Introduction to Model Predictive Control

Lectures 7-8: Optimization

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Summarizing...

Need

1. A discrete-time model of the system (Matlab, Simulink)
2. A state observer
3. Set up an Optimization Problem (Matlab, MPT toolbox/Yalmip)
4. Solve an optimization problem (Matlab/Optimization Toolbox, NPSOL)
5. Verify that the closed-loop system performs as desired (avoid infeasibility/stability)
6. Make sure it runs in real-time and code/download for the embedded platform

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Optimality Conditions

Optimality Conditions

$$
\begin{array}{lll}
\min_z & f(z) & \\
\text{such that} & g_i(z) \leq 0 & \text{for } i = 1, \ldots, m \\
 & h_i(z) = 0 & \text{for } i = 1, \ldots, p \\
 & z \in Z &
\end{array}
$$

- In general, an analytical solution does not exist.
- Solutions are usually computed by recursive algorithms which start from an initial guess z_0 and at step k generate a point z_k such that $\{f(z_k)\}_{k=0,1,2,\ldots}$ converges to J^*.
- These algorithms recursively use and/or solve analytical **conditions for optimality**

Optimality Conditions

Necessary optimality condition for **unconstrained** optimization problems.

Theorem

$f : \mathbb{R}^s \to \mathbb{R}$ differentiable at $\bar{z}$. If there exists a vector $\mathbf{d}$ such that $\nabla f(\bar{z})'\mathbf{d} < 0$, then there exists a $\delta > 0$ such that $f(\bar{z} + \lambda\mathbf{d}) < f(\bar{z})$ for all $\lambda \in (0, \delta)$.

- The vector $\mathbf{d}$ in the theorem above is called *descent direction*.
- The direction of *steepest descent* $\mathbf{d}_s$ at $\bar{z}$ is defined as the normalized direction where $\nabla f(\bar{z})'\mathbf{d}_s < 0$ is minimized.
- The direction $\mathbf{d}_s$ of steepest descent is $\mathbf{d}_s = -\frac{\nabla f(\bar{z})}{\|\nabla f(\bar{z})\|}$.

Optimality Conditions

Necessary optimality condition for **unconstrained** optimization problems.

Corollary

$f : \mathbb{R}^s \rightarrow \mathbb{R}$ is differentiable at $\bar{z}$. If $\bar{z}$ is a local minimizer, then $\nabla f(\bar{z}) = 0$.

Sufficient condition for **unconstrained** optimization problems.

Theorem

Suppose that $f : \mathbb{R}^s \rightarrow \mathbb{R}$ is twice differentiable at $\bar{z}$. If $\nabla f(\bar{z}) = 0$ and the Hessian of $f(z)$ at $\bar{z}$ is positive definite, then $\bar{z}$ is a local minimizer

Necessary and sufficient conditions for **unconstrained** optimization problems.

Theorem

Suppose that $f : \mathbb{R}^s \rightarrow \mathbb{R}$ is differentiable at $\bar{z}$. If f is convex, then $\bar{z}$ is a global minimizer if and only if $\nabla f(\bar{z}) = 0$.

KKT optimality conditions

The primal and dual optimal pair z^*, (u^*, v^*) of an optimization problem with differentiable cost and constraints and zero duality gap, have to satisfy the following conditions:

$$
\begin{aligned}
\nabla f(z^*) + \sum_{i=1}^{m} u_i^* \nabla g_i(z^*) + \sum_{j=1}^{p} v_j^* \nabla h_i(z^*) &= 0, && && \text{(1a)}\\
u_i^* g_i(z^*) &= 0, && i = 1, \ldots, m && \text{(1b)}\\
u_i^* &\geq 0, && i = 1, \ldots, m && \text{(1c)}\\
g_i(z^*) &\leq 0, && i = 1, \ldots, m && \text{(1d)}\\
h_j(z^*) &= 0 && j = 1, \ldots, p && \text{(1e)}
\end{aligned}
$$

Conditions (1a)-(1e) are called the *Karush-Kuhn-Tucker* (KKT) conditions.

Linear and Quadratic Optimization

Linear Programming

$$\begin{array}{ll} \inf_z & c'z \\ \text{such that} & Gz \leq W \end{array}$$

where $z \in \mathbb{R}^s$.

- Convex optimization problems.
- Other common forms:

$$\begin{array}{ll} \inf_z & c'z \\ \text{such that} & Gz \leq W \\ & G_{eq}z = W_{eq} \end{array}$$

or

$$\begin{array}{ll} \inf_z & c'z \\ \text{such that} & G_{eq}z = W_{eq} \\ & z \geq 0 \end{array}$$

- Always possible to convert one of the three forms into the other.

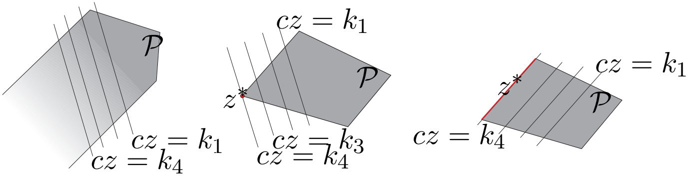
Graphical Interpretation and Solutions Properties

- Let $\mathcal{P}$ be the feasible set. $\mathcal{P}$ is a polyhedron.
- If $\mathcal{P}$ is empty, then the problem is infeasible.
- Denote by J^* the optimal value

Case 1. The LP solution is unbounded, i.e., $J^* = -\infty$.

Case 2. The LP solution is bounded, i.e., $J^* > -\infty$ and the optimizer is unique.

Case 3. The LP solution is bounded and there are multiple optima.



(a) Case 1 (b) Case 2 (c) Case 3

Figure: Graphical Interpretation of the Linear Program Solution, $k_i < k_{i-1}$

Convex Piecewise Linear Optimization

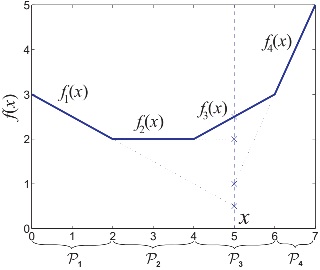
Consider

$$J^* = \begin{array}{ll} \min_z & J(z) \\ \text{such that} & Gz \leq W \end{array} \tag{2}$$

where the cost function has the form

$$J(z) = \max_{i=1,\dots,k} \{c_i z + d_i\} \tag{3}$$

where $c_i \in \mathbb{R}^s$ and $d_i \in \mathbb{R}$.



Convex Piecewise Linear Optimization

The cost function $J(z)$ in (3) is a convex PWA function. The optimization problem (2)-(3) can be solved by the following linear program:

$$
\begin{array}{rll}
J^* = & \min_{z,\varepsilon} & \varepsilon \\
 & \text{such that} & Gz \leq W \\
 & & c_i z + d_i \leq \varepsilon, \quad i = 1, \ldots, k
\end{array}
$$

Convex Piecewise Linear Optimization

Consider

$$
\begin{array}{rll}
J^* = & \min_z & J_1(z_1) + J_2(z_2) \\
& \text{such that} & G_1 z_1 + G_2 z_2 \leq W
\end{array}
\tag{4}
$$

where the cost function has the form

$$
\begin{array}{l}
J_1(z_1) = \max_{i=1,\ldots,k}\{c_i z_1 + d_i\} \\
J_2(z_2) = \max_{i=1,\ldots,j}\{m_i z_2 + n_i\}
\end{array}
\tag{5}
$$

The optimization problem (4)-(5) can be solved by the following linear program:

$$
\begin{array}{rll}
J^* = & \min_{z,\varepsilon_1,\varepsilon_2} & \varepsilon_1 + \varepsilon_2 \\
& \text{such that} & G_1 z_1 + G_2 z_2 \leq W \\
& & c_i z_1 + d_i \leq \varepsilon_1, \quad i = 1, \ldots, k \\
& & m_i z_2 + n_i \leq \varepsilon_2, \quad i = 1, \ldots, j
\end{array}
$$

Example

The optimization problem:

$$
\begin{aligned}
\min_{z_1,z_2} f(x,u) = \quad & \min |z_1+5| + |z_2-3| \\
\text{subject to} \quad & \quad 2.5 \le z_1 \le 5 \\
& \quad -1 \le z_2 \le 1
\end{aligned}
$$

can be solved in Matlab by using "v=linprog(f,A,b)" where $v = [\varepsilon_1^*, \varepsilon_2^*, z_1^*, z_2^*]$, $f = [1 \;\; 1 \;\; 0 \;\; 0]$,

$$
A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & -1 & 0 \\ 0 & -10 & 0 & 1 \\ 0 & -10 & 0 & -1 \end{bmatrix}, \quad b = \begin{bmatrix} 5 \\ -2.5 \\ 1 \\ 1 \\ -5 \\ 5 \\ 3 \\ -3 \end{bmatrix}
$$

Solution is $v = [7.5 \;\; 0.2 \;\; 2.5 \;\; 1]$

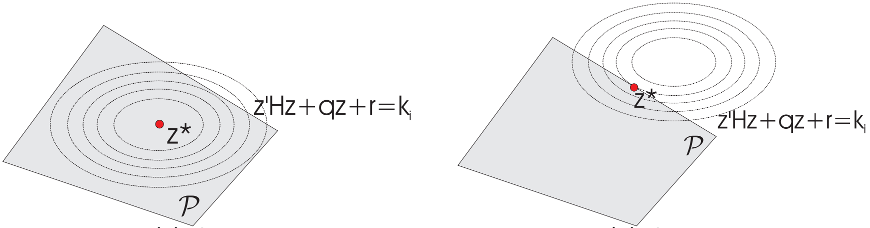
Quadratic Programming

$$\begin{array}{ll} \min_z & \frac{1}{2}z'Hz + q'z + r \\ \text{such that} & Gz \le W \end{array} \tag{6}$$

where $z \in \mathbb{R}^s$, $H = H' > 0 \in \mathbb{R}^{s \times s}$.

- Other QP forms often include equality and inequality constraints.
- Let $\mathcal{P}$ be the feasible set. Two cases can occur if $\mathcal{P}$ is not empty:

Case 1. The optimizer lies strictly inside the feasible polyhedron (Figure 2(a)).
Case 2. The optimizer lies on the boundary of the feasible polyhedron (Figure 2(b)).



(a) Case 1 (b) Case 2

Constrained Least-Squares Problems

The problem of minimizing the convex quadratic function arises in many fields and has many names, e.g., linear regression or least-squares approximation.

$$\|Az - b\|_2^2 = z'A'Az - 2b'Az + b'b$$

The minimizer is

$$z^* = (A'A)^{-1}A'b \triangleq A^\dagger b$$

When linear inequality constraints are added, the problem is called constrained linear regression or *constrained least-squares*, and there is no longer a simple analytical solution. As an example we can consider regression with lower and upper bounds on the variables, i.e.,

$$\begin{array}{ll} \min_z & \|Az - b\|_2^2 \\ \text{such that} & l_i \leq z_i \leq u_i, \;\; i = 1, \ldots, n, \end{array}$$

which is a QP.

Example

Consider

$$\min_{z} \quad \|Az - b\|_2^2$$

where

$$A = \begin{bmatrix} 0.7513 & 0.5472 & 0.8143 \\ 0.2551 & 0.1386 & 0.2435 \\ 0.5060 & 0.1493 & 0.9293 \\ 0.6991 & 0.2575 & 0.3500 \\ 0.8909 & 0.8407 & 0.1966 \\ 0.9593 & 0.2543 & 0.2511 \end{bmatrix}, \quad b = \begin{bmatrix} 0.6160 \\ 0.4733 \\ 0.3517 \\ 0.8308 \\ 0.5853 \\ 0.5497 \end{bmatrix}$$

- Unconstrained Least-Squares: in Matlab "$z = A\backslash b$" or "z=quadprog($A'A$,$-b'A$)". $z^* = [0.7166, -0.0205, 0.1180]$,
- Assume $z_2 \geq 0$. Constrained Least-Squares in Matlab: "z=quadprog($A'A$,$-b'A$,[0 -1 0],0)". $z^* = [0.7045,\ 0,\ 0.1194]$.

Nonlinear Programming

Consider

$$
\begin{array}{lll}
\min_z & f(z) & \\
\text{such that} & g_i(z) \leq 0 & \text{for } i = 1, \ldots, m \\
 & h_i(z) = 0 & \text{for } i = 1, \ldots, p \\
 & z \in Z &
\end{array}
$$

- A variety of softwares exists
- In general, global optimality not guaranteed
- Solutions are usually computed by recursive algorithms which start from an initial guess z_0 and at step k generate a point z_k such that $\{f(z_k)\}_{k=0,1,2,\ldots}$ converges to J^*.
- These algorithms recursively use and/or solve analytical **conditions for optimality**
- In this class we will use "NPSOL"

Nonlinear Programming - NPSOL

- Possible syntax:

  [INFORM,ITER,ISTATE,C,CJAC,CLAMDA,OBJF,OBJGRAD,R,X] = npsol(X0,A,L,U,'funobj','funcon',OPTION);

- Solving

$$
\begin{array}{ll}
\min_z & \text{funobj}(z) \\
\text{such that} & L \leq \begin{bmatrix} z \\ Az \\ \text{funcon}(z) \end{bmatrix} \leq U
\end{array}
$$

- NPSOL Manual and Example on bSpace
- Note it is a mex-function

Nonlinear Programming - Matlab Optimization toolbox

- Possible syntax:

  [X,FVAL,EXITFLAG] = fmincon(funobj,X0,A,B,Aeq,Beq,LB,UB,NONLCON);

- Solving

$$\begin{array}{ll} \min_z & \text{funobj}(z) \\ \text{such that} & A\ z \leq B \\ & Aeq\ z = Beq \\ & LB \leq\ z \leq UB \\ & Ceq(z) = 0 \\ & C(z) \leq 0 \end{array}$$

- The function NONLCON accepts z and returns the vectors $C(z)$ and $Ceq(z)$
- help "fmincon" in Matlab

Homework: will be graded!

Due on Friday October 30th, at beginning of lecture

Homework 1/2

1. Consider a discrete time model: $x_{k+1} = 0.5x_k + u_k$, with initial state $x_0 = 2$.
2. Consider the optimization problem:

$$\begin{array}{ll} \min\limits_{x,u} f(x,u) = & \min \frac{1}{2}(x_1^2 + x_2^2 + u_0^2 + u_1^2) \\ \text{subject to} & 2.5 \leq x_1 \leq 5 \\ & -1 \leq x_2 \leq 1 \\ & -2 \leq u_0 \leq 2 \\ & -2 \leq u_1 \leq 2 \end{array}$$

   Compute the optimal solution using MATLAB.
3. Consider the optimization problem:

$$\begin{array}{ll} \min\limits_{x,u} f(x,u) = & \min |x_1| + 0.5|x_2| + 0.5|u_0| + |u_1| \\ \text{subject to} & 2.5 \leq x_1 \leq 5 \\ & -1 \leq x_2 \leq 1 \\ & -2 \leq u_0 \leq 2 \\ & -2 \leq u_1 \leq 2 \end{array}$$

   Compute the optimal solution using MATLAB.

Homework 2/2

1. Consider the optimization problem:

$$\begin{array}{rl}\min\limits_{x,u} f(x,u) = & \min |x_1| + 0.5(x_2)^3 + 0.5|u_0 u_1| \\ \text{subject to} & 2.5 \le x_1 \le 5 \\ & -1 \le x_2 \le 1 \\ & -2 \le u_0 \le 2 \\ & (u_0)^3 + (u_1)^2 \le 8\end{array}$$

Compute the optimal solution using MATLAB.