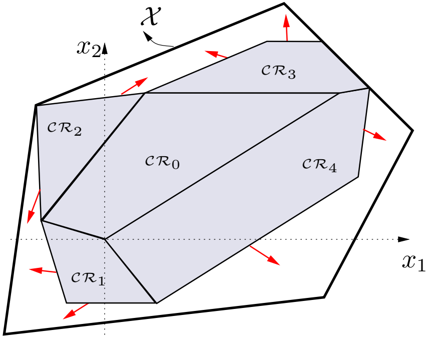
$$\mathcal{R}_i = \{x \in \mathcal{K}^* : \ T_i x \leq Z_i\}$$

$$\begin{array}{ll} \max\limits_{x,z,\epsilon} & \epsilon, \\ \text{subj. to} & T_j^i x + \epsilon \|T_j^i\|_2 \leq Z_j, \ j = 1, \ldots, n_{T^i} \\ & Gz - Sx \leq W \end{array}$$

mLP. Partitioning the parameters space

II) Crossing the facets





- for each of the facets of $\mathcal{CR}_0$ a point outside the region but close to the facet is selected and the procedure is repeated
- critical regions are computed “in one piece”, no artificial splitting
- heuristics: how far to step over the facet?
- no formal proof that whole $\mathcal{X}^*$ is covered
- however: in practice usually outperforms the strategy based on reversing inequalities

mpLP. Global properties of the solution

The following theorem summarizes the properties of the solution to the mpLP:

Theorem

i) *Feasible set $\mathcal{K}^*$ is closed and convex ,*

ii) *If the optimal solution z^* is unique $\forall x \in \mathcal{X}^*$, the optimizer function $z^*(x) : \mathcal{X}^* \to \mathbb{R}^m$ is:*

- *continuous*
- *polyhedral piecewise affine (PPWA) over $\mathcal{K}^*$, affine in each $\mathcal{CR}_i$*

Otherwise, it is always possible to choose a continuous and PPWA optimizer function $z^*(x)$.

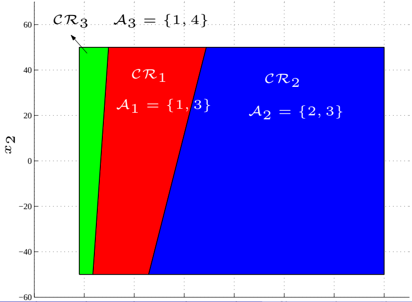
iii) *The value function $J^*(x) : \mathcal{X}^* \to \mathbb{R}$ is:*

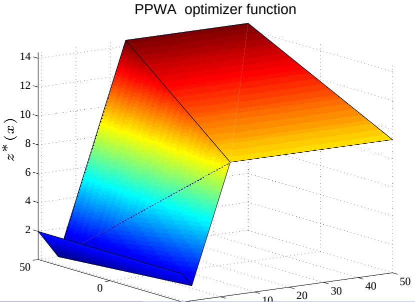
- *convex*
- *PPWA over $\mathcal{X}^*$, affine in each $\mathcal{CR}_i$*

mpLP. Example

$$
\begin{array}{ll}
\min_{z} & -3z_1 - 8z_2, \\
\text{s.t.} & \left\{
\begin{array}{lll}
z_1 + z_2 & \leq & 13 + x_1 \\
5z_1 - 4z_2 & \leq & 20 \\
-8z_1 + 22z_2 & \leq & 121 + x_2 \\
-4z_1 - z_2 & \leq & -8 \\
-z_1 & \leq & 0 \\
-z_2 & \leq & 0
\end{array}
\right.
\end{array}
$$

Critical regions and the optimizer





mpLP. Example

$$z^*(x) = \begin{cases} \begin{bmatrix} 0.733 & -0.0333 \\ 0.267 & 0.0333 \end{bmatrix} x + \begin{bmatrix} 5.50 \\ 7.50 \end{bmatrix} & \text{if } x \in \mathcal{CR}_1 \\ \begin{bmatrix} 0 & 0.0513 \\ 0.0641 & \end{bmatrix} x + \begin{bmatrix} 11.8 \\ 9.81 \end{bmatrix} & \text{if } x \in \mathcal{CR}_2 \\ \begin{bmatrix} -0.333 & 0 \\ 1.33 & 0 \end{bmatrix} x + \begin{bmatrix} -1.67 \\ 14.7 \end{bmatrix} & \text{if } x \in \mathcal{CR}_3 \end{cases}$$

mpLP. Example

$$
\begin{array}{ll}
\min & z_1 + z_2 + z_3 + z_4 \\
\text{subj. to} & \left\{
\begin{array}{rcl}
-z_1 + z_5 & \leq & 0 \\
-z_1 - z_5 & \leq & 0 \\
-z_2 + z_6 & \leq & 0 \\
-z_2 - z_6 & \leq & 0 \\
-z_3 & \leq & x_1 + x_2 \\
-z_3 - z_5 & \leq & x_2 \\
-z_3 & \leq & -x_1 - x_2 \\
-z_3 + z_5 & \leq & -x_2 \\
-z_4 - z_5 & \leq & x_1 + 2x_2 \\
-z_4 - z_5 - z_6 & \leq & x_2 \\
-z_4 + z_5 & \leq & -1x_1 - 2x_2 \\
-z_4 + z_5 + z_6 & \leq & -x_2 \\
z_5 & \leq & 1 \\
-z_5 & \leq & 1 \\
z_6 & \leq & 1 \\
-z_6 & \leq & 1
\end{array}
\right.
\end{array}
$$

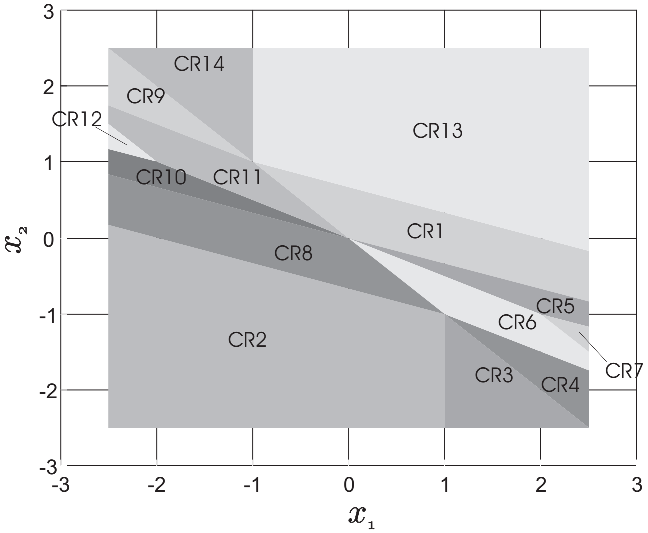
where $\mathcal{K}$ is given by

$$-2.5 \leq x_1 \leq 2.5, -2.5 \leq x_2 \leq 2.5.$$

mpLP. Example

Critical Region	Value function
CR1= $CR_{\{2,3,4,7,10,11\}}$	$2x_1$+$3x_2$
CR2= $CR_{\{1,3,4,5,9,13\}}$	-$2x_1$-$3x_2$
CR3= $CR_{\{1,3,4,6,9,13\}}$	-x_1-$3x_2$-1
CR4= $CR_{\{1,3,6,9,10,13\}}$	-$2x_2$-1
CR5= $CR_{\{1,2,3,7,10,11\}}$	x_1
CR6= $CR_{\{1,3,6,7,9,10,11,12\}}$	x_1
CR7= $CR_{\{1,3,7,10,11,15\}}$	x_1
CR8= $CR_{\{1,3,4,5,9,12\}}$	-$2x_1$-$3x_2$
CR9= $CR_{\{2,4,8,11,12,14\}}$	$2x_2$-1
CR10= $CR_{\{1,2,4,5,9,12\}}$	-x_1
CR11= $CR_{\{2,4,5,8,9,10,11,12\}}$	-x_1
CR12= $CR_{\{2,4,5,9,12,16\}}$	-x_1
CR13= $CR_{\{2,3,4,7,11,14\}}$	$2x_1$+$3x_2$
CR14= $CR_{\{2,3,4,8,11,14\}}$	x_1+$3x_2$-1

mpLP. Example



mpLP. Non-unique optimizer

Assume $Z^*(x)$ is not a singleton in $CR_{A(x_0)} \Rightarrow CR_{A(x_0)}$ is a dual degenerate critical region, i.e., in $CR_{A(x_0)}$ less than s constraints are active.

How do we chose a particular optimizer? Partition $CR_{A(x_0)}$ by searching over all the possible combinations of active constraints.

- **Step 1-** Chose a set of constraints $\widehat{A}(x_0)$ such that $\widehat{A}(x_0) \supset A(x_0)$ and $\text{rank}(G_{\widehat{A}(x_0)}) = s$.
- **Step 2-** Compute the subset $\widehat{CR}_{\widehat{A}(x_0)} \subset CR_{A(x_0)}$ and find the expression of the *unique* optimizer $z^*(x)$ therein.
- **Step 3-** Continue partitioning until $CR_{A(x_0)}$ is fully covered.

Note. The subset $\widehat{A}(x_0)$ is not a *critical region* in the sense we defined.
Note. The generated subset may overlap. To avoid overlapping, intersect the subset with the current partition.

mpLP. Non-unique optimizer. Example

Consider the mpLP problem

$$
\begin{array}{ll}
\min & -2z_1 - z_2 \\
\text{subj. to} & \left\{ \begin{array}{rcl}
z_1 + 3z_2 & \leq & 9 - 2x_1 + x_2 \\
2z_1 + z_2 & \leq & 8 + x_1 - 2x_2 \\
z_1 & \leq & 4 + x_1 + x_2 \\
-z_1 & \leq & 0 \\
-z_2 & \leq & 0
\end{array} \right.
\end{array}
$$

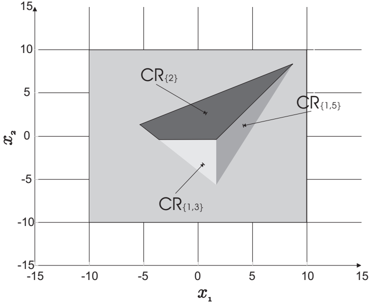
where $\mathcal{K}$ is given by:

$$
\begin{array}{c}
-10 \leq x_1 \leq 10 \\
-10 \leq x_2 \leq 10.
\end{array}
$$

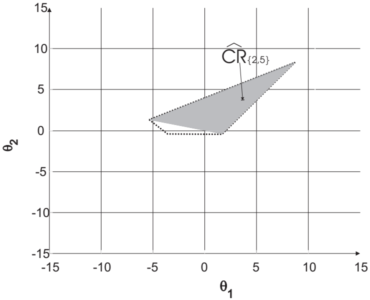
mpLP. Non-unique optimizer. Example

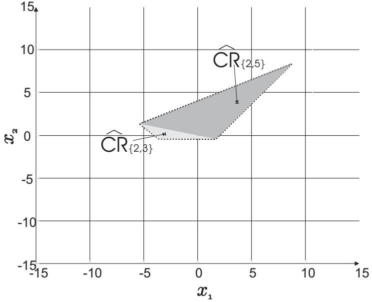
Expressions of optimizer and the value function are given for each critical region are

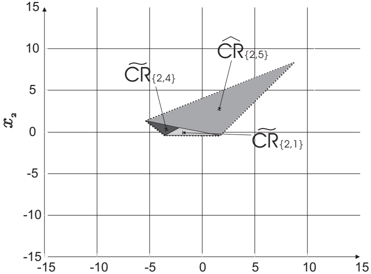
Region	Optimizer	Value Function
$CR_{\{2\}}$	not single valued	$-x_1 + 2x_1 - 8$
$CR_{\{1,5\}}$	$z_1^* = -2x_1 + x_2 + 9$, $z_2^* = 0$	$4x_1 - 2x_2 - 18$
$CR_{\{1,3\}}$	$z_1^* = x_1 + x_2 + 4$, $z_2^* = -x_1 + 5/3$	$-x_1 - 2x_2 - 29/3$



mpLP. Non-unique optimizer. Example







mpQP. Problem formulation

We consider multi-parametric quadratic programs of the form

$$
\begin{array}{rl}
J^*(x) = \min\limits_{z} & \left\{ J(z,x) = \frac{1}{2} z^T H z \right\}, \\
\text{subj. to} & Gz \le W + Sx
\end{array}
$$

where $z \in \mathcal{Z} \subseteq \mathbb{R}^s$, and $x \in \mathcal{X} \subseteq \mathbb{R}^n$, $G \in R^{m \times s}$.

Assumptions: no degeneracy

The algorithm is conceptually identical to the geometric algorithm for mpLP.
A QP solver is required.

For a given $\bar{x} \in \mathcal{K}^*$ let $(A, NA) \triangleq (A(\bar{x}), NA(\bar{x}))$, and let

$$CR_A \triangleq \{x \in \mathcal{K}^* : A(x) = A\}.$$

mpQP. Computation of the critical region, optimizer and value function

Consider the Dual Problem (DP)

$$\begin{array}{ll} \min_u & \frac{1}{2}u'(GH^{-1}G')u + u'(W + GH^{-1}q) + \frac{1}{2}q'H^{-1}q \\ \text{subj. to} & u \geq 0 \end{array}$$

The KKT Optimality Conditions

- *optimality condition*:

$$Hz + G^T u = 0$$

- *primal feasibility*:

$$Gz \leq W + Sx$$

- *dual feasibility*:

$$u \geq 0$$

- *complementarity slackness*:

$$u_i(G_i z - S_i x - W_i) = 0, \quad \forall\, i \in \mathcal{I}$$

mpQP. Computation of the critical region, optimizer and value function

- Consider a point $x_0 \in \mathcal{K}^*$.
- Be z_0^* and u_0^* the solutions of the PP and DP, respectively, at x_0 (use a QP solver).
- Determine $(A, NA) \triangleq (A(x_0), NA(x_0))$ for the PP at the optimum.
- From the optimality conditions:

$$z_0^* = -H^{-1}G'u_0^*.$$

- By substituting z_0^* in the complementarity slackness conditions:

$$u_0^*(-GH^{-1}G'u_0^* - W - Sx_0) = 0.$$

mpQP. Computation of the critical region, optimizer and value function

- Be u^*_{NA} and u^*_A the Lagrange multipliers corresponding to inactive and active constraints:

$$\left(-G_A H^{-1} G'_A\right) u^*_{0,A} - W_A - S_A x = 0.$$

- **Assume rows of G_A are linearly independent** $\Rightarrow\ -G_A H^{-1} G'_A$ is square full rank, hence invertible

$$u^*_{0,A} = -\left(G_A H^{-1} G'_A\right)^{-1} \left(W_A + S_A x\right).$$

- By substituting $u^*_{0,A}$ into the expression of the optimizers:

$$z^*_0 = H^{-1} G'_A \left(G_A H^{-1} G'_A\right)^{-1} \left(W_A + S_A x\right).$$

mpQP. Computation of the critical region, optimizer and value function

The critical region CR_A is given as the intersection of the two polyhedra $\mathcal{P}_p$ and $\mathcal{P}_d$

$$CR_A = \{x \in \mathcal{K}^* : \ x \in \mathcal{P}_p, \ x \in \mathcal{P}_d\},$$

where $\mathcal{P}_p$ is obtained by substituting z_0^* in the primal feasibility conditions

$$\mathcal{P}_p = \{x \in \mathcal{K}^* : \ GH^{-1}G_A' \left(G_A H^{-1} G_A'\right)^{-1} (W_A + S_A x) < (W + Sx)\},$$

and $\mathcal{P}_d$ is obtained by substituting u_0^* in the dual feasibility conditions

$$\mathcal{P}_d = \{x \in \mathcal{K}^* : \ -\left(G_A H^{-1} G_A'\right)^{-1} (W_A + S_A x) \geq 0\}.$$

NOTE. The case when the rows of G_A are **NOT** linearly independent is reported in the textbook.

mpQP. Global properties of the solution

Under the assumption that no degeneracy occurs, the following theorem summarizes the global properties of the solution to the mpQP:

Theorem

i) *feasible set* $\mathcal{K}^*$ *is closed and convex*

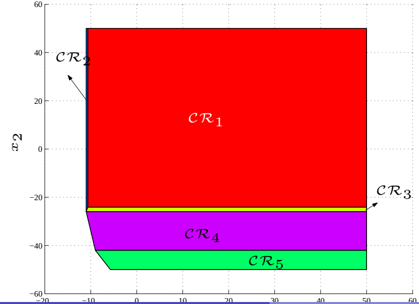
ii) *the optimizer function* $z^*(x) : \mathcal{K}^* \to \mathbb{R}^m$ *is:*
- *continuous*
- *PPWA over the critical regions* $\mathcal{CR}_i$

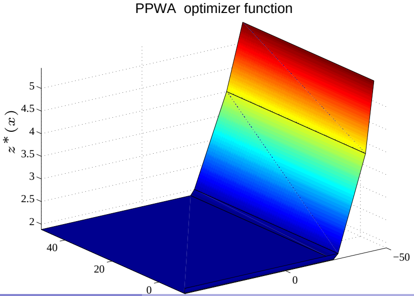
iii) *the value function* $J^*(x) : \mathcal{K}^* \to \mathbb{R}$ *is:*
- *continuous*
- *convex*
- *polyhedral piecewise quadratic (PPWQ) (piecewise quadratic over polyhedra), in particular over the critical regions* $\mathcal{CR}_i$

mpQP. Example

$$
\begin{array}{ll}
\min_{z} & \frac{1}{2}(z_1^2 + z_2^2), \\
\text{s.t.} & \left\{ \begin{array}{lcl}
z_1 + z_2 & \leq & 13 + x_1 \\
5z_1 - 4z_2 & \leq & 20 \\
-8z_1 + 2z_2 & \leq & 10 + x_2 \\
-4z_1 - z_2 & \leq & -8 \\
-z_1 & \leq & 0 \\
-z_2 & \leq & 0
\end{array} \right.
\end{array}
$$

Critical regions and the optimizer





mpQP. Example

$$z^*(x) = \begin{cases} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 1.88 \\ 0.47 \end{bmatrix} & \text{if } x \in \mathcal{CR}_1 \\ \begin{bmatrix} -0.3333 & 0 \\ 1.3333 & 0 \end{bmatrix} x + \begin{bmatrix} -1.67 \\ 14.67 \end{bmatrix} & \text{if } x \in \mathcal{CR}_2 \\ \begin{bmatrix} 0 & -0.0625 \\ 0 & 0.25 \end{bmatrix} x + \begin{bmatrix} 0.38 \\ 6.5 \end{bmatrix} & \text{if } x \in \mathcal{CR}_3 \\ \begin{bmatrix} 0 & -0.125 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} -1.25 \\ 0 \end{bmatrix} & \text{if } x \in \mathcal{CR}_4 \\ \begin{bmatrix} 0 & -0.1818 \\ 0 & -0.2273 \end{bmatrix} x + \begin{bmatrix} -3.64 \\ -9.55 \end{bmatrix} & \text{if } x \in \mathcal{CR}_5 \end{cases}$$

mpQP. Example

Consider the mpQP problem

$$
\begin{array}{rll}
J^*(x) = & \min\limits_{z} & \frac{1}{2}z'Hz + x'Fz \\
& \text{subj. to} & Gz \leq W + Sx,
\end{array}
$$

with

$$
\begin{array}{c} -8 \leq x_1 \leq 8 \\ -8 \leq x_2 \leq 8. \end{array}, \quad H = \begin{bmatrix} 8.18 & -3.00 & 5.36 \\ -3.00 & 5.00 & -3.00 \\ 5.36 & -3.00 & 10.90 \end{bmatrix}, \quad F = \begin{bmatrix} 0.60 & 0.00 & 1.80 \\ 5.54 & -3.00 & 8.44 \end{bmatrix}
$$

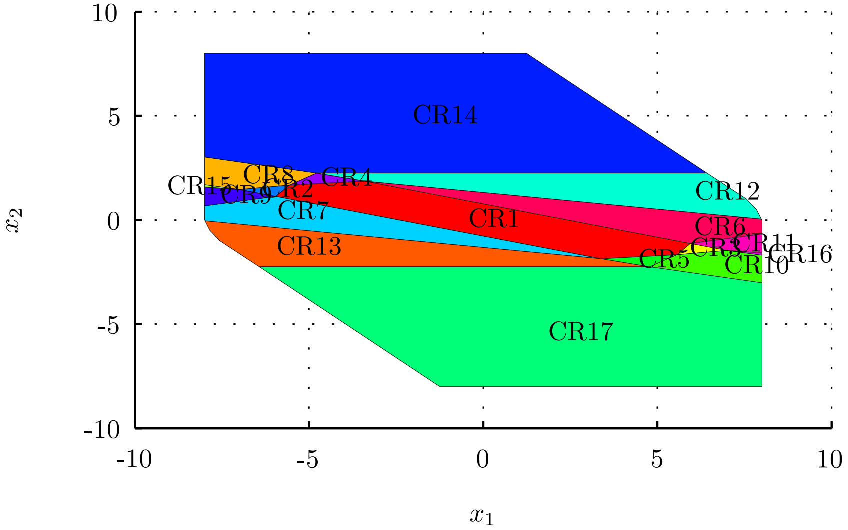
$$
G = \begin{bmatrix} 1.00 & -1.00 & 1.00 \\ -1.00 & 1.00 & -1.00 \\ 0.00 & 0.00 & -0.30 \\ -1.00 & 0.00 & -1.00 \\ 0.00 & 0.00 & 0.30 \\ 1.00 & 0.00 & 1.00 \\ -0.30 & 0.00 & -0.60 \\ 0.00 & -1.00 & 0.00 \\ 0.30 & 0.00 & 0.60 \\ 0.00 & 1.00 & 0.00 \\ -1.00 & 0.00 & 0.00 \\ 1.00 & 0.00 & 0.00 \\ 0.00 & 0.00 & 0.00 \\ 0.00 & 0.00 & -1.00 \\ 0.00 & 0.00 & 0.00 \\ 0.00 & 0.00 & 1.00 \\ 0.00 & 0.00 & -1.00 \\ 0.00 & 0.00 & 1.00 \end{bmatrix}, \quad E = \begin{bmatrix} 0.00 & -1.00 \\ 0.00 & 1.00 \\ 1.00 & 0.60 \\ 0.00 & 1.00 \\ -1.00 & -0.60 \\ 0.00 & -1.00 \\ 1.00 & 0.90 \\ 0.00 & 0.00 \\ -1.00 & -0.90 \\ 0.00 & 0.00 \\ 0.00 & 0.00 \\ 0.00 & 0.00 \\ 1.00 & 0.30 \\ 0.00 & 1.00 \\ -1.00 & -0.30 \\ 0.00 & -1.00 \\ 0.00 & 0.00 \\ 0.00 & 0.00 \end{bmatrix}, \quad W = \begin{bmatrix} 0.50 \\ 0.50 \\ 8.00 \\ 8.00 \\ 8.00 \\ 8.00 \\ 8.00 \\ 8.00 \\ 8.00 \\ 8.00 \\ 0.50 \\ 0.50 \\ 8.00 \\ 8.00 \\ 8.00 \\ 8.00 \\ 0.50 \\ 0.50 \end{bmatrix}
$$

mpQP. Example

The active constraints corresponding to the critical regions are

Critical Region	Active Constraints
CR1	{}
CR2	{1}
CR3	{2}
CR4	{11}
CR5	{12}
CR6	{17}
CR7	{18}
CR8	{1,11}
CR9	{1,18}
CR10	{2,12}
CR11	{2,17}
CR12	{11,17}
CR13	{12,18}
CR14	{1,11,17}
CR15	{1,11,18}
CR16	{2,12,17}
CR17	{2,12,18}

mpQP. Example



MILP. Problem formulation

Consider the mp-LP

$$
\begin{array}{rl}
J^*(x) = \min_{z} & \{J(z,x) = c'z\} \\
\text{subj. to} & Gz \le W + Sx,
\end{array}
$$

where $z \in \mathbb{R}^s$, $x \in \mathbb{R}^n$, $z \triangleq \{z_c, z_d\}$, $z_c \in \mathbb{R}^{s_c}$, $z_d \in \{0,1\}^{s_d}$ and $s \triangleq s_c + s_d$, $G' = [{G_1}' \dots {G_m}']$ and $G_j \in \mathbb{R}^n$ denotes the j-th row of G, $c \in \mathbb{R}^s$, $W \in \mathbb{R}^m$, and $S \in \mathbb{R}^{m \times n}$.
The set of parameters is defined as

$$
\mathcal{K} \triangleq \{x \in \mathbb{R}^n : \ Tx \le Z\}.
$$

Objective. Calculating the

- region $\mathcal{K}^* \subseteq \mathcal{K}$ of feasible parameters x,
- the expression of the value function $J^*(x)$,
- the expression of an optimizer function $z^*(x) \in Z^*(x)$.

mpMILP. Initialization

Solve the following problem

$$\begin{array}{ll} \min\limits_{\{z,x\}} & c'z \\ \text{subj. to} & Gz - Sx \leq W \\ & x \in \mathcal{K} \end{array}$$

where x is an independent variable. If the problem is infeasible then $\mathcal{K}^* = \emptyset$.

Initialization

Otherwise, z^* provides a feasible integer variable $\bar{z}_d$.
Set $j = 0$, $N_0 = 1$, $CR_1 = \mathcal{K}$, $Z_1 = \emptyset$, $\bar{J}_1 = +\infty$, $\bar{z}_{d_1}^1 = \bar{z}_d$.

mpLP subproblem

In CR_1 (i.e., $\mathcal{K}$), solve the mpLP problem

$$\begin{array}{rll} \tilde{J}_1(x) = & \min\limits_{z} & c'z \\ & \text{subj. to} & Gz \leq W + Sx \\ & & z_d = \bar{z}^1_{d_1} \\ & & x \in CR_1, \end{array}$$

and obtain

- a partition
$$CR_1 = \bigcup_{k=1}^{N_{R_1}} R_1^k$$
- a PPWA cost defined over the polyhedral partition $R_1^k,\ k = 1, \ldots, N_{R_1}$
$$\tilde{J}_1(x) = (\tilde{JR}_1^k(x) \triangleq c_1^k x + p_1^k) \text{ if } x \in \{R_1^k\},\ k = 1, \ldots, N_{R_1}$$

Note that if $\exists j : \tilde{JR}_1^j(x) = +\infty,\ \Rightarrow \bar{z}^1_{d_1}$ is infeasible in R_1^j.

mpLP subproblem

Update the costs $\bar{J}_1$ as follows

1. $\bar{J}_1(x) = \tilde{JR}_1^k(x)\ \forall x \in R_1^k$ if $(\tilde{JR}_1^k(x) \leq \bar{J}_1(x)\ \forall\ x \in R_1^k)$
2. $\bar{J}_1(x) = \bar{J}_1(x)\ \forall x \in R_1^k$ (if $\tilde{JR}_1^k(x) \geq \bar{J}_1(x)\ \forall\ x \in R_1^k)$
3. $\bar{J}_1(x) = \begin{cases} \bar{J}_1(x) & \forall x \in (R_1^k)_1 \triangleq \{x \in R_1^k :\ \tilde{JR}_1^k(x) \geq \bar{J}_1(x)\} \\ \tilde{JR}_1^k(x) & \forall x \in (R_1^k)_2 \triangleq \{x \in R_1^k :\ \tilde{JR}_1^k(x) \leq \bar{J}_1(x)\} \end{cases}$

$$k = 1, \ldots, N_{R_1}$$

MILP subproblem

Refer to the polyhedra of the new partition as CR_i, i.e., set $CR_k = \{R_1^k\},\ k = 1, \ldots, N_{R_1}$. Set $Z_i = \{\bar{z}_{d_1}^1\}$

At step 2 for each critical region CR_i solve the MILP problem.

$$
\begin{array}{ll}
\min\limits_{\{z,x\}} & c'z \\
\text{subj. to} & Gz - Sx \leq W \\
& c'z \leq \bar{J}_1(x) \\
& z_d \neq \bar{z}_{d_1} \\
& x \in CR_i,
\end{array}
$$

- if the problem is infeasible, CR_i is excluded from further explorations
- otherwise, z_d^* is a feasible integer variable and is stored $\bar{z}_{d_i}^{Nb_i+1} = z_d^*$.

MIQP

The geometric approach does ***NOT*** work for generic mpMIQPs.

In general the critical region cannot be decomposed in convex polyhedra (see example in Section 6.5 of the textbook).

Methods for solving mpMIQP problems derived from optimal control problem for hybrid systems exist.