Introduction to Model Predictive Control

Lectures 5-6: Optimization

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Summarizing...

Need

1. A discrete-time model of the system (Matlab, Simulink)
2. A state observer
3. Set up an Optimization Problem (Matlab, MPT toolbox/Yalmip)
4. Solve an optimization problem (Matlab/Optimization Toolbox, NPSOL)
5. Verify that the closed-loop system performs as desired (avoid infeasibility/stability)
6. Make sure it runs in real-time and code/download for the embedded platform

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Fundamental Concepts

Abstract Optimization Problems

- Optimization problems are abundant in economical daily life
- General abstract features of problem formulation:
  - Decision set Z
  - **Constraints** on decision and subset $S \subseteq Z$ of **feasible** decisions.
  - Assign to each decision a **cost** $f(x) \in \mathbb{R}$.
  - Goal is to **minimize** the cost by choosing a suitable decision.
- Concrete features of problem formulation:
  - Z is real vector space: Continuous problem.
  - Z is discrete set: Discrete or combinatorial problem.
  - if $dim(Z)$ is finite: Finite Dimensional problem.
  - if $dim(Z)$ is infinite: Infinite Dimensional problem.

Concrete Optimization Problems

Generally formulated as

$$\begin{array}{ll} \inf_z & f(z) \\ \text{such that} & z \in S \subseteq Z \end{array} \tag{1}$$

- The **vector** z collects the decision variables
- Z is the **domain** of the decision variables,
- $S \subseteq Z$ is the set of feasible or admissible decisions.
- The function $f : Z \to \mathbb{R}$ assigns to each decision z a cost $f(z) \in \mathbb{R}$.

Shorter form of problem (1)

$$\inf_{z \in S \subseteq Z} f(z) \tag{2}$$

Problem (2) is a called nonlinear mathematical program or simply nonlinear program.

Concrete Optimization Problems

Solving problem (2) means

- Compute the least possible cost J^* (called the **optimal value**)

$$J^* \triangleq \inf_{z \in S} f(z) \tag{3}$$

J^* is the greatest lower bound of $f(z)$ over the set S:

$$f(z) \geq J^*, \ \forall z \in S$$

AND

1. $$\exists \bar{z} \in S : f(\bar{z}) = J^*$$

OR

2. $$\forall \varepsilon > 0 \ \exists z \in S | \ f(z) \leq J^* + \varepsilon$$

- Compute the **optimizer**, $z^* \in S \text{ with} f(z^*) = J^*$. If z^* exists, then rewrite (3) as

$$J^* = \min_{z \in S} f(z) \tag{4}$$

Concrete Optimization Problems

Consider the Nonlinear Program (NLP)

$$J^* = \min_{z \in S} f(z)$$

Notation:

- If $J^* = -\infty$ the problem is unbounded below.
- If the set S is empty then the problem is said to be infeasible (we set $J^* = +\infty$).
- If $S = Z$ the problem is said to be unconstrained.
- The set of all optimal solutions is denoted by

$$\text{argmin}_{z \in S} f(z) \triangleq \{z \in S : f(z) = J^*\}$$

Continuous Problems

- The domain Z is a subset of $\mathbb{R}^s$ (the finite-dimensional Euclidian vector-space)
- The subset of admissible vectors defined through a list of real valued functions:

$$\begin{array}{lll} \inf_z & f(z) & \\ \text{such that} & g_i(z) \leq 0 & \text{for } i = 1, \dots, m \\ & h_i(z) = 0 & \text{for } i = 1, \dots, p \\ & z \in Z & \end{array} \tag{5}$$

- The inequalities $g_i(z) \leq 0$ are called inequality constraints and the equations $h_i(z) = 0$ are called equality constraints.
- A point $\bar{z} \in \mathbb{R}^s$ is feasible for problem (5) if it satisfies all inequality and equality constraints
- The set of feasible vectors is

$$S = \{z \in \mathbb{R}^s : g_i(z) \leq 0,\ i = 1, \dots, m,\\ h_i(z) = 0,\ i = 1, \dots, p\}.$$

Local and Global Optimizer

- Let J^* be the optimal value of problem (5). A **global optimizer**, if it exists, is a feasible vector z^* with $f(z^*) = J^*$.
- A feasible point $\bar{z}$ is a **local optimizer** for problem (5) if there exists an $R > 0$ such that

$$\begin{array}{rll} f(\bar{z}) = & \inf_z & f(z) \\ & \text{such that} & g_i(z) \leq 0 \quad \text{for } i = 1, \ldots, m \\ & & h_i(z) = 0 \quad \text{for } i = 1, \ldots, p \\ & & \|z - \bar{z}\| \leq R \\ & & z \in Z \end{array} \tag{6}$$

Active, Inactive and Redundant Constraints

Consider

$$\begin{array}{lll} \inf_z & f(z) & \\ \text{such that} & g_i(z) \le 0 & \text{for } i = 1, \dots, m \\ & h_i(z) = 0 & \text{for } i = 1, \dots, p \\ & z \in Z & \end{array}$$

- The i-th inequality constraint $g_i(z) \le 0$ is active at $\bar{z}$ if $g_i(\bar{z}) = 0$.
- If $g_i(\bar{z}) < 0$ we say that the constraint $g_i(z) \le 0$ is inactive at $\bar{z}$.
- Equality constraints are always active for all feasible points.
- We say that a constraint is redundant if removing it from the list of constraints does not change the feasible set S. This implies that removing a redundant constraint from the optimization problem does not change its solution.

Problem Description

- The functions f, g_i and h_i can be available in **analytical form** or can be described through an **oracle model** (also called "black box" or "subroutine" model).
- In an oracle model f, g_i and h_i are not known explicitly but can be evaluated by querying the oracle. Often the oracle consists of subroutines which, called with the augment z, return $f(z)$, $g_i(z)$ and $h_i(z)$ and their gradients $\nabla f(z)$, $\nabla g_i(z)$, $\nabla h_i(z)$.

Integer and Mixed-Integer Problems

- If the decision set Z in the optimization problem is finite, then the optimization problem is called combinatorial or discrete.
- If $Z \subseteq \{0,1\}^s$, then the problem is said to be integer.
- If Z is a subset of the Cartesian product of an integer set and a real Euclidian space, i.e., $Z \subseteq \{[z_c, z_b] : z_c \in \mathbb{R}^{s_c}, z_b \in \{0,1\}^{s_b}\}$, then the problem is said to be mixed-integer.

The standard formulation of a mixed-integer non-linear program is

$$\begin{array}{ll} \inf_{[z_c,z_b]} & f(z_c, z_b) \\ \text{such that} & g_i(z_c, z_b) \leq 0 \quad \text{for } i = 1, \ldots, m \\ & h_i(z_c, z_b) = 0 \quad \text{for } i = 1, \ldots, p \\ & z_c \in \mathbb{R}^{s_c},\ z_b \in \{0,1\}^{s_b} \end{array} \tag{7}$$

Convexity

A set $S \in \mathbb{R}^s$ is convex if

$$\lambda z_1 + (1-\lambda) z_2 \in S \text{ for all } z_1, z_2 \in S, \lambda \in [0,1].$$

A function $f : S \to \mathbb{R}$ is convex if S is convex and

$$\begin{aligned} f(\lambda z_1 + (1-\lambda) z_2) \quad &\leq \quad \lambda f(z_1) + (1-\lambda) f(z_2) \\ &\quad\quad \text{for all } z_1, z_2 \in S, \lambda \in [0,1]. \end{aligned}$$

A function $f : S \to \mathbb{R}$ is strictly convex if S is convex and

$$\begin{aligned} f(\lambda z_1 + (1-\lambda) z_2) \quad &< \quad \lambda f(z_1) + (1-\lambda) f(z_2) \\ &\quad\quad \text{for all } z_1, z_2 \in S, \lambda \in (0,1). \end{aligned}$$

A function $f : S \to \mathbb{R}$ is concave if S is convex and $-f$ is convex.

Operations preserving convexity

1. The intersection of an arbitrary number of convex sets is a convex set:

$$\text{if } S_1, S_2, \ldots, S_k \text{ are convex, then } S_1 \cap S_2 \cap \ldots \cap S_k \text{ is convex.}$$

2. The sub-level sets of a convex function f on S are convex:

$$\text{if } f(z) \text{ is convex then } S_\alpha \triangleq \{z \in S : f(z) \leq \alpha\} \text{ is convex } \forall\alpha.$$

3. If $f_1, \ldots, f_N$ are convex functions, then $\sum_{i=1}^{N} \alpha_i f_i$ is a convex function for all $\alpha_i \geq 0$, $i = 1, \ldots, N$.
4. The composition of a convex function $f(z)$ with an affine map $z = Ax + b$ generates a convex function $f(Ax + b)$ of x.

Operations preserving convexity

- The pointwise maximum of a set of convex functions is a convex function:

$$f_1(z), \dots, f_k(z) \text{ convex functions} \Rightarrow$$
$$f(z) = \max\{f_1(z), \dots, f_k(z)\} \text{ is a convex function.}$$

- A linear function $f(z) = c'z + d$ is both convex and concave.
- A quadratic function $f(z) = z'Qz + 2s'z + r$ is convex if and only if $Q \in \mathbb{R}^{s\times s}$ is positive semidefinite.
- A quadratic function $f(z) = z'Qz + 2s'z + r$ is strictly convex if and only if $Q \in \mathbb{R}^{s\times s}$ is positive definite.

Convex optimization problems

- The optimization problem (5) is said to be convex if the cost function f is convex on Z and S is a convex set.
- A fundamental property of convex optimization problems is that local optimizers are also global optimizers. It suffices to compute a local minimum to problem (5) to determine its global minimum.
- Convexity also plays a major role in most non-convex optimization problems: solved by iterating between the solutions of convex sub-problems.
- Difficult to determine whether the feasible set S is convex or not except in special cases. For example, if the functions $g_1, \ldots, g_m$ are convex and all the $h_i(z)$ (if any) are affine in z, then the feasible region S is an intersection of convex sets and therefore convex.
- Moreover there are non-convex problems which can be transformed into convex problems through a change of variables and manipulations on cost and constraints.

Polyhedra, Polytopes and Simplices

General Set Definitions and Operations

- An n-**dimensional ball** $B(x_0, \rho)$ is the set $B(x_0, \rho) = \{x \in \mathbb{R}^n | \sqrt{\|x - x_0\|_2} \leq \rho\}$. The vector x_0 is the center of the ball and ρ is the radius.
- Affine sets are sets described by the solutions of a system of linear equations:

$$F = \{x \in \mathbb{R}^n \ : \ Ax = b, \ \text{ with } \ A \in \mathbb{R}^{m \times n}, \ b \in \mathbb{R}^m\}.$$

- The affine combination of a finite set of points $x_1, \ldots, x_k$ belonging to $\mathbb{R}^n$ is defined as the point $\lambda_1 x_1 + \ldots + \lambda_k x_k$ where $\sum_{i=1}^k \lambda_i = 1$.
- The affine hull of $K \subseteq \mathbb{R}^n$ is the set of all affine combinations of points in K and it is the smallest affine set that contains K.
- The dimension of an affine set is the dimension of the largest ball of radius $\rho > 0$ included in the set.

General Set Definitions and Operations

- The convex combination of a finite set of points $x_1, \ldots, x_k$ is defined as the point $\lambda_1 x_1 + \ldots + \lambda_k x_k$ where $\sum_{i=1}^{k} \lambda_i = 1$ and $\lambda_i \geq 0$, $i = 1, \ldots, k$.
- The convex hull of a set $K \subseteq \mathbb{R}^n$ is the set of all convex combinations of points in K and it is denoted as $\mathrm{conv}(K)$:

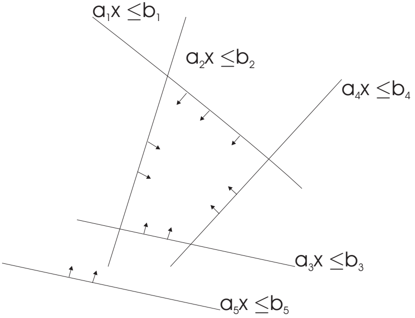
$$\mathrm{conv}(K) \triangleq \{\lambda_1 x_1 + \ldots + \lambda_k x_k \mid x_i \in K,\ \lambda_i \geq 0,\ \sum_{i=1}^{k} \lambda_i = 1\}.$$

Polyhedra Definitions and Representations

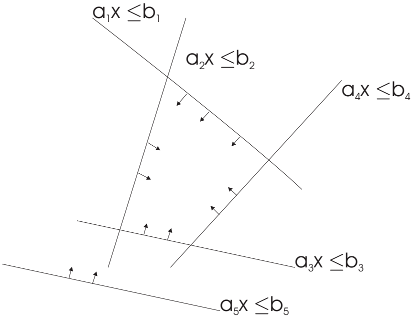
An $\mathcal{H}$-polyhedron $\mathcal{P}$ in $\mathbb{R}^n$ denotes an intersection of a finite set of closed halfspaces in $\mathbb{R}^n$:

$$\mathcal{P} = \{x \in \mathbb{R}^n : \ Ax \le b\}$$

where $Ax \le b$ is $a_i x \le b_i, \ i = 1, \ldots, m$, where $a_1, \ldots, a_m$ are the rows of A, and $b_1, \ldots, b_m$ are the components of b.



Polyhedra Definitions and Representations



- Inequalities which can be removed without changing the polyhedron are called redundant.
- The representation of an $\mathcal{H}$-polyhedron is minimal if it does not contain redundant inequalities.

Polyhedra Definitions and Representations

1. An $\mathcal{H}$-polytope is a bounded $\mathcal{H}$-polyhedron.
2. A $\mathcal{V}$-polytope is the convex hull of a finite set of points $V = \{V_1, \ldots, V_k\}$

$$\mathcal{P} = \mathrm{conv}(V)$$

3. Any $\mathcal{H}$-polytope is a $\mathcal{V}$-polytope.
4. A polytope $\mathcal{P} \subset \mathbb{R}^n$, is full-dimensional if it is possible to fit a non-empty n-dimensional ball in $\mathcal{P}$
   Otherwise, we say that polytope $\mathcal{P}$ is lower-dimensional.
5. If $\|P_i^x\|_2 = 1$, where P_i^x denotes the i-th row of a matrix P^x, we say that the polytope $\mathcal{P}$ is normalized.

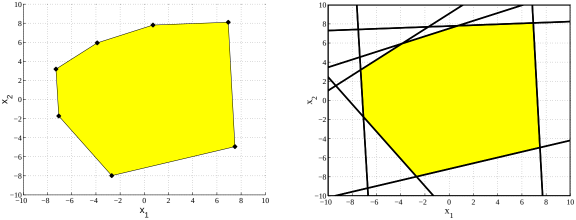
Polyhedra Definitions and Representations

1. A face of $\mathcal{P}$ is any nonempty set of the form

$$\mathcal{F} = \mathcal{P} \cap \{z \in \mathbb{R}^s |\ cz = c_0\}$$

where $cz \leq c_0$ is satisfied for all points in $\mathcal{P}$.

2. The faces of dimension 0,1, and dim($\mathcal{P}$)-1 are called vertices, edges and facets, respectively.
3. A d-simplex is a polytope of $\mathbb{R}^d$ with $d+1$ vertices.



(a) $\mathcal{V}$-representation. (b) $\mathcal{H}$-representation.

Basic Operations on Polytopes

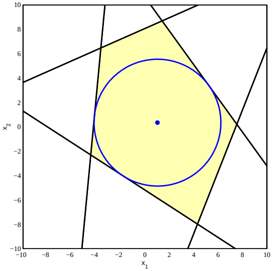
- Convex Hull of a set of points $V = \{V_i\}_{i=1}^{N_V}$, $\mathcal{P} = \mathrm{conv}(V)$ is used to switch from a $\mathcal{V}$-representation of a polytope to an $\mathcal{H}$-representation.

- Vertex Enumeration of a polytope $\mathcal{P}$ given in $\mathcal{H}$-representation. (dual of the convex hull operation)

Basic Operations on Polytopes

- Polytope reduction is the computation of the minimal representation of a polytope.

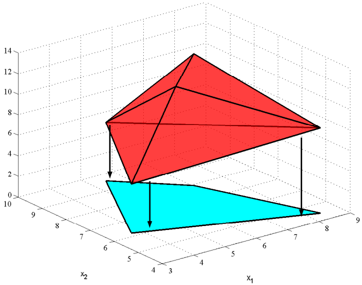
- The Chebychev Ball of a polytope $\mathcal{P}$ corresponds to the largest radius ball $\mathcal{B}(x_c, R)$ with center x_c, such that $\mathcal{B}(x_c, R) \subset \mathcal{P}$.



Basic Operations on Polytopes

- Projection Given a polytope $\mathcal{P} = \{[x'y']' \in \mathbb{R}^{n+m} \ : \ P^x x + P^y y \leq P^c\} \subset \mathbb{R}^{n+m}$ the projection onto the x-space $\mathbb{R}^n$ is defined as

$$\mathrm{Proj}_x(\mathcal{P}) := \{x \in \mathbb{R}^n \mid \exists y \in \mathbb{R}^m : P^x x + P^y y \leq P^c\}.$$



Affine Mappings and Polyhedra

Consider a polyhedron $\mathcal{P} = \{x \in \mathbb{R}^n \mid P^x x \leq P^c\}$, and an affine mapping $f(z)$

$$f: \ z \in \mathbb{R}^n \mapsto Az + b, \ A \in \mathbb{R}^{n \times n}, \ b \in \mathbb{R}^n$$

Define the composition of $\mathcal{P}$ and f as

$$\mathcal{P} \circ f \triangleq \{z \in \mathbb{R}^n \mid Az + b \in \mathcal{P}\}$$

It can be obtained by computing the following polyhedron

$$\mathcal{P} \circ f = \{z \in \mathbb{R}^n \mid P^x f(z) \leq P^c\} = \{z \in \mathbb{R}^n \mid P^x Az \leq P^c - P^x b\}$$

Affine Mappings and Polyhedra

Define the composition of f and $\mathcal{P}$ as the following polyhedron

$$f \circ \mathcal{P} \triangleq \{y \in \mathbb{R}^n \mid y = Az + b \ \forall z \in \mathcal{P}\}$$

The polyhedron $f \circ \mathcal{P}$ in (29) can be computed as follows.

1. Write $\mathcal{P}$ in $\mathcal{V}$-representation $\mathcal{P} = \mathrm{conv}(V)$
2. Map the vertices $V = \{V_1, \ldots, V_k\}$ through the transformation f.
3. Because the transformation is affine, the set $f \circ \mathcal{P}$ is the convex hull of the transformed vertices

$$f \circ \mathcal{P} = \mathrm{conv}(F), \ \ F = \{AV_1 + b, \ldots, AV_k + b\}.$$

Note: If f is invertible $z = A^{-1}y - A^{-1}b$ and therefore

$$f \circ \mathcal{P} = \{y \in \mathbb{R}^n \mid P^x A^{-1} y \leq P^c + P^x A^{-1} b\}$$

Homework

Homework

1. Installation the Multiparametrix Toolbox (http://control.ee.ethz.ch/ mpt/)
2. run 'mpt_demo1' in Matlab to study the demo of MPT package, skip operation not shown during lecture. Also run 'help mpt/polytope' to learn the list of commands for working with polytopes
3. What is the V-representation of the following polyhedron?

$$H = \begin{bmatrix} 0 & 1 \\ -1 & 0 \\ -1 & -1 \\ 1 & 1 \end{bmatrix}, \quad K = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

4. Compute the projection on the (x_1, x_2) space of the following polyhedron where $x = (x_1, x_2, x_3)$. Plot the projected 2 dimensional polyhedron.

$$P := \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix} x \leq \begin{bmatrix} 2 \\ 2 \\ 2 \\ 2 \\ 1 \\ 2 \\ 2 \end{bmatrix}$$

5. Consider the equality $f(x) = Ax$ with $A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$. Compute $f \circ \mathcal{X}_0$ and $\mathcal{X}_0 \circ f$ where $\mathcal{X}_0$ is

$$\mathcal{X}_0 := \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & -1 \\ -1 & 0 \end{bmatrix} x \leq \begin{bmatrix} 2 \\ 2 \\ 2 \\ 2 \end{bmatrix}$$

6. Read the help of linprog and quadprog commands (Type "help linprog" and "help quadprog" in Matlab)