Complex Number

$C =$ Real no + imaginary no.

$\underline{i = \sqrt{-1}}$.

Sum of the terms containing four consecutive powers of $i = 0$

$\sqrt{a} \times \sqrt{b} = \sqrt{ab}$ {Only when at least one of the a or b is non-negative}

any no. in the simplest form of $x + iy$ is complex no. $(x, y \in R)$

Purely Real $=$ $Im(z) = 0$, $x \in R$, $y = 0$

Purely Imaginary $=$ $Re(z) = 0$, $x = 0$, $y \in R$

Imaginary $=$ $x = R$, $y \in R - \{0\}$

$0 =$ Purely Real, Purely Imaginary and Both.

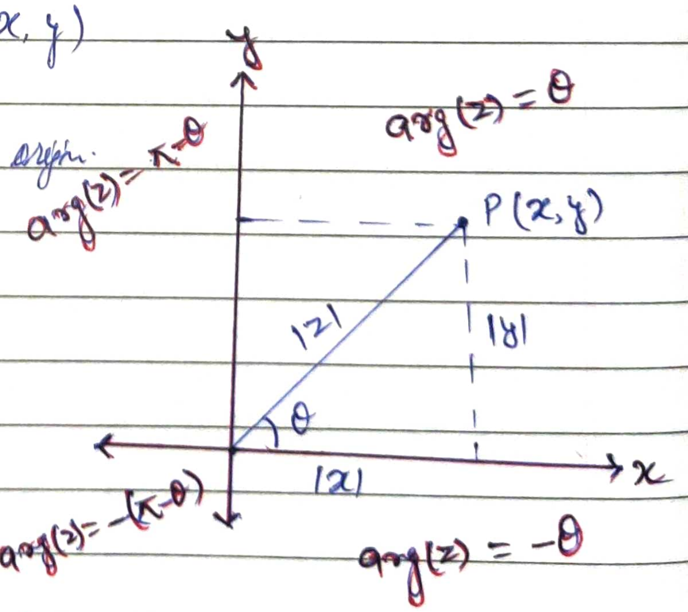
Inequality sign b/w complex no. is meaningless.

Complex no is a representation of a point on a argand plane.

$$z = x + iy \equiv (x, y)$$

→ $|z| =$ distance of point P from origin.

$$|z| = \sqrt{x^2 + y^2}$$

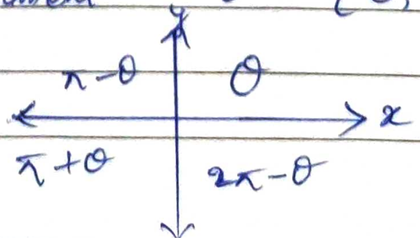


→ argument of z (θ)

$$\tan\theta = \frac{|y|}{|x|} \; ; \; \theta = \tan^{-1}\left(\frac{|y|}{|x|}\right)$$

i) Principal value (amplitude of z) : $-\pi < \theta \leq \pi$

ii) Least positive argument : $\theta \in [0, 2\pi)$



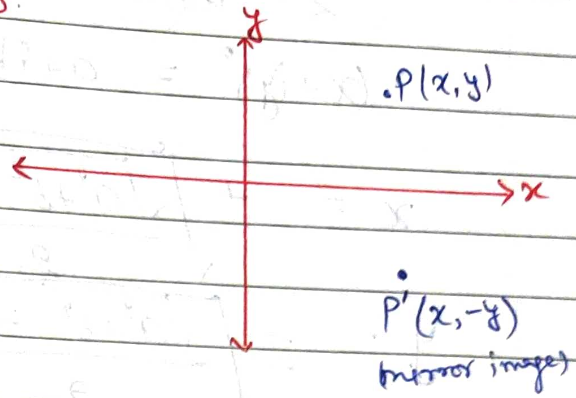
iii) Gen. value of argument : $2k\pi$ + principal value $(k \in I)$

→ Conjugate of a complex no.

$z = x+iy$ (P)

then,

$\bar{z} = x-iy$ (P')



Quadrilateral formed by $z, \bar{z}, -z, -\bar{z}$ is a rectangle.

→ Forms of Complex no.

① $\boxed{z = x+iy}$ (Cartesian form)

* Used to locate point. * To solve eqⁿ * To find locus

② Polar form (Trigonometric form)

$\boxed{z = |z|(\cos\theta + i\sin\theta)}$ θ = argument

* Uses:-
  * Higher power form * To find min & max value
  * n^{th} root of unity

③ Euler's form $\boxed{\cos\theta + i\sin\theta = e^{i\theta}}$

$|e^{i\theta}| = \sqrt{\cos^2\theta + \sin^2\theta} = 1 \ \forall \theta \in R$

$\boxed{\cos z = \dfrac{e^{iz} + e^{-iz}}{2}}$

$\boxed{\sin z = \dfrac{e^{iz} - e^{-iz}}{2i}}$

→ Square root of a given complex no.

$z = a+ib$

$\therefore (x+iy)^2 = a+ib \quad \left\{ (x+iy) = \sqrt{a+ib} \right\}$

$$x = \pm\sqrt{\frac{\sqrt{a^2+b^2}+a}{2}}$$

$$y = \pm\sqrt{\frac{\sqrt{a^2+b^2}-a}{2}}$$

$\therefore$ Sq. root of $z = a+ib$ is

$$= \pm\left(\sqrt{\frac{|z|+a}{2}} \pm i\sqrt{\frac{|z|-a}{2}}\right)$$

→ Properties

$(z = x+iy)$

* $z+\bar{z} = 2\,Re(z)$
* $z-\bar{z} = 2\,Im(z)$
* $z\bar{z} = |z|^2$
* $\overline{(\bar{z})} = z$
* $\frac{z}{\bar{z}} = \frac{z^2}{|z|^2}$; $z \neq 0$
* $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$

* $|z| = |-z| = |\bar{z}| = |-\bar{z}| = |iz| = |-iz|$
* $|z^n| = (|z|)^n$
* $\arg(z^n) = n\arg(z) + 2k\pi$ $(k \in I)$
* $\overline{z_1 \pm z_2} = \bar{z}_1 \pm \bar{z}_2$
* $\overline{z_1 z_2} = \bar{z}_1 \cdot \bar{z}_2$
* $|z_1+z_2| \leq |z_1|+|z_2|$
* $|z_1-z_2| \geq \big||z_1|-|z_2|\big|$
* $|z_1 z_2| = |z_1||z_2|$

* $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2) + 2k\pi$
* $\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2) + 2k\pi$
* $\big||z_1|-|z_2|\big| \leq |z_1 \pm z_2| \leq |z_1|+|z_2|$

→ Demolvre's theorem

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta) \qquad n \in I$$

$$(\cos\theta + i\sin\theta)^{p/q} = \left\{\cos(2n\pi + p\theta) + i\sin(2n\pi + p\theta)\right\}^{1/q}$$

p/q is rational

where $n \to 0$ to $q-1$

→ use in n^{th} root of unity

→ has q values.

→ Logarithim of complex no

$$z = |z| e^{i\theta} \Rightarrow \ln z = \ln|z| + i\theta$$

$$\left[\ln(x+iy) = \frac{1}{2}\ln(x^2+y^2) + i(\arg(z) + 2k\pi)\right]$$

→ Cubic root of unity

$$x^3 - 1 = 0$$

↳ $\boxed{x = 1}$ $\boxed{x = \frac{-1 + i\sqrt{3}}{2}}$ $\boxed{x = \frac{-1 - i\sqrt{3}}{2}}$

(ω) (ω^2)

∴ Cubic roots of unity are:-

$1, \omega, \omega^2$

\# $1 + \omega + \omega^2 = 0$

\# $1 \cdot \omega \cdot \omega^2 = 1$

→ n^{th} roots of unity

$x^n - 1 = 0$ has n roots.

$$x = \left\{\cos\left(\frac{2\pi}{n}\right) + i\sin\left(\frac{2\pi}{n}\right)\right\}^k \qquad k = 0 \text{ to } n-1$$

So, roots of $x^n - 1 = 0$ are α^k where, $k = 0$ to $n-1$.

and

$$\alpha = \cos\left(\frac{2\pi}{n}\right) + i\sin\left(\frac{2\pi}{n}\right)$$

$$\alpha = e^{i\left(\frac{2\pi}{n}\right)}$$

Roots :- $1, \alpha, \alpha^2, \alpha^3, \ldots \alpha^{n-1}$

⇒ Properties

→ all the roots are in G.P.

→ $1 + \alpha + \alpha^2 + \ldots + \alpha^{n-1} = 0$

→ $1 \cdot \alpha \cdot \alpha^2 \ldots \alpha^{n-1} = (-1)^{n-1}$

→ $|\alpha| = |\alpha^2| = |\alpha^3| = \ldots = |\alpha^{n-1}| = 1$

→ α^{n-p} & α^p are conjugate of each other

→ all roots of unity lie on unit circle. $\{C(0,0), 1$

→ $1, \alpha, \alpha^2, \ldots \alpha^{n-1}$ are vertices of n sided regular polygon.

→ diff of arg. of any two consecutive roots of unity is $\left(\frac{2\pi}{n}\right)$.

→ $\alpha^1, \alpha^2, \alpha^3 \ldots$ are the n^{th} roots of unity.

$(x-\alpha^1)(x-\alpha^2)(x-\alpha^3)\ldots(x-\alpha^{n-1}) = \frac{x^n - 1}{x-1}$ $(x \neq 1)$

$(x-\alpha^1)(x-\alpha^2)(x-\alpha^3)\ldots(x-\alpha^{n-1}) = 1 + x + x^2 + \ldots + x^{n-1}$ $(x=1)$

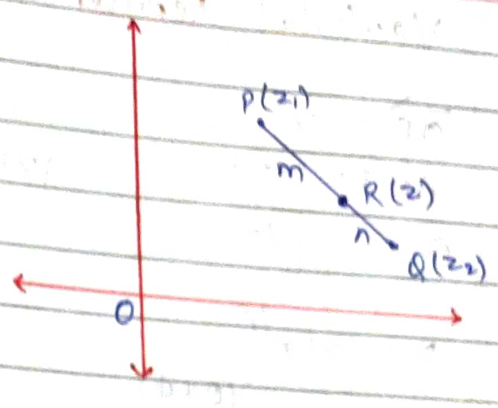
$\frac{1}{x-\alpha^1} + \frac{1}{x-\alpha^2} + \frac{1}{x-\alpha^3} + \ldots + \frac{1}{x-\alpha^{n-1}} = \frac{nx^{n-1}}{x^n - 1} - \frac{1}{x-1}$

$\frac{1}{x-\alpha^1} + \frac{1}{x-\alpha^2} + \frac{1}{x-\alpha^3} + \ldots + \frac{1}{x-\alpha^{n-1}} = \frac{1 + 2x + 3x^2 + \ldots + (n-1)x^{n-2}}{1 + x + x^2 + x^3 + \ldots x^{n-1}}$

Geometry of complex no.

* Distance formula :-

$$PQ = |z_1 - z_2|$$



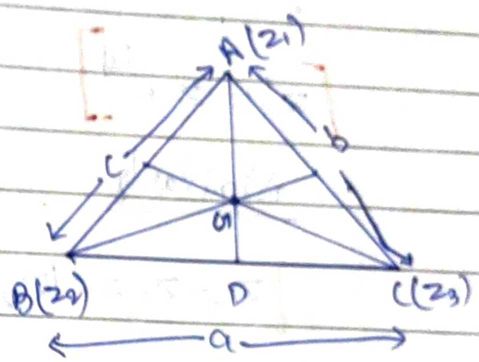
* Section formulae

$$z = \frac{mz_2 + nz_1}{m+n}$$

also mid point of PQ

$$z = \frac{z_1 + z_2}{2}$$

* Centroid (G)



$$D = \frac{z_2 + z_3}{2}$$

$$G = \frac{z_1 + z_2 + z_3}{3}$$

* ar ΔAGB = ar ΔBGC = ar ΔAGC

* Incentre (I)

$$I = \frac{az_1 + bz_2 + cz_3}{a+b+c}$$

* Orthocentre (P)

$$P = \frac{z_1 \tan A + z_2 \tan B + z_3 \tan C}{\tan A + \tan B + \tan C}$$

* Circumcentre (O)

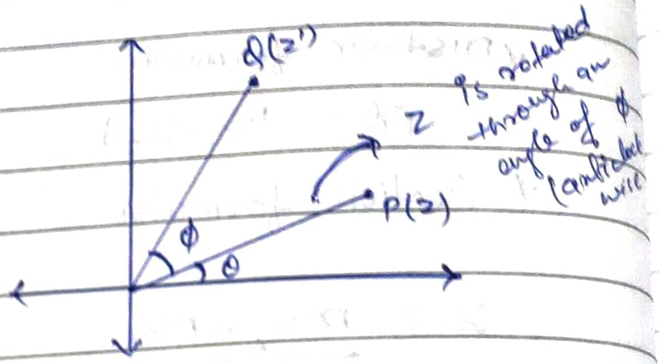
$$O = \frac{z_1 \sin 2A + z_2 \sin 2B + z_3 \sin 2C}{\sin 2A + \sin 2B + \sin 2C}$$

→ Vectorial Representation of a complex no.

$\vec{OP} = z$

$|\vec{OP}| = |z|$ $\qquad |z| = |z'|$

$\theta = \arg(z)$

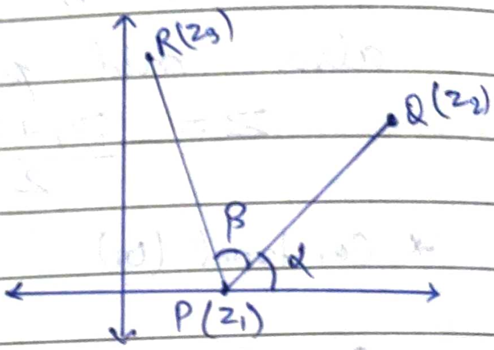


$z = |z| e^{i\theta}$

$z' = |z'| e^{i(\theta+\phi)}$

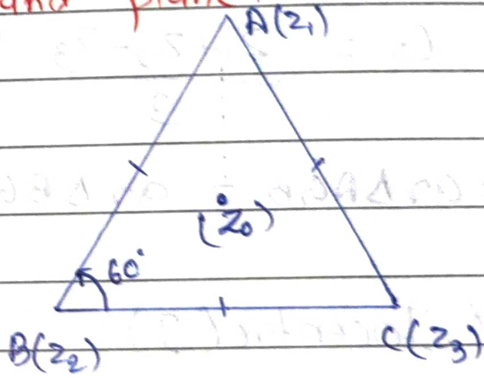
$$\left[z' = z e^{i\phi} \right]$$

<u>Similarily</u>



$$\left[z_3 - z_1 = (z_2 - z_1) e^{i\beta} \right]$$

Equilateral triangle on argand plane



$$\pm 60^\circ = \arg\left(\frac{z_1 - z_2}{z_3 - z_2}\right)$$

$$\frac{z_1 - z_2}{z_3 - z_2} = \left|\frac{z_1 - z_2}{z_3 - z_2}\right| e^{i\pi/3}$$

$$\left[\frac{z_1 - z_2}{z_3 - z_2} = e^{i\pi/3} \right]$$

z_0 = G, O, P, I (basically centre

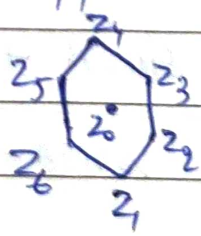
mainly circumcentre

$$* \left[z_1^2 + z_2^2 + z_3^2 = z_1 z_2 + z_2 z_3 + z_3 z_1 \right]$$

$$* \left[\frac{1}{z_1 - z_2} + \frac{1}{z_2 - z_3} + \frac{1}{z_3 - z_1} = 0 \right]$$

$$* \left[z_1^2 + z_2^2 + z_3^2 = 3 z_0^2 \right]$$ → applicable in any regular p

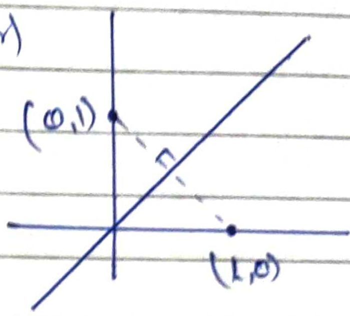
like in Regular hexagon



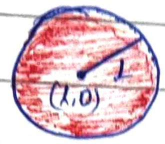
$$\Rightarrow \left[z_1^2 + z_2^2 + z_3^2 + z_4^2 + z_5^2 + z_6^2 = 6 z_0^2 \right]$$

Some standard loci

* $|z-(1+2i)| = 3 \Rightarrow$ circle $c(1,2)$; $R = 3$
* $|z-1| = |z-i| \Rightarrow$ st. line (⊥ur bisector)



* $|z-1| < 1 \Rightarrow$ Interior of circle



* $|z-z_1| + |z-z_2| = k$

case ① :

$k > |z_1 - z_2| \Rightarrow$ ellipse ; foci $\{z_1 \,\&\, z_2\}$

case (ii) :

$k = |z_1 - z_2| \Rightarrow$ line segment

case (iii) :

$k < |z_1 - z_2| \Rightarrow$ no locus.

Componento - dividendo

if $\frac{a}{b} = \frac{c}{d}$ then, $\left[\frac{a-b}{a+b} = \frac{c-d}{c+d}\right]$

also, $\left[\frac{ab}{(a+b)^2} = \frac{cd}{(c+d)^2}\right]$

~~Relati~~ **Quadratic Eqn.**

Quadratic eqn is,

$$ax^2 + bx + c = 0$$

have roots α & β

$$\boxed{\alpha + \beta = \frac{-b}{a} \qquad \alpha\beta = \frac{c}{a}}$$

& Eqn. be

$$\boxed{x^2 - (\text{sum of Roots})x + (\text{Product of Roots}) = 0}$$

$$\boxed{x^2 - Sx + P = 0}$$

$$\boxed{\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta}$$

$$\boxed{(\alpha - \beta)^2 = (\alpha + \beta)^2 - 4\alpha\beta = \frac{D}{a^2}}$$

$$\boxed{\alpha^3 + \beta^3 = (\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta)}$$

$$\boxed{\alpha^4 + \beta^4 = (\alpha^2 + \beta^2)^2 - 2\alpha^2\beta^2}$$

$$\boxed{\alpha^5+\beta^5=(\alpha^2+\beta^2)(\alpha^3+\beta^3)-\alpha^2\beta^2(\alpha+\beta)}$$

— x — x — x — x —

If $ax^2+bx+c=0 \rightarrow \alpha, \beta$

& if

$S_n = \alpha^n + \beta^n$ or $\alpha^n - \beta^n$

then

$$aS_{n+2}+bS_{n+1}+cS_n=0$$

Nature of Roots

for $ax^2+bx+c=0$

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}=\frac{-b\pm\sqrt{D}}{2a}$$

1.) If $D>0$ & is a perfect square, $\rightarrow$ rational Roots

2.) If $D>0$ but not a perfect square, $\Rightarrow$ Irrational Roots

★ If coefficient of quadratic Equation are real & one root is $a+ib$ then other root is $a-ib$

3) If $D=0 \Rightarrow$ roots are equal

4.) If $D<0 \Rightarrow$ Roots are complex

$\alpha + i\beta \qquad \alpha - i\beta$

Common Roots

$$a_1x^2 + b_1x + c_1 = 0$$
$$a_2x^2 + b_2x + c_2 = 0$$

1.) If one root is common

$$\boxed{(c_1a_2 - c_2a_1)^2 = (a_1b_2 - b_1a_2)(b_1c_2 - b_2c_1)}$$

Condition for 1 common Root

2) If both the roots are common :-

$$\boxed{\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}}$$

Condition for 2 common Roots

Range of Quadratic Equation

for $f(x) = ax^2 + bx + c$, $a \neq 0$

If $a > 0$ Range is $\left[-\frac{D}{4a}, \infty\right)$

If $a < 0$ Range is $\left(-\infty, -\frac{D}{4a}\right)$

Polynomial Eqn.

e.g., for $ax^3 + bx^2 + cx + d = 0$ → α, β, γ

$$\alpha + \beta + \gamma = \frac{-b}{a}$$

$$\alpha\beta + \beta\gamma + \alpha\gamma = \frac{c}{a}$$

$$\alpha\beta\gamma = \frac{-d}{a}$$

$$\boxed{1 < \frac{a^2+b^2+c^2}{ab+bc+ac} < 2}$$