Model Predictive Control for Linear and Hybrid Systems
Constrained Linear Optimal Control

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Outline

1. Constrained Linear Optimal Control
   - Problem Formulation
   - Feasible Sets
2. Constrained Optimal Control: 2-Norm Case
3. Constrained Optimal Control: 1-norm and ∞-norm
4. Infinite Horizon
5. Minimum Time Control

Constrained Linear Optimal Control

Consider the cost function

$$J_0(x(0), U_0) \triangleq p(x_N) + \sum_{k=0}^{N-1} q(x_k, u_k)$$

and the constrained finite time optimal control problem (CFTOC)

$$\begin{array}{lll} J_0^*(x(0)) = & \min_{U_0} & J_0(x(0), U_0) \\ & \text{subj. to} & x_{k+1} = Ax_k + Bu_k,\ k = 0, \ldots, N-1 \\ & & x_k \in \mathcal{X},\ u_k \in \mathcal{U},\ k = 0, \ldots, N-1 \\ & & x_N \in \mathcal{X}_f \\ & & x_0 = x(0) \end{array} \tag{1}$$

where N is the time horizon and $\mathcal{U},\ \mathcal{X},\ \mathcal{X}_f$ are polyhedral regions.

- Denote by $U_0 \triangleq [u_0', \ldots, u_{N-1}']' \in \mathbb{R}^s$, $s \triangleq mN$ the optimization vector.
- If the 1-norm or ∞-norm is used in the cost function (1), then $p(x_N) = \|Px_N\|_p$ and $q(x_k, u_k) = \|Qx_k\|_p + \|Ru_k\|_p$.
- If the squared euclidian norm is used in the cost function (1), then $p(x_N) = x_N' P x_N$ and $q(x_k, u_k) = x_k' Q x_k + u_k' R u_k$.

Feasible Sets

Denote by $\mathcal{X}_0 \subseteq \mathcal{X}$ the set of initial states $x(0)$ for which the optimal control problem (1) is feasible, i.e.,

$$\begin{aligned}\mathcal{X}_0 = \ & \{x_0 \in \mathbb{R}^n | \ \exists (u_0, \dots, u_{N-1}) \text{ such that } x_k \in \mathcal{X}, \ u_k \in \mathcal{U}, \\ & k = 0, \dots, N-1, \ x_N \in \mathcal{X}_f, \text{ where } x_{k+1} = Ax_k + Bu_k, \\ & k = 0, \dots, N-1\},\end{aligned}$$

We denote with $\mathcal{X}_i$ the set of states x_i at time i for which (1) is feasible

$$\begin{aligned}\mathcal{X}_i = \ & \{x_i \in \mathbb{R}^n | \ \exists (u_i, \dots, u_{N-1}) \text{ such that } x_k \in \mathcal{X}, \ u_k \in \mathcal{U}, \\ & k = i, \dots, N-1, \ x_N \in \mathcal{X}_f, \text{ where } x_{k+1} = Ax_k + Bu_k\},\end{aligned}$$

- The sets $\mathcal{X}_i$ for $i = 0, \dots, N$ play an important role in the the solution of the CFTOC. They are ***independent*** on ***the cost***.
- We will study the properties of these sets in the next lectures... Let's first show how to solve the problem.

Outline

1. Constrained Linear Optimal Control
2. Constrained Optimal Control: 2-Norm Case
   - Batch Approach
   - Recursive Approach
   - Example
3. Constrained Optimal Control: 1-norm and ∞-norm
4. Infinite Horizon
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2-Norm Constrained Linear Optimal Control

Consider the cost function

$$J_0(x(0), U_0) \triangleq x_N' P x_N + \sum_{k=0}^{N-1} x_k' Q x_k + u_k' R u_k \tag{2}$$

with $P \succeq 0$, $Q \succeq 0$, $R \succ 0$ and the constrained finite time optimal control problem (CFTOC)

$$\begin{array}{rll} J_0^*(x(0)) = & \min\limits_{U_0} & J_0(x(0), U_0) \\ & \text{subj. to} & x_{k+1} = Ax_k + Bu_k,\ k = 0, \ldots, N-1 \\ & & x_k \in \mathcal{X},\ u_k \in \mathcal{U},\ k = 0, \ldots, N-1 \\ & & x_N \in \mathcal{X}_f \\ & & x_0 = x(0) \end{array} \tag{3}$$

where N is the time horizon and $\mathcal{U},\ \mathcal{X},\ \mathcal{X}_f$ are polyhedral regions.

- Let's try to compute the state-feedback solution to (2)–(3) by using mp-QP
- Recall we had two approaches: batch approach and recursive approach.

Feasible Sets -Batch Approach

Be $A_x x \leq b_x$, $A_f x \leq b_f$, $A_u u \leq b_u$ the $\mathcal{H}$-representations of sets $\mathcal{X}$, $\mathcal{X}_f$ and $\mathcal{U}$, respectively. Define the polyhedron $\mathcal{P}_i$ for $i = 0, \ldots, N-1$ as follows

$$\mathcal{P}_i = \{(U_i, x_i) \in \mathbb{R}^{m(N-i)+n} |\ G_i U_i - E_i x_i \leq W_i\}$$

where G_i, E_i and W_i are defined as follows

$$G_i = \begin{bmatrix} A_u & 0 & \ldots & 0 \\ 0 & A_u & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & A_u \\ 0 & 0 & \ldots & 0 \\ A_x B & 0 & \ldots & 0 \\ A_x A B & A_x B & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ A_f A^{N-i-1} B & A_x A^{N-i-2} B & \ldots & A_x B \end{bmatrix}$$

Feasible Set - Batch Approach

$$E_i = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ -A_x \\ -A_x A \\ -A_x A^2 \\ \vdots \\ -A_f A^{N-i} \end{bmatrix} \quad W_i = \begin{bmatrix} b_u \\ b_u \\ \vdots \\ b_u \\ b_x \\ b_x \\ b_x \\ \vdots \\ b_f \end{bmatrix}$$

Then set $\mathcal{X}_i$ is a **polyhedron** and can be computed by **projecting** the polyhedron $\mathcal{P}_i$ on the x_i space.

Batch Approach

Rewrite the problem as

$$
\begin{aligned}
J_0^*(x(0)) &= \min_{U_0} J_0(x(0), U_0) = U_0' H U_0 + 2x'(0) F U_0 + x'(0) Y x(0) \\
&= \min_{U_0} J_0(x(0), U_0) = (U_0' \ x'(0))' \left[\begin{smallmatrix} H & F' \\ F & Y \end{smallmatrix}\right] (U_0 \ x(0))
\end{aligned}
$$

$$
\text{subj. to } \ G_0 U_0 \le W_0 + E_0 x(0)
$$

Observe that $\left[\begin{smallmatrix} H & F' \\ F & Y \end{smallmatrix}\right] \succeq 0$ since $J_0(x(0), U_0) \ge 0$ by assumption.

Define $z \triangleq U_0 + H^{-1} F' x(0)$ and transform the problem into

$$
\begin{aligned}
\hat{J}^*(x(0)) = \quad & \min_{z} && z' H z \\
& \text{subj. to} && G_0 z \le W_0 + S_0 x(0),
\end{aligned}
$$

where $S_0 \triangleq E_0 + G_0 H^{-1} F'$, and
$\hat{J}^*(x(0)) = J_0^*(x(0)) - x(0)'(Y - F H^{-1} F') x(0)$.

Batch Approach

The CFTOC problem can be recast as a ***multiparametric quadratic program***.

Main Results

The ***Open loop optimal control function*** can be obtained by solving the mp-QP problem and calculating $U_0^*(x(0))$, $\forall x(0) \in \mathcal{X}_0$ as
$U_0{}^* = z^*(x(0)) - H^{-1}F'x(0)$.

Batch Approach

Corollary

The control law $u^*(0) = f_0(x(0))$, $f_0 : \mathbb{R}^n \to \mathbb{R}^m$, *obtained as a solution of the CFTOC (2)–(3) is continuous and piecewise affine on polyhedra*

$$f_0(x) = F_0^i x + g_0^i \quad \textit{if} \quad x \in CR_0^i, \quad i = 1, \ldots, N_0^r$$

where the polyhedral sets $CR_0^i = \{x \in \mathbb{R}^n | H_0^i x \leq K_0^i\}$, $i = 1, \ldots, N_0^r$ *are a partition of the feasible polyhedron* $\mathcal{X}_0$.

Corollary

The value function $J_0^*(x(0))$ *is convex and piecewise quadratic on polyhedra. Moreover, if the mp-QP problem is not degenerate, then the value function* $J_0^*(x(0))$ *is* $C^{(1)}$.

Batch Approach. State Feedback Solution

Consider the same CFTOC over the shortened time horizon $[i, N]$

$$\begin{array}{ll} \min\limits_{U_i} & \|Px_N\|_2 + \sum\limits_{k=i}^{N-1} \|Qx_k\|_2 + \|Ru_k\|_2 \\ \text{subj. to} & x_{k+1} = Ax_k + Bu_k,\ k = i, \ldots, N-1 \\ & x_k \in \mathcal{X},\ u_k \in \mathcal{U},\ k = i, \ldots, N-1 \\ & x_N \in \mathcal{X}_f \\ & x_i = x(i) \end{array}$$

where $U_i \triangleq [u_i', \ldots, u_{N-1}']$. The problem can be translated into the mp-QP

$$\begin{array}{ll} \min\limits_{U_i} & {U_i}' H_i U_i + 2x'(i) F_i U_i + x'(i) Y_i x(i) \\ \\ \text{subj. to} & G_i U_i \leq W_i + E_i x(i). \end{array}$$

Batch Approach. State Feedback Solution

Main Results

1. The ***Open loop optimal control function over*** $[i, N]$ can be obtained by solving the corresponding mp-QP problem and calculating $U_i^*(x(i))$, $\forall x(i) \in \mathcal{X}_i$ as $U_i^* = z^*(x(i)) - H_i^{-1} F_i' x(0)$.
2. The first component of the multiparametric solution has the form

$$u_i^*(x(i)) = f_i(x(i)), \ \forall x(i) \in \mathcal{X}_i,$$

where the control law $f_i : \mathbb{R}^n \to \mathbb{R}^m$, is ***continuous and PPWA***

$$f_i(x) = F_i^j x + g_i^j \quad \text{if} \quad x \in CR_i^j, \ j = 1, \ldots, N_i^r$$

and where the polyhedral sets $CR_i^j = \{x \in \mathbb{R}^n | H_i^j x \leq K_i^j\}$, $j = 1, \ldots, N_i^r$ are a ***partition of the feasible polyhedron*** $\mathcal{X}_i$.

Batch Approach. State Feedback Solution

The feedback solution $u^*(k) = f_k(x(k)),\ k = 0, \ldots, N-1$ of the CFTOC with $p = 2$ is obtained by ***solving N mp-QP problems of decreasing size***.

Corollary

The state-feedback control law $u^*(k) = f_k(x(k)),\ f_k : \mathcal{X}_k \subseteq \mathbb{R}^n \to \mathcal{U} \subseteq \mathbb{R}^m$, obtained as a solution of the CFTOC (2)–(3) for $k = 0, \ldots, N-1$ is time-varying, continuous and piecewise affine on polyhedra

$$f_k(x) = F_k^i x + g_k^i \quad \textit{if} \quad x \in CR_k^i, \quad i = 1, \ldots, N_k^r$$

where the polyhedral sets $CR_k^i = \{x \in \mathbb{R}^n \mid H_k^i x \leq K_k^i\}$, $i = 1, \ldots, N_k^r$ are a partition of the feasible polyhedron $\mathcal{X}_k$.

Recursive Approach. State Feedback Solution

Consider the ***dynamic programming formulation*** of the CFTOC

$$
\begin{array}{rlll}
J_j^*(x_j) & \triangleq & \min\limits_{u_j} & x_j'Qx_j + u_j'Ru_j + J_{j+1}^*(Ax_j + Bu_j) \\
 & & \text{subj. to} & x_j \in \mathcal{X},\ u_j \in \mathcal{U}, \\
 & & & Ax_j + Bu_j \in \mathcal{X}_{j+1}
\end{array}
$$

for $j = 0, \ldots, N-1$, with boundary conditions

$$
\begin{gathered}
J_N^*(x_N) = x_N'Px_N \\
\mathcal{X}_N = \mathcal{X}_f,
\end{gathered}
$$

Observe that $J_{j+1}^*(Ax_j + Bu_j)$ is piecewise quadratic for $j < N-1$ and the problem is not simply an mp-QP.

Example

Consider the double integrator

$$\begin{cases} x(t+1) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \\ y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(t) \end{cases}$$

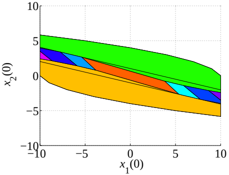
subject to constraints

$$-1 \le u(k) \le 1,\ \ k = 0, \ldots, 5$$

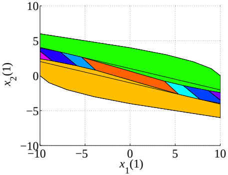
$$\begin{bmatrix} -10 \\ -10 \end{bmatrix} \le x(k) \le \begin{bmatrix} 10 \\ 10 \end{bmatrix},\ \ k = 0, \ldots, 5$$

Compute the ***state feedback*** optimal controller solving the CFTOC problem with $N = 6$, $Q = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$, $R = 0.1$, P the solution of the ARE, $\mathcal{X}_f = \mathbb{R}^2$.

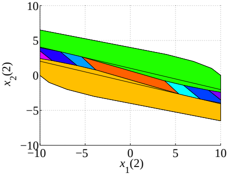
Example



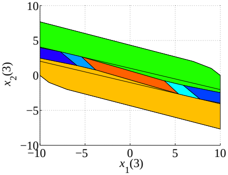
(a) Partition of the state space for the affine control law $u^*(0)$ ($N_0^r = 13$)



(b) Partition of the state space for the affine control law $u^*(1)$ ($N_1^r = 13$)

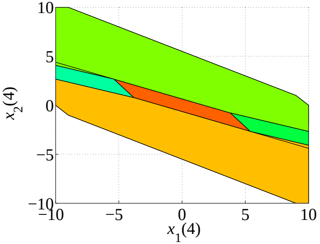


(c) Partition of the state space for the affine control law $u^*(2)$ ($N_2^r = 13$)

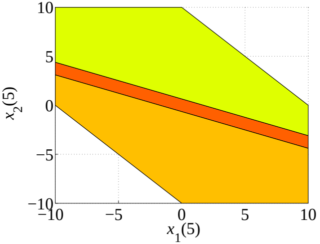


(d) Partition of the state space for the affine control law $u^*(3)$ ($N_3^r = 11$)

Example



(e) Partition of the state space for the affine control law $u^*(4)$ ($N_4^r = 7$)



(f) Partition of the state space for the affine control law $u^*(5)$ ($N_5^r = 3$)

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Problem Formulation

Consider the cost function

$$J_0(x(0), U_0) \triangleq \|Px_N\|_p + \sum_{k=0}^{N-1} \|Qx_k\|_p + \|Ru_k\|_p \tag{4}$$

with $p = 1$ or $p = \infty$, P, Q, R full column rank matrices, and the constrained finite time optimal control problem (CFTOC)

$$\begin{array}{lll} J_0^*(x(0)) = & \min\limits_{U_0} & J_0(x(0), U_0) \\ & \text{subj. to} & x_{k+1} = Ax_k + Bu_k,\ k = 0, \ldots, N-1 \\ & & x_k \in \mathcal{X},\ u_k \in \mathcal{U},\ k = 0, \ldots, N-1 \\ & & x_N \in \mathcal{X}_f \\ & & x_0 = x(0) \end{array} \tag{5}$$

where N is the time horizon and $\mathcal{U},\ \mathcal{X},\ \mathcal{X}_f$ are polyhedral regions.

- Let's to compute the state-feedback solution to (4)–(5) by using mp-LP
- Recall we had two approaches: batch approach and recursive approach.

Problem Formulation

Recall that the problem can be equivalently formulated as

$$
\begin{array}{ll}
\min\limits_{z_0} & \varepsilon_0^x + \ldots + \varepsilon_N^x + \varepsilon_0^u + \ldots + \varepsilon_{N-1}^u \\
\text{subj. to} & -\mathbf{1}_n \varepsilon_k^x \le \pm Q \left[A^k x_0 + \sum_{j=0}^{k-1} A^j B u_{k-1-j} \right], \\
& -\mathbf{1}_r \varepsilon_N^x \le \pm P \left[A^N x_0 + \sum_{j=0}^{N-1} A^j B u_{N-1-j} \right], \\
& -\mathbf{1}_m \varepsilon_k^u \le \pm R u_k, \\
& A^k x_0 + \sum_{j=0}^{k-1} A^j B u_{k-1-j} \in \mathcal{X},\ u_k \in \mathcal{U}, \\
& A^N x_0 + \sum_{j=0}^{N-1} A^j B u_{N-1-j} \in \mathcal{X}_f, \\
& k = 0, \ldots, N-1 \\
& x_0 = x(0)
\end{array}
$$

Batch Approach

The the problem results in the following standard mp-LP

$$\begin{array}{ll} \min\limits_{z_0} & c_0' z_0 \\ \text{subj. to} & \bar{G}_0 z_0 \leq \bar{W}_0 + \bar{S}_0 x(0) \end{array}$$

where $z_0 \triangleq \{\varepsilon_0^x, \ldots, \varepsilon_N^x, \varepsilon_0^u, \ldots, \varepsilon_{N-1}^u, u_0', \ldots, u_{N-1}'\} \in \mathbb{R}^s$,
$s \triangleq (m+1)N + N + 1$ and

$$\bar{G}_0 = \left[\begin{array}{cc} G_\varepsilon & 0 \\ 0 & G_0 \end{array} \right],\ \bar{S}_0 = \left[\begin{array}{c} S_\varepsilon \\ S_0 \end{array} \right],\ \bar{W}_0 = \left[\begin{array}{c} W_\varepsilon \\ W_0 \end{array} \right]$$

Batch Approach: State Feedback Solution

Main Results

1. ***Open loop input trajectory***.
   1. Solve the mp-LP and find $z_0^*(x(0))$ as a continuous piecewise affine function of $x(0)$.
   2. Calculate
   $$U_0^* = [0 \ \ldots 0 \ I_m \ 0 \ \ldots \ 0] z_0^*(x(0)).$$
   3. Properties and structure of $z_0^*(x(0))$ inherited by U_0^*.

2. ***State feedback loop input trajectory***.
   1. Solve a sequence of mp-LPs
   $$\begin{array}{ll} \min\limits_{z_i} & c_i' z_i \\ \text{subj. to} & \bar{G}_i z_i \le \bar{W}_i + \bar{S}_i x(i), \end{array}$$
   obtained by rewriting the original problem over the finite time horizon $[i, N]$, and find $z_i^*(x(i))$ as a continuous piecewise affine function of $x(i)$.
   2. Calculate
   $$u_i^*(x(i)) = [0 \ \ldots 0 \ I_m \ 0 \ \ldots \ 0] z_i^*(x(i)).$$
   3. Properties and structure of $z_i^*(x(i))$ inherited by $u_i^*(x(i))$.

Batch Approach. State Feedback Solution

The feedback solution $u^*(k) = f_k(x(k)),\ k = 0, \ldots, N-1$ of the CFTOC (4)–(5) is obtained by ***solving N mp-LP problems of decreasing size***.

Corollary

The state-feedback control law $u^*(k) = f_k(x(k)),\ f_k : \mathcal{X}_k \subseteq \mathbb{R}^n \to \mathcal{U} \subseteq \mathbb{R}^m$, obtained as a solution of the CFTOC (4)–(5) for $k = 0, \ldots, N-1$ is time-varying, continuous and piecewise affine on polyhedra

$$f_k(x) = F_k^i x + g_k^i \quad \textit{if} \quad x \in CR_k^i, \quad i = 1, \ldots, N_k^r$$

where the polyhedral sets $CR_k^i = \{x \in \mathbb{R}^n \mid H_k^i x \le K_k^i\}$, $i = 1, \ldots, N_k^r$ are a partition of the feasible polyhedron $\mathcal{X}_k$.

Recursive Approach. State Feedback Solution

Consider the ***dynamic programming formulation***

$$
\begin{aligned}
J_j^*(x_j) \quad \triangleq \quad & \min_{u_j} \quad && \|Qx_j\|_p + \|Ru_j\|_p + J_{j+1}^*(Ax_j + Bu_j) \\
& \text{subj. to} && x_j \in \mathcal{X},\ u_j \in \mathcal{U}, \\
& && Ax_j + Bu_j \in \mathcal{X}_{j+1}
\end{aligned}
$$

for $j = 0, \ldots, N-1$, with boundary conditions

$$
\begin{gathered}
J_N^*(x_N) = \|Px_N\|_p \\
\mathcal{X}_N = \mathcal{X}_f,
\end{gathered}
$$

Theorem

The state feedback piecewise affine solution of the CFTOC for $p = 1$ or $p = \infty$ is obtained by solving the above problem via N mp-LPs.

Recursive Approach. State Feedback Solution

Consider the first step $j = N-1$ of the dynamic programming recursion

$$
\begin{aligned}
J^*_{N-1}(x_{N-1}) \triangleq \quad & \min_{u_{N-1}} \|Qx_{N-1}\|_p + \|Ru_{N-1}\|_p \\
& \qquad\qquad + J^*_N(Ax_{N-1} + Bu_{N-1}) \\
& \text{subj. to} \\
& x_{N-1} \in \mathcal{X},\ u_{N-1} \in \mathcal{U}, \\
& Ax_{N-1} + Bu_{N-1} \in \mathcal{X}_f
\end{aligned}
$$

$J^*_{N-1}(x_{N-1})$, $u^*_{N-1}(x_{N-1})$ and $\mathcal{X}_{N-1}$ can be calculated by solving the following mp-LP

$$
\begin{aligned}
J^*_{N-1}(x_{N-1}) \quad \triangleq \quad & \min_{\mu, u_{N-1}} \quad \mu \\
& \text{subj. to} \\
& \mu \geq \|Qx_{N-1}\|_p + \|Ru_{N-1}\|_p \\
& \qquad\qquad + \|P(Ax_{N-1} + Bu_{N-1})\|_p \\
& x_{N-1} \in \mathcal{X},\ u_{N-1} \in \mathcal{U}, \\
& Ax_{N-1} + Bu_{N-1} \in \mathcal{X}_f
\end{aligned}
$$

Recursive Approach. State Feedback Solution

At step $j = N-2$ of the dynamic programming recursion

$$
\begin{aligned}
J^*_{N-2}(x_{N-2}) \triangleq \quad & \min_{u_{N-2}} \|Qx_{N-2}\|_p + \|Ru_{N-2}\|_p \\
& \qquad\qquad + J^*_{N-1}(Ax_{N-2} + Bu_{N-2}) \\
& \text{subj. to} \\
& x_{N-2} \in \mathcal{X},\ u_{N-2} \in \mathcal{U}, \\
& Ax_{N-2} + Bu_{N-2} \in \mathcal{X}_f
\end{aligned}
$$

Recall that $J^*_{N-1}(x_{N-1})$ is a convex and piecewise affine function of x_{N-1}, i.e.,

$$
J^*_{N-1}(x_{N-1}) = \max_{i=1,\dots,n_{N-1}} \{c_i x_{N-1} + d_i\}
$$

Recursive Approach. State Feedback Solution

Rewrite the problem at step $j = N - 2$ as

$$
\begin{array}{rcl}
J^*_{N-2}(x_{N-2}) & \triangleq & \min\limits_{\mu, u_{N-2}} \;\; \mu \\
& & \text{subj. to} \\
& & \mu \geq \|Qx_{N-2}\|_p + \|Ru_{N-2}\|_p \\
& & \qquad\qquad +c_i(Ax_{N-2} + Bu_{N-2}) + d_i \\
& & \qquad\qquad i = 1, \ldots, n_{N-1}, \\
& & x_{N-2} \in \mathcal{X},\; u_{N-2} \in \mathcal{U}, \\
& & Ax_{N-2} + Bu_{N-2} \in \mathcal{X}_{N-1}
\end{array}
$$

and solve it to calculate $J^*_{N-2}(x_{N-2})$, $u^*_{N-2}(x_{N-2})$ and $\mathcal{X}_{N-2}$.

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   - A CLQR Algorithm
   - Infinite Horizon Solution: 1-norm and ∞-norm
5. Minimum Time Control

Infinite Horizon: 2-norm

Consider the following ***infinite-horizon constrained linear quadratic*** regulation problem (CLQR)

$$
\begin{array}{lll}
J_\infty^*(x(0)) = & \min\limits_{u_0,u_1,\dots} & \sum\limits_{k=0}^{\infty} x_k' Q x_k + u_k' R u_k \\
& \text{subj. to} & x_{k+1} = Ax_k + Bu_k,\ k = 0,\dots,\infty \\
& & x_k \in \mathcal{X},\ u_k \in \mathcal{U},\ k = 0,\dots,\infty \\
& & x_0 = x(0)
\end{array}
$$

Consider the feasible set

$$\mathcal{X}_\infty = \{x(0) \in \mathbb{R}^n |\ \text{Problem is feasible and } J_\infty^*(x(0)) < +\infty\}.$$

Observe that

- $\mathcal{X}_\infty = \mathcal{K}_\infty(\mathcal{O})$ with $\mathcal{O} = 0$
- If $x(0)$ is close to the origin, then the constraints will never become active and the solution of the problem will yield the *unconstrained* LQR.

Infinite Horizon

Definition (Maximal LQR Invariant Set $\mathcal{O}_\infty^{\text{LQR}}$)

Consider the system $x(k+1) = Ax(k) + Bu(k)$. $\mathcal{O}_\infty^{\text{LQR}} \subseteq \mathbb{R}^n$ denotes the maximal positively invariant set for the autonomous constrained linear system:

$$x(k+1) = (A + BF_\infty)x(k),\ x(k) \in \mathcal{X},\ u(k) \in \mathcal{U},\ \forall\ k \geq 0$$

where $u(k) = F_\infty x(k)$ is the unconstrained LQR control law obtained from the solution of the ARE.

We guess that there is some finite time $\bar{N}(x_0)$ at which the state enters $\mathcal{O}_\infty^{\text{LQR}}$. After $\bar{N}(x_0)$ the system evolves in an unconstrained manner ($x_k^* \in \mathcal{X},\ u_k^* \in \mathcal{U}, \forall k > \bar{N}$).

Infinite Horizon

Use the ***optimality principle*** and split the problem into ***two subproblems***.

1. up to time $k = \bar{N}$, where the constraints may be active
2. $k > \bar{N}$ where there are no constraints active.

Up to time $k = \bar{N}$

$$
\begin{array}{lll}
J^*_\infty(x(0)) = & \displaystyle\min_{u_0, u_1, \dots} & \displaystyle\sum_{k=0}^{\bar{N}-1} x_k' Q x_k + u_k' R u_k + J^*_{\bar{N}\to\infty}(x_{\bar{N}}) \\
& \text{subj. to} & x_k \in \mathcal{X},\ u_k \in \mathcal{U},\ k = 0, \dots, \bar{N}-1 \\
& & x_{k+1} = A x_k + B u_k,\ k \geq 0 \\
& & x_0 = x(0).
\end{array}
$$

Infinite Horizon

At time $k > \bar{N}$

$$
\begin{aligned}
J^*_{\bar{N}\to\infty}(x_{\bar{N}}) = & \min_{u_{\bar{N}}, u_{\bar{N}+1}, \ldots} & \sum_{k=\bar{N}}^{\infty} x_k' Q x_k + u_k' R u_k \\
& \text{subj. to} & x_{k+1} = A x_k + B u_k,\ k \geq \bar{N} \\
= & \; x_{\bar{N}}' P_\infty x_{\bar{N}} &
\end{aligned}
$$

Theorem (Equality of Finite and Infinite Optimal Control)

For any given initial state $x(0)$, the solution to the two subproblems is equal to the infinite-time solution of, if the terminal state $x_{\bar{N}}$ of subproblem 1 lies in the positive invariant set $\mathcal{O}_\infty^{LQR}$ and no terminal set constraint is applied, i.e. the state 'voluntarily' enters the set $\mathcal{O}_\infty^{LQR}$ after $\bar{N}$ steps.

Infinite Horizon

Q: *How to determine* $\bar{N}(x_0)$*?*

Theorem (Explicit solution of CLQR)

Assume that (A, B) *is a stabilizable pair and* $(Q^{1/2}, A)$ *is an observable pair,* $R \succ 0$*. The state-feedback solution to the (infinite time) CLQR problem in a compact set of the initial conditions* $\mathcal{S} \subseteq \mathcal{X}_\infty = \mathcal{K}_\infty(\mathbf{0})$ *is time-invariant, continuous and piecewise affine on polyhedra*

$$u^*(k) = f_\infty(x(k)), \quad f_\infty(x) = F^j x + g^j \quad \textit{if} \quad x \in CR^j_\infty, \quad j = 1, \ldots, N^r_\infty$$

where the polyhedral sets $CR^j_\infty = \{x \in \mathbb{R}^n \ : \ H^j x \leq K^j\}$, $j = 1, \ldots, N^r_\infty$ *are a finite partition of the feasible compact polyhedron* $\mathcal{S} \subseteq \mathcal{X}_\infty$.

A CLQR Algorithm

Consider the constrained finite time optimal control problem

$$
\begin{array}{lll}
J_0^*(x(0)) = & \min\limits_{U_0} & J_0(x(0), U_0) \\
& \text{subj. to} & x_{k+1} = Ax_k + Bu_k,\ k = 0, \ldots, N-1 \\
& & x_k \in \mathcal{X},\ u_k \in \mathcal{U},\ k = 0, \ldots, N-1 \\
& & x_N \in \mathbb{R}^n \quad \textbf{no terminal constraints} \\
& & x_0 = x(0)
\end{array}
$$

with

$$
J_0(x(0), U_0) \triangleq \underbrace{\|P_\infty x_N\|_p}_{P_\infty \ \textbf{solution of the ARE}} + \sum_{k=0}^{N-1} \|Qx_k\|_p + \|Ru_k\|_p
$$

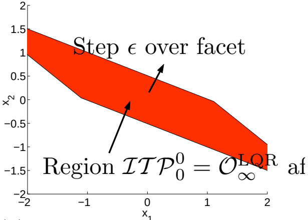
All states entering the invariant set $\mathcal{O}_\infty^{\mathrm{LQR}}$ in N steps, through the computed control law are ***infinite-horizon optimal***.

A CLQR Algorithm

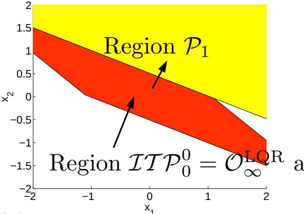
1. Compute the Maximal LQR Invariant Set $\mathcal{O}_\infty^{\mathrm{LQR}}$. Be

$$\mathcal{P}_0 \triangleq \mathcal{O}_\infty^{\mathrm{LQR}} = \{x \in \mathbb{R}^n | H_0 x \leq K_0\}$$

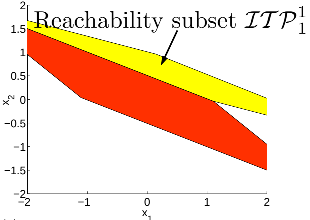
2. Find a point $\bar{x}$ by stepping over a facet of $\mathcal{O}_\infty^{\mathrm{LQR}}$ with a small step ϵ.
3. Solve the CFTOC for $x(0) = \bar{x}$, $\mathcal{X}_f = \mathbb{R}^n$, $P = P_\infty$, $N = 1$.



(g) Compute positive invariant region $\mathcal{O}_\infty^{\mathrm{LQR}}$ after $\bar{N}$ and step over facet with step-size ϵ.



(h) Solve QP for new point with horizon $N = 1$ to create the first constrained region $\mathcal{P}_1$.



(i) Compute reachability subset of $\mathcal{P}_1$ to obtain $\mathcal{ITP}_1^1$.

A CLQR Algorithm

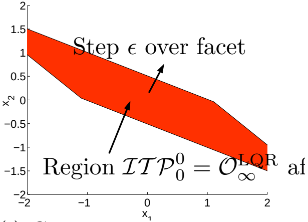
4. If the problem is feasible, be

$$\mathcal{P}_1 = \{x \in \mathbb{R}^n | H_1 x \leq K_1\}$$

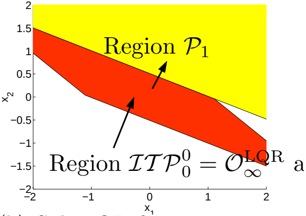
the polyhedron defined by the active constraints at $\bar{x}$ and $U_1^* = F_1 x(0) + G_1$ the associated control law.

5. Find the points in $\mathcal{P}_1$ evolving in one step to $\mathcal{O}_\infty^{\mathrm{LQR}}$ as

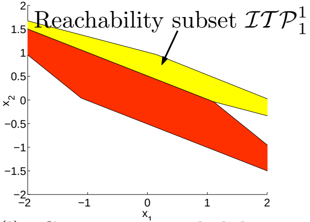
$$x_1 \in \mathcal{O}_\infty^{\mathrm{LQR}},\ x_1 = Ax_0 + BU_1^*,$$
$$x_0 \in \mathcal{P}_1$$



(j) Compute positive invariant region $\mathcal{O}_\infty^{\mathrm{LQR}}$ after $\bar{N}$ and step over facet with step-size ϵ.



(k) Solve QP for new point with horizon $N = 1$ to create the first constrained region $\mathcal{P}_1$.



(l) Compute reachability subset of $\mathcal{P}_1$ to obtain $\mathcal{ITP}_1^1$.

A CLQR Algorithm

At a generic step r of the algorithm

1. Step over a facet to a new point $\bar{x}$ and determine the polyhedron $\mathcal{P}_r$ and the associated control law ($U_N^* = F_r x(0) + G_r$) with the horizon N.
2. Extract from $\mathcal{P}_r$ the set of points entering $\mathcal{O}_\infty^{\mathrm{LQR}}$ in N time-steps by applying U_N^*.

$$
\begin{aligned}
x_N &\in \mathcal{O}_\infty^{\mathrm{LQR}} \\
x_0 &\in \mathcal{P}_r
\end{aligned}
$$

3. Continue exploring the facets increasing N. The algorithm terminates when $\mathcal{S}$ is covered or when we can no longer find a new feasible polyhedron $\mathcal{P}_r$.

Theorem (Exact Computation of $\bar{N}_{\mathcal{S}}$)

If we explore any given compact set $\mathcal{S}$ with the proposed algorithm, the largest resulting horizon is equal to $\bar{N}_{\mathcal{S}}$, i.e.,

$$\bar{N}_{\mathcal{S}} = \max_{\mathcal{ITP}_r^N \quad r=0,\ldots,R} N$$

Infinite Horizon

Consider the following ***infinite-horizon*** problem with constraints

$$
\begin{array}{rll}
J_\infty^*(x(0)) = & \min\limits_{u_0,u_1,\ldots} & \sum\limits_{k=0}^{\infty} \|Qx_k\|_p + \|Ru_k\|_p \\
& \text{subj. to} & x_{k+1} = Ax_k + Bu_k,\ k = 0,\ldots,\infty \\
& & x_k \in \mathcal{X},\ u_k \in \mathcal{U},\ k = 0,\ldots,\infty \\
& & x_0 = x(0)
\end{array}
\tag{9}
$$

with Q and R full column rank and the constraint sets $\mathcal{X}$ and $\mathcal{U}$ containing the origin in their interior and the set

$$\mathcal{X}_\infty = \{x(0) \in \mathbb{R}^n |\ \text{Problem (9) is feasible and } J_\infty^*(x(0)) < +\infty\}.$$

Infinite Horizon

Observe that

1. ***Full rank assumption*** on Q and R implies $u_k^* \rightarrow 0$ and $x_k^* \rightarrow 0$.
2. If $x(0)$ close enough to the origin, the ***problem is unconstrained***
3. Splitting the problem into a constrained and unconstrained still works. ***But*** the calculation of the maximal invariant set is *not trivial* since the ***unconstrained controller is a PPWA***.
4. The DP approach is straightforward here (recall that in the 2-norm case it was not because of the PPWQ structure of the cost-to-go), since the cost-to-go is PPWA.

Outline

1 Constrained Linear Optimal Control

2 Constrained Optimal Control: 2-Norm Case

3 Constrained Optimal Control: 1-norm and ∞-norm

4 Infinite Horizon

5 Minimum Time Control
- Example

Minimum Time Control

Consider the ***minimum-time*** constrained optimal control problem

$$
\begin{array}{lll}
J_0^*(x(0)) = & \min\limits_{U_0,N} & N \\
& \text{subj. to} & x_{k+1} = Ax_k + Bu_k,\ k = 0,\dots,N-1 \\
& & x_k \in \mathcal{X},\ u_k \in \mathcal{U},\ k = 0,\dots,N-1 \\
& & x_N \in \mathcal{X}_f \\
& & x_0 = x(0)
\end{array}
$$

IDEA

1. ***Offline phase*** Solve a sequence of 1-step problems to enter $\mathcal{X}_f$ in $1, 2, \dots, N$ steps. Recall that the result for each problem is a state feedback controller along with a feasibility set.
2. ***Online phase*** Given the current state, use the controller leading to $\mathcal{X}_f$ in minimum time.

Minimum Time Control. Offline Phase

1. Solve the following multiparametric program

$$\begin{array}{ll}\min\limits_{u_0} & c(x_0,u_0)\\ \text{subj. to} & x_1 = Ax_0 + Bu_0\\ & x_0 \in \mathcal{X},\ u_0 \in \mathcal{U}\\ & x_1 \in \mathcal{X}_f\end{array}$$

   with $c(x_0,u_0)$ any quadratic function. The solution is a PPWA controller and $\mathcal{X}_0 = \mathcal{K}_1(\mathcal{X}_f)$

2. Continue setting up and solving 1-step mp programs

$$\begin{array}{ll}\min\limits_{u_0} & c(x_0,u_0)\\ \text{subj. to} & x_1 = Ax_0 + Bu_0\\ & x_0 \in \mathcal{X},\ u_0 \in \mathcal{U}\\ & x_1 \in \mathcal{K}_{j-1}(\mathcal{X}_f)\end{array}$$

   where $\mathcal{X}_0 = \mathcal{K}_j(\mathcal{X}_f)$

3. Obtain $\mathcal{K}_1(\mathcal{X}_f), \ldots, \mathcal{K}_N(\mathcal{X}_f)$

Minimum Time Control. Online Phase

Algorithm (Minimum-Time Controller: On-Line Application)

1. *Obtain state measurement* x*.*
2. *Find controller partition* $c_{min} = \min_{c \in \{0,\ldots,N\}} c, \quad s.t.\ x \in \mathcal{K}_c(\mathcal{X}_f)$.
3. *Find controller region* r*, such that* $x \in \mathcal{P}_r^{c_{min}}$ *and compute* $u_0 = F_r^{c_{min}} x + G_r^{c_{min}}$.
4. *Apply input* u_0 *to system and go to Step 1.*

Minimum Time Control. Example

Consider the double integrator

$$\begin{cases} x(t+1) & = & \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \\ y(t) & = & \begin{bmatrix} 1 & 0 \end{bmatrix} x(t) \end{cases}$$

subject to constraints

$$-1 \leq u(k) \leq 1, \ k = 0, \ldots, 5$$

$$\begin{bmatrix} -10 \\ -10 \end{bmatrix} \leq x(k) \leq \begin{bmatrix} 10 \\ 10 \end{bmatrix}, \ k = 0, \ldots, 5$$

Minimum Time Control. Example