Model Predictive Control for Linear and Hybrid Systems
Reachability and Controllability

Francesco Borrelli

Department of Mechanical Engineering,
University of California at Berkeley,
USA

fborrelli@berkeley.edu

February 8, 2011

References

From my Book:

- Initial part of **Section 10.1** - Nominal Case
- Initial part of **Section 12.2** - Robust Case

Outline

Set Definition

We consider the following two types of systems **autonomous systems:**

$$x(t+1) = f_a(x(t)), \tag{1}$$

and **systems subject to external inputs:**

$$x(t+1) = f(x(t), u(t)). \tag{2}$$

Both systems are subject to state and input constraints

$$x(t) \in \mathcal{X},\ u(t) \in \mathcal{U},\ \forall\ t \geq 0.$$

The sets $\mathcal{X}$ and $\mathcal{U}$ are polyhedra and contain the origin in their interior.

Reach Set Definition

For the autonomous system (1) we denote the one-step reachable set as

$$\text{Reach}(\mathcal{S}) \triangleq \{x \in \mathbb{R}^n : \; \exists \; x(0) \in \mathcal{S} \text{ s.t. } x = f_a(x(0))\}$$

For the system (2) with inputs we denote the one-step reachable set as

$$\text{Reach}(\mathcal{S}) \triangleq \{x \in \mathbb{R}^n : \; \exists \; x(0) \in \mathcal{S}, \; \exists \; u(0) \in \mathcal{U} \text{ s.t. } x = f(x(0), u(0))\}$$

Pre Set Definition

"Pre" sets are the dual of one-step reachable sets. The set

$$\text{Pre}(\mathcal{S}) \triangleq \{x \in \mathbb{R}^n \ : \ f_a(x) \in \mathcal{S}\}$$

defines the set of states which evolve into the target set $\mathcal{S}$ in one time step for the system (1).

Similarly, for the system (2) the set of states which can be driven into the target set $\mathcal{S}$ in one time step is defined as

$$\text{Pre}(\mathcal{S}) \triangleq \{x \in \mathbb{R}^n \ : \ \exists u \in \mathcal{U} \text{ s.t. } f(x,u) \in \mathcal{S}\}$$

Pre Set Computation -Autonomous Systems

Assume the system is linear and autonomous

$$x(t+1) = Ax(t)$$

Let

$$\mathcal{X} = \{x \ : \ Hx \leq h\}, \tag{3}$$

Then the set $\mathrm{Pre}(\mathcal{X})$ is

$$\mathrm{Pre}(\mathcal{X}) = \{x \ : \ HAx \leq h\}$$

Note that by using polyhedral notation, the set $\mathrm{Pre}(\mathcal{X})$ is simply $\mathcal{X} \circ A$.

Reach Set Computation - Autonomous Systems

The set $\text{Reach}(\mathcal{X})$ is obtained by applying the map A to the set $\mathcal{X}$.
Write $\mathcal{X}$ in $\mathcal{V}$-representation

$$\mathcal{X} = \text{conv}(V) \tag{4}$$

and map the set of vertices V through the transformation A.
Because the transformation is linear, the reach set is simply the convex hull of the transformed vertices

$$\text{Reach}(\mathcal{X}) = A \circ \mathcal{X} = \text{conv}(AV) \tag{5}$$

Pre Set Computation

Consider the system

$$x(t+1) = Ax(t) + Bu(t)$$

Let

$$\mathcal{X} = \{x \mid Hx \le h\}, \quad \mathcal{U} = \{u \mid H_u u \le h_u\}, \tag{6}$$

The Pre set is

$$\text{Pre}(\mathcal{X}) = \left\{ x \in \mathbb{R}^n \,|\, \exists u \in \mathbb{R} \text{ s.t. } \begin{bmatrix} HA & HB \\ 0 & H_u \end{bmatrix} \begin{pmatrix} x \\ u \end{pmatrix} \le \begin{bmatrix} h \\ h_u \end{bmatrix} \right\}$$

Note that by using the definition of the Minkowski we can compactly write the set as:

$$\begin{array}{rl} \text{Pre}(\mathcal{X}) = & \{x \ : \ \exists u \in \mathcal{U} \text{ s.t. } Ax + Bu \in \mathcal{X}\} \\ & \{x \ : \ \exists y \in \mathcal{X}, \ \exists u \in \mathcal{U}, \ Ax = y - Bu\} \\ & \{x \ : \ Ax = \mathcal{X} \oplus (-B) \circ \mathcal{U}\} \\ = & (\mathcal{X} \oplus (-B) \circ \mathcal{U}) \circ A \end{array} \tag{7}$$

Reach Set Computation

The set $\text{Reach}(\mathcal{X})$ to the set $\mathcal{X}$ and then considering the effect of the input $u \in \mathcal{U}$.
Recall

$$A \circ \mathcal{X} = \text{conv}(AV) \tag{8}$$

and therefore

$$\text{Reach}(\mathcal{X}) = \{y + Bu \ : \ y \in A \circ \mathcal{X}, \ u \in \mathcal{U}\}$$

and therefore

$$\text{Reach}(\mathcal{X}) = (A \circ \mathcal{X}) \oplus (B \circ \mathcal{U})$$

Summary

In summary, the sets $\text{Pre}(\mathcal{X})$ and $\text{Reach}(\mathcal{X})$ are the results of linear operations on the polyhedra $\mathcal{X}$ and $\mathcal{U}$ and therefore are polyhedra. By using the definition of the Minkowski sum and of affine operation on polyhedra the Pre and Reach operations on linear systems are summarized as

	$x(t+1) = Ax(t)$	$x(k+1) = Ax(t) + Bu(t)$
$\text{Pre}(\mathcal{X})$	$\mathcal{X} \circ A$	$(\mathcal{X} \oplus (-B \circ \mathcal{U})) \circ A$
$\text{Reach}(\mathcal{X})$	$A \circ \mathcal{X}$	$(A \circ \mathcal{X}) \oplus (B \circ \mathcal{U})$

Table: Pre and Reach operations for linear systems subject to polyhedral input and state constraints $x(t) \in \mathcal{X},\ u(t) \in \mathcal{U}$

Outline

1. Reachability and Controllability – Nominal Case

2. Reachability and Controllability – Robust Case
   - Pre and Reach Sets Definition
   - Pre and Reach Sets Computation
   - Summary

Set Definition

We consider the following two types of systems **autonomous systems:**

$$x(t+1) = f_a(x(t), w(t)) \tag{9}$$

and **systems subject to external inputs:**

$$x(t+1) = f(x(t), u(t), w(t)) \tag{10}$$

Both systems are subject to disturbance $w(t)$ and to the constraints

$$x(t) \in \mathcal{X},\ u(t) \in \mathcal{U},\ w(t) \in \mathcal{W}\ \forall\ t \geq 0. \tag{11}$$

The sets $\mathcal{X}$ and $\mathcal{U}$ and $\mathcal{W}$ are polytopes and contain the origin in their interior.

Reach Set Definition

For the autonomous system (9) we denote the one-step robust reachable set as

$$\text{Reach}(\mathcal{S},\mathcal{W}) \triangleq \{x \in \mathbb{R}^n \ : \ \exists\ x(0) \in \mathcal{S},\ \exists\ w \in \mathcal{W} \text{ such that } x = f_a(x(0), w)\}$$

For the system (10) with inputs we denote the one-step robust reachable set as

$$\begin{aligned}\text{Reach}(\mathcal{S},\mathcal{W}) \triangleq \quad & \{x \in \mathbb{R}^n \ : \ \exists\ x(0) \in \mathcal{S},\ \exists\ u \in \mathcal{U}, \\ & \exists\ w \in \mathcal{W},\ \text{such that } x = f(x(0), u, w)\}\end{aligned}$$

Pre Set Definition

"Pre" sets are the dual of one-step reachable sets. The set

$$\text{Pre}(\mathcal{S}, \mathcal{W}) \triangleq \{x \in \mathbb{R}^n \ : \ f_a(x, w) \in \mathcal{S}, \ \forall w \in \mathcal{W}\}$$

defines the set of system states which evolve into the target set $\mathcal{S}$ in one time step for all possible disturbances $w \in \mathcal{W}$.

Similarly, the set of states which can be robustly driven into the target set $\mathcal{S}$ in one time step is defined as

$$\text{Pre}(\mathcal{S}, \mathcal{W}) \triangleq \{x \in \mathbb{R}^n \ : \ \exists u \in \mathcal{U} \text{ s.t. } f(x, u, w) \in \mathcal{S}, \ \forall w \in \mathcal{W}\}. \tag{12}$$

Pre Set Computation -Autonomous Systems

Assume the system is linear and autonomous

$$x(t+1) = Ax(t) + w(t)$$

Let

$$\mathcal{X} = \{x \ : \ Hx \leq h\}, \tag{13}$$

Then the set $\text{Pre}(\mathcal{X}, \mathcal{W})$ is

$$\text{Pre}(\int, \mathcal{W}) = \{x \ : \ HAx \leq h - Hw, \ \forall w \in \mathcal{W}\}.$$

which can be represented as

$$\text{Pre}(\mathcal{X}, \mathcal{W}) = \{x \in \mathbb{R}^n \ : \ HAx \leq \tilde{h}\}$$

with

$$\tilde{h}_i = \min_{w \in \mathcal{W}} (h_i - H_i w).$$

Note that by using polyhedral notation, the Pre set can be written as

$$\begin{aligned} \text{Pre}(\mathcal{X}, \mathcal{W}) &= \{x \in \mathbb{R}^n \ : \ Ax + w \in \mathcal{S}, \ \forall w \in \mathcal{W}\} = \{x \in \mathbb{R}^n \ : \ Ax \in \mathcal{S} \ominus \mathcal{W}\} = \\ &= (\mathcal{X} \ominus \mathcal{W}) \circ A. \end{aligned}$$

Reach Set Computation - Autonomous Systems

The set

$$\text{Reach}(\mathcal{X}, \mathcal{W}) = \{y \; : \; \exists x \in \mathcal{X}, \; \exists w \in \mathcal{W} \text{ such that } y = Ax + w\}$$

is obtained by applying the map A to the set $\mathcal{X}$ and then considering the effect of the disturbance $w \in \mathcal{W}$.
Write $\mathcal{X}$ in $\mathcal{V}$-representation

$$\mathcal{X} = \text{conv}(V) \tag{14}$$

Because the transformation is linear, the composition of the map A with the set $\mathcal{X}$, denoted as $A \circ \mathcal{X}$, is simply the convex hull of the transformed vertices

$$A \circ \mathcal{X} = \text{conv}(AV). \tag{15}$$

Rewrite the set

$$\text{Reach}(\mathcal{X}, \mathcal{W}) = \{y \in \mathbb{R}^n \; : \; \exists \; z \in A \circ \mathcal{X}, \; \exists w \in \mathcal{W} \text{ such that } y = z + w\}.$$

We can use the definition of Minkowski sum and rewrite the Reach set as

$$\text{Reach}(\mathcal{X}, \mathcal{W}) = (A \circ \mathcal{X}) \oplus \mathcal{W}.$$

Pre Set Computation

Consider the system

$$x(t+1) = Ax(t) + Bu(t) + w(t)$$

Let

$$\mathcal{X} = \{x \mid Hx \leq h\}, \quad \mathcal{U} = \{u \mid H_u u \leq h_u\}, \tag{16}$$

The Pre set is

$$\text{Pre}(\mathcal{X}, \mathcal{W}) = \left\{ x \in \mathbb{R}^n \;:\; \exists u \in \mathbb{R}^m \text{ s.t. } \begin{bmatrix} HA & HB \\ 0 & H_u \end{bmatrix} \begin{pmatrix} x \\ u \end{pmatrix} \leq \begin{bmatrix} h - Hw \\ h_u \end{bmatrix}, \; \forall \, w \in \mathcal{W} \right\}$$

which can be compactly written as

$$\text{Pre}(\mathcal{X}, \mathcal{W}) = \left\{ x \in \mathbb{R}^n \;:\; \exists u \in \mathbb{R}^m \text{ s.t. } \begin{bmatrix} HA & HB \\ 0 & H_u \end{bmatrix} \begin{pmatrix} x \\ u \end{pmatrix} \leq \begin{bmatrix} \tilde{h} \\ h_u \end{bmatrix} \right\}.$$

where

$$\tilde{h}_i = \min_{w \in \mathcal{W}} (h_i - H_i w).$$

Note that one can use polyhedral operations and rewrite the set as:

$$\text{Pre}(\mathcal{X}, \mathcal{W}) = ((\mathcal{X} \ominus \mathcal{W}) \oplus (-B \circ \mathcal{U})) \circ A \tag{17}$$

Reach Set Computation

The set Reach($\mathcal{X}$)

$$\text{Reach}(\mathcal{X}, \mathcal{W}) = \{y : \ \exists x \in \mathcal{X}, \ \exists u \in \mathcal{U}, \ \exists w \in \mathcal{W} \text{ s.t. } y = Ax + Bu + w\}$$

is obtained by applying the map A to the set $\mathcal{X}$ and then considering the effect of the input $u \in \mathcal{U}$ and of the disturbance $w \in \mathcal{W}$.
We can use the polyhedral operations and rewrite Reach($\mathcal{X}, \mathcal{W}$) as

$$\text{Reach}(\mathcal{X}, \mathcal{W}) = (A \circ \mathcal{X}) \oplus (B \circ \mathcal{U}) \oplus \mathcal{W}.$$

Summary

In summary, for linear systems with additive disturbances the sets $\text{Pre}(\mathcal{X}, \mathcal{W})$ and $\text{Reach}(\mathcal{X}, \mathcal{W})$ are the results of linear operations on the polytopes $\mathcal{X}$, $\mathcal{U}$ and $\mathcal{W}$ and therefore are polytopes. By using the definition of Minkowski sum, Pontryagin difference and affine operation on polyhedra we obtain the following.

	$x(t+1) = Ax(t) + w(t)$	$x(k+1) = Ax(t) + Bu(t) + w(t)$
$\text{Pre}(\mathcal{X}, \mathcal{W})$	$(\mathcal{X} \ominus \mathcal{W}) \circ A$	$(\mathcal{X} \ominus \mathcal{W} \oplus -B \circ \mathcal{U}) \circ A$
$\text{Reach}(\mathcal{X}), \mathcal{W}$	$(A \circ \mathcal{X}) \oplus \mathcal{W}$	$(A \circ \mathcal{X}) \oplus (B \circ \mathcal{U}) \oplus \mathcal{W}$

Table: Pre and Reach operations for uncertain linear systems subject to polyhedral input and state constraints $x(t) \in \mathcal{X},\ u(t) \in \mathcal{U}$ with additive polyhedral disturbances $w(t) \in \mathcal{W}$

Note that the summary applies also to the class of systems $x(k+1) = Ax(t) + Bu(t) + E\tilde{d}(t)$ where $\tilde{d} \in \tilde{\mathcal{W}}$. This can be transformed into $x(k+1) = Ax(t) + Bu(t) + w(t)$ where $w \in \mathcal{W} \triangleq E \circ \tilde{\mathcal{W}}$.