Introduction to Model Predictive Control

Lectures 12-14: Model Predictive Control

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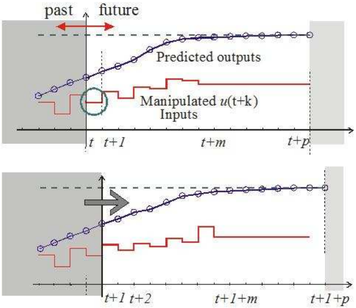
ME190M-Fall 2009

Summarizing...

Need

1. A discrete-time model of the system (Matlab, Simulink)
2. A state observer
3. Set up an Optimization Problem (Matlab, MPT toolbox/Yalmip)
4. Solve an optimization problem (Matlab/Optimization Toolbox, NPSOL)
5. Implement a MPC Controller and Verify that the closed-loop system performs as desired (avoid infeasibility/stability)
6. Make sure it runs in real-time and code/download for the embedded platform

Introduction



- At each sampling time, starting at the current state, an open-loop optimal control problem is solved over a finite horizon (top diagram).
- The computed optimal manipulated input signal is applied to the process only during the following sampling interval $[t, t+1]$.
- At the next time step $t+1$ a new optimal control problem based on new measurements of the state is solved over a shifted horizon (bottom diagram).
- The resultant controller is referred to as *Model Predictive Control*

Introduction

- The basic idea of receding horizon control was already indicated by the theoretical work of Propoi in 1963
- Gained attention in the mid-1970s, when Richalet proposed the MPC technique (they called it "Model Predictive Heuristic Control (MPHC)").
- Shortly thereafter, Cutler and Ramaker introduced the predictive control algorithm called Dynamic Matrix Control (DMC) which has been hugely successful in the petro-chemical industry.
- A vast variety of different names and methodologies followed, such as Quadratic Dynamic Matrix Control (QDMC), Adaptive Predictive Control (APC), Generalized Predictive Control (GPC), Sequential Open Loop Optimization (SOLO), and others.
- They all share the same structural features: a model of the plant, the receding horizon idea, and an optimization procedure to obtain the control action by optimizing the system's predicted evolution.
- Some of the first industrial MPC algorithms like IDCOM and DMC developed for constrained MPC with quadratic performance indices. However, input and output constraints were treated in an indirect ad-hoc fashion.
- Only later, algorithms like QDMC overcame this limitation by employing quadratic programming to solve constrained MPC problems with quadratic performance indices.
- Later an extensive theoretical effort devoted to provide conditions for guaranteeing feasibility and closed-loop stability,

RHC Notation

Consider the problem of regulating to the origin the discrete-time linear time-invariant system

$$\begin{cases} x(t+1) = Ax(t) + Bu(t) \\ y(t) = Cx(t) \end{cases} \quad (1)$$

subject to the constraints

$$x(t) \in \mathcal{X},\ u(t) \in \mathcal{U},\ \forall t \geq 0 \quad (2)$$

where the sets $\mathcal{X} \subseteq \mathbb{R}^n$ and $\mathcal{U} \subseteq \mathbb{R}^m$ are polyhedra. Solve the finite time optimal control problem

$$\begin{array}{lll} J_t^*(x(t)) = & \min_{U_{t\to t+N|t}} & J_t(x(t), U_{t\to t+N|t}) \\ & \text{such that} & x_{t+k+1|t} = Ax_{t+k|t} + Bu_{t+k|t},\ k = 0, \ldots, N-1 \\ & & x_{t+k|t} \in \mathcal{X},\ u_{t+k|t} \in \mathcal{U},\ k = 0, \ldots, N-1 \\ & & x_{t+N|t} \in \mathcal{X}_f \\ & & x_{t|t} = x(t) \end{array} \quad (3)$$

is solved at time t

RHC Notation

- $U_{t \rightarrow t+N|t} = \{u_{t|t}, \ldots, u_{t+N-1|t}\}$
- $x_{t+k|t}$ denotes the state vector at time $t+k$ predicted at time t obtained by starting from the current state $x_{t|t} = x(t)$ and applying to the system model

$$x_{t+k+1|t} = Ax_{t+k|t} + Bu_{t+k|t} \quad (4)$$

  the input sequence $u_{t|t}, \ldots, u_{t+N-1|t}$.
- The symbol $x_{t+k|t}$ is read as "the state x at time $t+k$ predicted at time t". Similarly $u_{t+k|t}$ is read as "the input u at time $t+k$ computed at time t".
- For instance, $x_{3|1}$ represents the predicted state at time 3 when the prediction is done at time $t = 1$ starting from the current state $x(1)$. It is different, in general, from $x_{3|2}$ which is the predicted state at time 3 when the prediction is done at time $t = 2$ starting from the current state $x(2)$.

RHC Notation

Let $U^*_{t\to t+N|t} = \{u^*_{t|t}, \dots, u^*_{t+N-1|t}\}$ be the optimal solution Then, the first element of $U^*_{t\to t+N|t}$ is applied to system

$$u(t) = u^*_{t|t}(x(t)). \tag{5}$$

The optimization is repeated at time $t+1$, based on the new state $x_{t+1|t+1} = x(t+1)$, yielding a *moving* or *receding horizon* control strategy.

Denote by $f_t(x(t)) = u^*_{t|t}(x(t))$ the receding horizon control law when the current state is $x(t)$. Then, the closed loop system obtained by controlling the system with the RHC (3)-(5) is

$$x(k+1) = Ax(k) + Bf_k(x(k)) \triangleq f_{cl}(x(k)),\ k \geq 0 \tag{6}$$

RHC Notation

Note that the system, the constraints and the cost function are time-invariant. For this reason, the solution to problem (3) is a time-invariant function of the initial state $x(t)$. Set $t = 0$ and remove the term "$|0$" since now redundant

$$
\begin{array}{rll}
J_0^*(x(t)) = & \min_{U_0} & J_0(x(t), U_0) \\
& \text{such that} & x_{k+1} = Ax_k + Bu_k,\ k = 0, \ldots, N-1 \\
& & x_k \in \mathcal{X},\ u_k \in \mathcal{U},\ k = 0, \ldots, N-1 \\
& & x_N \in \mathcal{X}_f \\
& & x_0 = x(t)
\end{array}
\tag{7}
$$

where $U_0 = \{u_0, \ldots, u_{N-1}\}$. The control law

$$u(t) = f_0(x(t)) = u_0^*(x(t)). \tag{8}$$

and closed loop system

$$x(k+1) = Ax(k) + Bf_0(x(k)) = f_{cl}(x(k)),\ k \geq 0 \tag{9}$$

are time-invariant as well.

RHC Algorithm

Compare problem (7) and the CFTOC. The **only** difference is that problem (7) is solved for $x_0 = x(t)$, $t \geq 0$ rather than for $x_0 = x(0)$. For this reason we can make use of **all** the results studied so far

- $\mathcal{X}_0$ denotes the set of feasible states $x(t)$ for problem (7).
- The procedure of this *on-line* optimal control technique is summarized in the following algorithm.

Algorithm (On-line receding horizon control)

1. *MEASURE the state* $x(t)$ *at time instance* t
2. *OBTAIN* $U_0^*(x(t))$ *by solving the optimization problem (7)*
3. *IF* $U_0^*(x(t)) = \emptyset$ *THEN 'problem infeasible' STOP*
4. *APPLY the first element* u_0^* *of* U_0^* *to the system*
5. *WAIT for the new sampling time* $t+1$, *GOTO (1.)*

RHC Example

Consider the double integrator system

$$\left\{ \begin{array}{rcl} x(t+1) & = & \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \\ y(t) & = & \begin{bmatrix} 1 & 0 \end{bmatrix} x(t) \end{array} \right. \tag{10}$$

The aim is to compute the receding horizon controller that solves the optimization problem (7) with $p = 2, N = 3, P = Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, R = 10,$ $\mathcal{X}_f = \mathbb{R}^2$ subject to the input constraints

$$-0.5 \leq u(k) \leq 0.5,\ k = 0, \ldots, 3 \tag{11}$$

and the state constraints

$$\begin{bmatrix} -5 \\ -5 \end{bmatrix} \leq x(k) \leq \begin{bmatrix} 5 \\ 5 \end{bmatrix},\ k = 0, \ldots, 3. \tag{12}$$

RHC Example

The QP problem associated with the RHC has

$$H = \begin{bmatrix} 13.50 & -10.00 & -0.50 \\ -10.00 & 22.00 & -10.00 \\ -0.50 & -10.00 & 31.50 \end{bmatrix},\; F = \begin{bmatrix} -10.50 & 10.00 & -0.50 \\ -20.50 & 10.00 & 9.50 \end{bmatrix},\; Y = \begin{bmatrix} 14.50 & 23.50 \\ 23.50 & 54.50 \end{bmatrix} \tag{13}$$

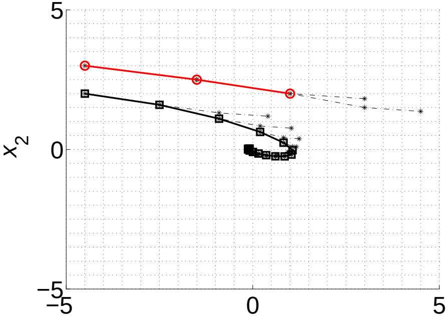
and

$$G_0 = \begin{bmatrix} 0.50 & -1.00 & 0.50 \\ -0.50 & 1.00 & -0.50 \\ -0.50 & 0.00 & 0.50 \\ -0.50 & 0.00 & -0.50 \\ 0.50 & 0.00 & -0.50 \\ 0.50 & 0.00 & 0.50 \\ -1.00 & 0.00 & 0.00 \\ 0.00 & -1.00 & 0.00 \\ 1.00 & 0.00 & 0.00 \\ 0.00 & 1.00 & 0.00 \\ 0.00 & 0.00 & -1.00 \\ 0.00 & 0.00 & 1.00 \\ 0.00 & 0.00 & 0.00 \\ -0.50 & 0.00 & 0.50 \\ 0.00 & 0.00 & 0.00 \\ 0.50 & 0.00 & -0.50 \\ -0.50 & 0.00 & 0.50 \\ 0.50 & 0.00 & -0.50 \\ 0.00 & 0.00 & 0.00 \\ 0.00 & 0.00 & 0.00 \\ 0.00 & 0.00 & 0.00 \\ 0.00 & 0.00 & 0.00 \end{bmatrix},\; E_0 = \begin{bmatrix} 0.50 & 0.50 \\ -0.50 & -0.50 \\ 0.50 & 0.50 \\ -0.50 & -0.50 \\ -0.50 & -0.50 \\ 0.50 & 0.50 \\ 0.00 & 0.00 \\ 0.00 & 0.00 \\ 0.00 & 0.00 \\ 0.00 & 0.00 \\ 0.00 & 0.00 \\ 0.00 & 0.00 \\ 1.00 & 1.00 \\ -0.50 & -0.50 \\ -1.00 & -1.00 \\ 0.50 & 0.50 \\ -0.50 & -1.50 \\ 0.50 & 1.50 \\ 1.00 & 0.00 \\ 0.00 & 1.00 \\ -1.00 & 0.00 \\ 0.00 & -1.00 \end{bmatrix},\; W_0 = \begin{bmatrix} 0.50 \\ 0.50 \\ 5.00 \\ 5.00 \\ 5.00 \\ 5.00 \\ 5.00 \\ 5.00 \\ 5.00 \\ 5.00 \\ 5.00 \\ 0.50 \\ 0.50 \\ 5.00 \\ 5.00 \\ 5.00 \\ 5.00 \\ 0.50 \\ 0.50 \\ 5.00 \\ 5.00 \\ 5.00 \\ 5.00 \end{bmatrix} \tag{14}$$

RHC Example

The RHC (7)-(8) algorithm becomes

1. MEASURE the state $x(t)$ at time instance t
2. COMPUTE $\tilde{F} = 2x'(t)F$ and $\tilde{W}_0 = W_0 + E_0 x(t)$
3. OBTAIN $U_0^*(x(t))$ by solving the optimization problem $[U_0^*, \mathrm{Flag}] = \mathrm{QP}(H, \tilde{F}, G_0, \tilde{W}_0)$
4. IF Flag='infeasible' THEN STOP
5. APPLY the first element u_0^* of U_0^* to the system
6. WAIT for the new sampling time $t+1$, GOTO (1.)



RHC Example

Consider the unstable system

$$\left\{ \begin{array}{rcl} x(t+1) & = & \begin{bmatrix} 2 & 1 \\ 0 & 0.5 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t)\,. \end{array} \right. \tag{15}$$

with the input constraints

$$-1 \le u(k) \le 1,\ k = 0, \ldots, N-1 \tag{16}$$

and the state constraints

$$\begin{bmatrix} -10 \\ -10 \end{bmatrix} \le x(k) \le \begin{bmatrix} 10 \\ 10 \end{bmatrix},\ k = 0, \ldots, N-1. \tag{17}$$

Solve the receding horizon control problem for different horizons N and weights R. (set $Q = I$, $\mathcal{X}_f = \mathbb{R}^2$, $P = 0$).

RHC Example

Clearly the state $x(0)$ lies outside the **maximum positive invariant set** $\mathcal{O}_\infty$ of the closed-loop system, and thus even if feasible at time 0 the trajectory exits the feasible set at time step 2: $x(2) \notin \mathcal{X}_0$.

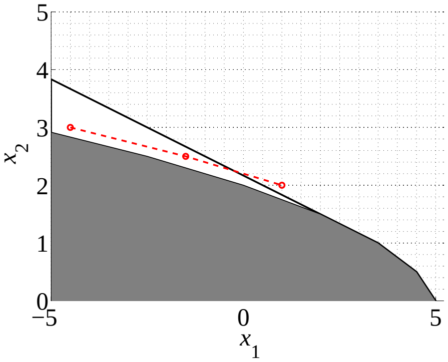
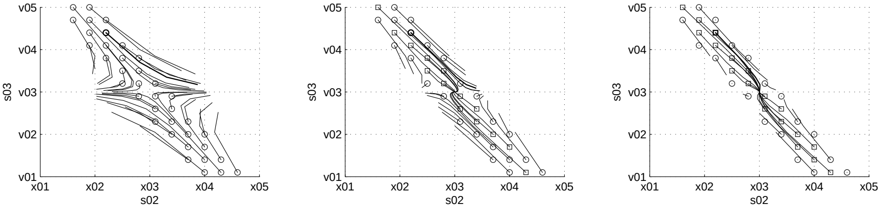


Figure: Maximal positive invariant set $\mathcal{O}_\infty$ (grey) and set of initial feasible states $\mathcal{X}_0$ (white and grey). The initial condition $x(0) = [-4.5, 3]$ belongs to $\mathcal{X}_0 \setminus \mathcal{O}_\infty$.

RHC Example



Closed-loop trajectories for receding horizon control loops that were obtained with the following parameter settings, respectively (from left to right)

Setting 1: $N = 2, R = 10$

Setting 2: $N = 3, R = 2$

Setting 3: $N = 4, R = 1$

Trajectories from circles diverges from squares converge

RHC Example

1. For *Setting 1* there is evidently no initial state that can be steered to the origin. Indeed, it turns out, that *all* non-zero initial states $x(0) \in \mathbb{R}^2$ diverge from the origin and eventually become infeasible.
2. Different from that, *Setting 2* leads to a receding horizon controller, that manages to get some of the initial states converge to the origin.
3. Finally *Setting 3* can expand the set of those initial states that can be brought to the origin.
4. These results indicate, that the choice of parameters for receding horizon control influences the behavior of the resulting closed-loop trajectories in a complex manner.

Feasibility of RHC

The next fundamental theorem provides a guideline for guaranteeing persistent feasibility.

Theorem

If $\mathcal{X}_f = 0$ then the RHC is persistently feasible for all feasible u.

Proof: Let $x(0) \in \mathcal{X}_0$ and let $U_0^* = \{u_0^*, \ldots, u_{N-1}^*\}$ be the optimizer of the RHC problem at time $t = 0$ and $\mathbf{x}_0 = \{x(0), x_1, \ldots, x_N\}$ be the corresponding optimal state trajectory. Because of absence of model mismatch and disturbances $x(1) = x_1 = Ax(0) + Bu_0^*$. Consider now problem the RHC problem for $t = 1$. Consider the sequence $\tilde{U}_1 = \{u_1^*, \ldots, u_{N-1}^*, 0\}$ and the corresponding state trajectory resulting from the initial state $x(1)$, $\tilde{\mathbf{x}}_1 = \{x_1, \ldots, x_N, Ax_N + B \cdot 0\}$. From the feasibility of the problem at time 0 we get $x_N = 0$, and therefore $Ax_N + B \cdot 0 = 0$. Therefore the RHC problem is feasible for $t = 1$ since $\tilde{U}_1$ is a feasible input.

□

Stability of RHC

Theorem

Consider system (1)-(2), the RHC law (7)-(8) and the closed-loop system (6). Assume that

(A0) $J_0(x(0), U_0) \triangleq x_N' P x_N + \sum_{k=0}^{N-1} x_k' Q x_k + u_k' R u_k$ *with* $Q = Q' \succ 0,\ R = R' \succ 0,\ P \succ 0,$

(A1) *The sets* $\mathcal{X}$*,and* $\mathcal{U}$ *contain the origin and are closed.*

(A2) $\mathcal{X}_f = 0$

Then, the state of the closed-loop system (6) converges to the origin, i.e., $\lim_{k\to\infty} x(k) = 0$ *for all* $x(0) \in \mathcal{X}_0$

RHC Extensions

In order to reduce the size of the optimization problem modify the problem as:

$$
\begin{array}{ll}
\min\limits_{U_{t\to t+N|t}} & \left\{ \|P{x'}_{N_y}\|_p + \displaystyle\sum_{k=0}^{N_y-1} \left[\|Qx_k\|_p + \|Ru_k\|_p\right] \right\} \\
\text{such that} & y_{\min} \leq y_k \leq y_{\max},\ k = 1, \ldots, N_c \\
& u_{\min} \leq u_k \leq u_{\max},\ k = 0, 1, \ldots, N_u \\
& x_0 = x(t) \\
& x_{k+1} = Ax_k + Bu_k,\ k \geq 0 \\
& y_k = Cx_k,\ k \geq 0 \\
& u_k = Kx_k,\ N_u \leq k < N_y
\end{array}
\tag{18}
$$

where K is some feedback gain, N_y, N_u, N_c are the output, input, and constraint horizons, respectively, with $N_u \leq N_y$ and $N_c \leq N_y - 1$.

RHC Reference Tracking

The MPC is designed as follows

$$\begin{array}{ll} \min_{u_0,\ldots,u_{N-1}} & \|x_N - \bar{x}_t\|_P^2 + \sum_{k=0}^{N-1} \|x_k - \bar{x}_t\|_Q^2 + \|u_k - \bar{u}_t\|_R^2 \\ \text{such that} & Ex_k + Lu_k \leq M,\ k = 0, \ldots, N \\ & x_{k+1} = Ax_k + Bu_k + B_d d_k,\ k = 0, \ldots, N \\ & d_{k+1} = d_k,\ k = 0, \ldots, N \\ & x_0 = \hat{x}(t) \\ & d_0 = \hat{d}(t), \end{array} \tag{19}$$

with $\bar{u}_t$ and $\bar{x}_t$ given by

$$\begin{bmatrix} A - I & B \\ HC & 0 \end{bmatrix} \begin{bmatrix} \bar{x}_t \\ \bar{u}_t \end{bmatrix} = \begin{bmatrix} -B_d\hat{d}(t) \\ r(t) - HC_d\hat{d}(t) \end{bmatrix} \tag{20}$$

and where $\|x\|_M^2 \triangleq x^T Mx$, $Q \succeq 0$, $R \succ 0$, and P satisfies the Riccati equation

RHC Extensions

- **Soft Constraints**
  In practice, output constraints are relaxed or softened [**?**] as $y_{\min} - M\varepsilon \leq y(t) \leq y_{\max} + M\varepsilon$, where $M \in \mathbb{R}^p$ is a constant vector ($M^i \geq 0$ is related to the "concern" for the violation of the i-th output constraint), and the term $\rho\varepsilon^2$ is added to the objective to penalize constraint violations (ρ is a suitably large scalar).
- **Variable Constraints**
  The bounds $y_{\min}$, $y_{\max}$, $\delta u_{\min}$, $\delta u_{\max}$, $u_{\min}$, $u_{\max}$ may change depending on the operating conditions, or in the case of a stuck actuator the constraints become $\delta u_{\min} = \delta u_{\max} = 0$. This possibility can again be built into the control law. The bounds can be treated as parameters in the QP and added to the vector x.