Model Predictive Control for Linear and Hybrid Systems.
Model Predictive Control

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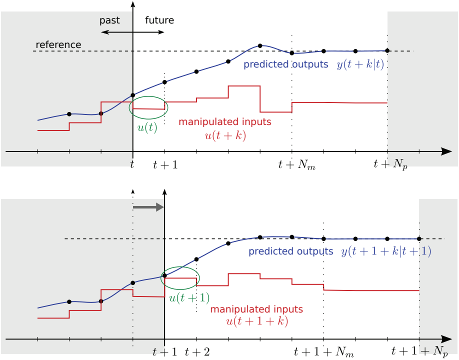
Outline

Introduction



1. At each sampling time, solve a ***CFTOC***.
2. Apply the optimal input ***only during*** $[t, t+1]$
3. At $t+1$ solve a CFTOC over a ***shifted horizon*** based on new state measurements
4. The resultant controller is referred to as ***Receding Horizon Controller (RHC)*** or ***Model Predictive Controller (MPC)***.

An Historical Perspective

1. ***A. I. Propoi, 1963***, "Use of linear programming methods for synthesizing sampled-data automatic systems", *Automation and Remote Control.*

2. ***J. Richalet et al., 1978*** "Model predictive heuristic control- application to industrial processes". *Automatica*, 14:413-428.
    - known as ***IDCOM (Identification and Command)***
    - impulse response model for the plant, linear in inputs or internal variables (***only stable plants***)
    - quadratic performance objective over a finite prediction horizon
    - future plant output behavior specified by a reference trajectory
    - ***ad hoc*** input and output constraints
    - optimal inputs computed using a heuristic iterative algorithm, interpreted as the dual of identification
    - controller was not a transfer function, hence called ***heuristic***

An Historical Perspective

3. ***C. R. Cutler, B. L. Ramaker, 1979*** "Dynamic matrix control – a computer control algorithm". *AICHE national meeting*, Houston, TX.
   - successful in the petro-chemical industry (***Shell Oil***)
   - linear step response model for the plant
   - quadratic performance objective over a finite prediction horizon
   - future plant output behavior specified by trying to follow the set-point as closely as possible
   - input and output constraints included in the formulation
   - optimal inputs computed as the solution to a least–squares problem
   - ***ad hoc*** input and output constraints. Additional equation added online to account for constraints. Hence a ***dynamic matrix*** in the least squares problem.
4. ***C. Cutler, A. Morshedi, J. Haydel, 1983***. "An industrial perspective on advanced control".*AICHE annual meeting*, Washington, DC.
   - Standard QP problem formulated in order to systematically account for constraints.

An Historical Perspective

In summary

5. A vast variety of different names and methodologies, such as Adaptive Predictive Control (APC), Generalized Predictive Control (GPC), Sequential Open Loop Optimization (SOLO) and others.
6. They all share the same structural features: a model of the plant, the idea and an optimization procedure to obtain the control action by optimizing the system's predicted evolution.
7. In ***IDCOM*** and ***DMC*** input and output constraints were treated in an indirect ad-hoc fashion.
8. ***QDMC*** overcame this limitation by employing ***quadratic programming*** to solve constrained MPC problems with quadratic performance indices.
9. Later an extensive theoretical effort devoted to provide conditions for guaranteeing ***feasibility*** and ***closed-loop stability***.

Simple RHC Design

Problem: regulating to the origin the discrete-time linear time-invariant system

$$\left\{ \begin{array}{rcl} x(t+1) & = & Ax(t) + Bu(t) \\ y(t) & = & Cx(t) \end{array} \right. , \qquad \begin{array}{c} x(t) \in \mathcal{X}, \quad u(t) \in \mathcal{U}, \\ \forall t \geq 0 \end{array}$$

where the sets $\mathcal{X} \subseteq \mathbb{R}^n$ and $\mathcal{U} \subseteq \mathbb{R}^m$ are polyhedra.

Assume $x(t)$ ***is available*** and solve the CFTOC problem

$$\begin{array}{rll} J_t^*(x(t)) = & \min\limits_{U_{t\rightarrow t+N|t}} & J_t(x(t), U_{t\rightarrow t+N|t}) \triangleq p(x_{t+N|t}) + \sum\limits_{k=0}^{N-1} q(x_{t+k|t}, u_{t+k|t}) \\ & \text{subj. to} & x_{t+k+1|t} = Ax_{t+k|t} + Bu_{t+k|t},\ k = 0, \ldots, N-1 \\ & & x_{t+k|t} \in \mathcal{X},\ u_{t+k|t} \in \mathcal{U},\ k = 0, \ldots, N-1 \\ & & x_{t+N|t} \in \mathcal{X}_f \\ & & x_{t|t} = x(t) \end{array}$$

with $U_{t\rightarrow t+N|t} = \{u_{t|t}, \ldots, u_{t+N-1|t}\}$.

Simple RHC Design

- Let $U^*_{t\rightarrow t+N|t} = \{u^*_{t|t}, \ldots, u^*_{t+N-1|t}\}$ be the optimal solution. The first element of $U^*_{t\rightarrow t+N|t}$ is applied to system

$$u(t) = u^*_{t|t}(x(t)).$$

- The CFTOC problem is reformulated and solved at time $t+1$, based on the new state $x_{t+1|t+1} = x(t+1)$.
- ***Closed loop trajectories***. Denote by $f_t(x(t)) = u^*_{t|t}(x(t))$ the receding horizon control law when the current state is $x(t)$. Then, the closed loop system obtained by controlling the system with the RHC state feedback control law

$$x(t+1) = Ax(t) + Bf_t(x(t)) \triangleq f_{cl}(x(t)),\ t \geq 0$$

Notation

- Note that $x_{t+k|t}$ is the state vector at time $t+k$, predicted at time t obtained by starting from the current state $x_{t|t} = x(t)$ and applying to the system model
$$x_{t+k+1|t} = Ax_{t+k|t} + Bu_{t+k|t}$$
the input sequence $u_{t|t}, \ldots, u_{t+N-1|t}$.
- For instance, $x_{3|1}$ represents the predicted state at time 3 when the prediction is done at time $t = 1$ starting from the current state $x(1)$. It is different, in general, from $x_{3|2}$ which is the predicted state at time 3 when the prediction is done at time $t = 2$ starting from the current state $x(2)$.
- Similarly $u_{t+k|t}$ is read as "the input u at time $t+k$ computed at time t".

Notation

Note that the *problem is time-invariant* $\Longrightarrow f_t(x(t))$ is a time-invariant function of the initial state $x(t)$. Rewrite the problem as

$$
\begin{array}{lll}
J_0^*(x(t)) = & \min\limits_{U_0} & J_0(x(t), U_0) \triangleq p(x_N) + \sum\limits_{k=0}^{N-1} q(x_k, u_k) \\
& \text{subj. to} & \\
& & x_{k+1} = Ax_k + Bu_k,\ k = 0, \ldots, N-1 \\
& & x_k \in \mathcal{X},\ u_k \in \mathcal{U},\ k = 0, \ldots, N-1 \\
& & x_N \in \mathcal{X}_f \\
& & x_0 = x(t)
\end{array}
$$

where $U_0 = \{u_0, \ldots, u_{N-1}\}$.

The control law and closed loop system are time-invariant as well.

Moreover, $\mathcal{X}_0$ denotes the set of feasible states $x(t)$ for the problem.

RHC Algorithm

Algorithm (On-line receding horizon control)

1. *MEASURE the state* $x(t)$ *at time instance* t
2. *OBTAIN* $U_0^*(x(t))$ *by solving the optimization problem (2)*
3. *IF* $U_0^*(x(t)) = \emptyset$ *THEN 'problem infeasible' STOP*
4. *APPLY the first element* u_0^* *of* U_0^* *to the system*
5. *WAIT for the new sampling time* $t+1$*, GOTO (1.)*

Note that, we can make use of **all** the results studied so far.

RHC Example

Consider the ***double integrator***

$$\begin{cases} x(t+1) &= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \\ y(t) &= \begin{bmatrix} 1 & 0 \end{bmatrix} x(t) \end{cases}$$

subject to the ***input constraints***

$$-0.5 \leq u(t) \leq 0.5,\ t = 0, \ldots, 3$$

and the ***state constraints***

$$\begin{bmatrix} -5 \\ -5 \end{bmatrix} \leq x(t) \leq \begin{bmatrix} 5 \\ 5 \end{bmatrix},\ t = 0, \ldots, 3.$$

Compute a receding horizon controller with

$$p = 2,\ N = 3,\ P = Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ R = 10,\ \mathcal{X}_f = \mathbb{R}^2$$

Example. Double Integrator

The QP problem associated with the RHC is

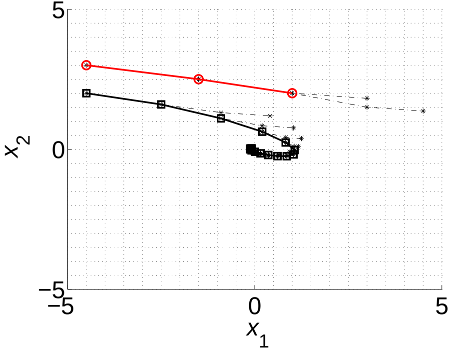
$$H = \begin{bmatrix} 13.50 & -10.00 & -0.50 \\ -10.00 & 22.00 & -10.00 \\ -0.50 & -10.00 & 31.50 \end{bmatrix},\ F = \begin{bmatrix} -10.50 & 10.00 & -0.50 \\ -20.50 & 10.00 & 9.50 \end{bmatrix},\ Y = \begin{bmatrix} 14.50 & 23.50 \\ 23.50 & 54.50 \end{bmatrix}$$

$$G_0 = \begin{bmatrix} 0.50 & -1.00 & 0.50 \\ -0.50 & 1.00 & -0.50 \\ -0.50 & 0.00 & 0.50 \\ -0.50 & 0.00 & -0.50 \\ 0.50 & 0.00 & -0.50 \\ 0.50 & 0.00 & 0.50 \\ -1.00 & 0.00 & 0.00 \\ 0.00 & -1.00 & 0.00 \\ 1.00 & 0.00 & 0.00 \\ 0.00 & 1.00 & 0.00 \\ 0.00 & 0.00 & -1.00 \\ 0.00 & 0.00 & 1.00 \\ 0.00 & 0.00 & 0.00 \\ -0.50 & 0.00 & 0.50 \\ 0.00 & 0.00 & 0.00 \\ 0.50 & 0.00 & -0.50 \\ -0.50 & 0.00 & 0.50 \\ 0.50 & 0.00 & -0.50 \\ 0.00 & 0.00 & 0.00 \\ 0.00 & 0.00 & 0.00 \\ 0.00 & 0.00 & 0.00 \\ 0.00 & 0.00 & 0.00 \end{bmatrix},\ E_0 = \begin{bmatrix} 0.50 & 0.50 \\ -0.50 & -0.50 \\ 0.50 & 0.50 \\ -0.50 & -0.50 \\ -0.50 & -0.50 \\ 0.50 & 0.50 \\ 0.00 & 0.00 \\ 0.00 & 0.00 \\ 0.00 & 0.00 \\ 0.00 & 0.00 \\ 0.00 & 0.00 \\ 0.00 & 0.00 \\ 1.00 & 1.00 \\ -0.50 & -0.50 \\ -1.00 & -1.00 \\ 0.50 & 0.50 \\ -0.50 & -1.50 \\ 0.50 & 1.50 \\ 1.00 & 0.00 \\ 0.00 & 1.00 \\ -1.00 & 0.00 \\ 0.00 & -1.00 \end{bmatrix},\ W_0 = \begin{bmatrix} 0.50 \\ 0.50 \\ 5.00 \\ 5.00 \\ 5.00 \\ 5.00 \\ 5.00 \\ 5.00 \\ 5.00 \\ 5.00 \\ 0.50 \\ 0.50 \\ 5.00 \\ 5.00 \\ 5.00 \\ 5.00 \\ 0.50 \\ 0.50 \\ 5.00 \\ 5.00 \\ 5.00 \\ 5.00 \end{bmatrix}$$

Example. Double Integrator

The RHC algorithm becomes

1. MEASURE the state $x(t)$ at time instance t
2. COMPUTE $\tilde{F} = 2x'(t)F$ and $\tilde{W}_0 = W_0 + E_0 x(t)$
3. OBTAIN $U_0^*(x(t))$ by solving the optimization problem $[U_0^*, \mathrm{Flag}] = \mathrm{QP}(H, \tilde{F}, G_0, \tilde{W}_0)$
4. IF Flag='infeasible' THEN STOP
5. APPLY the first element u_0^* of U_0^* to the system
6. WAIT for the new sampling time $t+1$, GOTO (1.)



Example. Double Integrator

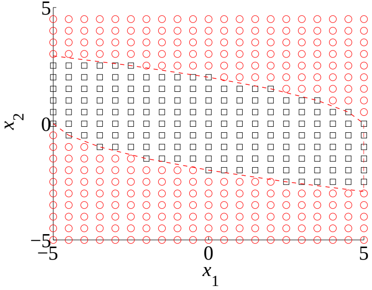


Figure: *Boxes* (*Circles*) are initial points leading (not leading) to feasible closed-loop trajectories

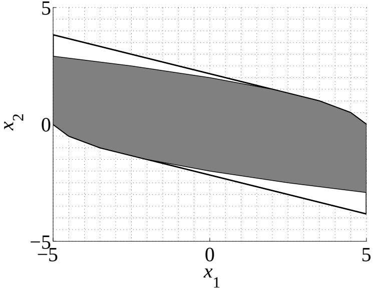


Figure: Maximal positive invariant set $\mathcal{O}_\infty$ (grey) and set of initial feasible states $\mathcal{X}_0$ (white and gray)

Example. Double Integrator

Note that $x(0) \notin \mathcal{O}_\infty \Longrightarrow x(2) \notin \mathcal{X}_0$ although the state is feasible at time 0. Because of the nonlinear nature of f_0, the computation of $\mathcal{O}_\infty$ is not an easy task. Therefore we will show how to choose a terminal invariant set $\mathcal{X}_f$ such that $\mathcal{O}_\infty = \mathcal{X}_0$ is guaranteed automatically.

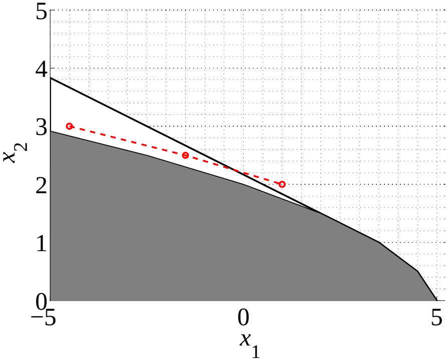


Figure: Maximal positive invariant set $\mathcal{O}_\infty$ (grey) and set of initial feasible states $\mathcal{X}_0$ (white and grey). The initial condition $x(0) = [-4.5, 3]$ belongs to $\mathcal{X}_0 \setminus \mathcal{O}_\infty$.

Example. Unstable System

Consider the ***unstable system***

$$\left\{ \begin{array}{rcl} x(t+1) & = & \begin{bmatrix} 2 & 1 \\ 0 & 0.5 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t) \end{array} \right.$$

subject to the ***input constraints***

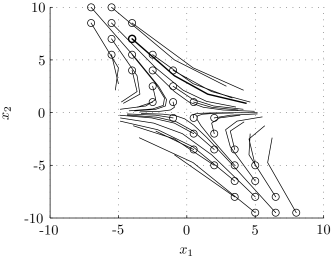
$$-1 \leq u(k) \leq 1,\ k = 0, \ldots, N-1$$

and the ***state constraints***

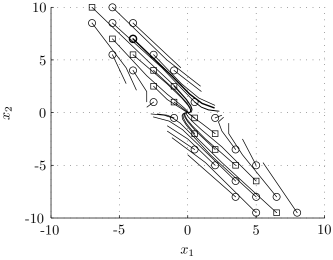
$$\begin{bmatrix} -10 \\ -10 \end{bmatrix} \leq x(k) \leq \begin{bmatrix} 10 \\ 10 \end{bmatrix},\ k = 0, \ldots, N-1.$$

Solve the receding horizon control problem with $p(x_N) = x_N' P x_N$, $q(x_k, u_k) = x_k' Q x_k + u_k' R u_k$, for different horizons N and weights R. (set $Q = I$, $\mathcal{X}_f = \mathbb{R}^2$, $P = 0$)

Example. Unstable System



$N = 2, R = 10$



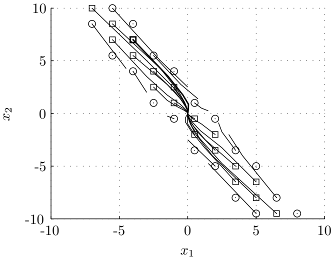
$N = 3, R = 2$

Boxes (*Circles*) are initial points leading (not leading) to feasible closed-loop trajectories

Setting 1: $N = 2, R = 10$

Setting 2: $N = 3, R = 2$

Setting 3: $N = 4, R = 1$



$N = 4, R = 1$

Example. Unstable System

1. ***Setting 1***. No initial state can be steered to the origin.
2. ***Setting 2***. Some of the initial states converge to the origin.
3. ***Setting 3*** expands the set of initial states that can be brought to the origin.

- The choice of parameters influences the behavior of the resulting closed-loop trajectories in a complex manner.
- For a better understanding of the effects of parameter changes inspect the maximal positive invariant sets $\mathcal{O}_\infty$ and the maximal control invariant $\mathcal{C}_\infty$

Example. Unstable System

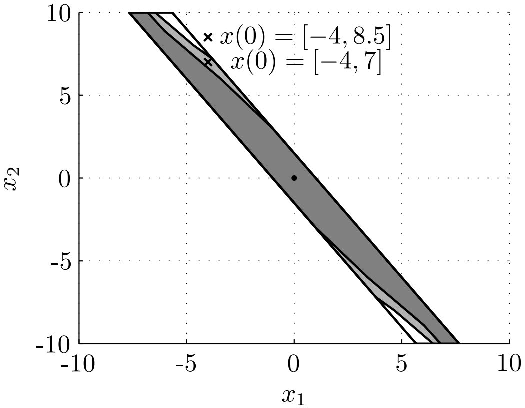


Figure: Maximal positive invariant sets $\mathcal{O}_\infty$ for different parameter settings: *Setting 1* (origin), *Setting 2* (dark-gray) and *Setting 3* (gray and dark-gray). Also depicted is the maximal control invariant set $\mathcal{C}_\infty$ (white and gray and dark-gray).

Summary

Remark. If we solve the RHC problem for $N = \infty$ (as done for LQR), then open loop trajectories are the same as the closed loop trajectories. Hence

- If the problem is feasible, the closed loop trajectories will be always feasible
- If the cost is finite, the states and inputs will converge asymptotically to the origin

The RHC is a "short-sight" strategy that mimics the infinite horizon controller. ***But***

1. ***Feasibility***. After a number of steps the finite horizon optimal control problem becomes infeasible.
   Note that infeasibility can occur even in absence of disturbances and model mismatch.
2. ***Stability***. Even if the feasibility problem does not occur, the generated control inputs may not lead to trajectories that converge to the origin,

Summary

Conditions will be derived on how the terminal weight P and the terminal constraint set $\mathcal{X}_f$ should be chosen such that **closed-loop stability and feasibility are ensured**.

Outline

Feasibility of RHC

Desirable Property:

Definition (*Persistent Feasibility*)

The RHC problem is persistently feasible if for all initial states $x(0) \in \mathcal{X}_0$, feasibility **for all future times** is guaranteed.

Recall Some Definitions

$\mathcal{C}_\infty$: only affected by the sets $\mathcal{X}$ and $\mathcal{U}$. It is the largest set over which we can expect *any* controller to work.

$\mathcal{X}_0$: The control problem is feasible, if $x(0) \in \mathcal{X}_0$. It depends on $\mathcal{X}$ and $\mathcal{U}$, on the controller horizon N and on the controller terminal set $\mathcal{X}_f$. It does not depend on the objective function and it has generally no relation with $\mathcal{C}_\infty$ (it can be larger, smaller, etc.).

$\mathcal{O}_\infty$: It depends on the controller and as such on all parameters affecting the controller, i.e., $\mathcal{X}$, $\mathcal{U}$, N, $\mathcal{X}_f$ and the objective function with its parameters P, Q and R. Clearly $\mathcal{O}_\infty \subseteq \mathcal{X}_0$. Clearly the closed-loop is persistently feasible for all states $x(0) \in \mathcal{O}_\infty$.

Feasibility of RHC

Lemma

Let $\mathcal{O}_\infty$ be the maximal positive invariant set for the closed-loop system $x(k+1) = f_{cl}(x(k))$ The RHC problem is persistently feasible if and only if $\mathcal{X}_0 = \mathcal{O}_\infty$

Proof: For the RHC problem to be persistently feasible $\mathcal{X}_0$ must be positive invariant for the closed-loop system. We argued above that $\mathcal{O}_\infty \subseteq \mathcal{X}_0$. As the positive invariant set $\mathcal{X}_0$ cannot be larger than the maximal positive invariant set $\mathcal{O}_\infty$, it follows that $\mathcal{X}_0 = \mathcal{O}_\infty$. □

Observe that

- $\mathcal{X}_0$ does not depend on the controller parameters P, Q and R but $\mathcal{O}_\infty$ does
- The requirement $\mathcal{X}_0 = \mathcal{O}_\infty$ implies that only some P, Q and R are allowed.
- The parameters P, Q and R affect the performance. This makes their choice extremely difficult for the design engineer.

In the following we will study *sufficient* conditions for persistent feasibility.

Feasibility of RHC

Theorem

If $\mathcal{X}_1$ is a control invariant set then the RHC with $N \geq 1$ is persistently feasible. Also, $\mathcal{O}_\infty$ is independent of P, Q and R.

Proof: If $\mathcal{X}_1$ is control invariant then, by definition, $\mathcal{X}_1 \subseteq \text{Pre}(\mathcal{X}_1)$. Also recall that $\text{Pre}(\mathcal{X}_1) = \mathcal{X}_0$. Pick some $x \in \mathcal{X}_0$ and some feasible control u for that x and define $x^+ = Ax + Bu \in \mathcal{X}_1$. Then $x^+ \in \mathcal{X}_1 \subseteq \text{Pre}(\mathcal{X}_1) = \mathcal{X}_0$. As u was arbitrary (as long as it is feasible) $x^+ \in \mathcal{X}_0$ for all feasible u. As $\mathcal{X}_0$ is positive invariant, $\mathcal{X}_0 = \mathcal{O}_\infty$ from Lemma 2. As $\mathcal{X}_0$ is positive invariant for all feasible u, $\mathcal{O}_\infty$ does not depend on P, Q and R. $\square$

- Use Theorem 3 as follows. For $N = 1$, $\mathcal{X}_1 = \mathcal{X}_f$. So if we choose $\mathcal{X}_f$ to be control invariant then $\mathcal{X}_0 = \mathcal{O}_\infty$ and RHC will be persistently feasible independent of chosen control objectives and parameters.
- A control horizon of $N = 1$ is often too restrictive...

Feasibility of RHC

Corollary

If $\mathcal{X}_f$ is a control invariant set then the RHC is persistently feasible.

Proof: If $\mathcal{X}_f$ is control invariant, then $\mathcal{X}_{N-1}, \mathcal{X}_{N-2}, \ldots, \mathcal{X}_1$ are control invariant and Lemma 3 establishes persistent feasibility. □

Recall the properties of the set $\mathcal{X}_0$ as N varies. Therefore, the previous results provide also guidelines on the choice of the horizon N for guaranteeing persistent feasibility.

Corollary

If N is greater than the determinedness index $\bar{N}$ of $\mathcal{K}_\infty(\mathcal{X}_f)$ then the RHC is persistently feasible.

Proof: The feasible set $\mathcal{X}_i$ for $i = 1, \ldots, N-1$ is equal to the $(N-i)$-step controllable set $\mathcal{X}_i = \mathcal{K}_{N-i}(\mathcal{X}_f)$. If the maximal controllable set is finitely determined then $\mathcal{X}_i = \mathcal{K}_\infty(\mathcal{X}_f)$ for $i \leq N - \bar{N}$. Note that $\mathcal{K}_\infty(\mathcal{X}_f)$ is control invariant. Then persistent feasibility follows from the previous Corollary. □

Feasibility of RHC

Persistent feasibility does not guarantee that the closed-loop trajectories converge towards the desired equilibrium point.
One of the most popular approaches to guarantee persistent feasibility and stability of the RHC law makes use of a control invariant terminal set $\mathcal{X}_f$ and a terminal cost P which drives the closed-loop optimal trajectories towards $\mathcal{X}_f$. A detailed discussion follows.

Outline

Introduction to Stability Issue

- Even if persistent feasibility is guaranteed. The closed loop system might be unstable
- Our objective is to find a Lyapunov function for the closed-loop system.
- We will show next that if terminal cost and constraint are appropriately chosen, then the value function $J_0^*(\cdot)$ is a Lyapunov function.

Stability of RHC. Main Idea

Assume zero terminal constraint $x_N \in \mathcal{X}_f = 0$.
Recall

$$\begin{array}{lll} J^*_{0\to N}(x_0) = & \min_{U_{0\to N}} & J_{0\to N}(x_0, U_{0\to N}) = p(x_N) + \sum_{k=0}^{N-1} q(x_k, u_k) \\ & \text{subj. to} & x_{k+1} = g(x_k, u_k),\ k = 0, \dots, N-1 \\ & & h(x_k, u_k) \leq 0,\ k = 0, \dots, N-1 \\ & & x_N \in \mathcal{X}_f \\ & & x_0 = x(0) \end{array}$$

Assume $p(x) \succ 0$, $q(x,u) \succ 0$

1. At $x(0)$, apply u_0^* and let the system to evolve to $x(1) = g(x(0), u_0^*)$.
2. At $x(1)$, use the *feasible* input sequence $u_1^*, \dots, u_{N-1}^*, 0$. The cost is

$$J^*_{0\to N}(x_0) - q(x_0, u_0) + \overbrace{q(x_{N+1}, 0)}^{=0}.$$

3. Since $u_1^*, \dots, u_{N-1}^*, 0$ is not necessarily optimal

$$J^*_{1\to N+1}(x_1) \leq J^*_{0\to N}(x_0) - q(x_0, u_0).$$

Stability of RHC. Main Idea

4. Since system and cost function are time invariant $J^*_{1\to N+1}(x_1) = J^*_{0\to N}(x_1)$ and

$$J^*_{0\to N}(x_1) \leq J^*_{0\to N}(x_0) - q(x_0, u_0).$$

5. $q \succ 0,\ \forall (x,u) \neq (0,0) \Rightarrow J^*_{0\to N}(x_{k+1}) - J^*_{0\to N}(x_k) < 0\ \forall (x_k) \neq 0$
6. $p(x)$ and $q(x,u) \succ 0 \Rightarrow J^*_{0\to N}(x) \succ 0$
7. $J^*_{0\to N}(x)$ is positive definite and decreasing along the closed loop trajectories. This implies asymptotic convergence to the origin, i.e., $\lim_{k\to\infty} x(k) = 0$.
8. Not yet asymptotic stability. What is missing? (see later)

Stability of RHC. Main Theorem

Theorem

Consider system

$$\left\{ \begin{array}{rcl} x(t+1) & = & Ax(t) + Bu(t) \\ y(t) & = & Cx(t) \end{array} \right. , \quad x(t) \in \mathcal{X},\ u(t) \in \mathcal{U}$$

the RHC law

$$\begin{array}{lll} J_0^*(x(t)) = & \min_{U_0} & J_0(x(t), U_0) \triangleq p(x_N) + \sum_{k=0}^{N-1} q(x_k, u_k) \\ & \text{subj. to} & \\ & & x_{k+1} = Ax_k + Bu_k,\ k = 0, \ldots, N-1 \\ & & x_k \in \mathcal{X},\ u_k \in \mathcal{U},\ k = 0, \ldots, N-1 \\ & & x_N \in \mathcal{X}_f \\ & & x_0 = x(t) \end{array}$$

$$u(t) = u^*_{t|t}(x(t)), \qquad U_0^* = \{u^*_{t|t}, \ldots, u^*_{t+N-1|t}\}$$

and the resulting closed-loop system.

Stability of RHC. Main Theorem

Theorem

Assume that

(A0) $Q = Q' \succ 0$, $R = R' \succ 0$, $P \succ 0$, *if squared Eucledian norm is used, or* Q, R, P *full column rank matrices if* 1 *or* ∞ *norm is used.*

(A1) *The sets* $\mathcal{X}$, $\mathcal{X}_f$ *and* $\mathcal{U}$ *contain the origin in their interior and are closed.*

(A2) $\mathcal{X}_f$ *is control invariant,* $\mathcal{X}_f \subseteq \mathcal{X}$.

(A3) $\min_{v \in \mathcal{U},\ Ax+Bv \in \mathcal{X}_f} (-p(x) + q(x,v) + p(Ax+Bv)) \leq 0, \ \forall x \in \mathcal{X}_f$.

Then,
(i) the state of the closed-loop system converges to the origin, i.e., $\lim_{k\to\infty} x(k) = 0$,
(ii) the origin of the closed-loop system is asymptotically stable with domain of attraction $\mathcal{X}_0$.

Stability of RHC. Main Points of the Proof

1. (A2) guarantees ***persistent feasibility***.

2. Show that $J_0^*(\cdot)$ as a ***Lyapunov function*** for the closed-loop system.

3. ***Time-invariant problem*** $\Longrightarrow$ we study J_0^* between $k = 0$ and $k + 1 = 1$.

4. Be $x(0) \in \mathcal{X}_0$, $U_0^* = \{u_0^*, \dots, u_{N-1}^*\}$, $\mathbf{x}_0 = \{x(0), x_1, \dots, x_N\}$ the corresponding optimal input and state trajectories, respectively. $x(1) = x_1 = Ax(0) + Bu_0^*$

5. At time $t = 1$, be $\tilde{U}_1 = \{u_1^*, \dots, u_{N-1}^*, v\}$ a feasible sequence with v satisfying (A3) and $\tilde{\mathbf{x}}_1 = \{x_1, \dots, x_N, Ax_N + Bv\}$ the corresponding state trajectory. v exists by (A2).

6. $J_0(x(1), \tilde{U}_1)$ is an upper bound on $J_0^*(x(1))$.

Stability of RHC. Main Points of the Proof

7. Trajectories generated by U_0^* and $\tilde{U}_1$ overlap

$$J_0^*(x(1)) \leq J_0(x(1), \tilde{U}_1) = \begin{array}{l} J_0^*(x(0)) - q(x_0, u_0^*) - p(x_N) + \\ +(q(x_N, v) + p(Ax_N + Bv)) \end{array}$$

8. Let $x = x_0 = x(0)$ and $u = u_0^*$. Under assumption (A3)

$$J_0^*(Ax + Bu) - J_0^*(x) \leq -q(x, u), \ \forall x \in \mathcal{X}_0.$$

9. (A0) (on the matrices R and Q) ensures that $J_0^*(x)$ strictly decreases along the state trajectories of the closed loop system for any $x \in \mathcal{X}_0$.

10. In addition to the fact that $J_0^*(x)$ decreases, $J_0^*(x)$ is lower-bounded by zero and since the state trajectories generated by the closed-loop system starting from any $x(0) \in \mathcal{X}_0$ lie in $\mathcal{X}_0$ for all $k \geq 0$, the state converges to zero if the initial state lies in $\mathcal{X}_0$. (i) is proven.

Stability of RHC. Main Points of the Proof

11. It is left to establish that $J_0^*(x)$ is a Lyapunov function.
12. Positivity holds by definition, decrease follows from previous results.
13. For continuity at the origin we will show that $J_0^*(x) \leq p(x),\ \forall x \in \mathcal{X}_f$ and as $p(x)$ is continuous at the origin $J_0^*(x)$ must be continuous as well.
14. From assumption (A2) exists $\{u_0, \ldots, u_{N-1}\}$ starting from the initial state $x_0 = x$ whose corresponding $\{x_0, x_1, \ldots, x_N\}$ stays in $\mathcal{X}_f$,
15. Among all the aforementioned input sequences, focus on the one where u_i satisfies assumption (A3). Such a sequence provides an upper bound on the function J_0^*:

$$J_0^*(x_0) \leq \left(\sum_{i=0}^{N-1} q(x_i, u_i) \right) + p(x_N),\ x_i \in \mathcal{X}_f,\ i = 0, \ldots, N \qquad (1)$$

Stability of RHC. Main Points of the Proof

16. which can be rewritten as

$$\begin{aligned} J_0^*(x_0) \leq & \left(\sum_{i=0}^{N-1} q(x_i, u_i)\right) + p(x_N), \\ & = p(x_0) + \left(\sum_{i=0}^{N-1} q(x_i, u_i) + p(x_{i+1}) - p(x_i)\right) \\ & \quad x_i \in \mathcal{X}_f,\ i = 0, \ldots, N \end{aligned} \tag{2}$$

17. From assumption (A3) we obtain

$$J_0^*(x) \leq p(x),\ \forall x \in \mathcal{X}_f. \tag{3}$$

18. Concluding: there exist a finite time in which any $x \in \mathcal{X}_0$ is steered to a level set of $J_0^*(x)$ contained in $\mathcal{X}_f$ after which convergence to and stability of the origin follows.

Stability of RHC. Remarks

- The assumption on the positive definiteness of Q can be relaxed as in standard optimal control: $Q \succeq 0$ with $(Q^{\frac{1}{2}}, A)$ observable.

- Terminal set $\mathcal{X}_f$ and terminal cost are used to guarantee persistent feasibility and stability. Requiring $x_N \in \mathcal{X}_f$ usually decreases the size of the region of attraction $\mathcal{X}_0 = \mathcal{O}_\infty$.

- In some literature the constraint $\mathcal{X}_f$ is not used. However, it is typically required that the horizon N is sufficiently large to ensure feasibility of the RHC. N has to be greater than the determinedness index $\bar{N}$.
- A function $p(x)$ satisfying assumption (A3) of the Theorem is often called control Lyapunov function.

Stability of RHC. 2-Norm case

- ***In general***
  - choose $\mathcal{X}_f$ as the maximal positive invariant set for $x(k+1) = (A + BF_\infty)x(k)$ where F_∞ is the unconstrained infinite-time LQR.
  - With this choice (A3) becomes

$$x'(A'(P - PB(B'PB + R)^{-1}BP)A + Q - P)x \le 0, \ \forall x \in \mathcal{X}_f$$

    which is satisfied if P is chosen as the solution P_∞ of the the standard algebraic Riccati equation.

- ***If system is asymptotically stable***, then
  - $\mathcal{X}_f$ can be chosen as the positively invariant set of the autonomous system $x(k+1) = Ax(k)$, $x \in \mathcal{X}$. Therefore in $\mathcal{X}_f$ the input $\mathbf{0}$ is feasible and the assumption
  - (A3) becomes

$$-x'Px + x'A'PAx + x'Qx \le 0, \forall x \in \mathcal{X}_f$$

    which is satisfied if P solves $x'(-P + A'PA + Q)x \le 0$, i.e. the standard Lyapunov equation.

Stability of RHC. 1, ∞-Norm case

Let $p = 1$ or $p = \infty$.

- ***In general***
  - If the unconstrained optimal controller exists, it is PPWA. The computation of the maximal invariant set $\mathcal{X}_f$ for the closed loop PWA system

    $$x(k+1) = (A + F^i)x(k) \quad \text{if} \quad H^i x \leq 0, \; i = 1, \ldots, N^r$$

    is more involved.
  - If $\mathcal{X}_f$ is used as terminal constraint, (A3) is satisfied by the infinite time unconstrained optimal cost P_∞.

- ***If system is asymptotically stable***, then
  - $\mathcal{X}_f$ can be chosen as the positively invariant set of the autonomous system $x(k+1) = Ax(k)$, $x \in \mathcal{X}$. Therefore in $\mathcal{X}_f$ the input $\mathbf{0}$ is feasible
  - (A3) becomes

    $$-\|Px\|_p + \|PAx\|_p + \|Qx\|_p \leq 0, \forall x \in \mathcal{X}_f$$

    which is the corresponding Lyapunov equation for the 1, ∞-norm case.

Outline

1. Fundamentals of Predictive Control
2. Feasibility
3. Stability
4. RHC Implementation
   - Reducing the Problem Size
   - Reference Tracking
   - Delta Input Formulation
   - Integral Action and Anti-Windup
   - RHC Extensions

RHC Implementation

1. ***Off-line:*** Setup cost and constraints of the optimization problem associated to the BATCH approach
   On-line: Solve the optimization problem for the current $x(t)$
2. ***Off-line:*** Compute the cost to go $J^*_{1\to N}(x)$ and the corresponding feasible set $\mathcal{X}_{1\to N}$
   On-line: Solve

$$
\begin{array}{lll}
J^*_0(x) = & \min_{u_0} q(x_0,u_0) + J^*_{1\to N}(x_1) & \\
 & \text{subj. to} & x_1 = Ax_0 + Bu_0 \\
 & & x_1 \in \mathcal{X}_{1\to N} \\
 & & x_0 = x(t)
\end{array}
$$

   for the current $x(t)$
3. ***Off-line:*** Compute the explicit solution by using multiparametric progarmming
   On-line: Evaluate the look-up table

$$
u^*_0 = F^i_0 x_0 + g^i_0 \quad \text{if} \quad H^i_0 x_0 \le K^i_0, \quad i = 1,\dots,N^r_0
$$

   for the current $x_0 = x(t)$. Note that critical regions with the same first component can be merged.

Reducing the Problem Size

In order to reduce the size of the optimization problem modify the problem as:

$$
\begin{array}{ll}
\min\limits_{U_{t\to t+N|t}} & \left\{ \|Px'_{N_y}\|_p + \sum\limits_{k=0}^{N_y-1} [\|Qx_k\|_p + \|Ru_k\|_p] \right\} \\
\text{subj. to} & y_{\min} \le y_k \le y_{\max},\ k = 1, \dots, N_c \\
& u_{\min} \le u_k \le u_{\max},\ k = 0, 1, \dots, N_u \\
& x_0 = x(t) \\
& x_{k+1} = Ax_k + Bu_k,\ k \ge 0 \\
& y_k = Cx_k,\ k \ge 0 \\
& u_k = Kx_k,\ N_u \le k < N_y
\end{array}
$$

where K is some feedback gain, N_y, N_u, N_c are the output, input, and constraint horizons, respectively, with $N_u \le N_y$ and $N_c \le N_y - 1$.
As long as the control task can be expressed as an mp-QP or mp-LP, a piecewise affine controller results

RHC Reference Tracking

Consider the discrete-time, time-invariant system

$$\begin{cases} x_m(t+1) &= f(x_m(t), u(t)) \\ y_m(t) &= g(x_m(t)) \\ z(t) &= H y_m(t) \end{cases}$$

where

- $x_m(t) \in \mathbb{R}^n$, state variables
- $u(t) \in \mathbb{R}^m$, control inputs
- $y_m(t) \in \mathbb{R}^p$, measured output
- $z(t) \in \mathbb{R}^r$, controlled variables.

Assume H to have full row rank.

Objective

Designing an RHC in order to have $z(t)$ tracking the reference signal $r(t)$, with $r(t) \to r_\infty$ as $t \to \infty$. Moreover, we require ***zero steady-state*** tracking error, i.e., $(z(t) - r(t)) \to 0$ for $t \to \infty$.

RHC Reference Tracking

Consider the augmented model

$$\begin{cases} x(t+1) &= Ax(t) + Bu(t) + B_d d(t) \\ d(t+1) &= d(t) \\ y(t) &= Cx(t) + C_d d(t) \end{cases}$$

with $d(t) \in \mathbb{R}^{n_d}$.

Theorem

The augmented system is observable if and only if (C, A) is observable and

$$\begin{bmatrix} A - I & B_d \\ C & C_d \end{bmatrix} \text{ has full column rank.}$$

Observe that, at steady state

$$\begin{bmatrix} A - I & B_d \\ C & C_d \end{bmatrix} \begin{bmatrix} x_\infty \\ d_\infty \end{bmatrix} = \begin{bmatrix} 0 \\ y_\infty \end{bmatrix}$$

that is, given y_∞, d_∞ must be uniquely determined. (Note that the steady state values are denoted with a subscript ∞ and the forcing term u has been omitted for simplicity)

RHC Reference Tracking

Based on the augmented model, design the state observer

$$\begin{bmatrix} \hat{x}(t+1) \\ \hat{d}(t+1) \end{bmatrix} = \begin{bmatrix} A & B_d \\ 0 & I \end{bmatrix} \begin{bmatrix} \hat{x}(t) \\ \hat{d}(t) \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u(t) + \begin{bmatrix} L_x \\ L_d \end{bmatrix} (-y_m(t) + C\hat{x}(t) + C_d\hat{d}(t))$$

Lemma

If the observer is stable, then $\mathrm{rank}(L_d) = n_d$.

Lemma

Suppose the observer is stable and $n_d = p$. *The observer steady state satisfies:*

$$\begin{bmatrix} A - I & B \\ C & 0 \end{bmatrix} \begin{bmatrix} \hat{x}_\infty \\ u_\infty \end{bmatrix} = \begin{bmatrix} -B_d\hat{d}_\infty \\ y_{m,\infty} - C_d\hat{d}_\infty \end{bmatrix}.$$

where $y_{m,\infty}$ *and* u_∞ *are the steady state measured outputs and inputs.*

RHC Reference Tracking

Observe that for offset-free tracking at steady state we want $z_\infty = r_\infty$. The observer condition

$$\begin{bmatrix} A-I & B \\ C & 0 \end{bmatrix} \begin{bmatrix} \hat{x}_\infty \\ u_\infty \end{bmatrix} = \begin{bmatrix} -B_d\hat{d}_\infty \\ y_{m,\infty} - C_d\hat{d}_\infty \end{bmatrix}$$

suggests that at steady state the MPC should satisfy

$$\begin{bmatrix} A-I & B \\ HC & 0 \end{bmatrix} \begin{bmatrix} x_\infty \\ u_\infty \end{bmatrix} = \begin{bmatrix} -B_d\hat{d}_\infty \\ r_\infty - HC_d\hat{d}_\infty \end{bmatrix}$$

RHC Reference Tracking

Formulate the RHC problem

$$
\begin{array}{ll}
\min\limits_{U_0} & (x_N - \bar{x}_t)' P (x_N - \bar{x}_t) \\
& + \sum\limits_{k=0}^{N-1} (x_k - \bar{x}_t)' Q (x_k - \bar{x}_t) + (u_k - \bar{u}_t)' R (u_k - \bar{u}_t) \\
\\
\text{subj. to} & x_{k+1} = A x_k + B u_k + B_d d_k, \; k = 0, \ldots, N \\
& x_k \in \mathcal{X}, \; u_k \in \mathcal{U}, \; k = 0, \ldots, N-1 \\
& x_N \in \mathcal{X}_f \\
& d_{k+1} = d_k, \; k = 0, \ldots, N \\
& x_0 = \hat{x}(t) \\
& d_0 = \hat{d}(t),
\end{array}
$$

with the targets $\bar{u}_t$ and $\bar{x}_t$ given by

$$
\begin{bmatrix} A - I & B \\ HC & 0 \end{bmatrix} \begin{bmatrix} \bar{x}_t \\ \bar{u}_t \end{bmatrix} = \begin{bmatrix} -B_d \hat{d}(t) \\ r(t) - HC_d \hat{d}(t) \end{bmatrix}
$$

RHC Reference Tracking

Denote by $c_0(\hat{x}(t), \hat{d}(t), r(t)) = u_0^*(\hat{x}(t), \hat{d}(t), r(t))$ the control law when the estimated state and disturbance are $\hat{x}(t)$ and $\hat{d}(t)$, respectively.

Theorem

Consider the case $n_d = p = r$. Assume the RHC persistent feasible and unconstrained for $t \geq j$ with $j \in \mathbb{N}^+$ and the closed-loop system

$$\begin{array}{rcl} x(t+1) & = & f(x(t), c_0(\hat{x}(t), \hat{d}(t), r(t))) \\ \hat{x}(t+1) & = & (A + L_x C)\hat{x}(t) + (B_d + L_x C_d)\hat{d}(t) \\ & + & Bc_0(\hat{x}(t), \hat{d}(t), r(t)) - L_x y_m(t) \\ \hat{d}(t+1) & = & L_d C\hat{x}(t) + (I + L_d C_d)\hat{d}(t) - L_d y_m(t) \end{array}$$

converges to $\hat{x}_\infty$, $\hat{d}_\infty$, $y_{m,\infty}$, i.e., $\hat{x}(t) \to \hat{x}_\infty$, $\hat{d}(t) \to \hat{d}_\infty$, $y_m(t) \to y_{m,\infty}$ as $t \to \infty$.

Then $z(t) = Hy_m(t) \to r_\infty$ as $t \to \infty$.

RHC Reference Tracking

Main idea of the proof:
The asymptotic values $\hat{x}_\infty$, $\bar{x}_\infty$, u_∞ and $\bar{u}_\infty$ satisfy the observer conditions

$$\begin{bmatrix} A-I & B \\ C & 0 \end{bmatrix} \begin{bmatrix} \hat{x}_\infty \\ u_\infty \end{bmatrix} = \begin{bmatrix} -B_d \hat{d}_\infty \\ y_{m,\infty} - C_d \hat{d}_\infty \end{bmatrix} \tag{4}$$

and the controller requirement

$$\begin{bmatrix} A-I & B \\ HC & 0 \end{bmatrix} \begin{bmatrix} \bar{x}_\infty \\ \bar{u}_\infty \end{bmatrix} = \begin{bmatrix} -B_d \hat{d}_\infty \\ r_\infty - HC_d \hat{d}_\infty \end{bmatrix} \tag{5}$$

Define $\delta x = \hat{x}_\infty - \bar{x}_\infty$, $\delta u = u_\infty - \bar{u}_\infty$ and the offset $\varepsilon = z_\infty - r_\infty$. **Need to prove all δ go to zero at steady-state.**

RHC Reference Tracking

Question: How do we choose the matrices B_d and C_d?

Corollary

The augmented system with $n_d = p$ and $C_d = I$ is observable if and only if (C, A) is observable and

$$det \begin{bmatrix} A - I & B_d \\ C & I \end{bmatrix} = det(A - I - B_d C) \neq 0.$$

Remark

If the plant has no integrators, then $\det(A - I) \neq 0$ and we can choose $B_d = 0$. If the plant has integrators then B_d has to be chosen specifically to make $\det(A - I - B_d C) \neq 0$.

Delta Input Formulation

Rewrite the system dynamics as

$$\begin{cases} x(t+1) &= Ax(t) + Bu(t) \\ u(t) &= u(t-1) + \delta u(t) \\ y(t) &= Cx(t) \end{cases}$$

The RHC problem becomes

$$\begin{array}{ll} \min\limits_{\delta u_0,\ldots,\delta u_{N-1}} & \|y_k - r_k\|_Q^2 + \|\delta u_k\|_R^2 \\ \text{subj. to} & Ex_k + Lu_k \le M,\ k = 0,\ldots,N-1 \\ & x_{k+1} = Ax_k + Bu_k,\ k \ge 0 \\ & y_k = Cx_k\ \ k \ge 0 \\ & u_k = u_{k-1} + \delta u_k,\ k \ge 0 \\ & u_{-1} = \hat{u}(t) \\ & x_0 = \hat{x}(t) \end{array}$$

The control input applied to the system is

$$u(t) = \delta u_0^* + u(t-1).$$

Delta Input Formulation

The actual control $\hat{u}(t)$ is estimated by the observer

$$\begin{bmatrix} \hat{x}(t+1) \\ \hat{u}(t+1) \end{bmatrix} = \begin{bmatrix} A & B \\ 0 & I \end{bmatrix} \begin{bmatrix} \hat{x}(t) \\ \hat{u}(t) \end{bmatrix} + \begin{bmatrix} B \\ I \end{bmatrix} \delta u(t) + \begin{bmatrix} L_x \\ L_u \end{bmatrix} (-y_m(t) + C\hat{x}(t))$$

Remark

This scheme inherently achieves offset-free control, there is no need to add a disturbance model.

Integral Action and Anti-Windup

Consider the system

$$\begin{aligned} x(t+1) &= ax(t) + u(t) + d(t), \\ y(t) &= x(t). \end{aligned}$$

with input constraints:

$$|u(t)| \leq 1.$$

The reference value is assumed to be constant and equal to 0. The goal is to design a controller which achieves zero offset, i.e. $y(t) \to 0$ as $t \to \infty$.

Integral Action and Anti-Windup

The MPC is formulated as follows

$$
\begin{array}{ll}
\min_{u_0} & (u_0 - \bar{u}_t)^2 + (x_1 - \bar{x}_t)^2 \\
\text{subj. to} & -1 \leq u_{0,t} \leq 1 \\
& x_1 = a\hat{x}(t) + u_{0,t} + \hat{d}(t) \\
& \begin{bmatrix} a-1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \bar{x}_t \\ \bar{u}_t \end{bmatrix} = \begin{bmatrix} -\hat{d}(t) \\ 0 \end{bmatrix}
\end{array}
$$

The closed form solution can be easily computed:

$$
u^\star(t) = \begin{cases} 1, & -\hat{d}(t) - \frac{1}{2}a\hat{x}(t) > 1, \\ -1, & -\hat{d}(t) - \frac{1}{2}a\hat{x}(t) < 1, \\ -\hat{d}(t) - \frac{a}{2}\hat{x}(t) & \text{otherwise} \end{cases}
$$

Note that $K_{MPC} = -\frac{a}{2}$.

Integral Action and Anti-Windup

Any stabilizing observer will achieve offset-free control. We choose

$$L = -\begin{bmatrix} a \\ 1/4 \end{bmatrix}.$$

The dynamics of the controller in the unconstrained case is thus given by the piecewise affine system

$$\tilde{x}(t+1) = \begin{cases} \tilde{A}_c\tilde{x}(t) - Ly(t) + f, & h^T\tilde{x}(t) > 1, \\ \tilde{A}_c\tilde{x}(t) - Ly(t) - f, & h^T\tilde{x}(t) < 1, \\ \tilde{A}_u\tilde{x}(t) - Ly(t), & otherwise \end{cases}$$

Integral Action and Anti-Windup

with

$$\tilde{A}_u = \begin{bmatrix} \frac{a}{2} & 0 \\ -\frac{1}{4} & 1 \end{bmatrix}, \quad \tilde{A}_c = \begin{bmatrix} 0 & 1 \\ -\frac{1}{4} & 1 \end{bmatrix}, \tag{6}$$

$$f = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad h = -\begin{bmatrix} \hat{d}(t) \\ \frac{1}{2}a\hat{x}(t) \end{bmatrix} \tag{7}$$

and $\tilde{x}(t) = [\ \hat{x}(t)^T \quad \hat{d}(t)^T\ ]^T$.
One can notice that the unconstrained dynamics $\tilde{A}_u$ contains an integrator, while the constrained dynamics $\tilde{A}_c$ is asymptotically stable (two poles in 0.5). Hence, when the system saturates, we obtain an anti-windup effect.

RHC Extensions

- ***Disturbances***. We distinguish between *measured* and *unmeasured* disturbances. Measured disturbances $v(t)$ can be included in the prediction model

$$x(k+1) = Ax(k) + Bu(k) + Vv(k)$$

  where $v(k)$ is the prediction of the disturbance at time k based on the measured value $v(t)$. Then $v(t)$ appears as a vector of additional parameters in the optimization problem.
- ***Soft Constraints***. In practice, output constraints are relaxed or softened as $y_{\min} - M\varepsilon \leq y(t) \leq y_{\max} + M\varepsilon$, where $M \in \mathbb{R}^p$ is a constant vector ($M^i \geq 0$ is related to the "concern" for the violation of the i-th output constraint), and the term $\rho\varepsilon^2$ is added to the objective to penalize constraint violations (ρ is a suitably large scalar).
- ***Variable Constraints***. The bounds $y_{\min}$, $y_{\max}$, $\delta u_{\min}$, $\delta u_{\max}$, $u_{\min}$, $u_{\max}$ may change depending on the operating conditions, or in the case of a stuck actuator the constraints become $\delta u_{\min} = \delta u_{\max} = 0$.

We Will not Cover

- Fast on-line implementation with explicit solution
- Fast on-line implementation with online optimization
- Robustness (in detail)
- Issue and results when using soft-constraints
- State estimation