Introduction to Model Predictive Control

Lecture 11: Constrained Linear Optimal Control

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Summarizing...

Need

1. A discrete-time model of the system (Matlab, Simulink)
2. A state observer
3. Set up an Optimization Problem (Matlab, MPT toolbox/Yalmip)
4. Solve an optimization problem (Matlab/Optimization Toolbox, NPSOL)
5. Verify that the closed-loop system performs as desired (avoid infeasibility/stability)
6. Make sure it runs in real-time and code/download for the embedded platform

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Constrained Linear Optimal Control

Define the cost function

$$J_0(x(0), U_0) \triangleq x_N' P x_N + \sum_{k=0}^{N-1} x_k' Q x_k + u_k' R u_k \tag{1}$$

and consider the constrained finite time optimal control problem (CFTOC)

$$\begin{array}{lll} J_0^*(x(0)) = & \min_{U_0} & J_0(x(0), U_0) \\ & \text{such that} & x_{k+1} = Ax_k + Bu_k,\ k = 0, \ldots, N-1 \\ & & x_k \in \mathcal{X},\ u_k \in \mathcal{U},\ k = 0, \ldots, N-1 \\ & & x_N \in \mathcal{X}_f \\ & & x_0 = x(0) \end{array} \tag{2}$$

where N is the time horizon and $\mathcal{U},\ \mathcal{X},\ \mathcal{X}_f$ are polyhedral regions.
$U_0 \triangleq [u_0', \ldots, u_{N-1}']' \in \mathbb{R}^s$, $s \triangleq mN$ the optimization vector.

Feasible Sets

- Denote with $\mathcal{X}_0 \subseteq \mathcal{X}$ the set of initial states $x(0)$ for which the optimal control problem (1)–(2) is feasible, i.e.,

$$\begin{aligned}\mathcal{X}_0 = \ & \{x_0 \in \mathbb{R}^n |\ \exists (u_0, \ldots, u_{N-1}) \text{ such that } x_k \in \mathcal{X},\ u_k \in \mathcal{U},\\ & k = 0, \ldots, N-1,\\ & x_N \in \mathcal{X}_f,\ \text{where } x_{k+1} = Ax_k + Bu_k,\ k = 0, \ldots, N-1\}\end{aligned}$$

Feasible Set

Let the state and input constraint sets $\mathcal{X}$, $\mathcal{X}_f$ and $\mathcal{U}$ be the $\mathcal{H}$-polyhedra $A_x x \leq b_x$, $A_f x \leq b_f$, $A_u u \leq b_u$, respectively. Define the polyhedron $\mathcal{P}_0$ as follows

$$\mathcal{P}_0 = \{(U_0, x_0) \in \mathbb{R}^{m(N)+n} |\ G_0 U_0 - E_0 x_0 \leq W_0\} \tag{3}$$

where G_0, E_0 and W_0 are defined as follows

$$G_0 = \begin{bmatrix} A_u & 0 & \dots & 0 \\ 0 & A_u & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & A_u \\ 0 & 0 & \dots & 0 \\ A_x B & 0 & \dots & 0 \\ A_x A B & A_x B & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ A_f A^{N-1} B & A_x A^{N-2} B & \dots & A_x B \end{bmatrix} \quad E_0 = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ -A_x \\ -A_x A \\ -A_x A^2 \\ \vdots \\ -A_f A^N \end{bmatrix} \quad W_0 = \begin{bmatrix} b_u \\ b_u \\ \vdots \\ b_u \\ b_x \\ b_x \\ b_x \\ \vdots \\ b_f \end{bmatrix} \tag{4}$$

Then set $\mathcal{X}_0$ is a polyhedron and can be computed by projecting the polyhedron $\mathcal{P}_0$ in (3)-(4) on the x_0 space.

Feasible Set

Projection method

Then set $\mathcal{X}_0$ is a polyhedron and can be computed by projecting the polyhedron $\mathcal{P}_0$ in (3)-(4) on the x_0 space.

Explicit solution of CFTOC

Rewrite the CFTOC problem with $p = 2$ as

$$
\begin{array}{rcll}
J_0^*(x(0)) & = & \min\limits_{U_0} & J_0(x(0), U_0) = U_0' H U_0 + 2x'(0) F U_0 + x'(0) Y x \\
 & = & \min\limits_{U_0} & J_0(x(0), U_0) = (U_0' \; x'(0))' \left[\begin{smallmatrix} H & F' \\ F & Y \end{smallmatrix}\right] (U_0 \; x(0)) \\
 & & \text{such that} & G_0 U_0 \le W_0 + E_0 x(0)
\end{array}
\qquad (5)
$$

with G_0, W_0 and E_0 defined in (4).

Problem (5) for a fixed $x(0)$ is a quadratic program (QP)

Homework 1

- Study the matlab file on bSpace: "CFTOCExample.m"