Model Predictive Control for Linear and Hybrid Systems
Feasible Sets and Invariant Sets

Francesco Borrelli

Department of Mechanical Engineering,
University of California at Berkeley,
USA

fborrelli@berkeley.edu

March 17, 2011

Constrained Linear Optimal Control

Consider the cost function

$$J_0(x(0), U_0) \triangleq p(x_N) + \sum_{k=0}^{N-1} q(x_k, u_k)$$

and the constrained finite time optimal control problem (CFTOC)

$$\begin{array}{rll} J_0^*(x(0)) = & \min_{U_0} & J_0(x(0), U_0) \\ & \text{subj. to} & x_{k+1} = Ax_k + Bu_k,\ k = 0, \ldots, N-1 \\ & & x_k \in \mathcal{X},\ u_k \in \mathcal{U},\ k = 0, \ldots, N-1 \\ & & x_N \in \mathcal{X}_f \\ & & x_0 = x(0) \end{array} \tag{1}$$

where N is the time horizon and $\mathcal{U},\ \mathcal{X},\ \mathcal{X}_f$ are polyhedral regions.

- Denote by $U_0 \triangleq [u_0', \ldots, u_{N-1}']' \in \mathbb{R}^s$, $s \triangleq mN$ the optimization vector.
- If the 1-norm or ∞-norm is used in the cost function (1), then $p(x_N) = \|Px_N\|_p$ and $q(x_k, u_k) = \|Qx_k\|_p + \|Ru_k\|_p$.
- If the squared euclidian norm is used in the cost function (1), then $p(x_N) = x_N' P x_N$ and $q(x_k, u_k) = x_k' Q x_k + u_k' R u_k$.

Feasible Sets

Denote by $\mathcal{X}_0 \subseteq \mathcal{X}$ the set of initial states $x(0)$ for which the optimal control problem (1) is feasible, i.e.,

$$\begin{array}{rl}\mathcal{X}_0 = & \{x_0 \in \mathbb{R}^n | \ \exists(u_0, \ldots, u_{N-1}) \text{ such that } x_k \in \mathcal{X},\ u_k \in \mathcal{U}, \\ & k = 0, \ldots, N-1,\ x_N \in \mathcal{X}_f,\ \text{where } x_{k+1} = Ax_k + Bu_k, \\ & k = 0, \ldots, N-1\},\end{array}$$

We denote with $\mathcal{X}_i$ the set of states x_i at time i for which (1) is feasible

$$\begin{array}{rl}\mathcal{X}_i = & \{x_i \in \mathbb{R}^n | \ \exists(u_i, \ldots, u_{N-1}) \text{ such that } x_k \in \mathcal{X},\ u_k \in \mathcal{U}, \\ & k = i, \ldots, N-1,\ x_N \in \mathcal{X}_f,\ \text{where } x_{k+1} = Ax_k + Bu_k\},\end{array}$$

The sets $\mathcal{X}_i$ for $i = 0, \ldots, N$ play an important role in the the solution of the CFTOC. They are ***independent*** on ***the cost*** and the ***algorithm*** used for solving the problem.

Feasible Sets. Batch Approach

Be $A_x x \leq b_x$, $A_f x \leq b_f$, $A_u u \leq b_u$ the $\mathcal{H}$-representations of sets $\mathcal{X}$, $\mathcal{X}_f$ and $\mathcal{U}$, respectively. Define the polyhedron $\mathcal{P}_i$ for $i = 0, \ldots, N-1$ as follows

$$\mathcal{P}_i = \{(U_i, x_i) \in \mathbb{R}^{m(N-i)+n} |\ G_i U_i - E_i x_i \leq W_i\}$$

where G_i, E_i and W_i are defined as follows

$$G_i = \begin{bmatrix} A_u & 0 & \ldots & 0 \\ 0 & A_u & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & A_u \\ 0 & 0 & \ldots & 0 \\ A_x B & 0 & \ldots & 0 \\ A_x A B & A_x B & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ A_f A^{N-i-1} B & A_f A^{N-i-2} B & \ldots & A_f B \end{bmatrix}$$

Feasible Sets. Batch Approach

$$E_i = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ -A_x \\ -A_x A \\ -A_x A^2 \\ \vdots \\ -A_f A^{N-i} \end{bmatrix} \quad W_i = \begin{bmatrix} b_u \\ b_u \\ \vdots \\ b_u \\ b_x \\ b_x \\ b_x \\ \vdots \\ b_f \end{bmatrix}$$

Then set $\mathcal{X}_i$ is a **polyhedron** and can be computed by **projecting** the polyhedron $\mathcal{P}_i$ on the x_i space.

Feasible Sets Recursive Approach

Define the set $\mathcal{X}_i$ as

$$\begin{aligned}
\mathcal{X}_i &= \{x \in \mathcal{X} |\ \exists u \in \mathcal{U} \text{ such that } Ax + Bu \in \mathcal{X}_{i+1}\ \},\\
&\quad\ i = 0, \ldots, N-1\\
\mathcal{X}_N &= \mathcal{X}_f.
\end{aligned}$$

- $\mathcal{X}_i$ as in the batch approach.
- more efficient approach for computing the sets $\mathcal{X}_i$, since at each time step a smaller polyhedron needs to be projected.
- Let $\mathcal{X}_i$ be the $\mathcal{H}$-polyhedra $A_{\mathcal{X}_i} x \leq b_{\mathcal{X}_i}$. Then the set $\mathcal{X}_{i-1}$ is the projection of the following polyhedron

$$\begin{bmatrix} A_u \\ 0 \\ A_{\mathcal{X}_i} B \end{bmatrix} u_i + \begin{bmatrix} 0 \\ A_x \\ A_{\mathcal{X}_i} A \end{bmatrix} x_{i-1} \leq \begin{bmatrix} b_u \\ b_x \\ b_{\mathcal{X}_i} \end{bmatrix} \tag{2}$$

  on the x_i space.
- Compactly

$$\mathcal{X}_i = \mathrm{Pre}(\mathcal{X}_{i+1}) \cap \mathcal{X}$$

Contorllable Sets

Definition (N-Step Controllable Set $\mathcal{K}_N(\mathcal{O})$)

For a given target set $\mathcal{O} \subseteq \mathcal{X}$, the N-step controllable set $\mathcal{K}_N(\mathcal{O})$ is defined as:

$$\mathcal{K}_N(\mathcal{O}) \triangleq \operatorname{Pre}(\mathcal{K}_{N-1}(\mathcal{O})) \cap \mathcal{X}, \quad \mathcal{K}_0(\mathcal{O}) = \mathcal{O}, \qquad N \in \mathbb{N}^+.$$

All states $x_0 \in \mathcal{K}_N(\mathcal{O})$ can be driven,through a time-varying control law, to the target set $\mathcal{O}$ in N steps, while satisfying input and state constraints.

Definition (Maximal Controllable Set $\mathcal{K}_\infty(\mathcal{O})$)

For a given target set $\mathcal{O} \subseteq \mathcal{X}$, the maximal controllable set $\mathcal{K}_\infty(\mathcal{O})$ for the system $x(t+1) = f(x(t), u(t))$ subject to the constraints $x(t) \in \mathcal{X},\ u(t) \in \mathcal{U}$ is the union of all N-step controllable sets contained in $\mathcal{X}$ ($N \in \mathbb{N}$).

N-Step Reachable Sets

Definition (N-Step Reachable Set $\mathcal{R}_N(\mathcal{X}_0)$)

For a given initial set $\mathcal{X}_0 \subseteq \mathcal{X}$, the N-step reachable set $\mathcal{R}_N(\mathcal{X}_0)$ is

$$\mathcal{R}_{i+1}(\mathcal{X}_0) \triangleq \mathrm{Reach}(\mathcal{R}_i(\mathcal{X}_0)), \quad \mathcal{R}_0(\mathcal{X}_0) = \mathcal{X}_0, \qquad i = 0, \ldots, N-1$$

All states $x_0 \in \mathcal{X}_0$ can will evolve to the N-step reachable set $\mathcal{R}_N(\mathcal{X}_0)$ in N steps

Same definition of Maximal Reachable Set $\mathcal{R}_\infty(\mathcal{X}_0)$ can be introduced.

Invariant Sets

Invariant sets

- are computed for ***autonomous systems***
- for a *given* feedback controller $u = g(x)$, provide the set of initial states whose trajectory will never violate the system constraints.

Definition (Positive Invariant Set)

A set $\mathcal{O}$ is said to be a positive invariant set for the autonomous system $x(t+1) = f_a(x(t))$ subject to the constraints $x(t) \in \mathcal{X}$, if

$$x(0) \in \mathcal{O} \quad \Rightarrow \quad x(t) \in \mathcal{O}, \quad \forall t \in \mathbb{N}^+$$

Definition (Maximal Positive Invariant Set $\mathcal{O}_\infty$)

The set $\mathcal{O}_\infty$ is the maximal invariant set if $0 \in \mathcal{O}_\infty$, $\mathcal{O}_\infty$ is invariant and $\mathcal{O}_\infty$ contains all invariant sets that contain the origin.

Invariant Sets

Theorem (Geometric condition for invariance)

A set $\mathcal{O}$ is a positive invariant set if and only if

$$\mathcal{O} \subseteq \text{Pre}(\mathcal{O})$$

Proof: We prove the contrapositive for both the necessary and sufficient parts.

- (necessary) If $\mathcal{O} \nsubseteq \text{Pre}(\mathcal{O})$ then $\exists \bar{x} \in \mathcal{O}$ such that $\bar{x} \notin \text{Pre}(\mathcal{O})$. From the definition of $\text{Pre}(\mathcal{O})$, $f_a(\bar{x}) \notin \mathcal{O}$ and thus $\mathcal{O}$ is not a positive invariant.
- (sufficient) If $\mathcal{O}$ is not a positive invariant set then $\exists \bar{x} \in \mathcal{O}$ such that $f_a(\bar{x}) \notin \mathcal{O}$. This implies that $\bar{x} \in \mathcal{O}$ and $\bar{x} \notin \text{Pre}(\mathcal{O})$ and thus $\mathcal{O} \nsubseteq \text{Pre}(\mathcal{O})$

$\square$

Clearly

$$\mathcal{O} \subseteq \text{Pre}(\mathcal{O}) \iff \text{Pre}(\mathcal{O}) \cap \mathcal{O} = \mathcal{O} \tag{3}$$

Invariant Sets

Algorithm

Input: f_a , $\mathcal{X}$

Output: $\mathcal{O}_\infty$

1 **let** $\Omega_0 = \mathcal{X}$,

2 **let** $\Omega_{k+1} = \text{Pre}(\Omega_k) \cap \Omega_k$

3 **if** $\Omega_{k+1} = \Omega_k$ **then** $\mathcal{O}_\infty \leftarrow \Omega_{k+1}$

4 **else go to** 2

The algorithm generates the set sequence $\{\Omega_k\}$ satisfying $\Omega_{k+1} \subseteq \Omega_k, \forall k \in \mathbb{N}$ and it terminates when $\Omega_{k+1} = \Omega_k$ so that Ω_k is the maximal positive invariant set $\mathcal{O}_\infty$ for $x(t+1) = f_a(x(t))$.

Control Invariant Sets

Control invariant sets

- are computed for systems ***subject to external inputs***
- provide the set of initial states for which *there exists* a controller such that the system constraints are never violated.

Definition (Control Invariant Set)

A set $\mathcal{C} \subseteq \mathcal{X}$ is said to be a control invariant set if

$$x(t) \in \mathcal{C} \quad \Rightarrow \quad \exists u(t) \in \mathcal{U} \text{ such that } f(x(t), u(t)) \in \mathcal{C}, \quad \forall t \in \mathbb{N}^+$$

Definition (Maximal Control Invariant Set $\mathcal{C}_\infty$)

The set $\mathcal{C}_\infty$ is said to be the maximal control invariant set for the system $x(t+1) = f(x(t), u(t))$ subject to the constraints in $x(t) \in \mathcal{X},\ u(t) \in \mathcal{U}$, if it is control invariant and contains all control invariant sets contained in $\mathcal{X}$.

Control Invariant Sets

Same geometric condition for control invariants holds: $\mathcal{C}$ is a control invariant set if and only if

$$\mathcal{C} \subseteq \mathrm{Pre}(\mathcal{C}) \tag{4}$$

Algorithm

Input: f, $\mathcal{X}$ and $\mathcal{U}$

Output: $\mathcal{C}_\infty$

1 **let** $\Omega_0 = \mathcal{X}$,

2 **let** $\Omega_{k+1} = \mathrm{Pre}(\Omega_k) \cap \Omega_k$

3 **if** $\Omega_{k+1} = \Omega_k$ **then** $\mathcal{C}_\infty \leftarrow \Omega_{k+1}$

4 **else go to** 2

The algorithm generates the set sequence $\{\Omega_k\}$ satisfying $\Omega_{k+1} \subseteq \Omega_k, \forall k \in \mathbb{N}$ and it terminates if $\Omega_{k+1} = \Omega_k$ so that Ω_k is the maximal control invariant set $\mathcal{C}_\infty$ for the constrained system.

Invariant Sets and Control Invariant Sets

- The set $\mathcal{O}_\infty$ ($\mathcal{C}_\infty$) is ***finitely determined*** if and only if $\exists\ i \in \mathbb{N}$ such that $\Omega_{i+1} = \Omega_i$.
- The smallest element $i \in \mathbb{N}$ such that $\Omega_{i+1} = \Omega_i$ is called the ***determinedness index***.
- For linear system with linear constraints the sets $\mathcal{O}_\infty$ and $\mathcal{C}_\infty$ are polyhedra if they are finitely determined.
- For autonomous systems, if the does not terminate then $\mathcal{O}_\infty = \bigcap_{k\geq 0} \Omega_k$. If $\Omega_k = \emptyset$ for some integer k then $\mathcal{O}_\infty = \emptyset$. The same holds true for non-autonomous systems.
- For all states contained in the maximal control invariant set $\mathcal{C}_\infty$ there exists a control law, such that the system constraints are never violated. This does not imply that there exists a control law which can drive the state into a user-specified target set.

Stabilizable Sets

Observe that controllable sets $\mathcal{K}_N(\mathcal{O})$ where the target $\mathcal{O}$ is a control invariant set are special sets

Definition (N-step (Maximal) Stabilizable Set)

For a given control invariant set $\mathcal{O} \subseteq \mathcal{X}$, the N-step (maximal) stabilizable set is the N-step (maximal) controllable set $\mathcal{K}_N(\mathcal{O})$ ($\mathcal{K}_\infty(\mathcal{O})$).

In addition to guaranteeing that from $\mathcal{K}_N(\mathcal{O})$ we reach $\mathcal{O}$ in N steps, one can ensure that once it has reached $\mathcal{O}$, the system can stay there at all future time instants.

Feasible Set Evolution

Theorem

Let the terminal constraint set $\mathcal{X}_f$ be equal to $\mathcal{X}$. *Then,*

1. *The feasible set $\mathcal{X}_i$, $i = 0, \dots, N-1$ is equal to the $(N-i)$-step controllable set:*
$$\mathcal{X}_i = \mathcal{K}_{N-i}(\mathcal{X})$$
2. *The feasible set $\mathcal{X}_i$, $i = 0, \dots, N-1$ contains the maximal control invariant set:*
$$\mathcal{C}_\infty \subseteq \mathcal{X}_i$$
3. *The feasible set $\mathcal{X}_i$ is control invariant if and only if the maximal control invariant set is finitely determined and $N-i$ is equal to or greater than its determinedness index $\bar{N}$, i.e.*
$$\mathcal{X}_i \subseteq Pre(\mathcal{X}_i) \Leftrightarrow \mathcal{C}_\infty = \mathcal{K}_{N-i}(\mathcal{X}) \ \ \textit{for all } i \leq N - \bar{N}$$

Feasible Set Evolution

Theorem

4. $\mathcal{X}_i \subseteq \mathcal{X}_j$ *if* $i < j$ *for* $i = 0, \ldots, N-1$. *The size of the feasible set* $\mathcal{X}_i$ *stops decreasing (with decreasing* i*) if and only if the maximal control invariant set is finitely determined and* $N - i$ *is larger than its determinedness index, i.e.*

$$\mathcal{X}_i \subset \mathcal{X}_j \text{ if } N - \bar{N} < i < j < N$$

Furthermore,

$$\mathcal{X}_i = \mathcal{C}_\infty \text{ if } i \leq N - \bar{N}$$

Feasible Set Evolution

Theorem

Let the terminal constraint set $\mathcal{X}_f$ be a control invariant subset of $\mathcal{X}$. *Then,*

1. *The feasible set $\mathcal{X}_i$, $i = 0, \ldots, N-1$ is equal to the $(N-i)$-step stabilizable set:*

$$\mathcal{X}_i = \mathcal{K}_{N-i}(\mathcal{X}_f)$$

2. *The feasible set $\mathcal{X}_i$, $i = 0, \ldots, N-1$ is control invariant and contained within the maximal control invariant set:*

$$\mathcal{X}_i \subseteq \mathcal{C}_\infty$$

Feasible Set Evolution

Theorem

3. $\mathcal{X}_i \supseteq \mathcal{X}_j$ *if* $i < j$, $i = 0, \ldots, N-1$. *The size of the feasible* $\mathcal{X}_i$ *set stops increasing (with decreasing* i*) if and only if the maximal stabilizable set is finitely determined and* $N - i$ *is larger than its determinedness index, i.e.*

$$\mathcal{X}_i \supset \mathcal{X}_j \text{ if } N - \bar{N} < i < j < N$$

Furthermore,

$$\mathcal{X}_i = \mathcal{K}_\infty(\mathcal{X}_f) \text{ if } i \leq N - \bar{N}$$

Feasible Set Evolution

- With $\mathcal{X}_f = \mathcal{X}$, $\mathcal{X}_i$ shrinks as i decreases and stops shrinking when it becomes the maximal control invariant set. Also, depending on i, either it is not a control invariant set or it is the maximal control invariant set.
- If a control invariant set is chosen as terminal constraint $\mathcal{X}_f$, the set $\mathcal{X}_i$ is always a control invariant. Moreover it grows as i decreases and stops growing when it becomes the maximal stabilizable set.

Interesting Example

Consider the simple constrained one-dimensional system

$$x(t+1) = 2x(t) + u(t), \tag{5a}$$

$$|x(t)| \le 1, \text{ and } |u(t)| \le 1 \tag{5b}$$

and the state-feedback control law

$$u(t) = \begin{cases} 1 & \text{if} \quad x(t) \in \left[-1,\ -\frac{1}{2}\right], \\ -2x(t) & \text{if} \quad x(t) \in \left[-\frac{1}{2},\ \frac{1}{2}\right], \\ -1 & \text{if} \quad x(t) \in \left[\frac{1}{2},\ 1\right] \end{cases} \tag{6}$$

- The closed-loop system has three equilibria at -1, 0, and 1 and the system is always feasible for all initial states in $[-1,\ 1] \Rightarrow \mathcal{C}_\infty = [-1,\ 1]$.
- It is, however,asymptotically stable only for the open set $(-1,\ 1)$.
- Therefore when $\mathcal{O} = 0$ then $\mathcal{K}_\infty(\mathcal{O}) = (-1,\ 1) \subset \mathcal{C}_\infty$.
- In this example the maximal stabilizable set is open. One can easily argue that, in general, if the maximal stabilizable set is closed then it is equal to the maximal control invariant set.