QUADRATIC EQUATION & EXPRESSION

Ans

(I) The expression of the type 'ax^2+bx+c' is called quadratic ~~equation~~ expression in variable 'x', where a, b, c are constants & $a \neq 0$.

(II) Quadratic Expression equated to zero is called quadratic Equation.

(III) $a \neq 0$, $ax^2+bx+c \equiv a\left(x^2+\frac{b}{a}x+\frac{c}{a}\right)$

$$\equiv a\left(\left(x+\frac{b}{2a}\right)^2+\frac{c}{a}-\frac{b^2}{4a^2}\right)$$

$$\equiv a\left(\left(x+\frac{b}{2a}\right)^2-\frac{b^2-4ac}{4a^2}\right), \text{ Let } b^2-4ac = D$$

$$\equiv a\left(\left(x+\frac{b}{2a}\right)^2-\frac{D}{4a^2}\right)$$

$$\equiv a\left(\left(x+\frac{b}{2a}\right)^2-\left(\frac{\sqrt{D}}{2a}\right)^2\right)$$

$$\equiv a\left(x+\frac{b}{2a}-\frac{\sqrt{D}}{2a}\right)\left(x+\frac{b}{2a}+\frac{\sqrt{D}}{2a}\right)$$

$$\equiv a\left(x+\frac{b-\sqrt{D}}{2a}\right)\left(x+\frac{b+\sqrt{D}}{2a}\right)$$

$$\equiv a\left(x-\underbrace{\frac{-b+\sqrt{D}}{2a}}_{\alpha}\right)\left(x-\underbrace{\frac{-b-\sqrt{D}}{2a}}_{\beta \text{ (say)}}\right)$$

$$\equiv a(x-\alpha)(x-\beta)$$

$\Rightarrow$ $\boxed{ax^2+bx+c \equiv a(x-\alpha)(x-\beta)}$ — (I), where

$\alpha, \beta = \frac{-b \pm \sqrt{D}}{2a}$, $a \neq 0$

NOTE: (a) Equation (I) is valid even if a, b, c are not constants.

(b) If a, b, c are constants, and $a \neq 0$ then
$a^2x^2 + bx + c = 0$.
$\Rightarrow a(x-\alpha)(x-\beta) = 0$ — (II)
as $a \neq 0$, hence only $x = \alpha$ & $x = \beta$ are the roots of equation (II).
Hence, Every quadratic equation has exactly '2' roots.
$\Rightarrow x = \alpha$ (or) β
roots $\rightarrow$ $x = \dfrac{-b+\sqrt{D}}{2a}$ (or) $\dfrac{-b-\sqrt{D}}{2a}$

(c) Let a, b, c are const. & $a \neq 0$, then
$ax^2 + bx + c \equiv a(x-\alpha)(x-\beta)$ is an identity, i.e, it will hold $\forall x$.

(d) Now as Equation (I) is identity (given a, b, c are const.)
$ax^2 + bx + c \equiv a(x-\alpha)(x-\beta) \equiv a(x^2 - (\alpha+\beta)x + \alpha\beta)$
$\equiv ax^2 - a(\alpha+\beta)x + a\cdot\alpha\beta$

- Identity:
Eg: $3x^2 + cx + 5\sin(x) + d + 0\cdot\cos(x)$
$\equiv ax^2 + 50x + b\sin(x) + e\cos(x) + 7$
$\Rightarrow$ compare coeff. $\Rightarrow a=3, b=5, c=50, d=7, e=0$
If Question: $3x^2 + ax + 7 = bx^2 + 3x - c$, ($\forall x$)
then it is an identity.

$\rightarrow$ Now as it is an identity hence coeff. can be compared.
$\Rightarrow a = a$, $b = -a(\alpha+\beta)$, $c = a\cdot\alpha\beta$
$\Rightarrow \alpha+\beta = -\dfrac{b}{a}$, $\alpha\beta = \dfrac{c}{a}$

Hence for equation,
$ax^2 + bx + c = 0$, where $a \neq 0$ & a, b, c are

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consts.

$\rightarrow$ Sum of roots $= \alpha + \beta = -\frac{b}{a} = S$

$\rightarrow$ Product of roots $= \alpha\beta = \frac{c}{a} = P$

$ax^2 + bx + c = 0$

$\Rightarrow a(x-\alpha)(x-\beta) = 0$

$\Rightarrow a(x^2 - (\alpha+\beta)x + \alpha\beta) = 0$

$\Rightarrow a(x^2 - Sx + P) = 0$

ⓔ From above, a quardratic equation with roots as α & β can be given as

$x^2 - (\alpha+\beta)x + \alpha\beta = 0$

$\Rightarrow \boxed{x^2 - Sx + P = 0}$

A general Quadratic Equation with roots as α & β can be given as

$\Rightarrow \lambda(x^2 - (\alpha+\beta)x + \alpha\beta) = 0$

$\Rightarrow \boxed{\lambda(x^2 - Sx + P) = 0}$, where λ is any non-zero constant.

Q1. Solve $3x^2 - 5x - 7 = 0$

Ans $x = \frac{-b \pm \sqrt{D}}{2a} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{5 \pm \sqrt{25 + 4 \cdot 3 \cdot 7}}{2 \cdot 3}$

$\Rightarrow x = \frac{5 \pm \sqrt{109}}{6} = \frac{5 + \sqrt{109}}{6}, \frac{5 - \sqrt{109}}{6}$

Q2. Find a quadratic equation with roots as 3 & −8.

Ans 2. $S = -5$, $P = -24$

Required Equation : $x^2 - Sx + P = 0$

$\Rightarrow \boxed{x^2 + 5x - 24 = 0}$ Ans

Q3. Let α, β are roots of $4x^2+3x-9=0$, then find (a) $\alpha^2+\beta^2$, (b) $\alpha^3+\beta^3$ (c) $\frac{\alpha}{\beta}+\frac{\beta}{\alpha}$

(d) $(3-\alpha)(3-\beta)$ (e) $(25-\alpha^2)(25-\beta^2)$

Ans 3. $\alpha+\beta = \frac{-3}{4}$, $\alpha\beta = -\frac{9}{4}$

(a) $\Rightarrow (\alpha+\beta)^2 = \frac{9}{16} \Rightarrow \alpha^2+\beta^2+2\alpha\beta = \frac{9}{16}$

$\Rightarrow \alpha^2+\beta^2 = \frac{9}{16}+\frac{9}{2} \Rightarrow \alpha^2+\beta^2 = \frac{81}{16}$ Ans

(b) $(\alpha+\beta)^3 = -\frac{27}{64}$

$\Rightarrow \alpha^3+\beta^3+3\alpha\beta(\alpha+\beta) = -\frac{27}{64}$

$\Rightarrow \alpha^3+\beta^3 = -\frac{27}{64} - 3\left(-\frac{9}{4}\right)\left(\frac{-3}{4}\right)$

$\Rightarrow \alpha^3+\beta^3 = -\frac{27}{64} - \frac{81}{16}$

$\Rightarrow \alpha^3+\beta^3 = \cancel{\frac{297}{16}}\ \frac{-351}{64}$ Ans

(c) $\Rightarrow \frac{\alpha}{\beta}+\frac{\beta}{\alpha} = \frac{\alpha^2+\beta^2}{\alpha\beta} = \frac{81}{16}\cdot\frac{4}{-9} = -\frac{9}{4}$ Ans

(d) $\Rightarrow (3-\alpha)(3-\beta) = 9-3(\alpha+\beta)+\alpha\beta = 9-3\left(\frac{-3}{4}\right)-\frac{9}{4} = 9$ Ans

OR

$4x^2+3x-9 \equiv 4(x-\alpha)(x-\beta)$

$x=3 \Rightarrow 36+9-9 \equiv 4(3-\alpha)(3-\beta)$

$\Rightarrow \frac{36}{4} = (3-\alpha)(3-\beta) \Rightarrow (3-\alpha)(3-\beta) = 9$ Ans

(e) $(25-\alpha^2)(25-\beta^2) = (5+\alpha)(5+\beta)(5-\alpha)(5-\beta)$

$(5+\beta)(5+\alpha) = 25+(\alpha+\beta)5+\alpha\beta$

$= 25+\left(\frac{-3}{4}\right)5+\left(\frac{-9}{4}\right) = \frac{100-15-9}{4}$

$= 19$

$4(5-\alpha)(5-\beta) = 4(5)^2+3(5)-9$

$\Rightarrow (5-\alpha)(5-\beta) = \frac{100+\cancel{24}6}{4} \Rightarrow (5-\alpha)(5-\beta) = \frac{106}{4}$

$\Rightarrow (5+\alpha)(5+\beta)(5-\alpha)(5-\beta) = 19\times\frac{\overset{53}{\cancel{106}}}{\cancel{4}\,2} = \cancel{5089}$

$= \frac{53\times19}{2}$ Ans

NOTE (i) With the help of $\alpha+\beta$ & $\alpha\beta$, we can calculate symmetric expressions in α, β.
An expression is called symmetric in α & β if interchanging α & β does not changes the expression.
Eg. $\alpha^2+\beta^2$, $\frac{\alpha}{\beta}+\frac{\beta}{\alpha}$, $\alpha^3+\beta^3+3\alpha\beta$ etc. are symmetric

Difference of Roots

$ax^2+bx+c=0, \; a\neq0$

$\hookrightarrow \alpha, \beta = \frac{-b\pm\sqrt{D}}{2a}$

Difference of roots $= \alpha\sim\beta$

$= |\alpha-\beta| = \left|\frac{-\cancel{b}+\sqrt{D}}{2a} - \left(\frac{-\cancel{b}-\sqrt{D}}{2a}\right)\right|$

$= \left|\frac{2\sqrt{D}}{2a}\right| = \left|\frac{\sqrt{D}}{a}\right|$

$\Rightarrow$ Difference of Roots $= \left|\frac{\sqrt{D}}{a}\right|$

NOTE: Presence of 'D' in a question may Imply application of difference of roots.

Q1. Let roots of $ax^2+2bx+c=0$ are α, β & that of $Ax^2+2Bx+C=0$ are ~~$Ax^2+2Bx+C=0$~~ $\alpha+\gamma$ & $\beta+\gamma$ then prove that
$A^2(b^2-ac) = a^2(B^2-AC)$

Ans:- $|\alpha-\beta| = |(\alpha+\gamma)-(\beta+\gamma)|$

$\Rightarrow |\alpha-\beta| = |(\alpha+\gamma)-(\beta+\gamma)|$

$\Rightarrow \left|\frac{\sqrt{D_1}}{a}\right| = \left|\frac{\sqrt{D_2}}{A}\right|$

$()^2 \Rightarrow \frac{D_1}{a^2} = \frac{D_2}{A^2}$

$\Rightarrow A^2(4b^2-4ac) = a^2(4B^2-4AC)$

$\Rightarrow \cancel{4}A^2(b^2-ac) = \cancel{4}a^2(B^2-AC)$

$\Rightarrow A^2(b^2-ac) = a^2(B^2-AC)$ Hence, proved.

NOTE (ii) If the expression ax^2+bx+c, where a, b, c are const., is not referred as quadratic, then we will analyse it in two cases:, One with $a=0$ & second with $a\neq 0$

(iii) If $ax^2+bx+c=0$ is satisfied for more than 2-distinct values of 'x' (given a, b, c are const.) then it will be an identity ie., it will be true for all 'x' & also in this case $\boxed{a=b=c=0}$, ie., given equation is
$0\cdot x^2+0\cdot x+0=0$

Q2. Prove that :

$$\frac{(x-a)(x-b)}{(c-a)(c-b)}.c + \frac{(x-b)(x-c)}{(a-b)(a-c)}a + \frac{(x-a)(x-c)}{(b-a)(b-c)}b \equiv x$$

Ans2. It is sufficient to prove that :

$$\underbrace{\frac{(x-a)(x-b)}{(c-a)(c-b)}c + \frac{(x-b)(x-c)}{(a-b)(a-c)}a + \frac{(x-a)(x-c)}{(b-a)(b-c)}b - x}_{= f(x) \text{ (let)}} \equiv 0$$

$\therefore$ To prove : $f(x) \equiv 0$

$f(x) \equiv Ax^2 + Bx + C$

$x = a \rightarrow$ $f(a) = 0 + \frac{\cancel{(a-b)}\cancel{(a-c)}}{\cancel{(a-b)}\cancel{(a-c)}}.a + 0 - \cancel{x}a$

$\Rightarrow f(a) = a - a = 0 \Rightarrow f(a) = 0$

$x = b \Rightarrow f(b) = 0$

$x = c \Rightarrow f(c) = 0$

$\rightarrow$ As $f(x)$ is of the type $Ax^2 + Bx + C$ & it is zero at 3- distinct values of x (i.e. a, b, c),

$\Rightarrow \boxed{f(x) \equiv 0}$ Hence, proved.

Note : If 'α' is a root of a quadratic equation, then any polynomial expression of 'α' can be reduced into linear expression of 'α'; using the fact that $a\alpha^2 + b\alpha + c = 0$, where α is the root of Quadratic equation $ax^2 + bx + c = 0$.

- Polynomial Expression $\equiv$ (Quotient)(Divisor) + Remainder

Divisor Polynomial) Polynomial Expression (Quotient

⋮ ⋮

Remainder

Q1. Let α is a root of $x^2+3x+5=0$ then express $\alpha^4+5\alpha^3+7\alpha^2+8\alpha-9$ as linear expression of 'α'.

Ans $\alpha^2+3\alpha+5=0$

$\alpha^4+5\alpha^3+7\alpha^2+8\alpha-9$

$\equiv (\alpha^2+\cancel{3\alpha}+5)(\alpha^2+2\alpha-4) + (10\alpha+11)$

$\equiv 10\alpha+11$ Ans

Q2. Let α, β are the roots of $x^2-3x-7=0$ then calculate

(I) $\dfrac{\alpha^2+5}{\alpha-3} + \dfrac{\beta^2+5}{\beta-3}$

$\alpha^2-3\alpha-7=0 \Rightarrow \alpha^2=3\alpha+7 \Rightarrow \alpha^2-3\alpha=7$

Similarly $\beta^2=3\beta+7 \Rightarrow \beta^2-3\beta=7$

$\Rightarrow \dfrac{\alpha^2+5}{\alpha-3} + \dfrac{\beta^2+5}{\beta-3} = \dfrac{3\alpha+12}{\alpha-3} + \dfrac{3\beta+12}{\beta-3}$

$= \dfrac{(3\alpha+12)\alpha}{\alpha^2-3\alpha} + \dfrac{(3\beta+12)\beta}{\beta^2-3\beta} = \dfrac{(3\alpha+12)\alpha}{7} + \dfrac{(3\beta+12)\beta}{7}$

$= \dfrac{3}{7}(\alpha^2+4\alpha+\beta^2+4\beta) = \dfrac{3}{7}((\alpha+\beta)^2+4(\alpha+\beta)-2\alpha\beta)$

$= \dfrac{3}{7}(9+\cancel{28}\,12+14) = \dfrac{3}{7}\times 35 = 15$ Ans.

or $= \dfrac{3}{7}(\alpha^2+4\alpha+\beta^2+4\beta) = \dfrac{3}{7}(7\alpha+7+7\beta+7)$

$= 3(\alpha+\beta+2) = 3\times 5 = 15$ Ans

(II) $\dfrac{\alpha^5-7\alpha^3}{3\alpha+7} + \dfrac{\beta^5-7\beta^3}{3\beta+7}$

$= \dfrac{\alpha^{\cancel{3}}(\alpha^2-7)}{\cancel{\alpha^2}} + \dfrac{\beta^{\cancel{3}}(\beta^2-7)}{\cancel{\beta^2}} = 3\alpha^2+3\beta^2$

telco Dt.: Pg.:

$= 3(3\alpha+7+3\beta+7)$
$= 3(3\times3+14) = 69$ Ans

TRANSFORMATION of ROOTS

Q1. Let $ax^2+bx+c=0$ has roots as α & β then form a Quadratic Equation with roots as

(i) $\alpha+3$ & $\beta+3$

Let $\gamma = \alpha+3$ & $\delta = \beta+3$

$\gamma + \cancel{\beta}\delta = -\frac{b}{a} + 6 = \frac{-b+6a}{a}$

$\gamma\delta = (\alpha+3)(\beta+3)$

$\because a(\alpha+3)(\beta+3) = 9a-3b+c$ (Using Identity)

$\Rightarrow \gamma\delta = \frac{9a-3b+c}{a}$

$\therefore$ Required eqn $= x^2 - \left(\frac{6a-b}{a}\right)x + \left(\frac{9a-3b+c}{a}\right) = 0$

$\Rightarrow (ax^2 + (b-6a)x + (9a-3b+c)) = 0$ Ans

(ii) $\frac{\alpha^2}{\beta}$ & $\frac{\beta^2}{\alpha}$

Let $\gamma = \frac{\alpha^2}{\beta}$, $\delta = \frac{\beta^2}{\alpha}$

$\Rightarrow S = \gamma+\delta = \frac{\alpha^3+\beta^3}{\alpha\beta} = \frac{(\alpha+\beta)^3 - 3\alpha\beta(\alpha+\beta)}{\alpha\beta}$

$\Rightarrow S = \frac{\left(\frac{-b}{a}\right)^3 - 3\left(\frac{c}{a}\right)\left(\frac{-b}{a}\right)}{\left(\frac{c}{a}\right)} = \frac{-b^3/a^3 + 3bc/a^2}{c/a}$

$\Rightarrow S = \frac{-b^3+3abc}{a^2c} = \frac{b(3ac-b^2)}{a^2c}$

$$P = \gamma\delta = \frac{\alpha^2}{\beta} \cdot \frac{\beta^2}{\alpha} = \alpha\beta = \frac{c}{a}$$

Required Eq^n: $\lambda\left(x^2 - \frac{b}{a^2c}(3ac - b^2)x + \frac{c}{a}\right) = 0$

$\Rightarrow \lambda(ca^2x^2 - b(3ac - b^2)x + ac^2) = 0$ Ans

Alt: Let α, β are roots of $ax^2 + bx + c = 0$, $a \neq 0$ then to form a quadratic equation with roots as $\frac{K_1\alpha + K_2}{K_3\alpha + K_4}$ & $\frac{K_1\beta + K_2}{K_3\beta + K_4}$

$\left(\frac{\text{Linear}}{\text{Linear}}, \frac{1}{\text{Linear}}\right.$, Linear type roots, α & β should be segregated)

Procedure: Assume: $y = \frac{K_1x + K_2}{K_3x + K_4}$

$\Rightarrow x = g(y) = \frac{\text{Linear of } y}{\text{Linear of } y}$

Now substitute 'x' in original given Quadratic equation.

The obtained quadratic in 'y' is required equation.

Q1. Let α, β are roots of $3x^2 + x + 5 = 0$ then find a quadratic equation with roots as $\frac{\alpha+1}{\alpha-3}$ & $\frac{\beta+1}{\beta-3}$

$y = \frac{x+1}{x-3} \Rightarrow xy - 3y = x + 1 \Rightarrow x(y-1) = 3y+1$

$\Rightarrow x = \frac{3y+1}{y-1}$

Dt.:
Pg.:

$\Rightarrow 3\left(\frac{3y+1}{y-1}\right)^2 + \left(\frac{3y+1}{y-1}\right) + 5 = 0$

$\Rightarrow 3(3y+1)^2 + (3y+1)(y-1) + 5(y-1)^2 = 0$

$\Rightarrow 3(9y^2+1+6y) + (3y^2 \cancel{+} -2y-1) + 5(y^2+1-2y) = 0$

$\Rightarrow 35y^2 + 6y + 7 = 0$: Required equation

$\Rightarrow \boxed{35x^2+6x+7=0}$ Ans

COMMON ROOT CONDITION

(I) $a_1x^2 + b_1x + c_1 = 0$ — (i)

$a_2x^2 + b_2x + c_2 = 0$ — (ii)

C-1 If $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{a_2}$ then equation (i) & (ii) will have both roots in common.

C-2 If equation (i) & (ii) have exactly one root in common (say 'α') ~~then~~ them.

$a_1\alpha^2 + b_1\alpha + c_1 = 0$ — (iii)

$a_2\alpha^2 + b_2\alpha + c_2 = 0$ — (iv)

$$\frac{\alpha^2}{b_1c_2 - c_1b_2} = \frac{-\alpha}{a_1c_2 - c_1a_2} = \frac{1}{a_1b_2 - b_1a_2}$$

$$\Rightarrow \alpha = \frac{a_2c_1 - a_1c_2}{a_1b_2 - b_1a_2}$$

$$\Rightarrow \alpha = \frac{b_1c_2 - c_1b_2}{a_2c_1 - a_1c_2}$$

$$\Rightarrow \alpha = \frac{a_2c_1 - a_1c_2}{a_1b_2 - b_1a_2} = \frac{b_1c_2 - c_1b_2}{a_2c_1 - a_1c_2}$$

$$\Rightarrow \boxed{(a_2c_1 - a_1c_2)^2 = (a_1b_2 - b_1a_2)(b_1c_2 - c_1b_2)}$$

or

$$\Rightarrow \boxed{(a_1c_2 - c_1a_2)^2 = (a_1b_2 - b_1a_2)(b_1c_2 - c_1b_2)}$$

$\hookrightarrow$ COMMON ROOT CONDITION

Dt.:
Pg.:

Q1. Find 'a' if the equations $x^2-(a+3)x+4=0$ & $x^2-7x+3a=0$ have a common root.

Ans 1. If both the roots are in common, then

$\Rightarrow \frac{1}{1} = \frac{-(a+3)}{-7} = \frac{4}{3a}$

$\Rightarrow 3a(a+3) = 28 \Rightarrow 3a^2+9a-28=0$

$\hookrightarrow 3a = 4 \Rightarrow a = 4/3$

$\hookrightarrow a+3=7 \Rightarrow a=4$ $\Bigg\} \Rightarrow$ Not possible

$\Rightarrow$ Both roots cannot be common.

If only one root is in common, then

$x^2-(a+3)x+4=0$

$x^2-7x+3a=0$

$(a_1c_2-c_1a_2)^2 = (a_1b_2-b_1a_2)(b_1c_2-c_1b_2)$

$\Rightarrow (1.3a-4.1)^2 = (1.(-7)+(a+3).1.)(-(a+3).3a+4.7)$

$\Rightarrow (3a-4)^2 = (a-4)(28-(a+3)3a)$

$\Rightarrow (3a-4)^2 = -(a-4)(3a^2+9a-28)$

$\Rightarrow 9a^2+16-24a = -(3a^3-3a^2-64a+112)$

$\Rightarrow 3a^3-3a^2-64a+112+9a^2+16-24a=0$

$\Rightarrow 3a^3+6a^2-88a+128=0$

$\Rightarrow (a-2)(3a^2+12a-64)=0$

Obtained by Hit & trial

$\Rightarrow a=2$ or $3a^2+12a-64=0$

$a=2$ or $\hookrightarrow a = \frac{-12 \pm \sqrt{12^2+4\times3\times64}}{3\times2}$

$\Rightarrow a = \frac{-12 \pm 4\sqrt{9+48}}{3\times2} = \frac{-6 \pm 2\sqrt{57}}{3}$

$\Rightarrow a=2$ or $\frac{-6+2\sqrt{57}}{3}$ or $\frac{-6-2\sqrt{57}}{3}$ Ans

Q2. Let equations $x^2+3x+5=0$ & $3x^2+ax+b=0$ have one root in common, where $a,b \in R$ then find a,b.

Ans2. If Both the roots are in common, then

$\frac{1}{3} = \frac{3}{a} = \frac{5}{b}$

This is because: As 1^{st} equation has imaginary roots $\Rightarrow$ common root will be imaginary. Now as coefficients of 2^{nd} equation are real hence imaginary roots will occur in pair $\Rightarrow$ Both roots of 2^{nd} equation are imaginary & should be common with 1^{st} equation. Hence, we compared coefficients.

$\Rightarrow$ $a=9$, $b=15$ Ans

Q3. Let the equation $x^2(a-b) + (b-c)x + (c-a) = 0$ has a root $2+\sqrt{3}$ then find required condition on a,b,c, given $a,b,c \in Q$.

Ans 3. As coefficients are rational hence $2-\sqrt{3}$ should also be a root. Now, from observation '1' is a root of the equation

$\Rightarrow$ Equation has $2+\sqrt{3}$, $2-\sqrt{3}$ & 1 as its roots.

$\Rightarrow$ It is an identity & $a-b=b-c=c-a=0$

$\Rightarrow$ $\boxed{a=b=c}$ Ans

• Alt. Method for Common Root Condition

(i) $f(x)=0 \to \alpha, \beta, \gamma$ $\Rightarrow f(\beta)=0, f(\alpha)=0, f(\gamma)=0$

(ii) $g(x)=0 \to \alpha, \beta, \delta, \theta$ $\Rightarrow g(\alpha)=0, g(\beta)=0$

teleco Dt.: Pg.:

Consider $k_1(i) + k_2(ii)$

$\Rightarrow k_1 f(x) + k_2 g(x) = 0$ — (iii), $k_1, k_2 \neq 0$ (or) $k_1 \cdot k_2 \neq 0$

$x = \alpha \Rightarrow k_1 f(\alpha) + k_2 g(\alpha) = 0$

$x = \gamma \Rightarrow k_1 f(\gamma) + k_2 g(\gamma) \neq 0$

In the above case (iii) equation is combination of 1^{st} two equations. In this case all common roots between 1^{st} two equations will also be roots of 3^{rd} equation & any root which satisfy exactly one of the first two equations will not be the root of 3^{rd} equation.

i.e., if any root satisfies the two of the three given equations then it will also satisfy remaining third equation.

Let $a_1x^2 + b_1x + c_1 = 0$ — (i) has roots α, β

& $a_2x^2 + b_2x + c_2 = 0$ — (ii) has roots α, γ.

Now consider $a_2(i) - a_1(ii)$

$(b_1a_2 - a_1b_2)x + a_2c_1 - c_2a_1 = 0$ — (iii)

$\Rightarrow$ Now as (iii) ~~is~~ equation is combination of 1^{st} two equations hence it should carry the common root b/w 1^{st} 2 equation. But it is linear hence will have only one root $\Rightarrow$ This root should be 'α' i.e, common root.

$\Rightarrow$ Common root b/w 1^{st} 2 equation should ~~will~~ be

$$\boxed{\alpha = \frac{c_2a_1 - a_2c_1}{b_1a_2 - b_2a_1}}$$

Substitute in equation (i)

$\Rightarrow$ Common Root condition

$$\boxed{a_1\left(\frac{c_2a_1 - a_2c_1}{b_1a_2 - b_2a_1}\right)^2 + b_1\left(\frac{c_2a_1 - a_2c_1}{b_1a_2 - b_2a_1}\right) + c_1 = 0}$$

Telco Dt.: Pg.:

Q1. Find 'a' if the equations $x^2-(a+2)x+3a=0$ (i) & $x^2-(a+3)x+5a=0$ (ii) has a common root.

Ans 1. Both roots cannot be common as $\frac{1}{1} \neq \frac{a+2}{a+3}$.

Now consider (i) − (ii)

$\Rightarrow (a+3)x-(a+2)x-2a=0$

$\Rightarrow x-2a=0 \Rightarrow \boxed{x=2a}$ → Common root

For common root condition substitute $x=2a$ in eq (i)

$\Rightarrow (2a)^2-(a+2)(2a)+3a=0$

$\Rightarrow 4a^2-2a^2-4a+3a=0$

$\Rightarrow 2a^2-a=0$

$\Rightarrow a(2a-1)=0$

$\Rightarrow \boxed{a=0, \frac{1}{2}}$ Ans   Common root $=0, 1$

Q2. Let α is negative real number & Satisfy all three equations $x^2-ax+15=0$, $x^2-bx+21=0$ & $x^2-(a+b)x+45=0$, then find α, a, b.

Ans 2. $x^2-ax+15=0$ — (i)

$x^2-bx+21=0$ — (ii)

$x^2-(a+b)x+45=0$ — (iii)

Consider (i) + (ii) − (iii)

$x^2-9=0$ — (iv)

$\Rightarrow$ As (iv) is combination of first three equations hence will have common root.

$\Rightarrow x = 3, -3$

$\Rightarrow$ Common root $= -3$

$\Rightarrow \boxed{\alpha = -3}$

Consider (i) $\qquad \beta = -5$

$x^2 - ax + 15 = 0 \longrightarrow -3, \beta$ $\quad$ Product of roots $= -3 \times \beta = 15$

$\Rightarrow 9 + 3a + 15 = 0$ $\quad$ (or) Sum of roots $= a = -3-5 = -8$

$\Rightarrow 3a = -24 \Rightarrow \boxed{a = -8}$

Consider (ii)

$\Rightarrow x^2 - bx + 21 = 0 \longrightarrow -3, \gamma$ (or) Product of roots $= 21 = \gamma \times -3$

$\Rightarrow 9 + 3b + 21 = 0$ $\quad \Rightarrow \gamma = -7$

$\Rightarrow 3b = -30 \Rightarrow \boxed{b = -10}$ $\quad \Rightarrow$ Sum of roots $= b = -7-3 = -10$

$\Rightarrow \boxed{\alpha = -3}$ $\boxed{a = -8}$ $\boxed{b = -10}$

NATURE of ROOTS

(I) Let $a \neq 0$ & a, b, c are constants then roots of $ax^2 + bx + c = 0$ will be

(a) Real, if $b^2 - 4ac \geq 0$

(b) Imaginary, if $b^2 - 4ac < 0$

$\rightarrow$ If $b^2 - 4ac > 0$, then roots will be real & distinct

$\rightarrow$ If $b^2 - 4ac = 0$, then roots will be real & repeated

Q Find α if roots of $x^2 - 4\alpha x + (\alpha + 5)$ are real & distinct.

$\Rightarrow D > 0 \Rightarrow (4\alpha)^2 - 4(\alpha + 5) > 0$

$\Rightarrow 4\alpha^2 - \alpha - 5 > 0 \Rightarrow (\alpha + 1)(4\alpha - 5) > 0$

$\Rightarrow \alpha \in (-\infty, -1) \cup \left(\frac{5}{4}, \infty\right)$ Ans

(II) If an equation is multiplied by a constant then its roots remain unchanged.

(III) If coefficients are real then, if exist, imaginary roots will occur in conjugate pair.

$\sqrt{-1} = i \Rightarrow i^2 = -1$

Q $x^2 + 2x + 4 = 0$: Solve

$\Rightarrow x = \frac{-2 \pm \sqrt{12}\sqrt{-1}}{2} = \frac{-2 \pm 2\sqrt{3}\,i}{2}$

$\Rightarrow x = -1 \pm 2\sqrt{3}i$

$\Rightarrow x = -1 + 2\sqrt{3}i, -1 - 2\sqrt{3}i.$

- $i = \sqrt{-1} \Rightarrow i^2 = -1 \Rightarrow i^3 = -i, i^4 = 1$
- $3+2i \xleftrightarrow{\text{Conjugate}} 3-2i$

  $5i \xleftrightarrow{\text{Conjugate}} -5i$

  $4i-8 \xleftrightarrow{\text{Conjugate}} -8-4i$

Q1. True / False:

(i) If roots of $ax^2+3x+5=0$ are imaginary then, they will be conjugate of each other (given $a \neq 0$).

Ans False.

(ii) If one root of $3x^2+5x+27=0$ is $\alpha+i\beta$ then other root will be $\alpha - i\beta$.

Ans: ~~True~~ False. because nature of α & β is not Known.

(IV) If coefficients of a Quadratic are rational number then if exists, irrational roots will occur in pair.

$\frac{(-b) \pm \sqrt{D}}{(2a)}$ → Responsible for irrationality

Rational

Q1. Let one root of $ax^2+3x+b=0$ is $2+\sqrt{3}$ then find $a, b (\in Q)$ & find other root

Ans. ∵ $a, b, 3 \in Q \Rightarrow$ Irrational roots occur in pair

$\Rightarrow$ other root $= 2-\sqrt{3}$

Sum of roots $= \frac{-3}{a} = 2-\sqrt{3}+2+\sqrt{3} \Rightarrow \frac{-3}{a} = 4$

$\Rightarrow \boxed{a = \frac{-3}{4}}$

Product of roots $= \frac{b}{a} = (2-\sqrt{3})(2+\sqrt{3}) = 4-3$

$\Rightarrow b = a = \frac{-3}{4} \Rightarrow \boxed{b = \frac{-3}{4}}$

$\Rightarrow \boxed{a = b = -\frac{3}{4}}$ (2) other root $= 2-\sqrt{3}$ Ans

Q2. Let $a, b \in Q$ & one root of equation $ax^2+(b+3a)x+(b+2a)=0$ is $2+\sqrt{5}$ then find a, b. Given $a \cdot b \neq 0$

Ans2. Coeff are rational $\Rightarrow$ Roots occur in pair.

$\Rightarrow$ Other root $= 2-\sqrt{5}$

$\Rightarrow S = \frac{-(b+3a)}{a} = 4 \Rightarrow b+7a=0$

$\Rightarrow P = \frac{(b+2a)}{a} = -1 \Rightarrow b+3a=0$

$\Rightarrow a = 0$ ×

∵ $a=0$, we need not to find value of 'b' as $ab \neq 0$

$\Rightarrow$ $\exists$ no such value of a & b.

Q3. Solve for $x, y (\in \mathbb{Q})$ if $x + (\sqrt{3}+2)y + (5+\sqrt{3})(2x+1) = 0$

$\Rightarrow$ $x + \sqrt{3}y + 2y + 5 + 10x + 2\sqrt{3}x + \sqrt{3} = 0$

$\Rightarrow$ $\sqrt{3}(y+2x+1) + (x+2y+5) = 0$

$\Rightarrow$ (Irrational)(Rational) + Rational $= 0$, iff Rationals are $= 0$

$\Rightarrow$ $\left.\begin{array}{l} y+2x+1=0 \\ x+2y+5=0 \end{array}\right] \Rightarrow y=-3, x=1$ Ans

- Important Points

(I) If sum of coefficients is zero then one of the roots will be '1'.
i.e., if $a+b+c=0$ then '1' will be one of the roots of the equation $ax^2+bx+c=0$.

Q1. Prove that following equations have rationals roots given $a, b, c \in \mathbb{Q}$.

(i) $(a-b)x^2 + (b-c)x + (c-a) = 0$

$x=1 \Rightarrow$ $0=0 \Rightarrow \boxed{x=1}$ is a solution

$\Rightarrow$ One root is 1, hence other root should also be rational, as coefficients are rational.
Product of roots $= 1 \cdot \alpha = \dfrac{c-a}{a-b}$

$\Rightarrow$ $\alpha = \dfrac{c-a}{a-b}$ → Other root

(ii) $(a-b)x^2 + (3a+2b)x + 2a+8b = 0$

$\Rightarrow$ By observation, -2 is one root, hence other root should be rational as coefficients are rational.

Telco Dt.: Pg.:

Products of roots $= \frac{2a+8b}{(a-b)} = -2\cdot\beta$

$\Rightarrow \boxed{\beta = -\frac{(a+4b)}{(a-b)}}$ Ans $\rightarrow$ Other root.

(II) With respect to the equation $ax^2+bx+c=0$
a,b,c are real constants & $a\neq 0$.

(a) If roots are imaginary then a,c should be of the same sign.
$D = b^2 - 4ac$ (+)

Note: Vice-versa of (a) is not true.

(b) If a,c are of opposite sign, then roots will be real & distinct & vice-versa is not true.

Q. Prove that the equation $(ax^2+bx+c)(ax^2+dx-c)=0$ will have at least 2 real roots. Given, $a,b,c,d \in R$, $a\neq 0$

Note: $E_1=0 \rightarrow \alpha,\beta$ $\quad E_2=0 \rightarrow \gamma,\delta,\theta$

$\Rightarrow \boxed{E_1 E_2 = 0} \longrightarrow \alpha,\beta,\gamma,\delta,\theta$

Ans. $ax^2+bx+c=0 \longrightarrow \alpha,\beta$

$ax^2+dx-c=0 \longrightarrow \gamma,\delta$

$\hookrightarrow D_1 = d^2+4ac$

$\hookrightarrow D_2 = d^2-4ac$

C-1 $\quad ac>0 \Rightarrow D_1>0 \Rightarrow \gamma,\delta$ are 2 real roots

C-2 $\quad ac<0 \Rightarrow D_2>0 \Rightarrow \alpha,\beta$ are 2 real roots

C-3 $\quad ac=0 \Rightarrow D_1\geq 0, D_2\geq 0 \Rightarrow \alpha,\beta,\gamma,\delta$ all are roots.

$\Rightarrow$ Hence, proved.

Alt. $\left.\begin{array}{l} D_2 = b^2 - 4ac \\ D_1 = b^2 + 4ac \end{array}\right] \Rightarrow D_1 + D_2 = b^2 + d^2$

$\Rightarrow D_1 + D_2 \geq 0 \Rightarrow D_1$ & D_2 can not be negative simultaneously $\Rightarrow$ Atleast 1 equation will have real roots. Hence, proved.

SIGN of QUADRATIC EXPRESSION

Let $f(x) \overset{y=}{=} ax^2 + bx + c,\ a \neq 0,\ a, b, c \in R$

$f(2) = 4a + 2b + c$

$9a - 3b + c = f(-3)$

$25a + 15b + 9c = 9\left(\frac{25}{9}a + \frac{15}{9}b + c\right)$

$= 9\left(\frac{25}{9}a + \frac{5}{3}b + c\right)$

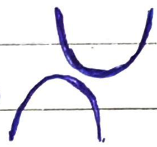
$= 9f\left(\frac{5}{3}\right)$

• Graph of Quadratic Expression

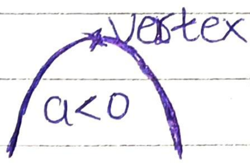
$y = f(x) = ax^2 + bx + c,\ a \neq 0,\ a, b, c = \text{const.}$

(I) $a > 0 \Rightarrow$ Parabola Upwards

$a < 0 \Rightarrow$ Parabola Downwards



(II)

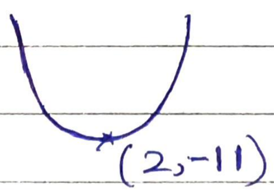


$\boxed{\text{Vertex} : \left(\frac{-b}{2a}, \frac{-D}{4a}\right)}$

Eg. $y = 2x^2 - 8x - 3$

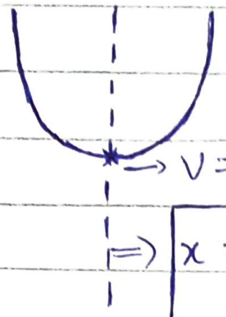
$a = 2,\ b = -8,\ c = -3$

$D = 64 + 4 \times 2 \times 3 = 88$



$\Rightarrow \text{Vertex} = \left(\frac{-b}{2a}, \frac{-D}{4a}\right) = (2, -11) = (2, -11)$

(III) Line of Symmetry

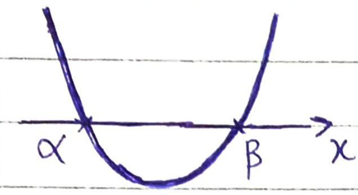


$$V = \left(\frac{-b}{2a}, \frac{-D}{4a}\right)$$

$$\Rightarrow \boxed{x = \frac{-b}{2a}}$$

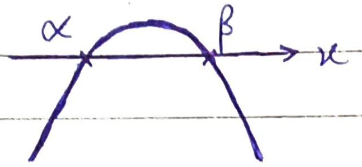
(IV)

C-1

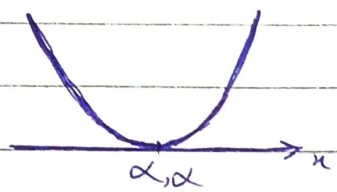


$\boxed{D>0}$

OR

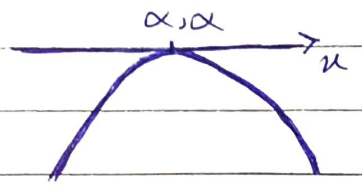


C-2

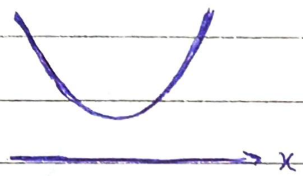


$\boxed{D=0}$

OR

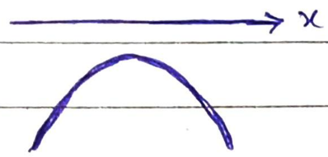


C-3



$\boxed{D<0}$

OR

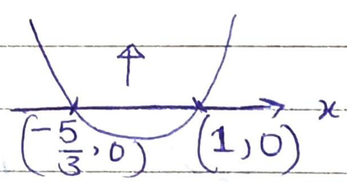


Eg. 1 $y = 3x^2 + 2x - 5$

$\hookrightarrow a > 0 \Rightarrow$ P.U.

$\hookrightarrow D = 4 + 4 \times 3 \times 5 = 64 > 0$

roots $= \frac{-2 \pm 8}{6} = 1, -\frac{5}{3}$

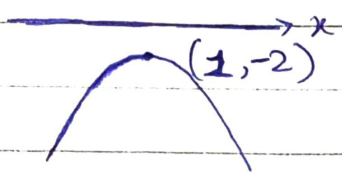


2. $y = -x^2 + 2x - 3$

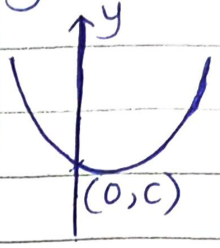
$\hookrightarrow a < 0 \Rightarrow$ P.D.

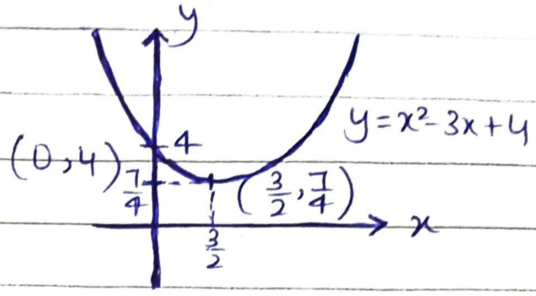
$\hookrightarrow D = 4 - 4 \times 3 = -8 < 0$

$\frac{-D}{4a} = \frac{8}{4(-1)} = -2, \quad \frac{-b}{2a} = \frac{-2}{2(-1)} = 1$



(V) $y = ax^2+bx+c \xrightarrow{x=0} y = c \Rightarrow (0,c)$



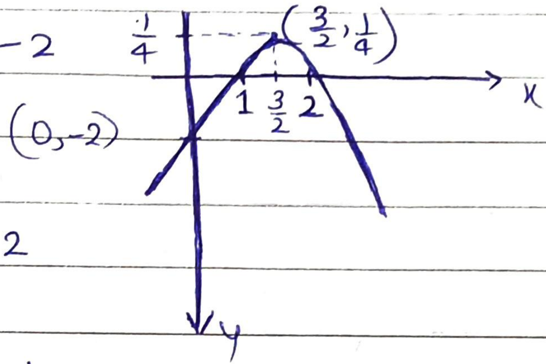


Ex:1 $y = x^2-3x+4$

↳ $a>0 \Rightarrow$ P.U

↳ $D = 9-4(4) = -7<0$

$\frac{-b}{2a} = \frac{3}{2}$, $\frac{-D}{4a} = \frac{7}{4}$



2. $y = 3x - x^2 - \cancel{4}\, 2 = -x^2+3x-2$

↳ $\cancel{a>0}$ $a<0 \longrightarrow$ P.D.

↳ $D = 9+4(2)(-1) = 1>0$

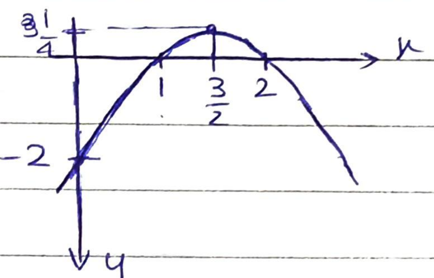
↳ Roots: $\frac{-3\pm 1}{2(-1)} = 1, 2$

↳ $\frac{-b}{2a} = \frac{-3}{2(-1)} = \frac{3}{2}$, $\frac{-D}{4a} = \frac{-1}{4(-1)} = \frac{1}{4}$

OR

$y = -x^2+3x-2 = -(x-1)(x-2)$

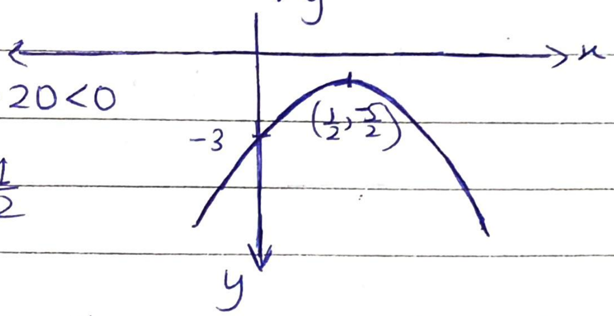
$x = \frac{3}{2} \Rightarrow y = -\frac{1}{2}\left(\frac{-1}{2}\right) = +\frac{1}{4}$



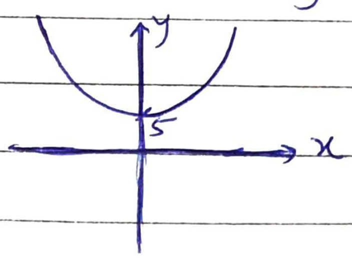
3. $y = -2x^2+2x-3$

↳ $D = 4-4\times(-2)(-3) = -20<0$

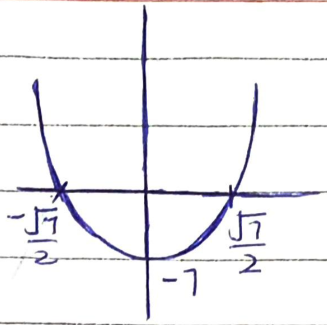
$\frac{-D}{4a} = \frac{20}{4(-2)} = \frac{-5}{2}$, $\frac{-b}{2a} = \frac{1}{2}$



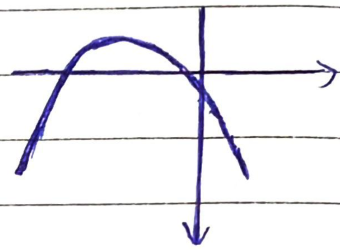
4. $y = 2x^2+5$



5. $y = 4x^2 - 7$



6. $y = ax^2 + bx + c$, Find sign of a, b, c, D.



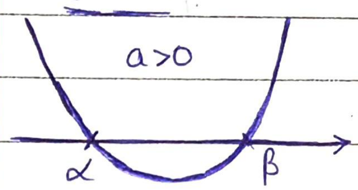
Ans $a < 0$, $-\frac{b}{2a} < 0$, $c < 0$, $D > 0$

$\Rightarrow$ $a < 0$, $b < 0$, $c < 0$, $D > 0$

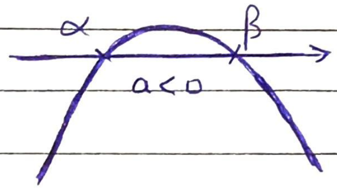
SIGN of QUADRATIC EXPRESSION

$y = ax^2 + bx + c$, $a \neq 0$ a, b, c = const.

C-1 $D > 0$



OR

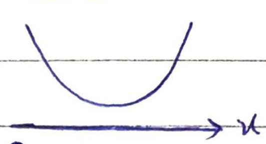


$f(x) < 0, \forall x \in (\alpha, \beta)$	$f(x) < 0, \forall x \in (-\infty, \alpha) \cup (\beta, \infty)$
$f(x) > 0, \forall x \in (-\infty, \alpha) \cup (\beta, \infty)$	$f(x) > 0, \forall x \in (\alpha, \beta)$
$f(x) = 0, x = \alpha, \beta$.	$f(x) = 0, x = \alpha, \beta$

$\Downarrow$

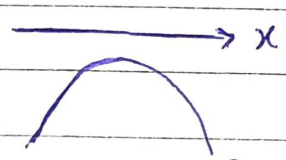
$$\boxed{D > 0 \Rightarrow \begin{cases} \bullet\ a \cdot f(x) < 0, & \forall x \in (\alpha, \beta) \\ \bullet\ a f(x) > 0, & \forall x \in (-\infty, \alpha) \cup (\beta, \infty) \\ \bullet\ a f(x) = 0, & x = \alpha, \beta \end{cases}}$$

C-2 $D < 0$



$f(x) > 0$, $x \in R$

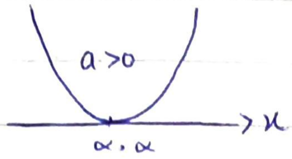
OR



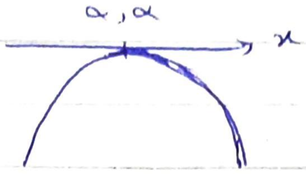
$f(x) < 0$, $x \in R$

$\Rightarrow$ $\boxed{D<0 \Rightarrow a f(x) > 0, \forall x \in R}$

C-3



(OR)
D=0



$f(x) \geq 0, \forall x \in R$ || $f(x) \leq 0, \forall x \in R$

$\boxed{D = 0 \Rightarrow a f(x) \geq 0, \forall x \in R}$

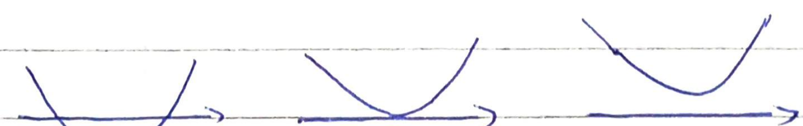
NOTE:- (I) $a \cdot f(x)$ can be negative only in first case i.e. for $\boxed{D>0}$
hence if $a \cdot f(k) < 0$, then roots will be real & distinct. Also k will lie b/w roots.

(II) If $a f(k) > 0$, then roots may or may not be real.

(III) If $f(k_1) \cdot f(k_2) < 0$, then roots should be real & distinct. & b/w k_1 & k_2 there should be exactly one root.
(OR) the interval (k_1, k_2) will contain exactly one root.

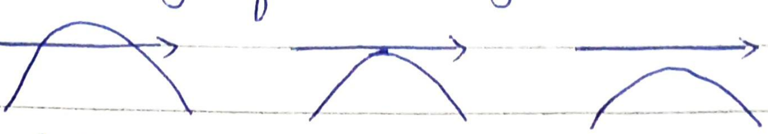
(IV) If $f(k_1) \cdot f(k_2) > 0$, then roots may (or) may not be real.

(V) (a)



Parabola upwards can be negative in an interval of finite length.

(b)



Parabola downwards can be positive in an interval of finite length.

Telco Dt.: Pg.:

Q1. In which of the following cases roots of $ax^2+bx+c=0$, $a\neq 0$ & $a,b,c \in R$, will be definitely real?

(i) $a(a+b+c)<0$

$\Rightarrow af(1)<0$

$\Rightarrow$ Roots are real & distinct.

(ii) $(a+3b+9c)(16a-4b+c)>0$

$\Rightarrow 9f\left(\frac{1}{3}\right)f(-4)>0$

$\Rightarrow f\left(\frac{1}{3}\right)\cdot f(-4)>0$

$\Rightarrow$ Roots may or may not be real.

(iii) $c(4a-2b+c)<0$

$\Rightarrow cf(-2)<0$

$\Rightarrow f(0)\cdot f(-2)<0 \quad (c=f(0))$

$\Rightarrow$ Roots are real & distinct as $f(k_1)\cdot f(k_2)<0$

(iv) $a(a+7b+49c)+c(a-b+c)<0$

$\Rightarrow \frac{a49}{\cancel{49}}\left(\frac{a}{49}+\frac{b}{7}+c\right)+cf(-1)<0$

$\Rightarrow 49af\left(\frac{1}{7}\right)+f(0)\cdot f(-1)<0$

As at least one of the two terms at LHS should be negative $\Rightarrow$ at least one of the : $af\left(\frac{1}{7}\right)$ or $f(0)\cdot f(-1)$ should be negative $\Rightarrow$ Roots are real & distinct.

Q2. Find 'a' if $(a+1)x^2-2ax+(a+1)>0 \;\forall\; x\in R$

Ans Let $f(x)=(a+1)x^2-2ax+(a+1)$

We need to find $f(x)>0, \forall x\in R$

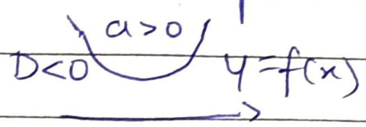
C-1 If $a+1=0 \Rightarrow \boxed{a=-1}$

$\Rightarrow f(x)>0 \Rightarrow -2(-1)(x)>0 \Rightarrow 2x>0 \ \forall \ x \in R$

which is not possible hence $a \neq -1$

C-2 If $a+1 \neq 0 \Rightarrow a \neq -1 \Rightarrow y=f(x)$ will be a parabola

$\Rightarrow f(x)>0, \forall \ x \in R$



↳ (1) $a+1>0 \Rightarrow a>-1$

↳ (2) $D = 4a^2 - 4(a+1)^2 < 0$

$\Rightarrow a^2-(a^2+1+2a)<0 \Rightarrow -2a-1<0$

$\Rightarrow 2a+1>0 \Rightarrow a>-\frac{1}{2}$

$\Rightarrow \boxed{a \in \left(-\frac{1}{2}, \infty\right)}$ Ans

Q1. Let the equation $4ax^2-bx-c=0$ has a root 'α' ($\in R$) & equation $5ax^2+bx+c=0$ has a real root 'β'. Then prove that the equation $ax^2+bx+c=0$ will have a real root b/w 'α' & 'β', ($a \neq 0$, $a, b, c, \in R$)

Ans Let $f(x)=ax^2+bx+c$

Given: $4a\alpha^2-b\alpha-c=0$ & $5a\beta^2+b\beta+c=0$

$4a\alpha^2=b\alpha+c$ ↳ $b\beta+c=-5a\beta^2$

$f(\alpha)=a\alpha^2+b\alpha+c$

$f(\alpha)=5a\alpha^2$

$f(\beta)=a\beta^2+b\beta+c=-4a\beta^2$

$\Rightarrow f(\alpha)\cdot f(\beta)=(5a\alpha^2)(-4a\beta^2)=-20a^2\alpha^2\beta^2<0$ always

$\Rightarrow f(\alpha)\cdot f(\beta) < 0$

$\therefore$ One root of $f(x)=0$ lies b/w α & β

Hence Proved.

LOCATION of ROOTS

Let $f(x) = ax^2 + bx + c,\ a \neq 0,\ a, b, c \in R$, then the wrt the equation $f(x) = 0$

C-1 If 'K' lies b/w the roots of $f(x) = 0$, then $\boxed{a \cdot f(K) < 0}$

Q. Find 'a' if '3' lies between the roots of $(a+4)x^2 - 3ax + 5 = 0$

Ans $(a+4) \cdot f(3) < 0$

$\Rightarrow (a+4)(9(a+4) - 9a + 5) < 0$

$\Rightarrow (a+4)(41) < 0$

$\Rightarrow a < -4 \Rightarrow a \in (-\infty, -4)$ Ans

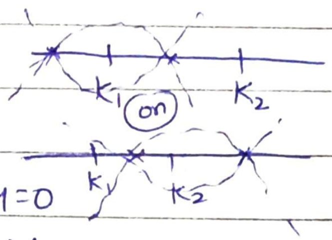
Q2 Find 'a' if roots of the equation $(a-5)x^2 + (4a-3)x + (7a-8) = 0$ are of opposite sign.

Ans $(a-5) f(0) < 0$

$\Rightarrow (a-5)(7a-8) < 0$

$\Rightarrow a \in \left(\frac{8}{7}, 5\right)$ Ans

C-2 If exactly one root of $f(x) = 0$ lies b/w K_1 & K_2 & K_1, K_2 themselves are not roots, then $\boxed{f(K_1) \cdot f(K_2) < 0}$.



Q1. Find 'a' if equation: $(a+3)x^2 - 3ax + 4 = 0$ have roots such that magnitude of one root is less than '2' & that of other is greater than '2'.

Ans 1. $f(-2) \cdot f(2) < 0$

$\Rightarrow (4(a+3) + 6a + 4)(4(a+3) - 6a + 4) < 0$

$\Rightarrow (10a+16)(16-2a) < 0$

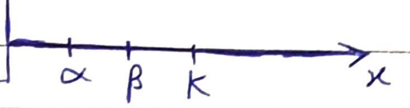
$\Rightarrow -2(5a+8)(a-8) < 0$

$\Rightarrow (5a+8)(a-8) > 0$

$\Rightarrow \boxed{a \in \left(-\infty, -\frac{8}{5}\right) \cup (8, \infty)}$ Ans

C-3 If k is greater than both the roots.

$\rightarrow \boxed{a \cdot f(k) > 0 \text{ \& } D \geq 0} \boxed{\text{\& } k > \frac{-b}{2a}}$



(i) $a \cdot f(k) > 0$
& (ii) $D \geq 0$
& (iii) $k > \frac{-b}{2a}$

} All are necessary & sufficient

C-4 If k is less than both ~~thea~~ roots.

(i) $a \cdot f(k) > 0$
& (ii) $D \geq 0$
& (iii) $k < \frac{-b}{2a}$

} All are necessary & sufficient.

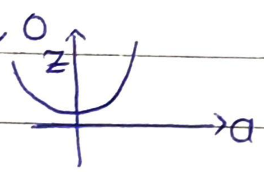
Q2. Find 'a' if both roots of $ax^2 - 2(a+2)x + 3 = 0$ are negative.

$\Rightarrow D \geq 0$ & $af(0) > 0$ & $0 > \frac{+2(a+2)}{a}$

(i) $\Rightarrow \cancel{4}(a+2)^2 - \cancel{4}a(3) \geq 0$

$\Rightarrow (a+2)^2 - 3a \geq 0 \Rightarrow a^2 + 4 + a \geq 0$

$\Rightarrow a^2 + 4 + a \geq 0$ always true



$\Rightarrow a \in R$

(ii) $af(0) > 0 \Rightarrow a > 0$

iii) $\frac{2(a+2)}{a} < 0 \Rightarrow \frac{a+2}{a} < 0 \Rightarrow a \in (-2, 0)$

$\Rightarrow a = (i) \cap (ii) \cap (iii) = R \cap (0, \infty) \cap (-2, 0)$

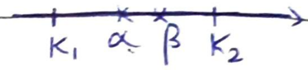
$\Rightarrow \boxed{a \in \phi}$ Ans

C-5 If (K_1, K_2) Contains both the roots

(i) $D \geq 0$

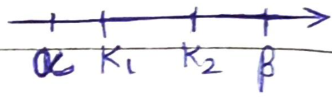
(ii) $af(K_2) > 0$ & $af(K_1) > 0$

(iii) $K_1 < -\frac{b}{2a} < K_2$

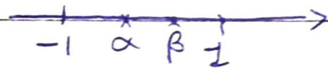


C-6 If the interval $[K_1, K_2]$ is contained b/w the roots, then

$af(K_1) < 0$ & $af(K_2) < 0$



Q1. Find 'a' if both roots of $(a+1)x^2 - 2ax - 3 = 0$ are less than '1' in magnitude



(i) $D \geq 0$

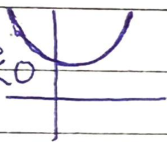
(ii) $(a+1)f(1) > 0$ & $(a+1)f(-1) > 0$

(iii) $-1 < +\frac{2a}{2(a+1)} < 1$

(i) $\Rightarrow 4a^2 + 4(a+1)(3) \geq 0$

$\Rightarrow a^2 + 3a + 3 \geq 0$

always true $D_1 < 0$



$\Rightarrow a \in R$

(ii) $(a+1)(a+1-2a-3) > 0$

$\Rightarrow (a+1)(a+2) < 0$

$\Rightarrow a \in (-2, -1)$

(iii) $(a+1)(a+1+2a-3) > 0$

$\Rightarrow (a+1)(3a-2) > 0$

$\Rightarrow a \in (-\infty, -1) \cup (\frac{2}{3}, \infty)$

(iv) $-1 < \frac{a}{a+1} < 1 \Rightarrow -1 < \frac{a}{a+1}$ & $\frac{a}{a+1} < 1$

$\Rightarrow \frac{a}{a+1} + 1 > 0$ & $\frac{a}{a+1} - 1 < 0$

$\Rightarrow \frac{2a+1}{a+1} > 0$ & $\frac{-1}{a+1} < 0 \Rightarrow \frac{1}{a+1} > 0$

$\Rightarrow a \in (-\infty,-1)\cup(-\frac{1}{2},\infty)$ & $a \in (-1,\infty)$

$\Rightarrow$ $\cap$

$a \in (-\frac{1}{2},\infty)$

$\Rightarrow a = (i)\cap(ii)\cap(iii)\cap(iv)$

$= R\cap(-2,-1)\cap(-\infty,-1)\cup(\frac{2}{3},\infty)\cap(-\frac{1}{2},\infty)$

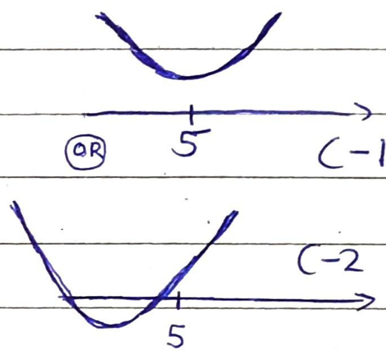
$\Rightarrow \boxed{a \in \phi}$ Ans

$\rightarrow a \neq 2$ as if $a=2 \Rightarrow x < \frac{1}{2}$ Not possible

Q2. Find 'a' if $(a+2)x^2-2x+1 > 0 \ \forall x \in [5,\infty)$

$\Rightarrow$ It has to be parabola upwards

$\Rightarrow a+2 > 0$



C-1 If roots are not real

$f(x) = (a+2)x^2-2x+1$

$\Rightarrow D = 4-4(a+2)(1) < 0$

$\Rightarrow 1-a-2 < 0$

$\Rightarrow a+1 > 0 \Rightarrow a > -1$

$\Rightarrow a \in (-2,\infty)\cap(-1,\infty) \Rightarrow a \in (-1,\infty)$ — (1)

C-2 $D \geq 0 \Rightarrow 1-a-2 \geq 0 \Rightarrow -a \geq +1 \Rightarrow a \leq -1$

& $af(5) > 0 \Rightarrow (a+2)(25a+50-10+1) > 0$ $\Rightarrow a \in (-\infty,-1]$

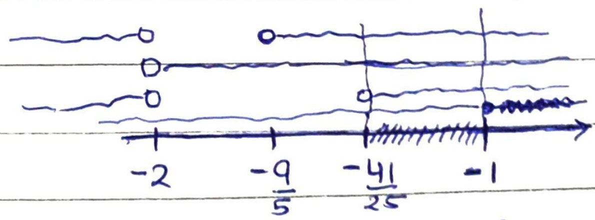
& $5 > \frac{-(-2)}{2(a+2)} \Rightarrow 5 > \frac{1}{a+2} \Rightarrow \frac{1}{a+2} - 5 < 0$

(i) $a \in (-\infty,-1]$

(ii) $(a+2)(25a+41) > 0 \Rightarrow \cancel{a \in (-2,-\frac{41}{25})}$ $a \in (-\infty,-2)\cup(-\frac{41}{25},\infty)$

(iii) $\frac{-(5a+9)}{(a+2)} < 0 \Rightarrow \frac{5a+9}{a+2} > 0$

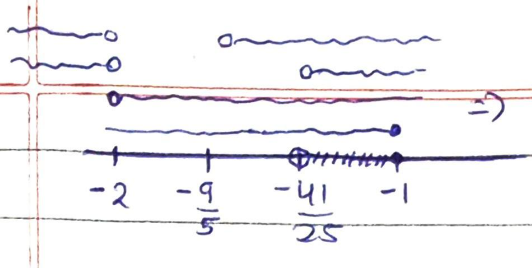
$\Rightarrow a \in (-\infty,-2)\cup(-\frac{9}{5},\infty)$



(iv) $a \in (-2,\infty)$

$\Rightarrow (i)\cap(ii)\cap(iii)\cap(iv) \Rightarrow a \in (-\infty,-1]\cap(-\infty,-2)\cup(-\frac{41}{25},\infty)\cap$

$(-2,\infty)\cap(-\infty,-2)\cup(-\frac{9}{5},\infty)$



$\Rightarrow a \in \left(-\frac{41}{25}, -1\right]$ — (2)

$\Rightarrow$ C-1 $\cup$ C-2 $\Rightarrow a \in \left(-\frac{41}{25}, -1\right] \cup (-1, \infty) \equiv \left(-\frac{41}{25}, \infty\right)$ Ans

Q3. Find 'a' if $(a+3)x^2 - 4x + 2 < 0 \ \forall \ x \in [-2, +1]$.

If $a+3 = 0 \Rightarrow -4x + 2 < 0$

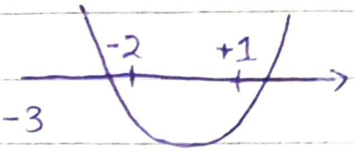
$\Rightarrow x > 1/2$

$\hookrightarrow$ Not possible

$\Rightarrow a \neq -3$

C-1 Parabola Upwards $(a+3 > 0)$

$\hookrightarrow a > -3$



$f(x) = (a+3)x^2 - 4x + 2$

$f(x) < 0, \forall x \in [-2, -1]$

$D \geq 0$

$\Rightarrow 4^2 - 4(a+3)(2) > 0$

$\Rightarrow 2 - a - 3 > 0$

$\Rightarrow a < -1$

$\Rightarrow a \cdot f(-2) < 0$ & $a f(1) < 0$

$\Rightarrow (a+3) f(-2) < 0$ & $(a+3) f(1) < 0$

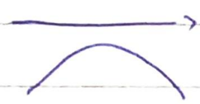
$\Rightarrow (a+3)(4a+22) < 0$ & $(a+3)(a+1) < 0$

$\Rightarrow a \in \left(-\frac{22}{4}, -3\right)$ & $a \in (-3, -1)$ & $a \in (-3, \infty)$

$a = \Phi$

C-2 Parabola Downwards $(a+3 < 0) \Rightarrow a \in (-3, \infty)$

SC-1 D



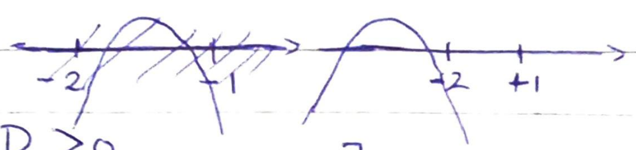
$D < 0$

$\Rightarrow 16 - 4(a+3)(2) < 0$

$\Rightarrow 2 - a - 3 < 0 \Rightarrow a > -1$

$\hookrightarrow a \in (-1, \infty)$

SC-2

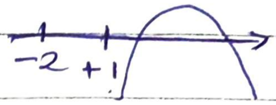


$D \geq 0$

$(a+3) f(-2) \geq 0$ → SC-2

$-2 > \frac{+4}{2(a+3)}$

S-C-3



$D \geq 0$

$(a+3) f(1) > 0$ $\cap$ $\rightarrow$ S-C-3

$1 < \frac{4}{2(a+3)}$

C-2 : SC-1 $\cup$ SC-2 $\cup$ SC-3

$\Rightarrow$ C-1 $\cup$ C-2 Ans

Q34. Find 'k' if $(k-2)\,9^x - 2\cdot 3^x + 3 > 0,\ \forall x > 0,\ k \neq 2$

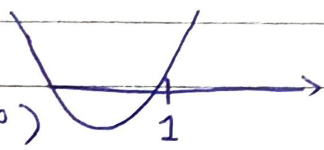
$\Rightarrow (k-2)\,9^x - 2\cdot 3^x + 3 > 0,\ \forall x > 0$

$3^x = t \Rightarrow$ if $x > 0 \Rightarrow 3^x > 1 \Rightarrow t > 1$

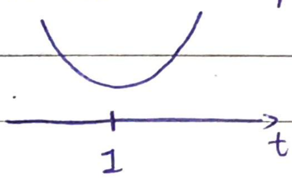
$\Rightarrow (k-2)t^2 - 2t + 3 > 0,\ \forall t > 1$

Parabola Upwards $(k-2 > 0)$

$\Rightarrow k \in (2, \infty)$

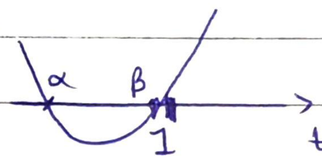


C-1



$D < 0$

C-2

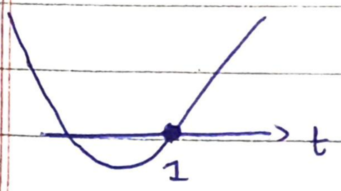


$D \geq 0$

$(k-2) f(1) > 0$

$1 > \frac{2}{2(k-2)}$

C-3



$f(1) = 0$

$\Rightarrow k-2-2+3 = 0$

$\Rightarrow k = 1$

$f(t) = -t^2 + 2t + 3 \Rightarrow$ Parabola downwards

$\Rightarrow$ Not Possible $\Rightarrow k \in \phi$

$\Rightarrow k \in (C-1) \cup (C-2)$

THEORY OF EQUATIONS

Let

$$a_n x^n + a_{n-1} \cdot x^{n-1} + a_{n-2} \cdot x^{n-2} + \ldots + a_1 x + a_0 = 0 \quad \text{(i)}$$

$a_0, a_1, a_2, \ldots, a_n \in R$

(or) $a_i \in R$, where $i = 0, 1, 2, \ldots, n$

N-1 If $a_n \neq 0$, then equation (i) is a n-degree polynomial equation which will have exactly n-roots, say $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n$

(a) $a_n x^n + a_{n-1} x^{n-1} + \ldots + a_0 \equiv a_n(x-\alpha_1)(x-\alpha_2)(x-\alpha_3)\ldots(x-\alpha_n)$ — (ii)

(b) (ii) is an identity, hence from both the sides coeff. can be compared.

$\Rightarrow a_n = a_n$

$\Rightarrow a_{n-1} = -a_n(\alpha_1 + \alpha_2 + \alpha_3 \ldots + \alpha_n)$

- $\Rightarrow \alpha_1 + \alpha_2 + \alpha_3 + \ldots + \alpha_n = -\dfrac{a_{n-1}}{a_n} = \Sigma \alpha_i$ = Sum of roots taken one at a time

- $\Rightarrow \alpha_1\alpha_2 + \alpha_1\alpha_3 + \ldots + \alpha_1\alpha_n + \alpha_2\alpha_3 + \ldots + \alpha_2\alpha_n + \alpha_3\alpha_4 + \ldots + \alpha_3\alpha_n$
  $= \dfrac{a_{n-2}}{a_n} = \Sigma \alpha_i \alpha_j$ = Sum of roots taken two at a time.

- $\Rightarrow \alpha_1\alpha_2\alpha_3 + \alpha_1\alpha_3\alpha_4 + \ldots + \ldots = -\dfrac{a_{n-3}}{a_n}$ = Sum of roots taken three at a time
  $= \Sigma \alpha_i \alpha_j \alpha_k$

- Sum of roots taken 'n' at a time
  $= \alpha_1 \cdot \alpha_2 \cdot \alpha_3 \cdot \alpha_4 \ldots \alpha_n$ = Product of roots $= (-1)^n \dfrac{a_0}{a_n}$

Eg. Find sum of roots of $3x^4 + 5x^3 - 7x^2 - 8x + 9 = 0$

$\Rightarrow \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = -\dfrac{5}{3}$ Ans

$\Rightarrow \alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_1\alpha_4 + \alpha_2\alpha_3 + \alpha_2\alpha_4 + \alpha_3\alpha_4 = +\dfrac{(-7)}{3} = -\dfrac{7}{3}$

$\Rightarrow \alpha_1\alpha_2\alpha_3 + \alpha_2\alpha_3\alpha_4 + \alpha_1\alpha_3\alpha_4 + \alpha_1\alpha_2\alpha_4 = -\dfrac{(-8)}{3} = \dfrac{8}{3}$

telco Dt.: Pg.:

$\Rightarrow \alpha_1\alpha_2\alpha_3\alpha_4 = (-1)^4\frac{9}{3} = \frac{9}{3}$

N-2 If equation (i) is satisfied for more 'n' distinct values of x, then it will become an identity i.e. it will be satisfied $\forall x$ & also in this case $a_n = a_{n-1} = a_{n-2} = \cdots = a_1 = a_0 = 0$

N-3 If $a_n \neq 0$ & if coeff. are real then imaginary roots (if exist) will occur in conjugate pair.

(a) Any polynomial equation with real coeff. will have even number of imaginary roots.

(b) Any odd degree polynomial equation with real coeff. will always have at least one real root. [(or) odd no. of real roots)

N-4 If $a_n \neq 0$, & coeff. are ~~real~~ rational then, if exist, irrational roots will occur in pair.

Q1. If roots of $ax^3 - bx^2 - cx + d = 0$ are in A.P. then find required condition.

Ans. Let roots are α, β, γ.

$\Rightarrow \alpha + \beta + \gamma = -\frac{(-b)}{a} = \frac{b}{a}$

$\Rightarrow \alpha\beta + \beta\gamma + \gamma\alpha = +\left(\frac{-c}{a}\right) = \frac{-c}{a}$

$\Rightarrow \alpha\beta\gamma = -\frac{(d)}{a} = -\frac{d}{a}$

According to Question:

$\Rightarrow 2\beta = \alpha + \gamma$

$\Rightarrow 3\beta = \alpha + \beta + \gamma$

$\Rightarrow 3\beta = \frac{b}{a} \Rightarrow \beta = \frac{b}{3a}$

$\Rightarrow$ β is a root of $ax^3 - bx^2 - cx + d = 0$, hence it will satisfy it.

$\Rightarrow a\left(\frac{b}{3a}\right)^3 - b\left(\frac{b}{3a}\right)^2 - c\left(\frac{b}{3a}\right) + d = 0$

$\Rightarrow \frac{b^3}{27a^2} - \frac{b^3}{9a^2} - \frac{bc}{3a} + d = 0$

$\Rightarrow d - \frac{2b^3}{27a^2} - \frac{bc}{3a} = 0$ Ans

Q2. Let $\alpha, \beta, \gamma, \delta$ are roots of $x^4 - 3x + 7 = 0$ then Calculate $(\alpha^4+4)(\beta^4+4)(\gamma^4+4)(\delta^4+4)$.

$\Rightarrow \alpha^4 - 3\alpha + 7 = 0$

$\Rightarrow \alpha^4 = 3\alpha - 7$

$\Rightarrow \alpha^4 + 4 = 3(\alpha - 1)$

$\Rightarrow (\alpha^4+4)(\beta^4+4)(\gamma^4+4)(\delta^4+4) = 3^4(\alpha-1)(\beta-1)(\gamma-1)(\delta-1)$

$\Rightarrow$ Required $= 81(\alpha-1)(\beta-1)(\gamma-1)(\delta-1)$

$\Rightarrow x^4 - 3x + 7 \equiv (x-\alpha)(x-\beta)(x-\gamma)(x-\delta)$

$x=1$ $\Rightarrow 1 - 3(1) + 7 \equiv \cancel{(x-\alpha)}\ (1-\alpha)(1-\beta)(1-\gamma)(1-\delta)$

$\Rightarrow 5 = (\alpha-1)(\beta-1)(\gamma-1)(\delta-1)$

$\Rightarrow$ Required $= 81 \times 5 = 405$ Ans.

Q3. Let roots of $x^3 - 7x^2 + 5x - 11 = 0$ are $\alpha, \beta, \gamma, \cancel{\delta}$ then find (a) $\Sigma \alpha^2$ (b) $\Sigma \alpha^3$

Ans $\alpha + \beta + \gamma = \frac{7}{1}$, $\alpha\beta + \beta\gamma + \gamma\alpha = \frac{5}{1}$

$\alpha\beta\gamma = (-1)^3(-11) = +11$

telco Dt.: Pg.:

ⓐ $\alpha^2+\beta^2+\gamma^2 = (\alpha+\beta+\gamma)^2 - 2(\alpha\beta+\beta\gamma+\gamma\alpha)$

$= (7)^2 - 2(5)$

$= 49 - 10 = 39$ Ans

ⓑ $\alpha^3+\beta^3+\gamma^3 = 7(\alpha^2+\beta^2+\gamma^2) - 5(\alpha+\beta+\gamma) + 11 \times 3$

$= 7 \times 39 - 5(7) + 33$

$= 7 \times 34 + 33 = 238 + 33 = 271$ Ans