Localization of a Mobile Robot Using Relative Bearing Measurements

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***Abstract*—In this paper, the problem of recursive robot localization based on relative bearing measurements is considered, where unknown but bounded measurement uncertainties are assumed. A common approach is to approximate the resulting set of feasible states by simple-shaped bounding sets such as, e.g., axis-aligned boxes, and calculate the optimal parameters of this approximation based on the measurements and prior knowledge. In the novel approach presented here, a nonlinear transformation of the measurement equation into a higher dimensional space is performed. This yields a tight, possibly complex-shaped, bounding set in a closed-form representation whose parameters can be determined analytically for the measurement step. It is shown that the new bound is superior to commonly used outer bounds.**

***Index Terms*—Robot localization, angle measurements, set-theoretic estimation, bounded uncertainty and errors in variables.**

I. Introduction

LOCALIZATION with respect to known features in the environment is one of the most important skills required for mobile robot navigation [1]–[3]. Based on measurements related to these features, the so-called landmarks, the position and orientation of a mobile robot, is determined with respect to a reference frame. We consider the relative bearing problem, where noisy bearing measurements to landmarks are obtained from the robot's sensors, which has been investigated in simulations [4] and experiments using omnidirectional vision sensors [5], [6]. Measurement errors due to sensor noise, landmark misidentification, or inaccurate world models are usually modeled in a statistical framework [7]–[9]. Standard estimation tools like linear least squares [10], extended Kalman filtering [11], [12], or more robust filters based on covariance intersection [13], [14] can then be applied. Nonlinear approaches to localization include particle filtering [9] and grid-based uncertainty representations [15]. However, the underlying statistical assumptions are often hard to verify [16] and parameters of the noise models have to be tuned.

To achieve a localization result which *guarantees* to contain all feasible robot poses consistent with the given measurements and prior knowledge, similar to [4], a bounded-error model is adopted. Within this framework, the only assumption on the measurement errors is that they are bounded in amplitude [17], [18]. We assume that the matching of measurements to the corresponding landmarks in the map has been successfully performed, e.g., by tree-search methods described in [19]. Furthermore, it is reasonable to assume that errors due to map inaccuracies are small compared to the errors in the relative bearing measurements [4]. An optimal localization algorithm for this error model yields the set $\tilde{\mathcal{X}}_k$ of all feasible robot positions compatible with all measurements, where $\tilde{\mathcal{X}}_k$ is guaranteed to contain the true but unknown robot position $\underline{x}_R$.

As the complexity of a straightforward geometric calculation of $\tilde{\mathcal{X}}_k$ by intersection of feasible sets defined by the measurements increases with the number of measurements, a conservative approximation $\mathcal{X}_k^e$ of the exact set $\tilde{\mathcal{X}}_k$ is required to solve the localization problem. This approximation should be as tight as possible, suitable for recursive filtering, and should be described by a fixed number of parameters.

In [4], two approximation techniques have been proposed based on previous research [20] in the set membership estimation area. These techniques are based on boxes and parallelotopes and yield a conservative approximation $\mathcal{X}_k^e$ of the exact set of feasible positions $\tilde{\mathcal{X}}_k$, which contains the true position of the robot. This approximation is calculated based on the relative bearing measurements and their associated error bounds. These simple-shaped sets can be computed with very little computational cost [4]. However, the drawback of this approach is that the complex-shaped exact sets $\tilde{\mathcal{X}}_k$, which bound the position of the robot, can only roughly be approximated. Recent approaches proposed in [21]–[23] calculate tighter bounds by set-inversion algorithms using interval analysis. The exact set of all feasible robot positions is approximated by inner and outer subpavings, each consisting of a list of nonoverlapping boxes.

This paper presents a simple, closed-form solution for the stated localization problem in the case of bounded errors. It is based on the key idea of reformulating the given localization problem as a nonlinear filtering problem and consists of two main contributions. The first contribution is a general novel nonlinear filtering concept that allows determining analytically tight bounding sets $\mathcal{X}_k$ of feasible states for a given nonlinear measurement equation. The second contribution is an *exact* transformation of the given localization problem into a form that is suitable for application of this proposed nonlinear filtering concept. Within this concept, a set of feasible robot poses $\mathcal{X}_k$ is determined, which is a guaranteed conservative approximation of the desired true set $\tilde{\mathcal{X}}_k$. This approximation can represent nonconvex or even nonconnected sets in a closed-form analytic way.

Manuscript received November 6, 2002. This paper was recommended for publication by Associate Editor J. Leonard and Editor S. Hutchinson upon evaluation of the reviewers' comments. This work was supported by the Bavarian Science Foundation, Germany, within the Project DIROKOL.

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Digital Object Identifier 10.1109/TRA.2003.820933

The proposed new filtering concept [24], [25] has been successfully applied to localization in global systems for mobile communications (GSM) networks [26] and has been modified for the application presented here. It extends a concept based on overparametrization presented in [16] and has been implemented for various types of nonlinear measurement equations like distance measurements and absolute bearing measurements. In this paper, the transformation for relative bearing measurements is presented as a special case. Only the measurement step is considered, which is equivalent to localization of a static observer. A solution for the prediction step required for dynamic setups also exists, but is not treated here.

The paper is structured as follows. In Section II, a brief formulation of the problem of robot localization based on relative bearing measurements is given. Section III-A reviews the bounding ellipsoid filter, which is a basic building block in the proposed concept for nonlinear filtering that is introduced in Section III-B. Section IV describes the exact transformation of the given localization problem into a form that is amenable to the proposed framework of nonlinear filtering. The new algorithm is applied to the given localization problem in Section V.

II. Localization Based on Relative Bearing Measurements

We consider the localization problem for a robot navigating in a two-dimensional (2-D) environment. The robot pose at discrete time instants t_k is given by

$$\underline{x}_k = [x_k, y_k]^T \tag{1}$$

where x_k, y_k are the position coordinates of the robot with respect to a given reference frame. The robot is equipped with sensors that provide relative bearing measurements $\gamma_k^{i,j}$ to pairs of landmarks [illegible] in the environment. These measurements are subject to additive noise $v_k^{i,j}$ and are related to the position $\underline{x}_k$ according to

[illegible] (2)

where $\hat{\gamma}_k^{i,j}$ is the given measurement at time instant k. The measurement noise $v_k^{i,j}$ is assumed to be unknown but bounded according to

[illegible] (3)

The coordinates $\underline{x}_{Li} = [x_{Li}, y_{Li}]^T$ of the N landmarks are assumed to be known. The localization problem as stated above is to determine the set $\tilde{\mathcal{X}}_k$ of all robot positions $\underline{x}_k$ compatible with the relative bearing measurements according to (2).

Each relative bearing measurement $\gamma_k^{i,j}$ to two landmarks L_i, L_j defines a circular arc $\tilde{\mathcal{C}}_k^{i,j}$, which constrains the position of the robot, provided that the measurements are exact. The endpoints of the arc are the two given landmarks. Given an uncertain relative bearing measurement $\hat{\gamma}_k^{i,j} = \gamma_k^{i,j} + v_k^{i,j}$, the set of feasible robot positions $\tilde{\mathcal{M}}_k^{i,j}$ under assumption (3) is given by

$$\tilde{\mathcal{M}}_k^{i,j} = \{\underline{x} \text{ [illegible]} \}.$$

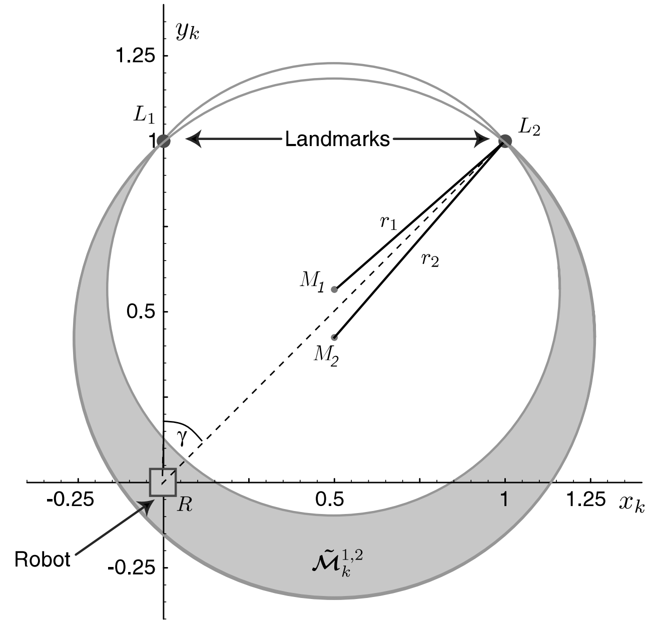

Fig. 1. Localization by a relative bearing measurement with bounded uncertainties. $\tilde{\mathcal{M}}_k^{1,2}$ is the exact set of all feasible robot positions defined by the relative bearing measurement. The relative bearing angle is $\gamma = 45°$ and the measurement uncertainty was chosen as $\epsilon_k^{v,1,2} = 4°$.

As outlined in [4], $\tilde{\mathcal{M}}_k^{i,j}$ can be described as a "thickened ring" from a geometric point of view. In Fig. 1, an example for a resulting measurement set $\tilde{\mathcal{M}}_k^{1,2}$ is shown for two landmarks, L_1 and L_2, and a robot, R, at position $\underline{x}_k = [0, 0]^T$. The measured difference angle in this example is $\gamma_k^{1,2} = 45°$ and the upper bound for the measurement noise was chosen as $\epsilon_k^{v,1,2} = 4°$.

Given a set of relative bearing measurements [illegible], it is obvious that the measurement set of feasible robot poses $\tilde{\mathcal{X}}_k^M$ is defined by the intersection of all sets [illegible]

[illegible]

Because $\tilde{\mathcal{X}}_k^M$ is, in general, a complex-shaped set, a conservative approximate description $\mathcal{X}_k^M$ of $\tilde{\mathcal{X}}_k^M$ with $\mathcal{X}_k^M \supset \tilde{\mathcal{X}}_k^M$ is required, which can be used in a recursive estimation scheme. This approximation should be described by a finite set of parameters and should degrade gracefully with a decreasing number of parameters. To find such an approximation, $\mathcal{X}_k^M$ of $\tilde{\mathcal{X}}_k^M$ is, in general, not a trivial task. In Fig. 8, an example for a resulting measurement set $\tilde{\mathcal{X}}_k^M$ is shown for the case of $N = 3$ relative bearing measurements to three landmarks L_1, L_2, and L_3. It can be seen that the gray-shaded set $\tilde{\mathcal{X}}_k^M$ results from the intersection of three sets $\tilde{\mathcal{M}}_k^{i,j}$. Note that this nonconvex set cannot be represented by polytopes.

In the novel approach presented in this paper, the problem of localization in the case of relative bearing measurements stated in this section is reformulated as a filtering problem in a system-theoretic framework. Application of a recursive nonlinear filtering algorithm allows sequentially fusing measurements with constant computational complexity, yielding an estimated set $\mathcal{X}_k$ that is guaranteed to contain the true state $\underline{x}_k$ under the given assumptions. The algorithm yields estimates with small remaining approximation errors even for complex shaped, nonconvex, or nonconnected sets.

The two main contributions of the proposed approach are described in detail in the following sections. Within the new general concept for nonlinear filtering presented in Section III-B, a complicated uncertainty set $\boldsymbol{\mathcal{X}}_k$ in the N-dimensional original space S is represented by a simpler shaped uncertainty set $\boldsymbol{\mathcal{X}}_k^*$ in an L-dimensional hyperspace S^* with $L > N$. The parameters of the so-called pseudoellipsoidal set $\boldsymbol{\mathcal{X}}_k^*$ are determined analytically for the measurement step by means of a linear bounding ellipsoid filter operating in the hyperspace S^*.

The *exact* transformation of the given localization problem into a form that is suitable for the application of the proposed new general framework for nonlinear filtering is derived in Section IV. Equivalent transformations have been found for other types of nonlinear measurement equations, such as absolute bearing measurements or distance measurements [26].

III. Framework for Nonlinear Filtering

Within the framework for nonlinear filtering in the case of bounded error models proposed in this paper, the measurement update equations of the bounding ellipsoid filter are used in a hyperspace S^* of transformed state variables to calculate a set $\boldsymbol{\mathcal{X}}_k^{e,*}$ in S^* that defines the complex-shaped estimated set of states $\boldsymbol{\mathcal{X}}_k^e$ in the original space S.

The equations of the bounding ellipsoid filter resemble the well-known Kalman filter equations. They were first derived in [18], and extensions for minimum volume ellipsoids were presented in [27]. These equations will be briefly reviewed in Section III-A. Section III-B derives the concept for nonlinear filtering using the bounding ellipsoid filter as a basic building block. This basic building block may be replaced by stochastic filtering algorithms [25] when uncertainties are modeled in a stochastic framework.

A. Bounding Ellipsoid Filter

At time step k, a prior estimate of the state $\underline{x}_k \in \mathbb{R}^{\mathrm{N}}$ described by the ellipsoidal set

$$\boldsymbol{\mathcal{X}}_k^p = \left\{ \underline{x}_k : \left(\underline{x}_k - \hat{\underline{x}}_k^p\right)^T \left(\boldsymbol{C}_k^p\right)^{-1} \left(\underline{x}_k - \hat{\underline{x}}_k^p\right) \leq 1 \right\} \tag{4}$$

where $\boldsymbol{C}_k^p$ is a positive, symmetric matrix and $\hat{\underline{x}}_k^p$ is the midpoint vector, and a linear time-varying measurement equation with uncertain measurement $\hat{\underline{z}}_k \in \mathbb{R}^{\mathrm{M}}$ according to

$$\hat{\underline{z}}_k = \boldsymbol{H}_k \underline{x}_k + \underline{v}_k \tag{5}$$

with $M \times N$-dimensional measurement matrix $\boldsymbol{H}_k$ and additive, bounded measurement noise $\underline{v}_k$ are given. Then an ellipsoidal conservative approximation for the set of all states compatible with the measurement and the prior estimated set is obtained as

$$\boldsymbol{\mathcal{X}}_k^e = \left\{ \underline{x}_k : \left(\underline{x}_k - \hat{\underline{x}}_k^e\right)^T \left(\boldsymbol{C}_k^e\right)^{-1} \left(\underline{x}_k - \hat{\underline{x}}_k^e\right) \leq 1 \right\} \tag{6}$$

where the midpoint vector $\hat{\underline{x}}_k^e$ of the bounding ellipsoid $\boldsymbol{\mathcal{X}}_k^e$ is given by

$$\hat{\underline{x}}_k^e = \hat{\underline{x}} \text{[illegible]} \underline{z}_k - \boldsymbol{H}_k \hat{\underline{x}}_k^p) .$$

The matrix $\boldsymbol{C}_k^e$ is given by

[illegible]

where

$$s_k = 1 + \lambda_k - \lambda_k \left(\hat{\underline{z}}_k - \boldsymbol{H}_k \hat{\underline{x}} \text{[illegible]} \underline{z}_k - \boldsymbol{H}_k \hat{\underline{x}}_k^p\right) . \tag{7}$$

The only assumption about the measurement uncertainties is that they are bounded according to $\underline{v}_k \in \boldsymbol{\mathcal{V}}_k$ with ellipsoidal sets $\boldsymbol{\mathcal{V}}_k$ given by

$$\boldsymbol{\mathcal{V}}_k = \left\{ \underline{v}_k : \underline{v}_k^T (\boldsymbol{V}_k)^{-1} \underline{v}_k \leq 1 \right\} . \tag{8}$$

$\boldsymbol{V}_k$ is the symmetric positive definite matrix defining the set of all possible measurements according to (5). λ_k is a *fusion parameter* with $\lambda_k \in [0, \infty)$ that is chosen such that the volume of the bounding set $\boldsymbol{\mathcal{X}}_k^e$ is minimized. The calculation of the matrix $\boldsymbol{C}_k^e$ is very similar to the well-known Kalman filter equations. The additional parameter s_k according to (7) modifies the intermediate matrix $\boldsymbol{P}_k$ such that the resulting size of the bounding ellipsoidal set $\boldsymbol{\mathcal{X}}_k^e$ depends upon the actual measurement $\hat{\underline{z}}_k$, which is not the case for the Kalman filter.

B. Nonlinear Filtering Algorithm

In the proposed framework for nonlinear filtering, the linear bounding ellipsoid filter is applied in a higher dimensional space S^*. A transformed state $\underline{x}_k^*$ at time k in the L-dimensional space S^* is related to the state $\underline{x}_k$ in the original N-dimensional space S via a nonlinear transformation $\boldsymbol{T}(\,\cdot\,)$ according to

$$\underline{x}_k^* = \boldsymbol{T}(\underline{x}_k) = \left[T_1(\underline{x}_k) \quad T_2(\underline{x}_k) \quad \ldots \quad T_L(\underline{x}_k)\right]^T . \tag{9}$$

Hence, the transformation $\boldsymbol{T}(\,\cdot\,)$ defines an N-dimensional manifold in the transformed space S^*, the so-called universal manifold U^*. The exact set of states $\tilde{\boldsymbol{\mathcal{X}}}_k$ in the original space S is then represented by an N-dimensional submanifold of U^*. This submanifold can now be bounded by the intersection of a simple-shaped L-dimensional set *and* the universal manifold U^*. We define pseudoellipsoidal sets $\boldsymbol{\mathcal{X}}^*$ as sets that are ellipsoidal in the transformed variables $\underline{x}^*$. An L-dimensional pseudoellipsoidal set according to

$$\boldsymbol{\mathcal{X}}_k^{e,*} = \left\{ \underline{x}_k^* : \left(\underline{x}_k^* - \hat{\underline{x}}_k^{e,*}\right)^T \left(\boldsymbol{C}_k^{e,*}\right)^{-1} \left(\underline{x}_k^* - \hat{\underline{x}}_k^{e,*}\right) \leq 1 \right\} \tag{10}$$

defines the resulting nonellipsoidal set $\boldsymbol{\mathcal{X}}_k^e$ in the original, N-dimensional space S, that is used to bound the true set $\tilde{\boldsymbol{\mathcal{X}}}_k$. $\hat{\underline{x}}_k^{e,*}$ is the midpoint of the pseudoellipsoid and $\boldsymbol{C}_k^{e,*}$ is a symmetric positive definite matrix. The advantage of this concept is that it yields a simple description for complicated uncertainties $\boldsymbol{\mathcal{X}}_k^e$ in S and allows nonlinear recursive filtering in a system-theoretic framework with little additional complexity, compared with a linear filtering problem. The most complicated problem within this framework is to calculate characteristic values of the uncertainty set $\boldsymbol{\mathcal{X}}_k^e$ in S from $\boldsymbol{\mathcal{X}}_k^{e,*}$ via an inverse transformation.

We consider estimating the state of a static nonlinear system based on measurements $\underline{z}_k$ that are subject to additive unknown but bounded errors $\underline{v}_k$. A measurement $\hat{\underline{z}}_k$ at time instant k is related to the state $\underline{x}_k$ via the measurement equation

$$\hat{\underline{z}}_k = \underline{h}_k(\underline{x}_k) + \underline{v}_k \tag{11}$$

which is, in general, nonlinear. To obtain the nonlinear transformation $\boldsymbol{T}(\cdot)$ that defines the hyperspace S^* according to (9), a transformation $\underline{\eta}_k(\cdot)$ is applied to both sides of the measurement equation (11), which results in

$$\underline{\eta}_k(\hat{\underline{z}}_k - \underline{v}_k) = \underline{\eta}_k(\underline{h}_k(\underline{x}_k)). \tag{12}$$

This transformation yields *nonlinear constraints* for the filtering result $\boldsymbol{\mathcal{X}}_k^e$, which are exploited to generate a tight approximation of the exact set $\tilde{\boldsymbol{\mathcal{X}}}_k$. In order to generate these constraints, the left-hand side (LHS) of (12) is exactly converted into

$$\underline{\eta}_k(\hat{\underline{z}}_k - \underline{v}_k) = \hat{\underline{z}}_k^* - \underline{v}_k^* \tag{13}$$

where $\hat{\underline{z}}_k^*$ represents the expected value of the transformed measurements in the transformed L-dimensional space S^*. $\underline{v}_k^* \in \tilde{\boldsymbol{\mathcal{V}}}_k^*$ depends on the given measurements $\hat{\underline{z}}_k$ and accounts for the respective transformed measurement uncertainties. The right-hand side (RHS) of (12) can be expanded in the transformed space S^* into

$$\underline{\eta}_k(\underline{h}_k(\underline{x}_k)) = \boldsymbol{H}_k^* \underline{x}_k^*. \tag{14}$$

(13) and (14) yield a pseudolinear form of the measurement equation (11) transformed by $\underline{\eta}_k(\cdot)$, resulting in

$$\hat{\underline{z}}_k^* = \boldsymbol{H}_k^* \underline{x}_k^* + \underline{v}_k^* \tag{15}$$

where $\boldsymbol{H}_k^*$ relates the transformed state vector *linearly* to the measurements in the transformed space S^*. The measurement uncertainty set $\boldsymbol{\mathcal{V}}_k^*$ with associated positive symmetric matrix $\boldsymbol{V}_k^*$ is given by

$$\boldsymbol{\mathcal{V}}_k^* = \{\underline{v}_k^* : (\underline{v}_k^* - \hat{\underline{v}}_k^*)^T (\boldsymbol{V}_k^*)^{-1} (\underline{v}_k^* - \hat{\underline{v}}_k^*) \leq 1\} \tag{16}$$

where the midpoint vector $\hat{\underline{v}}_k^*$ and the matrix $\boldsymbol{V}_k^*$ is calculated such that $\hat{\underline{z}}_k^* - \boldsymbol{H}_k^* \underline{x}_k^* \in \boldsymbol{\mathcal{V}}_k^*$ for all possible $\underline{v}_k^*$.

Example 1: Consider the distance measurement equation

$$\hat{z}_k = x_k^2 + y_k^2 + v_k$$

with the 2-D state vector $\underline{x}_k = [x_k, y_k]^T$, bounded measurement noise $v_k \in [-1, 1]$ and a scalar measurement $\hat{z}_k = 1$. Applying the nonlinear transformation $\underline{\eta}_k(z) = [z, z^2]^T$ yields two nonlinear measurement equations

[illegible] (17)

These can be expanded according to (15) in an $L = 5$-dimensional space S^* with a transformed state vector $\underline{x}_k^*$, 2-D measurement vector $\hat{\underline{z}}_k^*$, and related additive noise vector $\underline{v}_k^*$ according to

$$\begin{bmatrix} \hat{z}_k \\ \hat{z}_k^2 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 1 \end{bmatrix}}_{[illegible]} \underbrace{[illegible]}_{\underline{x}_k^*} + \underbrace{\begin{bmatrix} v_k \\ 2\hat{z}_k v_k - v_k^2 \end{bmatrix}}_{\underline{v}_k^*}.$$

To determine the exact transformed measurement uncertainty $\tilde{\boldsymbol{\mathcal{V}}}_k^*$, we consider the transformed random variable $\bar{z}_k = \hat{z}_k - v_k$, where $\hat{z}_k$ is a given constant value. Obviously, the random vector

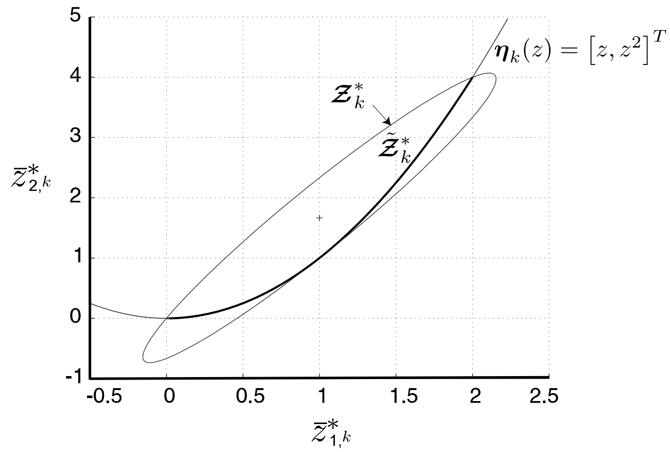

Fig. 2. Example for the application of a nonlinear transformation $\underline{\eta}_k(z) = [z, z^2]^T$ to generate additional nonlinear constraints for the estimated set of states $\boldsymbol{\mathcal{X}}_k^e$. The transformed, uncertain measurement $\bar{\underline{z}}_k^* = [\bar{z}_k, \bar{z}_k^2]^T$ is constrained to lie on a parabola $\tilde{\boldsymbol{\mathcal{Z}}}_k^*$ that can be bounded by the ellipsoidal set $\boldsymbol{\mathcal{Z}}_k^*$. The desired measurement uncertainty set $\boldsymbol{\mathcal{V}}_k^*$ is then directly obtained from $\boldsymbol{\mathcal{Z}}_k^*$.

$\bar{\underline{z}}_k^* = [\bar{z}_k, \bar{z}_k^2]^T$, which constitutes the LHS of (17), is constrained to a 1-D set $\tilde{\boldsymbol{\mathcal{Z}}}_k^*$, more precisely, a segment of a parabola in a 2-D measurement space with $\bar{z}_k \in [0, 2]$. Fig. 2 shows the parabola $\underline{\eta}_k(z) = [z, z^2]^T$ and the set $\tilde{\boldsymbol{\mathcal{Z}}}_k^*$, marked by a thick line. Exploiting the relation between $\bar{z}_k$ and v_k, the exact measurement uncertainty set $\tilde{\boldsymbol{\mathcal{V}}}_k^*$ is directly obtained from $\tilde{\boldsymbol{\mathcal{Z}}}_k^*$. In order to apply the proposed nonlinear filter in the hyperspace S^*, a conservative ellipsoidal approximation $\boldsymbol{\mathcal{V}}_k^*$ for $\tilde{\boldsymbol{\mathcal{V}}}_k^*$ has to be found, corresponding to the approximation $\boldsymbol{\mathcal{Z}}_k^*$ for $\tilde{\boldsymbol{\mathcal{Z}}}_k^*$ that is depicted in Fig. 2. Note that there exist infinitely many valid conservative approximations $\boldsymbol{\mathcal{V}}_k^*$ with $\boldsymbol{\mathcal{V}}_k^* \supset \tilde{\boldsymbol{\mathcal{V}}}_k^*$, where the optimal one is characterized by the fact that the state estimate $\boldsymbol{\mathcal{X}}_k^e$ becomes least conservative. A closed-form solution is available for the transformation $\underline{\eta}_k(z) = [z, z^2]^T$, yielding the positive matrix

[illegible]

and $\hat{\underline{v}}_k^* = [0, -0.667]^T$ that together define the measurement uncertainty according to (16), see Fig. 2. The effect of the additional nonlinear measurement equation that generates a tighter estimated conservative approximation $\boldsymbol{\mathcal{X}}_k^e$ for the exact set of states $\tilde{\boldsymbol{\mathcal{X}}}_k$ is demonstrated in the example for localization in the case of relative bearing measurements presented in Section V.□

Using (15), the measurement update equations of the bounding ellipsoid filter (7) can directly be applied in the hyperspace S^* to calculate a bound for the complicated-shaped estimated set of states $\boldsymbol{\mathcal{X}}_k^e$ in the original space S. This results in

$$\hat{\underline{x}}_k^{e,*} = \hat{\underline{x}}[illegible] \; [illegible] \; \hat{\underline{z}}_k^* - \boldsymbol{H}_k^* \hat{\underline{x}}_k^{p,*})$$

and

[illegible] (18)

where

$$\boldsymbol{s}_k^* = 1 + \lambda_k^* - \lambda_k^* \left(\underline{\hat{z}}_k^* - \boldsymbol{H}_k^* \underline{\hat{x}} \text{[illegible]} \right.$$
$$\left. \text{[illegible]} \; \underline{z}_k^* - \boldsymbol{H}_k^* \underline{\hat{x}}_k^{p,*}\right).$$

The fusion parameter λ_k^* is selected such that the volume of the bounding set in the *original* space S is minimized. The resulting pseudoellipsoidal set $\boldsymbol{\mathcal{X}}_k^{e,*}$ is completely defined by its midpoint vector $\underline{\hat{x}}_k^{e,*}$ and the matrix $\boldsymbol{C}_k^{e,*}$, according to (10).

Using this form to calculate the complex-shaped bounding set in the original space S possesses the following appealing properties.

- No additional uncertainty. The resulting set $\boldsymbol{\mathcal{X}}_k^e$ is a subset of the union of the predicted set $\boldsymbol{\mathcal{X}}_k^p$ and the measurement set $\boldsymbol{\mathcal{X}}_k^M$.
- Conservativeness. The resulting estimated set $\boldsymbol{\mathcal{X}}_k^e$ is a *guaranteed* upper bound for $\tilde{\boldsymbol{\mathcal{X}}}_k$.
- Tighter bound. The resulting nonlinear, implicit polynomial bounding set is a better approximation for the exact set of states $\tilde{\boldsymbol{\mathcal{X}}}_k$ than a single box-shaped set or an ellipsoidal bounding set which can, for example, be obtained by linearization of the original nonlinear problem. This means, the volume of the calculated approximation $\boldsymbol{\mathcal{X}}_k^e$ is smaller than the volume of the ellipsoidal bounding set or box-shaped sets.

IV. Transformation of the Localization Problem

To apply the described framework for nonlinear filtering to the problem of localization in the case of relative bearing measurements, the measurement equation given by (2) for two landmarks L_i, L_j is first transformed into a nonlinear measurement equation according to

$$\text{[illegible]} \tag{19}$$

where $\hat{\gamma}_k^{i,j} = \gamma_k^{i,j} + v_k^{i,j}$ is the observed uncertain relative bearing measurement. Equation (19) can be derived from (2) exploiting the fact that all robot positions, from which a constant difference angle is observed, lie on a circular arc $\tilde{\boldsymbol{\mathcal{C}}}_k^{i,j}$ [4], [10]. Hence, (19) is the definition of the circle $\boldsymbol{\mathcal{C}}_k^{i,j}$ corresponding to $\tilde{\boldsymbol{\mathcal{C}}}_k^{i,j}$ whose midpoint $\underline{M}(\hat{\gamma}_k^{i,j}) = [x_M, y_M]^T$ and radius $r(\hat{\gamma}_k^{i,j})$ are functions of the observed difference angle $\hat{\gamma}_k^{i,j}$ and are given by

$$\underline{M}\left(\hat{\gamma}_k^{i,j}\right) = \underline{x}_{\mathrm{LM}_i} + \frac{1}{2}\left(\underline{x}_{\mathrm{LM}_j} - \underline{x}_{\mathrm{LM}_i}\right) - \frac{1}{2}\cot\left(\hat{\gamma}_k^{i,j}\right)\underline{\Delta}_{i,j}^{\perp} \tag{20}$$

and

$$r\left(\hat{\gamma}_k^{i,j}\right) = \frac{\left\|\underline{x}_{\mathrm{LM}_j} - \underline{x}_{\mathrm{LM}_i}\right\|}{\left|2\sin\left(\hat{\gamma}_k^{i,j}\right)\right|} \tag{21}$$

where $\underline{x}_{\mathrm{LM}_i}$ is the position vector of landmark i and $\underline{\Delta}_{i,j}^{\perp}$ is the vector orthogonal to $\underline{x}_{\mathrm{LM}_j} - \underline{x}_{\mathrm{LM}_i}$ with $\|\underline{\Delta}_{i,j}^{\perp}\| = \|\underline{x}_{\mathrm{LM}_j} - \underline{x}_{\mathrm{LM}_i}\|$. The example given in Fig. 1 shows the set $\tilde{\boldsymbol{\mathcal{M}}}_k^{1,2}$ resulting from the union of all circular arcs $\tilde{\boldsymbol{\mathcal{C}}}_k^{1,2}(\gamma_k^{1,2})$ with [illegible]. The nonlinear measurement equation (19) can now be rewritten by two simple nonlinear measurement equations. Each of the two simple measurement equations describes a distance measurement. This transformation is possible for the assumed bounded error model because the intersection of the sets defined by the two distance measurements yields exactly the original crescent-shaped measurement set $\tilde{\boldsymbol{\mathcal{M}}}_k^{i,j}$ resulting from the uncertain relative bearing measurement (19). The measurement equations can be written in state-space form according to (11), where

$$\underline{h}_k(\underline{x}\text{[illegible]}) = \left[\text{[illegible]}\right] \tag{22}$$

and

$$\underline{\hat{z}}_k = \begin{bmatrix} \left(r_{1,k} + \frac{d_k}{2}\right)^2 \\ \left(r_{2,k} - \frac{d_k}{2}\right)^2 \end{bmatrix}. \tag{23}$$

$\underline{M}_{l,k} = [x_{M_{l,k}}, y_{M_{l,k}}]^T, l = 1, 2$ are the midpoints of the outer ring R_1 and the inner ring R_2 containing the exact measurement set $\tilde{\boldsymbol{\mathcal{M}}}_k^{i,j}$, and $r_{l,k}$ are the related radii at time step k, calculated from (20) and (21) when the bounds of the admissible interval for $\hat{\gamma}_k^{i,j}$ are inserted. The parameter d_k corresponds to the uncertainty of the transformed distance measurements, and can be directly obtained from $\underline{M}_{l,k}$ and $r_{l,k}, l = 1, 2$. For the example given in Fig. 1, a transformation into two distance measurement equations according to (11) with (22) and (23) is graphically illustrated in Fig. 3. The transformed measurement equations correspond to rings R_1, R_2 with midpoints $\underline{M}_{1,k}, \underline{M}_{2,k}$ and radii $r_{1,k} + (d_k/2)$ and $r_{2,k} - (d_k/2)$. It can be seen that the intersection of the two rings is equivalent to the original measurement set $\tilde{\boldsymbol{\mathcal{M}}}_k^{1,2}$ depicted in Fig. 1.

Expanding these equations into an intermediate pseudolinear form in an $L = 4$-dimensional hyperspace $\bar{S}^*$ yields two scalar measurement equations

$${}^l\underline{\bar{z}}_k^* = {}^l\bar{\boldsymbol{H}}_k^* \underline{\bar{x}}_k^* + {}^l\bar{v}_k^* \tag{24}$$

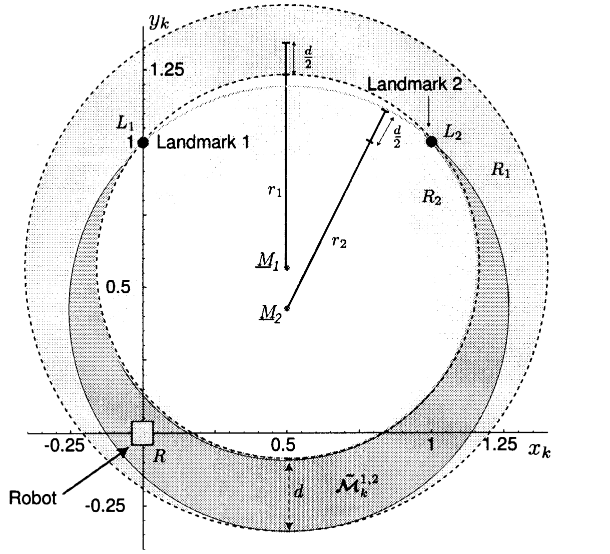

Fig. 3. Localization by a relative bearing measurement with bounded uncertainties. Graphical illustration of the transformation of the measurement equation. Each grey-shaded ring R_1 and R_2 corresponds to a transformed, pseudolinear measurement equation. The intersection of R_1 and R_2 is equivalent to the desired set of circular arcs $\tilde{\boldsymbol{\mathcal{M}}}_k^{i,j}$.

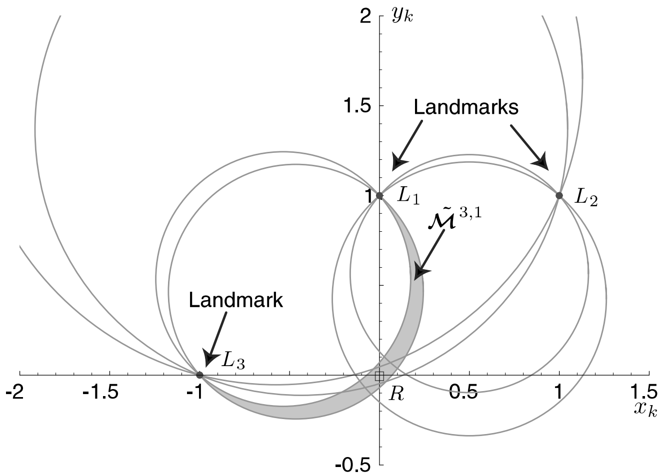

Fig. 4. Three relative bearing measurements with bounded uncertainties. The shaded set is the resulting, exact measurement set $\tilde{\boldsymbol{\mathcal{M}}}_k^{3,1}$ after the first filtering step.

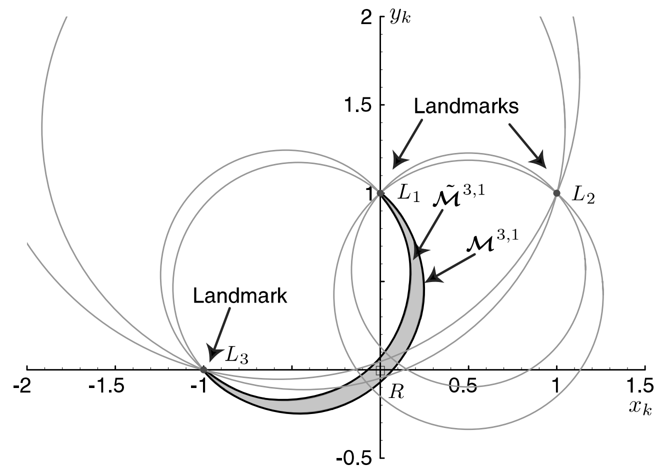

Fig. 5. Three relative bearing measurements with bounded uncertainties. The shaded resulting exact measurement set $\tilde{\boldsymbol{\mathcal{M}}}_k^{3,1}$ and the implicit polynomial approximation $\boldsymbol{\mathcal{M}}_k^{3,1}$ (thick black line) after the first filtering step with the new nonlinear pseudoellipsoid filter. The nonlinear transformation $\underline{\eta}_k(z) = [z, z^2]^T$ was applied to generate additional constraints.

$l = 1, 2$ with the new state vector $\bar{\underline{x}}_k^* = \bar{\boldsymbol{T}}(\underline{x}_k)$, measurement matrices ${}^l\bar{\boldsymbol{H}}_k^*$, and measurements ${}^l\bar{\hat{\underline{z}}}_k^*$ according to

$$\bar{\underline{x}}_k^* = \bar{\boldsymbol{T}}(\underline{x} \text{[illegible]}$$

[illegible]

$$\text{[illegible]} - \frac{k}{\text{[illegible]}} \text{[illegible]}$$

$$\text{[illegible]} - \frac{\text{[illegible]}}{2}\Big)^2 - \left(x_{M_{2,k}}^2 + y_{M_{2,k}}^2\right)\Big]$$

where the bar denotes variables related to the intermediate, pseudolinear form. The corresponding bounded measurement noise $\bar{v}_k^{*,l}$ is constrained by the intervals

[illegible]

where $d_k = d_k^{i,j}$ corresponds to the "thickness" of the rings belonging to $\tilde{\boldsymbol{\mathcal{M}}}_k^{i,j}$ (see Fig. 3). The pseudolinear measurement equations (24) can directly be used for filtering with the described pseudoellipsoidal approach, as will be demonstrated in an example in Section V.

Application of the nonlinear transformation $\underline{\eta}_k(\cdot)$, according to (12), generates an even tighter approximation of the exact measurement set $\tilde{\boldsymbol{\mathcal{X}}}_k^M$. Here the transformation $\underline{\eta}_k(z) = [z, z^2]^T$ is selected to obtain the final pseudolinear measurement equation. Transformations of higher order yield increasingly tighter approximations, but require higher computational effort. This transformation is applied to each (24), $l = 1, 2$ separately, which results in two measurement equations with the new state vector $\underline{x}_k^*$ given by

$$\underline{x} \text{[illegible]}^T \qquad (25)$$

and measurement matrices ${}^l\boldsymbol{H}_k^*$ with

$$\text{[illegible]} \qquad \text{[illegible]}\, 2 \qquad (26)$$

in the $L = 8$-dimensional space S^*. The associated measurement vectors ${}^l\hat{\underline{z}}_k^*$ are given by

$$ {}^1\hat{\underline{z}}_k^* = \Big[\big(r_{1,k} + \tfrac{d_k}{\text{[illegible]}} \text{[illegible]} \Big] \qquad (27)$$

$$ {}^2\hat{\underline{z}}_k^* = \Big[\big(r_{2,k} - \tfrac{d_k}{\text{[illegible]}} \text{[illegible]} \Big] \qquad (28)$$

The measurement uncertainty sets ${}^l\boldsymbol{\mathcal{V}}_k^*, l = 1, 2$ with associated definition matrices ${}^l\boldsymbol{V}_k^*$ are given by (16) for the transformed measurement equations (15). Similar to the example in Section III-B, their parameters given by the midpoint vectors ${}^l\hat{\underline{v}}_k^*$ and the matrices ${}^l\boldsymbol{V}_k^*$ are calculated such that ${}^l\hat{\underline{z}}_k^* - {}^l\boldsymbol{H}_k^*\underline{x}_k^* \in {}^l\boldsymbol{\mathcal{V}}_k^*$ for all possible $\underline{v}_k^*$. The calculation of an arbitrary conservative pseudoellipsoidal set ${}^l\boldsymbol{\mathcal{V}}_k^*$ is simple, yet to find the minimum volume ellipsoidal set ${}^l\boldsymbol{\mathcal{V}}_k^*$ is, in general, not a trivial task. A closed-form solution is available and has been used for the transformation $\underline{\eta}_k(z) = [z, z^2]^T$, as demonstrated in the example in Section III-B. For application of the filtering algorithm (18) $\hat{\underline{v}}_k^*$ is subtracted from $\hat{\underline{z}}_k^*$ to obtain zero-mean measurement noise.

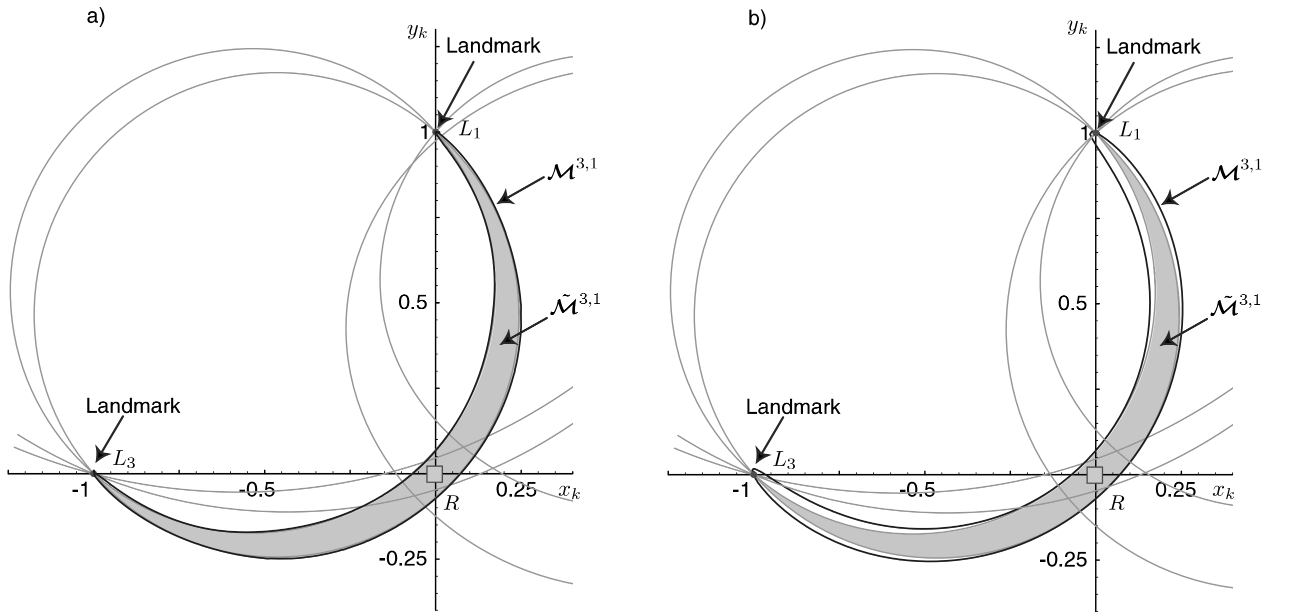

Fig. 6. Three relative bearing measurements with bounded uncertainties. The shaded resulting exact measurement set $\tilde{\mathcal{M}}_k^{3,1}$ and the implicit polynomial approximation $\mathcal{M}_k^{3,1}$ (thick black line) after the first filtering step with the new nonlinear pseudoellipsoid filter. (a) Zoom from Fig. 5. The nonlinear transformation $\underline{\eta}_k(z) = [z, z^2]^T$ was applied to generate additional constraints. (b) *No* additional nonlinear transformation $\underline{\eta}_k(\cdot)$ was applied. Please note: The resulting approximation $\mathcal{M}_k^{3,1}$ is more conservative than the one obtained in Fig. 5, but obviously still much tighter than the smallest possible box-shaped approximation using a single box.

V. Solution of the Localization Problem Using Nonlinear Filtering

To demonstrate the performance of the proposed approach to relative bearing localization, consider the typical scenario depicted in Fig. 4. A robot R at position $\underline{x}_k = [0, 0]^T$ conducts relative bearing measurements to $N = 3$ landmarks located at $L_1 = [0, 1]^T, L_2 = [1, 1]^T$, and $L_3 = [-1, 0]^T$. These measurements define three circular arcs $\tilde{\mathcal{C}}_k^{i,j}$ and associated measurement sets $\tilde{\mathcal{M}}_k^{i,j}$. Note that the true position of the robot is contained in the intersection of all three measurement sets.

To solve the given localization problem shown in Fig. 4, a noninformative prior $\mathcal{X}_k^p$ is assumed, i.e., $\mathcal{X}_k^p \cap \mathcal{X}_k^M = \mathcal{X}_k^M$. This means that position estimation is based on the measurements only. Consecutive application of the update equations (18) for each pair (i, j) of landmarks L_i, L_j yields the desired estimate $\hat{\underline{x}}_k^e$ of the robot position, together with the associated set $\mathcal{X}_k^e$ of all robot positions compatible with the given measurements. To calculate these desired quantities, the pseudoellipsoidal bounding set $\mathcal{X}_k^{e,*}$, according to (10), is evaluated on the 2-D manifold U^* defined in an 8-D space by the nonlinear transformation $\boldsymbol{T}(\cdot)$ given by (25). This yields a higher order implicit polynomial description of the uncertainty set $\mathcal{X}_k^e$ in the original space S. From this implicit description, characteristic values of $\mathcal{X}_k^e$ like the boundary of the set are calculated numerically by means of the inverse transformation.

For the first simulation, the intermediate measurement equations in pseudolinear form according to (24) are directly used with the proposed new filter algorithm, which is equivalent to choosing $\underline{\eta}_k(\cdot)$ as the identity transformation $\underline{\eta}_k(z) = z$. This means an $L = 4$-dimensional hyperspace S^* is used to describe the $N = 2$-dimensional uncertainty set $\mathcal{X}_k^M$ in the original space S. The resulting exact measurement set $\tilde{\mathcal{M}}_k^{3,1}$ after including the first relative bearing measurement between landmark L_3 and L_1 is shown in Fig. 4 as a shaded crescent-shaped area. The related approximation $\mathcal{M}_k^{3,1}$ calculated with the proposed filter algorithm is marked by a thick black line in Fig. 6(b).

For the following simulations, the additional transformation $\underline{\eta}_k(z) = [z, z^2]^T$ is applied, yielding measurement equation (15) with $\underline{x}_k^*, \boldsymbol{H}_k^*$, and $\hat{\underline{z}}_k^*$ given by (25)–(28) in an $L = 8$-dimensional space. Fig. 5 and the zoomed clipping depicted in Fig. 6(a) show the result obtained after including the first relative bearing measurement between landmark L_3 and L_1. It can clearly be seen that the shaded exact measurement set $\tilde{\mathcal{M}}_k^{3,1}$ in Fig. 4 is very tightly approximated by the implicit polynomial bound obtained with the proposed nonlinear pseudoellipsoid filter. The bound is marked by a thick black line in Fig. 5. Note that this approximation obtained with the additional transformation $\underline{\eta}_k(\cdot)$ is even better than the result obtained by straightforward application of the expanded measurement equations (24), which is depicted in Fig. 6(b). Fig. 7 shows the final result after all measurements have been included. From the zoomed clipping shown in Fig. 8, it can be seen that all $N = 3$ measurement sets $\tilde{\mathcal{M}}_k^{i,j}$ have been taken into account and a tight upper bound for the exact set has been achieved. For reference purposes, the tightest possible axis-aligned box-shaped set $\mathcal{X}_k^{M,B}$ is also shown, which is obviously a much more conservative approximation.

VI. Summary and Future Work

In this paper, a new approach to localization of a mobile robot for the case of relative bearing measurements has been presented, which consists of two main contributions. It is based on

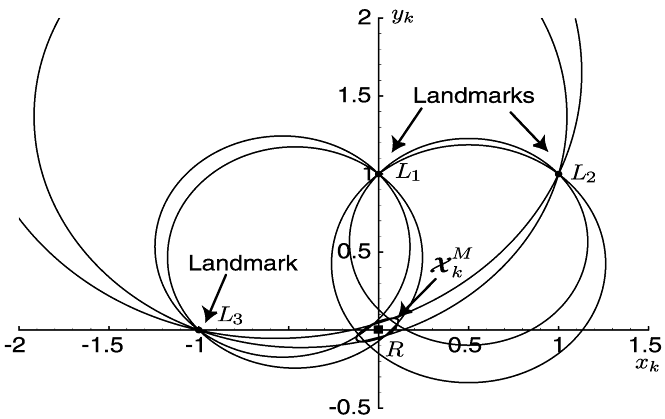

Fig. 7. Three relative bearing measurements with bounded uncertainties. Resulting exact measurement set $\tilde{\mathcal{X}}_k^M$ and implicit polynomial approximation $\mathcal{X}_k^M$ resulting from the new nonlinear pseudoellipsoid filter. The nonlinear transformation $\underline{\eta}_k(z) = [z, z^2]^T$ was applied to generate additional constraints.

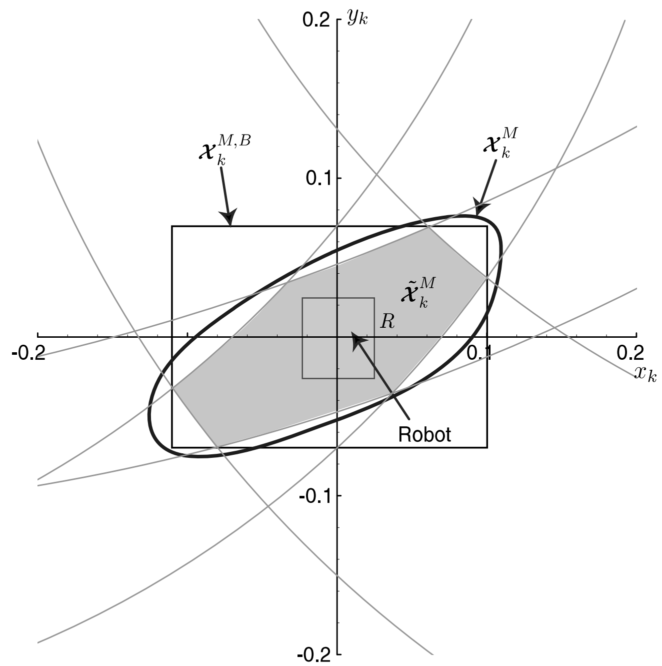

Fig. 8. Zoomed clipping from Fig. 7. Three relative bearing measurements with bounded uncertainties. Resulting exact measurement set $\tilde{\mathcal{X}}_k^M$ and implicit polynomial approximation $\mathcal{X}_k^M$ resulting from the new nonlinear pseudoellipsoid filter. The nonlinear transformation $\underline{\eta}_k(z) = [z, z^2]^T$ was applied to generate additional constraints. $\mathcal{X}_k^{M,B}$ is the tightest possible axis-aligned box-shaped set, which is much more conservative. Please note: The shaded exact set $\tilde{\mathcal{X}}_k^M$ cannot exactly be represented by polytopes.

the key idea to reformulate the given localization problem as a nonlinear filtering problem.

The first contribution is a *novel framework for nonlinear filtering* which is based on overparametrization. It yields a simple closed-form description of complex-shaped bounding sets in a higher dimensional hyperspace S^* and allows determining the corresponding parameters of the bounding sets analytically for the measurement step. A complex-shaped bounding set $\mathcal{X}_k$ in the original space S is represented by a simpler shaped set $\mathcal{X}_k^*$ and an associated transformation $\boldsymbol{T}(\cdot)$ in the higher dimensional hyperspace S^*. The most challenging problem in terms of computational complexity is the determination of characteristic values of the complicated uncertainty $\mathcal{X}_k$ by means of an inverse transformation, which, in general, requires numerical calculations.

The second contribution is an *exact* transformation of the given problem into a form which is suitable for the application of the proposed nonlinear filtering concept. For this transformation, the assumption of a bounded-error model was exploited to obtain two simple nonlinear measurement equations for each given measurement equation describing a relative bearing measurement to two landmarks.

In a practical localization example, it was shown that a transformation of the original 2-D state space to an 8-D hyperspace yields an approximation $\mathcal{X}_k^e$ for the true set $\tilde{\mathcal{X}}_k$ of all feasible robot poses that is very close to optimal and outperforms simple approximation schemes based on axis-aligned boxes. Increasing the order of the hyperspace results in even better approximations, but requires additional computational effort.

Future research on localization by means of the proposed nonlinear concept will focus on the question of how to obtain a closed-form solution for nonlinear system models and how system noise can be incorporated in the proposed approach.

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