Model Predictive Control for Linear and Hybrid Systems
Optimal Control

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Outline

General Problem Formulation

Consider the nonlinear time-invariant system

$$x(t+1) = g(x(t), u(t)),$$

subject to the constraints

$$h(x(t), u(t)) \leq 0, \ \forall t \geq 0$$

with $x(t) \in \mathbb{R}^n$ and $u(t) \in \mathbb{R}^m$ the state and input vectors. ***Assume that*** $g(0,0) = 0$, $h(0,0) < 0$.

Consider the following *objective* or *cost* function

$$J_{0\to N}(x_0, U_{0\to N}) \triangleq p(x_N) + \sum_{k=0}^{N-1} q(x_k, u_k)$$

where

- N is the time *horizon*,
- $x_k = g(x_{k-1}, u_{k-1}), \ k = 1, \ldots, N-1$ and $x_0 = x(0)$,
- $U_{0\to N} \triangleq [u_0', \ldots, u_{N-1}']' \in \mathbb{R}^s$, $s \triangleq mN$,
- $q(x_k, u_k)$ and $p(x_N)$ are the *stage cost* and *terminal cost*, respectively.

General Problem Formulation

Consider the **C**onstrained **F**inite **T**ime **O**ptimal **C**ontrol (CFTOC) problem.

$$
\begin{array}{lll}
J^*_{0\to N}(x_0) = & \min_{U_{0\to N}} & J_{0\to N}(x_0, U_{0\to N}) \\
& \text{subj. to} & x_{k+1} = g(x_k, u_k),\ k = 0, \ldots, N-1 \\
& & h(x_k, u_k) \leq 0,\ k = 0, \ldots, N-1 \\
& & x_N \in \mathcal{X}_f \\
& & x_0 = x(0)
\end{array}
$$

- $\mathcal{X}_f \subseteq \mathbb{R}^n$ is a *terminal region*,
- $\mathcal{X}_{0\to N} \subseteq \mathbb{R}^n$ to is the set of feasible initial conditions $x(0)$
- the optimal cost $J^*_{0\to N}(x_0)$ is called *value function*,
- assume that there exists a minimum
- denote by $U^*_{0\to N}$ one of the minima

Objectives

- ***Solution.***
  1. a general nonlinear programming problem (*batch approach*),
  2. recursively by invoking Bellman's Principle of Optimality (*recursive approach*).

- ***Infinite horizon.*** We will investigate if
  1. a solution exists as $N \rightarrow \infty$,
  2. the properties of this solution.
  3. approximation of the solution by using a *receding horizon* technique.

- ***Uncertainty.*** We will discuss how to extend the problem description and consider uncertainty so that a *robust controller* results from the solution.

Outline

Solution via Batch Approach. NLP formulation

Write the equality constraints from system constraints as

$$\begin{aligned} & x_1 = g(x(0), u_0) \\ & x_2 = g(x_1, u_1) \\ & \vdots \\ & x_N = g(x_{N-1}, u_{N-1}) \end{aligned}$$

then the optimal control problem

$$\begin{array}{lll} J^*_{0\to N}(x_0) = & \min_{U_{0\to N}} & p(x_N) + \sum_{k=0}^{N-1} q(x_k, u_k) \\ & \text{subj. to} & x_1 = g(x_0, u_0) \\ & & x_2 = g(x_1, u_1) \\ & & \vdots \\ & & x_N = g(x_{N-1}, u_{N-1}) \\ & & h(x_k, u_k) \leq 0, \; k = 0, \ldots, N-1 \\ & & x_N \in \mathcal{X}_f \\ & & x_0 = x(0) \end{array}$$

is a general Non Linear Programming (NLP) problem with variables $u_0, \ldots, u_{N-1}$ and $x_1, \ldots, x_N$.

NLP

Eliminate the state variables and equality constraints by successive substitutions

$$x_2 = g(x_1, u_1)$$
$$x_2 = g(g(x(0), u_0), u_1).$$

- The solution of the NLP is a sequence of present and future inputs $U^*_{0\to N} = [u_0^{*'}, \ldots, u_{N-1}^{*'}]'$ determined for the particular initial state $x(0)$.
- Except for linear systems, successive substitution may become complex.

Outline

Solution via Recursive Approach

Principle of optimality

For a trajectory $x_0, x_1^*, \ldots, x_N^*$ to be optimal, the trajectory starting from any intermediate point x_j^*, i.e. $x_j^*, x_{j+1}^*, \ldots, x_N^*$, $0 \le j \le N-1$, must be optimal.

Define the cost from j to N

$$J_{j \to N}(x_j, u_j, u_{j+1}, \ldots, u_{N-1}) \triangleq p(x_N) + \sum_{k=j}^{N-1} q(x_k, u_k),$$

also called the j-step *cost-to-go*. Then the *optimal cost-to-go* $J_{j \to N}^*$ is

$$\begin{array}{lll} J_{j \to N}^*(x_j) \triangleq & \min_{u_j, u_{j+1}, \ldots, u_{N-1}} & J_{j \to N}(x_j, u_j, u_{j+1}, \ldots, u_{N-1}) \\ & \text{subj. to} & x_{k+1} = g(x_k, u_k),\ k = j, \ldots, N-1 \\ & & h(x_k, u_k) \le 0,\ k = j, \ldots, N-1 \\ & & x_N \in \mathcal{X}_f \end{array}$$

Note that $J_{j \to N}^*(x_j)$ depends only on the initial state x_j.

Solution via Recursive Approach

By the ***principle of optimality*** the cost $J^*_{j-1\to N}$ can be found by solving

$$
\begin{array}{rll}
J^*_{j-1\to N}(x_{j-1}) = & \min\limits_{u_{j-1}} & \overbrace{q(x_{j-1},u_{j-1})}^{\text{stage cost}} + \overbrace{J^*_{j\to N}(x_j)}^{\text{optimal cost}-\text{to}-\text{go}} \\
& \text{subj. to} & x_j = g(x_{j-1},u_{j-1}) \\
& & h(x_{j-1},u_{j-1}) \le 0 \\
& & x_j \in \mathcal{X}_{j\to N}.
\end{array}
\tag{1}
$$

Note that

- the only decision variable is u_{j-1},
- the inputs $u^*_j,\ldots,u^*_{N-1}$ have already been selected optimally to yield the optimal cost-to-go $J^*_{j\to N}(x_j)$.
- in $J^*_{j\to N}(x_j)$, the state x_j can be replaced by $g(x_{j-1},u_{j-1})$

Solution via Recursive Approach

The following (recursive) ***dynamic programming*** algorithm can be used to compute the optimal control law.

$$
\begin{aligned}
J^*_{N\to N}(x_N) &= p(x_N)\\
\mathcal{X}_{N\to N} &= \mathcal{X}_f,
\end{aligned}
$$

$$
\begin{array}{llll}
J^*_{N-1\to N}(x_{N-1}) & = & \min\limits_{u_{N-1}} & q(x_{N-1},u_{N-1}) + \ J^*_{N\to N}(g(x_{N-1},u_{N-1}))\\
 & & \text{subj. to} & h(x_{N-1},u_{N-1}) \le 0,\\
 & & & g(x_{N-1},u_{N-1}) \in \mathcal{X}_{N\to N}\\
\vdots & & & \\
J^*_{0\to N}(x_0) & = & \min\limits_{u_0} & q(x_0,u_0) + \ J^*_{1\to N}(g(x_0,u_0))\\
 & & \text{subj. to} & h(x_0,u_0) \le 0,\\
 & & & g(x_0,u_0) \in \mathcal{X}_{1\to N}\\
 & & & x_0 = x(0).
\end{array}
$$

Solution via Recursive Approach: Comments

- DP algorithm is appealing because at each step j only u_j is computed.
- Need to construct the optimal cost-to-go $J^*_{N-j}(x_j)$, a *function* defined over $\mathcal{X}_{j\to N}$.
- In a few special cases we know the type of function and we can find it efficiently.
- "brute force" approach. Construct $J_{j-1\to N}$ by griding the set $\mathcal{X}_{j-1\to N}$ and computing the optimal cost-to-go function on each grid point.
- A nonlinear *feedback* (time varying) control law is implicitly defined:

$$\begin{array}{rll} u_j^*(x_j) = & \arg\min\limits_{u_j} & q(x_j,u_j) + J^*_{j+1\to N}(g(x_j,u_j)) \\ & \text{subj. to} & h(x_j,u_j) \le 0, \\ & & g(x_j,u_j) \in \mathcal{X}_{j+1\to N} \end{array}$$

Outline

Optimal Control Problem with Infinite Horizon

Consider the problem:

$$\begin{array}{lll} J^*_{0\to\infty}(x_0) = & \min_{u_0,u_1,\ldots} & \sum_{k=0}^{\infty} q(x_k,u_k) \\ & \text{subj. to} & x_{k+1} = g(x_k,u_k),\ k=0,\ldots,\infty \\ & & h(x_k,u_k) \leq 0,\ k=0,\ldots,\infty \\ & & x_0 = x(0) \end{array}$$

The set of feasible initial conditions is

$$\mathcal{X}_{0\to\infty} = \{x(0) \in \mathbb{R}^n |\ \text{the problem is feasible and } J^*_{0\to\infty}(x(0)) < +\infty\}.$$

Boundedness of $J^*_{0\to\infty}(x_0)$ implies that

$$\lim_{k\to\infty} q(x_k,u_k) = 0$$

and because $q(x_k,u_k) > 0\ \forall\ x_k, u_k \neq 0$

$$\lim_{k\to\infty} x_k = 0, \quad \lim_{k\to\infty} u_k = 0.$$

The system must be stabilizable.

Outline

Receding Horizon Control

Main Idea:

- If $J^*_{0\to N}(x_0)$ converges to $J^*_{0\to\infty}(x_0)$ as $N \to \infty$ then the effect of increasing N on the value of u_0 should diminish as $N \to \infty$.
- Intuitively, instead of making the horizon infinite we can use a long, but finite horizon N and repeat this optimization at each time step moving the horizon forward (*moving horizon* or *receding horizon* control).

Implementation:

- ***Batch approach***. set the horizon to N and calculate $u_0^*, \ldots, u_{N-1}^*$. Implement only u_0^*. At the next time step, reformulate and solve the problem with the current state $x(t)$ as new initial condition x_0.
- ***Dynamic programming approach***. Implement the control u_0 obtained by solving

$$
\begin{array}{rcll}
J^*_{0\to N}(x_0) & = & \min\limits_{u_0} & q(x_0,u_0) + J^*_{1\to N}(g(x_0,u_0)) \\
 & & \text{subj. to} & h(x_0,u_0) \le 0, \\
 & & & g(x_0,u_0) \in \mathcal{X}_{1\to N}, \\
 & & & x_0 = x(t)
\end{array}
$$

Notation

For the sake of simplicity we will use the following shorter notation

$$
\begin{aligned}
&J_j^*(x_j) \triangleq J_{j\to N}^*(x_j),\ j = 0, \ldots, N \\
&J_\infty^*(x_0) \triangleq J_{0\to\infty}^*(x_0) \\
&\mathcal{X}_j \triangleq \mathcal{X}_{j\to N},\ j = 0, \ldots, N \\
&\mathcal{X}_\infty \triangleq \mathcal{X}_{0\to\infty} \\
&U_0 \triangleq U_{0\to N}
\end{aligned}
$$

Outline

Linear Quadratic (LQ) optimal control

Consider the system

$$x(t+1) = Ax(t) + Bu(t),$$

the ***quadratic*** cost function

$$J_0(x_0, U_0) \triangleq x_N' P x_N + \sum_{k=0}^{N-1} x_k' Q x_k + u_k' R u_k,$$

and the ***finite time*** optimal control problem

$$\begin{array}{rll} J_0^*(x(0)) = & \min_{U_0} & J_0(x(0), U_0) \\ & \text{subj. to} & x_{k+1} = Ax_k + Bu_k,\ k = 0, 1, \ldots, N-1 \\ & & x_0 = x(0), \end{array}$$

with $U_0 \triangleq [u_0', \ldots, u_{N-1}']' \in \mathbb{R}^s$, $s \triangleq mN$ and $Q = Q' \succeq 0$, $P = P' \succeq 0$, $R = R' \succ 0$.

Solution via Batch Approach

The state trajectory $x_1, \ldots, x_N$ as function of the initial state $x(0)$ and the input trajectory U_0 is

$$\underbrace{\begin{bmatrix} x(0) \\ x_1 \\ \vdots \\ \vdots \\ x_N \end{bmatrix}}_{\mathcal{X}} = \underbrace{\begin{bmatrix} I \\ A \\ \vdots \\ \vdots \\ A^N \end{bmatrix}}_{\mathcal{S}^x} x(0) + \underbrace{\begin{bmatrix} 0 & \ldots & \ldots & 0 \\ B & 0 & \ldots & 0 \\ AB & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ A^{N-1}B & \ldots & \ldots & B \end{bmatrix}}_{\mathcal{S}^u} \underbrace{\begin{bmatrix} u_0 \\ \vdots \\ \vdots \\ u_{N-1} \end{bmatrix}_{U_0}} .$$

Rewrite as

$$\mathcal{X} = \mathcal{S}^x x(0) + \mathcal{S}^u U_0 \tag{2}$$

The objective function can be rewritten as

$$J(x(0), U_0) = \mathcal{X}'\bar{Q}\mathcal{X} + U_0{}'\bar{R}U_0 \tag{3}$$

where $\bar{Q} = \text{blockdiag}\{Q, \cdots, Q, P\}, \bar{Q} \succeq 0$, and $\bar{R} = \text{blockdiag}\{R, \cdots, R\}, \bar{R} \succ 0$.

Solution via Batch Approach

Substituting (2) into the objective function (3) yields

$$
\begin{aligned}
J_0(x(0),U_0) &= (\mathcal{S}^x x(0)+\mathcal{S}^u U_0)'\,\bar{Q}\,(\mathcal{S}^x x(0)+\mathcal{S}^u U_0)+{U_0}'\bar{R}U_0\\
&= {U_0}'\underbrace{(\mathcal{S}^{u\prime}\bar{Q}\mathcal{S}^u+\bar{R})}_{H}U_0+2x'(0)\underbrace{(\mathcal{S}^{x\prime}\bar{Q}\mathcal{S}^u)}_{F}U_0+x'(0)\underbrace{(\mathcal{S}^{x\prime}\mathcal{S}^x)}_{Y}x(0)\\
&= {U_0}'HU_0+2x(0)'FU_0+x'(0)Yx(0)
\end{aligned}
$$

Optimal vector

$$
\begin{aligned}
\underline{{U_0}^*} &= -H^{-1}F'x(0)\\
&= \underline{-\left(\mathcal{S}^{u\prime}\bar{Q}\mathcal{S}^u+\bar{R}\right)^{-1}\mathcal{S}^{u\prime}\bar{Q}\mathcal{S}^x x(0)}
\end{aligned}
$$

Optimal cost

$$
\begin{aligned}
\underline{J_0^*(x(0))} &= -x(0)'FH^{-1}F'x(0)+x(0)'Yx(0)\\
&= \underline{x(0)'\left[\mathcal{S}^{x\prime}\mathcal{S}^x-\mathcal{S}^{x\prime}\bar{Q}\mathcal{S}^u\left(\mathcal{S}^{u\prime}\bar{Q}\mathcal{S}^u+\bar{R}\right)^{-1}\mathcal{S}^{u\prime}\bar{Q}\mathcal{S}^x\right]x(0)}
\end{aligned}
$$

Note that U_0^* and $J_0^*(x(0))$ are linear and quadratic functions, respectively, of the initial state $x(0)$

Solution via Recursive Approach

At step $N-1$, by the principle of optimality

$$J^*_{N-1}(x_{N-1})) = \min_{u_{N-1}} \overbrace{x'_N P_N x_N}^{\text{cost}-\text{to}-\text{go}} + \overbrace{x'_{N-1} Q x_{N-1} + u'_{N-1} R u_{N-1}}^{\text{stage cost}}$$

$$\begin{array}{rcl} x_N & = & A x_{N-1} + B u_{N-1} \\ P_N & = & P \end{array}$$

Define P_j as the optimal cost-to-go $x_j{}' P_j x_j$ from time j to the end of the horizon N.

Solution via Recursive Approach

Substitute system dynamics into $J^*_{N-1}(x_{N-1})$

$$
\begin{aligned}
J^*_{N-1}(x_{N-1}) = \quad \min_{u_{N-1}} \quad & \{x'_{N-1}(A'P_NA + Q)x_{N-1} \\
& +2x'_{N-1}A'P_NBu_{N-1} \\
& +u'_{N-1}(B'P_NB + R)u_{N-1}\}
\end{aligned}
$$

The optimal control vector is

$$
u^*_{N-1} = \underbrace{-(B'P_NB + R)^{-1}B'P_NA}_{F_{N-1}}x_{N-1}
$$

and the one-step optimal cost-to-go

$$
J^*_{N-1}(x_{N-1}) = x'_{N-1}P_{N-1}x_{N-1},
$$

where we have defined

$$
P_{N-1} = A'P_NA + Q - A'P_NB(B'P_NB + R)^{-1}B'P_NA
$$

Solution via Recursive Approach

At step $N-2$, consider the problem

$$J^*_{N-2}(x_{N-2}) = \min_{u_{N-2}} \overbrace{x'_{N-1}P_{N-1}x_{N-1}}^{\text{cost-to-go}} + \overbrace{\left[x'_{N-2}Qx_{N-2} + u'_{N-2}Ru_{N-2}\right]}^{\text{stage cost}}$$

$$x_{N-1} \quad = \quad Ax_{N-2} + Bu_{N-2}$$

As at step $N-1$,

$$u^*_{N-2} = \underbrace{-(B'P_{N-1}B + R)^{-1}B'P_{N-1}A}_{F_{N-2}}x_{N-2}$$

The optimal ***two-step cost-to-go*** is

$$J^*_{N-2}(x_{N-2}) = x'_{N-2}P_{N-2}x_{N-2},$$

where

$$P_{N-2} = A'P_{N-1}A + Q - A'P_{N-1}B(B'P_{N-1}B + R)^{-1}B'P_{N-1}A$$

Solution via Recursive Approach

At step k, the optimal control is

$$\begin{aligned} u^*(k) &= -(B'P_{k+1}B + R)^{-1}B'P_{k+1}Ax(k), \\ &= F_k x(k), \qquad \text{for } k = 0, \ldots, N-1, \end{aligned}$$

where

$$P_k = A'P_{k+1}A + Q - A'P_{k+1}B(B'P_{k+1}B + R)^{-1}B'P_{k+1}A \tag{4}$$

and the optimal cost-to-go starting from the measured state $x(k)$ is

$$J_k^*(x(k)) = x'(k)P_k x(k)$$

Equation (4) (called *Discrete Time Riccati Equation* or *Riccati Difference Equation* - RDE) is initialized with $P_N = P$ and is solved backwards, i.e., starting with P_N and solving for P_{N-1}, etc.

Comparison Of The Two Approaches

- In ***batch approach*** the we calculate

$$U_0^* = -\left(\mathcal{S}^{u\prime}\bar{Q}\mathcal{S}^u + \bar{R}\right)^{-1}\mathcal{S}^{u\prime}\bar{Q}\mathcal{S}^x x(0) \tag{5}$$

while in the ***recursive dynamic programming approach***

$$u^*(k) = F_k x(k), \text{ for } k = 0, \ldots, N-1 \tag{6}$$

- Under no model mismatch (5) and (6) are identical.
- Same feedback effect can be obtained with batch approach if

$$J_j^*(x(j)) = \min_{u_j, \cdots, u_{N-1}} \left\{ \sum_{k=j}^{N-1} [x_k' Q x_k + u_k' R u_k] + x_N' P x_N \right\}$$

- The dynamic programming approach is clearly a more efficient way to generate the feedback policy.

Outline

Infinite Horizon Problem

Consider the ***infinite time*** cost

$$J_\infty^*(x(0)) = \min_{u(0),u(1),\ldots} \sum_{k=0}^{\infty} [x_k' Q x_k + u_k' R u_k]$$

Note that using the batch approach is impossible.

Assume $P_k \to P_\infty$, satisfying the ***Algebraic Riccati Equation***

$$P_\infty = A' P_\infty A + Q - A' P_\infty B (B' P_\infty B + R)^{-1} B' P_\infty A.$$

The optimal feedback control law is the ***Linear Quadratic Regulator (LQR)***

$$u^*(k) = \underbrace{-(B' P_\infty B + R)^{-1} B' P_\infty A}_{F_\infty} x(k), \qquad k = 0, \cdots, \infty$$

and the optimal infinite horizon cost is

$$J_\infty^*(x(0)) = x(0)' P_\infty x(0).$$

Infinite Horizon Problem: Convergence

Theorem

If (A, B) is a stabilizable pair and $(Q^{1/2}, A)$ is an observable pair, the RDE with $P_0 \geq 0$ converges to a unique positive definite solution P_∞ of the ARE (3) and all the eigenvalues of $(A + BF_\infty)$ lie inside the unit disk.

Outline

Lyapunov Stability. Definitions

Consider the autonomous system

$$x_{k+1} = f(x_k) \tag{7}$$

with $f(0) = 0$.
The equilibrium point $x = 0$ of system (7) is

- *stable* (in the sense of Lyapunov) if, for all$\varepsilon > 0$, there is $\delta = \delta(\varepsilon) > 0$ such that
$$\|x_0\| < \delta \Rightarrow \|x_k\| < \varepsilon,\ \forall k \geq 0 \tag{8}$$
- *unstable* if not stable
- *asymptotically stable* if it is stable and δ can be chosen such that
$$\|x_0\| < \delta \Rightarrow \lim_{k\to\infty} x_k = 0 \tag{9}$$
- *globally asymptotically stable* if it is asymptotically stable for all $x(0) \in \mathbb{R}^n$
- *exponentially stable* if it is stable and $\exists$ constants $\alpha > 0$ and $\gamma \in (0, 1)$ such that
$$\|x_0\| < \delta \Rightarrow \|x_k\| \leq \alpha \|x_0\| \gamma^k,\ \forall k \geq 0 \tag{10}$$

Lyapunov Stability

Lyapunov stability of the origin shown through a *Lyapunov function*, i.e. a function satisfying the conditions of the following theorem.

Theorem

Consider the equilibrium point $x = 0$ *of system (7). Let* $\Omega \subset \mathbb{R}^n$ *be a closed and bounded set containing the origin. Let* $V : \mathbb{R}^n \to \mathbb{R}$ *be a function, continuous at the origin, such that*

$$V(0) = 0 \text{ and } V(x) > 0, \ \forall x \in \Omega \setminus \{0\} \tag{11a}$$

$$V(x_{k+1}) - V(x_k) < 0 \ \forall x_k \in \Omega \setminus \{0\} \tag{11b}$$

then $x = 0$ *is asymptotically stable in* Ω*.*

Lyapunov Stability

Definition (Lyapunov Function)

A function $V(x)$ satisfying conditions (11a)-(11b) is called a *Lyapunov Function.*

Note that

- We can think of V as an energy function
- Condition (11b) requires that for any arbitrary state $x_k \neq 0$ the energy decreases as the system evolves.
- Note that Theorem 2 is only ***sufficient***.
- Condition (11b) can be relaxed as follows:

$$V(x_{k+1}) - V(x_k) \leq 0, \ \forall x_k \neq 0 \tag{12}$$

Condition (12) along with condition (11a) are sufficient to guarantee stability of the origin as long as the set $\{x_k : V(f(x_k)) - V(x_k) = 0\}$ contains no trajectory of the system $x_{k+1} = f(x_k)$ except for $x_k = 0$ for all $k \geq 0$. (Barbashin-Krasovski-LaSalle principle)

Global Lyapunov Stability

Global Lyapunov stability of the origin shown through a *Global Lyapunov function*, i.e. a function satisfying the conditions of the following theorem.

Theorem

Consider the equilibrium point $x = 0$ *of system*

$$x_{k+1} = f(x_k).$$

Let $V : \mathbb{R}^n \to \mathbb{R}$ *be a function such that*

$$\|x\| \to \infty \Rightarrow V(x) \to \infty \tag{13a}$$

$$V(0) = 0 \textit{ and } V(x) > 0, \ \forall x \neq 0 \tag{13b}$$

$$V(x_{k+1}) - V(x_k) < 0 \ \forall x_k \neq 0 \tag{13c}$$

then $x = 0$ *is globally asymptotically stable.*

Definition (Radially Unbounded Function)

A function $V(x)$ satisfying condition (13a) is said to be *radially unbounded.*

Lyapunov Stability. Linear Systems

Theorem

A linear system $x_{k+1} = Ax_k$ is globally asymptotically stable in the sense of Lyapunov if and only if all its eigenvalues are inside the unit circle.

Note that stability is always "global" for linear systems.

Quadratic Lyapunov functions A simple effective Lyapunov function for linear systems is

$$V(x) = x'Px, \;\; P > 0$$

Satisfies conditions (13a)-(13b) of Theorem 2. Test (13c)

$$\begin{array}{rcl} V(x_{k+1}) - V(x_k) & = & x'_{k+1}Px_{k+1} - x'_kPx_k \\ & = & x'_kA'PAx_k - x'_kPx_k = x'_k(A'PA - P)x_k \end{array}$$

(11b) is satisfied if $P > 0$ can be found such that

$$A'PA - P = -Q, \;\; Q > 0, \;\; \textbf{discrete-time Lyapunov equation}$$

Lyapunov Stability

Theorem

Consider the linear system $x_{k+1} = Ax_k$. The Lyapunov equation has a unique solution $P > 0$ for any $Q > 0$ if and only if A has all eigenvalues inside the unit circle.

In summary

- ***Linear systems***. A quadratic form $x'Px$ is always a suitable Lyapunov function and an appropriate P can be found if the system's eigenvalues lie inside the unit circle.
- ***Nonlinear systems***. Determining a suitable form for $V(x)$ is generally difficult.

Theorem

Consider the linear system $x_{k+1} = Ax_k$. The Lyapunov equation has a unique solution $P > 0$ for any $Q = C'C \geq 0$ if and only if A has all eigenvalues inside the unit circle and (C, A) is observable.

Exercise: Stability of the Infinite Horizon LQR

Consider the linear system (6) and the ∞-horizon LQR solution Prove that the closed loop system

$$x(t+1) = (A + BF_\infty)x(t) \tag{14}$$

is asymptotically stable for any F_∞ by showing that the ∞-horizon cost

$$J_\infty^*(x) = x' P_\infty x \tag{15}$$

is a Lyapunov function for the closed loop system.

Receding Horizon Control. Convergence

Q: How can we guarantee convergence of the closed loop system?

Main focus later in this class. For now assume zero terminal constraint $x_N \in \mathcal{X}_f = 0$.
Recall

$$\begin{array}{lll} J^*_{0\to N}(x_0) = & \min_{U_{0\to N}} & J_{0\to N}(x_0, U_{0\to N}) = p(x_N) + \sum_{k=0}^{N-1} q(x_k, u_k) \\ & \text{subj. to} & x_{k+1} = g(x_k, u_k),\ k = 0, \ldots, N-1 \\ & & h(x_k, u_k) \le 0,\ k = 0, \ldots, N-1 \\ & & x_N \in \mathcal{X}_f \\ & & x_0 = x(0) \end{array}$$

Assume $p(x) \succ 0$, $q(x, u) \succ 0$

1. At $x(0)$, apply u_0^* and let the system to evolve to $x(1) = g(x(0), u_0^*)$.
2. At $x(1)$, use the (*not optimal*) input sequence $u_1^*, \ldots, u_{N-1}^*, 0$. The cost is

$$J^*_{0\to N}(x_0) - q(x_0, u_0) + \overbrace{q(x_{N+1}, 0)}^{=0}.$$

3. Since $u_1^*, \ldots, u_{N-1}^*, 0$ is not optimal

$$J^*_{1\to N+1}(x_1) \le J^*_{0\to N}(x_0) - q(x_0, u_0).$$

Receding Horizon Control. Convergence

4. Since system and cost function are time invariant
$J^*_{1\to N+1}(x_1) = J^*_{0\to N}(x_1)$ and

$$J^*_{0\to N}(x_1) \leq J^*_{0\to N}(x_0) - q(x_0, u_0).$$

5. $q \succ 0,\ \forall (x,u) \neq (0,0) \Rightarrow J^*_{0\to N}(x_{k+1}) - J^*_{0\to N}(x_k) < 0\ \forall (x_k) \neq 0$
6. $p(x)$ and $q(x,u) \succ 0 \Rightarrow J^*_{0\to N}(x) \succ 0$
7. $J^*_{0\to N}(x)$ is a Lyapunov Function and the closed loop system is locally asymptotically stable.

Outline

$1/\infty$ Norm Optimal Control

Consider the cost function

$$J_0(x_0, U_0) \triangleq \|Px_N\|_p + \sum_{k=0}^{N-1} \|Qx_k\|_p + \|Ru_k\|_p$$

with $p = 1$ or $p = \infty$ and $Q \in \mathbb{R}^{n\times n}$, $R \in \mathbb{R}^{m\times m}$ and $P \in \mathbb{R}^{r\times n}$.

Consider the finite time optimal control problem

$$\begin{array}{rll} J_0^*(x(0)) = & \min_{U_0} & J_0(x(0), U_0) \\ & \text{subj. to} & x_{k+1} = Ax_k + Bu_k,\ k = 0, 1, \ldots, N-1 \\ & & x_0 = x(0). \end{array}$$

Observe that there does not exist a simple closed-form solution of the problem as in the 2-norm case ($p = 2$)

Next we consider the case of ∞***-norm***

Solution via Batch Approach

Recall that

$$x_k = \quad A^k x_0 + \sum_{j=0}^{k-1} A^j B u_{k-1-j}$$

The optimal control problem with $p = \infty$ can be rewritten as

$$\min_{z} \qquad \varepsilon_0^x + \ldots + \varepsilon_N^x + \varepsilon_0^u + \ldots + \varepsilon_{N-1}^u$$

$$\begin{aligned}
\text{subj. to} \qquad & -\mathbf{1}_n \varepsilon_k^x \leq \pm Q \left[A^k x_0 + \sum_{j=0}^{k-1} A^j B u_{k-1-j} \right], \\
& -\mathbf{1}_r \varepsilon_N^x \leq \pm P \left[A^N x_0 + \sum_{j=0}^{N-1} A^j B u_{N-1-j} \right], \\
& -\mathbf{1}_m \varepsilon_k^u \leq \pm R u_k, \\
& k = 0, \ldots, N-1 \\
& x_0 = x(0)
\end{aligned}$$

Solution via Batch Approach

The LP problem can be rewritten in the more compact form

$$\begin{array}{ll} \min_{z} & c'z \\ \text{subj. to} & G_{\varepsilon} z \leq W_{\varepsilon} + S_{\varepsilon} x_0 \end{array}$$

where $z = \left[\varepsilon_0^x, \dots, \varepsilon_N^x, \varepsilon_0^u, \dots, \varepsilon_{N-1}^u, u_0', \dots, u_{N-1}'\right]$.

Observe that, by treating x_0 as a vector of parameters, the problem becomes a *multiparametric linear program* (mp-LP)

Solution via Batch Approach

Observe that any combination of 1- and ∞-norms leads to a linear program.

The cost $J_0(x_0, U_0)$ with $p = \infty$ consists of a 1-norm over time and ∞-norm over state space.
The dual is

$$J_0(x(0), U_0) \triangleq \max_{k=0,\dots,N} \{\|Qx_k\|_1 + \|Ru_k\|_1\}.$$

In general, ∞-norm over time could result in a poor closed-loop performance while 1-norm over space leads to an LP with a larger number of variables. I.e., slack variables for the terms $\|Qx_k\|_1$,
$\varepsilon_{k,i} \geq \pm Q^i x_k \; k = 0, 2, \dots, N-1, \; i = 1, 2, \dots, n$, and so on...

Solution via Batch Approach

Be $z^*(x_0)$ the parametric solution of the mpLP problem. The optimal control input is

$$U^*(0) = [0 \ \ldots 0 \ I_m \ I_m \ \ldots \ I_m] z^*(x_0).$$

The controller $U^*(0)$ inherits the properties of $z^*(x_0)$

Corollary

There exists a control law $U^*(0) = \bar{f}_0(x_0)$, $\bar{f}_0 : \mathbb{R}^n \to \mathbb{R}^m$, obtained as a solution of the optimal control problem with $p = 1$ or $p = \infty$, which is continuous and PPWA

$$\bar{f}_0(x) = \bar{F}_0^i x + \bar{g}_0^i \quad \textit{if} \quad x \in CR_0^i, \ i = 1, \ldots, N_0^r \tag{17}$$

where the polyhedral sets $CR_0^i \triangleq \{H_0^i x \leq k_0^i\}$, $i = 1, \ldots, N_0^r$, are a partition of $\mathbb{R}^n$.

Solution via Recursive Approach

Define the optimal cost $J_j^*(x_j)$ at the step $N-j$

$$J_j^*(x_j) \triangleq \min_{u_j,\cdots,u_{N-1}} \|Px_N\|_p + \sum_{k=j}^{N-1} (\|Qx_k\|_p + \|Ru_k\|_p)$$

By the principle of optimality

$$J_{N-1}^*(x_{N-1}) = \min_{u_{N-1}} \|P_N x_N\|_p + \|Qx_{N-1}\|_p + \|Ru_{N-1}\|_p$$

$$\begin{array}{rcl} x_N & = & Ax_{N-1} + Bu_{N-1} \\ P_N & = & P \end{array}$$

We find u_{N-1}^* by solving the mp-LP

$$\begin{array}{cl} \min\limits_{\varepsilon_{N-1}^x,\varepsilon_N^x,\varepsilon_{N-1}^u,u_{N-1}} & \varepsilon_{N-1}^x + \varepsilon_N^x + \varepsilon_{N-1}^u \\ \text{subj. to} & -\mathbf{1}_n \varepsilon_{N-1}^x \leq \pm Qx_{N-1} \\ & -\mathbf{1}_r \varepsilon_N^x \leq \pm P_N \left[Ax_{N-1} + Bu_{N-1}\right], \\ & -\mathbf{1}_m \varepsilon_{N-1}^u \leq \pm Ru_{N-1}. \end{array}$$

Solution via Recursive Approach

J^*_{N-1} is a convex and piecewise linear function of x_{N-1}. Hence,we use the equivalence of representation between convex and PPWA functions and infinity norm and to write the cost-to-go as

$$J^*_{N-1}(x_{N-1}) = \|P_{N-1}x(N-1)\|_p.$$

By the principle of optimality, at step $N-2$

$$J^*_{N-2}(x_{N-2}) = \min_{u_{N-2}} \|P_{N-1}x_{N-1}\|_p + \|Qx_{N-2}\|_p + \|Ru_{N-2}\|_p$$

$$x_{N-1} \quad = \quad Ax_{N-2} + Bu_{N-2}$$

As at step $N-1$, u^*_{N-2} is found by solving the mp-LP

$$\begin{array}{cl} \displaystyle\min_{\varepsilon^x_{N-2},\varepsilon^x_{N-1},\varepsilon^u_{N-2},u_{N-2}} & \varepsilon^x_{N-2} + \varepsilon^x_{N-1} + \varepsilon^u_{N-2} \\ \text{subj. to} & -\mathbf{1}_n\varepsilon^x_{N-2} \leq \pm Qx_{N-2} \\ & -\mathbf{1}_r\varepsilon^x_{N-1} \leq \pm P_{N-1}\left[Ax_{N-2} + Bu_{N-2}\right], \\ & -\mathbf{1}_m\varepsilon^u_{N-2} \leq \pm Ru_{N-2}. \end{array}$$

Solution via Recursive Approach

Iterate to get

$$u^*(k) = f_k(x(k))$$

where $f_k(x)$ is continuous and PPWA

$$f_k(x) = F_k^i x \quad \text{if} \quad H_k^i x \leq 0, \; i = 1, \ldots, N_k^r$$

The optimal cost-to-go starting from the state $x(k)$ is

$$J_k^*(x(k)) = \|P_k x(k)\|_p$$

- P_k expresses the optimal cost-to-go $J_k^*(x(k)) = \|P_k x(j)\|_p$ from k to N.
- The rows of P_k correspond to the different affine functions in J_k^* and thus their number varies with the time index k.
- We do not have the equivalent closed form of the 2-norm Riccati Difference Equation.

Infinite Horizon Problem

Consider the cost function

$$J_{\infty}^{*}(x(0)) = \min_{u(0),u(1),\ldots} \sum_{k=0}^{\infty} \|Qx_k\|_p + \|Ru_k\|_p$$

A dynamic programming iteration is

$$\|P_j x_j\|_p = \min_{u_j} \|P_{j+1} x_{j+1}\|_p + \|Qx_j\|_p + \|Ru_j\|_p$$

$$x_{j+1} \quad = \quad Ax_j + Bu_j.$$

Set the terminal cost matrix $P_0 = Q$ and solve it backwards for $k \to -\infty$. Assume $P_k \to P_{\infty}$. Then

$$u^*(k) = F^i x(k) \quad \text{if} \quad H^i x \leq 0, \; i = 1, \ldots, N^r$$

and the optimal infinite horizon cost is

$$J_{\infty}^{*}(x(0)) = \|P_{\infty} x(0)\|_p.$$