Model Predictive Control for Linear and Hybrid Systems
Robust Constrained Optimal Control

Francesco Borrelli

Department of Mechanical Engineering,
University of California at Berkeley,
USA

fborrelli@berkeley.edu

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Outline

Introduction

- LQR, is a robust controller, in the following sense
  - infinite gain amplification in any input,
  - 1/2 gain reduction in any input,
  - phase margin of $\pi/3$
- In MPC, controller robustness means to require
  - feasibility of the controller for all time, in the presence of persistent perturbations (disturbances).
  - convergence to a set $\mathcal{X}_f$ that contains the desired equilibrium point, in the presence of persistent perturbations (disturbances).
- in the next lectures we will show how to address the effect of disturbances on a constrained systems and synthesize robust model predictive control

System Model

- Consider the uncertain discrete–time dynamical system:

$$x(t+1) = A(w^p(t))x(t) + B(w^p(t))u(t) + Ew^a(t)$$

  where

$$x(t) \in \mathcal{X},\ u(t) \in \mathcal{U},\ \forall t \geq 0.$$

  The sets $\mathcal{X} \subseteq \mathbb{R}^n$ and $\mathcal{U} \subseteq \mathbb{R}^m$ are polytopes.
- Vectors $w^a(t) \in \mathbb{R}^{n_a}$ and $w^p(t) \in \mathbb{R}^{n_p}$ are unknown additive disturbances and parametric uncertainties, respectively.
- The disturbance vector is $w(t) = [w^a(t);\ w^p(t)] \in \mathbb{R}^{n_w}$ with $n_w = n_a + n_p$.
- Only bounds on $w^a(t)$ and $w^p(t)$ are known, namely that $w \in \mathcal{W} = \mathcal{V} \times \mathcal{W}$ with $w^a(t) \in \mathcal{V}$

$$\mathcal{V} = \text{conv}(\{w^{a,1}, \ldots, w^{a,n_{\mathcal{V}}}\})$$

  and $w^p(t) \in \mathcal{W}$,

$$\mathcal{W} = \text{conv}(\{w^{p,1}, \ldots, w^{p,n_{\mathcal{W}}}\})$$

System Model

- Consider the uncertain discrete–time dynamical system:

$$x(t+1) = A(w^p(t))x(t) + B(w^p(t))u(t) + Ew^a(t)$$

- $A(\cdot)$, $B(\cdot)$ are affine functions of w^p

$$A(w^p) = A^0 + \sum_{i=1}^{n_p} A^i w_c^{p,i},\ B(w^p) = B^0 + \sum_{i=1}^{n_p} B^i w_c^{p,i}$$

  where $A^i \in \mathbb{R}^{n\times n}$ and $B^i \in \mathbb{R}^{n\times m}$ are given matrices

- For $i = 0\ldots, n_p$ and $w_c^{p,i}$ is the i-th component of the vector w^p, i.e., $w^p = [w_c^{p,1}, \ldots, w_c^{p,n_p}]$.

Outline

Constrained Robust Optimal Control. Batch Approach

Define the worst case cost function

$$J_0(x(0), U_0) \triangleq \max_{w_0,\dots,w_{N-1}} \left[p(x_N) + \sum_{k=0}^{N-1} q(x_k, u_k) \right]$$

$$\text{subj. to} \begin{cases} x_{k+1} = A(w_k^p)x_k + B(w_k^p)u_k + Ew_k^a \\ w_k^a \in \mathcal{V}, w_k^p \in \mathcal{W}, \\ k = 0, \dots, N-1 \end{cases}$$

where N is the time horizon and $U_0 \triangleq [u_0', \dots, u_{N-1}']' \in \mathbb{R}^s$, $s \triangleq mN$ the vector of the input sequence.
The robust optimal control problem is

$$J_0^*(x_0) = \min_{U_0} J_0(x_0, U_0)$$

$$\text{subj. to} \left\{ \begin{array}{l} x_k \in \mathcal{X},\ u_k \in \mathcal{U} \\ x_{k+1} = A(w_k^p)x_k + B(w_k^p)u_k + Ew_k^a \\ x_N \in \mathcal{X}_f \\ k = 0, \dots, N-1 \end{array} \right\} \begin{array}{l} \forall w_k^a \in \mathcal{V}, w_k^p \in \mathcal{W} \\ \forall k = 0, \dots, N-1 \end{array}$$

Constrained Robust Optimal Control. Batch Approach

- x_k denotes the state vector at time k obtained by starting from the state $x_0 = x(0)$ and applying to the system model

$$x_{k+1} = A(w_k^p)x_k + B(w_k^p)u_k + Ew_k^a$$

  the input sequence $u_0, \ldots, u_{k-1}$ and the disturbance sequences $\mathbf{w}^a \triangleq \{w_0^a, \ldots, w_{N-1}^a\}$, $\mathbf{w}^p \triangleq \{w_0^p, \ldots, w_{N-1}^p\}$.
- We denote with $\mathcal{X}_i^{OL} \subseteq \mathcal{X}$ the set of states x_i for which the robust optimal control problem is feasible, i.e.,

$$\begin{array}{rl} \mathcal{X}_i^{OL} = & \{x_i \in \mathcal{X} \text{such that } \exists (u_i, \ldots, u_{N-1}) \text{ such that} \\ & x_k \in \mathcal{X},\ u_k \in \mathcal{U},\ k = i, \ldots, N-1,\ x_N \in \mathcal{X}_f \forall\ w_k^a \in \mathcal{V}, w_k^p \in \mathcal{W} \\ & k = i, \ldots, N-1 \text{ where } x_{k+1} = A(w_k^p)x_k + B(w_k^p)u_k + Ew_k^a\}. \end{array}$$

- The reason for including constraints in the minimization problem and not in the maximization problem is that w_j^a and w_j^p are free to act regardless of the state constraints. On the other hand, the input u_j has the duty of keeping the state within the constraints for all possible disturbance realization.

Constrained Robust Optimal Control. Batch Approach

- The optimal control problem looks for the worst value $J(x_0, U)$ of the performance index and the corresponding worst sequences $\mathbf{w}^{p*}$, $\mathbf{w}^{a*}$ as a function of x_0 and U_0.
- It minimizes such a worst performance subject to the constraint that the input sequence must be feasible *for all* possible disturbance realizations.
- Note that worst sequences $\mathbf{w}^{a*}$, $\mathbf{w}^{p*}$for the performance are not necessarily worst sequences in terms of constraints satisfaction.
- The min-max formulation is based on an *open-loop* prediction and thus referred to as Constrained Robust Optimal Control with open-loop predictions (CROC-OL).
- The optimal control problem can be viewed as a deterministic zero-sum dynamic game between two players: the controller U and the disturbance W.

Constrained Robust Optimal Control. Batch Approach or Open-Loop Predictions

- The player U plays first. Given the initial state $x(0)$, U chooses his action over the whole horizon $\{u_0, \dots, u_{N-1}\}$, reveals his plan to the opponent W, who decides on his actions next $\{w_0^a, w_0^p, \dots, w_{N-1}^a, w_{N-1}^p\}$.
- For this reason the player U has the duty of counteracting *any* feasible disturbance realization with just *one* single sequence $\{u_0, \dots, u_{N-1}\}$.
- This prediction model does not consider that at the next time step, the payer can measure the state $x(1)$ and "adjust" his input $u(1)$ based on the current measured state.
- By not considering this fact, the effect of the uncertainty may grow over the prediction horizon and may easily lead to infeasibility of the min problem

Constrained Robust Optimal Control. Recursive Approach or Closed-Loop Predictions

The constrained robust optimal control problem based on closed-loop predictions (CROC-CL) is defined as:

$$J_j^*(x_j) \triangleq \min_{u_j} J_j(x_j, u_j)$$

$$\text{s.t.} \quad \left\{ \begin{array}{l} x_j \in \mathcal{X},\ u_j \in \mathcal{U} \\ A(w_j^p)x_j + B(w_j^p)u_j + Ew_j^a \in \mathcal{X}_{j+1} \end{array} \right\} \ \forall w_j^a \in \mathcal{V}, w_j^p \in \mathcal{W}$$

$$J_j(x_j, u_j) \triangleq \max_{w_j^a \in \mathcal{V},\ w_j^p \in \mathcal{W}} \left\{ q(x_j, u_j) + J_{j+1}^*(A(w_j^p)x_j + B(w_j^p)u_j + Ew_j^a) \right\},$$

for $j = 0, \ldots, N-1$ and with boundary conditions

$$J_N^*(x_N) = p(x_N)$$
$$\mathcal{X}_N = \mathcal{X}_f,$$

Constrained Robust Optimal Control. Closed-Loop Predictions

$\mathcal{X}_j$ denotes the set of states x for which the CROC-CL is feasible

$$\begin{array}{rl}\mathcal{X}_j = & \{x \in \mathcal{X} \ \text{ such that } \exists u \in \mathcal{U} \\ & \text{s.t. } A(w^p)x + B(w^p)u + Ew^a \in \mathcal{X}_{j+1} \ \forall w^a \in \mathcal{V}, w^p \in \mathcal{W}\}.\end{array}$$

- The reason for including constraints in the minimization problem and not in the maximization problem is that w_j^a and w_j^p are free to act regardless of the state constraints. On the other hand, the input u_j has the duty of keeping the state within the constraints for all possible disturbance realization.

Constrained Robust Optimal Control. Closed-Loop Predictions

- The optimal control problem can be viewed as a deterministic zero-sum dynamic game between two players: the controller U and the disturbance W.
- The game is played as follows. At the generic time j player U observes x_j and responds with $u_j(x_j)$. Player W observes $(x_j, u_j(x_j))$ and responds with w_j^a and w_j^p.
- Note that player U does *not* need to reveal his action u_j to player W (the disturbance). This happens for instance in games where U and W play at the same time, e.g. rock-paper-scissors.
- The player W will always play the worst case action only if it has knowledge of both x_j and $u_j(x_j)$. In fact, w_j^a and w_j^p are a function of x_j and u_j.
- If U does *not* reveal his action to player W, then we can only claim that the player W *might* play the worst case action.
- Robust constraint satisfaction and worst case minimization will always be guaranteed.

Constrained Robust Optimal Control. Closed-Loop Predictions

Consider the system

$$x_{k+1} = x_k + u_k + w_k$$

where x, u and w are state, input and disturbance, respectively.

- Let $u_k \in \{-1, 0, 1\}$ and $w_k \in \{-1, 0, 1\}$ be feasible input and disturbance.
- Here $\{-1, 0, 1\}$ denotes the set with three elements: -1, 0 and 1.
- Let $x(0) = 0$ be the initial state. The objective for player U is to play two moves in order to keep the state x_2 at time 2 in the set $[-1, 1]$.
- If U is able to do so for any possible disturbance, then he will win the game.
- The open-loop formulation is infeasible. In fact, in open-loop U can choose from nine possible sequences: (0,0), (1,1), (-1,-1), (-1,1) (1,-1), (-1,0), (1,0), (0,1) and (0,-1). For any of those sequence there will always exist a disturbance sequence w_0, w_1 which will bring x_2 outside the feasible set [-1,1].
- The closed-loop formulation is feasible and has a simple solution: $u_k = -x_k$. In this case the system becomes $x_{k+1} = w_k$ and $x_2 = w_1$ lies in the feasible set $[-1, 1]$ for all admissible disturbances w_1.

Outline

Basic Results

Proposition

Let $g : \mathbb{R}^{n_z} \times \mathbb{R}^{n} \times \mathbb{R}^{n_w} \to \mathbb{R}^{n_g}$ *be a function of* (z, x, w) *convex in* w *for each* (z, x)*. Assume that the variable* w *belongs to the polytope* $\mathcal{W}$ *with vertices* $\{\bar{w}_i\}_{i=1}^{n_{\mathcal{W}}}$*. Then, the constraint*

$$g(z, x, w) \ \leq 0 \ \forall w \in \mathcal{W} \tag{1}$$

is satisfied if and only if

$$g(z, x, \bar{w}_i) \leq 0, \ i = 1, \ldots, n_{\mathcal{W}}. \tag{2}$$

Basic Results

Proposition

Assume $g(z,x,w) = g^1(z,x) + g^2(w)$*. Then, the constraint (1) can be replaced by* $g^1(z,x) \leq -\bar{g}$*, where* $\bar{g} \triangleq \left[\bar{g}_1, \ldots, \bar{g}_{n_g}\right]'$ *is a vector whose i-th component is*

$$\bar{g}_i = \max_{w \in \mathcal{W}} \; g_i^2(w), \tag{3}$$

and $g_i^2(w)$ *denotes the i-th component of* $g^2(w)$*.*

Basic Results

Consider the second order autonomous system

$$x(t+1) \quad = \quad A(w^p(t))x(t) + w^a(t) = \left[\begin{array}{cc} 0.5 + w^p(t) & 0 \\ 1 & -0.5 \end{array} \right] x(t) + w^a(t) \tag{4}$$

subject to the state constraints

$$\begin{aligned} x(t) \in \mathcal{X} = \left\{ x \text{ s.t. } \left[\begin{array}{c} -10 \\ -10 \end{array} \right] \leq x \leq \left[\begin{array}{c} 10 \\ 10 \end{array} \right] \right\}, \ \forall t \geq 0 \\ w^a(t) \in \mathcal{W}^a = \left\{ w^a \text{ s.t. } \left[\begin{array}{c} -1 \\ -1 \end{array} \right] \leq w^a \leq \left[\begin{array}{c} 1 \\ 1 \end{array} \right] \right\}, \ \forall t \geq 0 \\ w^p(t) \in \mathcal{W}^p = \left\{ w^p \text{ s.t. } 0 \leq w^p \leq 0.5 \right\}, \ \forall t \geq 0. \end{aligned} \tag{5}$$

Let $w = [w^a;\ w^p]$ and $\mathcal{W} = \mathcal{W}^a \times \mathcal{W}^p$.

Basic Results

The set $\mathcal{X}$ is a polytope and it can be represented as an $\mathcal{H}$-polytope

$$\mathcal{X} = \{x \text{s.t. } Hx \leq h\}, \tag{6}$$

The set $\text{Pre}(\mathcal{X}, \mathcal{W})$ can be rewritten as

$$\begin{aligned} \text{Pre}(\mathcal{X}, \mathcal{W}) &= \left\{x \text{ s.t. } \quad Hf_a(x, w) \quad \leq h, \ \forall w \in \mathcal{W} \ \right\} &(7)\\ &= \{x \text{ s.t. } HA(w^p)x \leq h - Hw^a, \ \forall w^a \in \mathcal{W}^a, \ w^p \in \mathcal{W}^p\}. &(8) \end{aligned}$$

By using the propositions on previous slides we obtain

$$x \in \text{Pre}(\mathcal{X}, \mathcal{W}) = \{x \in \mathbb{R}^n \text{ s.t. } \begin{bmatrix} HA(0) \\ HA(0.5) \end{bmatrix} x \leq \begin{bmatrix} \tilde{h} \\ \tilde{h} \end{bmatrix}\} \tag{9}$$

with

$$\tilde{h}_i = \min_{w^a \in \mathcal{W}^a} (h_i - H_i w^a). \tag{10}$$

Feasible Set Computation. Batch Approach

Consider $\mathcal{X}_i^{OL} \subseteq \mathcal{X}$ the set of states x_i for which the robust optimal control problem is feasible, i.e.,

$$\begin{aligned}\mathcal{X}_i^{OL} = \ & \{x_i \in \mathcal{X} \text{such that } \exists (u_i, \dots, u_{N-1}) \text{ such that} \\ & x_k \in \mathcal{X},\ u_k \in \mathcal{U},\ k = i, \dots, N-1,\ x_N \in \mathcal{X}_f,\ \forall\ w_k^a \in \mathcal{V},\ w_k^p \in \mathcal{W} \\ & k = i, \dots, N-1 \text{ where } x_{k+1} = A(w_k^p)x_k + B(w_k^p)u_k + Ew_k^a\}.\end{aligned}$$

- For any initial state $x_i \in \mathcal{X}_i^{OL}$ there exists a feasible sequence of inputs $U_i \triangleq [u_i', \dots, u_{N-1}']$ which keeps the state evolution in the feasible set $\mathcal{X}$ at future time instants $k = i+1, \dots, N-1$ and forces x_N into $\mathcal{X}_f$ at time N for all feasible disturbance sequence $w_k^a \in \mathcal{V},\ w_k^p \in \mathcal{W},\ k = i, \dots, N-1$.
- Clearly $\mathcal{X}_N^{OL} = \mathcal{X}_f$.
- How to compute $\mathcal{X}_i^{OL}$ for $i = 0, \dots, N-1$?

Feasible Set Computation. Batch Approach

- Let the state and input constraint sets $\mathcal{X}$, $\mathcal{X}_f$ and $\mathcal{U}$ be the $\mathcal{H}$-polyhedra $A_x x \leq b_x$, $A_f x \leq b_f$, $A_u u \leq b_u$.
- Recall that the disturbance sets are defined as $\mathcal{V} = \text{conv}\{w^{a,1}, \ldots, w^{a,n_\mathcal{V}}\}$ and $\mathcal{W} = \text{conv}\{w^{p,1}, \ldots, w^{p,n_\mathcal{W}}\}$.
- Define $U_i \triangleq [u_i', \ldots, u_{N-1}']$ and the polyhedron $\mathcal{P}_i$ of robustly feasible states and input sequences at time i, defined as

$$\mathcal{P}_i = \{(U_i, x_i) \in \mathbb{R}^{m(N-i)+n} \text{s.t. } G_i U_i - E_i x_i \leq W_i\}.$$

- G_i, E_i and W_i are obtained by collecting all the following inequalities:
  - Input Constraints

$$A_u u_k \leq b_u, \quad k = i, \ldots, N-1$$

  - State Constraints

$$A_x x_k \leq b_x, \;\; k = i, \ldots, N-1 \text{ for all } \; w_l^a \in \mathcal{V}, \; w_l^p \in \mathcal{W}, \; l = i, \ldots, k-1. \quad (11)$$

  - Terminal State Constraints

$$A_f x_N \leq b_f, \quad \text{for all } \; w_l^a \in \mathcal{V}, \; w_l^p \in \mathcal{W}, \; l = i, \ldots, N-1. \quad (12)$$

- Constraints are enforced for all feasible disturbance sequences.

Feasible Set Computation. Batch Approach

- To enforce constraints for all feasible disturbance sequences. rewrite them as a function of x_i and the input sequence U_i.
- Since we assumed that that $A(\cdot)$, $B(\cdot)$ are affine functions of w^p and since the composition of a convex constraint with an affine map generates a convex constraint we can use the Propositions presented in "basic results" and impose the constraints at all the vertices of the sets $\underbrace{\mathcal{V} \times \mathcal{V} \times \ldots \times \mathcal{V}}_{i,\ldots,N-1}$ and $\underbrace{\mathcal{W} \times \mathcal{W} \times \ldots \times \mathcal{W}}_{i,\ldots,N-1}$.
- Note that the resulting constraints are now linear in x_i and U_i.
- Once the matrices G_i, E_i and W_i have been computed, the set $\mathcal{X}_i^{OL}$ is a polyhedron and can be computed by projecting the polyhedron $\mathcal{P}_i$ on the x_i space.

Feasible Set Computation. Revursive Approach

In the *recursive approach* we have

$$\begin{aligned} \mathcal{X}_i &= \{x \in \mathcal{X} \text{s.t. } \exists u \in \mathcal{U} \text{ such that } A(w_i^p)x + B(w_i^p)u + Ew_i^a \in \mathcal{X}_{i+1} \\ &\quad \forall w_i^a \in \mathcal{V},\ w_i^p \in \mathcal{W}\},\ i = 0, \ldots, N-1 \\ \mathcal{X}_N &= \mathcal{X}_f. \end{aligned} \tag{13}$$

- The definition of $\mathcal{X}_i$ is recursive and it requires that for any feasible initial state $x_i \in \mathcal{X}_i$ there exists a feasible input u_i which keeps the next state $A(w_i^p)x + B(w_i^p)u + Ew_i^a$ in the feasible set $\mathcal{X}_{i+1}$ for all feasible disturbances $w_i^a \in \mathcal{V},\ w_i^p \in \mathcal{W}$.
- Recall that $\mathcal{X}_0$ is different from $\mathcal{X}_0^{OL}$.
- Let $\mathcal{X}_i$ be the $\mathcal{H}$-polyhedron $A_{\mathcal{X}_i} x \leq b_{\mathcal{X}_i}$.
- The set $\mathcal{X}_{i-1}$ is the projection of the following polyhedron on the x_i space

$$\begin{bmatrix} A_u \\ 0 \\ A_{\mathcal{X}_i} B(w_i^p) \end{bmatrix} u_i + \begin{bmatrix} 0 \\ A_x \\ A_{\mathcal{X}_i} A(w_i^p) \end{bmatrix} x_i \leq \begin{bmatrix} b_u \\ b_x \\ b_{\mathcal{X}_i} - Ew_i^a \end{bmatrix} \tag{14}$$

$$\text{for all} \quad w_i^a \in \{w^{a,i}\}_{i=1}^{n_{\mathcal{V}}},\ w_i^p \in \{w^{p,i}\}_{i=1}^{n_{\mathcal{W}}}$$

Feasible Set Computation. Example

Consider the system

$$x_{k+1} = x_k + u_k + w_k \tag{15}$$

Let $u_k \in [-1, 1]$ and $w_k \in [-1, 1]$ be the feasible input and disturbance. The objective for player U is to play two moves in order to keep the state at time three x_3 in the set $\mathcal{X}_f = [-1, 1]$.

Batch approach

We rewrite the terminal constraint as

$$\begin{aligned} x_3 = x_0 + u_0 + u_1 + u_2 + w_0 + w_1 + w_2 \in [-1, 1] \\ \text{for all } \; w_0 \in [-1, 1], \; w_1 \in [-1, 1], \; w_2 \in [-1, 1] \end{aligned} \tag{16}$$

which becomes

$$\begin{aligned} -1 \le x_0 + u_0 + u_1 + u_2 + 3 \le 1 \\ -1 \le x_0 + u_0 + u_1 + u_2 + 1 \le 1 \\ -1 \le x_0 + u_0 + u_1 + u_2 - 1 \le 1 \\ -1 \le x_0 + u_0 + u_1 + u_2 - 3 \le 1 \end{aligned} \tag{17}$$

which by removing redundant constraints becomes the (infeasible) constraint

$$2 \le x_0 + u_0 + u_1 + u_2 \le -2$$

Feasible Set Computation. Example

The set $\mathcal{X}_0^{OL}$ is the projection on the x_0 space of the polyhedron $\mathcal{P}_0$

$$\mathcal{P}_0 = \{(u_0, u_1, u_2, x_0) \in \mathbb{R}^4 \text{s.t.} \begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \\ 1 & 1 & 1 \\ -1 & -1 & -1 \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} x_0 \leq \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ -2 \\ -2 \end{bmatrix} \tag{18}$$

which is empty since the terminal state constraint is infeasible.

Feasible Set Computation. Example

Recursive approach

By using the recursive we have $\mathcal{X}_3 = \mathcal{X}_f = [-1, 1]$.
Rewrite the terminal constraint as

$$x_3 = x_2 + u_2 + w_2 \in [-1, 1] \text{ for all } w_2 \in [-1, 1] \tag{19}$$

which becomes

$$\begin{array}{c} -1 \le x_2 + u_2 + 1 \le 1 \\ -1 \le x_2 + u_2 - 1 \le 1 \end{array} \tag{20}$$

which by removing redundant constraints becomes

$$0 \le x_2 + u_2 \le 0$$

The set $\mathcal{X}_2$ is the projection on the x_2 space of the polyhedron

$$\begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 1 & 1 \\ -1 & -1 \end{bmatrix} [u_2,\ x_2] \le \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \tag{21}$$

which yields $\mathcal{X}_2 = [-1, 1]$. Since $\mathcal{X}_2 = \mathcal{X}_3$ one can conclude that $\mathcal{X}_2$ is the maximal controllable robust invariant set and $\mathcal{X}_0 = \mathcal{X}_1 = \mathcal{X}_2 = [-1, 1]$.

Feasible Set Computation. Example

Batch approach with closed-loop predictions

- The horizon is three and therefore we consider the disturbance set over the horizon 3-1=2, $\mathcal{W} \times \mathcal{W} = [-1, 1] \times [-1, 1]$.
- Such a set has four vertices: $\{\tilde{w}_0^1 = 1,\ \tilde{w}_1^1 = 1\}$, $\{\tilde{w}_0^2 = -1,\ \tilde{w}_1^2 = 1\}$, $\{\tilde{w}_0^1 = 1,\ \tilde{w}_1^3 = -1\}$, $\{\tilde{w}_0^2 = -1,\ \tilde{w}_1^4 = -1\}$.
- We introduce an input sequence $u_0,\ \tilde{u}_1^i,\ \tilde{u}_2^j$, where the index $i \in \{1, 2\}$ is associated with the vertices $\tilde{w}_0^1$ and $\tilde{w}_0^2$ and the index $j \in \{1, 2, 3, 4\}$ is associated with the vertices $\tilde{w}_1^1,\ \tilde{w}_1^2,\ \tilde{w}_1^3,\ \tilde{w}_1^4$.
- The terminal constraint is thus rewritten as

$$x_3 = x_0 + u_0 + \tilde{u}_1^i + \tilde{u}_2^j + \tilde{w}_0^i + \tilde{w}_1^j + w_2 \in [-1, 1],\ i = 1, 2,\ j = 1, \ldots, 4, \quad (22)$$

$\forall\ w_2 \in [-1, 1]$

Feasible Set Computation. Example

which becomes

$$
\begin{array}{l}
-1 \leq x_0 + u_0 + \tilde{u}_1^1 + \tilde{u}_2^1 + 0 + w_2 \leq 1, \quad \forall\ w_2 \in [-1, 1] \\
-1 \leq x_0 + u_0 + \tilde{u}_1^2 + \tilde{u}_2^2 + 0 + w_2 \leq 1, \quad \forall\ w_2 \in [-1, 1] \\
-1 \leq x_0 + u_0 + \tilde{u}_1^1 + \tilde{u}_2^3 + 2 + w_2 \leq 1, \quad \forall\ w_2 \in [-1, 1] \\
-1 \leq x_0 + u_0 + \tilde{u}_1^2 + \tilde{u}_2^4 - 2 + w_2 \leq 1, \quad \forall\ w_2 \in [-1, 1] \\
-1 \leq u_0 \leq 1 \\
-1 \leq \tilde{u}_1^1 \leq 1 \\
-1 \leq \tilde{u}_1^2 \leq 1 \\
-1 \leq \tilde{u}_2^1 \leq 1 \\
-1 \leq \tilde{u}_2^2 \leq 1 \\
-1 \leq \tilde{u}_2^3 \leq 1 \\
-1 \leq \tilde{u}_2^4 \leq 1
\end{array}
\tag{23}
$$

The set $\mathcal{X}_0$ can be obtained by using Proposition 2 for the polyhedron (23) and projecting the resulting polyhedron in the $(x_0,\ u_0,\ \tilde{u}_1^1, \tilde{u}_1^2,\ \tilde{u}_2^1,\ \tilde{u}_2^2,\ \tilde{u}_2^3,\ \tilde{u}_2^4)$-space on the x_0 space. This yields $\mathcal{X}_0 = [-1, 1]$.

Outline

Basic Result

Lemma

Let $f : \mathbb{R}^s \times \mathbb{R}^n \times \mathbb{R}^{n_w} \to \mathbb{R}$ and $g : \mathbb{R}^s \times \mathbb{R}^n \times \mathbb{R}^{n_w} \to \mathbb{R}^{n_g}$ be functions of (z, x, w) convex in w for each (z, x). Assume that the variable w belongs to the polyhedron $\mathcal{W}$ with vertices $\{\bar{w}_i\}_{i=1}^{N_{\mathcal{W}}}$. Then the min-max multiparametric problem

$$\begin{array}{lll} J^*(x) = & \min_z & \max_{w \in \mathcal{W}} f(z, x, w) \\ & \textit{subj. to} & g(z, x, w) \ \leq 0 \ \forall w \in \mathcal{W} \end{array} \tag{24}$$

is equivalent to the multiparametric optimization problem

$$\begin{array}{lll} J^*(x) = & \min_{\mu, z} & \mu \\ & \textit{subj. to} & \mu \geq f(z, x, \bar{w}_i), \ i = 1, \ldots, N_{\mathcal{W}} \\ & & g(z, x, \bar{w}_i) \leq 0, \ i = 1, \ldots, N_{\mathcal{W}}. \end{array} \tag{25}$$

Basic Result

Corollary

If f is convex and piecewise affine in (z,x), i.e. $f(z,x,w) = \max_{i=1,\ldots,n_f}$ $\{L_i(w)z + H_i(w)x + K_i(w)\}$ and g is linear in (z,x) for all $w \in \mathcal{W}$, $g(z,x,w) = K_g(w) + L_g(w)x + H_g(w)z$ (with $K_g(\cdot)$, $L_g(\cdot)$, $H_g(\cdot)$, $L_i(\cdot)$, $H_i(\cdot)$, $K_i(\cdot)$, $i = 1,\ldots,n_f$, convex functions), then the min-max multiparametric problem (24) is equivalent to the mp-LP problem

$$\begin{array}{lll} J^*(x) = & \min_{\mu,z} & \mu \\ & \text{subj. to} & \mu \geq K_j(\bar{w}_i) + L_j(\bar{w}_i)z + H_j(\bar{w}_i)x, \\ & & i = 1,\ldots,N_{\mathcal{W}},\ j = 1,\ldots,n_f \\ & & L_g(\bar{w}_i)x + H_g(\bar{w}_i)z \leq -K_g(\bar{w}_i),\ i = 1,\ldots,N_{\mathcal{W}} \end{array} \tag{26}$$

Min-Max Constrained Robust Optimal Control-Recursive Approach

Theorem

There exists a state-feedback control law $u^*(k) = f_k(x(k))$, $f_k : \mathcal{X}_k \subseteq \mathbb{R}^n \to \mathcal{U} \subseteq \mathbb{R}^m$, *solution of the CROC-CL with cost based on one or infinity norm which is time-varying, continuous and piecewise affine on polyhedra*

$$f_k(x) = F_k^i x + g_k^i \quad \text{if} \quad x \in CR_k^i, \quad i = 1, \dots, N_k^r \tag{27}$$

where the polyhedral sets $CR_k^i = \{x \in \mathbb{R}^n \ \text{s.t.}\ H_k^i x \leq K_k^i\}$, $i = 1, \dots, N_k^r$ *are a partition of the feasible polyhedron* $\mathcal{X}_k$. *Moreover* f_i, $i = 0, \dots, N-1$ *can be found by solving* N *mp-LPs.*

Min-Max Constrained Robust Optimal Control-Recursive Approach

Consider the first step $j = N-1$ of dynamic programming

$$J^*_{N-1}(x_{N-1}) \triangleq \min_{u_{N-1}} J_{N-1}(x_{N-1}, u_{N-1}) \tag{28}$$

$$\text{subj. to} \left\{ \begin{array}{l} Fx_{N-1} + Gu_{N-1} \leq f \\ A(w^p_{N-1})x_{N-1} + B(w^p_{N-1})u_{N-1} + Ew^a_{N-1} \in \mathcal{X}_f \\ \forall w^a_{N-1} \in \mathcal{V}, w^p_{N-1} \in \mathcal{W} \end{array} \right. \tag{29}$$

$$J_{N-1}(x_{N-1}, u_{N-1}) \triangleq \max_{w^a_{N-1} \in \mathcal{V},\ w^p_{N-1} \in \mathcal{W}} \left\{ \begin{array}{l} \|Qx_{N-1}\|_p + \|Ru_{N-1}\|_p + \\ +\|P(A(w^p_{N-1})x_{N-1} + \\ +B(w^p_{N-1})u_{N-1} + Ew^a_{N-1})\|_p \end{array} \right\}. \tag{30}$$

Min-Max Constrained Robust Optimal Control-Recursive Approach

- The cost function in the maximization problem (30) is piecewise affine and convex with respect to the optimization vector w_{N-1}^a, w_{N-1}^p and the parameters u_{N-1}, x_{N-1}.
- The constraints in the minimization problem linear in (u_{N-1}, x_{N-1}) for all vectors w_{N-1}^a, w_{N-1}^p.
- Therefore, by "basic Results" $J_{N-1}^*(x_{N-1})$, $u_{N-1}^*(x_{N-1})$ and $\mathcal{X}_{N-1}$ are computable via the mp-LP:

$$
\begin{aligned}
J_{N-1}^*(x_{N-1}) \quad \triangleq \quad & \min_{\mu, u_{N-1}} \ \mu && \text{(31a)} \\
& \text{subj. to } \ \mu \geq \|Qx_{N-1}\|_p + \|Ru_{N-1}\|_p + \\
& \qquad + \|P(A(\bar{w}_h^p)x_{N-1} + B(\bar{w}_h^p)u_{N-1} + E\bar{w}_i^a)\|_p && \text{(31b)} \\
& \qquad Fx_{N-1} + Gu_{N-1} \leq f && \text{(31c)} \\
& \qquad A(\bar{w}_h^p)x_{N-1} + B(\bar{w}_h^p)u_{N-1} + E\bar{w}_i^a \in \mathcal{X}_N && \text{(31d)} \\
& \qquad \forall i = 1, \ldots, n_{\mathcal{V}}, \ \forall h = 1, \ldots, n_{\mathcal{W}}.
\end{aligned}
$$

The convexity and linearity arguments still hold for $j = N-2, \ldots, 0$ and the procedure can be iterated backwards in time j.

We Will not Cover

- Robust Invariant Set
- Robust invariant Set Evolution
- Robust MPC Theorem
- Robust Estimation