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d) $\frac{d}{dx}(a^x) = a^x \log a$

e) $\frac{d}{dx}(\log x) = \frac{1}{x}$

f) $\frac{d}{dx}(\sin ax) = a\cos ax$

g) $\frac{d}{dx}(\cos ax) = -a\sin ax$

h) $\frac{d}{dx}(\tan ax) = a\sec^2 ax$

i) $\frac{d}{dx}(\cot ax) = -a\,\text{cosec}^2\, ax$

j) $\frac{d}{dx}(\sec ax) = a\sec ax . \tan ax$

k) $\frac{d}{dx}(\text{cosec}\, ax) = -a\,\text{cosec}\, ax . \cot ax$

l) $\frac{d}{dx}\left(\sin^{-1}\frac{x}{a}\right) = \frac{1}{\sqrt{a^2-x^2}}$

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n) $\frac{d}{dx}\left(\tan^{-1}\frac{x}{a}\right) = \frac{a}{a^2+x^2}$

o) $\frac{d}{dx}\left(\cot^{-1}\frac{x}{a}\right) = -\frac{a}{a^2+x^2}$

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Exercise 1

1. Find from the first principles, the derivatives of the following functions.

i) $\sqrt{x}$ ii) $\tan^{-1} x$ iii) $\log(\cos x)$

iv) $e^{\sqrt{x}}$ v) $e^{\cos x}$ vi) x^x

Soln: i) $\sqrt{x}$

Let, $f(x) = \sqrt{x}$

$\therefore \quad f(x+h) = \sqrt{x+h}$

By definition, we have,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h} = \lim_{h \to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h} \times \frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}$$

$$= \lim_{h \to 0} \left[\frac{x+h-x}{h(\sqrt{x+h}+\sqrt{x})}\right] = \lim_{h \to 0} \left[\frac{h}{h(\sqrt{x+h}+\sqrt{x})}\right]$$

$$= \lim_{h \to 0} \left[\frac{1}{(\sqrt{x+h}+\sqrt{x})}\right]$$

$$= \frac{1}{\sqrt{x+h}+\sqrt{x}}$$

$$\therefore \quad \frac{d}{dx}(\sqrt{x}) = \frac{1}{2\sqrt{x}}$$

ii) $\tan^{-1} x$

Let, $f(x) = \tan^{-1} x$

$\therefore \quad f(x+h) = \tan^{-1}(x+h)$

By definition, we have,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\tan^{-1}(x+h)-\tan^{-1} x}{h} \quad \dots (i)$$

(Because to remove $\frac{0}{0}$ form in the case of trigonometric function we have to use $\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$)

Put $y = \tan^{-1} x$

or, $x = \tan y$

and $y + k = \tan^{-1}(x + h)$

or, $x + h = \tan(y + k)$

or, $h = \tan(y + k) - x$

or, $h = \tan(y + k) - \tan y$ ... (ii)

From (ii) when $h \to 0$, then $k \to 0$

$\therefore$ (i) will be;

$$f'(x) = \lim_{h \to 0} \frac{y+k-y}{h} = \lim_{h \to 0} \frac{k}{h}$$

$$= \lim_{h \to 0} \left[\frac{k}{\tan(y+k)-\tan y}\right] = \lim_{h \to 0} \left[\frac{k}{\frac{\sin(y+k)}{\cos(y+k)} + \frac{\sin y}{\cos y}}\right]$$

$$= \lim_{h \to 0} \left[\frac{k}{\frac{\sin(y+k)-\cos y-\cos(y+k)\sin y}{\cos(y+k).\cos y}}\right]$$

$$= \lim_{h \to 0} \left[\frac{k\cos(y+k).\cos y}{\sin(y+k-y)}\right]$$

$[\because \sin A \cos B - \cos A \sin B = \sin(A - B)]$

$$= \lim_{h \to 0} \frac{k}{\sin k} \times \lim_{h \to 0} \cos(y + k) . \cos y$$

$$= \lim_{k \to 0} \frac{k}{\sin k} \times \lim_{k \to 0} \cos(y + k) . \cos y$$

$[\because k \to 0 \text{ as } h \to 0]$

$$= \frac{1}{\lim_{k \to 0} \frac{\sin k}{k}} \times \lim_{k \to 0} \cos(y + k) . \cos y$$

$$= \frac{1}{1} \times \cos(y + 0) . \cos y$$

$$= \cos^2 y = \frac{1}{\sec^2 y} = \frac{1}{1+\tan^2 y} \qquad \left[\because \lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1\right]$$

$$= \frac{1}{1+x^2} \qquad [\because \tan y = x]$$

$$\therefore \frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$$

iii) log(cos x)

Let, $f(x) = \log(\cos x)$

$\therefore$ $f(x + h) = \log\{\cos(x + h)\}$

By definition, we have,

$$f'(x) = \lim_{h \to 0} \left[\frac{\log\{\cos(x+h)\}-\log(\cos x)}{h}\right] \text{ ... (i)}$$

Put $\cos x = y$ and $\cos(x + h) = y + k$

$k = \cos(x + h) - y$

$k = \cos(x + h) - \cos x$ ... (ii)

From (ii) $k \to 0$ as $h \to 0$

Then (i) will be;

$$f'(x) = \lim_{h \to 0} \frac{\log(y+k)-\log y}{h}$$

$$= \lim_{h \to 0} \left[\frac{\log\left(\frac{y+k}{y}\right)}{h}\right] \qquad \left[\because \log m - \log n = \log\frac{m}{n}\right]$$

$$= \lim_{h \to 0} \left[\frac{\log\left(\frac{y}{y}+\frac{k}{y}\right)}{h}\right]$$

$$= \lim_{h \to 0} \left[\frac{\log\left(1+\frac{k}{y}\right)}{\frac{k}{y}} \times \frac{\frac{k}{y}}{h}\right]$$

$$= \lim_{h \to 0} \frac{\log\left(1+\frac{k}{y}\right)}{\frac{k}{y}} \times \lim_{h \to 0} \frac{k}{yh}$$

$$= \lim_{h \to 0} \frac{\log\left(1+\frac{k}{y}\right)}{\frac{k}{y}} \times \lim_{h \to 0} \frac{\cos(x+h)-\cos x}{yh}$$

$[\because k \to 0 \text{ as } h \to 0 \text{ \& } k = \cos(x + h) - \cos x \text{ from (ii)}]$

$$= \lim_{\frac{k}{y} \to 0} \frac{\log\left(1+\frac{k}{y}\right)}{\frac{k}{y}} \times \frac{1}{y} \times \lim_{h \to 0} \left[\frac{\cos(x+h)-\cos x}{h}\right]$$

$$= 1 \times \frac{1}{y} \times \lim_{h \to 0} \frac{2\sin\left(\frac{x+h+x}{2}\right)\sin\left(\frac{x-x-h}{2}\right)}{h}$$

$$\left[\because \lim_{x \to 0} \frac{\log(1+x)}{x} = 1\right]$$

$$= \frac{1}{y} \lim_{h \to 0} \frac{2\sin\left(\frac{2x+h}{2}\right)\sin\left(\frac{-h}{2}\right)}{h}$$

$$= \frac{1}{y} \lim_{h \to 0} \frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}} \times \sin\left(\frac{2x+h}{2}\right) \quad [\because \sin(-\theta) = -\sin\theta]$$

$$= -\frac{1}{y} \times \lim_{\frac{h}{2} \to 0} \frac{\sin\frac{h}{2}}{\frac{h}{2}} \times \lim_{h \to 0} \sin\left(\frac{2x+h}{2}\right)$$

$$= -\frac{1}{y} \times 1 \times \sin\left(\frac{2x+0}{2}\right) \quad \left[\because \lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1\right]$$

$$= -\frac{1}{y} \times \sin x$$

$$= -\frac{1}{\cos x} \times \sin x$$

$$\therefore \frac{d}{dx}\{\log(\cos x)\} = -\tan x$$

iv) $e^{\sqrt{x}}$

Let, $f(x) = e^{\sqrt{x}}$

$\therefore \quad f(x+h) = e^{\sqrt{x+h}}$

By definition, we have,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$\text{or,} \quad f'(x) = \lim_{h \to 0} \frac{e^{\sqrt{x+h}} - e^{\sqrt{x}}}{h} \ldots \text{(i)}$$

Put $\sqrt{x} = y$ and $\sqrt{x+h} = y + k$

$k = \sqrt{x+h} - y$

$k = \sqrt{x+h} - \sqrt{x} \ldots$ (ii)

From (ii), $k \to 0$ as $h \to 0$ Now, (i) will be;

$$f'(x) = \lim_{h \to 0} \frac{e^{y+k} - e^{y}}{h}$$

$$= \lim_{h \to 0} \frac{e^{y}e^{k} - e^{y}}{h}$$

$$= \lim_{h \to 0} \frac{e^{y}e^{k} - e^{y}}{h}$$

$$= e^{y} \cdot \lim_{h \to 0} \frac{e^{k} - 1}{h}$$

$$= e^{y} \cdot \lim_{h \to 0} \left[\frac{e^{k} - 1}{k} \times \frac{k}{h}\right]$$

$$= e^{y} \cdot \lim_{h \to 0} \frac{e^{k} - 1}{k} \times \lim_{h \to 0} \frac{k}{h}$$

$$= e^{y} \cdot \lim_{k \to 0} \frac{e^{k} - 1}{k} \times \lim_{h \to 0} \frac{k}{h} \quad [\because k \to 0 \text{ as } h \to 0]$$

$$= e^{y} \times 1 \times \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}$$

$$\left[\because \lim_{x \to 0} \frac{e^{x} - 1}{x} = 1 \;\&\; \textit{using (ii)}\right]$$

$$= e^{y} \cdot \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \times \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}}$$

$$= e^{y} \cdot \lim_{h \to 0} \frac{x + h - x}{h(\sqrt{x+h} + \sqrt{x})}$$

$$= e^{y} \cdot \lim_{h \to 0} \left[\frac{h}{h(\sqrt{x+h} + \sqrt{x})}\right]$$

$$= e^{y} \cdot \lim_{h \to 0} \left[\frac{1}{\sqrt{x+h} + \sqrt{x}}\right]$$

$$= e^{y} \cdot \frac{1}{2\sqrt{x}}$$

$$\therefore \frac{d}{dx}\left(e^{\sqrt{x}}\right) = \frac{e^{\sqrt{x}}}{2\sqrt{x}}$$

v) $e^{\cos x}$

Let, $f(x) = e^{\cos x}$

$\therefore \quad f(x+h) = e^{\cos(x+h)}$

By definition, we have,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$\text{or,} \quad f'(x) = \lim_{h \to 0} \frac{e^{\cos(x+h)} - e^{\cos x}}{h} \ldots \text{(i)}$$

Put $\cos x = y$ and $\cos(x+h) = y + k$

$k = \cos(x+h) - y$

$k = \cos(x+h) - \cos x \ldots$ (ii)

From (ii) $k \to 0$, as $h \to 0$

Then (i) can be written as;

$$f'(x) = \lim_{h \to 0} \frac{e^{y+k} - e^{y}}{h}$$

$$= \lim_{h \to 0} \frac{e^{y}e^{k} - e^{y}}{h}$$

$$= e^{y} \cdot \lim_{h \to 0} \frac{e^{k} - 1}{h}$$

$$= e^y . \lim_{h \to 0} \left[\frac{e^k - 1}{k} \times \frac{k}{h}\right]$$

$$= e^y . \lim_{h \to 0} \frac{e^k - 1}{k} \times \lim_{h \to 0} \frac{k}{h}$$

$$= e^y . \lim_{h \to 0} \frac{e^k - 1}{k} \times \lim_{h \to 0} \frac{\cos(x+h) - \cos x}{h}$$

$[\because k \to 0$ as $h \to 0$ & *using* (ii)$]$

$$= e^y \times 1 \times \lim_{h \to 0} \frac{2 \sin\left(\frac{x+h+x}{2}\right) . \sin\left(\frac{x-x-h}{2}\right)}{h}$$

$$\left[\because \lim_{x \to 0} \frac{e^x - 1}{.x} = 1\right]$$

$$f'(x) = e^y . \lim_{h \to 0} 2 \times \frac{\sin\frac{h}{2}}{\frac{h}{2}} \times \lim_{h \to 0} \sin\left(\frac{2x+h}{2}\right)$$

$$[\because \sin(-\theta) = -\sin\theta]$$

$$= -e^y \times 1 \times \sin\left(\frac{2x+0}{2}\right) \qquad \left[\because \lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1\right]$$

$$= -e^y . \sin x$$

$$\therefore \quad \frac{d}{dx}(e^{\cos x}) = -e^{\cos x} . \sin x$$

vi) x^x

Let, $f(x) = x^x = e^{\log x^x}$

($\because$ Exponential and logarithmic functions are inverse each other i.e. $e^{\log a} = a$)

or, $f(x) = e^{x \log x}$ $\qquad [\because \log m^n = n \log m]$

$\therefore f(x+h) = e^{(x+h)\log(x+h)}$

By definition, we have,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{e^{(x+h)\log(x+h)} - e^{x \log x}}{h} \ldots \text{(i)}$$

Put $x \log x = y$ and $(x+h)\log(x+h) = y + k$

or, $k = (x+h)\log(x+h) - y$

or, $k = (x+h)\log(x+h) - x \log x \ldots$ (ii)

From (ii) $k \to 0$ as $h \to 0$

Now, (i) can be written as;

$$f'(x) = \lim_{h \to 0} \frac{e^{y+k} - e^y}{h} = \lim_{h \to 0} \frac{e^y . e^k - e^y}{h}$$

$$= e^y . \lim_{h \to 0} \frac{e^k - 1}{h} = e^y . \lim_{h \to 0} \left[\frac{e^k - 1}{k} \times \frac{k}{h}\right]$$

$$= e^y . \lim_{h \to 0} \frac{e^k - 1}{k} \times \lim_{h \to 0} \frac{k}{h}$$

$$= e^y \times 1 \times h \xrightarrow{\lim} 0 \left[\frac{(x+h)\log(x+h) - x \log x}{h}\right]$$

$$\left[\because \lim_{x \to 0} \frac{e^x - 1}{x} = 1 \text{ and using (ii)}\right]$$

$$= e^y \times \lim_{h \to 0} \left[\frac{x \log(x+h) + h \log(x+h) - x \log x}{h}\right]$$

$$= e^y \times \lim_{h \to 0} \left[\frac{x \log(x+h) - x \log x + h \log(x+h)}{h}\right]$$

$$= e^y \times \lim_{h \to 0} \left[\frac{x\{\log(x+h) - \log x\}}{h} + \frac{h \log(x+h)}{h}\right]$$

$$= e^y \times \lim_{h \to 0} \left[\frac{x \log\left(\frac{x+h}{x}\right)}{h} + \log(x+h)\right]$$

$$\left[\because \log m - \log n = \log\frac{m}{n}\right]$$

$$= e^y \times \left[\lim_{h \to 0} \frac{x \log\left(\frac{x}{x} + \frac{h}{x}\right)}{h} + \lim_{h \to 0} \log(x+h)\right]$$

$$= e^y \left[\lim_{h \to 0} \frac{x \log\left(1 + \frac{h}{x}\right)}{h} + \lim_{h \to 0} \log(x+h)\right]$$

$$= e^y \left[\lim_{h \to 0} \frac{\log\left(1 + \frac{h}{x}\right)}{\frac{h}{x}} + \lim_{h \to 0} \log(x+h)\right]$$

$$= e^y \left[\lim_{\frac{h}{x} \to 0} \frac{\log\left(1 + \frac{h}{x}\right)}{\frac{h}{x}} + \lim_{h \to 0} \log(x+h)\right]$$

$$= e^{x \log x}[1 + \log(x+0)] \left[\because \lim_{x \to 0} \frac{\log(1+x)}{x} = 1\right]$$

$$= e^{\log x^x}[1 + \log x][\because n \log m = \log m^n]$$

$$\therefore \quad \frac{d}{dx}(x^x) = x^x(1 + \log x) \qquad [\because e^{\log m} = m]$$

Note:
If $y = f(x)$ is differentiable in the interval n times, then its n^{th} derivative is denoted by $\frac{d^n y}{dx^n}$ or y^n or y_n or $f^n(x)$.

n^{th} derivatives of some special functions:

a) If $y = x^m, m \in z^+$

Then,

$$y_n = \begin{Bmatrix} \frac{m!}{(m-n)!} x^{m-n} & \text{for } m > n \\ 0 & \text{for } m < n \\ n! & \text{for } m = n \end{Bmatrix}$$

b) If $y = (ax + b)^m$, then,

$$y_n = a^n \frac{m!}{(m-n)!} (ax + b)^{m-n} \text{ for } m > n \text{ and } m \in z^+$$

c) If $y = e^{ax}$, then $y_n = a^n e^{ax}$

d) If $y = \frac{1}{ax+b}$, then $y_n = \frac{(-1)^n n! a^n}{(ax+b)^{n+1}}$

e) If $y = \frac{1}{x}$, then $y_n = \frac{(-1)^n n!}{x^{n+1}}$

f) If $y = \frac{1}{x+a}$, then $y_n = \frac{(-1)^n n!}{(x-a)^{n+1}}$

h) If $y = \sin(ax + b)$, then $y_n = a^n \sin\left(ax + b + \frac{n\pi}{2}\right)$

i) If $y = \cos(ax + b)$, then $y_n = a^n \cos\left(ax + b + \frac{n\pi}{2}\right)$

j) If $y = e^{ax} \sin bx$, then $y_n = (a^2 + b^2)^{\frac{n}{2}} e^{ax} \sin\left(bx + n \tan^{-1}\frac{b}{a}\right)$

k) If $y = e^{ax} \cos bx$, then $y_n = (a^2 + b^2)^{\frac{n}{2}} e^{ax} \cos\left(bx + n \tan^{-1}\frac{b}{a}\right)$

l) If $y = \frac{1}{x^2-a^2}$, then $y_n = \frac{(-1)^n n!}{2a}\left[\frac{1}{(x-a)^{n+1}} - \frac{1}{(x\ldots}\right.$

m) If $y = \frac{1}{x^2+a^2}$, then $y_n = \frac{(-1)^n n!}{a^{n+2}} \sin^{n+1}\theta \sin(n + 1)\theta$

Where, $\theta = \tan^{-1}\frac{a}{x}$

Leibnitz's Theorem:
If u and v are two functions of x, then the n^{th} derivative of their product is;

$$(uv)_n = u_n v + c(n, 1)u_{n-1}v_1 + c(n, 2)u_{n-2}v_2 + \ldots\ldots\ldots + c(n, r)u_{n-r}v_r + \ldots\ldots\ldots + uv_n$$

Where, the suffices of u and v denotes the order of the differentiation of u and v with respect to x.

Exercise 2

1. Find the n^{th} derivative of the following:

i) $y = (a - bx)^m$

Sol^n: We have $y = (a - bx)^m \ldots$ (i)

Differentiating (i) successively with respect to x, we get,

$y_1 = m(a - bx)^{m-1}(-b)$

or, $y_2 = m(m-1)(a-bx)^{m-2}(-b)^2$

$y_3 = m(m-1)(m-2)(a-bx)^{m-3}(-b)^3$

or, $\vdots$

$y_n = m(m-1)(m-2)\cdots\{m-(n-1)\}(-b)^n(a-bx)^{m-n}$

$\therefore$ $y_n = (-1)^n m(m-1)(m-2)\cdots(m-n+1)b^n(a-bx)^{m-n}$

or, $y_n = \dfrac{(-1)^n m(m-1)(m-2)\cdots\{m-(n+1)\}(m-n)\cdots 2.1}{(m-n)\cdots 2.1} b^n (a-bx)^{m-n}$

or, $y_n = \dfrac{(-1)^n b^n m!}{(m-n)!}(a-bx)^{m-n}$

ii) $y = (2 - 3x)^n$

Sol^n: We have,

$y = (2-3x)^n$

Differentiating (i) successively with respect to x, we get,

$y_1 = n(2-3x)^{n-1}(-3)$

or, $y_2 = n(n-1)(2-3x)^{n-2}(-3)^3$

$y_3 = n(n-1)(n-2)(2-3x)^{n-3}(-3)^3$

or, $\vdots$

$y_n = n(n-1)(n-2)\cdots\{n-(n-1)\}(2-3x)^{n-n}(-3)^n$

$= n(n-1)(n-2)\cdots 1(-1)^n 3^n$

$\therefore$ $y_n = (-1)^n 3^n n!$

iii) $y = \dfrac{1}{\sqrt{x}}$

Sol^n: We have,

$y = \dfrac{1}{\sqrt{x}}$

or, $y = x^{-\frac{1}{2}} \ldots$ (i)

Differentiating (i) successively with respect to x, we get,

$y_1 = -\frac{1}{2}x^{-\frac{1}{2}-1} = -\frac{1}{2}x^{-\frac{3}{2}}$

$y_2 = \left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)x^{-\frac{1}{2}-1-1}$

$= \left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)x^{-\frac{1}{2}-2}$

$= (-1)^2\frac{1}{2}\times\frac{3}{2}x^{-\frac{5}{2}}$

$y_3 = \left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)x^{-\frac{1}{2}-3} = (-1)^3\times\frac{1}{2}\times\frac{3}{2}\times\frac{5}{2}\times x^{-\frac{7}{2}}$

$\vdots \qquad = \dfrac{(-1)^3\times 1\times 3\times 5}{2^3}x^{-\frac{1}{2}-3}$

$\vdots$

$y_n = \left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\left(-\frac{7}{2}\right)\cdots\left\{-\frac{1}{2}-(n-1)\right\}x^{-\frac{1}{2}-n}$

or, $y_n = (-1)^n\cdot\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}\cdot\frac{7}{2}\cdots\frac{(2n-1)}{2}x^{-\frac{1}{2}-n}$

$\therefore$ $y_n = \dfrac{(-1)^n 1\cdot 3\cdot 5\cdot 7\cdots(2n-1)}{2^n x^{\frac{1}{2}+n}}$

iv) $y = x^{2n}$

Sol^n: Given that,

$y = x^{2n} \ldots$ (i)

Differentiating (i) successively with respect to x, we get,

$y_1 = 2nx^{2n-1}$

$y_2 = 2n(2n-1)x^{2n-2}$

$y_3 = 2n(2n-1)(2n-2)x^{2n-3}$

$\vdots$

$y_n = 2n(2n-1)(2n-2)\cdots\{2n-(n-1)\}x^{2n-n}$

or, $y_n = 2n(2n-1)(2n-2)\cdots(n+1)x^n$

or, $y_n = \dfrac{n(2n-1)(2n-2)(2n-3)\cdots(n+1)n(n-1)\cdots 2\cdot 1x^n}{n(n-1)\cdots 2\cdot 1}$

or, $y_n = \dfrac{\{2n\cdot(2n-2)(2n-4)\cdots 6\cdot 4\cdot 2\}\{(2n-1)(2n-3)\cdots 5\cdot 3\cdot 1\}x^n}{n!}$

or, $y_n = \dfrac{\{2^n n(n-1)(n-2)\cdots 3\cdot 2\cdot 1\}\{1\cdot 35\cdots(2n-3)(2n-1)\}x^n}{n!}$

or, $y_n = \dfrac{2^n n!\{1\cdot 3\cdot 5\cdots(2n-1)\}x^n}{n!}$

$\therefore$ $y_n = 2^n\{1\cdot 3\cdot 5\cdots(2n-1)\}x^n$

v) $y = \frac{1}{a-x}$

Sol^n: Given that,

$$y = \frac{1}{a-x}$$

or, $y = (a-x)^{-1} \ldots$ (i)

Differentiating (i) successively with respect to x, we get,

$$y_1 = (-1)(a-x)^{-2}(-1)$$
$$y_2 = (-1)(-2)(a-x)^{-(2+1)}(-1)^2$$
$$y_3 = (-1)(-1)(-3)(a-x)^{-(3+1)}(-1)^3$$

or, $y_3 = (-1)^3 1\cdot 2\cdot 3(a-x)^{-(3+1)}(-1)^3$

$$\vdots$$
$$y_n = (-1)^n 1\cdot 2\cdot 3\cdots n(a-x)^{-(n+1)}(-1)^n$$
$$= (-1)^n n!\,(a-x)^{-(n+1)}(-1)^n$$

$\therefore \quad y_n = \frac{(-1)^n n!(-1)^n}{(a-x)^{n+1}}$

Thus,

$$y_n = \frac{n!}{(a-x)^{n+1}} \qquad [\because (-1)^{2n} = 1, \text{ for n even or odd}]$$

vi) $y = \frac{x^n}{x-1}$

Sol^n: Given that

$$y = \frac{x^n}{x-1}$$
$$y = (x^{n-1} + x^{n-2} + \ldots\ldots\ldots + 1) + \frac{1}{x-1} \ldots \text{(i)}$$

Differentiating (i) successively with respect to x, we get,

$$y_n = 0 + D^n\left[\frac{1}{x-1}\right] \qquad [\because D^n(x^m) = 0 \text{ for } n > m]$$
$$= D^n\left[\frac{1}{x-1}\right]$$

$\therefore \quad y_n = \frac{(-1)^n n!}{(x-1)^{n+1}}$

vii) $y = \frac{x}{a+bx}$

Here,

$$y = \frac{x}{a+bx} = \frac{1}{b}\left[\frac{a+bx-a}{a+bx}\right] = \frac{1}{b}\left[1 - \frac{a}{a+bx}\right]$$
$$= \frac{1}{b} - \frac{a}{b}(a+bx)^{-1} \ldots \text{(i)}$$

Differentiating successively with respect to x, we get,

$$y_1 = 0 - \frac{a}{b}(-1)(a+bx)^{-2}b$$
$$y_2 = -\frac{a}{b}(-1)(-2)(a+bx)^{-3}b^2$$
$$y_3 = -\frac{a}{b}(-1)(-2)(-3)(a+bx)^{-(3+1)}b^3$$
$$y_3 = (-1)^{3+1}\frac{a}{b}1\cdot 2\cdot 3\cdot(a+bx)^{-(3+1)}b^3$$
$$\vdots$$
$$y_n = (-1)^{n+1}\frac{a}{b}\,1\cdot 2\cdot 3\cdot\cdots n(a+bx)^{-(n+1)}b^n$$

$\therefore \quad y_n = \frac{(-1)^{n+1}ab^{n-1}n!}{(a+bx)^{n+1}}$

viii) $y = \frac{x}{x^2+a^2}$

Sol^n: Here,

$$y = \frac{x}{x^2+a^2} = \frac{x}{(x-ia)(x+ia)} = \frac{1}{2}\left[\frac{1}{x+ia} + \frac{1}{x-ia}\right]$$

Differentiating up to n steps, we get,

$$y_n = \frac{1}{2}\left[D^n\left(\frac{1}{x+ia}\right) + D^n\left(\frac{1}{x-ia}\right)\right]$$
$$= \frac{1}{2}\left[\frac{(-1)^n n!}{(x+ia)^{n+1}} + \frac{(-1)^n n!}{(x-ia)^{n+1}}\right]$$
$$y_n = \frac{1}{2}(-1)^n n!\left[(x+ia)^{-(n+1)} + (x-ia)^{-(n+1)}\right] \ldots \text{(i)}$$

Put, $x = r\cos\theta$, $a = r\sin\theta$

$\therefore \quad \tan\theta = \frac{a}{x}$

$$\theta = \tan^{-1}\frac{a}{x}$$

Then (i) becomes;

$$y_n = \frac{(-1)^n n!}{2}\left[(r\cos\theta + ir\sin\theta)^{-(n+1)} + (r\cos\theta - ir\sin\theta)^{-(n+1)}\right]$$
$$= \frac{(-1)^n n!}{2}r^{-(n+1)}\left[(\cos\theta + i\sin\theta)^{-(n+1)} + (\cos\theta - i\sin\theta)^{-(n+1)}\right]$$
$$= \frac{(-1)^n n!}{2}r^{-(n+1)}[\cos\{-(n+1)\}\theta + i\sin\{-(n+1)\}\theta$$
$$+ \cos\{-(n+1)\}\theta - i\sin\{-(n+1)\}\theta]$$

($\therefore$ using De' Moivre's Theorem)

$$= \frac{(-1)^n n!}{2} r^{-(n+1)} [\cos(n+1)\theta - i\sin(n+1)\theta + \cos(n+1)\theta + i\sin(n+1)\theta]$$

$$= \frac{(-1)^n n!}{2} r^{-(n+1)} \cdot 2\cos(n+1)\theta$$

$$y_n = \frac{(-1)^n n! \cos(n+1)\theta}{r^{n+1}} \ldots \text{(ii)}$$

But $a = r\sin\theta$

$\therefore \quad r = \frac{a}{\sin\theta}$

Then from (ii),

$$y_n = \frac{(-1)^n n!}{\left(\frac{a}{\sin\theta}\right)^{n+1}} \cos(n+1)\theta$$

$$\therefore \quad y_n = \frac{(-1)^n n!}{a^{n+1}} \times \sin^{(n+1)}\theta \cos(n+1)\theta$$

Where, $\theta = \tan^{-1}\frac{a}{x}$

ix) $\mathbf{y = \tan^{-1}\left(\frac{1+x}{1-x}\right)}$

Solⁿ: Here,

$$y = \tan^{-1}\left(\frac{1+x}{1-x}\right)$$

We know that,

$$\tan^{-1}A + \tan^{-1}B = \tan^{-1}\left(\frac{A+B}{1-AB}\right)$$

$$\therefore \quad \tan^{-1}1 + \tan^{-1}x = \tan^{-1}\left(\frac{1+x}{1-x}\right)$$

$$\therefore \quad y = \tan^{-1}1 + \tan^{-1}x \ldots \text{(i)}$$

Differentiating (i) successively with respect to x, we get,

$$\therefore \quad y_1 = 0 + \frac{1}{1+x^2}$$

$$y_1 = \frac{1}{x^2+1^2} \ldots \text{(ii)}$$

Differentiating (ii) up to $(n-1)$ steps, we get,

$$y_n = D^{n-1}\left[\frac{1}{x^2+1^2}\right] = \frac{(-1)^{n-1}(n-1)!}{(1)^{n-1+2}} \sin^n\theta \sin n\theta$$

Where, $\theta = \tan^{-1}\frac{1}{x}$

$$\therefore \quad y_n = (-1)^{n-1}(n-1)! \sin^n\theta \sin n\theta$$

Where, $\theta = \tan^{-1}\frac{1}{x}$

{Using: $D^n\left[\frac{1}{x^2+1^2}\right] = \frac{(-1)^n n!}{a^{n+2}} \sin^{n+1}\theta \sin(n+1)\theta$}

x) $\mathbf{y = \tan^{-1}\frac{\sqrt{1+x^2}-1}{x}}$

Solⁿ: Here,

$$y = \tan^{-1}\left[\frac{\sqrt{1+x^2}-1}{x}\right] \ldots \text{(i)}$$

Put $x = \tan\theta$

$$\therefore \quad y = \tan^{-1}\left[\frac{\sqrt{1+\tan^2\theta}-1}{\tan\theta}\right]$$

$$= \tan^{-1}\left(\frac{\sec\theta-1}{\tan\theta}\right) = \tan^{-1}\left[\frac{\frac{1}{\cos\theta}-1}{\frac{\sin\theta}{\cos\theta}}\right] = \tan^{-1}\left[\frac{1-\cos\theta}{\sin\theta}\right]$$

$$= \tan^{-1}\left[\frac{2\sin^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cdot\cos\frac{\theta}{2}}\right] = \tan^{-1}\left(\tan\frac{\theta}{2}\right)$$

$$y = \frac{\theta}{2}$$

But $\theta = \tan^{-1}x$

$$\therefore \quad y = \frac{1}{2}\tan^{-1}x \ldots \text{(i)}$$

Differentiating (i) with respect to x, we get,

$$y_1 = \frac{1}{2}\frac{1}{x^2+1} \ldots \text{(ii)}$$

Again, differentiating (ii) up to $(n-1)$ steps.

$$\therefore \quad y_n = \frac{1}{2}D^{n-1}\left[\frac{1}{x^2+1}\right]$$

$$= \frac{1}{2}\frac{(-1)^{n-1}(n-1)!}{1^{n+1}} \sin^n\theta \sin n\theta$$

Where, $\theta = \tan^{-1}\frac{1}{x}$

$$\therefore \quad y_n = \frac{1}{2}(-1)^{n-1}(n-1)! \sin^n\theta \sin n\theta$$

Where, $\theta = \tan^{-1}\frac{1}{x}$

2. If $y = \frac{1}{x^x}$, then show that $y_1(1) = 0$

Sol^n: We have,

$$y = \frac{1}{x^x} = x^{-x}$$

Taking log on both sides, we get,

$$\log y = -x \log x$$

Differentiating with respect to x,

$$\therefore \quad \frac{1}{y}\frac{dy}{dx} = -\left[x \cdot \frac{1}{x} + \log x\right]$$

$$\therefore \quad \frac{dy}{dx} = -y[1 + \log x]$$

i.e. $y_1 = -y[1 + \log x]$

i.e. $y_1 = -x^{-x}(1 + \log x)$

$\therefore \quad y_1(1) = -1^{-1}(1 + \log 1) = -1(1 + 0) = -1$

Note:

Leibnitz's Theorem:

If u and v are two functions of x, then the n^{th} derivative of their product is;

$$D^n(uv) = c(n,0)u_n v + c(n,1)u_{n-1}v_1 + c(n,2)u_{n-2}v_2 + \dots\dots\dots + c(n,r)u_{n-r}v_r + \dots\dots\dots\dots + c(n,n)uv_n$$

Where, the suffices of u and v denotes the order of the derivative of u and v with respect to x.

3. Find y_n if

i) $\mathbf{y = e^{ax}x^3}$

Sol^n: Here,

$$y = e^{ax}x^3$$

Applying Leibnitz rule taking $u = e^{ax}$ and $v = x^3$, we get,

$$\begin{aligned} y_n &= D^n[e^{ax}x^3] \\ &= c(n,0)D^n(e^{ax})x^3 + c(n,1)D^{n-1}(e^{ax})D(x^3) + c(n,2)D^{n-2}(e^{ax})D^2(x^3) + c(n,3)D^{n-3}(e^{ax})D^3(x^3) + c(n,4)D^{n-4}(e^{ax})D^4(x^3) + \dots\dots + c(n,n)e^{ax}D^n(x^3) \\ &= \frac{n!}{(n-0)!0!}a^n e^{ax}x^3 + \frac{n!}{(n-1)!1!}a^{n-1}e^{ax}3x^2 + \frac{n!}{(n-2)!2!}a^{n-2}e^{ax}3.2.x + \frac{n!}{(n-3)!3!}a^{n-3}e^{ax}3.2.1 + 0 + 0 \dots + 0 \end{aligned}$$

$(\because D^n(x^m) = 0 \text{ for } n > m)$

$$= a^n e^{ax}x^3 + \frac{n(n-1)!}{(n-1)!1!}a^{n-1}e^{ax}3x^2 + 3n(n-1)a^{n-2}e^{ax}.x + n(n-1)(n-2)a^{n-3}e^{ax}$$

$$\therefore \quad y_n = e^{ax}a^{n-3}[a^3x^3 + 3na^2x^2 + 3n(n-1)ax + n(n-1)(n-2)]$$

ii) $\mathbf{y = x^3 \log x}$

Sol^n: Here,

$$y = x^3 \log x$$

Differentiating n times, we get,

$$\begin{aligned} y_n &= D^n[x^3 \log x] \\ &= D^{n-1}[D(x^3 \log x)] \\ &= D^{n-1}\left[3x^2.\log x + x^3\frac{1}{x}\right] \\ &= 3D^{n-1}[x^2.\log x] + D^{n-1}(x^2) \\ &= 3D^{n-1}[x^2.\log x] \qquad [\because D^n(x^m) = 0, \text{for } n > m] \\ &= 3D^{n-2}[D(x^2.\log x)] \\ &= 3D^{n-2}\left[2x\log x + x^2.\frac{1}{x}\right] \\ &= 6D^{n-2}[x \log x] + 3D^{n-2}(x) = 6D^{n-3}[D(x.\log x)] \\ &= 6D^{n-3}\left(x.\frac{1}{x} + \log x.1\right) = 6D^{n-3}(\log x + 1) \\ &= 6D^{n-4}[D(\log x + 1)] \\ &= 6D^{n-4}\left(\frac{1}{x}\right) \\ &= \frac{6(-1)^{n-4}(n-4)!}{x^{n-4+1}} \qquad \left[\because D^n\left(\frac{1}{4}\right) = \frac{(-1)^n.n!}{x^{n+1}}\right] \end{aligned}$$

$$\therefore \quad y_n = \frac{(-1)^{n-4}.\ 6(n-4)!}{x^{n-3}}$$

4. **If $y = \tan^{-1}x$, then prove that $(1+x^2)y_{n+1} + 2nxy_n + n(n+1)y_{n-1} = 0$**

Sol^n: Here,

$$y = \tan^{-1}x \dots (i)$$

Differentiating with respect to x, we get,

$$y_1 = \frac{1}{1+x^2}$$

or, $(1+x^2)y_1 = 1 \dots (ii)$

Differentiating n times by using Leibnitz's rule and taking $u = y_1, v = 1 + x^2$

$\therefore \quad c(n,0)y_{n+1}(1+x^2) + c(n,1)y_n 2x + c(n,2)y_{n-1}2 + c(n,3)y_{n-2} + 0 + 0 \ldots\ldots\ldots + 0 = 0$

or, $\quad y_{n+1}(1+x^2) + 2nxy_n + n(n-1)y_{n-1} = 0$

$\therefore \quad (1+x^2)y_{n+1} + 2nxy_n + n(n-1)y_{n-1} = 0$ Proved

5. **If $y = e^{a\tan^{-1}x}$, then prove that:**
$(1+x^2)y_{n+2} + (2nx + 2x - a)y_{n+1} + n(n+1)y_n = 1$

Sol^n: Given that,

$$y = e^{a\tan^{-1}x} \ldots \text{(i)}$$

Differentiating (i) with respect to x, we get,

$$y_1 = \frac{d\left(e^{a\tan^{-1}x}\right)}{dx}$$

$$= \frac{d\left(e^{a\tan^{-1}x}\right)}{dx\left(a\tan^{-1}x\right)} \times \frac{d\left(a\tan^{-1}x\right)}{dx} = e^{a\tan^{-1}x}\frac{a}{1+x^2}$$

or, $\quad (1+x^2)y_1 = ae^{a\tan^{-1}x}$

or, $\quad (1+x^2)y_1 = ay \qquad (\because \text{from (i) } y = e^{a\tan^{-1}x})$

Differentiating both sides with respect to x, we get,

$$(1+x^2)y_2 + 2xy_1 = ay_1$$

or, $\quad (1+x^2)y_2 + (2x-a)y_1 = 0$

Differentiating n times using Leibnitz's theorem, we get,

$$C(n,0)(1+x^2)y_{n+2} + c(n,1)2xy_{n+1} + c(n,2)2y_n + c(n,0)(2x-a)y_{n+1} + c(n,1)2y_n = 0$$

or, $\quad (1+x^2)y_{n+2} + 2nxy_{n+1} + n(n-1)y_n + (2x-a)y_{n+1} + 2ny_n = 0$

or, $\quad (1+x^2)y_{n+2} + (2nx + 2x - a)y_{n+1} + \{n(n-1) + 2n\}y_n = 0$

or, $\quad (1+x^2)y_{n+2} + (2nx + 2x - a)y_{n+1} + n(n-1+2)y_n = 0$

$\therefore \quad (1+x^2)y_{n+2} + (2nx + 2x - a)y_{n+1} + n(n+1)y_n = 0$

6. **If $y = ae^{mx} + be^{-mx}$, then prove that $y_2 - m^2y = 0$**

Sol^n: Given that,

$$y = ae^{mx} + be^{-mx} \ldots \text{(i)}$$

Differentiating (i) with respect to x,

$\therefore \quad y_1 = mae^{mx} + mbe^{-mx}$

Again, differentiating both sides with respect to x, we get,

$$y_2 = m^2ae^{mx} + m^2be^{-mx}$$

or, $\quad y_2 = m^2(ae^{mx} + be^{-mx})$

or, $\quad y_2 = m^2y \qquad [\because \text{from (i)}]$

$\therefore \quad y_2 - m^2y = 0$

Hence proved

7. **If $y = e^{ax}\sin bx$, then prove that:**
$y_{n+1} = 2ay_n - (a^2+b^2)y_{n-1} = 0$

Sol^n: Given that,

$$y = e^{ax}\sin bx \ldots \text{(i)}$$

Differentiating both sides with respect to x, we get,

$$y_1 = ae^{ax}\sin bx + e^{ax}b\cos bx$$

$$= a(e^{ax}\sin bx) + be^{ax}\cos bx \ldots \text{(ii)}$$

Again, differentiating (ii) with respect to x, we get,

$$y_2 = ay_1 + b[ae^{ax}\cos bx - e^{ax}b\sin bx]$$

or, $\quad y_2 = ay_1 + a(be^{ax}\cos bx) - b^2(e^{ax}\sin bx)$

or, $\quad y_2 = ay_1 + a(y_1 - ay) - b^2y$

(Using (i) and also from (ii) $y_1 - ay = be^{ax}\cos bx$)

or, $\quad y_2 - 2ay_1 + (a^2+b^2)y_{n-1} = 0 \ldots$ (iii)

Differentiating equation (iii) up to $(n-1)$ steps, we get,

$\therefore \quad y_{n+1} = 2ay_n - (a^2+b^2)y_{n-1} = 0$ proved

8. **If $y = x^{n-1}\log x$, then prove that $xy_n = (n-1)!$**

Sol^n: Given that,

$$y = x^{n-1}\log x \ldots \text{(i)}$$

Differentiating with respect to x, up to n steps, we get,

$$y_n = D^n[x^{n-1}\log x]$$
$$= D^{n-1}[D(x^{n-1}\log x)]$$
$$= D^{n-1}\left[(n-1)x^{n-2}\log x + x^{n-1}\frac{1}{x}\right]$$
$$= (n-1)D^{n-1}(x^{n-2}\log x) + D^{n-1}(x^{n-2})$$
$$= (n-1)D^{n-2}\{D(x^{n-2}\log x)\} + 0 \qquad (\because D^n(x^m) = 0 \text{ for } n > m)$$
$$= (n-1)D^{n-2}\left[(n-2)x^{n-3}\log x + x^{n-2}\frac{1}{x}\right]$$
$$= (n-1)(n-2)D^{n-2}(x^{n-3}\log x) + D^{n-2}(x^{n-3})$$
$$= (n-1)(n-2)D^{n-2}(x^{n-3}\log x) \quad [\because D^n(x^m) = 0 \text{ for } n > m]$$
$$\vdots$$
$$= (n-1)(n-2)(n-3)\cdots\{n-(n-1)\}D^{n-(n-1)}[x^{n-n}\log x]$$
$$= (n-1)(n-2)(n-3)\cdots 1D(\log x)$$

$\therefore \quad y_n = (n-1)!\frac{1}{x}$

So that, $xy_n = (n-1)!$

Hence proved

9. **If $y = a\cos(\log x) + b\sin(\log x)$, then prove that:**
$x^2y_{n+2} + (2n+1)xy_{n+1} + (n^2+1)y_n = 0$

Sol^n: Given that,

$$y = a\cos(\log x) + b\sin(\log x) \;..\; \text{(i)}$$

Differentiating both sides with respect to x, we get,

$$y_1 = -a\sin(\log x)\frac{1}{x} + b\cos(\log x)\frac{1}{x}$$

or, $xy_1 = -a\sin(\log x) + b\cos(\log x)$

Again, differentiating both sides with respect to x, we get,

$$xy_2 + y_1 = -a\cos(\log x)\frac{1}{x} - b\sin(\log x)\frac{1}{x}$$

$$= -\frac{1}{x}[a\cos(\log x) + b\sin(\log x)]$$

or, $xy_2 + y_1 = -\frac{1}{x}y$ $\quad$ ($\because$ by using (i))

$\therefore$ $x^2y_2 + xy_1 + y = 0 \ldots \text{(ii)}$

Differentiating (ii) up to n steps, by using Leibnitz's theorem, we get,

$$D^n(x^2y_2) + D^n(xy_1) + D^n(y) = 0$$

or, $x^2y_{n+2} + c(n,1)y_{n+1}2x + c(n,2)y_n2 + xy_{n+1} + c(n,1)y_n1 + y_n = 0$

or, $x^2y_{n+2} + \{2c(n,1)+1\}xy_{n+1} + \{2c(n,2) + c(n,1) + 1\}y_n = 0$

or, $x^2y_{n+2} + (2n+1)xy_{n+1} + [n(n-1) + n + 1]y_n = 0$

$$\left(\because c(n,1) = n \;\&\; c(n,2) = \frac{n(n-1)}{2}\right)$$

$\therefore$ $x^2y_{n+2} + (2n+1)xy_{n+1} + (n^2+1)y_n = 0$ $\quad$ Proved

10. **If $y = \sin^{-1}x$, then prove that:**
$(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - n^2y_n = 0$

Sol^n: Given that,

$$y = \sin^{-1}x \ldots \text{(i)}$$

Differentiating with respect to x, we get,

$$y_1 = \frac{1}{\sqrt{1-x^2}}$$

Squaring on both sides,

$\therefore$ $y_1^2 = \frac{1}{1-x^2}$

or, $y_1^2(1-x^2) = 1$

Again, differentiating with respect to x, we get,

$$2y_1y_2(1-x^2) + y_1^2(-2x) = 0$$

or, $y_2(1-x^2) - xy_1 = 0 \ldots \text{(ii)}$

Differentiating (ii) up to n steps by applying Leibnitz's theorem, we get,

$$c(n,0)y_{n+2}(1-x^2) + c(n,1)y_{n+1}(-2x) + c(n,2)y_n(-2)$$
$$-[c(n,0)y_{n+1}x + c(n,1)] = 0$$

or, $(1-x^2)y_{n+2} - 2nxy_{n+1} - n(n-1)y_n - [xy_{n+1} + ny_n] = 0$

or, $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2 - n + n)y_n = 0$

or, $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - n^2y_n = 0$

Hence proved

11. **If $y = e^{a\sin^{-1}x}$, then prove that:**
$(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2+a^2)y_n = 0$

Sol^n: Given that,

$$y = e^{a\sin^{-1}x}$$

Differentiating with respect to x, we get,

$$y_1 = e^{a\sin^{-1}x}\frac{a}{\sqrt{1-x^2}} \ldots \text{(i)} \quad (\because \text{using (i)})$$

$$y_1 = \frac{ay}{\sqrt{1-x^2}}$$

Squaring on both sides,

$\therefore$ $y_1^2 = \frac{a^2y^2}{1-x^2}$

or, $(1-x^2)y_1^2 - a^2y^2 = 0 \ldots \text{(ii)}$

Differentiating (ii) with respect to x, we get,

$$(1-x^2)2y_1y_2 + y_1^2(-2x) - 2a^2yy_1 = 0$$

or, $(1-x^2)y_2 - xy_1 - a^2y = 0 \ldots \text{(iii)}$

Differentiating up to n steps by applying Leibnitz's theorem, we get,

$$c(n,0)(1-x^2)y_{n+2} + c(n,1)y_{n+1}(-2x) + c(n,2)y_n(-2)$$
$$-[c(n,0)y_{n+1}x + c(n,1)y_n1] - a^2y_n = 0$$

or, $(1-x^2)y_{n+2} - 2nxy_{n+1} - n(n-1)y_n - xy_{n+1} - ny_n - a^2y_n = 0$

or, $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - [n(n-1) + n + a^2]y_n = 0$

or, $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2+a^2)y_n = 0$

Hence proved

12. If $y = e^{x^2}$, then show that $y_{n+1} - 2xy_n - 2ny_{n-1} = 0$

Sol^n: Given that,

$$y = e^{x^2} \ldots \text{(i)}$$

Differentiating with respect to x, we get,

$$y_1 = e^{x^2} \cdot 2x$$

or, $y_1 = y \cdot 2x$ (∵ by using (i))

or, $y_1 = y2x$

Differentiating up to n steps by applying Leibnitz's theorem, we get,

$$y_{n+1} = 2[y_n \cdot x + c(n,1)y_{n-1} \cdot 1]$$

or, $y_{n+1} = 2[xy_n + ny_{n-1}]$

or, $y_{n+1} - 2xy_n - 2ny_{n-1} = 0$

Hence proved

13. If $y = (\sin^{-1} x)^2$, then show that:
$(1 - x^2)y_{n+2} - (2n + 1)xy_{n+1} - n^2y_n = 0$

Sol^n: Given that,

$$y = (\sin^{-1} x)^2 \ldots \text{(i)}$$

Differentiating with respect to x, we get,

$$y_1 = 2\sin^{-1} x \cdot \frac{1}{\sqrt{1-x^2}}$$

Squaring both sides, we get

$$y_1^2 = \left(2\sin^{-1} x \cdot \frac{1}{\sqrt{1-x^2}}\right)^2$$

or, $(1 - x^2)y_1^2 = 4(\sin^{-1} x)^2$

or, $(1 - x^2)y_1^2 - 4y = 0$ (∵ using (i))

Again, differentiating with respect to x, we get

$$(1 - x^2)2y_1y_2 + y_1^2(-2x) - 4y_1 = 0$$

or, $(1 - x^2)y_2 - xy_1 - 2 = 0$

Differentiating up to n steps by using Leibnitz's rule, we get,

$$c(n,0)(1 - x^2)y_{n+2} + c(n,1)y_{n+1}(-2x) + c(n,2)y_n(-2)$$
$$-[c(n,0)xy_{n+1} + c(n,1)y_n \cdot 1] - 0 = 0$$

or, $(1 - x^2)y_{n+2} - 2nxy_{n+1} - n(n - 1)y_n - xy_{n+1} - ny_n = 0$

or, $(1 - x^2)y_{n+2} - (2n + 1)xy_{n+1} - (n^2 - n + n)y_n = 0$

or, $(1 - x^2)y_{n+2} - (2n + 1)xy_{n+1} - n^2y_n = 0$

Hence proved

14. If $y = \sin(m \sin^{-1} x)$, then show that:
$(1 - x^2)y_{n+2} - (2n + 1)xy_{n+1} + (m^2 - n^2)y_n = 0$

Sol^n: Given that,

$$y = \sin(m\sin^{-1} x) \ldots \text{(i)}$$

Differentiating with respect to x, we get,

$$y_1 = \cos(m\sin^{-1} x) \cdot \frac{m}{\sqrt{1-x^2}}$$

Squaring on both sides, we get,

$$y_1^2 = m^2\frac{\cos^2\left(m\sin^{-1} x\right)}{1-x^2}$$

or, $(1 - x^2)y_1^2 = m^2[1 - \sin^2(m\sin^{-1} x)]$

or, $(1 - x^2)y_1^2 = m^2(1 - y^2)$ (∵ by using (i))

or, $(1 - x^2)y_1^2 - m^2 + m^2y^2 = 0$

Again, differentiating with respect to x, we get,

$$(1 - x^2)2y_1y_2 + y_1^2(-2x) - 0 + m^2 2yy_1 = 0$$

or, $(1 - x^2)y_2 - xy_1 + m^2y = 0 \ldots$ (ii)

Differentiating (ii) up to n steps by using Leibnitz's rule,

$$c(n,0)(1 - x^2)y_{n+2} + c(n,1)y_{n+1}(-2x) + c(n,2)y_n(-2)$$
$$-[c(n,0)xy_{n+1} + c(n,1)y_n] + m^2y_n = 0$$

or, $(1 - x^2)y_{n+2} - 2nxy_{n+1} - n(n - 1)y_n - xy_{n+1} - ny_n + m^2y_n = 0$

or, $(1 - x^2)y_{n+2} - (2n + 1)xy_{n+1} - [n(n - 1) + n - m^2]y_n = 0$

∴ $(1 - x^2)y_{n+2} - (2n + 1)xy_{n+1} + (m^2 - n^2)y_n = 0$

Hence proved

Exercise 3

1. Verify Rolle's theorem for the function:

i) $f(x) = x^2, x \in [-1, 1]$

Sol^n: Given that,

$f(x) = x^2, x \in [-1,1]$

a) Since every polynomial function is continuous, so that $f(x) = x^2$ is continuous for $x \in [-1,1]$.

b) Also, $f'(x) = 2x$ which exists for all $x \in [-1,1]$

c) $f(1) = 1^2 = 1$ & $f(-1) = (-1)^2 = 1$

$\therefore$ $f(1) = f(-1)$

Hence from (a), (b) and (c) all the conditions of Rolle's theorem is satisfied by f(x)

Now,

$f'(x) = 0$ gives,

$2x = 0$

$\therefore$ $x = 0$

Hence there exists $0 \in (-1,1)$ such that $f'(0) = 0$

Thus, the Rolle's theorem is verified.

ii) $f(x) = e^{-x} \sin x,\ x \in [0, \pi]$

Sol^n: Given that,

$f(x) = e^{-x} \sin x,\ x \in [0, \pi]$

Now,

$f(0) = e^{-0} \sin 0 = 0$

$f(\pi) = e^{-\pi} \sin \pi = e^{-\pi} \times 0 = 0$

$\therefore$ $f(0) = f(\pi)$

Clearly, f(x) is differentiable in $(0, \pi)$ and continuous in $[0, \pi]$.

Also,

$f(x) = \frac{\sin x}{e^x}$ ... (i)

$\therefore$ $f'(x) = \frac{e^x \cos x - e^x \sin x}{(e^x)^2} = \frac{e^x(\cos x - \sin x)}{e^{2x}}$

Now,

$f'(x) = 0$ gives,

$\frac{\cos x - \sin x}{e^x} = 0$

or, $\cos x - \sin x = 0$

or, $\cos x = \sin x$

or, $x = \frac{\pi}{4}$ and $\frac{\pi}{4} \in (0, \pi)$

$\therefore$ $f'\left(\frac{\pi}{4}\right) = 0$

Hence verified

iii) $f(x) = \left(\frac{\sin x - \cos x}{e^{-x}}\right),\ x \in \left[\frac{\pi}{4}, \frac{5\pi}{\mathrm{t}}\right]$

Sol^n: Clearly, f(x) is differentiable in $\left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$ and continuous in $\left[\frac{\pi}{4}, \frac{5\pi}{4}\right]$

Also,

$f'^{(x)} = \frac{e^{-x}(\cos x + \sin x) + (\sin x - \cos x)e^{-x}}{(e^{-x})^2} = \frac{2e^{-x} \sin x}{e^{-2x}}$

or, $f'(x) = \frac{2 \sin x}{e^{-x}}$

Here,

$f\left(\frac{\pi}{4}\right) = \left(\frac{\sin\frac{\pi}{4} - \cos\frac{\pi}{4}}{e^{-\frac{\pi}{4}}}\right) = 0$

and $f\left(\frac{5\pi}{4}\right) = \left(\frac{\sin\frac{5\pi}{4} - \cos\frac{5\pi}{4}}{e^{-\frac{5\pi}{4}}}\right)$

$\therefore$ $f\left(\frac{\pi}{4}\right) = f\left(\frac{5\pi}{4}\right)$

Also,

$f'(x) = 0$ gives;

$\frac{2 \sin x}{e^{-x}} = 0$

or, $\sin x = 0$

$\therefore$ $\sin x = \sin \pi$

$\therefore$ $x = \pi$

which shows that there exists $\pi \in \left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$ such that $f^1(\pi) = 0$

Verified

iv) $(x) = \log\left\{\frac{x^2 + ab}{(a+b)x}\right\},\ x \in [a, b]$

Sol^n: Given that,

$f(x) = \log\left\{\frac{x^2 + ab}{(a+b)x}\right\},\ x \in [a, b]$

Clearly $f(x)$ is differential in (a,b) and continuous in the interval $[a,b]$

Also,

$$f'(x) = \frac{1}{\frac{x^2+ab}{(a+b)x}} \times \frac{(a+b)x \cdot 2x-(x^2+ab)(a+b)}{(a+b)^2x^2}$$

$$= \frac{(a+b)x}{x^2+ab} \times \frac{(a+b)(x^2-ab)}{(a+b)^2x^2}$$

$$f'(x) = \frac{x^2-ab}{x(x^2+ab)}$$

Which exists for all $x \in (a,b)$

Also,

$$f(a) = \log\left\{\frac{a^2+ab}{(a+b)a}\right\} = \log 1 = 0$$

and $f(b) = \log\left\{\frac{b^2+ab}{(a+b)b}\right\} = \log 1 = 0$

$\therefore$ $f(a) = f(b)$

Now,

$f'(x) = 0$ gives,

$$\frac{x^2-ab}{x(x^2+ab)} = 0$$

or, $x^2 - ab = 0$

or, $x = \pm\sqrt{ab}$

$\therefore$ There exists $\sqrt{ab} \in (a,b)$ such that $f^1(\sqrt{ab}) = 0$

Hence verified

v) $\mathbf{f(x) = \frac{x(x+3)}{e^{\frac{x}{2}}}, x \in [-3,0]}$

Sol^n: Given that,

$$f(x) = \frac{x(x+3)}{e^{\frac{x}{2}}} \ldots \text{(i)}, x \in [-3,0]$$

$$\therefore \quad f(-3) = \frac{-3(-3+3)}{e^{-\frac{3}{2}}} = 0$$

$$f(0) = \frac{0(0+3)}{e^{\frac{0}{2}}} = 0$$

So that $f(-3) = f(0)$

Clearly, $f(x)$ is differentiable in $(-3,0)$ and continuous in $[-3,0]$.

Differentiating (i) with respect to x, we get,

$$f'(x) = \frac{e^{\frac{x}{2}}(2x+3)-(x^2+3x)\cdot\frac{1}{2}e^{\frac{x}{2}}}{\left(e^{\frac{x}{2}}\right)^2}$$

Now,

$f'(x) = 0$ gives,

$$e^{\frac{x}{2}}(2x+3) - \frac{(x^2+3x)}{2}e^{\frac{x}{2}} = 0$$

or, $2e^{\frac{x}{2}}(2x+3) - (x^2+3x)e^{\frac{x}{2}} = 0$

or, $2(2x+3) - (x^2+3x) = 0$

or, $4x + 6 - x^2 - 3x = 0$

or, $-x^2 + x + 6 = 0$

or, $x^2 - x - 6 = 0$

or, $x^2 + 3x + 2x - 6 = 0$

or, $x(x-3) + 2(x-3) = 0$

or, $(x-3)(x-2) = 0$

or, $x = -2, 3$

$\therefore$ There exists $-2 \in (-3,0)$ such that $f'(-2) = 0$.

2. Verify Lagrange's mean value theorem for the function.

i) $f(x) = e^x, x \in [1,2]$

Sol^n: Given that,

$f(x) = e^x \ldots$ (i), $x \in [1,2]$

Here, $a = 1, b = 2$

$\therefore$ $f(a) = f(1) = e^1 = e$

$\therefore$ $f(b) = f(2) = e^2 = e^2$

Differentiating (i) with respect to x, we get,

$f'(x) = e^x$

$\therefore$ $f'(c) = e^c$

So that,

$f(2) - f(1) = (2-1)f'(c)$

or, $e^2 - e = 1e^c$

or, $e(e-1) = e^c$

or, $c = \log\{e(e-1)\} = \log e + \log(e-1) = 1 + \log(e-1)$

$[\because e \in (2,3) \therefore (e-1) < 2 \therefore \log(e-1) = 0 + \text{decimal}]$

$\in (1,2)$

Verified

ii) $f(x) = x^2, x \in [1,2]$

Sol^n: Given that,

$$f(x) = x^2 \ldots (i), x \in [1,2]$$

Clearly, f(x) is continuous in [1,2] and differentiable in (1,2)

Differentiating (i) with respect to x,

$$\therefore \quad f'(x) = 2x \ldots (ii)$$

Here, $a = 1,\ b = 2$

$$\therefore \quad f(a) = f(1) = 1^2 = 1$$
$$f(b) = f(2) = 2^2 = 4$$
$$\frac{f(b)-f(a)}{b-a} = \frac{4-1}{2-1} = 3$$

Now,

$$f'(c) = 2c$$

and $f(c) = \frac{f(b)-f(a)}{b-a}$ gives,

$$2c = 3$$
$$\therefore \quad c = \frac{3}{2}$$

Clearly $\frac{3}{2} \in (1,2)$

Hence verified

iii) $\mathbf{f(x) = \log x, x \in [1, e]}$

Sol^n: Given that,

$$f(x) = \log x \ldots (i),\ x \in [1, e]$$

Here,

$$a = 1, b = e$$
$$\therefore \quad f(a) = \log 1 = 0$$
$$f(b) = \log e = 1$$

So that,

$$\frac{f(b)-f(a)}{b-a} = \frac{1-0}{e-1} = \frac{1}{e-1}$$

Differentiating (i) with respect to x, we get,

$$f'(x) = \frac{1}{x}$$
$$\therefore \quad f'(c) = \frac{1}{c}$$

Also,

$$f'(c) = \frac{f(b)-f(a)}{b-a} \text{ gives}$$

or, $\frac{1}{c} = \frac{1}{e-1}$

$\therefore \quad c = e - 1 \in (1, e) \qquad \because e = 2.718$

Hence verified

iv) $\mathbf{f(x) = Ax^2 + Bx + c, x \in [a, b]}$

Sol^n: Given that,

$$f(x) = Ax^2 + Bx + c \ldots (i)\ \&\ x \in [a, b]$$
$$\therefore \quad f(a) = Aa^2 + Ba + c$$
$$f(b) = Ab^2 + Bb + c$$
$$f(b) - f(a) = Ab^2 - Aa^2 + Bb - Ba + c - c$$
$$= a(b^2 - a^2) + B(b - a)$$
$$= (b - a)\{A(b + a) + B\}$$

So that,

$$\frac{f(b)-f(a)}{b-a} = \frac{(b-a)\{A(b+a)+B\}}{(b-a)} = A(b + a) + B$$

Clearly, f(x) is differentiable in (a, b) and continuous in [a, b].

Now, differentiating (i) with respect to x, we get,

$$f'(x) = 2Ax + b$$
$$\therefore \quad f'(c_1) = 2Ac_1 + B$$

Here,

$$f'(c_1) = \frac{f(b)-f(a)}{b-a} \text{ gives,}$$

or, $2Ac_1 + B = A(b + a) + B$

or, $2Ac_1 = A(b + a)$

$\therefore \quad c_1 = \frac{b+a}{2} \in (a, b)$

Hence verified

v) $\mathbf{f(x) = \frac{x^2+1}{x}, x \in \left[\frac{1}{2}, 2\right]}$

Sol^n: Given that,

$$f(x) = \frac{x^2+1}{x} \ldots (i)\ x \in \left[\frac{1}{2}, 2\right]$$

Clearly, f(x) is differentiable in $\left(\frac{1}{2}, 2\right)$ and continuous in $\left[\frac{1}{2}, 2\right]$.

Here,

$$f\left(\frac{1}{2}\right) = \frac{\left(\frac{1}{2}\right)^2+1}{\frac{1}{2}} = \frac{\frac{1}{4}+1}{\frac{1}{2}} = \frac{5}{4} \times \frac{2}{1} = \frac{5}{2}$$
$$f(2) = \frac{2^2+1}{2} = \frac{5}{2}$$

$$\therefore \quad \frac{f(2)-f\left(\frac{1}{2}\right)}{2-\frac{1}{2}} = \frac{\frac{5}{2}-\frac{5}{2}}{\frac{3}{2}} = 0$$

From (i),

$$f(x) = \frac{x^2+1}{x}$$

$$f(x) = x + x^{-1}$$

$$\therefore \quad f'(x) = 1 - \frac{1}{x^2}$$

So that $f'(c) = 1 - \frac{1}{c^2}$

$$\therefore \quad f'(c) = f\frac{(2)-f\left(\frac{1}{2}\right)}{2-\frac{1}{2}} \text{ gives,}$$

$$1 - \frac{1}{c^2} = 0$$

or, $\frac{c^2-1}{c^2} = 0$

or, $c^2 - 1 = 0$

or, $c^2 = 1$

$\therefore \quad c = \pm 1$

So that there exists $1 \in \left(\frac{1}{2}, 2\right)$ such that

$$f'(1) = \frac{f(2)-f\left(\frac{1}{2}\right)}{2-\frac{1}{2}}$$

Hence verified

vi) $\mathbf{f(x) = x^2 + 3x + 2, x \in [0, 2]}$

Sol^n: Given that,

$$f(x) = x^2 + 3x + 2 \ldots \text{(i)}, x \in [0,2]$$

Clearly, f(x) is differentiable in (0,2) and continuous in [0,2]

Here,

$$f(0) = 0^2 + 3 \times 0 + 2 = 2$$

and $f(2) = 2^2 + 3 \times 2 + 2 = 12$

$$\therefore \quad \frac{f(2)-f(0)}{2-0} = \frac{12-2}{2} = 5$$

Differentiating (i) with respect to x, we get,

$$f'(x) = 2x + 3$$

$$\therefore \quad f'(c) = 2c + 3$$

Now,

$$f'(c) = \frac{f(2)-f(0)}{2-0} \text{ gives,}$$

$$2c + 3 = 5$$

or, $2c = 2$

$\therefore \quad c = 1 \in (0,2)$

Hence verified

3. **Verify Cauchy's mean value theorem:**

i) $\mathbf{f(x) = x^4, g(x) = 3x^2 \quad x \in [1, 2]}$

Sol^n: Given that,

$$f(x) = x^4 \ldots \text{(i)}$$

and $g(x) = 3x^2 \ldots \text{(ii)}, x \in [1,2]$

$\therefore \quad f(1) = 1^4 = 1 g(1) = 3 \times 1^2 = 3$

$f(2) = 2^4 = 16 g(2) = 3 \times 2^2 = 12$

Now,

$$f'(x) = 4x^3 \quad g'(x) = 6x$$

Since, f(x) and g(x) are differentiable in (1,2) and continuous in [1,2].

So that by Cauchy's mean value theorem, there exists a point $c \in (1,2)$ such that;

$$\frac{f'(c)}{g'(c)} = \frac{f(2)-f(1)}{g(2)-g(1)}$$

or, $\frac{4c^3}{6c} = \frac{16-1}{12-3}$

or, $\frac{2c^2}{3} = \frac{15}{9}$

or, $c^2 = \frac{15}{9} \times \frac{3}{2}$

or, $c^2 = \frac{5}{2}$

$\therefore \quad c = \pm\sqrt{\frac{5}{2}}$

Clearly $\sqrt{\frac{5}{2}} \in (1,2)$

Hence verified

ii) $\mathbf{f(x) = x^2, g(x) = x^3 \quad x \in [0, 1]}$

Sol^n: Given that,

$$f(x) = x^2 \ldots \text{(i)}$$

and $g(x) = x^3 \ldots \text{(ii)}, x \in [0,1]$

$\therefore$ $f(0) = 0^2 = 0 g(0) = 0^3 = 0$
$f(1) = 1^2 = 1 g(1) = 1^3 = 1$

Now,

$f'(x) = 2x$ and $g'(x) = 3x^2$

Since, f(x) and g(x)are differentiable in (0,1)and continuous [0,1], then by Cauchy's mean value theorem, there exists a point $c \in (0,1)$ such that,

$$\frac{f'(c)}{g'(c)} = \frac{f(1)-f(0)}{g(1)-g(0)}$$

or, $\frac{2c}{3c^2} = \frac{1-0}{1-0}$

or, $\frac{2}{3c} = 1$

$\therefore$ $c = \frac{2}{3}$

Clearly $\frac{2}{3} \in (0,1)$

Hence verified

4. In the mean value theorem $f(x+h) = f(x) + hf'(x+\theta h)$ find θ i

i) $f(x) = \frac{1}{x}$

Sol^n: Given that,

$f(x) = \frac{1}{x} \ldots$ (i)

$\therefore$ $f'(x) = -\frac{1}{x^2}$

Now,

$f(x+h) = \frac{1}{x+h}$

$f'(x+\theta h) = -\frac{1}{(x+\theta h)^2}$

So that,

$f(x+h) = f(x) + hf'(x+\theta h)$ gives,

$\frac{1}{x+h} = \frac{1}{x} + h\left\{-\frac{1}{(x+\theta h)^2}\right\}$

or, $x(x+\theta h)^2 = (x+h)\{x^2 + 2x\theta h + \theta^2 h^2 - hx\}$

or, $x\{x^2 + 2x\theta h + \theta^2 h^2\} = x^3 + 2x^2\theta h + x\theta^2 h^2 - hx^2 + hx^2 + 2x\theta h^2 + \theta^2 h^3 - h^2 x$

or, $x^3 + 2x^2\theta h + x\theta^2 h^2 = x^3 + 2x^2\theta h + x\theta^2 h^2 + 2x\theta h^2 + \theta^2 h^3 - h^2 x$

or, $\theta(2xh^2 + \theta h^3) = h^2 x$

or, $\theta h^2(2x + \theta h) = h^2 x$

or, $\theta^2 h + 2\theta x - x = 0$

$\therefore$ $\theta = -\frac{2x \pm \sqrt{(2x)^2 - 4 \cdot h \cdot (-x)}}{2 \cdot h} = -\frac{2x \pm 2\sqrt{x^2 + xh}}{2h}$

$\theta = -\frac{x \pm \sqrt{x^2 + xh}}{h}$

ii) $\mathbf{f(x) = e^x}$

Sol^n: Given that,

$f(x) = e^x \ldots$ (i)

$\therefore$ $f(x+h) = e^{x+h}$

$f'(x) = e^x$

$\therefore$ $f'(x+\theta h) = e^{x+\theta h}$

Now,

$f(x+h) = f(x) + hf'(x+\theta h)$ gives,

$e^{x+h} = e^x + he^{x+\theta h}$

or, $e^h = 1 + he^{\theta h}$

or, $he^{\theta h} = e^h - 1$

or, $e^{\theta h} = \frac{e^h - 1}{h}$

$\therefore$ $\theta h = \log\left(\frac{e^h - 1}{h}\right)$

$\therefore$ $\theta = \frac{1}{h}\log\left(\frac{e^h - 1}{h}\right)$

iii) $\mathbf{f(x) = \log x}$

Sol^n: Given that,

$f(x) = \log x \ldots$(i)

$\therefore$ $f'(x) = \frac{1}{x}$

So that,

$f(x+h) = \log(x+h)$

and $f'(x+\theta h) = \frac{1}{x+\theta h}$

Now,

$f(x+h) = f(x) + hf'(x+\theta h)$

or, $\log(x+h) = \log x + h \cdot \frac{1}{x+\theta h}$

or, $\log\left(\frac{x+h}{x}\right) = \frac{h}{x+\theta h}$

or, $\frac{x+\theta h}{h} = \frac{1}{\log\left(\frac{x+h}{x}\right)}$

or, $\frac{x}{h} + \theta = \frac{1}{\log\left(1+\frac{h}{x}\right)}$

$\therefore \quad \theta = \frac{1}{\log\left(1+\frac{h}{x}\right)} - \frac{x}{h}$

5. **In the mean value theorem $f(a+h) = f(a) + hf'(a+\theta h)$, find θ if $f(x) = \sqrt{x}$, $a = 1$, $h = 3$.**

Sol^n: We have,

$f(x) = \sqrt{x}$

$\therefore \quad f'(x) = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$

Also,

$f(a+h) = \sqrt{a+h}$

and $f'(a+\theta h) = \frac{1}{2\sqrt{a+\theta h}}$

Now,

$f(a+h) = f(a) + hf'(a+\theta h)$ gives,

$\sqrt{a+h} = \sqrt{a} + h \cdot \frac{1}{2\sqrt{1+\theta \cdot 3}}$

or, $2 = 1 + \frac{3}{2\sqrt{3\theta+1}}$

or, $\frac{1}{\sqrt{3\theta+1}} = \frac{2}{3}$

or, $\sqrt{3\theta+1} = \frac{3}{2}$

$\therefore \quad 3\theta + 1 = \frac{9}{4}$

or, $3\theta = \frac{9}{4} - 1$

or, $3\theta = \frac{5}{4}$

$\therefore \quad \theta = \frac{5}{12}$

6. **Assuming the validity of expansion, find the expansion of the following functions very Maclaurin's theorem.**

i) $\log(1+x)$

Sol^n: Let,

$f(x) = \log(1+x) \ldots$ (i)

$\therefore \quad f(0) = \log(1+0) = \log 1 = 0$

$f'(x) = \frac{1}{x+1}$

$\therefore \quad f'(0) = \frac{1}{0+1} = 1$

$f''(x) = \frac{-1}{(x+1)^2}$

$\therefore \quad f''(0) = \frac{-1}{(0+1)^2} = -1$

$f'''(x) = \frac{2}{(x+1)^3}$

$\therefore \quad f'''(0) = \frac{2}{(0+1)^3} = 2$

$f^{iv}(x) = \frac{-6}{(x+1)^4}$

$\therefore \quad f^{iv}(0) = \frac{-6}{(0+1)^4} = -6$

We have Maclaurin's series of $f(x)$ is given by,

$$f(x) = f(0) + \frac{x}{1!}f'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \frac{x^4}{4!}f^{iv}(0) + \ldots\ldots$$

$$\therefore \quad \log(1+x) = 0 + \frac{x}{1!}.1 + \frac{x^2}{2!}(-1) + \frac{x^3}{3!}(2) + \frac{x^4}{4!}(-6) + \ldots\ldots\ldots$$

$$\therefore \quad \log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\ldots\ldots\ldots$$

ii) $\sin hx$

Sol^n: Let,

$f(x) = \sin hx$

$\therefore \quad f(0) = \sin h(0) = 0$

$f'(x) = \cos hx$

$\therefore \quad f'(0) = \cos h0 = 1$

$f''(x) = \sin hx$

$\therefore \quad f''(0) = 0$

$f'''(x) = \cos hx$

$\therefore \quad f'''(0) = 1$

$f^{iv}(x) = \sinh x$

$\therefore \quad f^{iv}(0) = 0$

$f^{v}(x) = \cosh x$

$\therefore \quad f^{v}(0) = 1$

By Maclaurin's Theorem, we have,

$$f(x) = f(0) + \frac{x}{1!}f'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \frac{x^4}{4!}f^{iv}(0) + \frac{x^5}{5!}f^{v}(0) + \dots\dots\dots\dots$$

$$\therefore \quad \sinh x = \frac{x}{1!} + \frac{x^2}{2!}\times 0 + \frac{x^3}{3!}\times 1 + \frac{x^4}{4!}\times 0 + \frac{x^5}{5!}\times 1 + \dots\dots\dots$$

Hence,

$$\sinh x = \frac{x}{1!} + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots\dots$$

$$\left[\begin{array}{l}\underline{\text{Note:}}\\ \sinh x = \dfrac{e^x - e^{-x}}{2}\\ \cosh x = \dfrac{e^x + e^{-x}}{2}\end{array}\right]$$

iii) **cos hx**

Solⁿ: Let,

$f(x) = \cosh x$

$\therefore \quad f(0) = 1$

$f'(x) = \sinh x$

$\therefore \quad f'(0) = 0$

$f''(x) = \cosh x$

$\therefore \quad f''(0) = 1$

$f'''(x) = \sinh x$

$\therefore \quad f'''(0) = 0$

$f^{iv}(x) = \cosh x$

$\therefore \quad f^{iv}(0) = 1$

By Maclaurin's Theorem, we have,

$$f(x) = f(0) + \frac{x}{1!}f'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \frac{x^4}{4!}f^{iv}(0) + \dots\dots$$

$$\therefore \quad \cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots\dots\dots\dots$$

iv) **tan hx**

Solⁿ: Let,

$f(x) = \tanh x$

$\therefore \quad f(0) = 0$

$f'(x) = \operatorname{sech}^2 x$

$\therefore \quad f'(0) = 1$

$f''(x) = -2\operatorname{sech} x \cdot \tanh x$

$\therefore \quad f''(0) = 0$

$f'''(x) = -2\operatorname{sech}^4 x + 4\operatorname{sech}^2 x \cdot \tanh^2 x$

$\therefore \quad f'''(0) = -2$

$$\begin{aligned} f^{iv}(x) &= -8\operatorname{sech}^3 x \cdot \operatorname{sech} x \cdot \tanh x + 8\operatorname{sech}^2 x \cdot \tanh x \cdot \operatorname{sech}^2 x \\ &\quad + 8\operatorname{sech} x \cdot \operatorname{sech} x \cdot \tanh x \cdot \tanh^2 x \\ &= 16\operatorname{sech}^4 x \tanh x + 8\operatorname{sech}^2 x \tanh^3 x \end{aligned}$$

$\therefore \quad f^{iv}(0) = 0$

Again,

$$\begin{aligned} f^{v}(x) &= 16\operatorname{sech}^4 x \cdot \operatorname{sech}^2 x + 64\operatorname{sech}^4 x \cdot \tanh^2 x + 24\operatorname{sech}^4 x \\ &\quad \tanh^2 x + 16\operatorname{sech}^2 x \cdot \tanh^4 x \end{aligned}$$

$\therefore \quad f^{v}(0) = 16$

We know, the Maclaurin's series in infinite form is;

$$f(x) = f(0) + \frac{x}{1!}f'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \frac{x^4}{4!}f^{iv}(0) + \frac{x^5}{5!}f^{v}(0) + \dots\dots\dots$$

$$\therefore \quad \tanh x = 0 + \frac{x}{1!}.1 + \frac{x^2}{2!}.0 + \frac{x^3}{3!}(-2) + \frac{x^4}{4!}.0 + \frac{x^5}{5!}.16$$

So that,

$$\tanh x = x - \frac{2x^3}{3!} + \frac{6x^5}{5!} \dots\dots\dots$$

v) **tan x**

Solⁿ: Let,

$f(x) = \tan x$

$\therefore \quad f(0) = 0$

$f'(x) = \sec^2 x$

$\therefore \quad f'(0) = 1$

$$\begin{aligned} f''(x) &= 2\sec x . \sec x . \tan x = 2\tan x(1 + \tan^2 x) \\ &= 2\tan x + 2\tan^3 x \end{aligned}$$

$\therefore \quad f''(0) = 0$

Again,

$$\begin{aligned} f'''(x) &= 2\sec^2 x + 6\tan^2 x\sec^2 x \\ &= 2(1 + \tan^2 x) + 6\tan^2 x(1 + \tan^2 x) \\ &= 2 + 8\tan^2 x + 6\tan^4 x \end{aligned}$$

$\therefore \quad f'''(0) = 2$

$$\begin{aligned} f^{iv}(x) &= 0 + 16\tan x . \sec^2 x + 24\tan^3 x . \sec^2 x \\ &= 16\tan x(1 + \tan^2 x) + 24\tan^3 x(1 + \tan^2 x) \\ &= 16\tan x + 40\tan^3 x + 24\tan^5 x \end{aligned}$$

$\therefore \quad f^{iv}(0) = 0$

Again,

$$f^{v}(x) = 16\sec^2 x + 120\tan^2 x\sec^2 x + 120\tan^4 x\sec^2 x$$

$\therefore \quad f^{v}(0) = 0$

$f^{vi}(x) = 24[\sec^4 x \sec^2 x + 4\sec^4 x \tan^2 x]$
$\qquad -8[2\sec^2 x \tan^2 x + \sec$

$\therefore \quad f^{vi}(0) = 24 - 8 = 16$

By Maclaurin's series, we have,

$$f(x) = f(0) + \frac{x}{1!}f'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \frac{x^4}{4!}f^{iv}(0) + \frac{x^5}{5!}f^{v}(0) + \frac{x^6}{6!}f^{vi}(0) + \ldots\ldots$$

$$\therefore \quad \log(\sec x) = \frac{x^2}{2!} + \frac{2x^4}{4!} + \frac{16x^6}{6!} + \ldots\ldots\ldots\ldots$$

ix) $\frac{e^x}{1+e^x}$

Sol^n: Let,

$$f(x) = \frac{e^x}{1+e^x}$$

$$\therefore \quad f(0) = \frac{1}{1+1} = \frac{1}{2}$$

$$f'(x) = \frac{e^x}{(1+e^x)^2}$$

$$\therefore \quad f'(0) = \frac{1}{(1+1)^2} = \frac{1}{4}$$

Again,

$$f''(x) = \frac{(1+e^x)^2 e^x - 2(1+e^x)\cdot e^x \cdot e^x}{(1+e^x)^4}$$

$$= (1+e^x)\left[\frac{(1+e^x)e^x - 2e^{2x}}{(1+e^x)^4}\right]$$

or, $$f''(x) = \frac{e^x - e^{2x}}{(1+e^x)^3}$$

$$\therefore \quad f''(0) = \frac{1-1}{2^3} = 0$$

Also,

$$f'''(x) = \frac{(1+e^x)^3(e^x - 2e^{2x}) - (e^x - e^{2x})3(1+e^x)e^x}{(1+e^x)^6}$$

$$= (1+e^x)\left[\frac{(1+e^x)^2(e^x - 2e^{2x}) - 3e^x(e^x - e^{2x})}{(1+e^x)^6}\right]$$

$$= \frac{(1+e^x)^2(e^x - 2e^{2x}) - 3e^x(e^x - e^{2x})}{(1+e^x)^5}$$

$$\therefore \quad f'''(0) = \frac{(1+1)^2(1-2) - 3\cdot 1(1-1)}{(1+1)^5} = \frac{2^2(-1)}{2^5} = -\frac{1}{2^3} = -\frac{1}{8}$$

Hence, by Maclaurin's expansion,

$$f(x) = f(0) + \frac{x}{1!}f'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \ldots\ldots\ldots\ldots$$

$$\therefore \quad \frac{e^x}{1+e^x} = \frac{1}{2} + x\frac{1}{4} + \frac{x^2}{2}\cdot 0 + \frac{x^3}{6}\left(-\frac{1}{8}\right) + \ldots\ldots\ldots\ldots$$

Hence,

$$\frac{e^x}{1+e^x} = \frac{1}{2} + \frac{x}{4} - \frac{x^3}{48} + \ldots\ldots\ldots$$

x) $\log(1+e^x)$

Sol^n: Let,

$f(x) = \log(1+e^x)$

$\therefore \quad f(0) = \log(1+1) = \log 2$

$$f'(x) = \frac{e^x}{1+e^x}$$

$$\therefore \quad f'(0) = \frac{1}{1+1} = \frac{1}{2}$$

$$f''(x) = \frac{(1+e^x)e^x - e^x\cdot e^x}{(1+e^x)^2}$$

$$\therefore \quad f''(0) = \frac{(1+1)-1}{(1+1)^2} = \frac{1}{4}$$

We know the Maclaurin's expansion of f(x) is:

$$f(x) = f(0) + \frac{x}{1!}f'(0) + \frac{x^2}{2!}f''(0) + \cdots$$

or, $$\log(1+e^x) = \log 2 + x\cdot\frac{1}{2} + \frac{x^2}{2}\cdot\frac{1}{4} + \cdots$$

$$\therefore \quad \log(1+e^x) = \log 2 + \frac{x}{2} + \frac{x^2}{8} + \cdots$$

xi) $e^{\sin x}$

Sol^n: Let,

$f(x) = e^{\sin x}$

$\therefore \quad f(0) = 1$

$f'(x) = e^{\sin x}\cos x = f(x)\cos x$

$\therefore \quad f'(0) = f(0)\cos 0 = 1 \times 1 = 1$

$f''(x) = f'(x)\cos x - f(x)\sin x$

$\therefore \quad f''(0) = f'(0)\cos 0 - f(0)\sin 0 = 1 \times 1 - 0 = 1$

Again,

$$f'''(x) = f''(x)\cos x - f(x)\sin x - f(x)\sin x - f(x)\cos x$$

$$\therefore \quad f'''(0) = f''(0)\cos 0 - f'(0)\sin 0 - f'(0)\sin 0 - f(0)\cos 0$$
$$= 1 \times 1 - 0 - 0 - 1 = 0$$

Now,

$$f'''(x) = f''(x)\cos x - 2f'(x)\sin x - f(x)\cos x$$
$$f'''(x) = [f''(x) - f'(x)]\cos x - 2f'(x)\sin x$$

Again, differentiating with respect to x, we get,

$$f^{iv}(x) = [f'''(x) - f'(x)]\cos x - [f''(x) - f(x)]\sin x - 2[f'(x)\cos x + f''(x)\sin x]$$

$$\therefore \quad f^{iv}(0) = [f'''(0) - f'(0)]\cos 0 - [f''(0) - f(0)]\sin 0 - 2[f'(0)\cos 0 + f''(0)\sin 0]$$
$$= (-0-1)\cdot 1 - (1-1)\cdot 0 - 2(1 \times 1 + 1 \times 0)$$
$$= -1 - 2$$
$$= -3$$

By Maclaurin's theorem

$$f(x) = f(0) + \frac{x}{1!}f'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \frac{x^4}{4!}f^{iv}(0) + \cdots$$

$$\therefore \quad e^{\sin x} = 1 + x + \frac{x^2}{2} - \frac{3x^4}{4!}$$

Hence,

$$e^{\sin x} = 1 + x + \frac{x^2}{2} - \frac{1}{8}x^4 + \ldots\ldots\ldots$$

xii) **$x \cot x$**

Sol^n: Let,

$$f(x) = x\cot x = x\frac{\cos x}{\sin x}$$

$$= \frac{x\left[1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots\ldots\ldots\right]}{x - \frac{x^3}{3!} + \frac{x^5}{5!}\ldots\ldots\ldots}$$

$$= \frac{1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots\ldots\ldots}{1 - \left(\frac{x^2}{3!} - \frac{x^4}{5!} + \ldots\ldots\ldots\right)}$$

$$= \left[1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots\ldots\ldots\right]\left[1 - \left(\frac{x^2}{3!} - \frac{x^4}{5!} + \ldots\ldots\ldots\right)\right]^{-1}$$

$$= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots\ldots\ldots\right)\left[1 + \left(\frac{x^2}{3!} - \frac{x^4}{5!} + \ldots\ldots\right) + \left(\frac{x^2}{3!} - \frac{x^4}{5!} + \ldots\ldots\right)^2 + \ldots\ldots\right]$$

$$= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots\ldots\right)\left[1 + \frac{x^2}{3!} - \frac{x^4}{5!} + \ldots\ldots + \frac{x^4}{(31)^2} + \ldots\ldots\right]$$

$$= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots\ldots\ldots\right)\left(1 + \frac{x^2}{6} + \frac{7}{360}x^4 + \ldots\ldots\right)$$

$$= 1 + \frac{x^2}{6} + \frac{7}{360}x^4 + \ldots\ldots \frac{x^2}{2!} - \frac{x^4}{6 \times 2!}\ldots\ldots + \frac{x^4}{4!} + \ldots\ldots$$

$$= 1 + \left(\frac{1}{6} - \frac{1}{2}\right)x^2 + \left(\frac{7}{360} - \frac{1}{12} + \frac{1}{24}\right)x^4 + \ldots\ldots\ldots$$

$$\therefore \quad f(x) = 1 - \frac{1}{3}x^2 - \frac{1}{45}x^4 + \ldots\ldots$$

7. Assuming the validity of expansion, prove the following series using Maclaurin's series.

i) $\sin^{-1}x = x + \frac{x^3}{3!} + \frac{9x^5}{5!} + \ldots\ldots\ldots\ldots$

And hence show that:

$\cos^{-1}x = \frac{\pi}{2} - x - \frac{x^3}{3!}\ldots\ldots\ldots\ldots$

Sol^n: Let,

$$f(x) = \sin^{-1}x$$
$$\therefore \quad f(0) = 0$$
$$f'(x) = \frac{1}{\sqrt{1-x^2}}$$
$$\therefore \quad f'(0) = 1$$

Squaring we get,

$$(f'(x))^2 = \frac{1}{1-x^2}$$

or, $(1-x^2)(f'(x))^2 = 1$

Differentiating with respect to x, we get,

$$(1-x^2)2f'(x)f''(x) + \{f'(x)\}^2(-2x) = 0$$

or, $(1-x^2)f''(x) - xf'(x) = 0 \ldots$ (i)

Putting $x = 0$

$$f''(0) - 0f'(0) = 0$$

or, $f''(0) = 0$

Differentiating n times by using Leibnitz's rule on (i), we get,

$$(1-x^2)f^{n+2}(x) + c(n,1)f^{n+1}(x)(-2x) + c(n,2)f^n(x)(-2) - [xf^{n+1}(x) + c(n,1)fn(x)] = 0$$

or, $(1-x^2)f^{n+2}(x) - nf^{n+1}(x)2x - n(n-1)f^n(x) - xf^{n+1}(x) - nfn(x) = 0$

or, $(1-x^2)f^{n+2}(x)-(2n+1)xf^{n+1}(x)-n^2f^n(x)=0 \dots$ (ii)

Putting $x = 0$ in (ii),

$\therefore \quad f^{n+2}(0)=n^2f^n(0) \dots$ (iii)

Putting $n = 1,2,3$ in (iii), we get,

$$f'''(0)=1^2f'(0)=1\times 1=1$$
$$f^{iv}(0)=2^2f''(0)=4\times 0=0$$
$$f^{v}(0)=3^2f'''(0)=9\times 1=9$$

We know that, the Maclaurin's expansion for infinite series is;

$$f(x)=f(0)+\frac{x}{1!}f'(0)+\frac{x^2}{2!}f''(0)+\frac{x^3}{3!}f'''(0)+\frac{x^4}{4!}f^{iv}(0)+\frac{x^5}{5!}f^{v}(0)+\dots\dots\dots$$

$$\therefore \quad \sin^{-1}x=0+x\cdot 1+\frac{x^2}{2}\times 0+\frac{x^3}{3!}\times 1+\frac{x^4}{4!}\times 0+\frac{x^5}{5!}\times 9+\dots$$

$$\therefore \quad \sin^{-1}x=x+\frac{x^3}{3!}+\frac{9x^5}{5!}+\dots\dots\dots$$

Again, we know that,

$$\sin^{-1}x+\cos^{-1}x=\frac{\pi}{2}$$

$$\therefore \quad \cos^{-1}x=\frac{\pi}{2}\sin^{-1}x$$

$$=\frac{\pi}{2}-x-\frac{x^3}{3!}-\frac{9x^5}{5!}\dots\dots\dots \text{ Proved}$$

ii) $\mathbf{\sec x = 1+\frac{x^2}{2}+\frac{5}{24}x^4+\dots}$

Sol^n: Let

$$f(x)=\sec x$$
$$\therefore \quad f(0)=1$$
$$f'(x)=\sec x\tan x$$
$$\therefore \quad f'(0)=0$$

Again,

$$f''(x)=\sec x\tan^2x+\sec^3x$$
$$f''(x)=2\sec^3x-\sec x$$
$$f''(0)=0$$

Differentiating with respect to x,

$$\therefore \quad f'''(x)=6\sec^2x\cdot\sec x\cdot\tan x-\sec x\cdot\tan x$$
$$=6\{(1+\tan^2x)\sec x\tan x\}-\sec x\tan x$$
$$=6\sec x\tan x+6\sec x\tan^3x-\sec x\tan x$$
$$f'''(x)=5\sec x\tan x+6\sec x\tan^3x$$

$$\therefore \quad f'''(0)=0$$

Again, differentiating with respect to x, we get,

$$f^{iv}(x)=5[\sec x\sec^2x+\sec x\tan^2x]+6[\sec^2x\tan^4x+3\sec x\tan^2x\sec^2x]$$

$$\therefore \quad f^{iv}(0)=5[1+0]+6[0+0]=5$$

Hence, by Maclaurin's expansion of f(x), we get,

$$f(x)=f(0)+\frac{x}{1!}f'(0)+\frac{x^2}{2!}f''(0)+\frac{x^3}{3!}f'''(0)+\frac{x^4}{4!}f^{iv}(0)+\dots\dots$$

or, $\sec x=1+x\times 0+\frac{x^2}{2}\times 1+\frac{x^3}{6}\times 0+\frac{x^4}{24}\times 5+\dots\dots\dots$

$$\therefore \quad \sec x=1+\frac{x^2}{2}+\frac{5}{24}x^4+\dots\dots\dots \text{ Proved}$$

iii) $\mathbf{\log(1+x+x^2)}$

Sol^n: Let,

$$f(x)=\log(1+x+x^2)$$
$$=\log\left\{\frac{(1-x)(1+x+x^2)}{(1-x)}\right\}$$
$$=\log\left\{\frac{(1-x^3)}{(1-x)}\right\}$$

$$\therefore \quad f(x)=\log(1-x^3)-\log(1-x) \dots \text{(i)}$$

But, we have,

$$\log(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\frac{x^5}{5}\dots\dots\dots$$

So that,

$$\log(1-x)=-x-\frac{x^2}{2}-\frac{x^3}{3}-\frac{x^4}{4}-\frac{x^5}{5}\dots\dots\dots$$

and $\log(1-x^3)=-x^3-\frac{1}{2}(x^3)^2-\frac{1}{3}(x^3)^3-\frac{1}{4}(x^3)^4-\frac{1}{5}(x^3)^5\dots\dots$

$$=-x^3-\frac{x^6}{2}-\frac{x^9}{3}-\frac{x^{12}}{4}\dots\dots\dots$$

Then from (i),

$$f(x)=\left(-x^3-\frac{x^6}{2}-\frac{x^9}{3}-\frac{x^{12}}{4}\dots\dots\right)-\left\{-x-\frac{x^2}{2}-\frac{x^3}{3}-\frac{x^4}{4}\dots\dots\right\}$$
$$=x+\frac{x^2}{2}+\left(\frac{x^3}{3}-x^3\right)+\frac{x^4}{4}+\dots\dots\dots$$
$$f(x)=x+\frac{x^2}{2}-\frac{2}{3}x^3+\frac{x^4}{4}+\dots\dots\dots \quad \text{Proved}$$

iv) x cosec x

Solⁿ: Let,

$$f(x) = x \operatorname{cosec} x \qquad (\because f(0) = 0 \times \infty)$$

Let us suppose,

$$x \operatorname{cosec} x = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \dots \dots \quad \dots (i)$$

Where, $a_0, a_1, a_2, a_3, a_4 \dots\dots\dots$ are constants to be determined

or, $x = \sin x\,(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \dots\dots\dots)$

or, $x = \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} \dots\right)(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \dots)$

or, $x = a_0x + a_1x^2 + \left(a_2 - \frac{a_0}{3!}\right)x^3 + \left(a_3 - \frac{a_1}{3!}\right)x^4 + \left(a_4 - \frac{a_2}{3!} + \frac{a_0}{5!}\right)x^5 + \dots\dots\dots$

Equating the coefficient of like terms from both sides, we get,

$$a_0 = 1, a_1 = 0, a_2 - \frac{a_0}{3!} = 0$$

or, $a_2 = \frac{a_0}{3!} = \frac{1}{6}$

$$a_3 - \frac{a_1}{3!} = 0$$

$$a_3 = \frac{a_1}{3!} = 0$$

$$a_4 - \frac{a_2}{3!} + \frac{a_0}{5!} = 0$$

or, $a_4 = \frac{a_2}{3!} - \frac{a_0}{5!} = \frac{1}{6\times3!} - \frac{1}{5!} = \frac{1}{3!}\left(\frac{1}{6} - \frac{1}{20}\right) = \frac{1}{3!}\left(\frac{20-6}{120}\right)$

$$= \frac{1}{3\times2} \times \frac{14}{120} = \frac{7}{360}$$

Substituting the values of a_0, a_1, a_2, a_3 and a_4 in (i),

$\therefore \quad x \operatorname{cosec} x = 1 + \frac{1}{6}x^2 + \frac{7}{360}x^4 + \dots\dots\dots$ Proved

v) $\frac{1}{\sqrt{x^2+1}}$

Solⁿ: Let,

$$f(x) = \frac{1}{\sqrt{x^2+1}}$$

$\therefore \quad f(0) = 1$

Squaring on both sides,

$\therefore \quad \{f(x)\}^2 = \frac{1}{x^2+1}$

or, $(x^2 + 1)\{f(x)\}^2 = 1 \dots (i)$

Differentiating with respect to x, we get,

$$(1 + x^2) \cdot 2f(x) \cdot f'(x) + 2x \cdot \{f(x)\}^2 = 0 \dots (ii)$$

Put $x = 0$,

$\therefore \quad 2f(0) \times f'(0) + 2 \times 0\{f(0)\}^2 = 0$

or, $2 \times 1f'(0) = 0$

$\therefore \quad f'(0) = 0$

Again from (ii),

$\therefore \quad (1 + x^2)f'(x) + xf(x) = 0$

Again, differentiating with respect to x, n times by Leibnitz's rule we get,

$$(1 + x^2)f^{n+1}(x) + c(n,1)f^n(x)2x + c(n,2)f^{n-1}(x)2 + xf^n(x) + c(n,1)f^{n-1}(x) = 0$$

or, $(1 + x^2)f^{n+1}(x) + nf^n(x)2x + n(n-1)f^{n-1}(x) + xf^n(x) + nf^{n+1}(x) = 0$

or, $(1 + x^2)f^{n+1}(x) + (2n + 1)xf^n(x) + (n^2 - n + n)f^{n-1}(x) = 0$

or, $(1 + x^2)f^{n+1}(x) + (2n + 1)xf^n(x) + n^2f^{n-1}(x) = 0$

Put $n = 0$,

$\therefore \quad f^{n+1}(0) + 0 + n^2f^{n-1}(0) = 0$

or, $f^{n+1}(0) = -n^2f^{n-1}(0)$

Put $n = 1, 2, 3, \dots\dots\dots\dots$

$\therefore \quad f''(0) = -1^2f(0) = -1 \times 1 = -1$

$f'''(0) = -2^2f'(0) = -4 \times 0 = 0$

$f^{iv}(0) = -3^2f''(0) = -9 \times (-1) = 9$

By Maclaurin's series, we have,

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \frac{x^4}{4!}f^{iv}(0) + \dots\dots$$

or $\frac{1}{\sqrt{1+x^2}} = 1 + 0 - \frac{x^2}{2!} + 0 + \frac{x^4}{4!}.9 \dots\dots\dots\dots$

Hence,

$$\frac{1}{\sqrt{1+x^2}} = 1 - \frac{x^2}{2!} + \frac{9}{4!}x^4 \dots\dots\dots\dots \qquad \text{Proved}$$

vi) $\frac{x}{e^x-1}$

Solⁿ: Let,

$$\frac{x}{e^x-1} = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \dots\dots \quad \dots (i)$$

$$\left(\text{if } f(x) = \frac{x}{e^x-1} \text{ then } f(0) = \frac{0}{0}\right)$$

or, $x = (e^x - 1)\{a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \dots\dots\} \dots (ii)$

But,

$$e^x - 1 = \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} \cdots\cdots\cdots\right) - 1$$

$$\therefore \quad e^x - 1 = x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} \cdots\cdots\cdots$$

Use it in equation (ii), we get,

$$\therefore \quad x = \left(x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} \cdots\right)(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \cdots$$

$$\text{or,} \quad x = a_0x + \left(a_1 + \frac{a_0}{2!}\right)x^2 + \left(a_2 + \frac{a_1}{2!} + \frac{a_0}{3!}\right)x^3$$

$$+\left(a_3 + \frac{a_2}{2!} + \frac{a_1}{3!} + \frac{a_4}{4!}\right)x^4 + \cdots$$

Equating the coefficient of like terms, we get,

$$a_0 = 1, a_1 + \frac{a_0}{2!} = 0, a_2 + \frac{a_1}{2!} + \frac{a_0}{3!} = 0$$

$$\text{and} \quad a_3 + \frac{a_2}{2!} + \frac{a_1}{3!} + \frac{a_4}{4!} = 0$$

Solving, these equations, we get,

$$a_0 = 1, a_1 = -\frac{1}{2}, a_2 = \frac{1}{12}, a_3 = 0, a_4 = \frac{1}{720}$$

$\therefore$ From (i)

$$\frac{x}{e^x - 1} = 1 - \frac{x}{2} + \frac{x^2}{12} - \frac{1}{720}x^4 + \cdots\cdots\cdots \qquad \text{Proved}$$

8. If $y = e^{a\sin^{-1}x} = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots\cdots$ then show th
   i) $(1 - x^2)y_2 - xy_1 - a^2y = 0$
   ii) $(n + 1)(n + 2)an + 2 - (n^2 + a^2)an = 0$

   And hence obtain the expansion of $e^{a\sin^{-1}x}$

Sol^n: Given that,

$$y = e^{a\sin^{-1}x} = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots$$

Put $x = 0$, then $e^0 = a_0$

or, $a_0 = 1$

$$\therefore \quad y_1 = \frac{ae^{a\sin^{-1}x}}{\sqrt{1-x^2}}$$

By squaring both side,

$$\text{or,} \quad (1 - x^2){y_1}^2 = a^2(e^{a\sin^{-1}x})^2$$

$$(1 - x^2){y_1}^2 = a^2y^2$$

Differentiating we get,

$$(1 - x^2)2y_1y_2 + (-2x){y_1}^2 = a^2 2yy_1$$

$$\text{or,} \quad (1 - x^2)y_2 - xy_1 = a^2y$$

$$\text{or,} \quad (1 - x^2)y_2 - xy_1 - a^2y = 0 \ldots (*)$$

Hence, (i) is proved

Now, differentiating $(*)$ n times by using Leibnitz's theorem we get,

$$(1 - x^2)y_{n+2} + c(n,1)y_{n+1}(-2x) + c(n,2)y_n(-2) - xy_{n+1}$$
$$-c(n,1)y_n - a^2y_n = 0$$

$$\text{or,} \quad (1 - x^2)y_{n+2} - 2nxy_{n+1} - n(n-1)y_n - xy_{n+1} - ny_n - a^2y_n = 0$$

$$\text{or,} \quad (1 - x^2)y_{n+2} - (2n + 1)xy_{n+1} - (n^2 - n + n + a^2)y_n = 0$$

$$\text{or,} \quad (1 - x^2)y_{n+2} = (2n + 1)xy_{n+1} + (n^2 + a^2)y_n = 0 \ldots \text{(i)}$$

Also,

$$y = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots\cdots\cdots \ldots \text{(ii)}$$

and $a_0 = 1$

Now,

$$y_1 = a_1 + 2a_2x + 3a_3x^2 + \cdots\cdots\cdots$$

Put $x = 0$,

$\therefore \quad (y_1)_0 = a_1$

But

$$y_1 = \frac{ae^{a\sin^{-1}x}}{\sqrt{1-x^2}}$$

$\therefore$ At $x = 0$,

$$y_1 = a$$

So that,

$$(y_1)_0 = a_1 = a$$
$$y_2 = 2a_2 + 2 \cdot 3 \cdot a_3 \cdot x + \cdots\cdots\cdots$$
$$\therefore \quad (y_2)_0 = 2a_2 = 2!\,a_2$$
$$y_3 = 1 \cdot 2 \cdot 3 \cdot a_3 + \cdots \quad \therefore (y_3)_0 = 3!\,a_3$$
$$\vdots$$
$$y_n = n!\,a_n \qquad \therefore (y_n)_0 = n!\,a_n$$

Again, putting $x = 0$ in (i), we get,

$$(y_n + 2)_0 = (n^2 + a^2)(y_n)_0$$

$$\text{or,} \quad (n + 2)!\,a_{n+2} = (n^2 + a^2)n!\,a_n$$

$$(n + 2)(n + 1)a_{n+2} = (n^2 + a^2)\frac{n!}{n!}a_n$$

$$(n + 2)(n + 1)a_{n+2} = (n^2 + a^2)a_n = 0 \ldots \text{(iii)}$$

which proves the second part

But,

$$a_0 = 1, a_1 = a, a_2 = \frac{a_2}{2!}$$

Put $n = 1, 2, 3, \cdots\cdots\cdots$ in (iii) we get, When,

n = 1, then,

$$3\cdot 2\cdot a_3 = (1^2+a^2)a_1 = (1+a^2)a_1$$

$$\therefore\quad a_3 = \frac{(1+a^2)}{1\cdot 3\cdot 2}a_1 = \frac{(1+a^2)}{3!}a$$

When n = 2 then,

$$4\cdot 3\cdot a_4 = (2^2+a^2)a_2 = (2^2+a^2)\frac{a^2}{2!}$$

$$\therefore a_4 = \frac{a^2\left(2^2+a^2\right)}{2!\times 3\times 4} = \frac{a^2\left(2^2+a^2\right)}{4!}$$

When n = 3 then,

$$5\cdot 4\cdot a_5 = (3^2+a^2)a_3 = (3^2+a^2)\frac{a(1+a^2)}{3!}$$

$$\therefore\quad a_5 = \frac{a(1+a^2)\left(3^2+a^2\right)}{5\cdot 4\times 3!} = \frac{a(1+a^2)\left(3^2+a^2\right)}{5!}$$

Hence,

$$e^{a\sin^{-1}x} = y$$
$$= a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \ldots\ldots\ldots$$
$$= 1 + ax + \frac{a^2}{2!}x^2 + \frac{a(1+a^2)}{3!}x^3 + \frac{a^2\left(2^2+a^2\right)}{4!}x^4 + \frac{a(1+a^2)\left(3^2+a^2\right)}{5!}x^5 + \ldots\ldots$$

Exercise 4

1. Evaluate the following limits:

i) $\lim\limits_{x\to 0}\left[\frac{e^x-e^{-x}-2x}{x-\sin x}\right]$

Soln: Here,

$$\lim_{x\to 0}\left[\frac{e^x-e^{-x}-2x}{x-\sin x}\right] \quad \left(\frac{0}{0}\right)\text{form}$$

Using L' Hospital rule we get,

$$\lim_{x\to 0}\left[\frac{e^x+e^{-x}-2x}{x-\cos x}\right] \quad \left(\frac{0}{0}\right)\text{form}$$

$$= \lim_{x\to 0}\left[\frac{e^x-e^{-x}}{\sin x}\right] \quad \left(\frac{0}{0}\right)\text{form}$$

$$= \lim_{x\to 0}\left[\frac{e^x+e^{-x}}{\cos x}\right] = \frac{e^0+e^{-0}}{\cos 0} = \frac{1+1}{1} = 2$$

ii) $\lim\limits_{x\to 2}\left[\frac{x^3-2x^2+2x-4}{x^2-5x+6}\right]$

Soln: Here,

$$\lim_{x\to 2}\left[\frac{x^3-2x^2+2x-4}{x^2-5x+6}\right] \quad \left(\frac{0}{0}\right)\text{form}$$

Applying L' Hospital rule we get,

$$= \lim_{x\to 2}\left[\frac{3x^2-4x+2}{2x-5}\right] = \frac{3\times 2^2-4\times 2+2}{2\times 2-5} = \frac{6}{-1} = 6 \text{ Ans.}$$

iii) $\lim\limits_{x\to 0}\left[\frac{\tan x-x}{x-\sin x}\right]$

Soln: Here,

$$\lim_{x\to 0}\left[\frac{\tan x-x}{x-\sin x}\right] \quad \left(\frac{0}{0}\right)\text{form}$$

$$= \lim_{x\to 0}\left[\frac{\sec^2 x-1}{1-\cos x}\right] \quad \left(\frac{0}{0}\right)\text{form}$$

$$= \lim_{x\to 0}\left[\frac{2\sec x\cdot\sec x\cdot\tan x}{\sin x}\right]$$

$$= \lim_{x \to 0} \left[2\sec^2 x \cdot \frac{1}{\sin x} \cdot \frac{\sin x}{\cos x}\right]$$

$$= \frac{2 \times (\sec 0)^2}{\cos 0}$$

$$= 2 \times \frac{1^2}{1} = 2$$

iv) $\lim\limits_{x \to 0} \dfrac{x\cos x - \log(1+x)}{x^2}$

Solⁿ: Here,

$$\lim_{x \to 0} \frac{x\cos x - \log(1+x)}{x^2} \quad \left(\frac{0}{0}\right)\text{form}$$

$$= \lim_{x \to 0} \left[\frac{x(\sin x) + 1\cos x - \frac{1}{1+x}}{2x}\right] \quad \left(\frac{0}{0}\right)\text{form}$$

$$= \lim_{x \to 0} \left[\frac{-x\sin x + \cos x - \frac{1}{1+x}}{2x}\right]$$

$$= \lim_{x \to 0} \left[\frac{-x(1+x)\sin x + (1+x)\cos x - 1}{2x(1+x)}\right]$$

$$= \lim_{x \to 0} \left[\frac{-(x+x^2)\sin x + (1+x)\cos x - 1}{2(x+x^2)}\right] \quad \left(\frac{0}{0}\right)\text{form}$$

$$= \lim_{x \to 0} \left[\frac{-(x+x^2)\cos x - (1+2x)\sin x + (1+x)(-\sin x) + (0+1)\cos x}{2(1+2x)}\right]$$

$$= \frac{-(0+0)\cos 0 - (1+2\times 0)\sin 0 + (1+0)(-\sin 0) + \cos 0}{2(1+2\times 0)}$$

$$= -\frac{1}{2}$$

v) $\lim\limits_{x \to 0} \left[\dfrac{x - \sin^{-1} x}{\sin^3 x}\right]$

Solⁿ: Here,

$$\lim_{x \to 0} \left[\frac{x - \sin^{-1} x}{\sin^3 x}\right] \quad \left(\frac{0}{0}\right)\text{form}$$

$$= \lim_{x \to 0} \left[\frac{1 - \frac{1}{\sqrt{1-x^2}}}{3\sin^2 x \cdot \cos x}\right] = \lim_{x \to 0} \left[\frac{1-(1-x^2)^{-\frac{1}{2}}}{3\sin^2 x \cdot \cos x}\right] \quad \left(\frac{0}{0}\right)\text{form}$$

$$= \lim_{x \to 0} \left[\frac{0 - x(1-x^2)^{-\frac{3}{2}}}{6\sin x\cos^2 x - 3\sin^3 x}\right] \quad \left(\frac{0}{0}\right)\text{form}$$

$$= \lim_{x \to 0} \left[\frac{-\{1-x^2\}^{-\frac{3}{2}} - 3x^2(1-x^2)^{-\frac{5}{2}}}{6\cos^3 x - 12\cos x\sin^2 x - 9\sin^2 x \cdot \cos x}\right]$$

$$= \frac{-(1)^{-\frac{3}{2}} - 0}{6-0-0} = -\frac{1}{6}$$

vi) $\lim\limits_{x \to 0} \dfrac{2\sin x - \sin 2x}{\tan^3 x}$

Solⁿ: Here,

$$\lim_{x \to 0} \frac{2\sin x - \sin 2x}{\tan^3 x} \quad \left(\frac{0}{0}\right)\text{form}$$

If we put $x = 0$, then the given function take $\frac{0}{0}$ form. So that we apply the L' Hospital rule as

$$\lim_{x \to 0} \frac{2\sin x - \sin 2x}{\tan^3 x}$$

$$= \lim_{x \to 0} \left[\frac{2\cos x - 2\cos 2x}{3\tan^2 x \cdot \sec^2 x}\right]$$

$$= \lim_{x \to 0} \left[\frac{2(\cos x - \cos 2x)\cos^4 x}{3\sin^2 x}\right]$$

$$= \lim_{x \to 0} \left[\frac{2(-\sin x + 2\sin 2x)\cos^4 x - 8\cos^3 x\sin x(\cos x - \cos 2x)}{6\sin x\cos x}\right]$$

$$= \lim_{x \to 0} \left[\frac{-2\sin x\cos^4 x + 4\sin 2x\cos^4 x - 8\sin x\cos 4x + 8\cos^3 x\sin x}{3\sin 2x}\right]$$

$$= \lim_{x \to 0} \left[\frac{-10\sin x\cos^4 x + 4\sin 2x\cos^4 x + 2\cos^2 x\sin 4x}{3\sin 2x}\right] \left(\frac{0}{0}\right)\text{form}$$

$$= \lim_{x \to 0} \left[\frac{-10\cos^5 x + 40\cos^3 x\sin^2 x + 8\cos 2x\cos 4x - 16\sin 2x\cos^3 x\sin x + 8\cos^2 x\cos 4x - 4\cos x\sin x\sin 4x}{6\cos 2x}\right]$$

$$= \frac{-10+0+0+8-0+8-0}{6}$$

$$= \frac{6}{6} = 1$$

vii) $\lim\limits_{x \to 0} \dfrac{\tan x - x}{x^2 \tan x}$

Solⁿ: Here,

$$\lim_{x \to 0} \frac{\tan x - x}{x^2 \tan x}$$

Given function takes $\frac{0}{0}$ form when $x = 0$, so applying L' Hospital rule as

$$\lim_{x \to 0} \left[\frac{\tan x - x}{x^2 \tan x}\right] \quad \left(\frac{0}{0}\right)\text{form}$$

$$= \lim_{x \to 0} \left[\frac{\sec^2 x - 1}{x^2 \sec^2 x + 2x \tan x}\right] \quad \left(\frac{0}{0}\right)\text{form}$$

$$= \lim_{x \to 0} \left[\frac{2\sec^2 x \tan x}{2x^2 \sec^2 x \tan x + 2x \sec^2 x + 2x \sec^2 x + 2\tan x}\right] \quad \left(\frac{0}{0}\right)\text{form}$$

$$= \lim_{x \to 0} \left[\frac{2\sec^2 x \tan x}{2x^2 \sec^2 x \tan x + 4x \sec^2 x + 2\tan x}\right] \quad \left(\frac{0}{0}\right)\text{form}$$

$$= \lim_{x \to 0} \left[\frac{\tan x}{x^2 \tan x + 2x \tan x \cos^2 x}\right]$$

$$= \lim_{x \to 0} \left[\frac{\sec^2 x}{x^2 \sec^2 x + 2x \tan x + 2 + \cos 2x}\right]$$

$$= \frac{1}{0+0+2+1} = \frac{1}{3}$$

viii) $\lim\limits_{x \to 1} \left[\dfrac{1 + \log x - x}{1 - 2x + x^2}\right]$

Solⁿ: Here,

$$\lim_{x \to 1} \left[\frac{1 + \log x - x}{1 - 2x + x^2}\right]$$

Given function takes $\frac{0}{0}$ form when $x = 0$. So applying L' Hospital rule as:

$$\lim_{x \to 1} \left[\frac{1 + \log x - x}{1 - 2x + x^2}\right] \quad \left(\frac{0}{0}\right)\text{form}$$

$$= \lim_{x \to 1} \left[\frac{0 + \frac{1}{x} - 1}{-2 + 2x}\right] = \lim_{x \to 1} \left[\frac{\frac{1}{x} - 1}{-2 + 2x}\right] \quad \left(\frac{0}{0}\right)\text{form}$$

$$= \lim_{x \to 1} \left[\frac{-\frac{1}{x^2} - 0}{2}\right] = \lim_{x \to 1} \left[\frac{-\frac{1}{x^2}}{2}\right] = \frac{-\frac{1}{1^2}}{2} = -\frac{1}{2}$$

ix) $\lim\limits_{x \to 0} \left[\dfrac{\cosh x - \cos x}{x \sin x}\right]$

Solⁿ: Here,

$$\lim_{x \to 0} \left[\frac{\cosh x - \cos x}{x \sin x}\right]$$

If we put $x = 0$ then the given function takes $\frac{0}{0}$ form, so applying L' Hospital rule as

$$\lim_{x \to 0} \left[\frac{\cosh x - \cos x}{x \sin x}\right]$$

$$= \lim_{x \to 0} \left[\frac{e^x + e^{-x} - 2\cos x}{2x \sin x}\right] \qquad \left(\because \cosh x = \frac{e^x + e^{-x}}{2}\right)$$

$$= \lim_{x \to 0} \left[\frac{e^x - e^{-x} + 2\sin x}{2(\sin x + x \cos x)}\right] \quad \frac{0}{0}\text{ form}$$

$$= \lim_{x \to 0} \left[\frac{e^x + e^{-x} + 2\cos x}{2(\cos x + \cos x - x \sin x)}\right]$$

$$= \lim_{x \to 0} \left[\frac{e^x + e^{-x} + 2\cos x}{4\cos x - 2x \sin x}\right]$$

$$= \frac{1+1+2}{4} = 1$$

x) $\lim\limits_{x \to 0} \left[\dfrac{e^x - e^{\sin x}}{x - \sin x}\right]$

Solⁿ: Here,

$$\lim_{x \to 0} \left[\frac{e^x - e^{\sin x}}{x - \sin x}\right] \quad \frac{0}{0}\text{ form}$$

$$= \lim_{x \to 0} \left[\frac{e^x - \cos x e^{\sin x}}{1 - \cos x}\right] \quad \frac{0}{0}\text{ form}$$

$$= \lim_{x \to 0} \left[\frac{e^x - \{\cos x \cdot \cos x e^{\sin x} + (-\sin x) e^{\sin x}\}}{0 + \sin x}\right]$$

$$= \lim_{x \to 0} \left[\frac{e^x - \cos^2 x e^{\sin x} - \sin x e^{\sin x}}{\sin x}\right]$$

$$= \lim_{x \to 0} \left[\frac{e^x - (\cos^2 x + \sin x) e^{\sin x}}{\sin x}\right] \quad \frac{0}{0}\text{ form}$$

$$= \lim_{x \to 0} \left[\frac{e^x - (-2\sin x \cos x + \cos x) e^{\sin x} - (\cos^2 x + \sin x) e^{\sin x} \cos x}{\cos x}\right]$$

$$= \frac{e^0 - (-2 \times 0 \times 1 + 1) e^0 - (1+0) e^0 \cdot 1}{1} = \frac{1-1-1}{1} = -1$$

xi) $\lim\limits_{x \to 0} \dfrac{x\cos x - \sin x}{x^3}$

Sol^n: Here,

$$\lim_{x \to 0} \frac{x\cos x - \sin x}{x^3}$$

Given function takes $\frac{0}{0}$ form at $x = 0$, so applying L' Hospital rule as;

$$\lim_{x \to 0} \left[\frac{x\cos x - \sin x}{x^3}\right] \quad \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{x \to 0} \left[\frac{x(-\sin x) + 1\cos x - \cos x}{3x^2}\right]$$

$$= \lim_{x \to 0} \left[\frac{-x\sin x}{3x^2}\right] = \lim_{x \to 0} \left[\frac{-\sin x}{3x^2}\right]$$

$$= \lim_{x \to 0} \left[\frac{-\cos x}{3}\right] \quad \left(\frac{0}{0} \text{ form}\right)$$

$$= -\frac{1}{3}$$

xii) $\lim\limits_{x \to 0} \left[\dfrac{x\sin x - \sin^2 x}{x^4}\right]$

Sol^n: Here,

$$\lim_{x \to 0} \left[\frac{x\sin x - \sin^2 x}{x^4}\right] \quad \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{x \to 0} \left[\frac{x\cos x + 1\sin x - 2\sin x\cos x}{4x^3}\right]$$

$$= \lim_{x \to 0} \left[\frac{x\cos x + \sin x - \sin 2x}{4x^3}\right] \quad \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{x \to 0} \left[\frac{x(-\sin x) + 1\cos x + \cos x + \cos x - 2\cos 2x}{12x^2}\right]$$

$$= \lim_{x \to 0} \left[\frac{-x\sin x + 2\cos x - 2\cos 2x}{12x^2}\right] \quad \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{x \to 0} \left[\frac{-x\cos x - 1\sin x - \sin x + 4\sin 2x}{24x}\right]$$

$$= \lim_{x \to 0} \left[\frac{-x\cos x - 3\sin x + 4\sin 2x}{24x}\right] \quad \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{x \to 0} \left[\frac{-x(-\sin x) - 1\cos x - 3\cos x + 8\cos 2x}{24}\right]$$

$$= \frac{0 - \cos o - 3\cos 0 + 8\cos 0}{24} = \frac{-1-3+8}{24} = \frac{4}{24} = \frac{1}{6}$$

xiii) $\lim\limits_{x \to 0} \dfrac{xe^x - (1+x)\log(1+x)}{x^2}$

Sol^n: Here,

$$\lim_{x \to 0} \frac{xe^x - (1+x)\log(1+x)}{x^2}$$

If we put $x = 0$, then the given function takes $\frac{0}{0}$ form. So we use L' Hospital rule.

$$\lim_{x \to 0} \left[\frac{xe^x - (1+x)\log(1+x)}{x^2}\right] \quad \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{x \to 0} \left[\frac{xe^x + e^x - (1+x)\frac{1}{1+x} - \log(1+x)}{2x}\right]$$

$$= \lim_{x \to 0} \left[\frac{xe^x + e^x - 1 - \log(1+x)}{2x}\right] \quad \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{x \to 0} \left[\frac{xe^x + e^x + e^x - 0 - \frac{1}{1+x}}{2}\right]$$

$$= \frac{0\cdot e^0 + e^0 + e^0 - \frac{1}{1+0}}{2}$$

$$= \frac{1+1-1}{2} = \frac{1}{2}$$

2. Evaluate the following limits:

i) $\lim\limits_{x \to 0} \left[\dfrac{\log\tan x}{\cot x}\right]$

Sol^n: Here,

$$\lim_{x \to 0} \left[\frac{\log\tan x}{\cot x}\right]$$

Since $\log 0$ is indeterminate in itself. So that given function takes an indeterminate form when $x = 0$. Applying L' Hospital rule as

$$\lim_{x \to 0} \left[\frac{\log\tan x}{\cot x}\right]$$

$$= \lim_{x \to 0} \left[\frac{\frac{1}{\tan x}\sec^2 x}{-\operatorname{cosec}^2 x}\right] = \lim_{x \to 0} \left[\frac{\cos x}{-\sin x \operatorname{cosec}^2 x}\sec^2 x\right]$$

$$= \lim_{x \to 0} \left[\frac{\sec x}{-\operatorname{cosec} x}\right] = \lim_{x \to 0} \left[\frac{-\sin x}{\cos x}\right]$$

$$= \lim_{x \to 0} [-\tan x] = \lim_{x \to 0} (\tan x) = -\tan 0$$

$$= -0 = 0$$

ii) $\lim_{x \to \frac{\pi}{2}} \left[\frac{\tan 5x}{\tan x}\right]$

Sol^n: Here,

$$\lim_{x \to \frac{\pi}{2}} \left[\frac{\tan 5x}{\tan x}\right] \quad \left(\frac{\infty}{\infty} \text{ form}\right)$$

$$= \lim_{x \to \frac{\pi}{2}} \left[\frac{\sec^2 5x \cdot 5}{\sec^2 x}\right]$$

$$= \lim_{x \to \frac{\pi}{2}} \left[\frac{5 \sec^2 5x}{\sec^2 x}\right] \quad \left(\frac{\infty}{\infty} \text{ form}\right)$$

$$= \lim_{x \to \frac{\pi}{2}} \left[\frac{5 \cos^2 x}{\cos^2 5x}\right] \quad \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{x \to \frac{\pi}{2}} \left[\frac{-10 \cos x \sin x}{-2 \cos 5x \sin 5x \cdot 5}\right]$$

$$= \lim_{x \to \frac{\pi}{2}} \left[\frac{\sin 2x}{\sin 10x}\right] \quad \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{x \to \frac{\pi}{2}} \left[\frac{2 \cos 2x}{10 \cos 10x}\right] = \frac{1}{5} \frac{\cos 2 \cdot \frac{\pi}{2}}{\cos 10 \cdot \frac{\pi}{2}} = \frac{1}{5} \frac{\cos \pi}{\cos 5\pi} = \frac{1}{5}\left(\frac{-1}{-1}\right)$$

$$= \frac{1}{5}$$

iii) $\lim_{x \to 0} [x \log(\tan x)]$

Sol^n: Here,

$$\lim_{x \to 0} [x \log(\tan x)] \quad (0 \times \infty \text{ form})$$

$$= \lim_{x \to 0} \left[\frac{\log(\tan x)}{\frac{1}{x}}\right] = \lim_{x \to 0} \left[\frac{\frac{\sec^2 x}{\tan x}}{-\frac{1}{x^2}}\right]$$

$$= \lim_{x \to 0} \left[\frac{-x^2 \cdot \sec^2 x}{\tan x}\right] = \lim_{x \to 0} \left[\frac{-x^2}{\sin x \cos x}\right]$$

$$= \lim_{x \to 0} \left[\frac{-2x^2}{2 \sin x \cos x}\right] = \lim_{x \to 0} \left[\frac{-2x^2}{\sin 2x}\right] \quad \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{x \to 0} \left[\frac{-4x}{2 \cos 2x}\right]$$

$$= \frac{-4 \times 0}{2 \times \cos 0} = \frac{0}{2 \times 1} = 0$$

iv) $\lim_{x \to a} (a - x) \tan\left(\frac{\pi x}{2a}\right)$

Sol^n: Here,

$$\lim_{x \to a} (a - x) \tan\left(\frac{\pi x}{2a}\right) \quad (0 \times \infty \text{ form})$$

$$= \lim_{x \to a} \left[\frac{a - x}{\cot\left(\frac{\pi x}{2a}\right)}\right] \quad \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{x \to a} \left[\frac{1}{\frac{\pi}{2a} \operatorname{cosec}^2\left(\frac{\pi x}{2a}\right)}\right]$$

$$= \lim_{x \to a} \left[\frac{2a}{\pi} \sin^2\left(\frac{\pi x}{2a}\right)\right] = \frac{2a}{\pi} \sin^2\left(\frac{\pi \cdot a}{2a}\right) = \frac{2a}{\pi}\left(\sin\frac{\pi}{2}\right)^2$$

$$= \frac{2a}{\pi} \times 1^2 = \frac{2a}{\pi}$$

v) $\lim_{x \to 0} (x \log x)$

Sol^n: Here,

$$\lim_{x \to 0} (x \log x) \quad (0 \times \infty \text{ form})$$

$$= \lim_{x \to 0} \left[\frac{\log x}{\frac{1}{x}}\right] \quad \left(\frac{\infty}{\infty} \text{ form}\right)$$

$$= \lim_{x \to 0} \left[\frac{\frac{1}{x}}{-\frac{1}{x^2}}\right] = \lim_{x \to 0} (-x) = 0$$

vi) $\lim_{x \to 0} x \log \sin^2 x$

Sol^n: Here,

$$\lim_{x \to 0} [x \log(\sin^2 x)] \quad (0 \times \infty \text{ form})$$

$$= \lim_{x \to 0} \left[\frac{\log(\sin^2 x)}{\frac{1}{x}}\right] \quad \left(\frac{\infty}{\infty} \text{ form}\right)$$

$$= \lim_{x \to 0} \left[\frac{\frac{2 \sin x \cos x}{\sin^2 x}}{-\frac{1}{x^2}}\right]$$

$$= \lim_{x \to 0} \left[\frac{2 \cot x}{-\frac{1}{x^2}}\right] \quad \left(\frac{\infty}{\infty} \text{ form}\right)$$

$$= \lim_{x \to 0}\left[\frac{-2x^2\cos x}{\sin x}\right] \quad \left(\frac{\infty}{\infty}\text{ form}\right)$$

$$= \lim_{x \to 0}\left[\frac{-2(2x\cos x - x^2\sin x)}{\cos x}\right] = \frac{-2(2\times 0.1 - 0\times 0)}{1} = \frac{0}{1} = \ldots$$

vii) $\lim_{x \to 0} \log_{\tan x}(\tan 2x)$

Solⁿ: Here,

$$\lim_{x \to 0} \log_{\tan x}(\tan 2x)$$

$$= \lim_{x \to 0}\left[\frac{\log(\tan 2x)}{\log(\tan x)}\right] \qquad \left[\because \log_x(y) = \frac{\log_e(y)}{\log_e(x)} = \ldots\right]$$

$$\left(\frac{\infty}{\infty}\text{ form}\right)$$

$$= \lim_{x \to 0}\left[\frac{\frac{2\sec^2 2x}{\tan 2x}}{\frac{\sec^2 x}{\tan x}}\right] = \lim_{x \to 0}\left[\frac{2\sec^2 2x \cdot \tan x}{\sec^2 x \tan 2x}\right]$$

$$= \lim_{x \to 0}\left[2 \cdot \frac{\tan x}{\sec^2 x} \cdot \frac{1}{\sin 2x \cos 2x}\right]$$

$$= \lim_{x \to 0}\left[\frac{2\sin 2x}{2\sin 2x \cos 2x}\right]$$

$$= \lim_{x \to 0}\left[\frac{2\sin 2x}{\sin 4x}\right] \quad \left(\frac{0}{0}\text{ form}\right)$$

$$= \lim_{x \to 0}\left[\frac{4\cos 2x}{4\cos 4x}\right] = \frac{\cos 0}{\cos 0} = \frac{1}{1} = 1$$

3. Evaluate the following limits:

i) $\lim_{x \to \frac{\pi}{2}} (\sec x - \tan x)$

Solⁿ: Here,

$$\lim_{x \to \frac{\pi}{2}} (\sec x - \tan x) \quad (\infty - \infty\text{ form})$$

$$= \lim_{x \to \frac{\pi}{2}}\left[\frac{1}{\cos x} - \frac{\sin x}{\cos x}\right]$$

$$= \lim_{x \to \frac{\pi}{2}}\left[\frac{1-\sin x}{\cos x}\right] \quad \left(\frac{0}{0}\text{ form}\right)$$

$$= \lim_{x \to \frac{\pi}{2}}\left[\frac{-\cos x}{-\sin x}\right] = \frac{\cos\frac{\pi}{2}}{\sin\frac{\pi}{2}} = \frac{0}{1} = 0$$

ii) $\lim_{x \to 0}\left(\frac{1}{x} - \frac{1}{e^x - 1}\right)$

Solⁿ: Here,

$$\lim_{x \to 0}\left(\frac{1}{x} - \frac{1}{e^x-1}\right) \quad (\infty - \infty\text{ form})$$

$$= \lim_{x \to 0}\left[\frac{e^x - 1 - x}{x(e^x-1)}\right] \quad \left(\frac{0}{0}\text{ form}\right)$$

$$= \lim_{x \to 0}\left[\frac{e^x - 0 - 1}{x(e^x-1) + 1(e^x-1)}\right]$$

$$= \lim_{x \to 0}\left[\frac{e^x - 1}{xe^x - x + e^x - 1}\right] \quad \left(\frac{0}{0}\text{ form}\right)$$

$$= \lim_{x \to 0}\left[\frac{e^x - 0}{xe^x + 1e^x - 1 + e^x - 0}\right]$$

$$= \lim_{x \to 0}\left[\frac{e^x}{xe^x - 1}\right] = \frac{e^0}{0e^0 - 1} = \frac{1}{-1} = -1$$

iii) $\lim_{x \to 0}\left[\frac{1}{x^2} - \cot^2 x\right]$

Solⁿ: Here,

$$\lim_{x \to 0}\left[\frac{1}{x^2} - \cot^2 x\right] \quad (\infty - \infty\text{ form})$$

$$= \lim_{x \to 0}\left[\frac{1}{x^2} - \frac{\cos^2 x}{\sin^2 x}\right]$$

$$= \lim_{x \to 0}\left[\frac{\sin^2 x - x^2\cos^2 x}{x^2\sin^2 x}\right] \quad \left(\frac{0}{0}\text{ form}\right)$$

$$= \lim_{x \to 0}\left[\frac{\sin^2 x - x^2\cos^2 x}{x^4} \times \frac{x^2}{\sin^2 x}\right]$$

$$= \lim_{x \to 0}\left[\frac{\sin^2 x - x^2\cos^2 x}{x^4}\right] \times \left(\lim_{x \to 0}\frac{x}{\sin x}\right)^2$$

$$= \lim_{x \to 0}\left[\frac{\sin^2 x - x^2\cos^2 x}{x^4}\right] \times 1^2$$

$$= \lim_{x \to 0}\left[\frac{\sin^2 x - x^2\cos^2 x}{x^4}\right] \quad \left(\frac{0}{0}\text{ form}\right)$$

$$= \lim_{x \to 0}\left[\frac{\sin 2x - 2x\cos^2 x + x^2\sin 2x}{4x^3}\right] \quad \left(\frac{0}{0}\text{ form}\right)$$

$$= \lim_{x \to 0}\left[\frac{2\cos 2x - 2\cos^2 x + 2x\sin 2x + 2x\sin 2x + 2x^2\cos 2x}{12x^2}\right]$$

$$= \lim_{x \to 0}\left[\frac{2(\cos^2 x - \sin^2 x) - 2\cos^2 x + 4x\sin 2x + 2x^2\cos 2x}{12x^2}\right]$$

$$= \lim_{x \to 0}\left[\frac{-2\sin^2 x + 4x\sin 2x + 2x^2\cos 2x}{12x^2}\right] \quad \left(\frac{0}{0}\text{ form}\right)$$

$$= \lim_{x \to 0}\left[\frac{-2\sin x + 4x\sin 2x + 8x\cos 2x + 4x\cos 2x - 4x^2\sin}{24x}\right]$$

$$= \lim_{x \to 0}\left[\frac{2\sin 2x + 12x\cos 2x - 4x^2\sin 2x}{24x}\right] \quad \left(\frac{0}{0}\text{ form}\right)$$

$$= \lim_{x \to 0}\left[\frac{4\cos 2x + 12\cos 2x - 24x\sin 2x - 8x\sin 2x - 8x^2\cos 2}{24}\right]$$

$$= \frac{4+12-0-0-0}{24} = \frac{16}{24} = \frac{2}{3}$$

iv) $\lim_{x \to 0}\left[\frac{1}{x} - \frac{1}{x^2}\log(1+x)\right]$

Sol^n: Here,

$$\lim_{x \to 0}\left[\frac{1}{x} - \frac{1}{x^2}\log(1+x)\right] \quad (\infty - \infty\text{ form})$$

$$= \lim_{x \to 0}\left[\frac{x - \log(1+x)}{x^2}\right] \quad \left(\frac{0}{0}\text{ form}\right)$$

$$= \lim_{x \to 0}\left[\frac{1 - \frac{1}{1+x}}{2x}\right] = \lim_{x \to 0}\left[\frac{1+x-1}{2x(1+x)}\right]$$

$$= \lim_{x \to 0}\left[\frac{x}{2x(1+x)}\right] = \lim_{x \to 0}\left[\frac{1}{2(1+x)}\right] = \frac{1}{2(1+0)} = \frac{1}{2}$$

v) $\lim_{x \to 1}\left[\frac{1}{x-1} - \frac{1}{\log x}\right]$

Sol^n: Here,

$$\lim_{x \to 1}\left[\frac{1}{x-1} - \frac{1}{\log x}\right] \quad (\infty - \infty\text{ form})$$

$$= \lim_{x \to 1}\left[\frac{\log x - x + 1}{(x-1)\log x}\right] \quad \left(\frac{0}{0}\text{ form}\right)$$

$$= \lim_{x \to 1}\left[\frac{\frac{1}{x} - 1 + 0}{(x-1)\frac{1}{x} + (1-0)\log x}\right] = \lim_{x \to 1}\left[\frac{1-x}{x} \times \frac{x}{(x-1) + x\log x}\right]$$

$$= \lim_{x \to 1}\left[\frac{1-x}{x-1+x\log x}\right] \quad \left(\frac{0}{0}\text{ form}\right)$$

$$= \lim_{x \to 1}\left[\frac{0-1}{1-0+x\cdot\frac{1}{x}+1\cdot\log x}\right] = \lim_{x \to 1}\left[\frac{-1}{2+\log x}\right]$$

$$= \frac{-1}{2+\log 1} = \frac{-1}{2+0} = \frac{-1}{2}$$

4. Evaluate:

i) $\lim_{x \to \pi}(\sin x)^{\tan x}$

Sol^n: Here,

$$\lim_{x \to \pi}(\sin x)^{\tan x} \quad (0^0\text{ form})$$

Let, $y = (\sin x)^{\tan x}$

Taking log on both sides, we get,

$$\log y = \log(\sin x)^{\tan x}$$

or, $\log y = \tan x \log \sin x$

Taking limit as $x \to \pi$ on both sides,

$$\therefore \lim_{x \to \pi}(\log y) = \lim_{x \to \pi}[\tan x \cdot \log \sin x]$$

or, $\log\left[\lim_{x \to \pi}(y)\right] = \lim_{x \to \pi}\left[\frac{\log \sin x}{\cot x}\right] \quad \left(\frac{\infty}{\infty}\text{ form}\right)$

or, $\log\left[\lim_{x \to \pi}(y)\right] = \lim_{x \to \pi}\left[\frac{\frac{1}{\sin x}\cdot\cos x}{-\operatorname{cosec}^2 x}\right]$

$$= \lim_{x \to \pi}\left[\frac{-\cos x}{\operatorname{cosec} x}\right] = \lim_{x \to \pi}[-\sin x \cos x]$$

$$= \lim_{x \to \pi}\left[-\frac{\sin 2x}{2}\right] = \frac{0}{2} = 0$$

$$\therefore \lim_{x \to \pi} y = e^0 = 1 \qquad (\because y = \log x \Rightarrow x = e^y)$$

Hence,

$$\lim_{x \to \pi}(\sin x)^{\tan x} = 1$$

ii) $\lim_{x \to 0} x^{2\sin x}$

Sol^n: Here,

$$\lim_{x \to 0} x^{2\sin x} \quad (0^0\text{ form})$$

Let, $y = x^{2\sin x}$

Taking log on both sides, we get,

$$\log y = \log x^{2\sin x}$$

or, $\log y = 2\sin x \log x$

Taking limit as $x \to 0$ on both sides

$$\therefore \lim_{x\to 0}(\log y) = \lim_{x\to 0}[2\sin x \log x]$$

$$\text{or, } \log\left(\lim_{x\to 0} y\right) = \lim_{x\to 0}\left[\frac{2\log x}{\operatorname{cosec} x}\right] \quad \left(\frac{\infty}{\infty}\text{ form}\right)$$

$$= \lim_{x\to 0}\left[\frac{2\cdot\frac{1}{x}}{-\operatorname{cosec} x \cot x}\right]$$

$$= \lim_{x\to 0}\left[\frac{-2\sin^2 x}{x\cos x}\right]$$

$$= \lim_{x\to 0}\left[\frac{\cos 2x - 1}{x\cos x}\right] \quad \left(\frac{0}{0}\text{ form}\right)$$

$$= \lim_{x\to 0}\left[\frac{-2\sin x - 0}{1\cdot\cos x - x\sin x}\right]$$

$$= \frac{-2\times 0}{1-0} = \frac{0}{1} = 0$$

$$\therefore \log\left(\lim_{x\to 0} y\right) = 0$$

$$\text{or, } \lim_{x\to 0} y = e^0$$

$$\therefore \lim_{x\to 0}\left(x^{2\sin x}\right) = 1$$

iii) $\lim_{x\to 0}\left(\frac{1}{x^2}\right)^{\tan x}$

Sol^n: Here,

$$\lim_{x\to 0}\left(\frac{1}{x^2}\right)^{\tan x}$$

At $x = 0$ given function takes 0^0 form. So that,

Let, $y = \left(\frac{1}{x^2}\right)^{\tan x}$

Taking log on both sides

$$\therefore \log y = \log\left(\frac{1}{x^2}\right)^{\tan x}$$

$$\text{or, } \log y = \tan x \log\left(\frac{1}{x^2}\right)$$

Taking limit as $x \to 0$ on both sides, we get

$$\lim_{x\to 0}\log y = \lim_{x\to 0}\left[\tan x \log\left(\frac{1}{x^2}\right)\right]$$

$$= \lim_{x\to 0}[\tan x \log(x^{-2})]$$

$$= \lim_{x\to 0}\left[\frac{\log(x^{-2})}{\cot x}\right]$$

$$= \lim_{x\to 0}\left[\frac{-2\log x}{\cot x}\right] \quad \left(\frac{\infty}{\infty}\text{ form}\right)$$

$$= \lim_{x\to 0}\left[\frac{-2\cdot\frac{1}{x}}{-\operatorname{cosec}^2 x}\right]$$

$$= \lim_{x\to 0}\left[\frac{2\sin^2 x}{x}\right] \quad \left(\frac{0}{0}\text{ form}\right)$$

$$= \lim_{x\to 0}\left[\frac{4\sin x\cos x}{1}\right] = 4\times 0\times 1$$

$$\therefore \lim_{x\to 0}[\log y] = 0$$

$$\text{or, } \log\left[\lim_{x\to 0} y\right] = 0$$

$$\therefore \lim_{x\to 0} y = e^0$$

Hence,

$$\lim_{x\to 0}\left(\frac{1}{x^2}\right)^{\tan x} = 1$$

iv) $\lim_{x\to 0}(\cot x)^{\frac{1}{\log x}}$

Sol^n: Here,

$$\lim_{x\to 0}(\cot x)^{\frac{1}{\log x}}$$

$$y = (\cot x)^{\frac{1}{\log x}}$$

Taking log on both sides, we get

$$\log y = \log(\cot x)^{\frac{1}{\log x}}$$

$$\text{or, } \log y = \frac{1}{\log x}\log(\cot x)$$

Taking limit as $x \to 0$ on both sides

$$\therefore \lim_{x\to 0}(\log y) = \lim_{x\to 0}\left[\frac{1}{\log x}\log(\cot x)\right] \quad \left(\frac{\infty}{\infty}\text{ form}\right)$$

$$= \lim_{x\to 0}\left[\frac{\frac{1}{\cot x}(-\operatorname{cosec}^2 x)}{\frac{1}{x}}\right]$$

$$= \lim_{x \to 0}\left[\frac{-x}{\sin x \cos x}\right] = \lim_{x \to 0}\left[\frac{-2x}{2\sin x \cos x}\right]$$

$$= \lim_{x \to 0}\left[\frac{-2x}{\sin 2x}\right] \quad \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{x \to 0}\left[\frac{-2x}{2\cos 2x}\right] = \lim_{x \to 0}\left[\frac{-1}{\cos 2x}\right]$$

$$= -\frac{1}{\cos 0} = -\frac{1}{1}$$

$\therefore \quad \log\left[\lim_{x \to 0} y\right] = -1$

or, $\lim_{x \to 0} y = e^{-1}$

or, $\lim_{x \to 0} (\cot x)^{\frac{1}{\log x}} = e^{-1}$

v) $\lim\limits_{x \to \infty} x^{\frac{1}{x}}$

Sol^n: Here,

$\lim\limits_{x \to \infty} x^{\frac{1}{x}}$

Let, $y = x^{\frac{1}{x}}$

Taking log on both sides, we get

$\log y = \log x^{\frac{1}{x}}$

or, $\log y = \frac{1}{x}\log x$

Taking limit as $x \to \infty$ on both sides, we get

$\lim\limits_{x \to \infty} (\log y) = \lim\limits_{x \to \infty}\left[\frac{1}{x}\log x\right]$

or, $\log\left[\lim\limits_{x \to \infty} y\right] = \lim\limits_{x \to \infty}\left[\frac{\log x}{x}\right] \quad \left(\frac{\infty}{\infty} \text{ form}\right)$

$$= \lim_{x \to \infty}\left[\frac{\frac{1}{x}}{1}\right] = \lim_{x \to \infty}\left[\frac{1}{x}\right] = \frac{1}{\infty}$$

$\therefore \quad \log\left[\lim\limits_{x \to \infty} y\right] = 0$

or, $\lim\limits_{x \to \infty} y = e^0 = 1$

Hence,

$\lim\limits_{x \to \infty}\left(x^{\frac{1}{x}}\right) = 1$

vi) $\lim\limits_{x \to 0} (\cot^2 x)^{\sin x}$

Sol^n: Here,

$\lim\limits_{x \to 0} (\cot^2 x)^{\sin x}$

Let, $y = (\cot^2 x)^{\sin x}$

Taking log on both sides, we get

$\log y = \log(\cot^2 x)^{\sin x}$

or, $\log y = \sin x \log(\cot^2 x)$

Taking limit as $x \to 0$ on both sides, we get

$$\lim_{x \to 0} (\log y) = \lim_{x \to 0}[\sin x \log(\cot^2 x)]$$

$$= \lim_{x \to 0}\left[\frac{\log(\cot^2 x)}{\operatorname{cosec} x}\right] \quad \left(\frac{\infty}{\infty} \text{ form}\right)$$

$$= \lim_{x \to 0}\left[\frac{\frac{1}{\cot^2 x}\cdot 2\cot x(-\operatorname{cosec}^2 x)}{-\operatorname{cosec} x \cot x}\right]$$

$$= \lim_{x \to 0}\left[\frac{2\operatorname{cosec}^2 x}{\operatorname{cosec} x \cot^2 x}\right]$$

$$= \lim_{x \to 0}[2\operatorname{cosec} x \tan^2 x]$$

$$= \lim_{x \to 0}\left[2\frac{\sin x}{\cos^2 x}\right] = \frac{2 \times 0}{1}$$

$\therefore \quad \lim\limits_{x \to 0} (\log y) = 0$

or, $\log\left[\lim\limits_{x \to 0} y\right] = 0$

$\therefore \quad \lim\limits_{x \to 0} y = e^0 = 1$

Hence,

$\lim\limits_{x \to 0} (\cot^2 x)^{\sin x} = 1$

5. **Evaluate:**

i) $\lim\limits_{x \to 0} (\cos x)^{\frac{1}{x^2}}$

Sol^n: Here,

$\lim\limits_{x \to 0} (\cos x)^{\frac{1}{x^2}}$

Let, $y = (\cos x)^{\frac{1}{x^2}}$

Taking log in both sides, we get,

$\therefore \quad \log y = \log(\cos x)^{\frac{1}{x^2}}$

or, $\log y = \dfrac{\log(\cos x)}{x^2}$

Taking limit as $x \to 0$ on both sides, we get

$$\lim_{x \to 0}(\log y) = \lim_{x \to 0}\left[\frac{\log(\cos x)}{x^2}\right] \quad \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{x \to 0}\left[\frac{\frac{1}{\cos x}(-\sin x)}{2x}\right]$$

$$= \lim_{x \to 0}\left[\frac{-\tan x}{2x}\right] \quad \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{x \to 0}\left[\frac{-\sec^2 x}{2}\right] = \frac{-(\sec 0)^2}{2}$$

$\therefore \quad \log\left[\lim_{x \to 0} y\right] = -\frac{1}{2}$

$\therefore \quad \lim_{x \to 0} y = e^{-\frac{1}{2}}$

Hence,

$$\lim_{x \to 0}(\cos x)^{\frac{1}{x^2}} = e^{-\frac{1}{2}}$$

ii) $\lim\limits_{x \to \frac{\pi}{2}} (\sin x)^{\tan x}$

Sol^n: Here,

$\lim\limits_{x \to \frac{\pi}{2}}(\sin x)^{\tan x} \quad (1^\infty \text{ form})$

Let, $y = (\sin x)^{\tan x}$

Taking log on both sides, we get,

$\therefore \quad \log y = \log(\sin x)^{\tan x} = \tan x \log(\sin x)$

$\log y = \dfrac{\log(\sin x)}{\cot x}$

Taking limit as $x \to \frac{\pi}{2}$ on both sides, we get,

$$\therefore \quad \lim_{x \to \frac{\pi}{2}} \log y = \lim_{x \to \frac{\pi}{2}}\left[\frac{\log(\sin x)}{\cot x}\right]$$

$$= \lim_{x \to \frac{\pi}{2}}\left[\frac{\frac{1}{\sin x}(\cos x)}{-\operatorname{cosec}^2 x}\right] = \lim_{x \to \frac{\pi}{2}}\left[\frac{-\cos x}{\operatorname{cosec} x}\right]$$

$$= \lim_{x \to \frac{\pi}{2}}[-\sin x \cos x] = -\sin\frac{\pi}{2} \times \cos\frac{\pi}{2} = -1 \times 0 = 0$$

So that,

$$\log\left[\lim_{x \to \frac{\pi}{2}} y\right] = 0$$

$\therefore \quad \lim\limits_{x \to \frac{\pi}{2}} y = e^0 = 1$

Hence,

$$\lim_{x \to \frac{\pi}{2}}(\sin x)^{\tan x} = 1$$

iii) $\lim\limits_{x \to 0}(\cos x)^{\cot^2 x}$

Sol^n: Here,

$\lim\limits_{x \to 0}(\cos x)^{\cot^2 x}$

Let, $y = (\cos x)^{\cot^2 x}$

Taking log on both sides, we get,

$\log y = \log(\cos x)^{\cot^2 x}$

or, $\log y = \cot^2 x \log(\cos x)$

or, $\log y = \dfrac{\log(\cos x)}{\tan^2 x}$

Taking limit as $x \to 0$ on both sides, we get,

$$\lim_{x \to 0}(\log y) = \lim_{x \to 0}\left[\frac{\log(\cos x)}{\tan^2 x}\right] \quad \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{x \to 0}\left[\frac{\frac{1}{\cos x}(-\sin x)}{2\tan x \sec^2 x}\right] = \lim_{x \to 0}\left[\frac{-1}{2\sec^2 x}\right]$$

$$= -\frac{1}{2} \times \frac{1}{(\sec 0)^2}$$

$\therefore \quad \log\left[\lim_{x \to 0} y\right] = -\frac{1}{2}$

So that,

$$\lim_{x \to 0} y = e^{-\frac{1}{2}}$$

Hence,

$$\lim_{x \to 0}(\cos x)^{\cot^2 x} = e^{-\frac{1}{2}}$$

iv) $\lim\limits_{x \to 0} \left(\frac{\tan x}{x}\right)^{\frac{1}{x^2}}$

Sol^n: Here,

$$\lim_{x \to 0} \left(\frac{\tan x}{x}\right)^{\frac{1}{x^2}}$$

Let, $y = \left(\frac{\tan x}{x}\right)^{\frac{1}{x^2}}$

Taking log on both sides, we get,

$$\log y = \log\left(\frac{\tan x}{x}\right)^{\frac{1}{x^2}}$$

or, $\log y = \frac{1}{x^2}\log\left(\frac{\tan x}{x}\right)$

Taking limit as $x \to 0$ on both sides, we get,

$$\lim_{x \to 0}(\log y) = \lim_{x \to 0}\left[\frac{1}{x^2}\log\left(\frac{\tan x}{x}\right)\right] \quad \left(\frac{0}{0}\text{ form}\right)$$

$$= \lim_{x \to 0}\left[\frac{\frac{x}{\tan x}\left\{\frac{x\sec^2 x - \tan x}{x^2}\right\}}{2x}\right]$$

$$= \lim_{x \to 0}\left[\frac{x\sec^2 x - \tan x}{2x^2 \tan x}\right] \quad \left(\frac{0}{0}\text{ form}\right)$$

$$= \lim_{x \to 0}\left[\frac{2x\sec^2 x\tan x + \sec^2 x - \sec^2 x}{2x^2\sec^2 x + 4x\tan x}\right]$$

$$= \lim_{x \to 0}\left[\frac{2x\sec^2 x\tan x}{2x(x\sec^2 x + 2\tan x)}\right]$$

$$= \lim_{x \to 0}\left[\frac{\sec^2 x\tan x}{x\sec^2 x + 2\tan x}\right]$$

$$= \lim_{x \to 0}\left[\frac{\tan x + \tan^3 x}{x\sec^2 x + 2\tan x}\right] \quad \left(\frac{0}{0}\text{ form}\right)$$

$$= \lim_{x \to 0}\left[\frac{\sec^2 x + 3\tan^2 x\sec^2 x}{x\cdot 2\sec^2 x\tan x + 1\cdot\sec^2 x + 2\sec^2 x}\right]$$

$$= \frac{1+0}{0+1+2} = \frac{1}{3}$$

$\therefore \quad \log\left[\lim\limits_{x \to 0} y\right] = \frac{1}{3}$

So that,

$$\lim_{x \to 0} y = e^{\frac{1}{3}}$$

Hence,

$$\lim_{x \to 0}\left(\frac{\tan x}{x}\right)^{\frac{1}{x^2}} = e^{\frac{1}{3}}$$

v) $\lim\limits_{x \to 0}\left(\frac{\tan x}{x}\right)^{\frac{1}{x}}$

Sol^n: Here,

$$\lim_{x \to 0}\left(\frac{\tan x}{x}\right)^{\frac{1}{x}}$$

Let, $y = \left(\frac{\tan x}{x}\right)^{\frac{1}{x}}$

Taking log on both sides, we get,

$$\log y = \log\left(\frac{\tan x}{x}\right)^{\frac{1}{x}}$$

or, $\log y = \frac{1}{x}\log\left(\frac{\tan x}{x}\right)$

Taking limit as $x \to 0$ on both sides, $\frac{\tan x}{x} = 1$ हो ?

$$\lim_{x \to 0}(\log y) = \lim_{x \to 0}\left[\frac{\log\left(\frac{\tan x}{x}\right)}{x}\right]$$

$$= \lim_{x \to 0}\left[\frac{\frac{x}{\tan x}\left\{\frac{x\sec^2 x - \tan x}{x^2}\right\}}{1}\right]$$

$$= \lim_{x \to 0}\left[\frac{x\sec^2 x - \tan x}{x\tan x}\right] \quad \left(\frac{0}{0}\text{ form}\right)$$

$$= \lim_{x \to 0}\left[\frac{2x\sec^2 x\tan x + \sec^2 - \sec^2}{x\sec^2 x + \tan x}\right]$$

$$= \lim_{x \to 0}\left[\frac{2x(1+\tan^2 x)\cdot\tan x}{x\sec^2 x + \tan x}\right]$$

$$= \lim_{x \to 0}\left[\frac{2x\tan x + 2x\tan^3 x}{x\sec^2 x + \tan x}\right] \quad \left(\frac{0}{0}\text{ form}\right)$$

$$= \lim_{x \to 0}\left[\frac{2x\sec^2 x + 2\tan x + 2\tan^3 x + 6x\sec^2 x\tan^2 x}{2x\sec^2 x\tan x + \sec^2 x + \sec^2 x}\right]$$

$$= \frac{0+0+0+0}{0+1+1}$$

$\therefore \quad \log\left[\lim_{x \to 0} y\right] = 0$

$\therefore \quad \lim_{x \to 0} y = e^0$

Hence,

$$\lim_{x \to 0}\left(\frac{\tan x}{x}\right)^{\frac{1}{x}} = 1$$

vi) $\lim_{x \to a}\left(2 - \frac{x}{a}\right)^{\tan\frac{\pi x}{2a}}$

Sol^n: Here,

$$\lim_{x \to a}\left(2 - \frac{x}{a}\right)^{\tan\frac{\pi x}{2a}}$$

Let, $y = \left(2 - \frac{x}{a}\right)^{\tan\frac{\pi x}{2a}}$

Taking log on both sides, we get,

$$\log y = \tan\left(\frac{\pi x}{2a}\right)\log\left(2 - \frac{x}{a}\right)$$

Taking limit as $x \to a$ on both sides, we get,

$$\lim_{x \to a}\log y = \lim_{x \to a}\left[\tan\left(\frac{\pi x}{2a}\right)\log\left(2 - \frac{x}{a}\right)\right]$$

$$= \lim_{x \to a}\left[\frac{\log\left(2 - \frac{x}{a}\right)}{\cot\left(\frac{\pi x}{2a}\right)}\right] \quad \left(\frac{0}{0}\text{ form}\right)$$

$$= \lim_{x \to a}\left[\frac{\frac{1}{2 - \frac{x}{a}}\left(-\frac{1}{a}\right)}{\frac{\pi}{2a}\operatorname{cosec}^2\left(\frac{\pi x}{2a}\right)}\right]$$

$$= \lim_{x \to a}\left[\frac{\frac{2a}{\pi}\cdot\frac{1}{\frac{2a-x}{a}} \times \frac{1}{a}}{\operatorname{cosec}^2\left(\frac{\pi x}{2a}\right)}\right]$$

$$= \lim_{x \to a}\left[\frac{1}{2a-x} \times \frac{2a}{\pi} \times \sin^2\left(\frac{\pi x}{2a}\right)\right]$$

$$= \frac{2a}{\pi} \times \frac{1}{2a-a}\cdot\sin^2\left(\frac{\pi a}{2a}\right) = \frac{2a}{\pi} \times \frac{1}{a} \times 1$$

$\therefore \quad \log\left[\lim_{x \to a} y\right] = \frac{2}{\pi}$

Thus,

$$\lim_{x \to a}\left(2 - \frac{x}{a}\right)^{\tan\frac{\pi x}{2a}} = e^{\frac{2}{\pi}}$$

vii) $\lim_{x \to 0}\left(\frac{\sin x}{x}\right)^{\frac{1}{x}}$

Sol^n: Here,

$$\lim_{x \to 0}\left(\frac{\sin x}{x}\right)^{\frac{1}{x}}$$

Let,

$$y = \left(\frac{\sin x}{x}\right)^{\frac{1}{x}}$$

Taking log on both sides, we get,

$\therefore \quad \log y = \frac{1}{x}\log\left(\frac{\sin x}{x}\right)$ $\quad \frac{\sin x}{x} = 1$

Taking limit as $x \to 0$ on both sides, we get,

$$\lim_{x \to 0}(\log y) = \lim_{x \to 0}\left[\frac{\log\left(\frac{\sin x}{x}\right)}{x}\right] \quad \left(\frac{0}{0}\text{ form}\right)$$

or, $\log\left[\lim_{x \to 0}(y)\right] = \lim_{x \to 0}\left[\frac{\frac{x}{\sin x}\left\{\frac{x\cos x - \sin x}{x^2}\right\}}{1}\right]$

$$= \lim_{x \to 0}\left[\frac{x\cos x - \sin x}{x\sin x}\right] \quad \left(\frac{0}{0}\text{ form}\right)$$

$$= \lim_{x \to 0}\left[\frac{x(-\sin x) + 1\cdot\cos x - \cos x}{x\cos x + 1\cdot\sin x}\right]$$

$$= \lim_{x \to 0}\left[\frac{-x\sin x}{x\cos x + \sin x}\right] \quad \left(\frac{0}{0}\text{ form}\right)$$

$$= \lim_{x \to 0}\left[\frac{-x\cos x - 1\cdot\sin x}{x(-\sin x) + 1\cdot\cos x + \cos x}\right]$$

$$= \frac{-0-0}{0+1+1}$$

$\therefore \quad \log\left[\lim_{x \to 0} y\right] = 0$

or, $\lim_{x \to 0} y = e^0 = 1$

Hence,

$$\lim_{x \to 0}\left(\frac{\sin x}{x}\right)^{\frac{1}{x}} = 1$$

6. i) If $\lim\limits_{x \to 0}\left[\dfrac{\sin 2x + a\sin x}{x^3}\right]$ be finite, then find the value of 'a' and the limit.

Solⁿ: Here,

$$\lim_{x \to 0}\left[\frac{\sin 2x+a\sin x}{x^3}\right] \quad \left(\frac{0}{0}\text{ form}\right)$$

$$= \lim_{x \to 0}\left[\frac{2\cos 2x+a\cos x}{3x^2}\right]$$

When $x = 0$, then it takes $\frac{2+a}{0}$ form. Since the limit is finite, we must have $\frac{2+a}{0} = \frac{0}{0}$ form.

$\therefore \quad 2 + a = 0$

$\quad\;\; a = -2$

Then the limit is

$$\lim_{x \to 0}\left[\frac{2\cos 2x+a\cos x}{3x^2}\right]$$

$$= \lim_{x \to 0}\left[\frac{2\cos 2x \mp 2\cos x}{3x^2}\right] \quad \left(\frac{0}{0}\text{ form}\right)$$

$$= \lim_{x \to 0}\left[\frac{-4\sin 2x+2\sin x}{6x}\right] \quad \left(\frac{0}{0}\text{ form}\right)$$

$$= \lim_{x \to 0}\left[\frac{-8\cos 2x+2\cos x}{6}\right]$$

$$= \frac{-8\times 1+2\times 1}{6} = \frac{-8+2}{6} = \frac{-6}{6} = -1$$

ii) If $\lim\limits_{x \to 0}\left[\dfrac{a\sin x - \sin 2x}{\tan^3 x}\right]$ be finite, then find the value of 'a' and the limit.

Solⁿ: Here,

$$\lim_{x \to 0}\left[\frac{a\sin x-\sin 2x}{\tan^3 x}\right] \quad \left(\frac{0}{0}\text{ form}\right)$$

$$= \lim_{x \to 0}\left[\frac{a\cos x-2\cos 2x}{3\tan^2 x\sec^2 x}\right]$$

$$= \lim_{x \to 0}\left[\frac{(a\cos x-2\cos 2x)}{3\sin^2 x}\cos^4 x\right]$$

When $x \to 0$, then $3\sin^2 x = 0$ but, $(a\cos x - 2\cos 2x)\cos^4 x$ is not zero at that case. Since the limit is finite. So we must have

$$\lim_{x \to 0}[(a\cos x - 2\cos 2x)\cos^4 x] = 0$$

i.e. $(a \times 1 - 2 \times 1)\cdot 1 = 0$

$\therefore \quad a = 2$

Then the limit is;

$$\lim_{x \to 0}\left[\frac{(2\cos x-2\cos 2x)\cos^4 x}{3\sin^2 x}\right]$$

$$= \lim_{x \to 0}\left[\frac{2\cos^5 x-2\cos 2x\cos^4 x}{3\sin^2 x}\right] \quad \left(\frac{0}{0}\text{ form}\right)$$

$$= \lim_{x \to 0}\left[\frac{-10\cos^4 x\sin x+4\sin 2x\cos^4 x+8\cos 2x\cos^3 x\sin x}{6\sin x\cos x}\right]$$

$$= \lim_{x \to 0}\left[\frac{\sin x\cos x\{-10\cos^3 x+8\cos^4 x+8\cos 2x\cos^2 x\}}{6\sin x\cos x}\right]$$

$$= \frac{-10+8+8}{6} = \frac{6}{6} = 1$$

iii) If $\lim\limits_{x \to 0}\left[\dfrac{(a+b\cos x)x - c\sin x}{x^5}\right] = 1$ be finite, then find the value of a, b and c.

Solⁿ: Here,

$$\lim_{x \to 0}\left[\frac{(a+b\cos x)x-c\sin x}{x^5}\right] = 1 \ldots \text{(i)}$$

WE know that

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} \cdots\cdots\cdots$$

and $\sin x = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} \cdots\cdots\cdots$

$$\therefore \quad \lim_{x \to 0}\left[\frac{(a+b\cos x)x-c\sin x}{x^5}\right] = 1$$

or, $$\lim_{x \to 0}\left[\frac{ax+bx\cos x-c\sin x}{x^5}\right] = 1$$

or, $$\lim_{x \to 0}\left[\frac{ax+bx\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}\cdots\cdots\cdots\right)-c\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}\cdots\cdots\cdots\right)}{x^5}\right] = 1$$

or, $$\lim_{x \to 0}\left[\frac{ax+b\left(x-\frac{x^3}{2!}+\frac{x^5}{4!}\cdots\cdots\cdots\right)-c\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}\cdots\cdots\cdots\right)}{x^5}\right] = 1$$

or, $$\lim_{x \to 0}\left[\frac{(a+b-c)x+\left(\frac{c}{3!}-\frac{b}{2!}\right)x^3+\left(\frac{b}{4!}-\frac{c}{5!}\right)x^5+\cdots\cdots\cdots}{x^5}\right] = 1$$

or, $\lim_{x \to 0} \left[\frac{(a+b-c)x + \left(\frac{c}{6}-\frac{b}{2}\right)x^3 + \left(\frac{b}{24}-\frac{c}{120}\right)x^5 + \cdots\cdots}{x^5} \right] = 1$

or, $\lim_{x \to 0} \left[(a+b-c)x^{-4} + \left(\frac{c}{6}-\frac{b}{2}\right)x^{-2} + \left(\frac{b}{24}-\frac{c}{120}\right) + \cdots \right] = 1$

Equating the coefficients of live power's of x we get,

$$a+b-c = 0, \frac{c}{6}-\frac{b}{2} = 0, \frac{b}{24}-\frac{c}{120} = 1$$

Solving these, we get,

$$a = 120, b = 60 \text{ and } c = 180$$

Exercise 5

1. Find the asymptotes of the following curve as:

i) $y = \frac{3x}{x-2}$

Sol^n: Given that,

$y = \frac{3x}{x-2}$

or, $xy - 2y = 3x$

or, $xy - 3x - 2y = 0$ … (i)

This equation is of second degree and the terms involving x^2 and y^2 are both absent. Therefore, there are asymptotes parallel to x and y axes.

To find the asymptote parallel to x–axis, we equate the coefficient of highest degree term of x to zero in (i)

$\therefore$ $y - 3 = 0$

i.e. $y = 3$ … (ii)

Again, equating the coefficient of highest degree term of y to zero, we get,

$x - 2 = 0$

i.e. $x = 2$ … (iii)

Hence, $x = 2$ and $y = 3$ are the required asymptotes.

ii) $y = \frac{4x^2+4x-3}{x^2-4x+3}$

Sol^n: Here,

$y = \frac{4x^2+4x-3}{x^2-4x+3}$

or, $x^2y - 4xy + 3y = 4x^2 + 4x - 3$

or, $x^2y - 4xy - 4x^2 - 4x + 3y + 3 = 0$ … (i)

This equation is of third degree. Also, x^3 and y^3 are both absent, therefore there are the asymptotes parallel to both axes.

To find the asymptotes, equation the coefficients of highest degree terms of x and y to zero we get

$y - 4 = 0$

i.e. $y = 4$

and $x^2 - 4x + 3 = 0$

or, $x^2 - 3x - x + 3 = 0$

or, $x(x-3) - 1(x-3) = 0$

or, $(x-3)(x-1) = 0$

$\therefore$ $x = 1, x = 3$

Thus, the required asymptotes are $x = 1, x = 3$ and $y = 4$

iii) $y = \frac{x^2+2x-1}{x}$

Sol^n: Here,

$$y = \frac{x^2+2x-1}{x}$$

or, $xy = x^2 + 2x - 1$

or, $x^2 - xy + 2x - 1 = 0 \ldots$ (i)

This equation is of second degree. Since the term x^2 is present a y^2 is absent. Hence the equation if asymptotes parallel to y–axis obtained by equating the coefficient of highest degree term of y zero.

i.e. coefficient of $y = 0$

i.e. $-x = 0$

$\therefore$ $x = 0$

But, since x^2 is present, so there is no asymptotes parallel to axis. To find the oblique asymptotes of the form $y = mx + c$ p $x = 1, y = m$ in second degree and first degree term, we get,

$\phi_2(m) = 1^2 - 1 \cdot m$

$\therefore$ $\phi_2(m) = 1 - m$

Now,

$\phi_2(m) = 0$ gives $m = 1$

& $\phi_1(m) = 2 \times 1 = 2$

Now, $[\because \phi_n(m) =$

$$c = \frac{-\phi_1(m)}{\phi_2(m)} \qquad \left(\because c = \frac{-\phi_{n-1}(m)}{\phi_n(m)}\right.$$

$$= \frac{-2}{-1} = 2$$

$\therefore$ The oblique asymptote is $y = mx + c$

i.e. $y = 1 \cdot x + 2$

$\therefore$ $y = x + 2$

Thus the required asymptotes are $x = 0$ and $y = x + 2$

iv) $y = \frac{x^2}{x^2+1}$

Sol^n: Here,

$$y = \frac{x^2}{x^2+1}$$

or, $x^2y + y = x^2$

or, $x^2y - x^2 + y = 0 \ldots$ (i)

This equation is of third degree. So, it has 3 asymptotes real or imaginary. In this equation x^3 and y^3 are both absent, therefore the asymptotes parallel to both axes.

The asymptotes parallel to x–axis are obtained by equating the coefficient of highest degree term of x to zero.

i.e. coefficient of $x^2 = 0$

i.e. $y - 1 = 0$

$\therefore$ $y = 1 \ldots$ (ii)

Again, equating the coefficient of highest degree term of y to zero, to get the asymptotes parallel to y–axis.

i.e. coefficient of $y = 0$

i.e. $x^2 + 1 = 0$

$\therefore$ $x = 0 \pm i$

which does not given the real asymptotes.

$\therefore$ The equation of asymptote is $y = 1$

v) $y = \frac{(x+2)^2(x-3)}{x-1}$

Sol^n: Here,

$$y = \frac{(x+2)^2(x-3)}{x-1}$$

or, $(x+2)^2(x-3) = y(x-1)$

or, $x^3 + x^2 - 8x - 12 - xy + y = 0$

or, $x^3 + x^2 - xy - 8x + y - 12 = 0 \ldots$ (i)

This equation is of third degree, so that there are three asymptotes real or imaginary. But the term x^3 is presents, and y^3 is absent. Hence there is no asymptote parallel to x–axis and there is asymptote parallel to y–axis.

Equation of asymptote parallel to y–axis is obtained by equating the coefficient of highest degree term of y to zero.

i.e. the coefficient of $y = 0$

i.e. $-x = 1$

$\therefore$ $x = 1$

Again, due to the present of x^3 in (i), there is asymptote in form $y = mx + c$. To find such we put $x = 1$ and $y = m$ in th and second degree terms as;

$\phi_3(m) = 1$ and equate $\phi_3(m) = 0$ gives $1 = 0$ which is imposs

$\therefore$ only the asymptote is $x = 1$

vi) $\mathbf{x^3 - y^3 = 3ax^2}$

Sol^n: Here,

$$x^3 - y^3 = 3ax^2$$

or, $x^3 - y^3 - 3ax^2 = 0 \ldots$ (i)

This equation is of third degree and x^3 and y^3 are both present. there are no asymptotes parallel to x and y axes. To find oblique asymptote of the form $y = mx + c$, put $x = 1$ and $y = m$ in third and second degree term, we get,

$$\phi_3(m) = 1 - m^3$$

and $\phi_2(m) = -3a$

Now,

$$\phi_3(m) = 0 \text{ gives}$$

$$1 - m^3 = 0$$

or, $(m - 1)(m^2 + m + 1) = 0$

$\therefore$ only real value is $m = 1$

Also,

$$\phi_3'(m) = -3m^2$$

$\therefore$ $c = \dfrac{-\phi_2(m)}{\phi_3(m)} = \dfrac{-(-3a)}{-3m^2} = \dfrac{-a}{m^2}$

or, $c = \dfrac{-a}{1^2} = -a$

$\therefore$ The asymptote is $y = mx + c$

i.e. $y = x - a$

Hence, $y = x - a$ is only an oblique asymptote

vii) $\mathbf{y(y - 1)^2 - x^2y = 0}$

Sol^n: Here, given equation is;

$$y(y - 1)^2 - x^2y = 0$$

or, $y\{y^2 - 2y + 1\} - x^2y = 0$

or, $y^3 - 2y^2 + y - x^2y = 0$

or, $y^3 - x^2y - 2y^2 + y = 0$

This equation is of third degree. So it has at must three asymptotes real or imaginary. Here, y^3 is present, so that the no asymptote parallel to y–axis. But x^3 is absent, therefore to find the asymptote parallel to x–axis we equate the coefficient of highest degree term of x to zero.

i.e. coefficient of $x^2 = 0$

i.e. $-y = 0$

$\therefore$ $y = 0 \ldots$ (i) is an asymptote.

Again, since y^3 is present, so let $y = mx + c \ldots$ (ii) is an oblique asymptote.

To find the values of m and c, put $x = 1$ and $y = m$ in third degree and second degree terms.

$\therefore$ $\phi_3(m) = m^3 - 1^2 \cdot m \quad \Rightarrow \phi_3'(m) = 3m^2 - 1$

$$= m^3 - m$$

$\therefore$ $\phi_3(m)' = 0$ gives;

$$m^3 - m = 0$$

or, $m(m^2 - 1) = 0$

$\therefore$ $m = 0, 1, -1$

Again,

$$\phi_2(m) = -2m^2$$

Now,

$$c = \frac{-\phi_2(m)}{\phi_3(m)} = \frac{-(-2m^2)}{3m^2-1} = \frac{2m^2}{3m^2-1}$$

When $m = 0$, then $c = \dfrac{2\times0}{3\times0-1} = 0$

When $m = 1$, then $c = \dfrac{2\times1^2}{3\times1^2-1} = \dfrac{2}{2} = 1$

When $m = -1$, then $c = \dfrac{2\times(-1)^2}{3\times(-1)^2-1} = \dfrac{2}{2} = 1$

$\therefore$ $y = mx + c$ gives;

$$y = 0 \cdot x + 0$$

i.e. $y = 0$

$$y = 1 \cdot x + 1$$

i.e. $y - x - 1 = 0$

$$y = -1 \cdot x + 1$$

i.e. $y + x - 1 = 0$

Thus, the required asymptotes are;

$$y = 0, y - x - 1 = 0, y + x - 1 = 0$$

viii) $xy^2 - a^2(x - a) = 0$

Solⁿ: Here,

$$xy^2 - a^2(x - a) = 0$$

or, $xy^2 - xa^2 + a^3 = 0 \ldots$ (i)

This equation is of third degree, so it has at most three asymptotes real or imaginary. Tin this equation, both x^3 and y^3 are absent so that the asymptotes parallel to both axes.

To find the asymptote parallel to x–axis, we equate the coefficient of highest degree term of x to zero

i.e. coefficient of $x = 0$

i.e. $y^2 - a^2 = 0$

$\therefore$ $y = \pm a$

Also, by equating the coefficient of highest degree term of y to zero, we get the asymptote parallel to y–axis

i.e. coefficient of $y^2 = 0$

i.e. $x = 0$

Thus, the required asymptotes are; $x = 0, y = \pm a$

ix) $\dfrac{a^2}{x^2} + \dfrac{b^2}{y^2} = 1$

Solⁿ: Here,

$$\frac{a^2}{x^2} + \frac{b^2}{y^2} = 1$$

or, $a^2y^2 + b^2x^2 = x^2y^2$

or, $x^2y^2 - b^2x^2 - a^2y^2 = 0 \ldots$ (i)

This equation is of fourth degree and the terms x^4 and y^4 are both absent. Hence there are only parallel asymptotes with coordinate axes.

Now, the equations of asymptote parallel to y–axis are obtained by equating the coefficient or highest degree term of y to zero.

i.e. coefficient of $y^2 = 0$

i.e. $x^2 - a^2 = 0$

$\therefore$ $x = \pm a$

Similarly, the equations of asymptote parallel to x–axis is obtained by equating the coefficient of highest degree term of x to zero

i.e. coefficient of $x^2 = 0$

i.e. $y^2 - b^2 = 0$

$\therefore$ $y = \pm b$

Thus the required asymptotes are; $x = \pm a$ and $y = \pm b$

x) $(y - a)^2(x^2 - a^2) = x^4 + a^4$

Solⁿ: Given equation is;

$$(y - a)^2(x^2 - a^2) = x^4 + a^4$$

or, $(y^2 - 2ay + a^2)(x^2 - a^2) = x^4 + a^4$

or, $x^2y^2 - a^2y^2 - 2ax^2y + 2a^3y + a^2x^2 - a^4 = x^4 + a^4$

or, $x^2y^2 - x^4 - 2ax^2y + a^2x^2 - a^2y^2 + 2a^3y - 2a^4 = 0$

or, $x^4 - x^2y^2 + 2ax^2y - a^2x^2 + a^2y^2 - 2a^3y + 2a^4 = 0$

This equation is of fourth degree. So there are at most four asymptotes. Also, x^4 is present and y^4 is absent. Hence, there is no asymptotes parallel to x–axis and has asymptotes parallel to y–axis. The equation of asymptotes parallel to y–axis are obtained by equating the coefficient of highest power of y to zero.

i.e. Coefficient of $y^2 = 0$

i.e. $-x^2 + a^2 = 0$

$\therefore$ $x = \pm a$

Again, to find oblique asymptotes of the form $y = mx + c$, put $x = 1$ and $y = m$ in fourth and third degree terms. We get,

$$\phi_4(m) = 1 - m^2;\ \phi_3(m) = 2am$$

Now,

$$\phi'_4(m) = -2m$$

and $\phi_4(m) = 0$ gives

$$1 - m^2 = 0$$

or, $m = \pm 1$

Now,

$$c = \frac{-\phi_3(m)}{\phi'_4(m)} = \frac{-2am}{-2m} = a$$

$\therefore$ Equation of asymptotes are;

$$y = \pm 1x + a$$

or, $y \pm x = a$

Thus, the equation of asymptotes are; $x = \pm a, y \pm x = a$

xi) $x^3 + xy^2 - ay^2 = 0$

Solⁿ: Given equation is;

$$x^3 + xy^2 - ay^2 = 0$$

This equation is third degree. Here x^3 is present and y^3 is absent. So there is no asymptote parallel to x–axis, but the asymptotes parallel to y–axis. To find the asymptote parallel to y–axis, we equate the coefficient of highest degree term of y to zero.

i.e. Coefficient of $y^2 = 0$

i.e. $x - a = 0$

$\therefore \quad x = a$

Again, to find oblique asymptote in the form $y = mx + c_1$ we put $x = 1$ and $y = m$ in third and second degree term, we get,

$\phi_3(m) = 1^3 + 1.m^2 = m^2 + 1$

$\phi_2(m) = -a.m^2$

Now,

$\phi_3(m) = 0$ gives;

$m^2 + 1 = 0$

From which it is not possible to find the real value of x and hence we can get only the imaginary values and corresponding imaginary asymptotes. Thus, the real asymptotes is $x = a$.

xii) $\mathbf{x^2y^2 = a^2(x^2 + y^2)}$

Sol^n: Given equation is;

$x^2y^2 = a^2(x^2 + y^2)$

or, $x^2y^2 = a^2x^2 + a^2y^2$

or, $x^2y^2 - a^2x^2 - a^2y^2 = 0$

This equation is of degree four. Here both x^4 and y^4 are absent, so there are asymptotes parallel to both axes.

To find the asymptotes parallel to x–axis, equating the coefficient of highest degree term of x to zero.

i.e. Coefficient of $x^2 = 0$

i.e. $y^2 - a^2 = 0$

$\therefore \quad y = \pm a$

Again to find the asymptotes parallel to y–axis, equating the coefficient of highest degree term of y to zero.

i.e. Coefficient of $y^2 = 0$

i.e. $x^2 - a^2 = 0$

$\therefore \quad x = \pm a$

Thus, the required asymptotes are; $x = \pm a, y = \pm a$

2. **Find the asymptotes of the following curves.**

i) $\mathbf{y^3 - x^2y + 2y^2 + 4y + x = 0}$

Sol^n: Given equation is;

$y^3 - x^2y + 2y^2 + 4y + x = 0$

This equation is of third degree. Here y^3 is present but x^3 is absent. So, that here is no asymptotes parallel to y–axis, but the asymptotes parallel to x–axis.

To find the asymptotes parallel to x–axis, equating the coefficient of highest degree term of x to zero.

i.e. Coefficient of $x^2 = 0$

i.e. $-y = 0$

$\therefore \quad y = 0$

Again, since y^3 is present so there is no asymptotes parallel to y–axis. To find oblique asymptote $y = mx + c$, put $x = 1$ and $y = m$ in third and second degree terms, we get,

$\phi_3(m) = m^3 - m$

and $\phi_2(m) = 2m^2$

$\therefore \quad \phi'_3(m) = 3m^2 - 1$

Now,

$\phi_3(m) = 0$ gives

$m^3 - m = 0$

or, $m(m^2 - 1) = 0$

or, $m = 0, m = 1, m = -1$

We have,

$$c = \frac{-\phi_2(m)}{\phi'_3(m)} = \frac{-2m^2}{3m^2-1}$$

When, $m = 0$

Then, $c = 0$

When, $m = 1$

Then, $c = \frac{-2}{3-1} = -1$

When, $m = -1$

Then, $c = \frac{-2}{3-1} = -1$

$\therefore$ The asymptotes be $y = -x - 1$

i.e. $x + y + 1 = 0$

Thus, the required asymptotes be; $y = 0, x + y + 1 = 0$

ii) $\mathbf{y^3 + x^2y + 2xy^2 - y + 1 = 0}$

Sol^n: Given equation is;

$y^3 + x^2y + 2xy^2 - y + 1 = 0$

This equation is third degree. So, there at most 3 asymptotes real or imaginary. Here, y^3 is present, but x^3 is absent. So that the asymptote parallel to x–axis.

To find the asymptotes parallel to x–axis, equation the coefficient of highest degree term of x to zero.

i.e. Coefficient of $x^2 = 0$

$$\frac{c^2}{2!}\phi''_4(m) + c.\phi'_3(m) + \phi_2(m) = 0$$

or, $\frac{c^2}{2!}2 + +c \times 0 - a^2 - a^2m^2 = 0$

or, $c^2 - a^2 - a^2.1^2 = 0$

or, $c^2 - 2a^2 = 0$

$\therefore$ $c = \pm\sqrt{2}a$

$\therefore$ $c = \pm a\sqrt{2}$

$\therefore$ Oblique asymptotes are;

$y = x \pm a\sqrt{2}$

i.e. $x - y = \pm a\sqrt{2}$

Thus, the required asymptotes be; $x = \pm a; x - y = \pm a\sqrt{2}$

v) $\mathbf{x^3 + y^3 - xy^2 - x^2y + x^2 - y^2 = 1}$

Sol^n: Given equation is;

$x^3 + y^3 - xy^2 - x^2y + x^2 - y^2 = 1$

or, $x^3 + y^3 - xy^2 - x^2y + x^2 - y^2 - 1 = 0$ ... (i)

This equation is of third degree. So there are at most 3 asymptotes real or imaginary. Here the terms x^3 and y^3 are both present. Hence, there is no asymptote parallel to x and y axes.

To find oblique asymptote of the form $y = mx + c$ put $x = 1$ and $y = m$ in third and second degree terms,

We get,

$\phi_3(m) = 1 + m^3 - m^2 - m$

$\phi_2(m) = 1 - m^2$

Now,

$\phi_3(m) = 0$ gives

$m^3 - m^2 - m + 1 = 0$

or, $m^2(m-1) - 1(m-1) = 0$

or, $(m-1)(m^2-1) = 0$

$\therefore$ $m = 1, 1, -1$

Also,

$\phi'_3(m) = 3m^2 - 2m - 1$

For $m = -1$,

$$c = \frac{-\phi_2(m)}{\phi'_3(m)} = \frac{-(1-m^2)}{3m^2-2m-1} = \frac{-(1-1)}{3+2-1} = \frac{-0}{4} = 0$$

$\therefore$ $y = -x + 0$

But,

$m = 1$ is repeated root. So to find the value of c, we use,

$$\frac{c^2}{2!}\phi''_3(m) + c.\phi'_2(m) + \phi_1(m) = 0 \quad \text{... (ii)}$$

But,

$\phi''_3(m) = 6m - 2$

$\phi'_2(m) = -2m$

$\phi_1(m) = 0$

$\therefore$ From (ii),

$\frac{c^2}{2!}(6m-2) + c.(-2m) + 0 = 0$

or, $\frac{c^2}{2!}(6m-2) - 2cm = 0$

or, $c^2(6-2) - 2c = 0$

or, $4c^2 - 2c = 0$

or, $c^2 - c = 0$

or, $c(c-1) = 0$

$\therefore$ $c = 0, 1$

$\therefore$ $y = 1.x + 0$

i.e. $y = x$

and $y = 1.x + 1$

i.e. $y = x + 1$

Thus, the required asymptotes are;

$y = -x, y = x, y = x + 1$

i.e. $x + y = 0; x - y = 0; x - y + 1 = 0$

vi) $\mathbf{y^3 - 6xy^2 + 11x^2y - 6x^3 + x + y = 0}$

Sol^n: Given equation is;

$y^3 - 6xy^2 + 11x^2y - 6x^3 + x + y = 0$

This equation is of third degree. Here, the terms x^3 and y^3 are both present. So there are no asymptotes parallel to the axes. To find oblique asymptote in the form $y = mx + c$, put $x = 1$ and $y = m$ in third and second degree terms, we get,

$\phi_3(m) = m^3 - 6m^2 + 11m - 6$

$\phi_2(m) = 0$

Now,

$\phi_3(m) = 0$ gives

$m^3 - 6m^2 + 11m - 6 = 0$

or, $m^3 - m^2 - 5m^2 + 5m + 6m - 6 = 0$

or, $m^2(m-1) - 5m(m-1) + 6(m-1) = 0$

or, $(m-1)(m^2-5m+6)=0$
or, $(m-1)(m^2-3m-2m+6)=0$
or, $(m-1)\{m(m-3)-2(m-3)\}=0$
or, $(m-1)(m-3)(m-2)=0$
$\therefore$ $m=1,2,3$

Also,

$$\phi'_3(m)=3m^2-12m+11$$

Now,

$$c=\frac{-\phi_2(m)}{\phi'_3(m)}=\frac{-0}{3m^2-12m+11}$$

which shows that $c=0$ for $m=1,2,3$

$\therefore$ $y=1.x; y=2x; y=3x$

i.e. $x-y=0; 2x-y=0; 3x-y=0$ are the required asymptotes.

vii) $\mathbf{x^3-2x^2y+xy^2+x^2-xy+2=0}$

Sol^n: Given equation is;

$$x^3-2x^2y+xy^2+x^2-xy+2=0$$

This equation is of third degree. Here, y^3 is absent and x^3 is present, so that there are asymptotes parallel to y–axis.

To find the asymptote parallel to y–axis, equating the coefficient of highest degree term of y to zero.

i.e. Coefficient of $y^2=0$

$\therefore$ $x=0$

Again, to find oblique asymptotes in the form $y=mx+c$, put $x=1$ and $y=m$ in third and second degree terms, we get,

$$\phi_3(m)=1-2m+m^2$$
$$\phi_2(m)=1-m$$

Now,

$\phi_3(m)=0$ gives

$m^2-2m+1=0$

or, $(m-1)^2=0$

$\therefore$ $m=1,1$

Here, the roots of m be repeated, so to find c, we use,

$$\frac{c^2}{2!}\phi''_3(m)+c.\phi'_2(m)+\phi_1(m)=0 \ldots(i)$$

But

$\phi'_3(m)=-2+2m$

$\therefore$ $\phi''_3(m)=2$

and $\phi'_2(m)=-1$

$\phi_1(m)=0$

From (i);

$$\frac{c^2}{2!}2+c.(-1)+0=0$$

or, $c^2-c=0$

or, $c(c-1)=0$

$\therefore$ $c=0,1$

Then,

$y=mx+c$ gives;

$y=x+0$

i.e. $y=x$

and $y=x+1$

i.e. $x-y+1=0$

Thus, the required asymptotes be; $x=0; x-y=0; x-y+1=0$

viii) $\mathbf{x^3-4y^3+3x^2y+y-x+3=0}$

Sol^n: Given equation is;

$$x^3-4y^3+3x^2y+y-x+3=0$$

This equation is of third degree. Here x^3 and y^3 are both present. Hence there are no asymptotes parallel to y and x–axis.

To find oblique asymptote in the form $y=mx+c$, put $x=1$ and $y=m$ in third and second degree terms, we get,

$$\phi_3(m)=1+3m-4m^3$$
$$\phi_2(m)=0$$

Now,

$\phi_3(m)=0$ gives

$-4m^3+3m+1=0$

or, $4m^3-3m-1=0$

or, $4m^3-4m^2+4m^2-4m+m-1=0$

or, $4m^2(m-1)+4m(m-1)+1(m-1)=0$

or, $(m-1)(4m^2+4m+1)=0$

or, $(m-1)(2m+1)^2=0$

or, $m=1; m=-\frac{1}{2},-\frac{1}{2}$

When,

$m=1,$

Then,

$$c=\frac{-\phi_2(m)}{\phi'_3(m)}=\frac{-0}{\phi'_3(m)}=0$$

and the asymptote is;

$y=1x+0$

i.e. $x - y = 0$

Again,

$m = -\frac{1}{2}$ is repeated root, so that, to find c, we use,

$$\frac{c^2}{2!}\phi''_3(m) + c.\phi'_2(m) + \phi_1(m) = 0 \quad \dots \text{(i)}$$

But,

$$\phi_2(m) = 0$$
$$\phi_1(m) = -1 + m$$

and $\phi''_3(m) = -24m$

Then, from (i);

$$\frac{c^2}{2!}(-24m) + c.0 + (m - 1) = 0$$

or, $-12\,mc^2 + m - 1 = 0$

When,

$$m = -\frac{1}{2},$$

$\therefore$ $-12\left(-\frac{1}{2}\right)c^2 - \frac{1}{2} - 1 = 0$

or, $6c^2 - \frac{3}{2} = 0$

or, $c^2 = \frac{3}{12}$

or, $c = \pm\frac{1}{2}$

$\therefore$ The asymptotes are;

$$y = -\frac{1}{2}x \pm \frac{1}{2}$$

or, $x + 2y = \pm 1$

Thus, the required asymptotes are; $y = x; x + 2y = \pm 1$

ix) $\mathbf{(x^2 - y^2)(x + 2y + 1) + x - y + 1 = 0}$

Sol^n: Given equation is;

$$(x^2 - y^2)(x + 2y + 1) + x - y + 1 = 0$$

or, $x^3 + 2x^2y + x^2 - xy^2 - 2y^3 - y^2 + x + y + 1 = 0$

or, $x^3 - 2y^3 + 2x^2y - xy^2 + x^2 - y^2 + x + y + 1 = 0$

This equation is of third degree and term x^3 and y^3 are both present so that there are no asymptotes parallel to the co-ordinate axes. To find oblique asymptotes in the term $y = mx + c$, put $x = 1$ and $y = m$ in third and second degree terms, we get,

$$\phi_3(m) = 1 + 2m - m^2 - 2m^3$$
$$\phi_2(m) = 1 - m^2$$

Now,

$\phi_3(m) = 0$ gives

$$1 + 2m - m^2 - 2m^3 = 0$$

or, $2m^3 + m^2 - 2m - 1 = 0$

or, $m^2(2m + 1) - 1(2m + 1) = 0$

or, $(m^2 - 1)(2m + 1) = 0$

or, $(m - 1)(m + 1)(2m + 1) = 0$

$\therefore$ $m = 1, -1, -\frac{1}{2}$

Also,

$$c = \frac{-\phi_2(m)}{\phi'_3(m)} = \frac{m^2 - 1}{2 - 2m - 6m^2} = 0$$

When, $m = 1$

Then, $c = 0$

When, $m = -\frac{1}{2}$

Then, $c = \dfrac{\frac{1}{4} - 1}{2 + 1 - \frac{6}{4}} = -\frac{1}{2}$

$\therefore$ The required asymptotes are;

$$y = 1.x + 0; y = -1.x + 0; y = \frac{1}{2}x - \frac{1}{2}$$

i.e. $x - y = 0; x + y = 0; x + 2y + 1 = 0$

x) $\mathbf{y^3 + 2xy^2 + x^2y - y + 1 = 0}$

Sol^n: Given equation is;

$$y^3 + 2xy^2 + x^2y - y + 1 = 0$$

This equation is of third degree. Here, y^3 is present and x^3 is absent, so there are no asymptotes parallel to y–axis, but the asymptotes parallel to x–axis.

To find the asymptotes parallel to x–axis equating the coefficient of highest degree term of x to zero.

i.e. Coefficient of $x^2 = 0$

$\therefore$ $y = 0$

Again, to find oblique asymptotes in the form $y = mx + c$, put $x = 1$ and $y = m$ in third and second degree terms, we get,

$$\phi_3(m) = m^3 + 2m^2 + m$$
$$\phi_2(m) = 0$$

Now,

$\phi_3(m) = 0$ gives

$m^3 + 2m^2 + m = 0$

i.e. $m(m^2 + 2m + 1) = 0$

or, $m(m+1)^2 = 0$

or, $m = 0, -1, -1$

When,

$m = 0$

$c = \dfrac{-\phi_2(m)}{\phi_3'(m)} = \dfrac{-0}{\phi_3'(m)} = 0$

$\therefore$ $y = 0$ is the asymptote.

Again,

$m = -1$ is repeated root. To find the value of c, we use,

$\dfrac{c^2}{2!}\phi''_3(m) + c.\phi'_2(m) + \phi_1(m) = 0 \ldots$ (i)

But

$\phi'_3(m) = 3m^2 + 4m + 1$

$\therefore$ $\phi''_3(m) = 6m + 4$

and $\phi'_2(m) = 0$

$\phi_1(m) = -m$

Then, from (i);

$\dfrac{c^2}{2!}(6m + 4) + c.0 + (-m) = 0$

or, $\{3(-1) + 2\}c^2 - (-1) = 0$

or, $-c^2 + 1 = 0$

or, $c^2 = 1$

$\therefore$ $c = \pm 1$

Then,

$y = -1.x \pm 1$ be the asymptotes.

i.e. $x + y = \pm 1$

Hence, the required asymptotes be; $y = 0; x + y = \pm 1$

3. **Find the asymptotes of the following curves.**

i) $\mathbf{r\cos\theta = a\sin\theta}$

Sol^n: Given curve is;

$r\cos\theta = a\sin\theta$

or, $\dfrac{1}{r} = \dfrac{1}{a}\dfrac{\cos\theta}{\sin\theta}$

or, $\dfrac{1}{r} = \dfrac{1}{a}\cot\theta$

or, $\dfrac{1}{r} = f(\theta)$ say … (i)

Where,

$f(\theta) = \frac{1}{a}\cot\theta$

Now,

$f(\theta) = 0$ implies

$\dfrac{1}{a}\cot\theta = 0$

or, $\cot\theta = 0$

or, $\theta = \dfrac{\pi}{2} = \alpha$ (say)

Also,

$f'(\theta) = \dfrac{-\text{cosec}^2\theta}{a}$

$\therefore$ $f'(\alpha) = \dfrac{-\text{cosec}^2\alpha}{a} = \dfrac{-\text{cosec}^2\frac{\pi}{2}}{a} = -\dfrac{1}{a}$

The required asymptote is;

$r\sin(\theta - \alpha) = \dfrac{1}{f'(\alpha)}$

i.e. $r\sin\left(\theta - \frac{\pi}{2}\right) = \dfrac{1}{-\frac{1}{a}}$

or, $-r\sin\left\{\dfrac{\pi}{2} - \theta\right\} = -a$

or, $-r\cos\theta = -a$

or, $r\cos\theta = a$

ii) $\mathbf{r\theta = a}$

Sol^n: Given curve is;

$r\theta = a$

or, $\dfrac{1}{r} = \dfrac{\theta}{a}$

or, $\dfrac{1}{r} = f(\theta)$

Where,

$f(\theta) = \dfrac{\theta}{a}$

Now,

$f(\theta) = 0$ implies

$\dfrac{\theta}{a} = 0$

$\therefore$ $\theta = 0$

$\therefore \quad \alpha = 0$ (say)

Also,

$$f'(\theta) = \frac{1}{a}$$

$$\therefore \quad f'(\alpha) = \frac{1}{a}$$

So that the equation of asymptote is;

$$r\sin(\theta - \alpha) = \frac{1}{f'(\alpha)}$$

i.e. $r\sin(\theta - 0) = \frac{1}{\frac{1}{a}}$

or, $r\sin\theta = a$

iii) $\mathbf{2r^2 = \tan 2\theta}$

Sol^n: Given curve is;

$$2r^2 = \tan 2\theta$$

or, $\frac{1}{r^2} = \frac{2}{\tan 2\theta}$

or, $\frac{1}{r^2} = 2\cot 2\theta$

or, $\frac{1}{r^2} = f(\theta)$

Where,

$$f(\theta) = 0 \text{ gives}$$

$$2\cot 2\theta = 0$$

or, $\frac{2\cos 2\theta}{\sin 2\theta} = 0$

$\therefore \quad \cos 2\theta = 0$

or, $2\theta = \frac{\pi}{2}$

or, $\theta = \frac{\pi}{4}$

$\therefore \quad \alpha = \frac{\pi}{4}$

Also,

$$f'(\theta) = -4\operatorname{cosec}^2 2\theta$$

So that the required asymptote is;

$$r\sin(\theta - \alpha) = \frac{1}{f'(\alpha)}$$

i.e. $r\sin\left(\theta - \frac{\pi}{4}\right) = \frac{1}{-4\operatorname{cosec}^2 2\frac{\pi}{4}}$

or, $r\sin\left(\theta - \frac{\pi}{4}\right) = \frac{1}{-4}\sin^2\frac{\pi}{2}$

or, $\sin\left(\theta - \frac{\pi}{4}\right) = \frac{1}{-4r}$ ... (*)

But,

$$\frac{1}{-4r} = \frac{1}{-4}\sqrt{\frac{2}{\tan 2\theta}}$$

$\therefore \quad$ At $\theta = \frac{\pi}{4}$

$$\frac{1}{-4r} = 0$$

So that from (*);

$$\sin\left(\theta - \frac{\pi}{4}\right) = 0$$

or, $\theta - \frac{\pi}{4} = 0$

$\therefore \quad \theta = \frac{\pi}{4}$

Exercise 6

Curvature:

The rate of change of direction of the curves with respect to the arc is called curvature at the point p along the curve.

Some formulae for radius of curvature:

1. For the Cartesian equation $y = f(x)$

$$\rho = \frac{(1+y_1^2)^{\frac{3}{2}}}{y_2} \text{ where } y_1 = \frac{dy}{dx}, y_2 = \frac{d^2y}{dx^2} \neq 0$$

But if the equation of the curve be $x = f(y)$

$$\text{Then } \rho = \frac{(1+x_1^2)^{\frac{3}{2}}}{x_2}, x_2 \neq 0$$

$$\text{where } x_1 = \frac{dx}{dy}, x_2 = \frac{d^2x}{dy^2}$$

2. For the parametric equations $x = \beta(t), y = \psi(t)$ The radius of curvature ρ is given by;

$$\rho = \frac{\left(x'^2+y'^2\right)^{\frac{3}{2}}}{x'y''-y'x''} \text{ where } x'y''-y'x'' \neq 0$$

Where, $x' = \frac{dx}{dt}, y' = \frac{dy}{dt}$ etc

3. For the polar equation $r = f(\theta)$

$$\rho = \frac{(r^2+r_1^2)^{\frac{3}{2}}}{r^2+2r_1^2-rr_2} \text{ where } r_1 = \frac{dr}{d\theta}, r_2 = \frac{d^2r}{d\theta^2}$$

and $r^2 + 2r_1^2 - rr_2 \neq 0$

4. For the pedal equation $p = f(r)$

The radius of curvature ρ is given by; $\rho = r\frac{dr}{dp}$

5. For due tangential polar equation $p = f(\psi)$

The radius of curvature ρ is given by; $\rho = p + \frac{d^2p}{d\psi^2}$

1. **Find the radius of curvature at any point (x, y) of the following curves.**

i) $ay^2 = x^3$

$y = \frac{x^3}{a}$

Solⁿ:

$$ay^2 = x^3 \ldots \text{(i)}$$

Differentiating equation (i) with respect to x, we get,

$$2ay\frac{dy}{dx} = 3x^2$$

$y = \sqrt{\frac{x^3}{a}} = \frac{x^{3/2}}{\sqrt{a}} = \frac{3}{2}a$

$$\text{or, } y_1 = \frac{dy}{dx} = \frac{3x^2}{2ay}$$

$$\text{or, } y_1^2 = \frac{9x^4}{4a^2y^2}$$

$$\text{or, } y_1^2 = \frac{9x^4 \cdot a}{4a^2x^3}$$

$$\text{or, } y_1^2 = \frac{9x}{4a} \ldots \text{(ii)}$$

Again, differentiating equation (ii), we get,

$$2y_1y_2 = \frac{9}{4a}$$

$$\text{or, } y_2 = \frac{9}{4a \cdot 2y_1} = \frac{9 \times 2ay}{8a \times 3x^2} = \frac{y}{4x^2} = \frac{3}{4}\frac{x^{\frac{3}{2}}}{a^{\frac{1}{2}}x^2} = \frac{3}{4}\frac{x^{-\frac{1}{2}}}{a^{\frac{1}{2}}}$$

$$\therefore \text{ Radius of curvature } (\rho) = \frac{(1+y_1^2)^{\frac{3}{2}}}{y_2} = \frac{\left(1+\frac{9x}{4a}\right)^{\frac{3}{2}}}{\frac{3}{4}\frac{x^{-\frac{1}{2}}}{a^{\frac{1}{2}}}}$$

$$= \frac{(4a+9x)^{\frac{3}{2}}}{(4a)^{\frac{3}{2}}} \times \frac{4a^{\frac{1}{2}}}{3x^{-\frac{1}{2}}}$$

$$= \frac{(4a+9x)^{\frac{3}{2}}}{a^{\frac{3}{2}} \times 8} \times \frac{4a^{\frac{1}{2}}}{3x^{-\frac{1}{2}}} = \frac{(4a+9x)^{\frac{3}{2}}x^{\frac{1}{2}}}{6a}$$

ii) $xy = c^2$

Solⁿ: Here,

$$y = \frac{c^2}{x} \ldots \text{(i)}$$

Differentiating (i) with respect to x, we get,

$$y_1 = -\frac{c^2}{x^2}, y_2 = -c^2(-2x^{-3}) = \frac{2c^2}{x^3}$$

We know by formula,

$$\rho = \frac{(1+y_1^2)^{\frac{3}{2}}}{y_2} = \frac{\left(1+\frac{c^4}{x^4}\right)^{\frac{3}{2}}}{\frac{2c^2}{x^3}} = \frac{(x^4+c^4)^{\frac{3}{2}}}{2c^2x^3}$$

But $c^2 = xy$

$\therefore \quad c^4 = x^2y^2$

$$\therefore \quad \rho = \frac{(x^4+x^2y^2)^{\frac{3}{2}}}{2c^2x^3} = \frac{x^3(x^2+y^2)^{\frac{3}{2}}}{2c^2x^3} = \frac{(x^2+y^2)^{\frac{3}{2}}}{2c^2}$$

iii) $\mathbf{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1}$

Sol^n: Here,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \ldots \text{(i)}$$

Differentiating equation (i) with respect to x, we get,

$$\frac{2x}{a^2} + \frac{2y}{b^2}\frac{dy}{dx} = 0$$

or, $\frac{dy}{dx} = y_1 = -\frac{2x}{a^2} \times \frac{2y}{b^2} = -\frac{b^2x}{a^2y}$

or, $\frac{d^2y}{dx^2} = y_2 = -\frac{b^2}{a^2}\left[\frac{y-xy_1}{y^2}\right]$

$$= -\frac{b^2}{a^2}\left[\frac{y - x\left(-\frac{b^2}{a^2}\right)\frac{x}{y}}{y^2}\right]$$

$$= -\frac{b^2}{a^2}\left[\frac{a^2y^2+b^2x^2}{a^2y^3}\right] = -\frac{b^2}{a^2}\left[\frac{a^2b^2}{a^2y^3}\right]$$

$$\therefore \quad \text{Radius of curvature } (\rho) = \frac{(1+y_1{}^2)^{\frac{3}{2}}}{y_2} = \frac{\left(1+\frac{b^4x^2}{a^4y^2}\right)^{\frac{3}{2}}}{\frac{b^2}{a^2}\left[\frac{a^2b^2}{a^2y^3}\right]}$$

$$= -\frac{\left(a^4y^2+b^4x^2\right)^{\frac{3}{2}}}{a^6y^3} \times \frac{a^2y^3}{b^4}$$

$$= -\frac{\left(a^4y^2+b^4x^2\right)^{\frac{3}{2}}}{a^4b^4}$$

$$= \frac{\left(a^4y^2+b^4x^2\right)^{\frac{3}{2}}}{a^4b^4}$$

iv) $\mathbf{x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}}$

Sol^n: Here,

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}} \ldots \text{(i)}$$

Differentiating (i) with respect to x, we get,

$$\frac{2}{3}x^{-\frac{1}{3}} + \frac{2}{3}y^{-\frac{1}{3}}\frac{dy}{dx} = 0$$

$$\therefore \quad \frac{dy}{dx} = \frac{-x^{-\frac{1}{3}}}{y^{-\frac{1}{3}}} = \frac{-y^{\frac{1}{3}}}{x^{\frac{1}{3}}}$$

Now,

$$1 + y_1^2 = 1 + \left(\frac{-y^{\frac{1}{3}}}{x^{\frac{1}{3}}}\right)^2 = \frac{x^{\frac{2}{3}}+y^{\frac{2}{3}}}{x^{\frac{2}{3}}} = \frac{a^{\frac{2}{3}}}{x^{\frac{2}{3}}}$$

$$\therefore \quad 1 + y_1^2 = a^{\frac{2}{3}}x^{-\frac{2}{3}} \ldots \text{(ii)}$$

Again, differentiating with respect to x, we get,

$$0 + 2y_1y_2 = a^{\frac{2}{3}}\left(-\frac{2}{3}x^{-\frac{5}{3}}\right) = -\frac{2}{3}a^{\frac{2}{3}}x^{-\frac{5}{3}}$$

$$\therefore \quad y_2 = -\frac{1}{3}a^{\frac{2}{3}}\frac{x^{-\frac{5}{3}}}{y_1}$$

But

$$y_1 = \frac{-y^{\frac{1}{3}}}{x^{\frac{1}{3}}}$$

$$\therefore \quad y_2 = \frac{1}{3}a^{\frac{2}{3}}\frac{x^{-\frac{5}{3}}}{y^{\frac{1}{3}}}x^{\frac{1}{3}} = \frac{1}{3}a^{\frac{2}{3}}x^{-\frac{4}{3}}y^{-\frac{1}{3}}$$

We know, the formula for radius of curvature ρ in Cartesian form

$$\rho = \frac{(1+y_1^2)^{\frac{3}{2}}}{y_2} = \frac{\left(a^{\frac{2}{3}}x^{\frac{2}{3}}\right)^{\frac{3}{2}}}{\frac{1}{3}a^{\frac{2}{3}}x^{-\frac{4}{3}}y^{-\frac{1}{3}}} = \frac{3ax^{-1}}{a^{\frac{2}{3}}x^{-1}x^{-\frac{1}{3}}y^{-\frac{1}{3}}} = 3a^{\frac{1}{3}}x^{\frac{1}{3}}y^{\frac{1}{3}} = 3(axy)^{\frac{1}{3}}$$

$\therefore \quad \rho = 3(axy)^{\frac{1}{3}}$

v) $\mathbf{y^2 = 4x}$ **at the vertex** $\mathbf{(0,0)}$

Sol^n: We have,

$y^2 = 4x \ldots$ (i)

Differentiating (i) with respect to x, we get

$$2y\frac{dy}{dx} = 4$$

$$\therefore \quad \frac{dy}{dx} = \frac{2}{y}$$

$$\therefore \quad \frac{dy}{dx} \text{ at } (0,0) = \infty$$

Thus, $\frac{dy}{dx}$ at the vertex is infinite, Hence the tangent at the vertex being y–axis.

Hence, the formula

$\rho = \frac{(1+y_1^2)^{\frac{3}{2}}}{y_2}$ is not applicable in this case

Hence, we apply

$$\rho = \frac{(1+x_1^2)^{\frac{3}{2}}}{x_2}$$

Now, from (i)

$$x = \frac{y^2}{4}$$

$$\therefore \quad \frac{dx}{dy} = \frac{1}{2}y$$

and

$$\frac{d^2x}{dy^2} = \frac{1}{2}$$

$\therefore$ At the vertex (0,0)

$x_1 = 0$ and $x_2 = \frac{1}{2}$

Hence,

$$\rho = \frac{(1+x_1^2)^{\frac{3}{2}}}{x_2} = \frac{(1+0)^{\frac{3}{2}}}{\frac{1}{2}} = \frac{1^{\frac{3}{2}}}{\frac{1}{2}} = \frac{1}{\frac{1}{2}} = 2$$

vi) $\mathbf{y = x^3 - 2x^2 + 7x}$ **at** $\mathbf{(0,0)}$

Sol^n: Let,

$y = x^3 - 2x^2 + 7x \ldots$ (i)

Differentiating (i) with respect to x, we get,

$y_1 = 3x^2 - 4x + 7$

y_1 at (0,0) is 7

and $y_2 = 6x - 4$

y_2 at $(0,0) = -4$

$$\therefore \quad \text{Radius of curvature } (\rho) = \frac{(1+y_1^2)^{\frac{3}{2}}}{y_2} = \frac{(1+49)^{\frac{3}{2}}}{-4} = \frac{(50)^{\frac{3}{2}}}{-4}$$

$$= -\frac{1}{2}\, 125\sqrt{2} \text{ (ignore negative)}$$

$$= -\frac{125\sqrt{2}}{2}$$

2. **Find the radius of curvature at any point** $\mathbf{(r,\theta)}$ **for the curve.**

i) $\mathbf{r^3 = a^3 \cos 3\theta}$

Sol^n: Here,

$r^3 = a^3 \cos 3\theta \ldots$ (i)

Differentiating equation (i) with respect to θ, we get,

$$3r^2\frac{dr}{d\theta} = -3a^3 \sin 3\theta$$

$$\text{or,} \quad \frac{dr}{d\theta} = \frac{-3a^3 \sin 3\theta}{3r^2} = \frac{-ra^3 \sin 3\theta}{r^3} = \frac{-ra^3 \sin 3\theta}{a^3 \cos 3\theta} \qquad \text{(from i)}$$

$$r_1 = \frac{dr}{d\theta} = -r\tan 3\theta \ldots \text{(ii)}$$

Again, differentiating equation (ii), we get,

$$r_2 = -[-r\tan 3\theta + 3r\sec^3 3\theta]$$
$$= -[\tan^2 3\theta + 3r\sec^2 3\theta]$$
$$= r\tan^2 3\theta - 3r\sec^2 3\theta$$

$$\text{Radius of curvature } (\rho) = \frac{(r^2+r_1{}^2)^{\frac{3}{2}}}{r^2+2r_1{}^2-rr_2}$$

$$\text{or,} \quad \rho = \frac{(r^2+r^2\tan^2 3\theta)^{\frac{3}{2}}}{r^2+2r^2\tan^2 3\theta - r^2\tan^2 3\theta + 3r^2\sec^2 3\theta}$$

$$= \frac{r^3\sec^3 3\theta}{4r^2\sec^3 3\theta} = \frac{1}{4}\, r\sec 3\theta = \frac{r}{4\cos 3\theta} \qquad \text{(from i)}$$

$$= \frac{r}{4\frac{r^3}{a^3}} = \frac{a^3}{4r^2}$$

$$\therefore \quad \rho = \frac{a^3}{4r^2}$$

ii) $\mathbf{r = a\cos\theta}$

Sol^n: Let,

$$r = a\cos\theta \ \ldots \text{(i)}$$

Differentiating (i) with respect to θ we get

$$r_1 = -a\sin\theta$$

and

$$r_2 = -a\cos\theta$$

We know, radius of curvature,

$$\rho = \frac{(r^2+r_1^2)^{\frac{3}{2}}}{r^2+2r_1^2-rr_2} = \frac{\left(a^2\cos^2\theta+a^2\sin^2\theta\right)^{\frac{3}{2}}}{a^2\cos^2\theta+a^2\sin^2\theta+a^2\cos^2\theta}$$

$$= \frac{(a^2)^{\frac{3}{2}}}{2a^2} = \frac{a^3}{2a^2} = \frac{1}{2}a$$

iii) $\mathbf{r^2\cos 2\theta = a^2}$

Sol^n: Given;

$$r^2\cos 2\theta = a^2 \ \ldots \text{(i)}$$

Differentiating (i) with respect to θ, we get,

$$2rr_1\cos 2\theta - 2r^2\sin 2\theta = 0$$

$$\therefore \quad r_1 = \frac{2r^2\sin 2\theta}{2r\cos 2\theta}$$

$$r_1 = r\tan 2\theta$$

Again, differentiating, we get,

$$r_2 = r_1\tan 2\theta + 2r\sec^2 2\theta$$

$\therefore$ By the formula,

$$\rho = \frac{(r^2+r_1^2)^{\frac{3}{2}}}{r^2+2r_1^2-rr_2}$$

$$\therefore \quad \rho = \frac{\left(r^2+r^2\tan^2 2\theta\right)^{\frac{3}{2}}}{r^2+2r^2\tan^2 2\theta-r^2\tan^2 2\theta-2r^2\sec^2 2\theta}$$

$$= \frac{\left[r^2\left(1+\tan^2 2\theta\right)\right]^{\frac{3}{2}}}{r^2+r^2\tan^2 2\theta-2r^2\sec^2 2\theta}$$

$$= \frac{r^3\sec^3 2\theta}{r^2\sec^2 2\theta-2r^2\sec^2 2\theta}$$

$$= \frac{r^3\sec^3 2\theta}{r^2\sec^2 2\theta} = r\sec 2\theta = \frac{r}{\cos 2\theta}$$

$\therefore$ By the question;

$$\cos 2\theta = \frac{a^2}{r^2}$$

$$\therefore \quad \rho = \frac{r}{\frac{a^2}{r^2}} = \frac{r^3}{a^2}$$

iv) $\mathbf{r^2 = a^2\cos 2\theta \text{ at } \theta = 0}$

Sol^n: We have,

$$r^2 = a^2\cos 2\theta$$

Differentiating with respect to θ, we get

$$2rr_1 = -a^2\sin 2\theta\cdot 2$$

$$\therefore \quad r_1 = \frac{-a^2\sin 2\theta}{r}$$

$$\therefore \quad r_1 = \frac{-a^2\sin 2\theta}{r^2} = \frac{-a^2 r\sin 2\theta}{a^2\cos 2\theta} = -r\tan 2\theta$$

Again, differentiating with respect to θ, we get

$$r_2 = -[r_1\tan 2\theta + 2r\sec^2 2\theta]$$
$$= -[-r\tan 2\theta\tan 2\theta + 2r\sec^2 2\theta]$$
$$= r\tan^2 2\theta - 2r\sec^2 2\theta$$
$$= r[\tan^2 2\theta - 2 - 2\tan^2 2\theta] = -r[2+\tan^2 2\theta]$$

We know, the formula,

$$\rho = \frac{(r^2+r_1^2)^{\frac{3}{2}}}{r^2+2r_1^2-rr_2}$$

$$= \frac{\left[r^2+r^2\tan^2 2\theta\right]^{\frac{3}{2}}}{r^2+2r^2\tan^2 2\theta+r^2\left[2+\tan^2 2\theta\right]} = \frac{r^3\sec^3 2\theta}{3r^2\left(1+\tan^2 2\theta\right)}$$

$$= \frac{r}{3}\sec 2\theta$$

$$\therefore \quad \rho = \frac{a\sqrt{\cos 2\theta}\cdot\sec 2\theta}{3}$$

Now,

At, $\theta = 0$;

$$\rho = \frac{a \cdot 1 \cdot 1}{3} = \frac{a}{3}$$

v) $\mathbf{r = a(\theta + \sin\theta)}$ **at** $\mathbf{\theta = 0}$

Sol^n: We have,

$$r = a(\theta + \sin\theta)$$

Differentiating with respect to θ, we get,

$$r_1 = a(1+\cos\theta), r_2 = -a\sin\theta$$

At $\theta = 0, r_1 = a(1+1) = 2a$

$$r_2 = -a\sin 0 = 0$$
$$r = a(0+0) = 0$$

We know,

$$\rho = \frac{(r^2+r_1^2)^{\frac{3}{2}}}{r^2+2r_1^2-rr_2} = \frac{(0+4a^2)^{\frac{3}{2}}}{0+8a^2-0} = \frac{8a^3}{8a^2} = a$$

$\therefore \quad \rho = a$

3. Find the radius of curvature at the point (r, θ) on the cardi[oid] $r = a(1-\cos\theta)$ and show that it varies $\sqrt{r}$.

Sol^n: We have,

$$r = a(1-\cos\theta) \ldots \text{(i)}$$

Differentiating with respect to θ, gives,

$$r_1 = a\sin\theta, r_2 = a\cos\theta$$

Applying formula,

$$\rho = \frac{(r^2+r_1^2)^{\frac{3}{2}}}{r^2+2r_1^2-rr_2}$$

$$= \frac{\left[a^2(1-\cos\theta)^2+a^2\sin^2\theta\right]^{\frac{3}{2}}}{a^2(1-\cos\theta)^2+2a^2\sin^2\theta-a^2(1-\cos\theta)\cdot\cos\theta}$$

$$= \frac{\left[a^2(1-2\cos\theta+1)\right]^{\frac{3}{2}}}{a^2\left(1-2\cos\theta+\cos^2\theta+2\sin^2\theta-\cos\theta+\cos^2\theta\right)}$$

$$= \frac{\left[2a^2(1-\cos\theta)\right]^{\frac{3}{2}}}{3a^2(1-\cos\theta)}$$

But,

$$r = a(1-\cos\theta)$$

$$\therefore \quad \rho = \frac{\left[2a\cdot a(1-\cos\theta)\right]^{\frac{3}{2}}}{3a\cdot a(1-\cos\theta)}$$

$$= \frac{(2ar)^{\frac{3}{2}}}{3ar} = \frac{2^{\frac{3}{2}}}{3}(ar)^{\frac{3}{2}-1} = \frac{2\sqrt{2}}{3}\sqrt{ar} = \frac{2\sqrt{2ar}}{3}$$

4. Show that the radius of curvature at any point (r, θ) of the curve $\mathbf{r^m = a^m \cos m\theta}$ is $\dfrac{a^m}{(m+1)r^{m-1}}$.

Sol^n: We have,

$$r^m = a^m \cos m\theta \ldots \text{(i)}$$

Taking log on both sides, we get

$$m\log r = m\log a + \log\cos m\theta$$

Differentiating with respect to θ gives;

$$m \cdot \frac{1}{r} \cdot r_1 = 0 - m\frac{\sin m\theta}{\cos m\theta}$$

$$\therefore \quad m\frac{r_1}{r} = -m\tan m\theta$$

$r_1 = -r\tan m\theta$ then UV

Again, differentiating with respect to θ, we get,

$$r_2 = -[r_1\tan m\theta + rm\sec^2 m\theta]$$
$$= -[-r\tan^2 m\theta + rm\sec^2 m\theta]$$
$$= r\tan^2 m\theta - mr\sec^2 m\theta$$

We know the formula

$$\rho = \frac{(r^2+r_1^2)^{\frac{3}{2}}}{r^2+2r_1^2-rr_2}$$

$$= \frac{(r^2+r^2\tan^2 m\theta)^{\frac{3}{2}}}{r^2+2r^2\tan^2 m\theta - r\left[r\tan^2 m\theta - mr\sec^2 m\theta\right]}$$

$$= \frac{(r^2\sec^2 m\theta)^{\frac{3}{2}}}{r^2\left[1+\tan^2 m\theta + m\sec^2 m\theta\right]}$$

$$= \frac{r^3\sec^3 m\theta}{r^2(\sec^2 m\theta + m\sec^2 m\theta)} = \frac{r^3\sec^3 m\theta}{r^2\sec^2 m\theta(1+m)} = \frac{r\sec m\theta}{1+m}$$

$$= \frac{r}{(1+m)\cos m\theta}$$

But,

$$\cos m\theta = \frac{r^m}{a^m}$$

$$\rho = \frac{r}{(1+m)}\cdot\frac{r^m}{a^m} = \frac{a^m}{(m+1)r^{m-1}} \text{ Proved}$$

5. Find the radius of curvature at any point for the curves.

i) $x = at^2, y = 2at$

Soln: We have,

$x = at^2, y = 2at$

$x' = 2at, y' = 2a$

$x'' = 2a, y'' = 0$

We know,

$$\text{Radius of curvature } (\rho) = \frac{\left[(x')^2+(y')^2\right]^{\frac{3}{2}}}{x'y''-y'x''}$$

$$= \frac{\left[4a^2t^2+4a^2\right]^{\frac{3}{2}}}{0-2a\cdot 2a}$$

$$= \frac{(4a^2)^{\frac{3}{2}}(1+t^2)^{\frac{3}{2}}}{4a^2}$$

$$= \frac{8a^3}{4a^2}(1+t^2)^{\frac{3}{2}} = -2a(1+t^2)^{\frac{3}{2}}$$

$$= 2a(1+t^2)^{\frac{3}{2}}$$

ii) $x = a\cos\theta, y = a\sin\theta$

Soln: Differentiating both sides with respect to θ, we get,

$x' = -a\sin\theta,\ y' = a\cos\theta$

$x'' = -a\cos\theta, y'' = -a\sin\theta$

We know, the radius of curvature for parametric form

$$(\rho) = \frac{\left[(x')^2+(y')^2\right]^{\frac{3}{2}}}{x'y''-y'x''}$$

$$\therefore\quad \rho = \frac{\left[a^2\sin^2\theta+a^2\cos^2\theta\right]^{\frac{3}{2}}}{(-a\sin\theta)(-a\sin\theta)-(a\cos\theta)(-a\cos\theta)}$$

$$= \frac{(a^2)^{\frac{3}{2}}}{a^2(\sin^2\theta+\cos^2\theta)}$$

$$= \frac{a^3}{a^2} = a$$

iii) $x = a(\theta+\sin\theta), y = a(1-\cos\theta)$ at $\theta = 0°$

Soln: We have,

$x = a(\theta+\sin\theta), y = a(1-\cos\theta)$

Differentiating with respect to θ, we get,

$x' = a(1+\cos\theta), y' = a\sin\theta$

Again, differentiating, we get,

$x'' = -a\sin\theta, y'' = a\cos\theta$

At $\theta = 0, x' = a(1+1), y' = 0$

$x'' = 0, y'' = a$

We know the formula,

$$(\rho) = \frac{\left[(x')^2+(y')^2\right]^{\frac{3}{2}}}{x'y''-y'x''}$$

$$= \frac{\left[(2a)^2+0^2\right]^{\frac{3}{2}}}{2a\cdot a-0\cdot 0}$$

$$= \frac{\left[4a^2\right]^{\frac{3}{2}}}{2a^2} = \frac{(2a)^3}{2a^2} = 4a$$

iv) $x = a\cos\theta, y = b\sin\theta$

Soln: Differentiating both sides with respect to θ, we get,

$x' = -a\sin\theta, y' = b\cos\theta$

Again differentiating we get

$x'' = -a\cos\theta, y'' = -b\sin\theta$

We know the formula for radius of curvature

$$(\rho) = \frac{\left[(x')^2+(y')^2\right]^{\frac{3}{2}}}{x'y''-y'x''}$$

$$= \frac{\left[a^2\sin^2\theta+b^2\cos^2\theta\right]^{\frac{3}{2}}}{(-a\sin\theta)(-b\sin\theta)-(-a\cos\theta)b\cos\theta}$$

$$= \frac{\left[a^2\sin^2\theta+b^2\cos^2\theta\right]^{\frac{3}{2}}}{ab}$$

6. Find the radius of curvature at any point (p, r) on the following curves whose pedal equations are;

i) $r^2 = 2ap$

Sol^n: Differentiating with respect to p, we get,

$$2r\frac{dr}{dp} = 2a$$

or, $\frac{dr}{dp} = \frac{2a}{2r} = \frac{a}{r}$

$\therefore$ Radius of curvature $(\rho) = r\frac{dr}{dp} = r\frac{a}{r} = a$

ii) $r^3 = a^2p$

Sol^n: Differentiating with respect to p we get,

$$3r^2\frac{dr}{dp} = a^2$$

$\therefore \frac{dr}{dp} = \frac{a^2}{3r^2}$

$$\rho = r\frac{dr}{dp} = \frac{a^2r}{3r^2}$$

$\therefore$ Radius of curvature $(\rho) = \frac{a^2}{3r}$

iii) $p^2 = ar$

Sol^n: Differentiating with respect to p we get,

$$2p = a\frac{dr}{dp}$$

or, $\frac{dr}{dp} = \frac{2p}{a}$

$$\rho = r\frac{dr}{dp} = r\frac{2p}{a} = \frac{2r}{a}\sqrt{ar}$$

Radius of curvature $(\rho) = 2\sqrt{\frac{r^3}{a}}$

7. Find the radius of curvature at the origin for the following curves:

i) $4x^2 - 3xy + y^2 - 3y = 0$

Sol^n: The curve passes through the origin. Here, we use substitution method to find the value of ρ.

$$8x - 3xy_1 - 3y + 2yy_1 - 3y_1 = 0$$

$\therefore y_1 = \frac{8x-3y}{3-2y+3x}$

y_1 at $(0,0) = 0$

Again,

$$8 - 3xy_2 - 6y_1 + 2yy_2 - 2y_1^2 - 3y_2 = 0$$

$\therefore y_2 = \frac{8-6y_1+2y_1^2}{3x-2y+3}$

y_2 at $(0,0) = \frac{8}{3}$

$\therefore \rho = \frac{(1+0)^{\frac{3}{2}}}{\frac{8}{3}} = \frac{3}{8}$

ii) $x^3 + y^3 - 2x^2 + 6y = 0$

Sol^n: The curve passes through the origin. Here, we use the substitution method to find the value of ρ

$$3x^2 + 3y^2\frac{dy}{dx} - 4x + 6y_1 = 0$$

$\therefore y_1 = \frac{4x-3x^2}{3y^2+6}$

y_1 at $(0,0)$ is 0

Again,

$$6x + 3y^2y_2 + 6yy_1^2 - 4 + 6y_2 = 0$$

$\therefore y_2 = \frac{4-6x-6yy_1^2}{3y^2+6}$

$\therefore y_2$ at $(0,0) = \frac{4}{6} = \frac{2}{3}$

$\therefore \rho = \frac{(1+y_1^2)^{\frac{3}{2}}}{y_2} = \frac{(1+0)^{\frac{3}{2}}}{\frac{2}{3}} = \frac{3}{2}$

8. Find the radius of curvature of the curve $y = x^2(x-3)$ at the points where the tangent is parallel to x–axis.

Proof:

We have,

$$y = x^2(x-3) \ \ldots \text{(i)}$$

$\therefore \frac{dy}{dx} = 2x(x-3) + x^2 \cdot 1 = 3x^2 - 6x$

If the tangent is parallel to x–axis, then $\frac{dy}{dx} = 0$

$$3x^2 - 6x = 0$$

$\therefore x = 0, x = 2$

Then,

If $x = 0, y = 0$, if $x = 2, y = -4$

$\therefore$ Points are (0,0) and (2, −4)

$\therefore$ y_1 at $(2,-4) = 3\cdot4 - 12 = 12 - 12 = 0$

y_2 at $(2,-4) = 6\cdot2 - 6 = 0$

$$\therefore \quad \rho = \frac{(1+y_1^2)^{\frac{3}{2}}}{y_2} = \frac{(1+0)^{\frac{3}{2}}}{6} = \frac{1}{6}$$

9. **Show that for the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, the radius of curvature at the extremity of the major axis is equal to half the latus rectum.**

Proof:

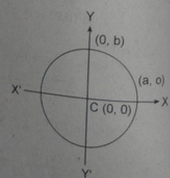
We have,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \dots \text{(i)}$$

Differentiating with respect to x, gives;

$$\frac{2x}{a^2} + \frac{2y}{b^2}\frac{dy}{dx} = 0$$

$$\therefore \quad \frac{dy}{dx} = -\frac{b^2x}{a^2y} \dots \text{(ii)}$$



Now, the end of major axis of the ellipse is (a, 0)

$\therefore$ $x = a, y = 0$

$\therefore$ $\frac{dy}{dx}$ at (a, 0) is ∞, i.e. infinite

So, we cannot find ρ from the formula,

$$\rho = \frac{(1+y_1^2)^{\frac{3}{2}}}{y_2}$$

Hence, we apply

$$\rho = \frac{(1+x_1^2)^{\frac{3}{2}}}{x_2}$$

Hence, from equation (ii), we have,

$$\frac{dx}{dy} = -\frac{a^2y}{b^2x}$$

$$\therefore \quad \frac{dx}{dy} \text{ at } (a,0) = 0$$

Again, differentiating with respect to y, we get

$$\frac{d^2x}{dy^2} = -\frac{a^2}{b^2}\left[\frac{x\cdot1 - y\cdot\frac{dx}{dy}}{x^2}\right]$$

$$= -\frac{a^2}{b^2}\left[\frac{x - y\left(-\frac{a^2y}{b^2x}\right)}{x^2}\right] = -\frac{a^2}{b^2}\frac{\left[b^2x^2 + a^2y^2\right]}{b^2x^3}$$

$$\therefore \quad \frac{d^2x}{dy^2} \text{ at } (a,0) = -\frac{a^2}{b^2}\frac{\left[b^2a^2 + a^2\cdot0\right]}{b^2a^3} = -\frac{a^2}{b^2}\frac{b^2a^2}{b^2a^3} = -\frac{a}{b^2}$$

We know, the formula,

$$\rho = \frac{(1+x_1^2)^{\frac{3}{2}}}{x_2} = \frac{(1+0)^{\frac{3}{2}}}{-\frac{a}{b^2}} = -\frac{b^2}{a} = \frac{b^2}{a}$$

$$\therefore \quad \rho = \frac{b^2}{a} = \frac{1}{2}\left[\frac{2b^2}{a}\right] = \frac{1}{2} \text{ latus rectum} = \text{semi latus rectum.}$$

10. **Find the chord of curvature through the pole for the following curves.**

i) $\mathbf{r = a(1 + \cos\theta)}$

Soln: Here,

$$r = a(1+\cos\theta) \dots \text{(i)}$$

The chord of curvature through the pole is $2p\frac{dr}{dp}$

First we change equation (i) into pedal form

$$r = a(1+\cos\theta)$$

$$\therefore \quad \frac{dr}{d\theta} = -a\sin\theta$$

Now,

$$\tan\phi = r\frac{d\theta}{dr} = \frac{r}{\frac{dr}{d\theta}} = \frac{r}{-a\sin\theta} = \frac{a(1+\cos\theta)}{-a\sin\theta}$$

$$\tan\phi = -\frac{2\cos^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}} = -\cot\frac{\theta}{2} = \tan\left(90° + \frac{\theta}{2}\right)$$

$$\therefore \quad \phi = 90° + \frac{\theta}{2}$$

Hence,

$$p = r\sin\phi$$

$$p = r\sin\left(90° + \frac{\theta}{2}\right) = r\cos\frac{\theta}{2}$$

$$p^2 = r^2\cos^2\frac{\theta}{2} = r^2\frac{(1+\cos\theta)}{2} = \frac{r^2}{2}\cdot\frac{r}{a} = \frac{r^3}{2a}$$

$\therefore \quad r^3 = 2ap^2 \ldots$ (ii)

which is the pedal equation of the given curve

Again, differentiating (ii) with respect to p, we get,

$$3r^2\frac{dr}{dp} = 4ap$$

$$\therefore \quad \frac{dr}{dp} = \frac{4ap}{3r^2}$$

Hence,

Chord of curvature through the pole $= 2p\dfrac{dr}{dp} = 2p\dfrac{4ap}{3r^2} = \dfrac{8ap^2}{3r^2}$

$$= \frac{4\cdot 2ap^2}{3r^2} = \frac{4r^3}{3r^2} = \frac{4r}{3}$$

ii) $\mathbf{r^2\cos 2\theta = a^2}$

Sol^n: Given curve is;

$$r^2\cos 2\theta = a^2$$

or, $r^2 = a^2\sec 2\theta$

or, $2r\dfrac{dr}{d\theta} = a^2\, 2\sec 2\theta + \tan 2\theta$

or, $r\dfrac{dr}{d\theta} = a^2\ \sec 2\theta + \tan 2\theta$

or, $\dfrac{d\theta}{r\,dr} = \dfrac{\cos 2\theta\ .\ \cot 2\theta}{a^2}$

or, $\dfrac{r^2}{r}\dfrac{d\theta}{dr} = \dfrac{r^2\cos 2\theta\ .\ \cot 2\theta}{a^2}$

or, $r\dfrac{d\theta}{dr} = \dfrac{a^2}{\cos 2\theta}\ \dfrac{\cos 2\theta\ .\ \cot 2\theta}{a^2} = \cot 2\theta$

or, $\tan\phi = \tan\left(\dfrac{\pi}{2} - 2\theta\right)$

or, $\phi = \dfrac{\pi}{2} - 2\theta$

or, $P = r\sin\left(\dfrac{\pi}{2} - 2\theta\right) = r\cos 2\theta$

or, $P = r\dfrac{a^2}{r^2}$

or, $P = \dfrac{a^2}{r}$

or, $Pr = a^2$

or, $P\dfrac{dr}{dP} + r = 0$

$\therefore$ Chord of curvature $= \dfrac{P\,dr}{dP} = 2r$

iii) $\mathbf{r = ae^{\theta\cot\alpha}}$

Sol^n: Here,

$$r = ae^{\theta\cot\alpha} \ldots \text{(i)}$$

Differentiating (i) with respect to θ, we get,

$$\therefore \quad \frac{dr}{d\theta} = ae^{\theta\cot\alpha}\cdot\cot\alpha = r\cot\alpha$$

Now,

$$\tan\phi = r\frac{d\theta}{dr} = \frac{r}{\frac{dr}{d\theta}} = \frac{r}{r\cot\alpha} = \tan\alpha$$

$\therefore \quad \phi = \alpha$

Hence,

$p = r\sin\phi$ gives $p = r\sin\alpha \ldots$ (ii)

which is pedal equation

Differentiating (ii) with respect to p, we get,

$$1 = \frac{dr}{dp}\cdot\sin\alpha$$

$$\therefore \quad \frac{dr}{dp} = \frac{1}{\sin\alpha}$$

Hence, the chord of curvature through the pole;

$$2p\frac{dr}{dp} = 2p\cdot\frac{1}{\sin\alpha} = \frac{2r\sin\alpha}{\sin\alpha} = 2r$$

11. **Show that the chord of curvature through the pole of the curve $r^m = a^m\cos m\theta$ is $\dfrac{2r}{m+1}$.**

Sol^n: Here,

$$r^m = a^m\cos m\theta$$

$$\therefore \quad \frac{d\theta}{dr} = -\frac{r^{m-1}}{a^m\sin m\theta}$$

$\therefore \quad r\frac{d\theta}{dr} = -\frac{r^m}{a^m \sin m\theta} = -\frac{a^m \cos m\theta}{a^m \sin m\theta}$

$\therefore \quad \tan\phi = r\frac{d\theta}{dr} = -\cot m\theta = \tan\left(\frac{\pi}{2} + m\theta\right)$

$\therefore \quad \phi = \frac{\pi}{2} + m\theta$

$\therefore \quad p = r\sin\phi = r\sin\left(\frac{\pi}{2} + m\theta\right) = r\cos m\theta$

or, $p = r \cdot \frac{r^m}{a^m} = \frac{r^{m+1}}{a^m}$

$\frac{dp}{dr} = \frac{(m+1)r^m}{a^m}$

Now,

The chord of curvature through the pole $= 2p\frac{dr}{dp}$

$$= 2 \cdot \frac{r^{m+1}}{a^m} \cdot \frac{a^m}{(m+1)r^m}$$

$$= \frac{2r}{m+1}$$

12. **Show that the chord of curvature through the pole for the curve $p = f(r)$ is given by $\frac{2f(r)}{f'(r)}$.**

Sol^n: Chord of curvature through the pole $= 2p\frac{dr}{dp}$

Since,

$p = f(r)$

So, $1 = f'(r)\frac{dr}{dp}$

$\therefore \quad \frac{dr}{dp} = \frac{1}{f'(r)}$

Now, chord of curvature through the pole $= 2p\frac{dr}{dp}$

$$= 2f(r) \cdot \frac{1}{f'(r)} = \frac{2f(r)}{f'(r)}$$

13. **Show that the chord of curvature parallel to y–axis for the curve $y = c\cosh\left(\frac{x}{c}\right)$ is double ordinate.**

Sol^n: The required chord of curvature parallel to y–axis $= 2\rho\cos\psi$

First we find ρ:

We have,

$y = c\cosh\left(\frac{x}{c}\right) \ldots$ (i)

Differentiating both sides with respect to x, we get,

$\frac{dy}{dx} = c\sinh\frac{x}{c} \cdot \frac{1}{c} = \sinh\frac{x}{c}$

$\therefore \quad \tan\psi = \sinh\frac{x}{c}$

Now,

$\sec\psi = \sqrt{1 + \tan^2\psi} = \sqrt{1 + \sinh^2\left(\frac{x}{c}\right)} = \cosh\left(\frac{x}{c}\right) = \frac{y}{c}$

$\therefore \quad y = c\sec\psi \ldots$ (ii)

Differentiating both sides with respect to s, we get,

$\frac{dy}{ds} = c\sec\psi\tan\psi\frac{d\psi}{ds}$

or, $\sin\psi = c\sec\psi\tan\psi\frac{1}{\rho}$

$\therefore \quad \rho = \frac{c\sec\psi\tan\psi}{\sin\psi} = c\sec^2\psi$

Hence, required chord of curvature parallel to y–axis

$= 2\rho\cos\psi$

$= 2c\sec^2\psi\cos\psi$

$= 2c\sec\psi$

$= 2y =$ double of the ordinate. Hence, proved

14. **Find the pedal equation of the following curves.**

i) $y^2 = 4a(x + a)$

Sol^n: We have,

$y^2 = 4a(x + a) \ldots$ (i)

Differentiating both sides with respect to x, we get,

$2y\frac{dy}{dx} = 4a$

$\therefore \quad \frac{dy}{dx} = \frac{2a}{y}$

Let us take (x, Y) as the current coordinates; the equation of the tangent at any point (x, y) is given by

$Y - y = \frac{2a}{y}(X - x)$

or, $2aX - Yy + y^2 - 2ax = 0$

Now, if the length of the perpendicular from the origin (0,0) on the tangent is p, then,

$p = \frac{2a\cdot 0 - 0\cdot y + y^2 - 2ax}{\sqrt{4a^2 + y^2}}$

$$\therefore \quad p^2 = \frac{(y^2-2ax)^2}{4a^2+y^2}$$

But,

$$y^2 = 4a(x+a)$$

$$\therefore \quad p^2 = \frac{\{4a(x+a)-2ax\}^2}{4a^2+4a^2+4ax} = \frac{\{2a(x+2a)\}^2}{4a(2a+x)}$$

$$= a(x+2a) = a\sqrt{(x+2a)^2}$$

or, $p^2 = a\sqrt{x^2+4ax+4a^2}$

or, $p^2 = a\sqrt{x^2+y^2}$ $\quad [\because y^2 = 4a(x+a)]$

$= a\sqrt{r^2} \quad (x^2+y^2 = r^2)$

$\therefore$ $p^2 = ar$ is the required pedal equation

ii) $\mathbf{y^2 = 4ax}$

Sol^n: Here,

$$y^2 = 4ax$$

Differentiating with respect to x, we get,

$$2y\,y_1 = 4a$$

or, $y_1 = \frac{2a}{y}$

Therefore, the equation of tangent at (x, y) is

$$Y - y = y_1(X - x)$$

or, $Y - y = \frac{2a}{y}(X - x)$

or, $yY - y^2 = 2aX - 2ax$

or, $yY - 2aX = 2ax$

or, $2aX - yY = -2ax$

or, $2aX - yY + 2ax = 0 \ldots$ (i)

Length of perpendicular $(0, 0)$ to equation (i) is;

$$p = \frac{2ax}{\sqrt{(2a)^2+y^2}}$$

or, $p = \frac{2ax}{\sqrt{4a^2+y^2}}$

or, $p^2 = \frac{4a^2x^2}{4a^2+y^2} = \frac{ax^2}{x+a}$

$\therefore ax^2 - p^2x - ap^2 = 0 \ldots$ (iii)

We have,

$$r^2 = x^2 + y^2$$

or, $r^2 = x^2 + 4ax$ $\quad (\because y^2 = 4ax)$

$\therefore$ $x^2 + 4ax - r^2 = 0 \ldots$ (iii)

Now, eliminating x from equation (ii) and (iii) by the rule of cross multiplication, we get,

$$\frac{x^2}{p^2r^2+4a^2p^2} = \frac{x}{ar^2+ap^2} = \frac{1}{4a^2+p^2}$$

or, $\frac{x^2}{p^2r^2+4a^2p^2} = \frac{1}{4a^2+p^2}$ and $\frac{x}{ar^2-ap^2} = \frac{1}{4a^2+p^2}$

or, $x^2 = \frac{p^2r^2+4a^2p^2}{4a^2+p^2}$ and $x = \frac{a(r^2-p^2)}{4a^2+p^2}$

$$\therefore \quad x^2 = \frac{(ar^2-ap^2)^2}{(4a^2+p^2)^2}$$

$$\therefore \quad \frac{(ar^2-ap^2)^2}{(4a^2+p^2)^2} = \frac{p^2r^2+4a^2p^2}{4a^2+p^2}$$

i.e. $(ar^2 - ap^2)^2 = (4a^2+p^2)(p^2r^2+4a^2p^2)$ which is the required equation.

iii) $\mathbf{x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}}$

Sol^n: We have given curve is;

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}} \ldots \text{(i)}$$

Differentiating (i) with respect to x, we get

$$\frac{2}{3}x^{-\frac{1}{3}} + \frac{2}{3}y^{-\frac{1}{3}}\frac{dy}{dx} = 0$$

$$\therefore \quad \frac{dy}{dx} = -\frac{y^{\frac{1}{3}}}{x^{\frac{1}{3}}}$$

Let us take (X, Y) as the current coordinates. Then, the wquation of tangent at any point (x, y) is

$$Y - y = -\frac{y^{\frac{1}{3}}}{x^{\frac{1}{3}}}(X - x)$$

or, $Yy^{-\frac{1}{3}} - y^{\frac{2}{3}} = -Xx^{-\frac{1}{3}} + x^{\frac{2}{3}}$

$$Xx^{-\frac{1}{3}} + Yy^{-\frac{1}{3}} = x^{\frac{2}{3}} + y^{\frac{2}{3}}$$

$$Xx^{-\frac{1}{3}} + Yy^{-\frac{1}{3}} = a^{\frac{2}{3}}$$

Now, if p is the length of perpendicular form the origin o the tangent, then,

$$p = \frac{0 \cdot x^{-\frac{1}{3}} + y^{-\frac{1}{3}} - a^{\frac{2}{3}}}{\sqrt{x^{-\frac{1}{3}} + y^{-\frac{2}{3}}}} = \frac{-a^{\frac{2}{3}} x^{\frac{1}{3}} y^{\frac{1}{3}}}{\sqrt{x^{\frac{2}{3}} + y^{\frac{2}{3}}}}$$

$$= \frac{-a^{\frac{2}{3}} x^{\frac{1}{3}} y^{\frac{1}{3}}}{a^{\frac{1}{3}}} = a^{\frac{1}{3}} x^{\frac{1}{3}} y^{\frac{1}{3}} \text{ (Numerically)}$$

Now,

$$r^2 + 3p^2 = x^2 + y^2 + 3a^{\frac{2}{3}} x^{\frac{2}{3}} y^{\frac{2}{3}}$$

$$= \left(x^{\frac{2}{3}}\right)^3 + \left(y^{\frac{2}{3}}\right)^3 + 3x^{\frac{2}{3}} y^{\frac{2}{3}} \left(x^{\frac{2}{3}} + y^{\frac{2}{3}}\right)$$

$$= \left(x^{\frac{2}{3}} + y^{\frac{2}{3}}\right)^3$$

But,

$$a^{\frac{2}{3}} = x^{\frac{2}{3}} + y^{\frac{2}{3}}$$

or, $r^2 + 3p^2 = \left(a^{\frac{2}{3}}\right)^3 = a^2$

$\therefore$ $\boxed{r^2 + 3p^2 = a^2}$ which is required pedal equation.

15. **Find the pedal equation of the following curves.**

i) $\mathbf{r = a(1 - \cos\theta)}$

Sol^n: We have,

$$r = a(1 - \cos\theta) \ldots \text{(i)}$$

Differentiating (i) with respect to θ, we get,

$$\frac{dr}{d\theta} = a\sin\theta$$

Now,

$$\tan\phi = r\frac{d\theta}{dr} = \frac{r}{\frac{dr}{d\theta}} = \frac{r}{a\sin\theta} = \frac{a(1-\cos\theta)}{a\sin\theta}$$

$$= \frac{2\sin^2\frac{\theta}{2}}{2a\sin\frac{\theta}{2}\cos\frac{\theta}{2}} = \tan\frac{\theta}{2}$$

$\therefore$ $\phi = \frac{\theta}{2}$

Also, we know,

$$p = r\sin\phi$$

or, $p = r\sin\frac{\theta}{2}$

or, $p^2 = r^2\sin^2\frac{\theta}{2} \ldots$ (ii)

But,

$$r = a(1 - \cos\theta)$$

or, $r = a \cdot 2\sin^2\frac{\theta}{2}$

$\therefore$ $\sin^2\frac{\theta}{2} = \frac{r}{2a} \ldots$ (iii)

Then from equation (ii), we have,

$\therefore$ $p^2 = r^2\frac{r}{2a}$

$\therefore$ $\boxed{r^3 = 2ap^2}$

ii) $\mathbf{r = a(1 + \cos\theta)}$

Sol^n: We have,

$$r = a(1 + \cos\theta) \ldots \text{(i)}$$

Differentiating (i) with respect to θ, we get,

$$\frac{dr}{d\theta} = a(1 - \sin\theta) = -a\sin\theta$$

Now,

$$\tan\phi = r\frac{d\theta}{dr} = \frac{r}{\frac{dr}{d\theta}} = \frac{r}{-a\sin\theta} = \frac{a(1+\cos\theta)}{-a\sin\theta}$$

$$= -a\frac{2\cos^2\frac{\theta}{2}}{2a\sin\frac{\theta}{2}\cos\frac{\theta}{2}} = -\cot\frac{\theta}{2}$$

$\therefore$ $\tan\phi = \tan\left(\frac{\pi}{2} + \frac{\theta}{2}\right)$

$\therefore$ $\phi = \frac{\pi}{2} + \frac{\theta}{2}$

Also, we know,

$$p = r\sin\phi$$

or, $p = r\sin\left(\frac{\pi}{2} + \frac{\theta}{2}\right)$

or, $p = r\cos\frac{\theta}{2}$

or, $p^2 = r^2\cos^2\frac{\theta}{2} \ldots$ (ii)

But,

$$r = a(1+\cos\theta)$$

$$\therefore \quad r = a \cdot 2\cos^2\frac{\theta}{2}$$

$$\therefore \quad \cos^2\frac{\theta}{2} = \frac{r}{2a} \dots \text{(iii)}$$

Then from equation (ii), we have,

$$\therefore \quad p^2 = r^2\frac{r}{2a}$$

$$\therefore \quad \boxed{r^3 = 2ap^2}$$

iii) $\mathbf{r^2 = a^2\cos 2\theta}$

Sol^n: We have,

$$r^2 = a^2\cos 2\theta$$

or, $2\log r = \log a^2 + \log\cos 2\theta$

or, $2\left(\frac{1}{r}\right)\frac{dr}{d\theta} = -\frac{2\sin 2\theta}{\cos 2\theta}$

$$\tan\phi = -\cot 2\theta$$

or, $\tan\phi = \tan\left(\frac{\pi}{2}+2\theta\right)$

or, $\phi = \frac{\pi}{2}+2\theta$

Also,

$$p = r\sin\phi$$

or, $p = r\sin\left(\frac{\pi}{2}+2\theta\right)$

or, $p = r\cos 2\theta$

or, $p = r\cdot\frac{r^2}{a^2}$

or, $r^3 = pa^2$ is the pedal equation

iv) $\mathbf{r^2\cos 2\theta = a^2}$

Sol^n: We have,

$$r^2\cos 2\theta = a^2$$

or, $r^2 = a^2\sec^2 2\theta \dots \text{(i)}$

Differentiating of (i) with respect to θ, we get,

$$2r\frac{dr}{d\theta} = a^2\sec 2\theta\tan 2\theta \cdot 2$$

$$\therefore \quad \frac{dr}{d\theta} = \frac{a^2\sec 2\theta\tan 2\theta}{2}$$

But,

$$\tan\phi = r\frac{d\theta}{dr} = \frac{r}{\frac{dr}{d\theta}} = \frac{r}{\frac{a^2\sec 2\theta\tan 2\theta}{r}} = \frac{r^2}{a^2\sec 2\theta\tan 2\theta}$$

But,

$$r^2 = a^2\sec 2\theta$$

$$\therefore \quad \tan\phi = \frac{r^2}{r^2\tan 2\theta} = \cot 2\theta = \tan\left(\frac{\pi}{2}-2\theta\right)$$

$$\therefore \quad \phi = \frac{\pi}{2}-2\theta$$

Again,

$p = r\sin\phi$ gives

$$p = r\sin\left(\frac{\pi}{2}-2\theta\right)$$

$$-p = r\cos 2\theta$$

But,

$$r^2\cos 2\theta = a^2$$

or, $\cos 2\theta = \frac{a^2}{r^2}$

$$\therefore \quad p = r\cdot\frac{a^2}{r^2}$$

$\therefore \quad \boxed{pr = a^2}$ which is required pedal equation.

v) $\mathbf{r^m = a^m\cos m\theta}$

Sol^n: We have,

$$r^m = a^m\cos m\theta \dots \text{(i)}$$

Taking log on both sides of (i) we get,

$$m\log r = m\log a + \log\cos m\theta$$

Differentiating both sides with respect to θ, we get,

$$m\cdot\frac{1}{r}\frac{dr}{d\theta} = \frac{1}{\cos m\theta}(-\cos m\theta)$$

or, $m\cdot\frac{1}{r}\frac{dr}{d\theta} = -m\tan m\theta$

or, $r\frac{d\theta}{dr} = -\cot m\theta$

But we know,

$$\tan\phi = r\frac{d\theta}{dr} = -\cot m\theta$$

or, $\tan\phi = \tan\left(\frac{\pi}{2} + m\theta\right)$

$\therefore \quad \phi = \frac{\pi}{2} + m\theta$

Again, from

$p = r\sin\phi$ gives

$p = r\sin\left(\frac{\pi}{2} + m\theta\right)$

or, $p = r\cos m\theta \ldots (*)$

But,

$r^m = a^m \cos m\theta$

$\therefore \quad \cos m\theta = \frac{r^m}{a^m}$

Then from (*), we have,

$p = r \cdot \frac{r^m}{a^m}$

or, $r^{m+1} = a^m p$ which is required pedal equation.

vi) $\mathbf{r = ae^{\theta\cot\alpha}}$

Sol^n: We have,

$r = ae^{\theta\cot\alpha} \ldots (i)$

Differentiating both sides, with respect to θ, we get,

$\frac{dr}{d\theta} = ae^{\theta\cot\alpha}\cot\alpha$

$\therefore \quad \frac{dr}{d\theta} = r\cot\alpha$

Now,

$\tan\phi = r\frac{d\theta}{dr} = \frac{r}{r\cot\alpha} = \tan\alpha$

$\therefore \quad \phi = \alpha$

Again, we know,

$p = r\sin\phi$

or, $p = r\sin\alpha$ which is required pedal equation.

Exercise 7

Indefinite integrals:

Definition:

Let f(x) be differentiable function and its derivative with respect to x is F(x).

i.e. $\frac{d}{dx}[f(x)] = F(x)$ then the integration of F(x) with respect to x is f(x).

i.e. $\int F(x)dx = f(x)$

Also, we have,

$\frac{d}{dx}[f(x) + C] = F(x)$

$\therefore \quad \int F(x)dx = f(x) + C$

Where,

C is called constant of integration

The constant of integration is usually omitted in general practice but it is always kept in mind that this constant exists in every case of integration. Thus, the integral $\int F(x)dx$ is called indefinite integral of F(x) with respect to x.

Some important formula:

1. $\int x^n dx = \frac{x^{n+1}}{n+1} (n \neq -1)$
2. $\int \frac{1}{x}dx = \log x$
3. $\int e^x dx = e^x$
4. $\int a^x dx = \frac{a^x}{\log a}$
5. $\int \sin x\,dx = -\cos x$
6. $\int \cos x\,dx = \sin x$
7. $\int \tan x\,dx = \log\sec x$
8. $\int \cot x\,dx = \log\sin x$

9. $\int \sec x\,dx = \log(\sec x + \tan x) = \log\tan\left(\frac{x}{2} + \frac{\pi}{4}\right)$

10. $\int \operatorname{cosec} x\,dx = \log(\operatorname{cosec} x - \cot x) = \log\tan\left(\frac{x}{2}\right)$

11. $\int \sec^2 x\,dx = \tan x$

12. $\int \operatorname{cosec}^2 x\,dx = -\cot x$

13. $\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}$

14. $\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\log\frac{a+x}{a-x}$

15. $\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\log\frac{x-a}{a+x}$

16. $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a}$

17. $\int \frac{dx}{\sqrt{a^2+x^2}} = \log\left(x+\sqrt{a^2+x^2}\right) = \sin h^{-1}\frac{x}{a}$

18. $\int \frac{dx}{\sqrt{x^2-a^2}} = \log\left(x+\sqrt{x^2-a^2}\right) = \cos h^{-1}\frac{x}{a}$

19. $\int uv\,dx = u\int v\,dx - \int\left(\frac{du}{dx}\cdot\int v\,dx\right)dx$ (integration by parts)

20. $\int \sqrt{a^2+x^2}\,dx = \frac{x\sqrt{a^2+x^2}}{2} + \frac{a^2}{2}\log\left(x+\sqrt{a^2+x^2}\right)$

21. $\int \sqrt{a^2-x^2}\,dx = \frac{x\sqrt{a^2-x^2}}{2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}$

22. $\int \sqrt{x^2-a^2}\,dx = \frac{x\sqrt{x^2-a^2}}{2} - \frac{a^2}{2}\log\left(x+\sqrt{x^2-a^2}\right)$

23. $\int e^{ax}\sin bx\,dx = \frac{e^{ax}(a\sin bx - b\cos bx)}{a^2+b^2}$

24. $\int e^{ax}\cos bx\,dx = \frac{e^{ax}(a\cos bx + b\sin bx)}{a^2+b^2}$

25. $\int_{-a}^{a} f(x) = 2\int_0^a f(x)dx$, if $f(x)$ is even function
$= 0$, if $f(x)$ is odd function.

Integrate the following:

1. $\int \frac{3x^2}{1+x^6}dx$

Sol[n]: Let,

$$I = \int \frac{3x^2}{1+x^6}dx$$

$$I = \int \frac{3x^2}{1^2+(x^3)^2}dx \ldots \text{(i)}$$

Put $x^3 = t$

$3x^2dx = dt$

Given from (i),

$$I = \int \frac{dt}{1+t^2}dt = \tan^{-1}t = \tan^{-1}(x^3)$$

2. $\int \frac{dx}{e^x+e^{-x}}$

Sol[n]: Let,

$$I = \int \frac{dx}{e^x+e^{-x}} = \int \frac{e^x\,dx}{1+e^{2x}}$$

Put $e^x = t$

$e^x dx = dt$

$$\therefore \quad I = \int \frac{dt}{1+t^2} = \tan^{-1}t = \tan^{-1}(e^x)$$

3. $\int \frac{dx}{e^x+1}$

Sol[n]: Let,

$$I = \int \frac{dx}{e^x+1} = \int \frac{e^{-x}}{e^{-x}(e^x+1)}dx = \int \frac{e^{-x}}{e^{-x}+1}dx$$

$$= -\int \frac{-e^{-x}\,dx}{e^{-x}+1}dx = -\log(e^{-x}+1) \quad \left[\because \int \frac{f'(x)}{f(x)}dx = \log f(x)\right]$$

4. $\int \sqrt{\frac{a+x}{a-x}}dx$

Sol[n]: Let,

$$I = \int \sqrt{\frac{a+x}{a-x}}dx$$

Put $x = a\cos\theta$, $dx = -a\sin\theta\,d\theta$

$$I = -\int \sqrt{\frac{a+a\cos\theta}{a-a\cos\theta}}\ a\sin\theta\, d\theta$$

$$= -\int \sqrt{\frac{(1+\cos\theta)}{(1-\cos\theta)}}\ a\sin\theta\, d\theta$$

$$= -\int \sqrt{\frac{2\cos^2\left(\frac{\theta}{2}\right)}{2\sin^2\left(\frac{\theta}{2}\right)}}\ 2\theta\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right) d\theta$$

$$= -a\int 2\theta\cos^2\left(\frac{\theta}{2}\right) d\theta = -a\int(1+\cos\theta)\,d\theta$$

$$= -a\theta - a\sin\theta = -a\cos^{-1}\frac{x}{a} - a\sqrt{1-\cos^2\theta}$$

$$= -a\cos^{-1}\frac{x}{a} - a\sqrt{1-\frac{x^2}{a^2}}$$

$$= -a\cos^{-1}\frac{x}{a} - a\sqrt{a^2-x^2}$$

6. $\displaystyle\int \frac{dx}{x\sqrt{x^4-1}}$

Sol^n^: Let,

$$I = \int \frac{dx}{x\sqrt{x^4-1}} = \int \frac{x}{x^2\sqrt{x^4-1}}\,dx$$

Put $x^2 = \sec\theta$, $2x\,dx = \sec\theta\tan\theta\, d\theta$

$$I = \frac{1}{2}\int \frac{\sec\theta\tan\theta}{\sec\theta\sqrt{\sec^2\theta-1}}\,d\theta$$

$$= \frac{1}{2}\int \frac{\tan\theta}{\tan\theta}\,d\theta = \frac{1}{2}d\theta = \frac{1}{2}\theta$$

$$= \frac{1}{2}\sec^{-1}x^2$$

7. $\displaystyle\int \frac{(x+1)dx}{\sqrt{4+8x-5x^2}}$

Sol^n^: Let,

$$I = \int \frac{(x+1)dx}{\sqrt{4+8x-5x^2}} = \int \frac{-\frac{1}{10}(8-10x)+1+\frac{4}{5}}{\sqrt{4+8x-5x^2}}\,dx$$

$$= -\frac{1}{10}\int \frac{(8-10x)\,dx}{\sqrt{4+8x-5x^2}} + \frac{9}{5}\int \frac{dx}{\sqrt{4+8x-5x^2}}$$

$$= -\frac{1}{10}I_1 + \frac{9}{5}I_2$$

Where,

$$I_1 = \int \frac{(8-10x)\,dx}{\sqrt{4+8x-5x^2}}$$

Put $4+8x-5x^2 = t^2$ in (I) $(8-10x)\,dx = 2t\,dt$

Thus,

$$I_1 = \int \frac{2t\,dt}{t} = 2\int dt = 2\sqrt{4+8x-5x^2}$$

Now,

$$I_2 = \int \frac{dx}{\sqrt{4+8x-5x^2}} = \int \frac{dx}{\sqrt{-5\left(x^2-\frac{8}{5}x-\frac{4}{5}\right)}}$$

$$= \frac{1}{\sqrt{5}}\int \frac{dx}{\sqrt{-\left\{(x)^2-2x\frac{4}{5}+\left(\frac{4}{5}\right)^2-\frac{4}{5}-\frac{16}{25}\right\}}}$$

$$= \frac{1}{\sqrt{5}}\int \frac{dx}{\sqrt{-\left\{\left(x-\frac{4}{5}\right)^2-\frac{36}{25}\right\}}} = \frac{1}{\sqrt{5}}\int \frac{dx}{\sqrt{\left(\frac{6}{5}\right)^2-\left(x-\frac{4}{5}\right)^2}}$$

$$= \frac{1}{\sqrt{5}}\sin^{-1}\frac{(5x-4)}{6}$$

So,

$$I = -\frac{1}{10}I_1 + \frac{9}{5}I_2$$

$$\therefore\ I = -\frac{1}{5}\sqrt{4+8x-5x^2} + \frac{9}{5\sqrt{5}}\sin^{-1}\frac{(5x-4)}{6}$$

$$= \frac{9}{5\sqrt{5}}\sin^{-1}\frac{(5x-4)}{6} - \frac{1}{5}\sqrt{4+8x-5x^2}$$

8. $\displaystyle\int \sqrt{\frac{1+x}{1-x}}\,dx$ (2060 B.E.)

Sol^n^: Let,

$$I = \int \sqrt{\frac{1+x}{1-x}}\,dx = \int \frac{1+x}{\sqrt{1-x^2}}\,dx = \int \frac{dx}{\sqrt{1-x^2}} + \int \frac{x\,dx}{\sqrt{1-x^2}}$$

$$= \sin^{-1}x + \int \frac{x}{\sqrt{1-x^2}}\,dx$$

Put $1-x^2=t^2$,

$-2x\,dx = 2t\,dt$

$$= \sin^{-1}x - \int \frac{t\,dt}{t} = \sin^{-1}x - \int dt = \sin^{-1}x - t$$

$$= \sin^{-1}x - \sqrt{1-x^2}$$

9. $\int \sqrt{18x-65-x^2}\,dx$

Sol^n: Let,

$$I = \int \sqrt{18x-65-x^2}\,dx$$

$$= \int \sqrt{-(x^2-18x+65)}\,dx$$

$$= \int \sqrt{-\{(x)^2-2.x.9+(9)^2+65-81\}}\,dx$$

$$= \int \sqrt{-\{(x)^2-2.x.9+(9)^2+65-81\}}\,dx$$

$$= \int \sqrt{-\{(x-9)^2-16\}}\,dx$$

$$= \int \sqrt{\{(9)^2-(x-9)^2\}}\,dx$$

$$= \int \sqrt{(4x)^2-(x-9)^2}\,dx$$

$$= \frac{(x-9)\sqrt{(4)^2-(x-9)^2}}{2} + \frac{16}{2}\sin^{-1}\frac{(x-9)}{4}$$

$$= \frac{(x-9)}{2}\sqrt{18x-65-x^2} + 8\sin^{-1}\frac{(x-9)}{4}$$

10. $\int \frac{\cos x\,dx}{\sqrt{2\sin^2 x+3\sin x+4}}$

Sol^n: Let,

$$I = \int \frac{\cos x\,dx}{\sqrt{2\sin^2 x+3\sin x+4}}$$

Put $\sin x = t, \cos x\,dx = dt$

$$\therefore \quad I = \int \frac{dt}{\sqrt{2t^2+3t+4}} = \frac{1}{\sqrt{2}} = \int \frac{dt}{\sqrt{t^2+\frac{3}{2}t+2}}$$

$$= \frac{1}{\sqrt{2}}\int \frac{dt}{\sqrt{(t)^2+2.t.\frac{3}{4}t+\left(\frac{3}{4}\right)^2+2-\frac{9}{16}}} = \frac{1}{\sqrt{2}}\int \frac{dt}{\sqrt{\left(t+\frac{3}{4}\right)^2+\left(\frac{\sqrt{23}}{4}\right)^2}}$$

$$= \frac{1}{\sqrt{2}}\sin h^{-1}\left(\frac{t+\frac{3}{4}}{\frac{\sqrt{23}}{4}}\right) = \frac{1}{\sqrt{2}}\sin h^{-1}\left(\frac{4t+3}{\sqrt{23}}\right)$$

$$= \frac{1}{\sqrt{2}}\sin h^{-1}\left(\frac{4\sin x+3}{\sqrt{23}}\right)$$

11. $\int \frac{dx}{\sqrt{1+\sin x}}$

Sol^n: Let,

$$I = \int \frac{dx}{\sqrt{1+\sin x}}$$

$$= \int \frac{dx}{\sqrt{\sin^2\left(\frac{x}{2}\right)+\cos^2\left(\frac{x}{2}\right)+2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)}} = \int \frac{dx}{\sin\left(\frac{x}{2}\right)+\cos\left(\frac{x}{2}\right)}$$

$$= \frac{1}{\sqrt{2}}\int \frac{dx}{\left(\frac{1}{\sqrt{2}}\sin\frac{x}{2}+\frac{1}{\sqrt{2}}\cos\frac{x}{2}\right)} = \frac{1}{\sqrt{2}}\int \frac{dx}{\left(\sin\frac{x}{2}\cos\frac{\pi}{4}+\cos\frac{x}{2}\sin\frac{\pi}{4}\right)}$$

$$= \frac{1}{\sqrt{2}}\int \frac{dx}{\sin\left(\frac{x}{2}+\frac{\pi}{4}\right)} = \frac{1}{\sqrt{2}}\int \operatorname{cosec}\left(\frac{x}{2}+\frac{\pi}{4}\right)dx$$

$$= \frac{1\times 2}{\sqrt{2}}\log\left\{\tan\left(\frac{x}{2}+\frac{\pi}{4}\right)\right\} = \sqrt{2}\log\tan\left(\frac{x}{4}+\frac{\pi}{8}\right)$$

12. $\int \frac{dx}{\sqrt{4+5\cos x}}$

Sol^n: Let,

$$I = \int \frac{dx}{\sqrt{4+5\cos x}}$$

$$= \int \frac{dx}{4+5\cos^2\left(\frac{x}{2}\right)-5\sin^2\left(\frac{x}{2}\right)} = \int \frac{\sec^2\left(\frac{x}{2}\right)dx}{4\sec^2\left(\frac{x}{2}\right)+5-5\tan^2\left(\frac{x}{2}\right)}$$

$$= \int \frac{\sec^2\left(\frac{x}{2}\right)dx}{4+4\tan^2\left(\frac{x}{2}\right)+5-5\tan^2\left(\frac{x}{2}\right)} = \int \frac{\sec^2\frac{x}{2}\,dx}{9-\tan^2\frac{x}{2}}$$

Put $\tan\left(\frac{x}{2}\right) = t, \sec^2\left(\frac{x}{2}\right) = 2\,dt$

$$\therefore \quad I = \frac{\sqrt{a^2-b^2}}{a\left(a^2-b^2\right)}\int \frac{dt}{1+t^2} = \frac{1}{a\left(a^2-b^2\right)}\tan^{-1} t$$

$$= \frac{1}{a\left(a^2-b^2\right)}\tan^{-1}\left(\frac{a\tan x}{\sqrt{a^2-b^2}}\right)$$

18. $\int \sqrt{\sec x - 1}\, dx$

Sol^n: Let,

$$I = \int \sqrt{\sec x - 1}\, dx = \int \sqrt{\frac{1}{\cos x} - 1}\, dx = \int \sqrt{\frac{1-\cos x}{\cos x}}\, dx$$

$$= \int \sqrt{\frac{2\sin^2\frac{x}{2}}{2\cos^2\frac{x}{2}-1}}\, dx = \sqrt{2}\int \frac{\sin\frac{x}{2}\, dx}{\sqrt{2\cos^2\frac{x}{2}-1}}$$

Put $\sqrt{2}\cos\frac{x}{2} = t$

$$-\frac{\sqrt{2}}{2}\sin\frac{x}{2} = dt$$

$$\therefore \quad I = -2\int \frac{dt}{\sqrt{t^2-1}} = -2\cosh^{-1} t = -2\cosh^{-1}\left(\sqrt{2}\cos\frac{x}{2}\right)$$

19. $\int \frac{\cos x\, dx}{5-3\cos x}$

Sol^n: Let,

$$I = \int \frac{\cos x\, dx}{5-3\cos x} = \int \frac{-\frac{1}{3}(5-3\cos x)+\frac{5}{3}}{5-3\cos x}\, dx$$

$$\therefore \quad I = -\frac{1}{3}\int dx + \frac{5}{3}\int \frac{dx}{5-3\cos x}$$

$$= -\frac{1}{3}x + \frac{5}{3}\int \frac{dx}{5-6\cos^2\frac{x}{2}+3}$$

$$= -\frac{1}{3}x + \frac{5}{3}\int \frac{\sec^2\frac{x}{2}}{8\sec^2\frac{x}{2}-6}\, dx - \frac{x}{3} + \frac{5}{3}\int \frac{\sec\frac{x}{2}}{8+8\tan^2\frac{x}{2}-6}$$

$$= -\frac{x}{3} + \frac{5}{3}\int \frac{\sec^2\frac{x}{2}\, dx}{2+8\tan^2\frac{x}{2}} = -\frac{x}{3} + \frac{5}{6}\int \frac{\sec^2\frac{x}{2}\, dx}{1+4\tan^2\frac{x}{2}}$$

Put $2\tan\frac{x}{2} = t, \sec^2\frac{x}{2}\, dx = dt$

$$= -\frac{x}{3} + \frac{5}{6}\int \frac{dt}{1+t^2} = -\frac{x}{3} + \frac{5}{6}\tan^{-1} t$$

$$= -\frac{x}{3} + \frac{5}{6}\tan^{-1}\left(2\tan\frac{x}{2}\right)$$

20. $\int \frac{dx}{1-\cos x+\sin x}$

Sol^n: Let,

$$I = \int \frac{dx}{1-\cos x+\sin x}$$

$$= \int \frac{dx}{2\sin^2\frac{x}{2}+2\sin\frac{x}{2}\cos\frac{x}{2}} = \int \frac{\sec^2\frac{x}{2}\, dx}{2\tan^2\frac{x}{2}+2\tan\frac{x}{2}}$$

$$= \int \frac{\frac{1}{2}\sec^2\frac{x}{2}\, dx}{\tan^2\frac{x}{2}+\tan\frac{x}{2}}$$

Put $\tan\frac{x}{2} = t, \frac{1}{2}\sec^2\frac{x}{2}\, dx = dt$

$$I = \int \frac{dt}{t^2+t}$$

$$= \int \frac{dt}{t(t+1)}$$

$$= \int \left(\frac{1}{t} - \frac{1}{t+1}\right) dt = \int \log t - \log(t+1)$$

$$= \int \log\frac{t}{t+1} = \int \log\frac{\tan\frac{x}{2}}{1+\tan\frac{x}{2}} = -\log\left(1+\cot\frac{x}{2}\right)$$

21. $\int \frac{5\cos x+6}{2\cos x+\sin x+3}\, dx$

Sol^n: Let,

$$I = \int \frac{5\cos x+6}{2\cos x+\sin x+3}\, dx$$

$$= \int \frac{2(2\cos x+\sin x+3)+\cos x-2\sin x}{2\cos x+\sin x+3}\, dx$$

$$= 2\int dx + \int \frac{\cos x-2\sin x}{2\cos x+\sin x+3}\, dx$$

$$= 2x + \log(2\cos x+\sin x+3)$$

22. $\displaystyle\int \frac{(x+\sin x)}{1+\cos x}dx$

Sol^n: Let,

$$I=\int \frac{(x+\sin x)}{1+\cos x}dx=\int \frac{x}{1+\cos x}dx+\int \frac{\sin x}{1+\cos x}dx$$

$$=\int \frac{x}{2\cos^2\frac{x}{2}}dx+\int \frac{2\sin\frac{x}{2}\cos\frac{x}{2}\,dx}{2\cos^2\frac{x}{2}}$$

$$=\frac{1}{2}\int x\sec^2\frac{x}{2}+\int \tan\frac{x}{2}\,dx$$

Integrating by parts to the first integrate

$$=\frac{1}{2}x.2\tan\frac{x}{2}-\frac{1}{2}\int 1\times 2\tan\frac{x}{2}\,dx+\int \tan\frac{x}{2}\,dx$$

$$=x\tan\frac{x}{2}$$

23. $\displaystyle\int \frac{\cos x}{2\sin x+3\cos x}dx$

Sol^n: Let,

$$I=\int \frac{\cos x}{2\sin x+3\cos x}dx$$

Here,

$\cos x = l$ (denominator) + m(diff^n coeff^n of denominator)

$\therefore\ \cos x = l(2\sin x+3\cos x)+m(2\cos x-3\sin x)$

Equating the coefficients of like terms,

$1=3l+2m,\ 0=2l-3m$

Solving these,

$$l=\frac{3}{13},\ m=\frac{2}{13}$$

$$\therefore\ I=\int \frac{l(2\sin x+3\cos x)+m(2\cos x-3\sin x)}{2\sin x+3\cos x}dx$$

$$=\frac{3}{13}\int dx+\frac{2}{13}\int \frac{2\cos x-3\sin x}{2\sin x+3\cos x}dx$$

$$=\frac{3}{13}x+\frac{2}{13}\log(2\sin x+3\cos x)$$

24. $\displaystyle\int \frac{e^x(1+\sin x)}{1+\cos x}dx$

Sol^n: Let,

$$I=\int \frac{e^x(1+\sin x)}{1+\cos x}dx$$

$$=\int \frac{e^x}{1+\cos x}dx+\int \frac{e^x\sin x}{1+\cos x}dx$$

$$=\int \frac{e^x}{2\cos^2\frac{x}{2}}dx+\int \frac{e^x.\ 2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)dx}{2\cos^2\left(\frac{x}{2}\right)}dx$$

$$\therefore\ I=\int \frac{1}{2}\sec^2\frac{x}{2}\,e^x\,dx+\int e^x\tan\frac{x}{2}dx$$

Integrating by parts to the first integration,

$$=e^x\tan\left(\frac{x}{2}\right)-\int e^x\tan\frac{x}{2}dx+\int e^x\tan\left(\frac{x}{2}\right)dx$$

$$=e^x\tan\left(\frac{x}{2}\right)$$

25. $\displaystyle\int \frac{dx}{\sin x\,(3+2\cos x)}$ [2057, 2062 B.E.]

Sol^n: Let,

$$I=\int \frac{dx}{\sin x\,(3+2\cos x)}=\int \frac{\sin x\,dx}{(1-\cos^2 x)(3+2\cos x)}$$

Put $\cos x=t,\ -\sin x\ dx=dt$

$$\therefore I=\int \frac{dt}{(1-t^2)(3+2t)}=\int \frac{dt}{(1+t)(1-t)(3+2t)}$$

Now,

$$\frac{1}{(1+t)(1-t)(3+2t)}=\frac{A}{1+t}+\frac{B}{1-t}+\frac{C}{3+2t}$$

$$1=A(1-t)(3+2t)+B(1+t)(3+2t)+C(1+t)(1-t)$$

Put, $t=-1,\ 1=2A$

$\therefore A=-\frac{1}{2}$

Put, $t=1,\ 1=10B$

$\therefore\ B=\frac{1}{10}$

Put, $t=0,\ 1=-3A+3B-C$

$\therefore\ B=\frac{4}{5}$

$$I=-\int \frac{1}{2(1+t)}dt+\int \frac{1}{10(1-t)}dt+\frac{4}{5}\int \frac{dt}{(3+2t)}$$

$$= \int \frac{\sin x\, dx}{\sin^2 x\,(1+\cos x)} + \int \frac{dx}{2\cos^2 \frac{x}{2}}$$

$$= \int \frac{\sin x\, dx}{(1-\cos^2 x)\,(1+\cos x)} + \frac{1}{2}\sec^2 \frac{x}{2}\, dx$$

Put $\cos x = t$ in the first integral,

$-\sin x\; dx = dt$

$$\therefore \quad I = \int \frac{dt}{(1-t^2)\,(1+t)} + \tan\frac{x}{2}$$

Now,

$$\frac{1}{(1-t)\,(1+t)^2} = \frac{A}{1-t} + \frac{B}{1+t} + \frac{C}{(1+t)^2}$$

or, $1 = A(1+t)^2 + B(1+t)(1-t) + C(1-t)$

Put $t = 0, 1 = A + B + C$

Put $t = -1, 1 = 2C$

$\therefore \quad C = \frac{1}{2}$

Put $t = 1, 1 = 4A$

$\therefore \quad A = \frac{1}{4}$ and $B = \frac{1}{4}$

$$I = -\frac{1}{4}\int \frac{dt}{1-t} - \frac{1}{4}\int \frac{dt}{1+t} - \frac{1}{2}\int \frac{dt}{(1+t)^2} + \tan\frac{x}{2}$$

$$= \frac{1}{4}\log(1-t) - \frac{1}{4}\log(1+t) + \frac{1}{2(1+t)} + \tan\frac{x}{2}$$

$$= \frac{1}{4}\log\frac{1-t}{1+t} + \frac{1}{2(1+t)} + \tan\frac{x}{2}$$

$$= \frac{1}{4}\log\frac{1-\cos x}{1+\cos x} + \frac{1}{2(1+\cos x)} + \tan\frac{x}{2}$$

$$= \frac{1}{2}\log\left(\tan\frac{x}{2}\right) + \frac{1}{4}\sec^2\frac{x}{2} + \tan\frac{x}{2}$$

30. $\displaystyle \int \sqrt{\frac{\sin(x-\alpha)}{\sin(x+\alpha)}}\,dx$

***Sol^n*:** Let,

$$I = \int \sqrt{\frac{\sin(x-\alpha)}{\sin(x+\alpha)}}\,dx$$

$$= \int \sqrt{\frac{\sin x\cos\alpha - \cos x\sin\alpha}{\sin x\cos\alpha + \cos x\sin\alpha}}\,dx$$

$$= \int \sqrt{\frac{(\sin x\cos\alpha - \cos x\sin\alpha)(\sin x\cos\alpha - \cos x\sin\alpha)}{(\sin x\cos\alpha + \cos x\sin\alpha)(\sin x\cos\alpha - \cos x\sin\alpha)}}\,dx$$

$$= \int \frac{(\sin x\cos\alpha - \cos x\sin\alpha)}{\sin^2 x\cos^2\alpha - \cos^2 x\sin^2\alpha}\,dx$$

$$= \int \frac{\sin x\cos\alpha\, dx}{\sqrt{\sin^2 x\cos^2\alpha - \cos^2 x\sin^2\alpha}} - \int \frac{\cos x\sin\alpha\, dx}{\sqrt{\sin^2 x\cos^2\alpha - \cos^2 x\sin^2\alpha}}$$

$$I = \cos\alpha \int \frac{\sin x\, dx}{\sqrt{(1-\cos^2 x)\cos^2\alpha - \cos^2 x\sin^2\alpha}}$$

$$- \sin\alpha \int \frac{\cos x\, dx}{\sqrt{\sin^2 x\cos^2\alpha - \left(1-\sin^2 x\right)\sin^2\alpha}}$$

$$= \cos\alpha \int \frac{\sin x\, dx}{\sqrt{\cos^2\alpha - \cos^2 x}} - \sin\alpha \int \frac{\cos x\, dx}{\sqrt{\sin^2 x - \sin^2\alpha}}$$

Put $\cos x = \cos\alpha\; u$ in the first integral

$-\sin x\; dx = \cos\alpha\; du$

Put $\sin x = \sin\alpha\, v$ in the second integral

$\cos x\; dx = \sin\alpha\; dv$

$$I = -\frac{\cos^2\alpha}{\cos^2\alpha}\int \frac{du}{\sqrt{1-u^2}} - \frac{\sin^2\alpha}{\sin\alpha}\int \frac{dv}{\sqrt{v^2-1}}$$

$$= \cos\alpha\cos^{-1}(u) - \sin\alpha\log\left(v + \sqrt{v^2-1}\right)$$

$$= \cos\alpha\cos^{-1}(\cos x\sec x) - \sin\alpha\log\left(\sin x + \sqrt{\sin^2 x - \sin^2\alpha}\right)$$

31. $\displaystyle \int \frac{\cos x\, dx}{(1+\sin x)(2+\sin x)}$

Sol^n: Let,

$$I = \int \frac{\cos x\, dx}{(1+\sin x)(2+\sin x)}$$

Put $\sin x = t$

Then,

$\cos x\; dx = dt$

$$\therefore \quad I = \int \frac{dt}{(1+t)(2+t)}$$

$$= \int \left(\frac{1}{1+t} - \frac{1}{2+t}\right) dt = \log(1+t) - \log(2+t)$$

$$= \log\left(\frac{1+t}{2+t}\right) = \log\left(\frac{1+\sin x}{2+\sin x}\right)$$

32. $\int \frac{\mathbf{dx}}{\mathbf{x+\sqrt{x}}}$

Sol^n: Let,

$$I = \int \frac{dx}{x+\sqrt{x}} = \int \frac{dx}{\sqrt{x}\,(1+\sqrt{x})}$$

Put $1+\sqrt{x} = t, \frac{1}{2\sqrt{x}} dx = dt$

$$= 2\int \frac{1}{t} dt = 2\log t = 2\log\left(1+\sqrt{x}\right)$$

33. $\int \frac{\mathbf{x^2\,dx}}{\mathbf{x^6-6x^3+5}}$

Sol^n: Let,

$$I = \int \frac{x^2\,dx}{x^6-6x^3+5} = \int \frac{x^2\,dx}{(x^3)^2-6x^3+5}$$

Put $x^3 = t$

$3x^2\,dx = dt$

$$= \frac{1}{3}\int \frac{1}{t^2-6t+5} dt = \frac{1}{3}\int \frac{1}{(t-5)(t-1)} dt$$

$$= \frac{1}{3}\cdot\frac{1}{4}\int \left(\frac{1}{t-5} - \frac{1}{t-1}\right) dt$$

$$= \frac{1}{12}[\log(t-5) - \log(t-1)]$$

$$= \frac{1}{12}\log\left(\frac{t-5}{t-1}\right) = \frac{1}{12}\log\left(\frac{x^3-5}{x^3-1}\right)$$

34. $\int \mathbf{x^2\sqrt{x^6-1}\,dx}$

Sol^n: Let,

$$I = \int x^2\sqrt{x^6-1}\,dx$$

Put $x^3 = t, 3x^2\,dx = dt$

$$= \frac{1}{3}\int \sqrt{t^2-1}\,dt$$

$$= \frac{1}{3}\left[\frac{t\sqrt{t^2-1}}{2} - \frac{1}{2}\log\left(t+\sqrt{t^2-1}\right)\right]$$

$$= \frac{1}{6}\left[x^3\sqrt{x^6-1} - \log\left(x^3\sqrt{x^6-1}\right)\right]$$

35. $\int \frac{\mathbf{dx}}{\mathbf{(x^2+a^2)(x^2+b^2)}}$

Sol^n: Let,

$$I = \int \frac{dx}{(x^2+a^2)(x^2+b^2)}$$

$$= \frac{1}{b^2-a^2}\int \left(\frac{1}{x^2+a^2} - \frac{1}{x^2+b^2}\right) dx$$

$$= \frac{1}{b^2-a^2}\left[\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) - \frac{1}{b}\tan^{-1}\left(\frac{x}{b}\right)\right]$$

36. $\int \frac{\mathbf{dx}}{\sqrt{\mathbf{(x-\alpha)(x-\beta)}}}\ \alpha > \beta$

Sol^n: Let,

$$I = \int \frac{dx}{(x-\alpha)(x-\beta)}\alpha > \beta$$

Put $x-\alpha = t^2$

Then,

$dx = 2t\,dt$

so, $I = \int \frac{2t\,dt}{t\sqrt{t^2+\alpha-\beta}}$

$$= \int \frac{2\,dt}{\sqrt{t^2+\alpha-\beta}} = \int \frac{2\,dt}{\sqrt{t^2+\left(\sqrt{\alpha-\beta}\right)^2}}$$

$$= 2\log\left(t+\sqrt{t^2+\alpha-\beta}\right)$$

$$= 2\log\left(\sqrt{x-\alpha}+\sqrt{x-\beta}\right)$$

37. $\displaystyle\int \frac{xe^x}{(x+1)^2}\,dx$ [2061 B[illegible]

Solⁿ: Let,

$$I = \int \frac{xe^x}{(x+1)^2}\,dx$$

$$= \int \frac{(x+1-1)e^x}{(x+1)^2}\,dx$$

$$= \int \frac{e^x}{x+1} - \int \frac{e^x}{(x+1)^2}\,dx$$

$$= \left(\frac{1}{x+1}e^x - \int \frac{-1}{(x+1)^2}e^x\,dx\right) - \int \frac{e^x}{(x+1)^2}\,dx$$

$$= \frac{e^x}{x+1}$$

38. $\displaystyle\int \frac{x^2\,dx}{(x+1)^2(x+2)}$

Solⁿ: Let,

$$I = \int \frac{x^2\,dx}{(x+1)^2(x+2)}$$

Now,

$$\frac{x^2\,dx}{(x+1)^2(x+2)} = \frac{A}{x+2} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$$

$\therefore \quad x^2 = A(x+1)^2 + B(x+1)(x+2) + C(x+2)$

Put $x = -1, 1 = C$

Put $x = -2, 4 = A$

Equating the coefficient of x^2 on both sides, we get,

$1 = A + B$

$\therefore \quad B = -3$

so, $\displaystyle I = \int \frac{4}{x+2}\,dx - \int \frac{3\,dx}{x+1} + \int \frac{1}{(x+1)^2}\,dx$

$$= 4\log(x+2) - 3\log(x+1) - \frac{1}{x+1}$$

$$= \frac{-1}{x+1} - 3\log(x+1) + 4\log(x+3)$$

39. $\displaystyle\int \frac{x^3\,dx}{x^4 - x^2 - 12}$

Solⁿ: Let,

$$I = \int \frac{x^3\,dx}{x^4 - x^2 - 12} = \int \frac{x^2.x\,dx}{x^4 - x^2 - 12}$$

Put $x^2 = t$,

$2x\,dx = dt$

$$= \frac{1}{2}\int \frac{t\,dt}{t^2 - t - 12} = \frac{1}{2}\int \frac{t\,dt}{(t-4)(t+3)}$$

$$= \frac{1}{2}\int \frac{t-4+4}{(t-4)(t+3)}\,dt$$

$$= \frac{1}{2}\int \left\{\frac{1}{t+3} + \frac{4}{(t-4)(t+3)}\right\}dt$$

$$= \frac{1}{2}\int \left\{\frac{1}{t+3} + \frac{4}{7}\left(\frac{1}{t-4} + \frac{1}{t+3}\right)\right\}dt$$

$$= \frac{1}{2}\int \left\{\frac{3}{t+3} + \frac{4}{7(t-4)}\right\}dt$$

$$= \frac{1}{2}\left\{3\log(t+3) + \frac{4}{7}\log(t-4)\right\}$$

$$= \frac{3}{2}\log(x^2+3) + \frac{2}{7}\log(x^2-4)$$

40. $\displaystyle\int \frac{dx}{1+x^4}$

Solⁿ:

$$= \frac{1}{2}\int \left(\frac{x^2+1}{1+x^4} - \frac{x^2-1}{1+x^4}\right)dx = \frac{1}{2}\int \frac{1+\frac{1}{x^2}}{x^2+\frac{1}{x^2}}\,dx - \frac{1}{2}\int \frac{1-\frac{1}{x^2}}{x^2+\frac{1}{x^2}}\,dx$$

$$= \frac{1}{2}(I_1 - I_2) \text{ say}$$

In I_1 put $x - \frac{1}{x} = t$, then $\left(1 + \frac{1}{x^2}\right)dx = dt$

$$\therefore \quad I_1 = \int \frac{dt}{\left(x - \frac{1}{x}\right)^2 + 2}$$

$$= \int \frac{dt}{t^2+2} = \frac{1}{\sqrt{2}}\tan^{-1}\frac{t}{\sqrt{2}}$$

$$= \frac{1}{\sqrt{2}}\tan^{-1}\frac{x^2-1}{\sqrt{2}x}$$

In I_2 put $x + \frac{1}{x} = z$, then $\left(1 - \frac{1}{x^2}\right) dx = dz$

$$\therefore \quad I_2 = \int \frac{dz}{\left(x+\frac{1}{x}\right)^2 - 2}$$

$$= \int \frac{dz}{z^2 - 2}$$

$$= \frac{1}{2\sqrt{2}} \log \frac{z - \sqrt{2}}{z + \sqrt{2}}$$

$$= \frac{1}{2\sqrt{2}} \log \frac{x^2 - \sqrt{2}x + 1}{x^2 + \sqrt{2}x + 1}$$

$$\therefore \quad I = \frac{1}{2\sqrt{2}} \tan^{-1} \frac{x^2 - 1}{\sqrt{2}x} - \frac{1}{4\sqrt{2}} \log \frac{x^2 - \sqrt{2}x + 1}{x^2 + \sqrt{2}x + 1}$$

Exercise 8

Definite integral:

The integral having the lower limit of x from $x = a$ to $x = b$ is defined by $\int_a^b f(x)dx$.

Now,

$$\int_a^b f(x)dx = [F(x) + c]_a^b = [F(b) + c - F(a) - c]$$

$$\therefore \quad \int_a^b f(x)dx = F(b) - F(a)$$

1. $\int_0^a \frac{x+a}{a^2+x^2} dx$

Sol^n: Here,

$$\int_0^a \frac{x\,dx}{a^2+x^2} + \int_0^a \frac{a\,dx}{a^2+x^2}$$

$$= \int_0^a \frac{1}{2} \frac{2x}{a^2+x^2} dx + a \int_0^a \frac{1}{a^2+x^2} dx$$

$$= \left[\frac{1}{2} \log(a^2 + x^2)\right]_0^a + a \left[\frac{1}{a} \tan^{-1} \frac{x}{a}\right]_0^a$$

$$= \frac{1}{2} [\log 2a^2 - \log a^2] + \tan^{-1} 1$$

$$= \frac{1}{2} \log \frac{2a^2}{a^2} + \frac{\pi}{4} = \frac{1}{2} \log 2 + \frac{\pi}{4}$$

2. $\int_0^1 \frac{\sin^{-1}x}{\sqrt{1-x^2}} dx$

Sol^n: Let,

$$I = \int_0^1 \frac{\sin^{-1}x}{\sqrt{1-x^2}} dx$$

Put $\sin^{-1}x = t$

$$\frac{1}{\sqrt{1-x^2}} dx = dt$$

7. $\displaystyle\int_0^1 \frac{dx}{(1+x^2)\sqrt{1-x^2}}$

Sol^n: Let,

$$I = \int_0^1 \frac{dx}{(1+x^2)\sqrt{1-x^2}}$$

Put $x = \tan\theta, dx = \sec^2\theta\, d\theta$

When $x = 0, \theta = 0$

When $x = 1, \theta = \frac{\pi}{4}$

$$\text{So, } I = \int_0^{\frac{\pi}{4}} \frac{\sec^2\theta\, d\theta}{(1+\tan^2\theta)\sqrt{1-\tan^2\theta}}$$

$$= \int_0^{\frac{\pi}{4}} \frac{\sec^2\theta\, d\theta}{(1+\tan^2\theta)\sqrt{1-\frac{\sin^2\theta}{\cos^2\theta}}} = \int_0^{\frac{\pi}{4}} \frac{\cos\theta\, d\theta}{\sqrt{\cos^2\theta - \sin^2\theta}}$$

$$= \int_0^{\frac{\pi}{4}} \frac{\cos\theta\, d\theta}{\sqrt{1-2\sin^2\theta}}$$

Put $\sqrt{2}\sin\theta = t,$

$\sqrt{2}\cos\theta\ d\theta = dt$

When $x = 0, \theta = 0$

When $x = \frac{1}{2}, \theta = \frac{\pi}{6}$

$$\text{So, } I = \frac{1}{\sqrt{2}}\int_0^1 \frac{dt}{\sqrt{1-t^2}} = \frac{1}{\sqrt{2}}\,\frac{\pi}{2} = \frac{\pi}{2\sqrt{2}}$$

8. $\displaystyle\int_0^{\frac{1}{2}} \frac{dx}{(1-2x^2)\sqrt{1-x^2}}$

Sol^n: Let,

$$I = \int_0^{\frac{1}{2}} \frac{dx}{(1-2x^2)\sqrt{1-x^2}}$$

Put $x = \sin\theta$

$dx = \cos\theta\ d\theta$

When $\theta = 0, t = 0$

When $\theta = \frac{1}{2}, t = \frac{\pi}{6}$

$$\therefore\quad I = \int_0^{\frac{\pi}{6}} \frac{\cos\theta\, d\theta}{\left(1+2\sin^2\theta\right)\sqrt{1-\cot\theta}}$$

$$= \int_0^{\frac{\pi}{6}} \frac{\cos\theta\, d\theta}{\cos 2\theta\cos\theta} = \int_0^{\frac{\pi}{6}} \sec 2\theta\ d\theta$$

$$= \frac{1}{2}[\log(\sec 2\theta + \tan 2\theta)]_0^{\frac{\pi}{6}}$$

$$= \frac{1}{2}\left[\log\left(\sec\left(2+\sqrt{3}\right) + \log 1\right)\right] = \frac{1}{2}\log\left(2+\sqrt{3}\right)$$

9. $\displaystyle\int_1^e \frac{dx}{x\sqrt{1-(\log x)^2}}$

Sol^n: Let,

$$I = \int_1^e \frac{dx}{x\sqrt{1-(\log x)^2}}$$

Put $\log x = t$

$\frac{1}{x}dx = dt$

When $x = 1, t = 0$

When $\theta = e, t = 1$

$$\therefore\quad I = \int_0^1 \frac{dt}{\sqrt{1-t^2}} = [\sin^{-1} t]_0^1 = \frac{\pi}{2}$$

10. $\displaystyle\int_0^1 \frac{1-x}{1+x}dx$ [2061 B.E.]

Sol^n: Let,

$$I = \int_0^1 \frac{1-x}{1+x}dx = \int_0^1 \frac{-(1+x)+2}{1+x}dx = -\int_0^1 dx + \int_0^1 \frac{2\,dx}{1+x}$$

$$= -[x]_0^1 + 2[\log(1+x)]_0^1 = -1 + 2\log 2$$

11. $\displaystyle\int_1^2 \frac{dx}{x(1+x^4)}$

Solⁿ: Let,

$$I = \int_1^2 \frac{dx}{x(1+x^4)} = \int_1^2 \frac{x\,dx}{x^2(1+x^4)}$$

Put $x^2 = t$

$2x\,dx = dt$

When $x = 1, t = 1$

When $x = 2, t = 4$

$$\therefore \quad I = \frac{1}{2}\int_1^4 \frac{dt}{t(1+t^2)}$$

$$= \frac{1}{2}\int_1^4 \left(\frac{1}{t} - \frac{t}{1+t^2}\right) dt = \frac{1}{2}\left[\log t - \frac{1}{2}\log(1+t^2)\right]_1^4$$

$$= \frac{1}{2}\left[\log 4 - \frac{1}{2} - \log 17 - 0 + \frac{1}{2}\log 2\right]$$

$$= \frac{1}{2}\left[\frac{1}{2}\log 4^2 - \frac{1}{2}\log 17 + \frac{1}{2}\log 2\right]$$

$$= \frac{1}{4}\log\left(\frac{16}{17}\times 2\right) = \frac{1}{4}\log\frac{32}{17}$$

12. $\displaystyle\int_0^1 \frac{1-4x+2x^2}{\sqrt{2x-x^2}}\,dx$

Solⁿ: Let,

$$I = \int_0^1 \frac{1-4x+2x^2}{\sqrt{2x-x^2}}\,dx = \int_0^1 \frac{-2(2x-x^2+1)}{\sqrt{2x-x^2}}\,dx$$

$$= -2\int_0^1 \sqrt{2x-x^2}\,dx + \int_0^1 \frac{dx}{\sqrt{2x-x^2}}$$

$$= -2\int_0^1 \sqrt{-(x^2-2x)}\,dx + \int_0^1 \frac{dx}{\sqrt{-[(x)^2-2.x.1+(1)^2-(1)^2]}}$$

$$= -2\int_0^1 \sqrt{-\{(x)^2-2.x.1+(1)^2-1\}}\,dx$$

$$= -2\int_0^1 \sqrt{(1)^2-(x-1)^2}\,dx + \int_0^1 \frac{dx}{\sqrt{(1)^2-(x-1)^2}}$$

$$= -2\left[(x-1)\sqrt{(1)^2-(x-1)^2} + \frac{1}{2}\sin^{-1}\frac{x-1}{1}\right]_0^1 + \left[\sin^{-1}\frac{x-1}{1}\right]_0^1$$

$$= -2\left[0 - \frac{1}{2}\frac{\pi}{2}\right] + \left[0 - \frac{\pi}{2}\right] = \frac{\pi}{2} - \frac{\pi}{2} = 0$$

13. $\displaystyle\int_0^1 \frac{1-x^2}{1+x^2}\,dx$

Solⁿ: Let,

$$I = \int_0^1 \frac{1-x^2}{1+x^2}\,dx$$

$$= \int_0^1 \frac{2-(1+x^2)}{1+x^2}\,dx = \int_0^1 \left(\frac{2}{1+x^2} - 1\right) dx$$

$$= [2\tan^{-1}x - x]_0^1 = (2\tan^{-1}1 - 1) - (2\tan^{-1}0 - 0)$$

$$= 2\left(\frac{\pi}{4}\right) - 1 = \frac{\pi}{2} - 1$$

14. $\displaystyle\int_0^a \frac{1}{x+\sqrt{a^2-x^2}}\,dx$

Solⁿ: Let,

$$I = \int_0^a \frac{1}{x+\sqrt{a^2-x^2}}\,dx$$

Put $x = a\sin\theta$

Then,

$dx = a\cos\theta\, d\theta$

When $x = a, \sin\theta = 1$ or, $\theta = \frac{\pi}{2}$

When $x = 0, \sin\theta = 0$ or, $\theta = 0$

$$\therefore \quad I = \int_0^{\frac{\pi}{2}} \frac{a\cos\theta\,d\theta}{a\sin\theta + a\cos\theta}$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{(\cos\theta+\sin\theta)+(\cos\theta-\sin\theta)}{(\sin\theta+\cos\theta)}\,d\theta$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} d\theta + \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{\cos\theta - \sin\theta}{\sin\theta + \cos\theta}\, d\theta$$

$$= \frac{1}{2}[\theta]_0^{\frac{\pi}{2}} + \frac{1}{2}[\log(\sin\theta + \cos\theta)]_0^{\frac{\pi}{2}}$$

$$= \frac{1}{2} \times \frac{\pi}{2} + \frac{1}{2}(\log 1 - \log 1) = \frac{\pi}{4}$$

15. $\displaystyle\int_1^2 \frac{1}{(x+1)\sqrt{x^2-1}}\, dx$

Sol^n: Let,

$$I = \int_1^2 \frac{1}{(x+1)\sqrt{x^2-1}}\, dx$$

Put $x + 1 = \frac{1}{t}$

Then,

$$dx = -\frac{1}{t^2}\, dt$$

When $x = 1, t = \frac{1}{x+1} = \frac{1}{2}$

When $x = 2, t = \frac{1}{2+1} = \frac{1}{3}$

$$\therefore \quad I = \int_{\frac{1}{2}}^{\frac{1}{3}} \frac{-\frac{1}{t^2}\, dt}{\frac{1}{t}\sqrt{\left(\frac{1}{t} - 1\right)^2 - 1}}$$

$$= -\int_{\frac{1}{2}}^{\frac{1}{3}} \frac{dt}{\sqrt{(1-t)^2 - t^2}}$$

$$= -\int_{\frac{1}{2}}^{\frac{1}{3}} \frac{dt}{\sqrt{1-2t}} = -\left[\frac{(1-2t)^{\frac{1}{2}}}{\frac{1}{2}(-2)}\right]_{\frac{1}{2}}^{\frac{1}{3}}$$

$$= \left[\sqrt{1-2t}\right]_{\frac{1}{2}}^{\frac{1}{3}} = \sqrt{1 - \frac{2}{3}} - \sqrt{1-1} = \frac{1}{\sqrt{3}}$$

16. $\int_0^1 \tan^{-1} x\, dx$

Sol^n: Let,

$$I = \int_0^1 \tan^{-1} x\, dx = \int_0^1 \tan^{-1} x . 1\, dx$$

By parts,

$$[x \tan^{-1} x]_0^1 - \int_0^1 \frac{1}{1+x^2} x\, dx$$

$$= \frac{\pi}{4} - \frac{1}{2}[\log(1+x^2)]_0^1 = \frac{\pi}{4} - \frac{1}{2}\log 2$$

17. $\int_0^{\frac{\pi}{2}} \sqrt{\cos\theta}\, \sin^3\theta\, d\theta$

Sol^n: Let,

$$I = \int_0^{\frac{\pi}{2}} \sqrt{\cos\theta}\, \sin^3\theta\, d\theta = \int_0^{\frac{\pi}{2}} \sqrt{\cos\theta}\,(1 - \cos^2\theta)\sin\theta\, d\theta$$

Put $\cos\theta = t^2$

$-\sin\theta\, d\theta = 2t\, dt$

When $\theta = 0, t = 1$

When $\theta = \frac{\pi}{2}, t = 0$

$$\therefore \quad I = -\int_1^0 t(1 - t^4)\, 2t\, dt$$

$$= -2\int_1^0 (t^2 - t^6)\, dt = -2\left[\frac{t^2}{3} - \frac{t^7}{7}\right]_1^0 = 2\left[0 - \frac{1}{3} - \frac{1}{7}\right] = \frac{8}{21}$$

18. $\int_0^{\frac{\pi}{2}} e^x(\sin x + \cos x)\, dx$

Sol^n: Let,

$$I = \int_0^{\frac{\pi}{2}} e^x(\sin x + \cos x)\, dx$$

$$= \int_0^{\frac{\pi}{2}} e^x \sin x + \int_0^{\frac{\pi}{2}} e^x \cos x\, dx$$

Integrating by parts to the first integral;

$$[\sin x\, e^x]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \cos x\, e^x\, dx + \int_0^{\frac{\pi}{2}} e^x \cos x\, dx = e^{\frac{\pi}{2}}$$

19. $\int_0^{\frac{\pi}{2}} \sin^5\theta \cos^3\theta\, d\theta$

Sol^n: Let,

$$I = \int_0^{\frac{\pi}{2}} \sin^5\theta \cos^3\theta\, d\theta = \int_0^{\frac{\pi}{2}} (\sin^2\theta)^2 (1 - \sin^2\theta)\sin\theta\cos\theta\, d\theta$$

Put $\sin^2\theta = t, 2\sin\theta\cos\theta \, d\theta = dt$

When $\theta = 0, t = 0$

When $\theta = \frac{\pi}{2}, t = 1$

$$\therefore \quad I = \int_0^1 t^2 (1-t)\frac{dt}{2}$$

$$= \frac{1}{2}\int_0^1 (t^2 - t^3)\, dt = \frac{1}{2}\left[\frac{t^3}{3} - \frac{t^4}{4}\right]_0^1 = \frac{1}{2}\left[\frac{1}{3} - \frac{t^4}{4} - 0\right] = \frac{1}{24}$$

20. $\displaystyle\int_0^{\frac{\pi}{2}} \frac{dx}{1+2\cos x}$

Sol^n: Let,

$$I = \int_0^{\frac{\pi}{2}} \frac{dx}{1+2\cos x}$$

$$= \int_0^{\frac{\pi}{2}} \frac{dx}{1+4\cos^2\left(\frac{x}{2}\right)+2} = \int_0^{\frac{\pi}{2}} \frac{\sec^2\left(\frac{x}{2}\right)}{3-\sec^2\left(\frac{x}{2}\right)}\, dx = \int_0^{\frac{\pi}{2}} \frac{\sec^2\left(\frac{x}{2}\right)}{3-\tan^2\frac{x}{2}}$$

Put $\tan\frac{x}{2} = t$,

$\frac{1}{2}\sec^2\frac{x}{2} = dt$

When $x = 0, t = 0$

When $x = \frac{\pi}{2}, t = 1$

$$\therefore \quad I = \int_0^{\frac{\pi}{2}} \frac{2\, dt}{3-t^2}$$

$$= 2 \times \frac{1}{2\sqrt{3}}\left[\log\frac{\sqrt{3}+t}{\sqrt{3}-t}\right]_0^1$$

$$= \frac{1}{\sqrt{3}}\log\frac{\sqrt{3}+1}{\sqrt{3}-1}$$

$$= \frac{1}{\sqrt{3}}\log\left(\frac{\sqrt{3}+1}{\sqrt{3}-1} \times \frac{\sqrt{3}+1}{\sqrt{3}+1}\right)$$

$$= \frac{1}{\sqrt{3}}\log(2+\sqrt{3})$$

21. $\displaystyle\int_0^{\frac{\pi}{2}} \frac{dx}{4+5\sin x}$

Sol^n: Let,

$$I = \int_0^{\frac{\pi}{2}} \frac{dx}{4+5\sin x}$$

$$= \int_0^{\frac{\pi}{2}} \frac{dx}{4+10\sin\frac{x}{2}\cos\frac{x}{2}} = \int_0^{\frac{\pi}{2}} \frac{\sec^2\frac{x}{2}}{4\sec^2\frac{x}{2}+10\tan\frac{x}{2}}$$

$$= \int_0^{\frac{\pi}{2}} \frac{\sec^2\left(\frac{x}{2}\right)}{4+4\tan^2\left(\frac{x}{2}\right)+10\tan\left(\frac{x}{2}\right)}$$

Put $\tan\frac{x}{2} = t$,

$\frac{1}{2}\sec^2\frac{x}{2} = dt$

When $x = 0, t = 0$

When $x = \frac{\pi}{2}, t = 1$

$$\therefore \quad I = \int_0^1 \frac{2\, dt}{4t^2+10t+4}$$

$$= \frac{1}{2}\int_0^1 \frac{dt}{4t^2+\frac{5}{2}t+1}$$

$$= \frac{1}{2}\int_0^1 \frac{dt}{(t)^2+2.t.\frac{5}{4}+\left(\frac{5}{4}\right)^2+1-\frac{25}{16}}$$

$$= \frac{1}{2}\int_0^1 \frac{dt}{\left(t+\frac{5}{4}\right)^2-\left(\frac{3}{4}\right)^2} = \frac{1}{2}\frac{1}{2\times\frac{3}{4}}\left[\log\frac{t+\frac{5}{4}-\frac{3}{4}}{t+\frac{5}{4}+\frac{3}{4}}\right]_0^1$$

$$= \frac{1}{3}\left[\log\frac{\frac{3}{2}}{3} - \log 4\right] = \frac{1}{3}\log\left(\frac{1}{2} - \log\frac{1}{4}\right) = \frac{1}{3}\log\left(\frac{1}{2}\times 4\right)$$

$$= \frac{1}{3}\log 2$$

22. $\int_0^{\pi} \frac{dx}{3+2\sin x+\cos x}$

Solⁿ: Let,

$$I = \int_0^{\pi} \frac{dx}{3+2\sin x+\cos x}$$

$$= \int_0^{\pi} \frac{dx}{3+4\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)+2\cos^2\left(\frac{x}{2}\right)-1}$$

$$= \int_0^{\pi} \frac{\sec^2\frac{x}{2}\,dx}{2\sec^2\frac{x}{2}+4\tan\frac{x}{2}+2} = \int_0^{\pi} \frac{\sec^2\frac{x}{2}\,dx}{2\tan\frac{x}{2}+4\tan\frac{x}{2}+4}$$

$$= \int_0^{\pi} \frac{\frac{1}{2}\sec^2\frac{x}{2}\,dx}{\tan^2\frac{x}{2}+2\tan\frac{x}{2}+2}$$

Put $\tan\frac{x}{2} = t$

$\frac{1}{2}\sec^2\frac{x}{2} = dt$

When $x = 0, t = 0$

When $x = \pi, t = \infty$

$$\therefore \quad I = \int_0^{\infty} \frac{dt}{t^2+2t+2}$$

$$= \int_0^{\infty} \frac{dt}{(t+1)^2+(1)^2} = [\tan^{-1}(t+1)]_0^{\infty} = \frac{\pi}{2}-\frac{\pi}{4} = \frac{\pi}{4}$$

23. $\int_0^{\frac{\pi}{4}} \frac{(\sin x+\cos x)}{(9+16\sin 2x)}\,dx$ [2061 B.E.]

Solⁿ: Let,

$$I = \int_0^{\frac{\pi}{4}} \frac{(\sin x+\cos x)}{(9+16\sin 2x)}\,dx$$

$$= \int_0^{\frac{\pi}{4}} \frac{(\sin x+\cos x)}{9+32\sin x\cos x}\,dx$$

$$= \int_0^{\frac{\pi}{4}} \frac{(\sin x+\cos x)}{1+8+32\sin x\cos x}\,dx$$

$$= \int_0^{\frac{\pi}{4}} \frac{(\sin x+\cos x)\,dx}{1+8\left(\sin^2 x+\cos^2 x+4\sin x\cos x\right)}$$

$$= \int_0^{\frac{\pi}{4}} \frac{(\sin x+\cos x)\,dx}{1+8\left\{(\sin x+\cos x)^2+6\sin x\cos x\right\}}$$

$$= \int_0^{\frac{\pi}{4}} \frac{(\sin x+\cos x)\,dx}{1+8(\sin x+\cos x)^2+48\sin x\cos x}$$

Put $\sin x\cos x = t,$

$(\cos x+\sin x)dx = dt$

$1-\sin x\cos x = t^2$

$1-t^2 = 2\sin x\cos x$

When $x = 0, t = -1$

When $x = \frac{\pi}{4}, t = 0$

$$\therefore \quad I = \int_{-1}^{0} \frac{dt}{1+8t^2+24-24t^2}$$

$$= \int_{-1}^{0} \frac{dt}{25-16t^2} = \frac{1}{16}\int_{-1}^{0} \frac{dt}{\left(\frac{4}{5}\right)^2-t^2} = \frac{1}{16}\frac{1}{2\times\frac{5}{4}}\left[\log\frac{\frac{5}{4}+t}{\frac{5}{4}-t}\right]_1^0$$

$$= \frac{1}{40}\left(0-\log\frac{1}{9}\right) = \frac{-1}{40}\log 3^{-2} = \frac{1}{20}\log 3$$

24. $\int_0^{\frac{\pi}{2}}\left(\sqrt{\tan x}+\sqrt{\cot x}\right)dx$

Solⁿ: Let,

$$I = \int_0^{\frac{\pi}{2}}\left(\sqrt{\tan x}+\sqrt{\cot x}\right)dx$$

$$= \int_0^{\frac{\pi}{2}}\left(\sqrt{\frac{\sin x}{\cos x}}+\sqrt{\frac{\cos x}{\sin x}}\right)dx = \int_0^{\frac{\pi}{2}} \frac{(\sin x+\cos x)}{\sqrt{\sin x\cos x}}\,dx$$

$$= \sqrt{2}\int_0^{\frac{\pi}{2}} \frac{(\sin x+\cos x)}{\sqrt{2\sin x\cos x}}\,dx$$

$$= \sqrt{2}\int_0^{\frac{\pi}{2}} \frac{(\sin x + \cos x)\,dx}{\sqrt{\sin^2 x + \cos^2 x - 2\sin x\cos x - 1 + 4\sin x\cos x}}$$

$$= \sqrt{2}\int_0^{\frac{\pi}{2}} \frac{(\sin x + \cos x)\,dx}{\sqrt{(\sin x - \cos x)^2 + 4\sin x\cos x - 1}}$$

Put $\sin x - \cos x = t$,

$$(\cos x + \sin x)dx = dt$$
$$1 - 2\sin x\cos x = t^2$$
$$1 - t^2 = 2\sin x\cos x$$

When $x = 0, t = -1$

When $x = \frac{\pi}{2}, t = 1$

$$\therefore \quad I = \sqrt{2}\int_{-1}^{1} \frac{dt}{\sqrt{t^2 + 2 - 2t^2 - 1}}$$

$$= \sqrt{2}\int_{-1}^{1} \frac{dt}{\sqrt{1-t^2}} = \sqrt{2}[\sin^{-1} t]_{-1}^{1} = \sqrt{2}\left[\frac{\pi}{2} + \frac{\pi}{2}\right] = \pi\sqrt{2}$$

25. $\displaystyle\int_0^{\frac{\pi}{2}} \frac{dx}{\left(a^2\cos^2 x + b^2\sin^2 x\right)^2}$

Sol^n: Let,

$$I = \int_0^{\frac{\pi}{2}} \frac{dx}{\left(a^2\cos^2 x + b^2\sin^2 x\right)^2}$$

$$= \int_0^{\frac{\pi}{2}} \frac{(\sec^2 x)^2}{\left(a^2 + b^2\tan^2 x\right)^2} = \int_0^{\frac{\pi}{2}} \frac{\left(1+\tan^2 x\right)\sec^2 x\,dx}{\left(a^2 + b^2\tan^2 x\right)^2}$$

Put $b\tan x = a\tan\theta$,

$$b\sec^2 x\,dx = a\sec^2\theta\,d\theta$$

When $x = 0, \theta = 0$

When $x = \frac{\pi}{2}, \theta = \frac{\pi}{2}$

$$\therefore \quad I = \int_0^{\frac{\pi}{2}} \frac{\left(1 + \frac{a^2}{b^2}\tan\theta\right)\frac{a}{b}\sec^2\theta\,d\theta}{a^4\left(1+\tan^2\theta\right)^2}$$

$$= \frac{1}{a^3b^3}\int_0^{\frac{\pi}{2}} \frac{\left(b^2 + a^2\tan^2\theta\right)\sec^2\theta\,d\theta}{\sec^2\theta\sec^2\theta}$$

$$= \frac{1}{a^3b^3}\int_0^{\frac{\pi}{2}} (a^2\cos^2\theta + a^2\sin^2\theta)\,d\theta$$

$$= \frac{1}{a^3b^3}\int_0^{\frac{\pi}{2}} \left[b^2\left(\frac{1+\cos 2\theta}{2}\right) + a^2\left(\frac{1-\cos 2\theta}{2}\right)\right] d\theta$$

$$= \frac{1}{a^3b^3}\left[\frac{b^2\theta}{2} + \frac{b^2\sin 2\theta}{4} + \frac{a^2\theta}{2} - \frac{a^2\sin 2\theta}{4}\right]_0^{\frac{\pi}{2}}$$

$$= \frac{1}{a^3b^3}\left[\frac{b^2}{2}\frac{\pi}{2} + \frac{a^2}{2}\frac{\pi}{2} - 0 - 0\right] = \frac{\pi\left(a^2+b^2\right)}{4a^3b^3}$$

26. $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\cos x\,dx}{(1+\sin x)(2+\sin x)}$

Sol^n: Let,

$$I = \int_0^{\frac{\pi}{2}} \frac{\cos x\,dx}{(1+\sin x)(2+\sin x)}$$

Put $1 + \sin x = t$,

$$\cos x\,dx = dt$$

When $x = 0, t = 1$

When $x = \frac{\pi}{2}, t = 2$

$$\therefore \quad I = \int_1^2 \frac{dt}{t(2+t-1)} = \int_1^2 \frac{dt}{t(t+1)} = \int_1^2 \left(\frac{1}{t} - \frac{1}{t+1}\right) dt$$

$$= [\log t - \log(t+1)]_1^2$$
$$= \log 2 - \log 3 - 0 + \log 2 = 2\log 2 - \log 3$$
$$= \log 4 - \log 3 = \log\frac{4}{3}$$

Exercise 9

1. $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin x\, dx}{\sin x + \cos x} = \frac{\pi}{4}$

Sol^n: Let,

$$I = \int_0^{\frac{\pi}{2}} \frac{\sin x\, dx}{\sin x + \cos x} \cdots \text{(i)}$$

By using properties of definite integral,

$$= \int_0^{\frac{\pi}{2}} \frac{\sin\left(\frac{\pi}{2} - x\right)}{\sin\left(\frac{\pi}{2} - x\right) + \cos\left(\frac{\pi}{2} - x\right)} dx$$

$$I = \int_0^{\frac{\pi}{2}} \frac{\cos x\, dx}{\cos x + \sin x} \cdots \text{(ii)}$$

Adding equation (i) and (ii), we get,

$$2I = \int_0^{\frac{\pi}{2}} \frac{\cos x + \sin x}{\cos x + \sin x} dx = \int_0^{\frac{\pi}{2}} dx = [x]_0^{\frac{\pi}{2}}$$

$\therefore \quad 2I = \frac{\pi}{2}$

Hence, $I = \frac{\pi}{4}$

2. $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx = \frac{\pi}{4}$

Sol^n: Let,

$$I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx \ldots \text{(i)}$$

By using properties of definite integral,

$$= \int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin\left(\frac{\pi}{2} - x\right)}}{\sqrt{\sin\left(\frac{\pi}{2} - x\right)} + \sqrt{\cos\left(\frac{\pi}{2} - x\right)}} dx$$

$$I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\cos x}}{\sqrt{\cos x} + \sqrt{\sin x}} dx \ldots \text{(ii)}$$

Adding equation (i) and (ii), we get,

$$2I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x} + \sqrt{\cos x}}{\sqrt{\cos x} + \sqrt{\sin x}} dx = \int_0^{\frac{\pi}{2}} dx = [x]_0^{\frac{\pi}{2}}$$

$\therefore \quad 2I = \frac{\pi}{2}$

Hence, $I = \frac{\pi}{4}$

3. $\displaystyle\int_0^{\pi} x \sin^2 x\, dx = \frac{\pi^2}{4}$

Sol^n: Let,

$$I = \int_0^{\pi} x \sin^2 x\, dx$$

By using properties of definite integral,

$$I = \int_0^{\pi} (\pi - x) \sin^2(\pi - x)\, dx$$

or, $I = \int_0^{\pi} (\pi - x) \sin^2 x\, dx$

or, $I = \pi \int_0^{\pi} \sin^2 x\, dx - \int_0^{\pi} x \sin^2 x\, dx$

or, $I = \pi \int_0^{\pi} \sin^2 x\, dx - I$

$\therefore \quad 2I = \pi \int_0^{\pi} \sin^2 x\, dx$

$$= \pi \int_0^{\pi} \left(\frac{1 - \cos 2x}{2}\right) dx = \frac{\pi}{2}[x + 2\sin 2x]_0^{\pi}$$

$$= \frac{\pi}{2}.\pi = 0 = \frac{\pi^2}{2}$$

$\therefore \quad I = \frac{\pi^2}{4}$

4. $\displaystyle\int_0^{\frac{\pi}{2}} \log(\tan x) dx = 0$

Sol^n: Let,

$$I = \int_0^{\frac{\pi}{2}} \log(\tan x) dx$$

By using properties of definite integral,

$$= \int_0^{\frac{\pi}{2}} \log\left\{\tan\left(\frac{\pi}{2} - x\right)\right\} dx$$

$$= \int_0^{\frac{\pi}{2}} \log(\cot x) dx$$

$$= + \int_0^{\frac{\pi}{2}} \log \frac{1}{\tan x} dx = \int_0^{\frac{\pi}{2}} \log(\tan x)^{-1} dx$$

$$I = -\int_0^{\frac{\pi}{2}} \log(\tan x)\,dx$$ $\log(\tan x)^{-1}$

$$I = -I$$ $-1 \log(\tan x)$

Hence, $I = 0$ $(\because 2I = 0)$

5. $\displaystyle\int_0^a \frac{\sqrt{x}}{\sqrt{x}+\sqrt{a-x}}\,dx = \frac{a}{2}$

Sol^n: Let,

$$I = \int_0^a \frac{\sqrt{x}}{\sqrt{x}+\sqrt{a-x}}\,dx \ldots \text{(i)}$$

By using properties of definite integral,

$$I = \int_0^a \frac{\sqrt{a-x}}{\sqrt{a-x}+\sqrt{a-a+x}}\,dx \ldots \text{(ii)}$$

Adding equation (i) and (ii), we get,

$$2I = \int_0^a \frac{\sqrt{x}+\sqrt{a-x}}{(\sqrt{a-x}+\sqrt{x})}\,dx = \int_0^a dx = [x]_0^a = a$$

$\therefore \quad 2I = a$

$\therefore \quad I = \frac{a}{2}$

6. $\displaystyle\int_0^{\frac{\pi}{2}} \sin 2x \log(\tan x)\,dx = 0$

Sol^n: Let,

$$I = \int_0^{\frac{\pi}{2}} \sin 2x \log(\tan x)\,dx$$
$$= \int_0^{\frac{\pi}{2}} 2\sin x \cos x \log(\tan x)\,dx$$

By using properties of definite integrals,

$$= 2\int_0^{\frac{\pi}{2}} \sin\left(\frac{\pi}{2}-x\right)\cos\left(\frac{\pi}{2}-x\right)\log\left\{\tan\left(\frac{\pi}{2}-x\right)\right\}dx$$
$$= 2\int_0^{\frac{\pi}{2}} \cos x \sin x \log(\cot x)\,dx$$

$$\therefore \quad I = \int_0^{\frac{\pi}{2}} \sin 2x \log(\cot x)\,dx$$

$$= \int_0^{\frac{\pi}{2}} \sin 2x \log\left(\frac{1}{\tan x}\right)dx \overset{0}{=} -\int_0^{\frac{\pi}{2}} \sin 2x \log(\tan x)\,dx$$

$\therefore \quad I = -I$

or, $\quad 2I = 0$

Hence, $I = 0$

7. $\displaystyle\int_0^1 \frac{\log x}{\sqrt{1-x^2}}\,dx = \frac{\pi}{2}\log\frac{1}{2}$

Sol^n: Let,

$$I = \int_0^1 \frac{\log x}{\sqrt{1-x^2}}\,dx$$

Put $x = \sin\theta$,

$$dx = \cos\theta\, d\theta$$

When $x = 0, \theta = 0$

When $x = 1, \theta = \frac{\pi}{2}$

$$\therefore \quad I = \int_0^{\frac{\pi}{2}} \frac{\log(\sin\theta)}{\cos\theta}\cos\theta\, d\theta$$

$$I = \int_0^{\frac{\pi}{2}} \log(\sin\theta)\,d\theta \ldots \text{(i)}$$

By using properties of definite integral,

$$I = \int_0^{\frac{\pi}{2}} \log\sin\left(\frac{\pi}{2}-\theta\right)d\theta$$

$$I = \int_0^{\frac{\pi}{2}} \log(\cos\theta)\,d\theta \ldots \text{(ii)}$$

Adding equation (i) and (ii), we get,

$$2I = \int_0^{\frac{\pi}{2}} \log(\sin\theta) + \log(\cos\theta)\,d\theta$$

$$= \int_0^{\frac{\pi}{2}} \log(\sin\theta\cos\theta)\,d\theta = \int_0^{\frac{\pi}{2}} \log\frac{\sin 2\theta}{2}\,d\theta$$

$$= \int_0^{\frac{\pi}{2}} \log\sin 2\theta\, d\theta - \log 2\int_0^{\frac{\pi}{2}} d\theta$$

Put $2\theta = t$,

$$2d\theta = dt$$

When $\theta = 0, t = 0$

When $\theta = \frac{\pi}{2}, t = \pi$

$$= \frac{1}{2}\int_0^{\pi} \log(\sin t)\,dt - \log 2\int_0^{\frac{\pi}{2}} d\theta$$

$$2I = \frac{1}{2}\int_0^{\pi} \log(\sin t)\, dt - \log 2\, [\theta]_0^{\frac{\pi}{2}}$$

$t = 2\theta$

$$2I = \frac{1}{2} \times 2\int_0^{\frac{\pi}{2}} \log(\sin t)\, dt - \frac{\pi}{2}\log 2$$

$\frac{dt}{2} = d\theta$

$$= \int_0^{\frac{\pi}{2}} \log(\sin\theta)\, d\theta - \frac{\pi}{2}\log 2$$

$$2I = I - \frac{\pi}{2}\log 2$$

$$I = -\frac{\pi}{2}\log 2 = \frac{\pi}{2}\log\frac{1}{2}$$

Hence, $I = \frac{\pi}{2}\log 2$

8. $\displaystyle\int_0^{\frac{\pi}{2}} \frac{dx}{1+\sqrt{\tan x}} = \frac{\pi}{4}$

Sol^n: Let,

$$I = \int_0^{\frac{\pi}{2}} \frac{dx}{1+\sqrt{\tan x}}$$

$$= \int_0^{\frac{\pi}{2}} \frac{dx}{1+\sqrt{\frac{\sin x}{\cos x}}}$$

$$= \int_0^{\frac{\pi}{2}} \frac{\sqrt{\cos x}\, dx}{\sqrt{\cos x}+\sqrt{\sin x}}$$

$$= \int_0^{\frac{\pi}{2}} \frac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}}\, dx \quad \ldots \text{(i)}$$

By using definite integral,

$$= \int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin\left(\frac{\pi}{2}-x\right)}}{\sqrt{\sin\left(\frac{\pi}{2}-x\right)}+\sqrt{\cos\left(\frac{\pi}{2}-x\right)}}\, dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}}\, dx \quad \ldots \text{(ii)}$$

Adding equation (i) and (ii), we get,

$$2I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}+\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}}\, dx = \int_0^{\frac{\pi}{2}} dx = [x]_0^{\frac{\pi}{2}} = \frac{\pi}{2}$$

$\therefore \quad 2I = \frac{\pi}{2}$

Hence, $I = \frac{\pi}{4}$

9. $\displaystyle\int_0^{1} \frac{\log(1+x)}{1+x^2}\, dx = \frac{\pi}{8}\log 2$

Sol^n: Let,

$$I = \int_0^{1} \frac{\log(1+x)}{1+x^2}\, dx$$

Put $x = \tan\theta$,

$$dx = \sec^2\theta\, d\theta$$

When $x = 0, \theta = 0$

When $x = 1, \theta = \frac{\pi}{4}$

$$\therefore \quad I = \int_0^{\frac{\pi}{4}} \frac{\log(1+\tan\theta)\sec^2\theta\, d\theta}{\sec^2\theta}$$

$$I = \int_0^{\frac{\pi}{4}} \log(1+\tan\theta)\, d\theta$$

By using properties of definite integral,

$$I = \int_0^{\frac{\pi}{4}} \log\left[1+\tan\left(\frac{\pi}{4}-\theta\right)\right] d\theta$$

$$= \int_0^{\frac{\pi}{4}} \log\left[1+\frac{\tan\frac{\pi}{4}-\tan\theta}{1+\tan\frac{\pi}{4}+\tan\theta}\right] d\theta$$

$$= \int_0^{\frac{\pi}{4}} \log\left(1+\frac{1-\tan\theta}{1+\tan\theta}\right) d\theta$$

$$= \int_0^{\frac{\pi}{4}} \log\left(\frac{2}{1+\tan\theta}\right) d\theta$$

$$= \int_0^{\frac{\pi}{4}} \log 2\, d\theta - \int_0^{\frac{\pi}{4}} \log(1+\tan\theta)\, d\theta$$

$$I = \log 2\,[\theta]_0^{\frac{\pi}{4}} - I$$

$$\therefore \quad 2I = \log 2\,\frac{\pi}{4}$$

$$\therefore \quad I = \frac{\pi}{8}\log 2$$

10. $\displaystyle\int_0^{\pi} \frac{x \tan dx}{\sec x + \tan x} = \frac{\pi}{2}(\pi - 2)$

Sol^n: Let,

$$I = \int_0^{\pi} \frac{x \tan dx}{\sec x + \tan x}$$

By using properties of definite integral,

$$I = \int_0^{\pi} \frac{(\pi - x)\tan(\pi - x)\, dx}{\sec(\pi - x) + \tan(\pi - x)}$$

$$= \int_0^{\pi} \frac{-(\pi - x)\tan x\, dx}{-\sec x - \tan x}$$

$$= \int_0^{\pi} \frac{(\pi - x)\tan x\, dx}{\sec x + \tan x} - \int_0^{\pi} \frac{x \tan x\, dx}{\sec x + \tan x}$$

$$\therefore \quad I = \pi \int_0^{\pi} \frac{\sin x}{1 + \sin x} dx - I$$

so, $$2I = \pi \int_0^{\pi} \frac{\sin x}{1 + \sin x} dx$$

$$= \pi \int_0^{2\times\frac{\pi}{2}} \frac{\sin x}{1 + \sin x} dx$$

$$= 2\pi \int_0^{\frac{\pi}{2}} \frac{\sin x}{1 + \sin x} dx$$

$$= 2\pi \int_0^{\frac{\pi}{2}} \frac{1 + \sin x - 1}{1 + \sin x} dx \qquad \left[\because \sin\left(2\frac{\pi}{2} - x\right) = \sin x\right]$$

$$= 2\pi \int_0^{\frac{\pi}{2}} dx - 2\pi \int_0^{\frac{\pi}{2}} \frac{dx}{1 + 2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)}$$

$$= 2\pi [x]_0^{\frac{\pi}{2}} - 2\pi \int_0^{\frac{\pi}{2}} \frac{\sec^2\left(\frac{x}{2}\right) dx}{\sec^2\left(\frac{x}{2}\right) + 2\tan\left(\frac{x}{2}\right)}$$

$$= \pi^2 - 2\pi \int_0^{\frac{\pi}{2}} \frac{\sec^2\frac{x}{2}\, dx}{\tan^2\left(\frac{x}{2}\right) + 2\tan\left(\frac{x}{2}\right) + 1}$$

$$= \pi^2 - 2\pi \int_0^{\frac{\pi}{2}} \frac{\sec^2\frac{x}{2}\, dx}{\left\{\tan\left(\frac{x}{2}\right) + 1\right\}^2}$$

Put $\tan\frac{x}{2} + 1 = t$,

$$\frac{1}{2}\sec^2\frac{x}{2}\, dx = dt$$

When $x = 0, t = 1$

When $x = \frac{\pi}{2}, t = 2$

$$= \pi^2 - 2\pi \int_0^{\frac{\pi}{2}} \frac{2\, dt}{t^2} = \pi^2 - 4\pi\left[-\frac{1}{t}\right]_1^2$$

$$= \pi^2 - 4\pi\left[\frac{-1}{2} + 1\right] = \pi^2 - 2\pi$$

$$\therefore \quad I = \pi(\pi - 2)$$

11. $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin^2 x\, dx}{\sin x + \cos x} = \frac{1}{\sqrt{2}}\log\left(\sqrt{2} + 1\right)$

Sol^n: Let,

$$I = \int_0^{\frac{\pi}{2}} \frac{\sin^2 x\, dx}{\sin x + \cos x}$$

By using properties of definite integral,

$$I = \int_0^{\frac{\pi}{2}} \frac{\sin^2\left(\frac{\pi}{2} - x\right) dx}{\sin\left(\frac{\pi}{2} - x\right) + \cos\left(\frac{\pi}{2} - x\right)} = \int_0^{\frac{\pi}{2}} \frac{\cos^2 x\, dx}{\cos x + \sin x}$$

$$I = \int_0^{\frac{\pi}{2}} \frac{\left(1 - \sin^2 x\right) dx}{\cos x + \sin x} = \int_0^{\frac{\pi}{2}} \frac{dx}{\cos x + \sin x} - \int_0^{\frac{\pi}{2}} \frac{\sin^2 x\, dx}{\cos x + \sin x}$$

$$= \frac{1}{\sqrt{2}} \int_0^{\frac{\pi}{2}} \frac{dx}{\frac{1}{\sqrt{2}}\cos x + \frac{1}{\sqrt{2}}\sin x} - I$$

or, $$2I = \frac{1}{\sqrt{2}} \int_0^{\frac{\pi}{2}} \frac{dx}{\sin\frac{\pi}{4}\cos x + \sin x \cos\frac{\pi}{4}} = \frac{1}{\sqrt{2}} \int_0^{\frac{\pi}{2}} \frac{dx}{\sin\left(\frac{\pi}{4} + x\right)}$$

$$= \frac{1}{\sqrt{2}} \int_0^{\frac{\pi}{2}} \operatorname{cosec}\left(\frac{\pi}{4} + x\right) dx$$

$$= \frac{1}{\sqrt{2}}\left[\left\{\log \operatorname{cosec}\left(\frac{\pi}{4}+x\right) - \cot\left(\frac{\pi}{4}+x\right)\right\}\right]_0^{\frac{\pi}{2}}$$

$$= \frac{1}{\sqrt{2}}\left[\log \operatorname{cosec}\left(\frac{\pi}{4}+\frac{\pi}{2}\right) - \cot\left(\frac{\pi}{4}+\frac{\pi}{2}\right) - \log\left(\operatorname{cosec}\frac{\pi}{4} - \cot\frac{\pi}{4}\right)\right]$$

$$= \frac{1}{\sqrt{2}}\left[\log(\sqrt{2}+1) - \log(\sqrt{2}-1)\right]$$

$$= \frac{1}{\sqrt{2}}\log\frac{\sqrt{2}+1}{\sqrt{2}-1}$$

or, $2I = \frac{1}{\sqrt{2}}\log\frac{\sqrt{2}+1}{\sqrt{2}-1} = \frac{1}{\sqrt{2}}\log\frac{(\sqrt{2}+1)(\sqrt{2}+1)}{(\sqrt{2}-1)(\sqrt{2}+1)}$

or, $2I = \frac{1}{\sqrt{2}}\log\frac{(\sqrt{2}+1)^2}{2-1} = \frac{1}{\sqrt{2}}\log(\sqrt{2}+1)^2$

$\therefore \quad I = \frac{1}{2\sqrt{2}}2\log(\sqrt{2}+1)$

Hence, $I = \frac{1}{\sqrt{2}}\log(\sqrt{2}+1)$

12. $\displaystyle\int_0^{\pi} \frac{\mathbf{x \sin x}}{\mathbf{1+\cos^2 x}}\,\mathbf{dx} = \frac{\boldsymbol{\pi}^2}{4}$

Sol^n: Let,

$$I = \int_0^{\pi} \frac{x \sin x}{1+\cos^2 x}\,dx$$

$$= \int_0^{\pi} \frac{(\pi-x)\sin(\pi-x)\,dx}{1+\cos^2(\pi-x)} = \int_0^{\pi} \frac{(\pi-x)\sin x\,dx}{1+\cos^2 x}$$

$$= \int_0^{\pi} \frac{x \sin x\,dx}{1+\cos^2 x} - \int_0^{\pi} \frac{x \sin x\,dx}{1+\cos^2 x} = \pi\int_0^{\pi} \frac{\sin x\,dx}{1+\cos^2 x} - I$$

$\therefore \quad 2I = \pi\int_0^{\pi} \frac{\sin x\,dx}{1+\cos^2 x}$

Put $\cos x = t$,

Then,

$-\sin x\,dx = dt$

Also,

When $x = 0, t = 1$

When $x = \pi, t = -1$

$\therefore \quad 2I = \pi\int_1^{-1} -\frac{dt}{1+t^2} = \pi\int_{-1}^{1}\frac{dt}{1+t^2}$

$\therefore \quad I = \frac{\pi}{2}[\tan^{-1} t]_{-1}^{1} = \frac{\pi}{2}[\tan^{-1} 1 - \tan^{-1}(-1)]$

$$= \frac{\pi}{2}\left[\frac{\pi}{4} - \left(-\frac{\pi}{4}\right)\right] = \frac{\pi^2}{4}$$

13. $\displaystyle\int_0^{\pi} \mathbf{x \log(\sin x)\,dx} = \frac{\boldsymbol{\pi}^2}{2}\log\frac{1}{2}$

Sol^n: Let,

$I = \int_0^{\pi} x\log(\sin x)dx$

By using properties of definite integral,

$I = \int_0^{\pi}(\pi - x)\log\sin(\pi - x)\,dx$

$= \int_0^{\pi}(\pi - x)\log\sin x\,dx$

$= \pi\int_0^{\pi}\log\sin x\,dx - \int_0^{\pi} x\log\sin x\,dx$

$I = \pi\int_0^{\pi}\log\sin x\,dx - I$

$\therefore \quad 2I = \pi\int_0^{\pi}\log\sin x\,dx$

$= \pi\int_0^{2\times\frac{\pi}{2}}\log\sin x\,dx$

or, $2I = \pi\int_0^{\frac{\pi}{2}}\log\sin x\,dx \qquad \left[\because \sin\left(\frac{2\pi}{2} - x\right) = \sin x\right]$

or, $I = \pi\int_0^{\frac{\pi}{2}}\log\sin x\,dx \ldots$ (i)

By using properties of definite integral,

$$I = \int_0^{\frac{\pi}{2}}\log\sin\left(\frac{\pi}{2} - x\right)dx$$

$= \int_0^{\frac{\pi}{2}}\log\cos x\,dx \ldots$ (ii)

Adding equation (i) and (ii), we get,

$2I = \int_0^{\frac{\pi}{2}}\log(\sin x\cos x)\,dx$

$$= \pi\int_0^{\frac{\pi}{2}}\log\left(\frac{\sin 2x}{2}\right)dx$$

$= \pi\int_0^{\frac{\pi}{2}}\log\sin 2x\,dx - \pi\int_0^{\frac{\pi}{2}}\log 2\,dx$

or, $2I = \pi\int_0^{\frac{\pi}{2}}\log\sin 2x\,dx - \pi\log 2\,[x]_0^{\frac{\pi}{2}}$

Put $2x = t, 2dx = dt$

When $x = 0, t = 0$

When $x = \frac{\pi}{2}, t = \pi$

$$= \frac{\pi}{2}\int_0^{\pi} \log \sin t \, dt - \pi \log 2 \left(\frac{\pi}{2} - 0\right)$$

$$= \frac{\pi}{2}\int_0^{\pi} \log \sin x \, dx - \frac{\pi^2}{2}\log 2 \left(\because \int_a^b f(x)dx = \int_a^b f(t)dt\right)$$

or, $$2I = \frac{\pi}{2}\int_0^{2\times\frac{\pi}{2}} \log \sin x \, dx - \frac{\pi^2}{2}\log 2$$

or, $$2I = \pi\int_0^{\frac{\pi}{2}} \log \sin x \, dx - \frac{\pi^2}{2}\log 2$$

or, $$2I = I - \frac{\pi^2}{2}\log 2$$

Hence, $I = \frac{\pi^2}{2}\log\frac{1}{2}$

14. $\displaystyle\int_0^{\frac{\pi}{2}} \frac{x\,dx}{\sin x + \cos x} = \frac{\pi}{2\sqrt{2}}\log\left(\sqrt{2}+1\right)$

Sol^n: Let,

$$I = \int_0^{\frac{\pi}{2}} \frac{x\,dx}{\sin x + \cos x}$$

By using properties of definite integral,

$$= \int_0^{\frac{\pi}{2}} \frac{\left(\frac{\pi}{2} - x\right)dx}{\sin\left(\frac{\pi}{2} - x\right) + \cos\left(\frac{\pi}{2} - x\right)}$$

$$= \int_0^{\frac{\pi}{2}} \frac{\left(\frac{\pi}{2} - x\right)dx}{\cos x + \sin x} = \frac{\pi}{2}\int_0^{\frac{\pi}{2}} \frac{dx}{\cos x + \sin x} - \int_0^{\frac{\pi}{2}} \frac{x\,dx}{\cos x + \sin x}$$

$$I = \frac{\pi}{2}\int_0^{\frac{\pi}{2}} \frac{dx}{\sqrt{2}\left(\frac{1}{\sqrt{2}}\cos x + \frac{1}{\sqrt{2}}\sin x\right)} - 1$$

or, $$2I = \frac{\pi}{2\sqrt{2}}\int_0^{\frac{\pi}{2}} \frac{dx}{\left(\sin\frac{\pi}{4}\cos x + \sin x \cos\frac{\pi}{4}\right)}$$

$$= \frac{\pi}{2\sqrt{2}}\int_0^{\frac{\pi}{2}} \frac{dx}{\sin\left(\frac{\pi}{4} + x\right)} = \frac{\pi}{2\sqrt{2}}\int_0^{\frac{\pi}{2}} \operatorname{cosec}\left(\frac{\pi}{4} + x\right)dx$$

$$= \frac{\pi}{2\sqrt{2}}\left[\log\left\{\operatorname{cosec}\left(\frac{\pi}{4} + x\right) - \cot\left(\frac{\pi}{4} + x\right)\right\}\right]_0^{\frac{\pi}{2}}$$

$$= \frac{\pi}{2\sqrt{2}}\left[\log\left\{\operatorname{cosec}\left(\frac{\pi}{4} + \frac{\pi}{2}\right) - \cot\left(\frac{\pi}{4} + \frac{\pi}{2}\right)\right\} - \log\left\{\operatorname{cosec}\frac{\pi}{4} - \cot\frac{\pi}{4}\right\}\right]$$

$$= \frac{\pi}{2\sqrt{2}}\log(\sqrt{2}+1) - \log(\sqrt{2}-1)$$

or, $$2I = \frac{\pi}{2\sqrt{2}}\log\frac{(\sqrt{2}+1)}{(\sqrt{2}-1)}$$

$$= \frac{\pi}{2\sqrt{2}}\log\frac{(\sqrt{2}+1)(\sqrt{2}+1)}{(\sqrt{2}-1)(\sqrt{2}+1)} = \frac{\pi}{2\sqrt{2}}\log\frac{(\sqrt{2}+1)^2}{2-1}$$

$$= \frac{\pi}{2\sqrt{2}} \times 2\log(\sqrt{2}+1)$$

Hence,

$$I = \frac{\pi}{2\sqrt{2}}\log(\sqrt{2}+1)$$

15. $\displaystyle\int_0^1 \cot^{-1}(1 - x + x^2)dx = \frac{\pi}{2} - \log 2$

Sol^n: Let,

$$I = \int_0^1 \cot^{-1}(1 - x + x^2)dx$$

$$= \int_0^1 \tan^{-1}\frac{1}{1-x+x^2}\,dx \left(\because \cot^{-1}x = \tan^{-1}\frac{1}{x}\right)$$

$$= \int_0^1 \tan^{-1}\frac{x+(1-x)}{1-x(1-x)}\,dx$$

$$= \int_0^1 [\tan^{-1}x + \tan^{-1}(1-x)]dx$$

$$\left(\because \tan^{-1}\frac{x+y}{1-xy} = \tan^{-1}x + \tan^{-1}y\right)$$

$$= \int_0^1 \tan^{-1}x\,dx + \int_0^1 \tan^{-1}(1-x)dx$$

By using properties of definite integral to the second integral,

$$= \int_0^1 \tan^{-1}x + \int_0^1 \tan^{-1}x\,dx$$

$$= 2\int_0^1 \tan^{-1}x\,dx$$

Integrating by parts,

$$= 2[\tan^{-1}x]_0^1 - 2\int_0^1 \frac{1}{1+x^2}\,dx$$

$$= 2\left(\frac{\pi}{4} - 0\right) - \log[\log(1 + x^2)]_0^1 = \frac{\pi}{2} - \log 2$$

Hence, $I = \frac{\pi}{2} - \log 2$

16. $\displaystyle\int_0^{\pi} \frac{x}{a^2\sin^2 x + b^2\cos^2 x}\, dx\ (a, b > 0,) = \frac{\pi^2}{2ab}$

Solⁿ: Let,

$$I = \int_0^{\pi} \frac{x}{a^2\sin^2 x + b^2\cos^2 x}\, dx$$

By using properties of definite integral,

$$= \int_0^{\pi} \frac{(\pi - x)\, dx}{a^2\sin^2(\pi - x) + b^2\cos^2(\pi - x)}$$

$$\text{or, } I = \pi \int_0^{\pi} \frac{dx}{a^2\sin^2 x + b^2\cos^2 x} - \int_0^{\pi} \frac{x}{a^2\sin^2 x + b^2\cos^2 x}\, dx$$

$$\text{or, } I = \pi \int_0^{\pi} \frac{\sec^2 x\, dx}{a^2\tan^2 x + b^2} - I$$

$$\text{or, } 2I = \pi \int_0^{2\times\frac{\pi}{2}} \frac{\sec^2 x\, dx}{a^2\tan^2 x + b^2}$$

$$= 2\pi \int_0^{\frac{2\pi}{2}} \frac{\sec^2 x\, dx}{a^2\tan^2 x + b^2} \qquad \left[\because f\left(\frac{2\pi}{2} - x\right) = f(x)\right]$$

Put $a\tan x = bt$,

$a\sec^2 x\, dx = b\, dt$

When $x = 0, t = 0$

When $x = \frac{\pi}{2}, t = \infty$

$$\therefore \quad I = \pi \int_0^{\infty} \frac{b\, dt}{ab^2(t^2 + 1)}$$

$$= \frac{\pi}{ab}[\tan^{-1} t]_0^{\infty} = \frac{\pi}{ab}\left(\frac{\pi}{2} - 0\right)$$

$$\therefore \quad I = \frac{\pi^2}{2ab}$$

17. $\displaystyle\int_0^{\pi} \frac{x\, dx}{\left(a^2\sin^2 x + b^2\cos^2 x\right)^2} = \frac{\pi^2\left(a^2 + b^2\right)}{4a^3b^3}$

Solⁿ: Let,

$$I = \int_0^{\pi} \frac{x\, dx}{\left(a^2\sin^2 x + b^2\cos^2 x\right)^2}$$

By using properties of definite integral,

$$= \int_0^{\pi} \frac{(\pi - x)\, dx}{\left[a^2\sin^2(\pi - x) + b^2\cos^2(\pi - x)\right]^2}$$

$$= \pi \int_0^{\pi} \frac{dx}{\left(a^2\sin^2 x + b^2\cos^2 x\right)^2} - \int_0^{\pi} \frac{x\, dx}{\left(a^2\sin^2 x + b^2\cos^2 x\right)^2}$$

$$I = \int_0^{\pi} \frac{\pi \sec^2 x \,.\, \sec^2 x\, dx}{\left(a^2\tan^2 x + b^2\right)^2} - I$$

$$\text{or, } 2I = \pi \int_0^{2\times\frac{\pi}{2}} \frac{(1 + \tan^2 x)\sec^2 x\, dx}{\left(a^2\tan^2 x + b^2\right)^2}$$

$$\text{or, } 2I = 2\pi \int_0^{\pi} \frac{(1 + \tan^2 x)\sec^2 x\, dx}{\left(a^2\tan^2 x + b^2\right)^2} \qquad \left[\because f\left(\frac{2\pi}{2} - x\right) = f(x)\right]$$

Put $a\tan x = b\tan\theta$,

$a\sec^2 x\, dx = b\sec^2\theta\, d\theta$

When $x = 0, \theta = 0$

When $x = \frac{\pi}{2}, \theta = \frac{\pi}{2}$

$$\therefore \quad 2I = 2\pi \int_0^{\frac{\pi}{2}} \frac{\left(1 + \frac{b^2\tan^2\theta}{a^2}\right)\frac{b}{a}\sec^2\theta\, d\theta}{b^4\left(1 + \tan^2\theta\right)^2}$$

or, $$I = \frac{\pi}{a^3b^3}\int_0^{\frac{\pi}{2}} \frac{\left(a^3 + b^2\tan^2\theta\right)\sec^2\theta\, d\theta}{\sec^2\theta}$$

$$= \frac{\pi}{a^3b^3}\int_0^{\frac{\pi}{2}} (a^2\cos^2\theta + b^2\sin^2\theta)\,d\theta$$

$$= \frac{\pi}{a^3b^3}\int_0^{\frac{\pi}{2}} \left[a^2\left(\frac{1+\cos 2\theta}{2}\right) + b^2\left(\frac{1-\cos 2\theta}{2}\right)\right] d\theta$$

$$= \frac{\pi}{a^3b^3}\int_0^{\frac{\pi}{2}} \left[\frac{a^2\theta}{2} + \frac{a^2\sin 2\theta}{4} + \frac{b^2\theta}{2} - \frac{a^2\sin 2\theta}{4}\right]_0^{\frac{\pi}{2}}$$

$$= \frac{\pi}{a^3b^3}\int_0^{\frac{\pi}{2}} \left[\frac{a^2\pi}{4} + \frac{b^2\pi}{2} - 0\right] = \frac{\pi^2\left(a^2+b^2\right)}{4a^3b^3}$$

$$\therefore \quad I = \frac{\pi^2\left(a^2+b^2\right)}{4a^3b^3}$$

Exercise 10

Evaluate the following:

1. $$\lim_{n\to\infty}\left(\frac{n}{n^2+1^2} + \frac{n}{n^2+2^2} + \dots\dots\dots + \frac{n}{2n^2}\right)$$

Sol^n: Here,

$$\lim_{n\to\infty}\left(\frac{n}{n^2+1^2} + \frac{n}{n^2+2^2} + \dots\dots\dots + \frac{n}{2n^2}\right)$$

The r^{th} term $\frac{n}{n^2+r^2}$ where $r = 1, 2 \dots\dots\dots n$

So, the limit

$$= \lim_{n\to\infty}\sum_{r=1}^{n}\frac{n}{n^2+r^2}$$

$$= \lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{\left(1+\frac{r^2}{n^2}\right)}$$

$$= \int_0^1 \frac{1}{1+x^2}\,dx = [\tan^{-1}x]_0^1 = \frac{\pi}{4}$$

2. $$\lim_{n\to\infty}\left(\frac{n+1}{n^2+1^2} + \frac{n+2}{n^2+2^2} + \dots + \frac{1}{n}\right)$$

Sol^n: Here,

$$\lim_{n\to\infty}\left(\frac{n+1}{n^2+1^2} + \frac{n+2}{n^2+2^2} + \dots + \frac{1}{n}\right)$$

The r^{th} term$= \frac{n+r}{n^2+r^2}$ where $r = 1, 2, \dots\dots\dots\dots, n$

So that the given series,

$$= \lim_{n\to\infty}\sum_{n=1}^{n}\frac{n+r}{n^2+r^2}$$

$$= \lim_{n\to\infty}\sum_{n=1}^{n}\frac{1}{n}\left(\frac{1+\frac{r}{n}}{1+\frac{r^2}{n^2}}\right)$$

$$= \int_0^1 \frac{1+x}{1+x^2}dx = \int_0^1 \frac{1}{1+x^2}dx + \frac{1}{2}\int_0^1 \frac{2x}{1+x^2}dx$$

$$= [\tan^{-1}x]_0^1 + \left[\frac{1}{2}(\log(1+x^2))\right]_0^1 = \frac{\pi}{4} + \frac{1}{2}\log 2$$

3. $\lim\limits_{n \longrightarrow \infty} \sum_{r=1}^{n} \frac{r^3}{r^4+n^4}$

Sol^n: Here,

$$\lim_{n \longrightarrow \infty} \sum_{r=1}^{n} \frac{r^3}{n^3\left(1+\frac{r^4}{n^4}\right)}\frac{1}{n}$$

$$= \int_0^1 \frac{x^3}{(1+x^4)}dx = \frac{1}{4}[\log(1+x^4)]_0^1 = \frac{1}{4}\log 2$$

4. $\lim\limits_{n \longrightarrow \infty}\left[\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)\left(1+\frac{3}{n}\right)\cdots\left(1+\frac{n}{2}\right)\right]^{\frac{1}{n}}$

Sol^n:

Let,

$$A = \lim_{n \longrightarrow \infty}\left[\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)\left(1+\frac{3}{n}\right)\cdots\cdots\cdots\left(1+\frac{n}{2}\right)\right]^{\frac{1}{n}}$$

$$= \lim_{n \longrightarrow \infty}\log A = \lim_{n\to\infty}\sum_{r=1}^{n}\log\left(1+\frac{r}{n}\right)\frac{1}{n}$$

$$= \int_0^1 \log(1+x)\cdot 1dx$$

$$= [\log(1+x)x]_0^1 - \int_0^1 \frac{1}{1+x}dx$$

$$= \log 2 - \log 1 - \int_0^1 \frac{1+x-1}{1+x}dx$$

$$= \log 2 - \int_0^1 x\,dx + \int_0^1 \frac{dx}{1+x} = \log 2 - [x]_0^1 + [\log(1+x)]_0^1$$

$$= \log 2 - 1 + \log 2 = 2\log 2 - 1 = \log 2^2 - \log e$$

$$= \log 4 - \log e = \log\frac{4}{e}$$

$$\therefore \lim_{n \longrightarrow \infty} A = \frac{4}{e}$$

5. $\lim\limits_{n \longrightarrow \infty}\left(\frac{n!}{n^n}\right)^{\frac{1}{n}}$

Sol^n:

Let,

$$A = \left(\frac{n!}{n^n}\right)^{\frac{1}{n}} = \left(\frac{1\cdot 2\cdot 3\cdots\cdots n}{n\cdot n\cdots\cdots n}\right)^{\frac{1}{n}} = \left(\frac{1}{n}\cdot\frac{2}{n}\cdot\frac{3}{n}\cdots\cdots\frac{(n-1)}{n}\frac{n}{n}\right)^{\frac{1}{n}}$$

$$\lim_{n \longrightarrow \infty}\log A = \lim_{n \longrightarrow \infty}\frac{1}{n}\left[\log\frac{1}{n} + \log\frac{2}{n} + \log\frac{3}{n} + \cdots\log\frac{5}{n}\right]$$

$$= \lim_{n \longrightarrow \infty}\frac{1}{n}\sum_{r=1}^{n}\log\frac{r}{n}$$

$$= \int_0^1 \log x\,dx$$

Integrating by parts,

$$[\log x - x]_0^1 = -1$$

So,

$$\lim_{n \longrightarrow \infty}\log A = -1$$

$$\therefore \lim_{n \longrightarrow \infty} A = e^{-1}$$

Exercise 11

Definition:

Infinite or improper integral:

The integral $\int_a^b f(x)\,dx$ is called an infinite integral or improper integral if the integrands function (fx) becomes infinite in the interval $a \le x \le b$ or the limit of integration a or b (or both) becomes infinite.

For example:

$$\int_{-\infty}^{\infty} \frac{1}{x^2+1}\,dx, \int_a^{\infty} \frac{dx}{x^2}$$

$$\int_{-\infty}^{b} \frac{dx}{x^2+1}, \int_a^{b} \frac{dx}{x(a-x)}$$

1. Evaluate, if possible, the following improper integral.

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$$

Sol^n: Given;

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$$

$$= \lim_{a\to\infty} \int_{-a}^{0} \frac{dx}{1+x^2} + \lim_{b\to\infty} \int_0^b \frac{dx}{1+x^2}$$

$$= \lim_{a\to\infty} [\tan^{-1} x]_0^{-a} + \lim_{b\to\infty} [\tan^{-1} x]_0^b$$

$$= \frac{\pi}{2} + \frac{\pi}{2} = \pi$$

2. $\displaystyle\int_0^{\infty} \frac{dx}{9+x^2}$

Sol^n: Here,

$$\int_0^{\infty} \frac{dx}{9+x^2}$$

$$= \lim_{b\to\infty} \int_0^b \frac{1}{3^2+x^2}\,dx = \lim_{b\to\infty} \frac{1}{3}\left[\tan^{-1}\frac{x}{3}\right]_0^b$$

$$= \lim_{b\to\infty} \frac{1}{3}\left[\tan^{-1}\frac{b}{3} - \tan^{-1} 0\right] = \frac{\pi}{2}\cdot\frac{1}{3} = \frac{\pi}{6}$$

3. $\displaystyle\int_1^{\infty} \frac{\log x}{x^2}\,dx$

Sol^n: Here,

$$\int_1^{\infty} \frac{\log x}{x^2}\,dx$$

$$= \lim_{b\to\infty} \int_1^b \frac{\log x}{x^2}\,dx$$

Integrating by parts,

$$= \lim_{b\to\infty} \left[\log x\left(-\frac{1}{x}\right)\right]_1^b - \lim_{b\to\infty} \int_1^b \frac{1}{x}\left(-\frac{1}{x}\right)dx$$

$$= -\lim_{b\to\infty} \frac{\log b}{b} + 0 + \lim_{b\to\infty} \left[-\frac{1}{x}\right]_1^b \left[\frac{\infty}{\infty}\text{ form}\right]$$

$$= -\lim_{b\to\infty} \frac{1}{b} - \lim_{b\to\infty} \frac{1}{b} + 1 = -0 - 0 + 1 = 1$$

$$\therefore \int_1^{\infty} \frac{\log x}{x^2}\,dx = 1$$

4. $\displaystyle\int_1^{\infty} \frac{x\,dx}{\left(1+x^2\right)^2}$

Sol^n: Here,

$$\int_1^{\infty} \frac{x\,dx}{\left(1+x^2\right)^2} = \lim_{b\to\infty} \int_1^b \frac{x\,dx}{\left(1+x^2\right)^2}$$

Put, $1 + x^2 = t$, $2x\,dx = dt$

When, $x = 1$, $t = 2$

When, $x = b$, $t = 1 + b^2$

$$= \lim_{b\to\infty} \int_2^{1+b^2} \frac{dt}{2t^2}$$

$$= \lim_{b\longrightarrow\infty} \frac{1}{2}\left[-\frac{1}{t}\right]_2^{1+b^2} = \lim_{b\longrightarrow\infty} \frac{1}{2}\left[-\frac{1}{1+b^2}+\frac{1}{2}\right]$$

$$= 0 + \frac{1}{4} = \frac{1}{4}$$

$$\therefore \int_0^\infty \frac{1}{x^3} = \frac{1}{4}$$

5. $\displaystyle\int_1^\infty \frac{1}{x^3}\,dx$

Sol^n: Here,

$$\int_1^\infty \frac{1}{x^3}\,dx$$

$$= \lim_{b\longrightarrow\infty}\left[-\frac{1}{2x^2}\right]_1^b = \lim_{b\longrightarrow\infty}\left(-\frac{1}{2b^2}+\frac{1}{2}\right) = \frac{1}{2}$$

6. $\displaystyle\int_1^\infty \frac{1}{\sqrt{x}}\,dx$

Sol^n: Here,

$$\int_1^\infty \frac{1}{\sqrt{x}}\,dx$$

$$= \lim_{b\longrightarrow\infty}\int_1^b \frac{dx}{\sqrt{x}} = \lim_{b\longrightarrow\infty}\left[2\sqrt{x}\right]_1^b$$

$$= \lim_{b\longrightarrow\infty} 2\left(\sqrt{b}-1\right) \text{ which is not finite}$$

$\therefore$ The integral doesn't exist.

7. $\displaystyle\int_{-1}^1 \frac{1}{x^2}\,dx$

Sol^n: Here,

$$\int_{-1}^1 \frac{1}{x^2}\,dx$$

0 is an interior point of the interval $(-1, 1)$ and $\frac{1}{x^2} \to \infty$ as $x \to 0$

$$\therefore \int_{-1}^1 \frac{dx}{x^2} = \lim_{h\longrightarrow\infty}\int_{-1}^{n}\frac{dx}{x^2} + \lim_{n^2\longrightarrow\infty}\int_{n'}^{1}\frac{dx}{x^2}$$

$$= \lim_{h\longrightarrow\infty}\left[-\frac{1}{x}\right]_{-1}^{-h} + \lim_{h'\longrightarrow\infty}\left[-\frac{1}{x}\right]_{h'}^{1}$$

$$= \lim_{h\longrightarrow\infty}\left(\frac{1}{n}-1\right) + \lim_{h'\longrightarrow\infty}\left(-1+\frac{1}{h}\right)$$

Since both the limit doesn't exist, the improper integral doesn't exist.

8. $\displaystyle\int_0^\infty \frac{x\,dx}{x^2+4}$

Sol^n: Here,

$$\int_0^\infty \frac{x\,dx}{x^2+4}$$

$$= \lim_{b\longrightarrow\infty}\int_0^b \frac{x\,dx}{x^2+4} = \lim_{b\longrightarrow\infty}\frac{1}{2}\left[\log(x^2+4)\right]_0^b$$

$$= \lim_{b\longrightarrow\infty}\frac{1}{2}\log(b^2+4) - \frac{1}{2}\log 4$$

$$= \lim_{b\longrightarrow\infty}\frac{1}{2}\log\frac{b^2+4}{4}$$

Since,

$$\frac{1}{2}\log\frac{\left(b^2+4\right)}{4} \to \infty \text{ as } b \to \infty$$

Thus, the integral doesn't exist.

9. $\displaystyle\int_{-\infty}^\infty \frac{e^x}{1+e^{2x}}\,dx$

Soln: Here,

$$\int_{-\infty}^\infty \frac{e^x}{1+e^{2x}}\,dx$$

Since 0 is an interior points of $(-\infty, \infty)$

$$\therefore \int_{-\infty}^\infty \frac{e^x}{1+e^{2x}}\,dx$$

$$= \int_{\infty}^0 \frac{e^x dx}{1+e^{2x}} + \int_0^\infty \frac{e^x dx}{1+e^{2x}}$$

$$= \lim_{a\longrightarrow\infty}\int_{-a}^0 \frac{e^x dx}{1+\left(e^{2x}\right)^2} + \lim_{b\longrightarrow\infty}\int_0^b \frac{e^x dx}{1+\left(e^{2x}\right)^2}$$

Put, $e^x = t$

$e^x dx = dt$

So, $\int \frac{dt}{1+t_2} = \tan^{-1} t$

$$= \tan^{-1}(e^x)$$

$$= \lim_{a \longrightarrow \infty} [\tan^{-1}(e^x)]_{-a}^{0} + \lim_{b \longrightarrow \infty} [\tan^{-1} e^x]_0^b$$

$$= \lim_{a \longrightarrow \infty} \left[\frac{\pi}{4} - \tan^{-1}(e^x)\right] + \lim_{b \longrightarrow \infty} \left[\tan^{-1} e^b - \frac{\pi}{4}\right]$$

$$= \frac{\pi}{4} - 0 + \frac{\pi}{2} - \frac{\pi}{4}$$

$$= \frac{\pi}{2}$$

$\therefore \int_{-\infty}^{\infty} \frac{e^x}{1+e^{2x}} dx = \frac{\pi}{2}$

10. $\int_0^{\pi} \frac{\sin x}{\cos^2 x} \mathbf{dx}$ [2055, 2061 B.E.]

Soln: Here,

$$\int_0^{\pi} \frac{\sin x}{\cos^2 x} dx$$

The point $\frac{\pi}{2}$ is the interior point of $(0, \pi)$

Thus,

$$\int_0^{\pi} \frac{\sin x}{\cos^2 x} dx$$

$$= \lim_{h \longrightarrow \infty} \int_0^{\frac{\pi}{2}-h} \frac{\sin x}{\cos^2 x} dx + \lim_{h' \longrightarrow \infty} \int_{\frac{\pi}{2}+h'}^{\pi} \frac{\sin x}{\cos^2 x} dx$$

$$= \lim_{h \longrightarrow \infty} \int_0^{\frac{\pi}{2}-h} \sec x \tan x \, dx + \lim_{h' \longrightarrow \infty} \int_{\frac{\pi}{2}+h'}^{\pi} \sec x \tan x \, dx$$

$$= \lim_{h \longrightarrow \infty} [\sec x]_0^{\frac{\pi}{2}-h} + \lim_{h' \longrightarrow \infty} [\sec x]_{\frac{\pi}{2}+h'}^{\pi}$$

$$= \lim_{h \longrightarrow \infty} \sec\left(\frac{\pi}{2} - h\right) - 1 + \lim_{h' \longrightarrow \infty} \left[-1 - \sec\left(\frac{\pi}{2} + h'\right)\right]$$

$$= \lim_{h \longrightarrow \infty} \operatorname{cosec} h + \lim_{nh' \longrightarrow \infty} \operatorname{cosec} h' - 2$$

Since, both limit don't exist, the improper integral doesn't exist.

11. $\int_0^a \sqrt{\frac{a-x}{x}} \mathbf{dx}$

Sol[n]: Here,

$$\int_0^a \sqrt{\frac{a-x}{x}} dx$$

Since,

$$\sqrt{\frac{a-x}{x}} \to \infty \text{ as } x \to 0$$

So, the infinite integral is defined as;

$$\int_0^a \sqrt{\frac{a-x}{x}} dx = \lim_{h \longrightarrow 0} \int_{0+n}^a \sqrt{\frac{a-x}{x}} dx$$

Put, $x = t^2$

$dx = 2t \, dt$

When $x = n, t = \sqrt{h}$

When $x = a, t = \sqrt{a}$

$$\int_0^a \sqrt{\frac{a-x}{x}} dx = \lim_{h \longrightarrow 0} \int_{\sqrt{h}}^{\sqrt{a}} \sqrt{\frac{(a-t^2)}{t^2}} 2t \, dt$$

$$= 2 \lim_{h \longrightarrow 0} \int_{\sqrt{h}}^{\sqrt{a}} \sqrt{a - t^2} \, dt$$

$$= 2 \lim_{h \longrightarrow 0} \left[\frac{t\sqrt{a-t^2}}{2} + \frac{a}{2} \sin^{-1} \frac{t}{\sqrt{a}}\right]_{\sqrt{h}}^{\sqrt{a}}$$

$$= 2 \lim_{h \longrightarrow 0} \left[0 + \frac{a}{2}\frac{\pi}{2} - \frac{\sqrt{n}\sqrt{a-h}}{2} - \frac{a}{2} \sin^{-1} \sqrt{\frac{h}{a}}\right]$$

$$= \frac{a\pi}{2} - 0 - 0 = \frac{a\pi}{2}$$

12. $\int_0^2 \frac{\mathbf{dx}}{(\mathbf{1-x})^2}$

Sol[n]: Here,

$$\int_0^2 \frac{dx}{(1-x)^2}$$

Since, 1 is interior point of (0,2)

So, the infinite integral is defined as;

$$\int_0^2 \frac{dx}{(1-x)^2} = \lim_{h \to 0}\left[\int_0^{1-h} \frac{dx}{(1-x)^2}\right] + \lim_{h' \to 0}\left[\int_{1+h'}^2 \frac{dx}{(1-x)^2}\right]$$

$$= \lim_{h \to 0}\left[\frac{1}{1-x}\right]_0^{1-h} + \lim_{h' \to 0}\left[\frac{1}{(1-x)}\right]_{1+h'}^2$$

$$= \lim_{h \to 0}\left[\frac{1}{h} - 1\right] + \lim_{h' \to 0}\left[-1 + \frac{1}{h'}\right]$$

$\frac{1}{h} \to \infty$ as $h \to 0$,

$\frac{1}{h'} \to \infty$ as $h' \to 0$

So, the integral doesn't exists.

13. $\int_{-1}^2 \frac{dx}{x^3} dx$ [2058. B.E]

Sol^n: Here,

$$\int_{-1}^2 \frac{dx}{x^3} dx$$

0 is an interior point of the interval $(-1,2)$ and $\frac{1}{x^3} \to \infty$ as $x \to 0$

$$\therefore \int_{-1}^2 \frac{dx}{x^3} = \lim_{h \to 0}\int_{-1}^{-h} \frac{dx}{x^3} + \lim_{h' \to 0}\int_{h'}^2 \frac{dx}{x^3}$$

$$= \lim_{h \to 0}\left(-\frac{1}{2x^2}\right)_{-1}^{-h} + \lim_{h' \to 0}\left(-\frac{1}{2x^2}\right)_{h'}^2$$

$$= \lim_{h \to 0}\left(\frac{1}{2h^2} + \frac{1}{2}\right) + \lim_{h' \to 0}\left(-\frac{1}{8} + \frac{1}{2h'^2}\right)$$

Since, both the limit don't exist, the integral doesn't exist.

If $h = h'$, the principle value

$$= \lim_{h \to 0}\left(-\frac{1}{2h^2} + \frac{1}{2} - \frac{1}{8} + \frac{1}{2h^2}\right)$$

$$= \frac{1}{2} - \frac{1}{8} = \frac{3}{8}$$

Thus, the principle value of the integral exists though its general value doesn't exist.

14. $\int_0^1 \sqrt{\frac{1+x}{1-x}} dx$

Sol^n: Here,

$$\int_0^1 \sqrt{\frac{1+x}{1-x}} dx$$

Clearly, $\sqrt{\frac{1+x}{1-x}} \to \infty$ as $x \to 1$

So, the infinite integral is defined as;

$$\int_0^1 \sqrt{\frac{1+x}{1-x}} dx = \lim_{h \to 0}\int_0^{1-h} \sqrt{\frac{1+x}{1-x}} dx$$

$$= \lim_{h \to 0}\int_0^{1-h} \frac{1+x}{\sqrt{1-x^2}} dx$$

$$= \lim_{h \to 0}\left[\int_0^{1-h} \frac{1}{\sqrt{1-x^2}} dx + \int_0^{1-h} \frac{x}{\sqrt{1-x^2}} dx\right]$$

$$= \lim_{h \to 0}\left[\sin^{-1} x - \sqrt{1-x^2}\right]_0^{1-h}$$

$$= \lim_{h \to 0}\left[\sin^{-1}(1-h) - \sqrt{1-h^2} - \sin^{-1} 0 + 1\right]$$

$$= \frac{\pi}{2} + 1$$

$$\therefore \int_0^1 \sqrt{\frac{1+x}{1-x}} dx = \frac{\pi}{2} + 1$$

15. Show that $\int_0^\infty \frac{\log(1+x^2)}{1+x^2} = \pi \log 2$

Sol^n: Here,

$$\int_0^\infty \frac{\log(1+x^2)}{1+x^2}$$

Put $x = \tan\theta$, $dx = \sec^2\theta\, d\theta$

$$\therefore I = \int_0^{\frac{\pi}{2}} \frac{\log \sec^2\theta}{\sec^2\theta} \sec^2\theta\, d\theta$$

$$= \int_0^{\frac{\pi}{2}} 2 \log \sec\theta \, d\theta$$

$$= -2 \int_0^{\frac{\pi}{2}} \log \cos\theta \, d\theta$$

$$= -2 \frac{\pi}{2} \log \frac{1}{2} \qquad \text{(See Exercise 9)}$$

$$= \pi \log 2$$

Differentiation under integral sign: Some important integrals:

1. **Evaluate:**

$$\int_0^\infty \frac{\log(1+a^2x^2)}{1+b^2x^2} dx$$

Sol^n: Let,

$$I = \int_0^\infty \frac{\log(1+a^2x^2)}{1+b^2x^2} dx \ldots \text{(i)}$$

Differentiating with respect to 'a', we get,

$$\frac{dI}{da} = \int_0^\infty \frac{1}{1+a^2x^2} \cdot \frac{2ax^2}{1+b^2x^2} dx$$

$$= \frac{2a}{a^2-b^2} \int_0^\infty \left(\frac{1}{1+b^2x^2} - \frac{1}{1+a^2x^2} \right) dx$$

$$= \frac{2a}{a^2-b^2} \left[\frac{1}{b} \tan^{-1} bx - \frac{1}{a} \tan^{-1} ax \right]_0^\infty$$

$$= \frac{2a}{a^2-b^2} \left(\frac{1}{b} - \frac{1}{a} \right) \frac{\pi}{2}$$

or, $\frac{dI}{da} = \frac{\pi}{b(a+b)}$

or, $dI = \frac{\pi}{b(a+b)} da$

Integrating,

$$I = \frac{\pi}{b} \log(a+b) + C \ldots \text{(ii)}$$

When, $a = 0$

From Equation (i), we get,

$I = 0$

Using this we get, from equation (ii),

$$0 = \frac{\pi}{b} \log b + C$$

or, $C = -\frac{\pi}{b} \log b$

Hence, again from equation (ii),

$$I = \frac{\pi}{b} \log(a+b) - \frac{\pi}{b} \log b = \frac{\pi}{b} \log(a+b)$$

2. **Evaluate:**

$$\int_0^{\frac{\pi}{2}} \log(1+\cos\theta.\cos x)\frac{dx}{\cos x}$$

Sol^n: Let,

$$I = \int_0^{\frac{\pi}{2}} \log(1+\cos\theta.\cos x)\frac{dx}{\cos x} \dots \text{(i)}$$

Differentiating with respect to θ, we get,

$$\frac{dI}{d\theta} = \int_0^{\frac{\pi}{2}} \frac{1}{1+\cos\theta.\cos x}(-\sin\theta.\cos x)\frac{dx}{\cos x}$$

$$= -\sin\theta \int_0^{\frac{\pi}{2}} \frac{1}{1+\cos\theta\left(\frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right)}dx$$

$$= -\sin\theta \int_0^{\frac{\pi}{2}} \frac{\sec^2\frac{x}{2}\,dx}{(1+\cos\theta)+(1-\cos\theta)\tan^2\frac{x}{2}}$$

$$= -\frac{\sin\theta}{1-\cos\theta}\int_0^{\frac{\pi}{2}} \frac{\sec^2\frac{x}{2}\,dx}{\frac{1+\cos\theta}{1-\cos\theta}+\tan^2\frac{x}{2}}$$

$$= -\cot\frac{\theta}{2}\int_0^{\frac{\pi}{2}} \frac{\sec^2\frac{x}{2}\,dx}{\cot^2\frac{\theta}{2}+\tan^2\frac{x}{2}}$$

Let,

$$\tan\frac{x}{2} = z$$

Then,

$$\frac{1}{2}\sec^2\frac{x}{2}dx = dz$$

When,

$$x = 0, z = 0 \text{ and } x = \frac{\pi}{2}, z = 1$$

$$\therefore \quad \frac{dI}{d\theta} = -\cot\frac{\theta}{2}\int_0^1 \frac{2\,dz}{\left(\cot\frac{\theta}{2}\right)^2+z^2}$$

$$= -2\cot\frac{\theta}{2}\cdot\frac{1}{\cot\frac{\theta}{2}}\left[\frac{z}{\cot\frac{\theta}{2}}\right]_0^1$$

$$= -2\tan^{-1}\left(\frac{1}{\cot\frac{\theta}{2}}\right) = -2\frac{\theta}{2}$$

or, $\frac{dI}{d\theta} = -\theta$

$\therefore \quad dI = -\theta\,d\theta$

Integrating,

$$I = -\frac{\theta^2}{2} + C \dots \text{(ii)}$$

Now, when,

$$\theta = \frac{\pi}{2}$$

From equation (i),

$$I = 0$$

Using this in equation (ii), we get,

$$0 = -\frac{\pi^2}{8} + C$$

$$\therefore \quad C = \frac{\pi^2}{8}$$

From equation (ii), again,

$$I = -\frac{\theta^2}{2} + \frac{\pi^2}{8}$$

3. **Evaluate:**

$$\int_0^{\infty} \frac{x^{\gamma}\log x}{1+x^2}dx$$

Sol^n: Let,

$$I = \int_0^{\infty} \frac{x^{\gamma}\log x}{1+x^2}dx \dots \text{(i)}$$

Integrating both sides with respect to γ, we get,

$$\int I\,d\gamma = \int_0^{\infty} \frac{x^{\gamma}\log x}{(1+x^2)\log x}dx \qquad \left(\because \int a^x dx = \frac{a^x}{\log a}\right)$$

$$= \int_0^{\infty} \frac{x^{\gamma}\,dx}{1+x^2}$$

Putting $x^2 = y$; we get,

$$2x\,dx = dy$$

or, $x\,dx = \frac{1}{2}\frac{dy}{x} = \frac{1}{2}\frac{dy}{y^{\frac{1}{2}}}$

Also,

$$x \to (0, \infty) \text{ and } y \to (0, \infty)$$

$$\therefore \quad \int I\, d\gamma = \int_0^\infty \frac{y^{\frac{\gamma}{2}}}{1+y} \cdot \frac{1}{2} y^{-\frac{1}{2}} dy$$

$$= \frac{1}{2} \int_0^\infty \frac{y^{\frac{\gamma-1}{2}}}{1+y} dy$$

$$= \frac{1}{2} \int_0^\infty \frac{y^{\frac{\gamma+1}{2}-1}}{1+y} dy$$

$$= \frac{1}{2} \pi \operatorname{cosec}\left(\frac{\gamma+1}{2}\right)\pi \qquad \left(\because \int_0^\infty \frac{x^{n-1}}{1+x} dx = \frac{\pi}{\sin n\pi}\right)$$

$$= \frac{1}{2} \pi \sec \frac{\gamma\pi}{2}$$

Differentiating with respect to γ; we get,

$$I = \frac{\pi}{2} \frac{\pi}{2} \sec \frac{\gamma\pi}{2} \tan \frac{\gamma\pi}{2}$$

4. **Evaluate:**

$$\int_0^\infty e^{-a^2x^2} \cos(2bx)\, dx$$

Sol^n: Let,

$$I = \int_0^\infty e^{-a^2x^2} \cos(2bx)\, dx \ldots \text{(i)}$$

Integrating both sides with respect to b, we get,

$$\frac{dI}{db} = \int_0^\infty e^{-a^2x^2} (-2x) \sin(2bx)\, dx$$

or, $$\frac{dI}{db} = \frac{1}{a^2} \int_0^\infty e^{-a^2x^2} (-2a^2x) \sin(2bx)\, dx$$

Integrating by parts,

$$\frac{dI}{db} = \frac{1}{a^2}\left[\left\{\sin(2bx) e^{-a^2x^2}\right\}_0^\infty - \int_0^\infty 2b \cos 2bx \,.\, e^{-a^2x^2} dx\right]$$

$$= \frac{1}{a^2}[0 - 2bI]$$

$$= -\frac{2b}{a^2} I$$

or, $$\frac{dI}{I} = -\frac{2b}{a^2} db$$

Integrating,

$$\log I = -\frac{b^2}{a^2} + \log C$$

or, $$I = Ce^{-\frac{b^2}{a^2}} \ldots \text{(ii)}$$

When,

$$b = 0$$

Then, from equation (i),

$$I = \int_0^\infty e^{-a^2x^2} dx$$

i.e. $I = \dfrac{\sqrt{\pi}}{2a}$ (From gamma function; standard form)

Using this condition in equation (ii), we get,

$$\frac{\sqrt{\pi}}{2a} = Ce^0 \qquad (\text{For } b = 0)$$

or, $$C = \frac{\sqrt{\pi}}{2a}$$

Hence, again from equation (ii), we have,

$$I = \frac{\sqrt{\pi}}{2a} e^{-\frac{b^2}{a^2}}$$

5. **Evaluate:**

$$\int_0^1 \frac{x^\alpha + x^{-\alpha} - 2}{(1-x)\log x} dx$$

Sol^n: Let,

$$I = \int_0^1 \frac{x^\alpha + x^{-\alpha} - 2}{(1-x)\log x} dx \ldots \text{(i)}$$

Integrating both sides with respect to α, we get,

$$\frac{dI}{d\alpha} = \int_0^1 \frac{x^\alpha - x^{-\alpha}}{\log x} \frac{\log x}{(1-x)} dx$$

or, $$\frac{dI}{d\alpha} = \int_0^1 \frac{x^\alpha - x^{-\alpha}}{1-x} dx \ldots \text{(ii)}$$

We know,

$$\pi \cot(\pi\alpha) = \int_0^\infty \frac{x^{\alpha-1}}{1-x} dx$$

$$= \int_0^1 \frac{x^{\alpha-1}}{1-x} dx + \int_1^\infty \frac{x^{\alpha-1}}{1-x} dx \ldots \text{(iii)}$$

Let,

$$I_1 = \int_1^{\infty} \frac{x^{\alpha-1}}{1-x}\, dx$$

Putting $x = \frac{1}{y}$; we have,

$$dx = -\frac{1}{y^2} dy$$

Also,

$$x \to (1, \infty) \text{ and } y \to (1, 0)$$

$$\therefore \quad I_1 = \int_1^0 y^{-\alpha+1} \frac{y}{y-1}\left(\frac{-dy}{y^2}\right)$$

$$= \int_0^1 \frac{y^{-\alpha}}{y-1}\, dy = \int_0^1 \frac{x^{-\alpha}}{x-1}\, dy$$

(Replacing y by x; property of definite integral)

Hence, from equation (iii), we have,

$$\pi \cot(\pi\alpha) = \int_0^1 \frac{x^{\alpha-1}}{1-x}\, dx + \int_0^1 \frac{x^{-\alpha}}{x-1}\, dx$$

$$= \int_0^1 \frac{x^{\alpha-1} - x^{-\alpha}}{1-x}\, dx = \int_0^1 \frac{x^{\alpha-1} - x^{\alpha} + x^{\alpha} - x^{-\alpha}}{1-x}\, dx$$

$$= \int_0^1 \frac{x^{\alpha}(x^{-1}-1) + (x^{\alpha} - x^{-\alpha})}{1-x}\, dx$$

$$= \int_0^1 \frac{x^{\alpha}}{x}\, dx + \int_0^1 \frac{x^{\alpha} - x^{-\alpha}}{1-x}\, dx$$

$$= \int_0^1 x^{\alpha-1} dx + \frac{dI}{d\alpha} \quad \text{(From eq}^n\text{. ii)}$$

or, $$\pi \cot(\pi\alpha) = \left[\frac{x^{\alpha-1+1}}{\alpha}\right]_0^1 + \frac{dI}{d\alpha}$$

$$= \frac{1}{\alpha} + \frac{dI}{d\alpha}$$

or, $$dI = \left[\pi \cot(\pi\alpha) - \frac{1}{\alpha}\right] d\alpha$$

Integrating,

$$I = \pi . \frac{1}{\pi} \log \sin(\pi\alpha) - \log \alpha + \log C$$

or, $$I = \log\left(C . \frac{\sin \pi\alpha}{\alpha}\right) \ldots \text{(iv)}$$

When,

$$\alpha = 0$$

Then, from equation (i), we find;

$$I = 0$$

Using this in equation (iv), we get,

$$0 = \log\left(C \lim_{\alpha \to 0} \frac{\sin \pi\alpha}{\alpha}\right) = \log \pi C$$

or, $$\pi C = e^0 = 1$$

or, $$C = \frac{1}{\pi}$$

Hence, from equation (iv),

$$I = \log\left(\frac{1}{\pi a} \sin \pi a\right)$$

6. Evaluate:

$$\int_0^1 \frac{x^{\alpha-1} - x^{-\alpha}}{1+x} \frac{dx}{\log x}$$

Sol[n]: Let,

$$I = \int_0^1 \frac{x^{\alpha-1} - x^{-\alpha}}{1+x} \frac{dx}{\log x} \ldots \text{(i)}$$

Integrating both sides with respect to α, we get,

$$\frac{dI}{d\alpha} = \int_0^1 (x^{\alpha-1} + x^{-\alpha}) \log x . \frac{1}{x} \frac{dx}{\log x}$$

$$= \int_0^1 \frac{x^{\alpha-1} + x^{-\alpha}}{1+x}\, dx \ldots \text{(ii)}$$

Now, we know that,

$$\pi \operatorname{cosec}(a\pi) = \int_0^{\infty} \frac{t^{\alpha-1}}{1+t}\, dt$$

$$= \int_0^1 \frac{t^{\alpha-1}}{1+t}\, dt + \int_1^{\infty} \frac{t^{\alpha-1}}{1+t}\, dt \ldots \text{(iii)}$$

Let,

$$I_1 = \int_1^{\infty} \frac{t^{\alpha-1}}{1+t}\, dt$$

Putting $t = \frac{1}{y}$; we have,

$$dt = -\frac{1}{y^2}$$

Also,

$$t \to (1, \infty) \text{ and } y \to (1, 0)$$

$$\therefore \quad I_1 = \int_1^0 \frac{y.y^{-\alpha+1}}{y+1} \frac{dy}{-y^2}$$

$$= \int_0^1 \frac{t^{-\alpha}}{1+t} dt$$

(Changing y by t; property of definite integral)

Hence, from equation (iii), we have,

$$\pi \operatorname{cosec}(a\pi) = \int_0^1 \frac{t^{\alpha-1}}{1+t} dt + \int_0^1 \frac{t^{-\alpha}}{1+t} dt$$

$$= \int_0^1 \frac{t^{\alpha-1} + t^{-\alpha}}{1+t} dt$$

$$= \frac{dI}{d\alpha} \quad \text{(From eq}^n\text{. ii)}$$

$$\therefore \quad dI = \pi \operatorname{cosec}(a\pi)\, d\alpha$$

Integrating,

$$I = \pi . \frac{1}{\pi} \log \tan\left(\frac{a\pi}{2}\right) + C$$

or, $I = \log \tan\left(\frac{a\pi}{2}\right) + C \ldots$ (iv)

When,

$$\alpha = \frac{1}{2}$$

Then, from equation (i), we find;

$$I = 0$$

Using this in equation (iv), we get,

$$\log \tan\left(\frac{\pi}{4}\right) + C$$

or, $C = 0$

Hence, from equation (iii),

$$I = \log \tan\left(\frac{a\pi}{2}\right)$$

7. Evaluate:

$$\int_0^\infty \frac{\tan^{-1} ax\, dx}{x(1+x^2)}$$

Sol^n: Let,

$$I = \int_0^\infty \frac{\tan^{-1} ax\, dx}{x(1+x^2)} \ldots \text{(i)}$$

Differentiating both sides with respect to α, we have,

$$\frac{dI}{d\alpha} = \int_0^\infty \frac{1}{1+a^2x^2} \frac{x}{x(1+x^2)} dx$$

or, $\frac{dI}{d\alpha} = \int_0^\infty \frac{dx}{(1+x^2)(1+a^2x^2)}$

$$= \frac{a^2}{1-a^2} \int_0^\infty \left[\frac{1}{a^2(1+x^2)} - \frac{1}{1+a^2x^2}\right] dx \quad \text{(By partial fraction)}$$

$$= \frac{a^2}{1-a^2} \left[\frac{1}{a^2} \tan^{-1} x - \frac{1}{a} \tan^{-1}(ax)\right]_0^\infty$$

$$= \frac{a^2}{1-a^2} \left(\frac{1}{a^2} - \frac{1}{a}\right) \frac{\pi}{2} = \frac{\pi}{2(1+a)}$$

$$\therefore \quad dI = \frac{\pi}{2(1+a)} da$$

Integrating,

$$I = \frac{\pi}{2} \log(1+a) + C \ldots \text{(ii)}$$

When,

$$a = 0$$

Then, from equation (i),

$$I = 0$$

Using this in equation (ii), we get,

$$0 = \frac{\pi}{2} \log 1 + C$$

or, $C = 0$

Hence, again from equation (ii),

$$I = \frac{\pi}{2} \log(1+a)$$

8. **Prove that:**

$$\int_0^\infty \frac{e^{-ax}\sin bx}{x}dx$$

Hence deduce that:

$$\int_0^\infty \frac{\sin bx}{x}\,dx = \begin{cases} \frac{\pi}{2} \text{ if } b > 0 \\ -\frac{\pi}{2} \text{ if } b < 0 \end{cases}$$

Sol^n: Let,

$$I = \int_0^\infty \frac{e^{-ax}\sin bx}{x}dx \ldots \text{(i)}$$

Differentiating both sides with respect to b, we have,

$$\frac{dI}{db} = \int_0^\infty \frac{e^{-ax}\,x\cos bx}{x}dx$$

or, $\frac{dI}{d\alpha} = \int_0^\infty e^{-ax}\cos bx\ dx$

$$= \left[\frac{-ae^{-ax}\cos bx - e^{-ax}(-b\sin bx)}{a^2+b^2}\right]_0^\infty$$

$$= \frac{a}{a^2+b^2} \qquad \left(\text{Using } \lim_{z\to\infty} e^{-z} = 0\right)$$

$\therefore \quad dI = \frac{a}{a^2+b^2}db$

Integrating,

$$I = a\frac{1}{a}\tan^{-1}\frac{b}{a} + C$$

or, $I = \tan^{-1}\frac{b}{a} + C \ldots$ (ii)

When,

$b = 0$

Then, from equation (i),

$I = 0$

Using this in equation (ii), we get,

$0 = \tan^{-1}0 + C$

$\therefore \quad C = 0$

Hence, again from equation (ii),

$I = \tan^{-1}\frac{b}{a}$

Deduction part:

We obtained that;

$$\int_0^\infty \frac{e^{-ax}\sin bx}{x}dx = \tan^{-1}\frac{b}{a}$$

Taking $a = 0$, we find,

$$\int_0^\infty \frac{\sin bx}{x}\,dx = \begin{cases} \frac{\pi}{2} \text{ if } b > 0 \\ -\frac{\pi}{2} \text{ if } b < 0 \end{cases} \text{ Proved}$$

9. **Evaluate:**

$$\int_0^{\frac{\pi}{2}} \log(a^2\cos^2\theta + b^2\cos^2\theta)d\theta$$

Sol^n: Let,

$$I = \int_0^{\frac{\pi}{2}} \log(a^2\cos^2\theta + b^2\cos^2\theta)d\theta \ldots \text{(i)}$$

Assume that;

$\alpha, \beta > 0$

Differentiating both sides with respect to a, we have,

$$\frac{dI}{da} = \int_0^{\frac{\pi}{2}} \frac{1}{a^2\cos^2\theta + b^2\cos^2\theta}\ 2a\cos^2\theta\,d\theta$$

or, $\frac{dI}{d\alpha} = 2a\int_0^{\frac{\pi}{2}} \frac{d\theta}{a^2+b^2\tan^2\theta}$

Let,

$\tan\theta = t$

or, $\sec^2\theta = dt$

Also,

$\theta \to \left(0, \frac{\pi}{2}\right)$

We get,

$t \to (0,\ \infty)$

so,

$$= 2a\int_0^\infty \frac{1}{a^2+b^2t^2}\,\frac{dt}{1+t^2} = \frac{2a}{b^2-a^2}\int_0^\infty \frac{b^2}{a^2+b^2t^2} - \frac{1}{1+t^2}$$

$$= \frac{2a}{b^2-a^2}\int_0^\infty \left\{\frac{b^2}{a^2+b^2t^2} - \frac{1}{1+t^2}\right\}dt \qquad \text{(By partial fraction)}$$

$$= \frac{2a}{b^2-a^2}\left[\frac{b^2}{a.b}\tan^{-1}\left(\frac{bt}{a}\right) - \tan^{-1}t\right]^\infty$$

$$= \frac{2a}{b^2-a^2}\left(\frac{b}{a}-1\right)\frac{\pi}{2} = \frac{\pi}{a+b}$$

$\therefore \quad dI = \frac{\pi}{a+b}da$

Integrating,

$$I = \pi \log(a+b) + C \ldots \text{(ii)}$$

Putting a = b in equation (i), we get,

$$I = \int_0^{\frac{\pi}{2}} \log a^2 \, d\theta = 2\log a \frac{\pi}{2} = \pi \log a$$

$$a = 0$$

Using this in equation (ii), we get,

$$\pi \log a = \pi \log 2a + C$$

or, $C = -\pi \log 2$

Using equation (ii) again with this value of C, we get,

$$I = \pi \log(a+b) - \pi \log 2 = \pi \log\left(\frac{a+b}{2}\right)$$

10. Evaluate:

$$\int_0^{\pi} \frac{\log(1+a\cos x)}{\cos x}dx$$

Solⁿ: Let,

$$I = \int_0^{\pi} \frac{\log(1+a\cos x)}{\cos x}dx \ldots \text{(i)}$$

Differentiating both sides with respect to a, we have,

$$\frac{dI}{da} = \int_0^{\pi} \frac{1}{1+a\cos x}.\cos x \frac{dx}{\cos x}$$

$$= \int_0^{\pi} \frac{dx}{1+a\cos x} = \int_0^{\pi} \frac{dx}{1+a\left(\frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right)}$$

$$= \int_0^{\pi} \frac{\sec^2\frac{x}{2}\,dx}{(1+a)+(1-a)\tan^2\frac{x}{2}}$$

$$= \frac{1}{1-a}\int_0^{\pi} \frac{\sec^2\frac{x}{2}\,dx}{\frac{1+a}{1-a}+\tan^2\frac{x}{2}}$$

Let,

$$\tan\frac{x}{2} = z$$

Then,

$$\frac{1}{2}\sec^2\frac{x}{2}\,dx = dz$$

Also,

$$x \to (0, \pi)$$

We get,

$$z \to (0, \infty)$$

$$\therefore \quad \frac{dI}{da} = \frac{2}{1-a}\int_0^{\infty} \frac{dz}{\frac{1+a}{1-a}+z^2} = \frac{2}{1-a}\frac{1}{\sqrt{\frac{1+a}{1-a}}}\left[\tan^{-1}\frac{z}{\sqrt{\frac{1+a}{1-a}}}\right]_0^{\infty} = \frac{2}{\sqrt{1-a^2}}\frac{\pi}{2}$$

or, $\frac{dI}{da} = \frac{\pi}{\sqrt{1-a^2}}$

$\therefore \quad dI = \frac{\pi}{\sqrt{1-a^2}}da$

Integrating,

$$I = \pi \sin^{-1} a + C \ldots \text{(ii)}$$

When,

$$a = 0$$

From equation (i), we get,

$$I = 0$$

Using this in equation (ii), we get,

$$I = \pi \sin^{-1}(a)$$

11. Evaluate:

$$\int_0^{\frac{\pi}{2}} \frac{1}{a^2\sin^2x+b^2\cos^2x}dx$$ **and hence deduce that:**

$$\int_0^{\frac{\pi}{2}} \frac{1}{\left(a^2\sin^2x+b^2\cos^2x\right)^2}dx = \frac{\pi\left(a^2+b^2\right)}{4a^3b^3}$$

Solⁿ: We shall first evaluate;

$$I = \int_0^{\frac{\pi}{2}} \frac{1}{a^2\sin^2x+b^2\cos^2x}dx$$

or, $$I = \int_0^{\frac{\pi}{2}} \frac{\sec^2 x\, dx}{a^2\tan^2 x + b^2}$$

Putting $\tan x = z$; we have,

$$\sec^2 x\, dx = dz$$

Also,

$$x \to \left(0, \frac{\pi}{2}\right)$$

and $z \to (0, \infty)$

$$I = \int_0^{\infty} \frac{dz}{a^2z^2 + b^2} = \frac{1}{ab}\left[\tan^{-1}\frac{az}{b}\right]_0^{\infty} = \frac{\pi}{2ab}$$

i.e. $$\int_0^{\frac{\pi}{2}} \frac{1}{a^2\sin^2 x + b^2\cos^2 x}\, dx = \frac{\pi}{2ab} \ldots \text{(i)}$$

Differentiating partially with respect to 'a' on both sides, we get,

$$\int_0^{\frac{\pi}{2}} \frac{\partial}{\partial a}(a^2\sin^2 x + b^2\cos^2 x)^{-1} = \frac{\partial}{\partial a}\left(\frac{\pi}{2ab}\right)$$

or, $$\int_0^{\frac{\pi}{2}} -(a^2\sin^2 x + b^2\cos^2 x)^{-2}.\, 2a\sin^2 x\, dx = \frac{\pi}{2ab}(-a^{-2})$$

or, $$\int_0^{\frac{\pi}{2}} \frac{\sin^2 x\, dx}{\left(a^2\sin^2 x + b^2\cos^2 x\right)^2} = \frac{\pi}{4a^3b} \ldots \text{(ii)}$$

Similarly, differentiating partially with respect to 'b' on both sides, then from equation (i), we get,

$$\int_0^{\frac{\pi}{2}} \frac{\cos^2 x\, dx}{\left(a^2\sin^2 x + b^2\cos^2 x\right)^2} = \frac{\pi}{4a^3b} \ldots \text{(iii)}$$

Adding equation (ii) and (iii), we have,

$$\int_0^{\frac{\pi}{2}} \frac{dx}{\left(a^2\sin^2 x + b^2\cos^2 x\right)^2} = \frac{\pi}{4a^3b} + \frac{\pi}{4ab^3} = \frac{\pi\left(a^2 + b^2\right)}{4a^3b^3} \text{ Proved}$$

12. **Evaluate:**

$$\int_0^1 \frac{x^{\alpha} - 1}{\log x}\, dx$$

Solⁿ: Let,

$$I = \int_0^1 \frac{x^{\alpha} - 1}{\log x}\, dx \ldots \text{(i)}$$

Differentiating both sides with respect to α, we have,

$$\frac{dI}{d\alpha} = \int_0^1 \frac{x^{\alpha}.\log x}{\log x}\, dx = \int_0^1 x^{\alpha} = \frac{1}{\alpha + 1}$$

or, $dI = \frac{1}{\alpha + 1}\, d\alpha$

Integrating,

$$I = \log(\alpha + 1) + C \ldots \text{(ii)}$$

When,

$$\alpha = 0$$

Then, from equation (i),

$$I = 0$$

Using this in equation (ii), we get,

$$0 = \log 1 + C$$

i.e. $C = 0$

Hence, again from equation (ii),

$$I = \log(\alpha + 1)$$

13. **Evaluate:**

$$\int_0^{\infty} \frac{\tan^{-1} ax}{x\left(1 + x^2\right)}\, dx$$

Solⁿ: Let,

$$I = \int_0^{\infty} \frac{\tan^{-1} ax}{x\left(1 + x^2\right)}\, dx \ldots \text{(i)}$$

Differentiating both sides with respect to a, we have,

$$\frac{dI}{da} = \int_0^{\infty} \frac{1}{1 + a^2x^2}.\, x.\, \frac{1}{x\left(1 + x^2\right)}\, dx$$

$$= \int_0^{\infty} \frac{dx}{(1 + a^2x^2)(1 + x^2)}$$

$$= \frac{1}{1-a^2}\int_0^{\infty}\left(\frac{1}{1-x^2} - \frac{a^2}{1+a^2x^2}\right)dx$$

$$= \frac{1}{1-a^2}\left[\tan^{-1}x - a^2\frac{1}{a}\tan^{-1}(ax)\right]_0^{\infty}$$

$$= \frac{1}{1-a^2}(1-a)\frac{\pi}{2}$$

$$= \frac{\pi}{2(1+a)}$$

$\therefore \quad dI = \frac{\pi}{2(1+a)}da$

Integrating,

$$I = \frac{\pi}{2}\log(1+a) + C \ldots \text{(ii)}$$

When,

$a = 0$

Then, from equation (i),

$I = 0$

Using this condition in equation (ii), we get,

$0 = \frac{\pi}{2}\log 1 + C$

i.e. $C = 0$

$\therefore$ Again from equation (ii),

$$I = \frac{\pi}{2}\log(1+a)$$

Exercise 13

Reduction formula:

The formula of the given integral which is connected with some integrals of lower order is called reduction formula. Most of the cases the reduction formula is obtained by the method of integration.

1. Obtain a reduction formula for $\int_0^{\frac{\pi}{2}}\cos^n x\,dx$ and hence evaluate $\int_0^{\frac{\pi}{2}}\cos^{10} x\,dx$ [2060, 2062 B.E.]

Solⁿ: Let,

$$J_n = \int_0^{\frac{\pi}{2}}\cos^n x\,dx$$

$$= \int_0^{\frac{\pi}{2}}\cos^{n-1} x\cos x\,dx$$

Integration by parts

$$J_n = [\cos^{n-1}x\sin x]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}}(n-1)\cos^{n-2}x(-\sin x)\sin x\,dx$$

$$= 0 + (n-1)\int_0^{\frac{\pi}{2}}\cos^{n-2}x\sin^2 x\,dx$$

$$= (n-1)\int_0^{\frac{\pi}{2}}\cos^{n-2}x\sin^2 x\,dx$$

$$= (n-1)\int_0^{\frac{\pi}{2}}\cos^{n-2}x(1-\cos^2 x)dx$$

$$= (n-1)\int_0^{\frac{\pi}{2}}\cos^{n-2}x\,dx - (n-1)\int_0^{\frac{\pi}{2}}\cos^n x\,dx$$

$J_n = (n-1)J_{n-2} - (n-1)J_n$

or, $(1+n-1)J_n = (n-1)J_{n-2}$

$\therefore \quad J_n = \frac{n-1}{n}J_{n-2}$

which is the reduction formula for $\int_0^{\frac{\pi}{2}}\cos^n x\,dx$

Now,

$$J_{10} = \int_0^{\frac{\pi}{2}}\cos^{10}x\,dx = \frac{(10-1)}{10}J_8 = \frac{9}{10}\cdot\frac{(8-1)}{8}J_6$$

$$= \frac{9}{10}\cdot\frac{7}{8}\cdot\frac{(6-1)}{6} J_4 = \frac{9}{10}\cdot\frac{7}{8}\cdot\frac{5}{6}\cdot\frac{3}{4}\cdot\frac{(2-1)}{2} J_0$$

$$= \frac{63}{256}\int_0^{\frac{\pi}{2}} dx = \frac{63}{256}[x]_0^{\frac{\pi}{2}} = \frac{63\pi}{512}$$

2. **Obtain a reduction formula for $\int \sec^n x\,dx$ and hence find $\int \sec^6 x\,dx$**

Sol^n: Let,

$$I_n = \int \sec^n x\,dx$$

$$= \int \sec^{n-2} x \sec^2 x\,dx$$

Integrating by parts,

$$= \sec^{n-2} x \tan x - \int (n-2)\sec^{n-3} x \sec x \tan x \tan x\,dx$$

$$= \sec^{n-2} x \tan x - (n-2)\int \sec^{n-2} x(\sec^2 x - 1)dx$$

$$I_n = \sec^{n-2} x \tan x - (n-2)\int \sec^n x\,dx + (n-2)\int \sec^{n-2} x\,dx$$

$$= \sec^{n-2} x \tan x - (n-2)I_n + (n-2)I_{n-2}$$

$$\therefore \quad I_n = \frac{\sec^{n-2} x \tan x}{n-1} + \frac{(n-2)}{(n-1)} I_{n-2}$$

which is the reduction formula for $\int \sec^n x\,dx$

3. **If $I_n = \int_0^{\frac{\pi}{4}} \tan^n x\,dx$, show that $I_n + I_{n-2} = \frac{1}{n-1}$ and hence deduce the value of I_5.**

Sol^n: Let,

$$I_n = \int_0^{\frac{\pi}{4}} \tan^n x\,dx$$

$$= \int_0^{\frac{\pi}{4}} \tan^{n-2} x \tan^2 x\,dx = \int_0^{\frac{\pi}{4}} \tan^{n-2} x(\sec^2 x - 1)dx$$

$$= \int_0^{\frac{\pi}{4}} \tan^{n-2} x \sec^2 x\,dx - \int_0^{\frac{\pi}{4}} \tan^{n-2} x\,dx$$

$$= \frac{1}{n-1}\int_0^{\frac{\pi}{4}} d(\tan^{n-1} x) - I_{n-2} = \frac{1}{n-1}[\tan^{-1} x]_0^{\frac{\pi}{4}} - I_{n-2}$$

$$\therefore \quad I_n = \frac{1}{n-1} - I_{n-2}$$

$$\therefore \quad I_n + I_{n-2} = \frac{1}{n-1}$$

Now,

$$I_5 = \frac{1}{5-1} - I_3$$

$$= \frac{1}{4} - \left(\frac{1}{3-1} - I_1\right) = \frac{1}{4} - \frac{1}{2} + \int_0^{\frac{\pi}{4}} \tan x\,dx$$

$$= \frac{1}{4} - \frac{1}{2} + [\log \sec x]_0^{\frac{\pi}{4}}$$

$$= -\frac{1}{4} + (\log\sqrt{2} - \log 1)$$

$$\therefore \quad I_5 = \frac{1}{2}\log 2 - \frac{1}{4}$$

4. **If $I_{m,n} = \int_0^{\frac{\pi}{2}} \cos^m x \sin x\,dx$, prove that :**

$I_{m,n} = \frac{1}{m+n} + \frac{m}{m+n} I_{m-1,n-1}$

Sol^n: Here,

$$I_{m,n} = \int_0^{\frac{\pi}{2}} \cos^m x \sin x\,dx$$

Integrating by parts,

$$= \frac{1}{n}[\cos^m x(-\cos nx)]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} m\cos^{m-1} x(-\sin x)\left(\frac{-\cos nx}{n}\right)dx$$

$$= \frac{1}{n}[0+1] - \frac{m}{n}\int_0^{\frac{\pi}{2}} \cos^{m-1} x \cos nx \sin x\,dx$$

But,

$$\sin(nx - x) = \sin x \cos x - \cos nx \sin x$$

$$\cos nx \sin x = \sin nx \cos x - \sin(nx - x)$$

$$= \frac{1}{n} - \frac{m}{n}\int_0^{\frac{\pi}{2}} \cos^{m-1} x[\sin nx \cos x - \sin(nx - x)]dx$$

$$= \frac{1}{n} - \frac{m}{n}\int_0^{\frac{\pi}{2}} \cos^m x \sin nx\,dx + \frac{m}{n}\int_0^{\frac{\pi}{2}} \cos^{m-1} x \sin(n-1)x\,dx$$

$$= \frac{1}{n} - \frac{m}{n} I_{m,n} + \frac{m}{n} I_{m-1,n-1}$$

or, $\left(1 + \frac{m}{n}\right) I_{m,n} = \frac{1}{n} + \frac{m}{n} I_{m-1,n-1}$

or, $\left(\frac{n+m}{n}\right) I_{m,n} = \frac{1}{n} + \frac{m}{n} I_{m-1,n-1}$

$$\therefore \quad I_{m,n} = \frac{1}{m+n} + \frac{m}{m+n} I_{m-1,n-1}$$

5. **Prove that $\int_0^{\frac{\pi}{2}} \cos^5 x \sin 3x \, dx = \frac{1}{3}$**

Sol^n: Let,

$$I = \int_0^{\frac{\pi}{2}} \cos^5 x \sin 3x \, dx$$

By using the formula for $\int_0^{\frac{\pi}{2}} \cos^m x \sin nx \, dx$

$$\int_0^{\frac{\pi}{2}} \cos^5 x \sin 3x \, dx = \frac{1}{5+3} + \frac{5}{5+3}\int_0^{\frac{\pi}{2}} \cos^4 x \sin 2x \, dx$$

$$= \frac{1}{8} + \frac{5}{8}\left[\frac{1}{4+2} + \frac{4}{4+2}\int_0^{\frac{\pi}{2}} \cos^3 x \sin x \, dx\right]$$

$$= \frac{1}{8} + \frac{5}{48} + \frac{5}{12}\left[\frac{1}{3+1} + \frac{3}{3+1}\int_0^{\frac{\pi}{2}} \cos^2 x \sin x \, dx\right]$$

$$= \frac{1}{8} + \frac{5}{48} + \frac{5}{48} + 0 = \frac{16}{48} = \frac{1}{3}$$

6. **Prove that $\int_0^{\frac{\pi}{2}} \cos^m x \sin mx \, dx = \frac{1}{2^{m+1}}\left(2 + \frac{2^2}{2} + \frac{2^3}{3} + \cdots + \frac{2^m}{m}\right)$**

Sol^n: Let,

$$\int_0^{\frac{\pi}{2}} \cos^m x \sin mx \, dx$$

By using the formula for

$$\int_0^{\frac{\pi}{2}} \cos^m x \sin mx \, dx$$

$$= \frac{1}{m+m} + \frac{m}{m+m}\int_0^{\frac{\pi}{2}} \cos^{m-1} x \sin(m-1)x \, dx$$

$$= \frac{1}{2m} + \frac{1}{2}\left[\frac{1}{m-1+m-1} + \frac{m-1}{m-1+m-1}\int_0^{\frac{\pi}{2}} \cos^{m-2} x \sin(m-2)x \, dx\right]$$

$$= \frac{1}{2m} + \frac{1}{2^2(m-1)} + \frac{1}{2^2}\int_0^{\frac{\pi}{2}} \cos^{m-2} x \sin(m-2)x \, dx$$

$$= \frac{1}{2m} + \frac{1}{2^2(m-1)} + \frac{1}{2^2}\frac{1}{(m-2+m-2)} + \frac{1}{2^2}\frac{m-2}{(m-2+m-2)}\int_0^{\frac{\pi}{2}} \cos^{m-3} x \sin(m-3)x \, dx$$

$$= \frac{1}{2m} + \frac{1}{2^2(m-1)} + \frac{1}{2^3(m-2)} + \frac{1}{2^3}\int_0^{\frac{\pi}{2}} \cos^{m-3} x \sin(m-3)x \, dx$$

By this process, we arrive in the form,

$$= \frac{1}{2m} + \frac{1}{2^2(m-1)} + \frac{1}{2^3(m-2)} + \cdots\cdots + \frac{1}{2^{m-1}(\ldots 1)}$$

$$+ \frac{1}{2^m(m-m+1)} + \frac{1}{2m}\cos^{m-m} x \sin(m-m)x \, dx$$

$$= \frac{1}{2m} + \frac{1}{2^2(m-1)} + \frac{1}{2^3(m-2)} + \cdots + \frac{1}{2^{m-1}\cdot 2} + \frac{1}{2^{m-1}\cdot 1} + 0$$

$$= \frac{1}{2^m\cdot 1} + \frac{1}{2^{m-1}\cdot 2} + \frac{1}{2^{m-2}\cdot 3} + \cdots + \frac{1}{2^3(m-2)} + \frac{1}{2^2(m-1)} + \frac{1}{2m}$$

$$= \frac{1}{2^{m+1}}\left[2 + \frac{2^2}{2} + \frac{2^3}{3} + \cdots + \frac{2^{m-2}}{m-2} + \frac{2^{m-1}}{m-1} + \frac{2^m}{m}\right]$$

7. **If m, n are positive integer, then show that:**

$$I_{m,n} = \int_a^b (x-a)^m (b-x)^n dx = \frac{n(b-a)}{m+n+1} I_{m,n-1}$$

Sol^n: Here,

$$I_{m,n} = \int_a^b (x-a)^m (b-x)^n dx$$

Integrating by parts,

$$\left[(b-x)^n \frac{(x-a)^{m+1}}{m+1}\right]_a^b - n\int_a^b (b-x)^{n-1}(-1)\frac{(x-a)^{m+1}}{m+1} dx$$

$$= 0 - 0 + \frac{n}{m+1}\int_a^b (b-x)^{n-1}(x-a)^{m+1} dx$$

$$= \frac{n}{m+1}\int_a^b (x-a)^m (b-x)^{n-1}(x-a) dx$$

$$= \frac{n}{m+1}\int_a^b (x-a)^m (b-x)^{n-1}(x-a-b+b) dx$$

$$= \frac{n}{m+1}\int_a^b (x-a)^m (b-x)^{n-1}[-(b-x)+(b-a)] dx$$

$$= \frac{-n}{m+1}\int_a^b (x-a)^m (b-x)^n dx + \frac{n(b-a)}{m+1}\int_a^b (x-a)^m (b-x)^{n-1} dx$$

$$\therefore \quad I_{m,n} = \frac{-n}{m+1} I_{m,n} + \frac{n(b-a)}{m+1} I_{m,n-1}$$

$$\text{or,} \quad \left(1 + \frac{n}{m+1}\right) I_{m,n} = \frac{n(b-a)}{m+1} I_{m,n-1}$$

$$\text{or,} \quad \frac{(m+n+1)}{m+1} I_{m,n} = \frac{n(b-a)}{m+1} I_{m,n-1}$$

$$\therefore \quad I_{m,n} = \frac{n(b-a)}{m+n+1} I_{m,n-1}$$

8. Obtain a reduction formula for $\int \cos^m x \cos nx\, dx$ and hence show that $\int_0^{\frac{\pi}{2}} \cos^n x \cos nx\, dx = \frac{\pi}{2^{n+1}}$

Sol^n: Let,

$$I_{m,n} = \int \cos^m x \cos nx\, dx$$

Integrating by parts,

$$= \frac{\cos^m x \sin nx}{n} - \int m \cos^{m-1} x (-\sin x) \frac{\sin nx}{n} dx$$

$$= \frac{\cos^m x \sin nx}{n} + \frac{m}{n} \int \cos^{m-1} x \sin nx \sin x\, dx$$

But,

$$\cos(nx - x) = \cos nx \cos x + \sin nx \sin x$$

$$\sin nx \sin x = \cos(n-1)x - \cos nx \cos x$$

$$= \frac{\cos^m x \sin nx}{n} + \frac{m}{n} \int \cos^{m-1} x \, [\cos(n-1)x - \cos nx \cos x]dx$$

$$= \frac{\cos^m x \sin nx}{n} + \frac{m}{n} \int \cos^{m-1} x \cos(n-1)x\, dx - \frac{m}{n} \int \cos^m x \cos nx\, dx$$

$$\therefore \quad I_{m,n} = \frac{\cos^m x \sin nx}{n} + \frac{m}{n} I_{m-1,n-1} - \frac{m}{n} I_{m,n}$$

or, $\left(1 + \frac{m}{n}\right) I_{m,n} = \frac{\cos^m x \sin nx}{n} + \frac{m}{n} I_{m-1,n-1}$

$$\therefore \quad I_{m,n} = \frac{\cos^m x \sin nx}{n} + \frac{m}{m+n} I_{m-1,n-1}$$

which is the required reduction formula

Let,

$$J_n = \int_0^{\frac{\pi}{2}} \cos^n x \cos nx\, dx$$

Using formula,

$$J_n = \left[\frac{\cos^n x \sin nx}{n+n}\right]_0^{\frac{\pi}{2}} + \frac{n}{n+n} J_{n-1}$$

or, $J_n = 0 + \frac{1}{2} J_{n-1} = \frac{1}{2}\left[0 + \frac{n-1}{n-1+n-1}\right] J_{n-2}$

or, $J_n = \frac{1}{2^2} J_{n-2}$

or, $J_n = \frac{1}{2^2}\left[0 + \frac{n-2}{n-2+n-2} J_{n-3}\right]$

$$= \frac{1}{2^3} J_{n-3} \text{ and so on}$$

$$J_n = \frac{1}{2^n} J_{n-n} = \frac{1}{2^n} J_0$$

But,

$$J_0 = \int_0^{\frac{\pi}{2}} dx$$

$$J_0 = [x]_0^{\frac{\pi}{2}}$$

$$= \frac{\pi}{2} - 0$$

$$= \frac{\pi}{2}$$

$$\therefore \quad J_n = \frac{1}{2^n} \cdot \frac{\pi}{2} = \frac{\pi}{2^{n+1}}$$

Exercise 14

Beta and Gamma functions

Definition:

1. The integral $\int_0^1 x^{m-1}(1-x)^{n-1}dx\,(m>0, n>0)$ is known as the first Eulerian Integral or Beta function and is written as

   $\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx\,(m>0, n>0)$

2. The integral $\int_0^\infty e^{-x}x^{n-1}dx\,(n>0)$ is known as the second Eulerian integral or Gamma function and is written as;

   $\Gamma(n) = \int_0^\infty e^{-x}x^{n-1}dx\,(n>0)$

Some important formula:

1. $\beta(m,n) = \beta(n,m)$
2. $\Gamma(1) = 1$
3. $\Gamma(n+1) = n\Gamma n$ for n is positive integer.
4. $\int_0^\infty e^{-kx}x^{n-1}dx = \frac{\Gamma n}{k^n}$
5. $\beta(m,n) = \frac{\Gamma m + \Gamma n}{\Gamma m+n}$
6. $\beta(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}dx = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}dx$
7. $\Gamma(n+1) = n!$; n is positive integer

$\frac{\Gamma}{m\pi}$

or,

$\theta = \frac{\Gamma\frac{(p+1)}{2}\Gamma\frac{q+1}{2}}{\frac{p+q+2}{2}}, p, q > -1$

or, $J_n =$

11. $\int_0^{\frac{\pi}{2}} \sin^p\theta\, d\theta = \frac{\sqrt{\pi}\,\Gamma\frac{(p+1)}{2}}{2\,\Gamma\frac{(p+2)}{2}}$

12. $\int_0^{\frac{\pi}{2}} \cos^q\theta\, d\theta = \frac{\sqrt{\pi}}{2}\frac{\Gamma\frac{(q+1)}{2}}{\Gamma\frac{(q+2)}{2}}$

---

1. Find the value of (i) $\Gamma\frac{7}{2}$

   i) $\Gamma\frac{7}{2} = \Gamma\left(\frac{5}{2}+1\right)$

Sol[n]:

$= \frac{5}{2}\Gamma\frac{5}{2}$

$= \frac{5}{2}\Gamma\frac{3}{2}+1$

$\therefore \quad \Gamma x + 1 = x\Gamma x = \frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma\frac{1}{2} = \frac{15}{8}\sqrt{\pi}$

   ii) $\frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{5}{2}\right)}{\Gamma 5}$

Sol[n]:

$\frac{\Gamma\frac{1}{2}+1\ \Gamma\frac{3}{2}+1}{\Gamma 4+1} = \frac{\frac{1}{2}\Gamma\frac{1}{2}\frac{3}{2}\frac{1}{2}\Gamma\frac{1}{2}}{4!} = \frac{3\pi}{8\times 24} = \frac{\pi}{64}$

2. Evaluate,

   i) $\int_0^{\frac{\pi}{2}} \sin^4 x\cos^2 x\, dx$

Sol[n]: Here,

$\int_0^{\frac{\pi}{2}} \sin^4 x\cos^2 x\, dx = \frac{\Gamma\frac{4+1}{2}\Gamma\frac{2+1}{2}}{2\,\Gamma\frac{4+2+2}{2}}$

$= \frac{\Gamma\frac{5}{2}\Gamma\frac{3}{2}}{2\,\Gamma 4} = \frac{\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}\cdot\frac{1}{2}\sqrt{\pi}}{2\cdot 3!} = \frac{3\pi}{2\times2\times2\times2\times3\times2} = \frac{\pi}{32}$

   ii) $\int_0^{\frac{\pi}{4}} (1-2\sin^2\theta)^{\frac{3}{2}}\cos\theta\, d\theta$

Sol[n]: Here,

$\int_0^{\frac{\pi}{4}} (1-2\sin^2\theta)^{\frac{3}{2}}\cos\theta\, d\theta$

Put,

$$\sqrt{2}\sin\theta = \sin x$$

Then,

$$\sqrt{2}\cos\theta\, d\theta = \cos x\, dx$$

Also,

When $\theta = 0, x = 0$ and when $\theta = \frac{\pi}{4}, x = \frac{\pi}{2}$

$$\therefore \quad I = \int_0^{\frac{\pi}{2}} (1-\sin^2 x)^{\frac{3}{2}} \frac{\cos x}{\sqrt{2}}\, dx = \frac{1}{\sqrt{2}} \int_0^{\frac{\pi}{2}} \cos^4 x\, dx$$

$$= \frac{1}{2} \cdot \frac{\sqrt{\pi}}{2} \frac{\Gamma\left(\frac{5}{2}\right)}{\Gamma\left(\frac{6}{2}\right)} = \frac{\sqrt{\pi}}{\sqrt{2}} \cdot \frac{\frac{3}{2}\cdot\frac{1}{2}\,\Gamma\left(\frac{1}{2}\right)}{2\cdot 1} = \frac{3}{16\sqrt{2}} \cdot \sqrt{\pi} \cdot \sqrt{\pi} = \frac{3\pi}{16\sqrt{2}}$$

3. Use beta function to evaluate

i) $\int_0^a \mathbf{x^2 (a^2 - x^2)^{\frac{3}{2}} dx}$

Sol^n: Here,

$$\int_0^a x^2 (a^2 - x^2)^{\frac{3}{2}} dx$$

Put,

why ? $x = a\sin\theta, dx = a\cos\theta\, d\theta$

When,

$$x = 0, \theta = 0, \text{ when } x = a, \theta = \frac{\pi}{2}$$

So,

$$\int_0^a x^2 (a^2 - x^2)^{\frac{3}{2}} dx = \int_0^{\frac{\pi}{2}} a^2 \sin^2\theta\, (a^2 - a^2\sin^2\theta)^{\frac{3}{2}}\, a\cos\theta\, d\theta$$

$$= a^4 \int_0^{\frac{\pi}{2}} a^2 \sin^2\theta \cos^3\theta \cos\theta\, d\theta$$

$$= a^6 \int_0^{\frac{\pi}{2}} a^2 \sin^2\theta \cos^4\theta\, d\theta$$

$$= a^6 \frac{\Gamma\left(\frac{5}{2}\right)\Gamma\left(\frac{3}{2}\right)}{2\Gamma 4}$$

$$= \frac{a^6 \cdot \frac{3}{2}\cdot\frac{1}{2}\,\Gamma\frac{1}{2}\cdot\frac{1}{2}\,\Gamma\frac{1}{2}}{2\cdot 3\cdot 2\cdot 1}$$

$$= \frac{a^6}{32}\sqrt{\pi}\cdot\sqrt{\pi} = \frac{\pi a^6}{32}$$

$$\therefore \quad \int_0^a a^2 (a^2 - x^2)^{\frac{3}{2}} = \frac{\pi a^6}{32}$$

ii) $\int_0^1 \mathbf{x^{\frac{3}{2}} (1-x)^{\frac{3}{2}} dx}$

Sol^n: Here,

$$\int_0^1 x^{\frac{3}{2}} (1-x)^{\frac{3}{2}} dx = \int_0^1 x^{\frac{5}{2}-1} (1-x)^{\frac{5}{2}-1} dx = \beta\left(\frac{5}{2}, \frac{5}{2}\right)$$

$$= \frac{\Gamma\left(\frac{5}{2}\right)\Gamma\left(\frac{5}{2}\right)}{\Gamma 5} = \frac{\frac{3}{2}\frac{1}{2}\sqrt{\pi}\times\frac{3}{2}\times\frac{1}{2}\times\sqrt{\pi}}{4\times 3\times 2\times 1} = \frac{3\pi}{128}$$

$$\therefore \quad \int_0^1 x^{\frac{3}{2}} (1-x)^{\frac{3}{2}} dx = \frac{3\pi}{128}$$

iii) $\int_0^{2a} \mathbf{x^5 \sqrt{2ax - x^2}\, dx}$

Sol^n: Here,

$$\int_0^{2a} x^5 \sqrt{2ax - x^2}\, dx$$

Put,

$$x = 2a\sin^2\theta, dx = 4a\sin\theta\cos\theta\, d\theta$$

When,

$$x = 0, \theta = 0, \text{ when } x = 2a, \theta = \frac{\pi}{2}$$

So,

$$\int_0^{2a} x^5 \sqrt{2ax - x^2}\, dx$$

$$= \int_0^{2a} (2a\sin^2\theta)^5 \sqrt{4a^2\sin^2\theta - 4a^2\sin^4\theta}\; 4a\sin\theta\cos\theta\, d\theta$$

$$= 256a^7 \int_0^{\frac{\pi}{2}} \sin^{10}\theta \sin\theta\cos\theta\sin\theta\cos\theta\, d\theta$$

$$= 256a^7 \int_0^{\frac{\pi}{2}} \sin^{12}\theta \cos^2\theta\, d\theta$$

$$= 256a^7 \frac{\Gamma\frac{(12+1)}{2}\,\Gamma\frac{(2+1)}{2}}{2\Gamma\frac{(12+2+2)}{2}}$$

$$= 256a^7 \frac{\Gamma\frac{13}{2}\,\Gamma\left(\frac{3}{2}\right)}{2\,\Gamma 8}$$

$$= 256a^7 \frac{\frac{11}{2}\cdot\frac{9}{2}\cdot\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\,\Gamma\frac{1}{2}\,\frac{1}{2}\,\Gamma\frac{1}{2}}{2\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}$$

$$= \frac{33a^7}{16}\sqrt{\pi}\sqrt{\pi}$$

$$= \frac{33\pi a^7}{16}$$

iv) $\int_0^{2a} x^{\frac{9}{2}}(2a-x)^{-\frac{1}{2}}dx$

Solⁿ: Here,

$$\int_0^{2a} x^{\frac{9}{2}}(2a-x)^{-\frac{1}{2}}dx$$

Put,

$$x = 2a\sin^2\theta,\ dx = 2\cdot 2a\cos\theta\cdot\sin\theta\ d\theta$$

When $x = 0, \theta = 0$, when $x = 3a, \theta = \frac{\pi}{2}$

So,

$$\int_0^{2a} x^{\frac{9}{2}}(2a-x)^{-\frac{1}{2}}dx$$

$$= \int_0^{\frac{\pi}{2}}(2a\sin^2\theta)^{\frac{9}{2}}(2a-2a\sin^2\theta)^{-\frac{1}{2}}\cdot 4a\sin\theta\cos\theta\, d\theta$$

$$= (2a)^{\frac{9}{2}-\frac{1}{2}}\int_0^{\frac{\pi}{2}}\sin^9\theta\frac{1}{\cos\theta}4a\sin\theta\cos\theta\, d\theta$$

$$= 64a^5\int_0^{\frac{\pi}{2}}\sin^{10}\theta\, d\theta = 64a^5\frac{\sqrt{\pi}\,\Gamma\left(\frac{10+1}{2}\right)}{2\,\Gamma\left(\frac{10+2}{2}\right)} = 64a^5\frac{\sqrt{\pi}\,\Gamma\left(\frac{\pi}{2}\right)}{2\lfloor 6}$$

$$= 64a^5\frac{\sqrt{\pi}\cdot\frac{9}{2}\cdot\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma\frac{1}{2}}{2\cdot5\cdot4\cdot3\cdot2\cdot1} = \frac{63\sqrt{\pi}\sqrt{\pi}a^5}{8} = \frac{63\pi a^5}{8}$$

$$\therefore \int_0^{2a} x^{\frac{9}{2}}(2a-x)^{-\frac{1}{2}}dx = \frac{63\pi a^5}{8}$$

v) $\int_0^a x^4\sqrt{a^2-x^2}\,dx$

Solⁿ: Here,

$$\int_0^a x^4\sqrt{a^2-x^2}\,dx$$

Put,

$$x = a\sin\theta,\ dx = a\cos\theta\, d\theta$$

When,

$$x = 0, \theta = 0 \text{ when } x = a, \theta = \frac{\pi}{2}$$

So,

$$\int_0^a x^4\sqrt{a^2-x^2}\,dx = \int_0^{\frac{\pi}{2}} a^4\sin^4\theta\cdot a\cos\theta\cdot a\cos\theta\, d\theta$$

$$= a^6\int_0^{\frac{\pi}{2}}\sin^4\theta\cos^2\theta\, d\theta = a^6\frac{\Gamma\frac{4+1}{2}\,\Gamma\frac{2+1}{2}}{2\,\Gamma\frac{4+2+2}{2}}$$

$$= \frac{a^6\Gamma\left(\frac{5}{2}\right)\Gamma\left(\frac{3}{2}\right)}{2\,\Gamma\left(\frac{8}{2}\right)} = \frac{a^6\frac{3}{2}\cdot\frac{1}{2}\cdot\sqrt{\pi}\times\frac{1}{2}\sqrt{\pi}}{2\times3\times2\times1} = \frac{\pi a^6}{32}$$

$$\therefore \int_0^a x^4\sqrt{a^2-x^2}\,dx = \frac{\pi a^6}{32}$$

vi) $\int_0^a \frac{x^6dx}{\sqrt{1-x^2}}$

Solⁿ: Here,

$$\int_0^a \frac{x^6dx}{\sqrt{1-x^2}}$$

Put,

$$x = \sin\theta,\ dx = \cos\theta\, d\theta$$

When,

$$x = 0, \theta = 0 \text{ when } x = 0, \theta = \frac{\pi}{2}$$

$$\therefore \int_0^a \frac{x^6dx}{\sqrt{1+x^2}} = \int_0^{\frac{\pi}{2}}\frac{\sin^6\theta}{\cos\theta}\cos\theta\, d\theta$$

$$= \int_0^{\frac{\pi}{2}}\sin^6\theta\, d\theta = \frac{\sqrt{\pi}\,\Gamma\frac{6+1}{2}}{2\,\Gamma\left(\frac{6+1}{2}\right)} = \frac{\sqrt{\pi}\,\Gamma\frac{7}{2}}{2\,\Gamma}$$

$$= \frac{\sqrt{\pi}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\cdot\Gamma\left(\frac{1}{2}\right)}{2\cdot3\cdot2\cdot1} = \frac{5\sqrt{\pi}\sqrt{\pi}}{32} = \frac{5\pi}{32}$$

$$\therefore \int_0^a \frac{x^6dx}{\sqrt{1-x^2}} = \frac{5\pi}{32}$$

vii) $\int_0^a \frac{x^4}{\sqrt{a^2-x^2}}dx$

Solⁿ: Here,

$$\int_0^a \frac{x^4}{\sqrt{a^2-x^2}}dx$$

Put,

$$x = a\sin\theta,\ dx = a\cos\theta\, d\theta$$

When,

$$x = 0, \theta = 0, \text{ when } x = a, \theta = \frac{\pi}{2}$$

$$\therefore \int_0^a \frac{x^4}{\sqrt{a^2-x^2}}dx = \int_0^{\frac{\pi}{2}} \frac{a^4 \sin^2\theta a\cos\theta d\theta}{\sqrt{a^2-a^2\sin^2\theta}}$$

$$= a^4 \int_0^{\frac{\pi}{2}} \frac{\sin^4\theta\cos\theta d\theta}{\cos\theta}$$

$$= a^4 \int_0^{\frac{\pi}{2}} \sin^4\theta\, d\theta = \sqrt{\pi}\frac{a^4\,\Gamma\left(\frac{4+1}{2}\right)}{2\,\Gamma\frac{4+1}{2}}$$

$$= \frac{a^4\,\Gamma\frac{5}{2}}{2\,\Gamma}\sqrt{\pi} = \frac{a^4\sqrt{\pi}\frac{3}{2}\cdot\frac{1}{2}}{2\cdot2\cdot1}\sqrt{\pi} = \frac{3\pi a^4}{16}$$

$$\therefore \int_0^a x^4\sqrt{a^2-x^2}\,dx = \frac{3\pi a^4}{16}$$

viii) $\displaystyle\int_0^a \mathbf{x^3(a^2-x^2)^{\frac{5}{2}}dx}$

Solⁿ: Here,

$$\int_0^a x^3(a^2-x^2)^{\frac{5}{2}}dx$$

Put,

$$x = a\sin\theta,$$

Then,

$$dx = a\cos\theta\, d\theta$$

Also, when $x = 0, \theta = 0$ and when $x = a, \theta = \frac{\pi}{2}$

$$\therefore \int_0^a x^3(a^2-x^2)^{\frac{5}{2}}dx = \int_0^{\frac{\pi}{2}} a^3\sin^3\theta\,(a^2\cos^2\theta)a\cos\theta\, d\theta$$

$$= a^9\int_0^{\frac{\pi}{2}} \sin^3\theta\cos^6\theta\, d\theta = \frac{a^9\,\Gamma\left(\frac{3+1}{2}\right)\Gamma\left(\frac{6+1}{2}\right)}{2\,\Gamma\left(\frac{3+6+2}{2}\right)}$$

$$= a^9\frac{\Gamma2\,\Gamma\frac{7}{2}}{2\,\Gamma\frac{11}{2}} = \frac{a^9}{2}\cdot\frac{1\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\,\Gamma\left(\frac{1}{2}\right)}{\frac{9}{2}\cdot\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\,\Gamma\frac{1}{2}} = \frac{2a^9}{63}$$

5. Using Beta Gamma function, show that

i) $\displaystyle\int_0^{\frac{\pi}{6}} \mathbf{\cos^4 3\theta\sin^2 6\theta\, d\theta = \frac{5\pi}{192}}$ [058/059 B.E]

Solⁿ: Here,

$$\int_0^{\frac{\pi}{6}} \cos^4 3\theta\sin^2 6\theta\, d\theta$$

$$= \int_0^{\frac{\pi}{6}} \cos^4 3\theta\,(2\sin3\theta\cos3\theta)^2\, d\theta$$

$$= \int_0^{\frac{\pi}{6}} 4\sin^2 3\theta\cos^6 3\theta\, d\theta$$

Put,

$3\theta = t, 3d\theta = dt$

When $\theta = 0, t = 0$

When $\theta = \frac{\pi}{6}, t = \frac{\pi}{2}$

$$\therefore \frac{4}{3}\int_0^{\frac{\pi}{2}} \cos^6 t\sin^2 t\, dt = \frac{4}{3}\frac{\Gamma\frac{(6+1)}{2}\,\Gamma\frac{(2+1)}{2}}{2\Gamma\frac{(6+2+2)}{2}}$$

$$= \frac{4}{3}\frac{\Gamma\frac{7}{2}\Gamma\frac{3}{2}}{2\,\Gamma5} = \frac{4}{3}\frac{\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma\frac{1}{2}\cdot\frac{1}{2}\Gamma\frac{1}{2}}{2\cdot4\cdot3\cdot2\cdot1}$$

$$= \frac{5\sqrt{\pi}\sqrt{\pi}}{192} = \frac{5\pi}{192}$$

$$\therefore \int_0^{\frac{\pi}{6}} \cos^4 3\theta\sin^2 6\theta\, d\theta = \frac{5\pi}{192}$$

ii) $\displaystyle\int_0^{\frac{\pi}{6}} \mathbf{\cos^2 6\theta\sin^4 3\theta\, d\theta = \frac{7\pi}{192}}$

Solⁿ: Here,

$$\int_0^{\frac{\pi}{6}} \cos^2 6\theta\sin^4 3\theta\, d\theta$$

$$= \int_0^{\frac{\pi}{6}} [\cos 2(3\theta)]^2\sin^4 3\theta\, d\theta$$

$$= \int_0^{\frac{\pi}{6}} (1-2\sin^2 3\theta)^2\sin^4 3\theta\, d\theta$$

$$= \int_0^{\frac{\pi}{6}} (1-4\sin^2 3\theta+4\sin^4 3\theta)\sin^4 3\theta\, d\theta$$

$$= \int_0^{\frac{\pi}{6}} (\sin^4 3\theta-4\sin^6 3\theta+4\sin^8 3\theta)\, d\theta$$

Put,

$$3\theta = t,$$

$$3d\theta = dt$$

when $\theta = 0, t - 0$

when $\theta = \frac{\pi}{6}, t = \frac{\pi}{2}$

$$= \frac{1}{3}\int_0^{\frac{\pi}{2}} (\sin^4 t - 4\sin^6 t + 4\sin^8 t)\, dt$$

Using Gamma function, we get

$$= \frac{1}{3}\frac{\sqrt{\pi}\,\Gamma\frac{(4+1)}{2}}{2\,\Gamma\frac{(4+2)}{2}} - \frac{4}{3}\frac{\sqrt{\pi}\,\Gamma\frac{(6+1)}{2}}{2\,\Gamma\frac{(6+2)}{2}} + \frac{4}{3}\frac{\sqrt{\pi}\,\Gamma\frac{(8+1)}{2}}{2\,\Gamma\frac{(8+2)}{2}}$$

$$= \frac{\sqrt{\pi}\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}}{3\times3\times2\times1} + \frac{2\sqrt{\pi}\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}}{3\cdot4\cdot3\cdot2\cdot1} = \frac{\pi}{16} - \frac{5\pi}{24} + \frac{35\pi}{192}$$

$$= \frac{12\pi - 40\pi + 35\pi}{192} = \frac{7\pi}{192}$$

$$\therefore \quad \int_0^{\frac{\pi}{6}} \cos^2 6\theta \sin^4 3\theta\, d\theta = \frac{7\pi}{192}$$

iii) $\displaystyle\int_0^{\pi} \sin^6\frac{x}{2}\cos^8\frac{x}{2} = \frac{5\pi}{2^{11}}$

Sol^n: Here,

$$\int_0^{\pi} \sin^6\frac{x}{2}\cos^8\frac{x}{2}\, d\theta$$

Put,

$$\frac{x}{2} = \theta, dx = 2d\theta$$

When $x = 0, \theta = 0$

When $x = \pi, \theta = \frac{\pi}{2}$

$$\therefore \quad \int_0^{\pi} \sin^6\frac{x}{2}\cos^8\frac{x}{2}\, dx = \int_0^{\frac{\pi}{2}} \sin^6\theta \cos^8\theta\, 2d\theta$$

$$= 2\int_0^{\frac{\pi}{2}} \sin^6\theta\cos^8\theta\, d\theta$$

$$= 2\frac{\Gamma\frac{(6+1)}{2}\,\Gamma\frac{(8+1)}{2}}{2\Gamma\frac{(6+8+2)}{2}}$$

$$= \frac{2\,\Gamma\frac{7}{2}\,\Gamma\frac{9}{2}}{2\,\Gamma 8}$$

$$= \frac{\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma\frac{1}{2}\cdot\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma\frac{1}{2}}{7\cdot6\cdot5\cdot4\cdot3\cdot2\cdot1}$$

$$= \frac{5\sqrt{\pi}\cdot\sqrt{\pi}}{2^{11}} = \frac{5\pi}{2^{11}}$$

$$\therefore \quad \int_0^{\pi} \sin^6\frac{x}{2}\cos^8\frac{x}{2}\, dx = \frac{5\pi}{2^{11}}$$

iv) $\displaystyle\int_0^{\frac{\pi}{4}} \sin^4 x\cos^2 x\, dx = \frac{3\pi - 4}{192}$

Sol^n: Here,

$$\int_0^{\frac{\pi}{4}} \sin^4 x\cos^2 x\, dx = \int_0^{\frac{\pi}{4}} (\sin^2 x)^2 \cos^2 x\, dx$$

$$= \int_0^{\frac{\pi}{4}} \left(\frac{1-\cos 2x}{2}\right)^2 \left(\frac{1+\cos 2x}{2}\right) dx$$

$$= \frac{1}{8}\int_0^{\frac{\pi}{4}} (1 - 2\cos 2x + \cos^2 2x)(1 + \cos 2x)dx$$

$$= \frac{1}{8}\int_0^{\frac{\pi}{4}} (1 - 2\cos 2x + \cos^2 2x + \cos 2x - 2\cos^2 2x + \cos^3 2x)\, dx$$

$$= \frac{1}{8}\int_0^{\frac{\pi}{4}} (1 - \cos 2x)\, dx - \frac{1}{8}\int_0^{\frac{\pi}{4}} (\cos^2 2x - \cos^3 2x)\, dx$$

$$= \frac{1}{8}\left[x - \frac{\sin 2x}{2}\right]_0^{\frac{\pi}{4}} - \frac{1}{8}\int_0^{\frac{\pi}{4}} (\cos^2 2x - \cos^3 2x)\, dx$$

Put,

$2x = \theta, 2dx = d\theta$

When $x = 0, \theta = 0$

When $x = \frac{\pi}{4}, \theta = \frac{\pi}{2}$

$$= \frac{1}{8}\left(\frac{\pi}{4} - \frac{1}{2}\right) - \frac{1}{8}\int_0^{\frac{\pi}{4}} (\cos^2\theta - \cos^3\theta)\frac{d\theta}{2}$$

$$= \frac{\pi}{32} - \frac{1}{16} - \frac{1}{16}\int_0^{\frac{\pi}{2}} (\cos^2\theta - \cos^3\theta)\, d\theta$$

$$= \frac{\pi}{32} - \frac{1}{16} - \frac{1}{16}\frac{\sqrt{\pi}\,\Gamma\frac{(2+1)}{2}}{2\,\Gamma\frac{(2+2)}{2}} + \frac{1}{16}\frac{\sqrt{\pi}\,\Gamma\frac{(3+1)}{2}}{2\,\Gamma\frac{(3+2)}{2}}$$

$$= \frac{\pi}{32} - \frac{1}{16} - \frac{1}{32}\frac{\sqrt{\pi}\cdot\frac{1}{2}\cdot\sqrt{\pi}}{1} + \frac{1}{16}\frac{\sqrt{\pi}}{2\cdot\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}}$$

$$= \frac{\pi}{32} - \frac{1}{16} - \frac{\pi}{64} + \frac{1}{24} = \frac{\pi}{64} - \frac{1}{48} = \frac{3\pi - 4}{192}$$

$$\therefore \int_0^{\frac{\pi}{4}} \sin^4 x \cos^2 x\, dx = \frac{3\pi - 4}{192}$$

6. **Prove that $\int_0^\infty x^2 e^{-x^4} dx \times \int_0^\infty e^{-x^4} dx = \frac{\pi}{8\sqrt{2}}$** [2062 B.E]

Sol^n: Here,

$$\int_0^\infty x^2 e^{-x^4} dx \times \int_0^\infty e^{-x^4} dx$$

Put,

$$x^4 = t$$

Then,

$$4x^3 dx = dt$$

or, $dx = \frac{dt}{4x^3} = \frac{dt}{4t^{\frac{3}{4}}}$

$$\therefore \int_0^\infty x^2 e^{-x^4} dx \times \int_0^\infty e^{-x^4} dx = \int_0^\infty e^{-t} \frac{t^{\frac{1}{2}} dt}{4t^{\frac{3}{4}}} \times \frac{1}{4}\int_0^\infty e^{-t} t^{\frac{3}{4}} dt$$

$$= \frac{1}{16}\int_0^\infty t^{\frac{3}{4}-1} e^{-t} dt \times \int_0^\infty t^{\frac{1}{4}-1} e^{-t} dt$$

$$= \frac{1}{16}\Gamma\frac{3}{4}\,\Gamma\frac{1}{4} = \frac{1}{16}\sqrt{2}\pi$$

$$= \frac{\pi}{8\sqrt{2}}$$

7. **Prove that $\Gamma\frac{1}{3}\,\Gamma\frac{2}{3} = \frac{2\pi}{\sqrt{3}}$**

Sol^n: Here,

$$\Gamma\frac{1}{3}\,\Gamma\frac{2}{3} = \Gamma\frac{1}{3}\,\Gamma 1 - \frac{1}{3}$$

$$\frac{\pi}{\sin\frac{\pi}{3}} = \frac{2\pi}{\sqrt{3}} \qquad \left(\because \Gamma m\, \Gamma 1 - m = \frac{\pi}{\sin \pi}\right)$$

8. **Show that $\beta(m, n), \beta(m + n, l) = \beta(n, l), \beta(n + l, m)$**

Sol^n:

$$L.H.S = \beta(m, n), \beta(m + n, l) = \frac{\Gamma m\, \Gamma n}{\Gamma m{+}n}\,\frac{\Gamma m{+}n\, \Gamma l}{\Gamma m{+}n{+}l}$$

$$= \frac{\Gamma m\, \Gamma l}{\Gamma n{+}l}\,\frac{\Gamma n{+}l\, \Gamma m}{\Gamma m{+}n{+}l} = \beta(n, l)\beta(n + l, m)$$

$$\therefore \beta(m, n), \beta(m + n, l) = \beta(n, l), \beta(n + l, m)$$

9. **Show that $\int_0^\infty e^{-x^2} x^\alpha dx = \frac{1}{2}\Gamma\frac{\alpha+1}{2}$**

Sol^n:

$$L.H.S = \int_0^\infty e^{-x^2} x^\alpha dx$$

Put

$$x^2 = t,\ 2x\, dx = dt,\ dx = \frac{dt}{2\sqrt{t}}$$

When $x = 0, t = \infty$

When $x = \infty, t = \infty$

So,

$$\int_0^\infty e^{-x^2} x^\alpha dx = \int_0^\infty \frac{e^{-t} t^{\frac{\alpha}{2}} dt}{2t^{\frac{1}{2}}}$$

$$= \int_0^\infty e^{-t} t^{\frac{\alpha}{2}-\frac{1}{2}} dt$$

$$= \frac{1}{2}\int_0^\infty e^{-t} t^{\frac{\alpha+1}{2}-1} dt$$

$$= \frac{1}{2}\Gamma\left(\frac{\alpha+1}{2}\right)$$

$$\therefore \int_0^\infty e^{-x^2} x^\alpha dx = \frac{1}{2}\Gamma\frac{\alpha+1}{2}$$

10. Prove that:

$$\int_a^b (x - a)^m (b - x)^n dx = (b - a)^{m+n+1} \frac{\Gamma m{+}1\, \Gamma n{+}1}{\Gamma m{+}n{+}2}, m > -1, n > -1$$

Sol^n: Here,

$$\int_a^b (x - a)^m (b - x)^n dx$$

Put,

$$x - a = (b - a)y;\ x = (b - a)y + a,\ dx = (b - a)dy$$

When $x = a, y = 0$

When $x = b, y = 1$

$$= \int_0^1 \{(b - a)y\}^m (b - by + ay - a)^n (b - a)dy$$

$$= \int_0^1 (b - a)^m y^m (b - a)^n (1 - y)^n (b - a)dy$$

$$= (b - a)^{m+n+1} \int_0^1 y^m (1 - y)^n dy$$

$$= (b - a)^{m+n+1} \int_0^1 y^{m+1-1} (1 -$$

$$= (b-a)^{m+n+1}\beta(m+1, n+1)$$

$$= (b-a)^{m+n+1}\frac{\Gamma m+1\,\Gamma n+1}{\Gamma m+n+2}$$

In particular, $a = -1, b = 1$, then,

$$\int_{-1}^{1}(x+1)^m(1-x)^n dx = 2^{m+n+1}\frac{\Gamma(m+1)\,\Gamma(n+1)}{\Gamma(m+n+2)}$$

Exercise 15

Quadrature

Introduction:

Quadrature is the method of for finding out the area bounded by the curve. The definite integral $\int_a^b f(x)\,dx = \int_a^b y\,dx$ represents the area bounded by the curve $y = f(x)$, the x-axis and two fixed ordinates $x = a$ and $x = b$.

1. **Find the area bounded by the curve $y^2 = x^3$ and the line $y = 2x$.**

Solⁿ:

$y^2 = x^3 \ldots$ (i)

and $y = 2x \ldots$ (ii)

So, from equation (i)

$4x^2 = x^3$

or, $4x^2 - x^3 = 0$

or, $x^2(x-4) = 0$

or, $x = 0; x = 4$

$\therefore$ $x = 0$ to 4

We know,

$$\int_0^4 (y_1 - y_2)\,dx$$

$$= \int_0^4 \left(2x - x^{\frac{3}{2}}\right) dx = \left[x^2 - \frac{x^{\frac{5}{2}}}{\frac{5}{2}}\right]_0^4 = 16 - \frac{4^{\frac{5}{2}}}{\frac{5}{2}} = 16 - \frac{32}{5}\times 2$$

$$= 16 - \frac{64}{5} = \frac{80-64}{5} = \frac{16}{5}$$

2. **Show that the area bounded by the curves $y^2 = 4ax$ and $x^2 = 4ay$ is $\frac{16}{3}a^2$.**

Solⁿ: Here, we have to find the area OApBO. Solving the two equations $y^2 = 4ax$ and $x^2 = 4ay$ we have,

$4ay^3 = 4ax^3$

or, $y = x$

$\therefore \quad x^2 = 4ax$

or, $x(x-4a) = 0$

$\therefore \quad x = 0$ at O and 4a at p

$\therefore$ Area OApBO = area OBpMO − area OApMO

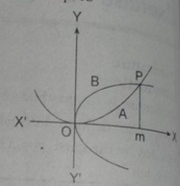
$$= \int_0^{4a} \sqrt{4ax}\,dx - \int_0^{4a} \frac{x^2}{4a}\,dx$$

$$= 2\sqrt{a} \times \frac{2}{3}\left[x^{\frac{3}{2}}\right]_0^{4a} - \frac{1}{4a}\left[\frac{x^3}{3}\right]_0^{4a}$$

$$= \frac{4}{3}\sqrt{a} \times (4a)^{\frac{3}{2}} - \frac{1}{12a}(4a)^3$$

$$= \frac{32}{3}a^2 - \frac{16}{3}a^2$$

$$= \frac{16}{3}a^2$$



3. **Find the area of the region bounded by the curve $y = \sin x$ and x-axis between $x = 0$ and $x = 2\pi$**

Sol^n: Here the curve is;

$$y = \sin x$$

Required area $= \int_0^{2\pi} y\,dx$

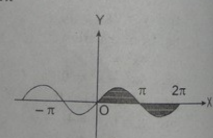
$$= \int_0^{2\pi} \sin x\,dx$$

$$= 2\int_0^{\pi} \sin x\,dx$$

$$= 2[-\cos x]_0^{\pi}$$

$$= 2[+2+2]$$

$$= 4 \text{ Square unit}$$



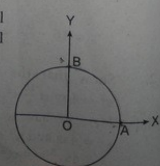
4. **Find the area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.**

Sol^n: The ellipse is divided into 4 symmetrical parts by the x and y axes. We will consider the area of the parts OAB.

From $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

We have,

$$y = \frac{b}{a}\sqrt{a^2 - x^2}$$



$\therefore$ Area of OAB $= \int_0^a y\,dx = \frac{b}{a}\int_0^a \sqrt{a^2 - x^2}\,dx$

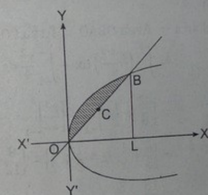
$$= \frac{b}{a}\left[\frac{x\sqrt{a^2 - x^2}}{2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_0^a$$

$$= \frac{b}{a}\left(\frac{a^2}{2}\cdot \sin^{-1} 1\right) = \frac{b}{a}\cdot\frac{a^2}{2}\cdot\frac{\pi}{2} = \frac{\pi ab}{4}$$

$\therefore$ Area of the ellipse $= 4 \times \frac{\pi ab}{4} = \pi ab$

5. **Find the area enclosed between the line $y = x$ and the parabola $y^2 = 16x$.**

Sol^n:



Equation parabola $y^2 = 16x \ldots$ (i)

and the line $y = x \ldots$ (ii)

Solving equation (i) and (ii) we get,

$x^2 = 16x$

or, $x = 0$,

$x = 16$

Required area = Area OABCO

$$= \text{Area OABLO} - \text{Area OCBL}$$

$$= \int_0^{16} \sqrt{16x}\,dx - \int_0^{16} x\,dx$$

$$= 4\left[\frac{2}{3}x^{\frac{3}{2}}\right]_0^{16} - \left[\frac{x^2}{2}\right]_0^{16}$$

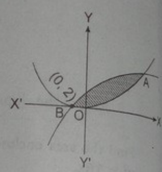
$$= \frac{512}{3} - 128$$

$$= \frac{128}{3} \text{ square unit}$$

6. **Find the area bounded by the curve $x^2 = 4y$ and the line $x = 4y - 2$.**

Sol^n:

Equation of the parabola $x^2 = 4y \dots$ (i)
and the line $x = 4y - 2 \dots$ (ii)
Solving equation (i) and (ii) we get,
from (ii),



$$y = \frac{x+2}{4}$$

$$x^2 = \frac{x+2}{4} \times 4$$

or, $x^2 - x - 2 = 0$

or, $(x-2)(x+1) = 0$

or, $x = 2, -1$

$\therefore$ Required area = Area OBAO = Area COABC − Area OABO

$$= \int_{-1}^{2} \left(\frac{x+2}{4}\right) dx - \int_{-1}^{2} \frac{x^2}{4} dx$$

$$= \left[\frac{x^2}{8} + \frac{1}{2}x\right]_{-1}^{2} - \left[\frac{x^3}{12}\right]_{-1}^{2}$$

$$= \left[\frac{4}{8} + \frac{1}{2}\cdot 2 - \frac{1}{8} + \frac{1}{2}\right] - \left[\frac{8}{12} + \frac{1}{12}\right]$$

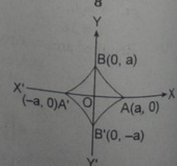
$$= \frac{15}{8} - \frac{3}{4}$$

$$= \frac{9}{8}$$

7. **Show that the area of the asteroid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ is $\frac{3}{8}\pi a^2$.**

Sol^n: The asteroid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$

or, $y = \left(a^{\frac{2}{3}} - x^{\frac{2}{3}}\right)^{\frac{3}{2}}$



It is symmetrical on the both sides
The curve meets x-axis at $(\pm a, 0)$
and y-axis at $(0, \pm a)$

Required area = 4 area of OAB = $4\int_0^a y\,dx$

$$= 4\int_0^a \left(a^{\frac{2}{3}} - x^{\frac{2}{3}}\right)^{\frac{3}{2}} dx$$

Put,
$x = a\sin^3\theta,$
$dx = 3a\sin^2\theta\cos\theta\, d\theta$

$$= 4\int_0^a \left(a^{\frac{2}{3}} - a^{\frac{2}{3}}\sin^2\theta\right)^{\frac{3}{2}} 3a\sin^2\theta\cos\theta\, d\theta$$

$$= 12\cdot a\cdot a\int_0^{\frac{\pi}{2}} \sin^2\theta\cos^4\theta\, d\theta$$

$$= 12a\cdot a\frac{\frac{1}{2}\sqrt{\pi}\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}}{2\cdot 3\cdot 2\cdot 1}$$

$$= \frac{3\pi}{8}a^2$$

8. **Find the area bounded by the curve $x^2y = a^2(a-y)$ and the x-axis.**

Sol^n: The equation of the curve is $x^2y = a^2(a-y)$ since x vanishes for $y = a$
So, the required area = $2\int_0^a x\,dy$

$$= 2\int_0^a a\sqrt{\frac{(a-y)}{y}}\,dy$$

Put,
$y = t^2, dy = 2t\,dt$
When $y = 0, t = 0$
When $y = a, t = \sqrt{a}$

Area $= 2a\int_0^{\sqrt{a}} \sqrt{a-t^2}\, 2dt$

$$= 4a\left[\frac{a\sqrt{a-t^2}}{2} + \frac{a}{2}\sin^{-1}\frac{t}{\sqrt{a}}\right]_0^{\sqrt{a}}$$

$$= 4a\left[0 + \frac{a}{2}\cdot\frac{\pi}{2}\right]$$

$= \pi a^2$ Square unit

9. **Find the area of the loop of the curve $y^2 = x^2(x+a)$**

Sol^n: Here,

$y^2 = x^2(x+a)$ since y vanishes for $x = -a$

So, the required area

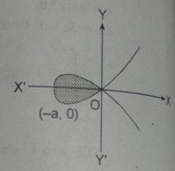
$$A = 2\int_0^{-a} y\,dx = 2\int_0^{-a} x\sqrt{x+a}\,dx$$

Put,

$x + a = t^2, dx = 2t\,dt$

When $x = 0, t = \sqrt{a}$

When $x = -a, t = 0$



$$= 2\int_{\sqrt{a}}^{0}(t^2 - a)\,t\cdot 2t\,dt$$

$$= 4\int_{\sqrt{a}}^{0}(t^4 - at^2)\,dt$$

$$= 4\left[\frac{t^5}{5} - \frac{at^3}{3}\right]_{\sqrt{a}}^{0}$$

$$= 4\left[0 - \frac{a^{\frac{5}{2}}}{5} + \frac{a^{\frac{5}{2}}}{3}\right] = 4a^{\frac{5}{2}}\left(\frac{1}{3} - \frac{1}{5}\right) = 4a^{\frac{5}{2}}\frac{(5-3)}{15}$$

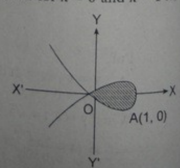
$$= \frac{8}{15}a^{\frac{5}{2}} \text{ square unit}$$

10. **Find the area of the curve $y^2 = x(x-1)^2$**

Sol^n: Here,

The curve $y^2 = x(x-1)^2$, since y vanishes for $x = 0$ and $x = 1$ so, the required area

$$A = 2\int_0^1 y\,dx$$



$$= 2\int_0^1 (x-1)\sqrt{x}\,dx$$

$$= 2\int_0^1\left(x^{\frac{3}{2}} - x^{\frac{1}{2}}\right)dx$$

$$= 2\left[\frac{2x^{\frac{5}{2}}}{5} - \frac{2x^{\frac{3}{2}}}{3}\right]_0^1$$

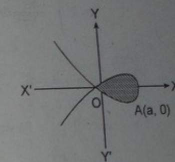
$$= 4\left[\frac{1}{5} - \frac{1}{3}\right] = 4\frac{(3-5)}{15}$$

$$= -\frac{8}{15}$$

$\therefore$ Area $= \frac{8}{15}$ square unit

11. **Find the area of the top of the curve $y^2(x+a) = x^2(a-x)$**

Sol^n: The curve $y^2(x+a) = x^2(a-x)$ since, y vanishes for $x = 0$, and $x = a$, so the required area



$$A = 2\int_0^a y\,dx$$

$$= 2\int_0^a x\sqrt{\frac{a-x}{a+x}}\,dx$$

$$= 2\int_0^a \frac{x(a-x)}{\sqrt{a^2-x^2}}\,dx$$

$$= 2\int_0^a \frac{ax\,dx}{\sqrt{a^2-x^2}} + 2\int_0^a \frac{(a^2-x^2-a^2)dx}{\sqrt{a^2-x^2}}$$

$$= -a\int_0^a \frac{-2x\,dx}{\sqrt{a^2-x^2}} + 2\int_0^a \sqrt{a^2-x^2}\,dx - 2a^2\int_0^a \frac{dx}{\sqrt{a^2-x^2}}$$

$$= -a\left[2\sqrt{a^2-x^2}\right]_0^a + 2\left[\frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_0^a - \left[2a^2\sin^{-1}\frac{x}{a}\right]_0^a$$

$$= [0 + 2a^2] + 2\left(0 + \frac{a^2}{2}.\frac{\pi}{2}\right) - 2a^2\left(\frac{\pi}{2} - 0\right)$$

$$= 2a^2 + a^2.\frac{\pi}{2} - 2a^2.\frac{\pi}{2}$$

$$= 2a^2 + \frac{\pi a^2}{2} - \pi a^2$$

$$= a^2\left(2 - \frac{\pi}{2}\right) \text{ square unit}$$

12. **Find the area between each of the following curve and its asymptotes.**

i) $y^2(a-x) = x^3$

Sol^n: The curve is;

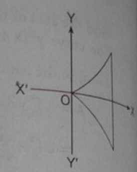
$y^2(a-x) = x^3$

or, $y = \frac{x^{\frac{3}{2}}}{\sqrt{a-x}}$

Equating the coefficient of highest power to zero, the equation of asymptotes is $x - a = 0, x = a$ so the required area bounded by its asymptotes

$$A = 2\int_0^a y\,dx$$

$$= 2\int_0^a \frac{x^{\frac{3}{2}}}{\sqrt{a-x}}\,dx$$



Put,

When $x = 0, \theta = 0$

When $x = a, \theta = \frac{\pi}{2}$

$x = a\sin^2\theta$, $dx = 2a\sin\theta\cos\theta\,d\theta$

$$A = 2\int_0^{\frac{\pi}{2}} \frac{a^{\frac{3}{2}}\sin^3\theta}{\sqrt{a\left(1-\sin^2\theta\right)}}\,2a\sin\theta\cos\theta\,d\theta$$

$$= 4a^2\int_0^{\frac{\pi}{2}} \sin^4\theta\,d\theta$$

$$= \frac{4a^2\sqrt{\pi}\,\Gamma\frac{4+1}{2}}{2\,\Gamma\left(\frac{4+1}{2}\right)}$$

$$= \frac{4a^2\sqrt{\pi}\cdot\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}}{2\cdot2\cdot1}$$

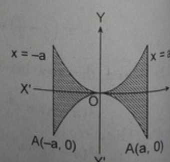
$$= \frac{3\pi a^2}{4} \text{ square unit}$$

ii) $\mathbf{a^2x^2 = y^2(a^2 - x^2)}$

Sol^n: The curve is;

$$a^2x^2 = y^2(a^2 - x^2)$$

The curve passes through origin and symmetrical on both axes; tangents at origin be $y = \pm x$. Equation of asymptotes is $x = \pm a$ which is obtained by equating the coefficient of highest power of y to zero.



So, the required area is given by;

$$A = 4\int_0^a y\,dx$$

$$= 4\int_0^a \frac{ax}{\sqrt{a^2-x^2}}\,dx = -4a\left[\sqrt{a^2-x^2}\right]_0^a = 4a^2 \text{ square unit}$$

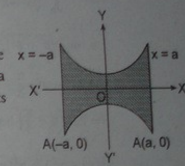
iii) $\mathbf{x^2y^2 + a^2b^2 = a^2y^2}$

Sol^n: The curve is;

$$x^2y^2 + a^2b^2 = a^2y^2$$

or, $a^2b^2 = (a^2 - x^2)y^2$

It is symmetrical on both axes. The equation of its asymptotes is $x = \pm a$ and passes through $(0, \pm b)$ tangents at $(0, -b)$ is $y = \pm b$.



So, the required area

$$A = 4\int_0^a y\,dx$$

$$= 4\int_0^{2a} \frac{ab}{\sqrt{a^2-x^2}}\,dx = 4ab\left[\sin^{-1}\frac{x}{a}\right]_0^a = 4ab\frac{\pi}{2}$$

$$= 2\pi ab \text{ square unit}$$

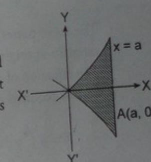
iv) $\mathbf{a(y^2 - x^2) = x(x^2 + y^2)}$

Sol^n: The curve is;

$$a(y^2 - x^2) = x(x^2 + y^2)$$

or, $(a - x)y^2 = x^2(x + a)$

The curve passes through origin and symmetrical on both axes, tangents at origin be $y = \pm x$. The equation of its asymptotes is $x = a$.



So, the required area

$$A = 2\int_0^a y\,dx$$

$$= 2\int_0^a x\sqrt{\frac{a+x}{a-x}}\,dx = 2\int_0^a \frac{ax+x^2}{\sqrt{a^2-x^2}}$$

$$= 2a\int_0^a \frac{x}{\sqrt{a^2-x^2}}\,dx + 2\int_0^a \frac{x^2dx}{\sqrt{a^2-x^2}}$$

$$= -a\int_0^a \frac{-2x\,dx}{\sqrt{a^2-x^2}} - 2\int_0^a \frac{\left(a^2-x^2-a^2\right)dx}{\sqrt{a^2-x^2}}$$

$$= -a\int_0^a \frac{-2x\,dx}{\sqrt{a^2-x^2}} - 2\int_0^a \sqrt{a^2-x^2}\,dx + 2a^2\int_0^a \frac{dx}{\sqrt{a^2-x^2}}$$

$$= -a\left[2\sqrt{a^2-x^2}\right]_0^a - 2\left[\frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_0^a + \left[2a^2\sin^{-1}\frac{x}{a}\right]_0^a$$

$$= 2a^2 - a^2\cdot\frac{\pi}{2} + 2a^2\cdot\frac{\pi}{2}$$

$$= 2a^2\left(1+\frac{\pi}{4}\right) \text{ Square unit}$$

v) $\mathbf{x^2(x^2+y^2) = a^2(y^2-x^2)}$

Sol^n: Here,

$$x^2(x^2+y^2) = a^2(y^2-x^2)$$

or, $x^4 + x^2y^2 = a^2y^2 - a^2x^2$

or, $x^4 + a^2x^2 = (a^2-x^2)y^2$

$\therefore$ $y = x\sqrt{\frac{(x^2+a^2)}{a^2-x^2}}$

The curve passes through origin and symmetrical on both axes and tangents at origin is $y = \pm x$.
The equation of its asymptotes is $x = \pm a$.
So, the required area

$$A = 4\int_0^a y\,dx = 4\int_0^a x\sqrt{\frac{a^2+x^2}{a^2-x^2}}\,dx$$

Put $x^2 = a^2\cos 2\theta$, $x\,dx = -a^2\sin 2\theta\,d\theta$

When $x = 0, \theta = \frac{\pi}{4}$

When $x = a, \theta = 0$

So,

$$A = 4\int_{\frac{\pi}{4}}^{0}\sqrt{\frac{a^2(1+\cos 2\theta)}{a^2(1-\cos 2\theta)}}\,(a^2\sin 2\theta)d\theta$$

$$= -4a^2\int_{\frac{\pi}{4}}^{0}\sqrt{\frac{2\cos^2\theta}{2\sin^2\theta}}\cdot 2\sin\theta\cos\theta\,d\theta$$

$$= -8a^2\int_{\frac{\pi}{4}}^{0}\cos^2\theta\,d\theta = -8a^2\int_{\frac{\pi}{4}}^{0}\frac{(1+\cos 2\theta)}{2}d\theta$$

$$= -4a^2\left[\theta + \frac{\sin 2\theta}{2}\right]_{\frac{\pi}{4}}^{0} = -4a^2\left[0 - \frac{\pi}{4} - \frac{1}{2}\right]$$

$$= a^2(\pi+2) \text{ Square unit}$$

13. **Find the area of a loop of the curve $r^2 = a^2\cos 2\theta$**

Sol^n: The equation of the curve is;

$$r^2 = a^2\cos 2\theta = a^2(\cos^2\theta - \sin^2\theta)$$

Put,

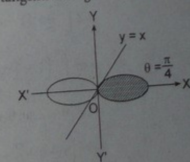
$$x = r\cos\theta, y = r\sin\theta, x^2+y^2 = r^2$$

or, $(x^2+y^2) = a^2(x^2-y^2)$

It is symmetrical on both axes and tangents at origin is $y = \pm x$

So that θ varies from 0 to $\frac{\pi}{4}$

The required area of a loop;



$$A = 2\int_0^{\frac{\pi}{4}}\frac{1}{2}r^2\,d\theta$$

$$= 2\int_0^{\frac{\pi}{4}}\frac{1}{2}a^2\cos 2\theta\,d\theta$$

$$= a^2\left[\frac{\sin 2\theta}{2}\right]_0^{\frac{\pi}{4}}$$

$$= a^2\left[\frac{1}{2} - 0\right]$$

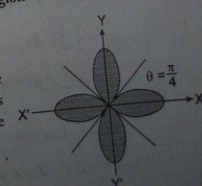
$$= \frac{a^2}{2} \text{ Square unit}$$

14. **Find the area of the entire region bounded by the curves $r = a\cos 2\theta$.**

Sol^n: Here,

$$r = a\cos 2\theta$$

It is symmetrical on the initial line and formed four equal loops required area of the loops. So, the required area of the loop,



$$A = 4\int_0^{\frac{\pi}{4}} 2\left(\frac{1}{2}r^2\,d\theta\right)$$

$$= 4\int_0^{\frac{\pi}{4}} a^2\cos^2 2\theta\, d\theta = 4a^2\int_0^{\frac{\pi}{4}}\left(\frac{1-\cos 4\theta}{2}d\theta\right)$$

$$= 2a^2\left[\theta - \frac{\sin 4\theta}{4}\right]_0^{\frac{\pi}{4}} = 2a^2\left[\frac{\pi}{4} - 0\right]$$

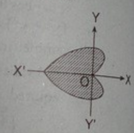
$$= \frac{\pi a^2}{2} \text{ Square unit}$$

15. Find the area bounded by the curve $r = a(1 - \cos\theta)$

Solⁿ: Here,

$$r = a(1-\cos\theta)$$

It is symmetrical on initial line and area of the cardiod,



$$A = 2\int_0^{\pi}\left(\frac{1}{2}r^2 d\theta\right)$$

$$= 2\int_0^{\pi}\frac{1}{2}a^2(1-\cos\theta)^2 d\theta$$

$$= a^2\int_0^{\pi}\left(1 - 2\cos\theta + \frac{1+\cos 2\theta}{2}\right)d\theta$$

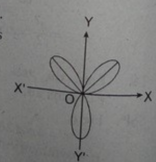
$$= a^2\left[\theta - 2\sin\theta + \tfrac{1}{2}\theta + \frac{\sin 2\theta}{4}\right]_0^{\pi}$$

$$= a^2\left[\pi - 0 + \frac{\pi}{2} + 0 - 0\right]$$

$$= \frac{3\pi a^2}{2} \text{ Square unit}$$

16. Find the area of a loop $r = a\sin 3\theta$

Solⁿ: The curve consists of three equal loops. One loop is obtained when θ increases from 0 to 60°.



$\therefore$ A = Area of the half loop

$$= \frac{1}{2}\int_0^{\frac{\pi}{6}} r^2\, d\theta$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{6}} a^2\sin^2 3\theta$$

Put,

$3\theta = t$, then $3d\theta = dt$

When $\theta = 0, t = 0$ and when $\theta = \frac{\pi}{6}, t = \frac{\pi}{2}$

$$\therefore\ A = \frac{1}{2}\int_0^{\frac{\pi}{2}} a^2\sin^2 t\frac{dt}{3} = \frac{a^2}{6}\int_0^{\frac{\pi}{2}}\frac{1-\cos 2t}{2}dt$$

$$= \frac{a^2}{12}\left[t - \frac{\sin 2t}{2}\right]_0^{\frac{\pi}{2}} = \frac{a^2}{12}\cdot\frac{\pi}{2}$$

$$\therefore \text{ Area of a loop} = 2\cdot\frac{a^2}{12}\cdot\frac{\pi}{2} = \frac{a^2\pi}{12}$$

Sometimes the area can be found conveniently if we transform a Cartesian equation in to polar and use the formula $\frac{1}{2}\int_{\alpha}^{\beta} r^2\, d\theta$.

17. Find the area common to the circle $r = a$ and the cardioids $r = a(1+\cos\theta)$.

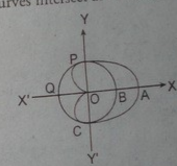
Solⁿ: Solving the two equations, we get,

$1 + \cos\theta = 1$

or, $\cos\theta = 0$

or, $\theta = \frac{\pi}{2}$

$\therefore$ The two curves intersect at P where $\theta = \frac{\pi}{2}$



The area common to them

$$= \text{Area OCBPQO}$$

$$= 2(\text{area OBP} + \text{area POQ})$$

$$= 2\left(\frac{1}{2}\int_0^{\frac{\pi}{2}} a^2\, d\theta + \frac{1}{2}\int_{\frac{\pi}{2}}^{\pi} a^2(1+\cos\theta)^2 d\theta\right)$$

$$= a^2\cdot\frac{\pi}{2} + a^2\int_{\frac{\pi}{2}}^{\pi}\left(1 + 2\cos\theta + \frac{1+\cos 2\theta}{2}\right)d\theta$$

$$= \frac{a^2\pi}{2} + \frac{3a^2\pi}{2} - \frac{3a^2\pi}{4} - 2a^2\sin\frac{\pi}{2}$$

$$= \frac{5a^2\pi}{4} - 2a^2 = a^2\left(\frac{5\pi}{4} - 2\right)$$

18. Find the area enclose by the following curves.

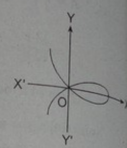
i) $x = a(1-t^2), y = at(1-t^2), -1 \le t \le 1$

Sol^n:

When $t = -1, x = y = 0$

Again,

When $t = 1, x = y = 0$



$\therefore$ Area of the loop

$$= \frac{1}{2}\int_{-1}^{1}\left(x\frac{dy}{dt} - y\frac{dx}{dt}\right)dt$$

$$= \frac{1}{2}\int_{-1}^{1}[at(1-t^2)a(1-3t^2) - at(1-t^2)a(-2t)]\,dt$$

$$= \frac{1}{2}a^2\int_{-1}^{1}(1-2t^2+t^4)\,dt = \frac{1}{2}a^2\left[t - \frac{2t^3}{3} + \frac{t^5}{5}\right]_{-1}^{1}$$

$$= \frac{1}{2}a^2\left(1 - \frac{2}{3} + \frac{1}{5} + 1 - \frac{2}{3} + \frac{1}{5}\right) = \frac{1}{2}a^2 \times \frac{16}{15} = \frac{8a^2}{15}$$

ii) $x = \frac{1-t^2}{1+t^2}, y = t\frac{1-t^2}{1+t^2} \quad -1 \le t \le 1$

Sol^n: Here,

$$x = \frac{1-t^2}{1+t^2}, y = \frac{t(1-t^2)}{1+t^2}$$

and t varies from $t = -1$ to $t = 1$, so that the required area

$$A = \frac{1}{2}\int_{-1}^{1}\left(x\frac{dy}{dt} - y\frac{dx}{dt}\right)dt$$

Now,

$$\frac{dx}{dt} = \frac{(1+t^2)(-2t)-(1-t^2)2t}{(1+t^2)^2} = \frac{-2t(1+t^2+1-t^2)}{(1+t^2)^2} = -\frac{4t}{(1+t^2)^2}$$

and $$\frac{dy}{dt} = \frac{(1+t^2)(1-3t^2)-(1-t^3)2t^4}{(1+t^2)^2} = \frac{1-4t^2-t^4}{(1+t^2)^2}$$

So,

$$A = \frac{1}{2}\int_{-1}^{1}\left[\frac{1-t^2}{1+t^2}\frac{(1+4t^2-t^4)}{(1+t^2)^2} - \frac{t-t^3}{1+t^2}\frac{-4t}{(1+t^2)^2}\right]dt$$

$$= \frac{1}{2}\int_{-1}^{1}\frac{1-t^2}{(1+t^2)^3}[1-4t^2-t^4+4t^2]dt$$

$$= \frac{1}{2}\int_{-1}^{1}\frac{1-t^2}{(1+t^2)^3}(1-t^4)dt = \frac{1}{2}\int_{-1}^{1}\frac{(1-t^2)(1-t^2)(1+t^2)}{(1+t^2)^3}dt$$

$$= \frac{1}{2}\int_{-1}^{1}\frac{(1-t^2)^2}{(1+t^2)^2}dt = \frac{1}{2}\int_{-1}^{1}\frac{(t^2-1)^2}{(t^2+1)^2}dt$$

$$= \frac{1}{2}\cdot 2\int_{0}^{1}\left(\frac{t^2-1}{t^2+1}\right)^2 dt$$

Put,

$t = \tan\theta, dt = \sec^2\theta$

When $t = 0, \theta = 0$, when $t = 1, \theta = \frac{\pi}{4}$

So,

$$A = \int_0^{\frac{\pi}{4}}\cos^2 2\theta\sec^2\theta\,d\theta$$

$$= \int_0^{\frac{\pi}{4}}(2\cos^2\theta - 1)^2\sec^2\theta\,d\theta$$

$$= \int_0^{\frac{\pi}{4}}(4\cos^4\theta - 4\cos^2\theta + 1)^2\sec^2\theta\,d\theta$$

$$= \int_0^{\frac{\pi}{4}}(2 + 2\cos 2\theta - 4 + \sec^2\theta)\,d\theta$$

$$= \left[2\theta + \frac{2\sin 2\theta}{2} - 4\theta + \tan\theta\right]_0^{\frac{\pi}{4}}$$

$$= \left[2\cdot\frac{\pi}{4} + 1 - 4\cdot\frac{\pi}{4} + 1\cdot 0\right]$$

$$= 2 - \frac{\pi}{2}$$

$\therefore \quad A = 2 - \frac{\pi}{2}$ Square unit

Exercise 16

Rectification:

Definition:

The process of finding the arc-length of the plane curves whose equations are given in Cartesian, parametric or polar form is called Rectification.

Some important formulae:

1. If the plane curves is $y = f(x)$, then the derivative of arc–length

$$\frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

So that arc-length is defined as;

$$s = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

Where the arc-length s measured from $x = a$ to $x = b$

Similarly,

$$\frac{ds}{dy} = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}$$

It follows that the arc-length is defined as;

$$s = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

Where, s measured from $y = c$ to $y = d$

2. If the plane curve is parametric, $x = f(t)$, and $y = g(t)$ then,

$$\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

It follows that the arc-length is defined as;

$$s = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Where, s measured from $t = t_1$ to $t = t_2$

3. If the equation of the curve is $r = f(\theta)$, then the derivative of arc-length is;

$$\frac{ds}{d\theta} = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}$$

Arc-length measured from $\theta = \alpha$ to $\theta = \beta$ is defined as;

$$s = \int_\alpha^\beta \sqrt{\gamma^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$$

4. If the equation of the curve is $\theta = f(r)$, then the derivative of arc-length is;

$$\frac{ds}{dr} = \sqrt{1 + \left(r\frac{d\theta}{dr}\right)^2}$$

Arc-length measured from $r = r_1$ to $r = r_2$ is defined as;

$$s = \int_{r_1}^{r_2} \sqrt{1 + \left(r\frac{d\theta}{dr}\right)^2}\, dr$$

---

1. **Find the length of the arc of the parabola the $x^2 = 4ay$ from the vertex to an extremity of the latus rectum.**

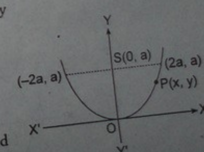
Solⁿ: Here,

The equation of parabola $x^2 = 4ay$

Differentiating yields;

$$2x\frac{dx}{dy} = 4a$$

$$\text{or,}\quad \frac{dx}{dy} = \frac{2a}{x}$$



Let (0,0) be the vertex and s(0, a) be the focus, so that an extremity of the latus rectum and arc-length is defined as;

$$s = \int_0^a \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy = \int_0^a \sqrt{1 + \frac{4a^2}{x^2}}\, dy$$

$$= \int_0^a \sqrt{\frac{x^2+4a^2}{x^2}}\, dy = \int_0^a \sqrt{\frac{4ay+4a^2}{4ay}}\, dy = \int_0^a \sqrt{\frac{y+a}{y}}\, dy$$

Put,

$y = t^2,\ dy = 2t\,dt$

When $y = 0,\ t = 0$

When $y = a,\ t = \sqrt{a}$

so,

$$s = \int_0^{\sqrt{a}} \frac{\sqrt{t^2+a}}{t}\, t\, dt = 2\int_0^{\sqrt{a}} \sqrt{t^2+a}\, dt$$

$$= 2\left[\frac{t\sqrt{t^2+a}}{2} + \frac{a}{2}\log\left(t + \sqrt{t^2+a}\right)\right]_0^{\sqrt{a}}$$

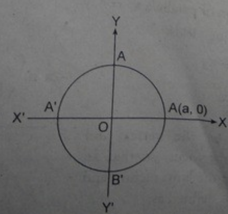
$$= 2\left[\frac{\sqrt{a}\sqrt{a+a}}{2} + \frac{a}{2}\log\left(\sqrt{a} + \sqrt{a+a}\right)\right]$$

$$= a\left(\sqrt{2} + \log\left(1 + \sqrt{2}\right)\right) \text{ unit}$$

2. Find the arc-length of the following:

i) $x^2 + y^2 = a^2$

Sol^n:



Here, the equation of the circle $x^2 + y^2 = a^2$

Differentiating yields;

$$2x + 2y\frac{dy}{dx} = 0$$

$$\therefore \quad \frac{dy}{dx} = -\frac{x}{y}$$

Its circumference = 4 times arc of OAB in the first quadrant

$$= 4\int_0^a \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = 4\int_0^a \sqrt{1 + \frac{x^2}{y^2}}\, dx$$

$$= 4\int_0^a \sqrt{\frac{x^2+y^2}{y^2}}\, dx = 4\int_0^a \sqrt{\frac{a^2}{y^2}}\, dx$$

$$= 4\frac{a}{y}dx \int_0^a \sqrt{\frac{a}{y}}\, dx = 4a\int_0^a \frac{1}{\sqrt{a^2-x^2}}\, dx$$

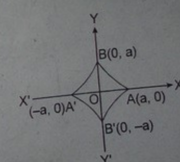
$$[\because x^2 + y^2 = a^2]$$

$$= 4a\left[\sin^{-1}\frac{x}{a}\right]_0^a = 4a \cdot \frac{\pi}{2} = 2\pi a \text{ unit}$$

ii) $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ [Astroid]

Sol^n: Here,

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$$



Differentiating yields,

$$\frac{2}{3}x^{-\frac{1}{3}} + \frac{2}{3}y^{-\frac{1}{3}}\frac{dy}{dx} = 0$$

$$\therefore \quad \frac{dy}{dx} = -\frac{y^{\frac{1}{3}}}{x^{\frac{1}{3}}}$$

Length of asteroid is defined as;

$$s = 4\int_0^a \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = 4\int_0^a \sqrt{1 + \frac{y^{\frac{2}{3}}}{x^{\frac{2}{3}}}}\, dx$$

$$= 4\int_0^a \sqrt{\frac{x^{\frac{2}{3}}+y^{\frac{2}{3}}}{x^{\frac{2}{3}}}}\, dx \qquad \left[\because x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}\right]$$

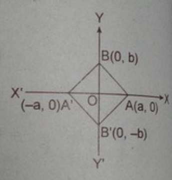
$$= 4\int_0^a \sqrt{\frac{a^{\frac{2}{3}}}{x^{\frac{2}{3}}}}\,dx = 4a^{\frac{1}{3}}\int_0^a \frac{1}{x^{\frac{1}{3}}}\,dx = 4a^{\frac{1}{3}}\frac{3}{2}\left[x^{\frac{2}{3}}\right]_0^a = \frac{12a^{\frac{1}{3}}}{2}\cdot a^{\frac{2}{3}}$$

$= 6a$ units

iii) $\frac{x^{\frac{2}{3}}}{a^{\frac{2}{3}}} + \frac{y^{\frac{2}{3}}}{b^{\frac{2}{3}}} = 1$ **[Hypocycloid]**

Sol^n: Here,

$$\frac{x^{\frac{2}{3}}}{a^{\frac{2}{3}}} + \frac{y^{\frac{2}{3}}}{b^{\frac{2}{3}}} = 1$$



Differentiating yields;

$$\frac{2}{3a^{\frac{2}{3}}}x^{-\frac{1}{3}} + \frac{2}{3b^{\frac{2}{3}}}y^{-\frac{1}{3}}\frac{dy}{dx} = 0$$

$$\therefore \quad \frac{dy}{dx} = -\frac{b^{\frac{2}{3}}}{a^{\frac{2}{3}}}\frac{y^{\frac{1}{3}}}{x^{\frac{1}{3}}}$$

Length of the hypocycloid is defined as;

$$s = 4\int_0^a \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$$

$$= 4\int_0^a \sqrt{1 + \frac{b^{\frac{4}{3}}\cdot y^{\frac{2}{3}}}{a^{\frac{4}{3}}\cdot x^{\frac{2}{3}}}}\,dx = 4\int_0^a \sqrt{\frac{a^2x^{\frac{2}{3}} + b^{\frac{4}{3}}\cdot\frac{b^{\frac{2}{3}}}{a^{\frac{2}{3}}}\left(a^{\frac{2}{3}} - x^{\frac{2}{3}}\right)}{a^{\frac{4}{3}}x^{\frac{2}{3}}}}\,dx$$

$$= 4\int_0^a \sqrt{\frac{a^2x^{\frac{2}{3}} + b^2a^{\frac{2}{3}} - b^2x^{\frac{2}{3}}}{a^{\frac{4}{3}}x^{\frac{2}{3}}}}\,dx$$

$$= 4\int_0^a \sqrt{\frac{\left(a^2 - b^2\right)x^{\frac{2}{3}} + b^2a^{\frac{2}{3}}}{a^2x^{\frac{2}{3}}}}\,dx \qquad \left[\because \frac{x^{\frac{2}{3}}}{a^{\frac{2}{3}}} + \frac{y^{\frac{2}{3}}}{b^{\frac{2}{3}}} = 1\right]$$

$$= \frac{4}{a}\int_0^a \sqrt{(a^2 - b^2)x^{\frac{2}{3}} + b^2a^{\frac{2}{3}}}\,x^{-\frac{1}{3}}dx$$

Put,

$$(a^2 - b^2)x^{\frac{2}{3}} + b^2a^{\frac{2}{3}} = t^2$$

$$\frac{2}{3}(a^2 - b^2)x^{-\frac{1}{3}}dx = 2t\,dt$$

When $x = 0,\ t = ba^{\frac{1}{3}}$

When $x = a,\ t = a^{\frac{4}{3}}$

$$= \frac{4}{a}\int_{ba^{\frac{1}{3}}}^{a^{\frac{4}{3}}} \frac{t\cdot 3t\,dt}{\left(a^2 - b^2\right)} = \frac{4}{a\left(a^2 - b^2\right)}\left[t^3\right]_{ba^{\frac{1}{3}}}^{a^{\frac{4}{3}}}$$

$$= \frac{4}{a\left(a^2 - b^2\right)}(a^4 - b^3a) = \frac{4a\left(a^3 - b^3\right)}{a(a+b)(a-b)}$$

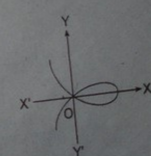
$$= \frac{4(a-b)\left(a^2 + ab + b^2\right)}{(a+b)(a-b)} = \frac{4(a^2 + ab + b^2)}{a+b} \text{ units}$$

3. **If s be the length of an arc of the curve $3ay^2 = x(x - a)^2$ measured from the origin to the point (x,y) show that $3s^2 = 4x^2 + 3y^2$. Show also that the entire length of the loop is $\frac{4}{3}\sqrt{3}a$.**

Sol^n: Here,

The equation of the curve is;

$$3ay^2 = x(x - a)^2$$



Differentiating yields,

$$6ay\frac{dy}{dx} = x2(x - a) + (x - a)^2$$

$$= (x - a)(2x + x - a)$$

$$\therefore \quad \frac{dy}{dx} = \frac{(x-a)(3x-a)}{6ay}$$

$$= \frac{(x-a)(3x-a)}{6a}\frac{\sqrt{3a}}{\sqrt{x}(x-a)} \qquad [\because 3ay^2 = x(x - a)^2]$$

$$\text{or,} \quad \frac{dy}{dx} = \frac{(3x-a)\sqrt{3a}}{6\sqrt{ax}}$$

Arc-length measured from the origin to the point (x, y) is;

$$s=\int_0^x\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx=\int_0^x\sqrt{1+\frac{(3x-a)^2}{12ax}}\,dx$$

$$=\int_0^x\sqrt{\frac{12ax+(3x-a)^2}{12ax}}\,dx=\int_0^x\sqrt{\frac{(3x+a)^2}{12ax}}\,dx$$

$$=\int_0^x\frac{3x+a}{\sqrt{12ax}}\,dx=\frac{1}{\sqrt{12a}}\int_0^x\left(3x^{\frac{1}{2}}+ax^{-\frac{1}{2}}\right)dx$$

$$=\frac{1}{\sqrt{12a}}\left[\frac{2}{3}3x^{\frac{3}{2}}+\frac{2}{1}ax^{\frac{1}{2}}\right]_0^x$$

or, $3=\dfrac{2x^{\frac{1}{2}}}{\sqrt{12a}}(x+a)$

Squaring both sides, we have,

$$s^2=\frac{4x}{12a}(x+a)^2$$

or, $s^2=\dfrac{x}{3a}\{(x-a)^2+4ax\}\Rightarrow 3as^2=x(x-a)^2+4ax^2$

or, $3s^2=3y^2+4x^2$

Entire length of the loop is defined as;

$$2\int_0^a\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$=2\int_0^a\frac{3x^{\frac{1}{2}}+ax^{-\frac{1}{2}}}{\sqrt{12a}}\,dx=2\left[\frac{2x^{\frac{1}{2}}}{\sqrt{12a}}(x+a)\right]_0^a$$

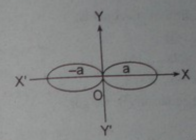
$$=\frac{2}{\sqrt{12a}}\left[\frac{3\times 2a^{\frac{3}{2}}}{3}+2a^{\frac{1}{2}}\right]=\frac{2}{2\sqrt{3a}}\left[2a^{\frac{3}{2}}+2a^{\frac{3}{2}}\right]$$

$$=\frac{1}{\sqrt{3a}}4a^{\frac{3}{2}}$$

$$=\frac{4}{3}\sqrt{3}$$

4. Show that the length of the curve $x^2(a^2-x^2)=8a^2y^2$ is $\pi a\sqrt{2}$.

Soln:



The curve cuts the x-axis at (0,0), (a, 0) and (−a, 0) and it consists of two loops.

$\therefore$ Whole length; $s=4\displaystyle\int_0^a\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$

Differentiating yields;

$$x^2(a^2-x^2)=8a^2y^2$$

We have,

$$2a^2x-4x^3=16a^2y\frac{dy}{dx}\qquad\because\left(\frac{dy}{dx}\right)^2=\frac{4x^2(a^2-2x^2)^2.\,8a^2}{256a^4x^2(a^2-x^2)}$$

$$=\frac{(a^2-2x^2)^2}{8a^2(a^2-x^2)}\qquad\because 1+\left(\frac{dy}{dx}\right)^2=1+\frac{a^4-4a^2x^2+4x^4}{8a^4-8a^2x^2}$$

$$=\frac{9a^4-12a^2x^2+4x^4}{8a^2(a^2-x^2)}$$

$$=\frac{(3a^2-2x^2)^2}{8a^2(a^2-x^2)}\qquad\because s=4\int_0^a\frac{3a^2-2x^2}{2\sqrt{2}a\sqrt{a^2-x^2}}\,dx$$

$$=4\int_0^a\frac{2(a^2-x^2)+a^2}{2\sqrt{2}\,a\sqrt{a^2-x^2}}\,dx$$

$$=\frac{2\sqrt{2}}{a}\int_0^a\sqrt{a^2-x^2}\,dx+\sqrt{2}a\int_0^a\frac{1}{\sqrt{a^2-x^2}}\,dx$$

$$=\frac{2\sqrt{2}}{a}\left[\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_0^a+\sqrt{2}a\left[\sin^{-1}\frac{x}{a}\right]_0^a$$

$$=\frac{2\sqrt{2}}{a}\cdot\frac{a^2}{2}\cdot\frac{\pi}{2}+\sqrt{2}a\frac{\pi}{2}=\frac{\pi a\sqrt{2}}{2}+\frac{\pi a\sqrt{2}}{2}=\pi a\sqrt{2}$$

5. **Find the arc–length of the parabola $y^2 = 4x$ cut off by the line $y = 2x$.**

Sol^n: Here, the equation of parabola is;

$y^2 = 4x$ ... (i)

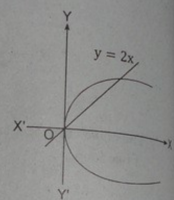
and the line $y = 2x$ ... (ii)

Solving these $x = 0,\ x = 1$

Differentiating equation (i), we get,

$$2y\frac{dy}{dx} = 4$$

$$\therefore\ \frac{dy}{dx} = \frac{2}{y}$$



The arc-length is defined as;

$$s = \int_0^1 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$$

$$= \int_0^1 \sqrt{1 + \frac{4}{y^2}}\,dx = \int_0^1 \sqrt{\frac{y^2+4}{y^2}}\,dx$$

$$= \int_0^1 \sqrt{\frac{4x+4}{4x}}\,dx$$

$$= 2\int_0^1 \sqrt{\frac{x+1}{x}}\,dx$$

Put $x = t^2,\ dx = 2t\,dt$

When $x = 0, t = 0$

When $x = 1, t = 1$

$$= \int_0^1 \frac{\sqrt{t^2+1}}{t}\,2t\,dt$$

$$= 2\int_0^1 \sqrt{t^2+1}\,dt$$

$$= 2\left[\frac{t\sqrt{t^2+1}}{2} + \frac{1}{2}\log\left(t + \sqrt{t^2+1}\right)\right]_0^1$$

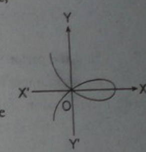
$$= \sqrt{2} + \log(1+\sqrt{2}) \text{ unit}$$

6. **Prove that the loop of the curve $x = t^2, y = t - \frac{t^3}{3}$ is length $4\sqrt{3}$.**

Sol^n: Here, the equation of the loop of the curve;

$$x = t^2, y = t - \frac{t^3}{3}$$

$$\text{or,}\ \ x = t^2, y = t\left(1 - \frac{t^2}{3}\right)$$



Eliminating t from these relations, we have,

$$y^2 = t^2\left(1 - \frac{t^2}{3}\right)^2$$

$$\text{or,}\ \ y^2 = x\left(1 - \frac{x}{3}\right)^2$$

$$\text{or,}\ \ 9y^2 = x(x-3)^2$$

Differentiating yields,

$$18y\frac{dy}{dx} = x - 2(x-3) + (x-3)^2$$

$$= (x-3)(2x + x - 3)$$

$$= (x-3)(3x-3)$$

$$\text{or,}\ \ \frac{dy}{dx} = \frac{3(x-3)(x-1)3}{18\sqrt{x}(x-3)}$$

$$\therefore\ \frac{dy}{dx} = \frac{x-1}{2\sqrt{x}}$$

Arc-length of the loop measured from $x = 0$ to $x = 3$ is given by

$$s = 2\int_0^3 \left[1 + \left(\frac{dy}{dx}\right)^2 dx\right]$$

$$= 2\int_0^3 \sqrt{1 + \frac{(x-1)^2}{4x}}\,dx = 2\int_0^3 \sqrt{\frac{4x+(x-1)^2}{4x}}\,dx$$

$$= 2\int_0^3 \sqrt{\frac{(x+1)^2}{4x}}\,dx = 2\int_0^3 \frac{x+1}{4x}\,dx = \int_0^3 \left(x^{\frac{1}{2}} + x^{-\frac{1}{2}}\right)dx$$

$$= \left[\frac{2}{3}x^{\frac{3}{2}} + x^{\frac{1}{2}}\right]_0^3 = \left[\frac{2}{3}(3)^{\frac{3}{2}} + 2(3)^{\frac{1}{2}}\right] = \left[2(3)^{\frac{1}{2}} + 2(3)^{\frac{1}{2}}\right]$$

$$= 4(3)^{\frac{1}{2}} = 4\sqrt{3} \text{ unit}$$

7. **Find the perimeter of cardioids $r = a(1 + \cos\theta)$ and show that the arc of upper half is bisected by $\theta = \frac{\pi}{3}$.**

Solⁿ: The equation of the curve is;

$$r = a(1 + \cos\theta)$$

$$\therefore \quad \frac{dr}{d\theta} = -a\sin\theta$$

The curve is symmetrical about the initial line and the limits for the upper half are 0 and π.

$$\therefore \quad \text{Length of the upper half} = \int_0^{\pi} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

$$= \int_0^{\pi} \sqrt{a^2 + (1+\cos\theta)^2 + a\sin^2\theta}\, d\theta$$

$$= \int_0^{\pi} a\sqrt{2(1+\cos\theta)}\, d\theta$$

$$= \int_0^{\pi} 2a\cos\frac{\theta}{2}\, d\theta = 4a\left[\sin\frac{\theta}{2}\right]_0^{\pi}$$

$$= 4a$$

$\therefore$ The entire length $= 2 \times 4a = 8a$

If we take the limits of integration 0 and $\frac{\pi}{3}$, the part of the upper half intercepted between $\theta = 0$ and $\theta = \frac{\pi}{3}$ is found to be;

$$4a\left[\sin\frac{\theta}{2}\right]_0^{\pi/3} = 4a \times \frac{1}{2} = 2a \text{ which is half of } 4a$$

$\therefore$ The upper half of the circle is bisected by $\theta = \frac{\pi}{3}$

8. **In the cycloid, $x = a(\theta + \sin\theta), y = a(1 - \cos\theta)$, show that $s^2 = 8ay$, s being measured from the vertex to any point.**

Solⁿ: Here,

$$\frac{dx}{d\theta} = a(1 + \cos\theta)$$

and $\frac{dy}{d\theta} = a\sin\theta$

At the vertex $0, \theta = 0$

$\therefore$ Length of the arc from $\theta = 0$ to any point θ is given by;

$$s = \int_0^{\theta} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$$

$$= \int_0^{\theta} \sqrt{a^2(1+\cos\theta)^2 + a^2\sin^2\theta\, d\theta}$$

$$= a\int_0^{\theta} \sqrt{2(1+\cos\theta)d\theta}$$

$$= a\int_0^{\theta} 2\cos\frac{\theta}{2}\, d\theta$$

$$= 4a\sin\frac{\theta}{2}$$

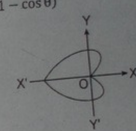
$$\therefore \quad s^2 = 16a^2\sin^2\frac{\theta}{2} = 8a \cdot a(1 - \cos\theta) = 8ay$$

9. **Find the perimeter of the cardioids $r = a(1 - \cos\theta)$ and show that the area of the upper half is bisected by $\theta = \frac{2\pi}{3}$.**

Solⁿ: Here, the equation of the cardioids $r = a(1 - \cos\theta)$

Differentiating yields,

$$\frac{dr}{d\theta} = a\sin\theta$$



Perimeter of the cardioids is defined by;

$$s = 2\int_0^{\pi} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

$$= 2\int_0^{\pi} \sqrt{a^2(1 - 2\cos\theta + \cos^2\theta + \sin^2\theta)}\, d\theta$$

$$= 2a\int_0^{\pi} \sqrt{(1 - 2\cos\theta + 1)}\, d\theta = 2a\int_0^{\pi} \sqrt{2(1-\cos\theta)}\, d\theta$$

$$= 2a\int_0^{\pi} \sqrt{4\sin^2\frac{\theta}{2}}\, d\theta = 4a\int_0^{\pi} \sin\frac{\theta}{2}\, d\theta$$

$$= 4a\left[-2\cos\frac{\theta}{2}\right]_0^{\pi} = 4a(-0 + 2) = 8a \text{ unit}$$

So, arc-length of upper half = 4a unit

Also, arc-length measured from $\theta = 0$ to $\theta = \frac{2\pi}{3}$ is given by;

$$s = \int_0^{\frac{2\pi}{3}} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}$$

$$= 2a\int_0^{\frac{2\pi}{3}} \sin\frac{\theta}{2}\,d\theta$$

$$= 2a\left[-\cos\frac{\pi}{2}\right]_0^{\frac{2\pi}{3}}$$

$$= 4a\left[-\frac{1}{2}+1\right]$$

$$= 2a \text{ unit}$$

Hence, arc-length of upper half is bisected by $\theta = \frac{2\pi}{3}$

Exercise 17

Volume and surface

Volume of solid of Revolution:

If the equation of the curve $y = f(x)$ is continuous in the interval (a, b), then the volume of the solid formed when the area bounded by the curve, the x–axis and the ordinates at $x = a$ and $x = b$ is revolved about the x–axis is $\int_a^b \pi y^2\,dx$.

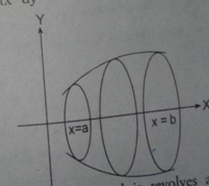
Some Standard Formulae:

1. If the area of a curve $y = f(x)$ from $x = a$ to $x = b$, is rotated through one revolution about the x-axis, then it generates a solid with plane circular ends as shown in the figure.

   Thus, the required volume is defined as $v = \int_a^b \pi y^2\,dx$

2. The volume of solid generated by revolution about y-axis of the area bounded by the curve $x = f(y)$, the y–axis and abscissa $y = c, y = d$ is defined as;

   $v = \int_c^d \pi x^2\,dy$



3. If the curve is $r = f(\theta)$ and it revolves about initial line, the volume generated is defined as;

   $v = \int_a^b \pi y^2\,dx$

   $= \pi\int_\alpha^\beta (r\sin\theta)^2 d(r\cos\theta)$

4. If the area bounded by the curve $r = f(\theta)$ $\theta = \theta_1, \theta = \theta_2$ is roted about initial line then the volume generated is defined as;

   $v = \frac{2}{3}\pi\int_{\theta_1}^{\theta_2} r^3 \sin\theta\,d\theta$

5. If the curve $y = f(x)$ is roted about x-axis from $x = a$ to $x = b$ then the whole surface is defined as;

$$s = \int_a^b 2\pi y \cdot ds = \int_a^b 2\pi y \frac{ds}{dx} dx$$

$$= \int_a^b 2\pi f(x) \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

6. If the curve $x = f(y)$ is roted about y-axis from $y = c$ to $y = d$ then the whole surface is defined as;

$$s = \int_c^d 2\pi x \cdot ds = \int_c^d 2\pi x \frac{ds}{dy} dy = \int_c^d 2\pi f(y) \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy$$

7. If the curve is polar form $r = f(\theta)$ then the surface is defined as;

$$s = \int_c^d 2\pi y \cdot ds = \int_\alpha^\beta 2\pi r \sin\theta \sqrt{r^2 + \left(\frac{dx}{d\theta}\right)^2} d\theta$$

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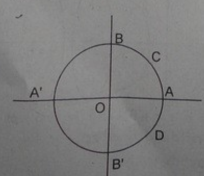
1. **Find the volume of ellipsoid formed by the revolution of an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ about x-axis.**

Sol^n: The equation of an ellipse is;

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$



The volume of the solid formed by the revolution of the ellipse about x-axis.

= volume formed by the revolution of two portion ABA'

=2 (volume formed by the revolution of the portion ACB)

$$= 2\pi \int_0^a y^2 dx = 2\pi \int_0^a b^2 \left(1 - \frac{x^2}{a^2}\right) dx = 2\pi b^2 \left[x - \frac{x^3}{3a^2}\right]_0^a$$

$$= 2\pi b^2 \left(a - \frac{a}{3}\right) = \frac{4\pi ab^2}{3}$$

2. **Prove that the volume and surface of a sphere of radius a is $\frac{4}{3}\pi a^3$ and $4\pi a^2$ respectively.**

Sol^n: Here, the equation of sphere of radius a is $x^2 + y^2 = a^2$ = 0

Differentiating yields,

$$2x + 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{x}{y} = -\frac{x}{\sqrt{a^2 - x^2}}$$

$\therefore$ The sphere is formed by revolution of the area of the circle $x^2 + y^2 = a^2$ about x–axis, so the required surface is defined as;

$$s = 2\int_0^a 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

$$= 2\int_0^a 2\pi y \sqrt{1 + \frac{x^2}{y^2}} dx = 2\int_0^a 2\pi y \sqrt{\frac{x^2+y^2}{y^2}} dx$$

$(\because x^2 + y^2 = a^2)$

$$= 4\pi \int_0^a y \cdot \frac{a}{y} dx$$

$$= 4\pi a[x]_0^a = 4\pi a^2 \text{ sq unit}$$

and the volume of the sphere is defined as;

$$v = 2\int_0^a \pi y^2 dx$$

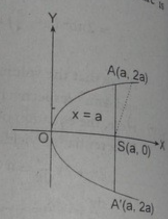
$$= 2\pi \int_0^a (a^2 - x^2) dx = 2\pi \left[a^2 x - \frac{x^3}{3}\right]_0^a$$

$$= 2\pi \left[a^3 - \frac{a^3}{3}\right]$$

$$= \frac{4\pi a^3}{3} \text{ cubic unit}$$

3. An arc of a parabola is bounded at both ends by the latus rectum of length 4a. Find the volume generated by when the arc is rotated about the latus rectum.

Sol^n: Here, the parabola having latus rectum is $y^2 = 4ax$. The volume generated by revolving the arc about latus rectum $x = a$ is given;



$$v = \int_{-2a}^{2a} \pi (a - x)^2 dy$$

$$= \int_{-2a}^{2a} \pi \left(a - \frac{y^2}{4a}\right)^2 dy$$

$$= 2\pi \int_{-2a}^{2a} \left(a^2 - \frac{y^2}{2} + \frac{y^4}{16a^2}\right) dy$$

$$= 2\pi \left[a^2 y - \frac{y^3}{6} + \frac{y^5}{80a^2}\right]_0^{2a}$$

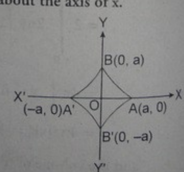
$$= 2\pi \left[2a^3 - \frac{4a^3}{3} + \frac{2}{5}a^3 - 0\right]$$

$$= \frac{32}{15}\pi a^3 \text{ cubic unit}$$

4. Find the volume and surface area of the solid generated by revolving the asteroid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ about the axis of x.

Sol^n: Here, the asteroid;

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$$



$$\therefore \quad y = \left(a^{\frac{2}{3}} - x^{\frac{2}{3}}\right)^{\frac{3}{2}}$$

Differentiating yields,

$$\frac{2}{3}x^{-\frac{1}{3}} + \frac{2}{3}y^{-\frac{1}{3}}\frac{dy}{dx} = 0$$

or, $\dfrac{dy}{dx} = -\dfrac{x^{-\frac{1}{3}}}{y^{-\frac{1}{3}}}$ and the volume of asteroid is defined as;

$$v = \int_{-a}^{a} \pi y^2 \, dx = 2\int_0^a \pi \left(a^{\frac{2}{3}} - x^{\frac{2}{3}}\right)^3 dx$$

Put $x = a\sin^3\theta$, $dx = 3a\sin^2\theta \cdot \cos\theta \, d\theta$

When $x = 0, \theta = 0$

When $x = a, \theta = \frac{\pi}{2}$

$$\therefore \quad v = 2\pi \int_0^{\frac{\pi}{2}} \left(a^{\frac{2}{3}} - a^{\frac{2}{3}}\sin^2\theta\right)^3 3a\sin^2\theta\cos^7\theta \, d\theta$$

$$= 6a^2\pi \int_0^{\frac{\pi}{2}} \sin^2\theta\cos^7\theta \, d\theta = 6a^2\pi \frac{\Gamma\left(\frac{2+1}{2}\right)\Gamma\left(\frac{7+1}{2}\right)}{2\Gamma\left(\frac{2+7+1}{7}\right)}$$

$$= \frac{3a^2 \frac{1}{2}\cdot\sqrt{\pi}\, 3\cdot 2\cdot 1}{\frac{9}{2}\cdot\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}} = \frac{32\pi a^3}{105} \text{ cubic unit}$$

and the surface of the given curve is defined as;

$$s = \int_{-a}^{a} 2\pi y \frac{ds}{dx} dx = 2\pi \int_{-a}^{a} y\sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

$$= 2\pi \int_{-a}^{a} y\sqrt{1 + \frac{y^{\frac{2}{3}}}{x^{\frac{2}{3}}}} dx = 2\pi \int_{-a}^{a} y\sqrt{\frac{x^{\frac{2}{3}} + y^{\frac{2}{3}}}{x^{\frac{2}{3}}}} dx$$

$$= 4\pi a^{\frac{1}{3}} \int_0^a \left(a^{\frac{2}{3}} - x^{\frac{2}{3}}\right)^{\frac{2}{3}} x^{-\frac{1}{3}} dx$$

Put $a^{\frac{2}{3}} - x^{\frac{2}{3}} = t^2$

$$-\frac{2}{3}x^{-\frac{1}{3}} dx = 2t \, dt$$

When $x = 0, t = a^{\frac{1}{3}}$

When $x = a, t = 0$

$$s = 4\pi a^{\frac{1}{3}} \int_1^0 t^3 (-3t \, dt) = -12\pi a^{\frac{1}{3}} \left[\frac{t^5}{5}\right]_{a^{\frac{1}{3}}}^0$$

$$= -12\pi a^{\frac{1}{3}} \left[0 - \frac{a^{\frac{5}{3}}}{5}\right] = \frac{12\pi a^2}{5} \text{ square unit}$$

5. The part of the parabola $y^2 = 4ax$ bounded by the latus rectum revolves about the tangent at the vertex. Find the volume and the area of the curved surface thus generated.

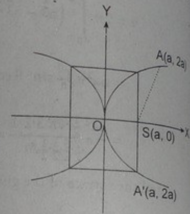
Sol^n: Here, the equation of the curve;

$$y^2 = 4ax$$

Differentiating yields,

$$2y\frac{dy}{dx} = 4a$$

or, $\frac{dy}{dx} = \frac{2a}{y}$



The volume generated by revolving the parabola about its tangents is defined as;

$$v = \int_{-2a}^{2a} \pi x^2\, dy$$

$$= \int_{-2a}^{2a} \frac{y^4}{16a^2}dy = \frac{\pi}{16a^2}\left[\frac{y^5}{5}\right]_{-2a}^{2a} = \frac{\pi}{16a^2}\left[\frac{(2a)^5}{5} + \frac{(2a)^5}{5}\right]$$

$$= \frac{4\pi a^3}{5} \text{ cubic unit}$$

The area of surface generated by revolving the parabola about its tangent is defined as;

$$s = 2\pi\int_{-2a}^{2a} x\,\frac{ds}{dy}dy = 2\pi\int_{-2a}^{2a} x\sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

$$= 2\pi\int_{-2a}^{2a} \frac{y^2}{4a}\sqrt{1 + \frac{y^2}{4a^2}}\, dy = \frac{\pi}{4a^2}\int_{-2a}^{2a} y^2\sqrt{4a^2 + y^2}\, dy$$

Put $y = 2a\tan\theta$

$dy = 2a\sec^2\theta\, d\theta$

When $y = -2a$, $\theta = -\frac{\pi}{4}$

When $y = 2a$, $\theta = \frac{\pi}{4}$

$$= \frac{\pi}{4a^2}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} 4a^2\tan^2\theta\sqrt{4a^2 + 4a^2\tan^2\theta} \times 2a\sec^2\theta\, d\theta$$

$$= 4\pi a^2\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}(\sec^2\theta - 1)\sec^2\theta\, d\theta$$

Using reduction formula,

$$= 4\pi a^2\left[\frac{\sec^3\theta\tan\theta}{4} - \frac{1}{8}\sec\theta\tan\theta - \frac{1}{8}\log(\sec\theta\tan\theta)\right]_{-\frac{\pi}{4}}^{\frac{\pi}{4}}$$

$$= 4\pi a^2\left[\frac{2\sqrt{2}}{4} - \frac{\sqrt{2}}{8} - \frac{1}{2}\log(\sqrt{2}+1) + \frac{2\sqrt{2}}{4} - \frac{\sqrt{2}}{8} + \frac{1}{8}\log(\sqrt{2}-1)\right]$$

$$= 4\pi a^2\left(\frac{3\sqrt{2}}{4} - \frac{1}{8}\log\frac{\sqrt{2}+1}{\sqrt{2}-1}\right) = 4\pi a^2\left[\frac{3\sqrt{2}}{2} - \frac{1}{2}\log(\sqrt{2}+1)\right]$$

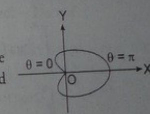
$$= \pi a^2\left[2\sqrt{2} - \log(\sqrt{2}+1)\right]$$

6. Find the volume of the solid formed by the revolution of the cardioids $r = a(1 + \cos\theta)$ about the initial line.

Sol^n: Here;

The equation of cardioids;

$$r = a(1 + \cos\theta)$$



The volume generated revolving the cardioids about the initial line is defined as;

$$v = \int_0^{\pi} y^2\, dx$$

Put $x = r\cos\theta$, $y = r\sin\theta$

$x = a(1 + \cos\theta)\cos\theta$, $y = a(1 + \cos\theta)\sin\theta$

Differentiating yields,

$dx = a(-\sin\theta - 2\cos\theta\sin\theta)d\theta$

$dy = a(\cos^2\theta + \cos^2\theta - \sin^2\theta)d\theta$

So, the volume generated by the cardioids revolving about initial line is defined as;

$$v = \int_0^{\pi} \pi y^2 dx$$

$$= \int_0^{\pi} \pi a^2(1 + \cos\theta)^2\sin^2\theta(1 + 2\cos\theta)(-a\sin\theta)d\theta$$

Put $\cos\theta = t$, $\sin\theta\, d\theta = dt$

When $\theta = 0$, $t = 1$

When $\theta = \pi$, $t = -1$

$$= \pi a^3\int_1^{-1}(1+t)^2(1-t^2)(1+2t)dt$$

$$= \pi a^3\int_1^{-1}(1 + 4t + 4t^2 - 2t^3 - 5t^4 - 2t^5)\, dt$$

$$= \pi a^3\left[t + 2t^2 + \frac{4}{3}t^3 - \frac{1}{2}t^4 - t^5 - \frac{1}{3}t^6\right]_1^{-1}$$

c unit

7. **Find the area of the surface of the solid generated by the revolution of the cardioids $r = a(1 - \cos\theta)$ about the initial line.**

Sol^n: Here, cardioids is;

$$r = a(1 - \cos\theta)$$

Differentiating yields,

$$\frac{dr}{d\theta} = a\sin\theta$$

The area of the surface generated by revolution of cardioids about initial line is defined as;

$$\therefore \quad s = \int_0^{\pi} 2\pi y \frac{ds}{d\theta} d\theta$$

Put $x = r\cos\theta$, $y = r\sin\theta$

We have,

$$x = a(1 - \cos\theta)\cos\theta, y = a(1 - \cos\theta)\sin\theta$$

$$s = \int_0^{\pi} 2\pi a(1 - \cos\theta)\sin\theta\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta$$

$$= \int_0^{\pi} 2\pi a(1 - \cos\theta)\sin\theta\sqrt{a^2(1 - 2\cos\theta + \cos^2\theta + \sin^2\theta)}d\theta$$

$$= \int_0^{\pi} 2\pi a(1 - \cos\theta)\sin\theta\sqrt{a^2 2(1 - \cos\theta)}d\theta$$

$$= 2\pi a^2\int_0^{\pi} 2\sin^2\left(\frac{\theta}{2}\right) 2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right) 2\sin\left(\frac{\theta}{2}\right) d\theta$$

$$= 16\pi a^2\int_0^{\pi} \sin^4\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right) d\theta$$

Put $\frac{\theta}{2} = t, d\theta = 2dt$

When $\theta = 0, t = 0$

When $\theta = \pi, t = \frac{\pi}{2}$

$$= 16\pi a^2\int_0^{\pi} \sin^4 t\cos t\, dt$$

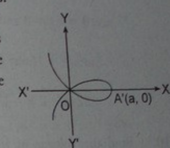
$$= 16\pi a^2\frac{2\Gamma\frac{(4+1)}{2}\Gamma\frac{(1+1)}{2}}{2\Gamma\frac{(4+1+2)}{2}}$$

$$= \frac{16\pi a^2 \cdot \frac{3}{2}\cdot\frac{1}{2}\cdot\sqrt{\pi}\cdot 1}{\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}}$$

$$= \frac{32}{5}\pi a^2 \text{ Square unit}$$

8. **Find the volume of the solids formed by the revolution of the curve $y^2 = x^2(a - x)$ about the x-axis.**

Sol^n: Here, $y^2 = x^2(a - x)$ it is symmetrical on x–axis and passes through the origin and $(a, 0)$. So the volume generated by revolving the curve about x-axis is defined as;



$$v = \int_0^a \pi y^2 dx$$

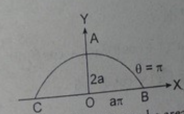
$$= \pi\int_0^a (ax^2 - x^3)\, dx$$

$$= \pi\left[\frac{ax^3}{3} - \frac{x^4}{4}\right]_0^a$$

$$= \frac{\pi a^4}{12} \text{ Cubic unit}$$

9. **Find the surface area of the solid generated by revolving the cycloid $x = a(\theta + \sin\theta), y = a(1 + \cos\theta)$ about its base.**

Sol^n:



The surface area required = 2 times the area of the surface generated by the revolution of the portion AOB

$$= 2\int_0^{\pi} 2\pi y \frac{ds}{d\theta} d\theta$$

But $$\frac{ds}{d\theta} = \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dr}{d\theta}\right)^2}$$

$$= \sqrt{a^2(1 + \cos\theta)^2 + a^2(-\sin\theta)^2}$$

$$= a\sqrt{2(1 + \cos\theta)} = 2a\cos\frac{\theta}{2}$$

$$\therefore \quad \text{Surface} = 4\pi\int_0^{\pi} a(1 + \cos\theta)2a\cos\frac{\theta}{2} d\theta$$

$$= 8\pi a^2\int_0^{\pi} 2\cos^3\frac{\theta}{2} d\theta$$

Put $\frac{\theta}{2} = t$ then $\frac{1}{2} d\theta = dt$

When $\theta = 0, t = 0$ and when $\theta = \pi, t = \frac{\pi}{2}$

$\therefore$ Surface $= 16\pi a^2 \int_0^{\frac{\pi}{2}} 2 \cos^3 \theta \, d\theta$

$= 32\pi a^2 \cdot \frac{2}{3} = \frac{64\pi a^2}{3}$

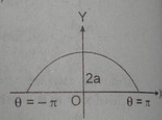
10. Find the volume of the solid formed by revolving the cycloid $x = a(\theta + \sin\theta), y = a(1 + \cos\theta)$ about its base.

Sol^n: Here, the cycloid;

$$x = a(\theta + \sin\theta) \; y = a(1 + \cos\theta),$$

Differentiating yields,

$$dx = a(1 + \cos\theta)d\theta,$$
$$dy = -a \sin d\theta$$



The required volume obtained by revolving the cycloid about its base is defined as;

$$v = \pi \int y^2 \, dx$$

$$= \pi \int_{-\pi}^{\pi} a^2(1 + \cos\theta)^2 a(1 + \cos\theta) d\theta$$

$$= \pi a^3 \cdot 2 \int_0^{\pi} (1 + \cos\theta)^3 \, d\theta$$

$$= 2\pi a^3 \int_0^{\pi} \left\{2 \cos^2\left(\frac{\theta}{2}\right)\right\}^3 d\theta$$

$$= \pi a^3 \cdot 2 \int_0^{\pi} 8 \cos^6 \frac{\theta}{2} \, d\theta$$

Put $\frac{\theta}{2} = t, d\theta = 2dt$

When $t = 0, \theta = \pi, t = \frac{\pi}{2}$

$$= 32\pi a^3 \int_0^{\frac{\pi}{2}} \cos^6 t \, dt$$

$$= 32\pi a^3 \sqrt{\pi} \frac{\Gamma\frac{(6+1)}{2}}{2\Gamma\frac{(6+2)}{2}}$$

$$= \frac{16\pi a^3 \sqrt{\pi} \cdot \frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2} \sqrt{\pi}}{3 \cdot 2 \cdot 1} = 5\pi^2 a^2 \text{ Cubic unit}$$

The surface of the given equation is defined as;

$$s = \int_{-a\pi}^{a\pi} 2\pi y \frac{ds}{dx} dx$$

$$= \int_{-a\pi}^{a\pi} 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx$$

$$= \int_{-\pi}^{\pi} 2\pi a (1 + \cos\theta) \sqrt{a^2(1 + \cos\theta)^2 + a^2 \sin^2\theta} \, d\theta$$

$$= 2\pi a^2 \cdot 2 \int_0^{\pi} (1 + \cos\theta) \sqrt{1 + 2\cos\theta + \cos^2\theta + \sin^2\theta} \, d\theta$$

$$= 4\pi a^2 \int_0^{\pi} 2 \cos^2\left(\frac{\theta}{2}\right) \sqrt{2(1 + \cos\theta) d\theta}$$

$$= 8\pi a^2 \int_0^{\pi} \cos^2\left(\frac{\theta}{2}\right) \sqrt{4 \cos^2 \frac{\theta}{2} d\theta}$$

$$= 16\pi a^2 \int_0^{\pi} \cos^3 \frac{\theta}{2} d\theta$$

Put $\frac{\theta}{2} = t, d\theta = 2dt$

When $\theta = 0, t = 0$

When $\theta = \pi, t = \frac{\pi}{2}$

$$= 32\pi a^2 \int_0^{\frac{\pi}{2}} \cos^3 t \, dt$$

$$= 32\pi a^2 \frac{\sqrt{\pi} \, \Gamma\frac{(3+1)}{2}}{2 \, \Gamma\frac{(3+2)}{2}}$$

$$= \frac{16\pi a^2 \sqrt{\pi} \cdot 1}{\frac{3}{2} \cdot \frac{1}{2} \cdot \sqrt{\pi}}$$

$$= \frac{64}{3} \pi d^2 \text{ Cubic unit}$$

Exercise 18

1. Determine the order and degree of each of the following differential equations.

i) $(x + 3y - 2)dx + (2x - 3y + 5) = 0$

Solⁿ: Here,

$$(x + 3y - 2)dx + (2x - 3y + 5) = 0$$

or, $\frac{dy}{dx} + \frac{x+3y-2}{2x-3y+5} = 0$ which is first order and first degree equation.

ii) $y = x\frac{d^2y}{dx^2} + \frac{k}{d^2y}$

Solⁿ: Here,

$$y = x\frac{d^2y}{dx^2} + \frac{k}{d^2y}$$

or, $y\frac{d^2y}{dx^2} = x\left(\frac{d^2y}{dx^2}\right)^2 + k$

or, $x\left(\frac{d^2y}{dx^2}\right)^2 - y\frac{d^2y}{dx^2} + k = 0$ which is second degree and second order equation.

iii) $\left[1 + \left(\frac{dy}{dx}\right)\right]^{\frac{3}{2}} = k\frac{d^2y}{dx^2}$

Solⁿ: Here,

$$\left[1 + \left(\frac{dy}{dx}\right)\right]^{\frac{3}{2}} = k\frac{d^2y}{dx^2}$$

or, $\left[1 + \left(\frac{dy}{dx}\right)^2\right]^3 = k^2\left(\frac{d^2y}{dx^2}\right)^2$ which is second order and second degree equation.

iv) $x^2\frac{d^2y}{dx^2} + 2xy\frac{dy}{dx} + \left(\frac{dy}{dx}\right)^3 = 0$

Solⁿ: Here,

$x^2\frac{d^2y}{dx^2} + 2xy\frac{dy}{dx} + \left(\frac{dy}{dx}\right)^3 = 0$ which is second order first degree equation.

2. Form the differential equations from the following equations.

i) $y = a\log x + b$

Solⁿ: Here,

$$y = a\log x + b$$

or, $\frac{dy}{dx} = \frac{a}{x}$

or, $\frac{d^2y}{dx^2} = \frac{-a}{x}$

or, $x\frac{d^2y}{dx^2} = \frac{-a}{x} = -\frac{dy}{dx}$

or, $x\frac{d^2y}{dx^2} + \frac{dy}{dx} = 0$ is the required equation.

ii) $xy = a + bx$

Solⁿ: Here,

$$y = \frac{a}{x} + b$$

or, $\frac{dy}{dx} = -\frac{a}{x^2}$

or, $\frac{d^2y}{dx^2} = \frac{2a}{x^3}$

or, $x\frac{d^2y}{dx^2} = \frac{2a}{x^2} = -2\frac{dy}{dx}$

or, $x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = 0$ is the required equation.

iii) $xy = Ae^x + Be^{-x}$

Solⁿ: Here,

$$x\frac{dy}{dx} + y\frac{dy}{dx} = Ae^x + Be^{-x}$$

or, $x\frac{d^2y}{dx^2} + \frac{dy}{dx} + \frac{dy}{dx} = Ae^x + Be^{-x}$

or, $x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = xy$

or, $\frac{d^2y}{dx^2} + \frac{2}{x}\frac{dy}{dx} - y = 0$ is the required solution.

iv) $\mathbf{y = ax^3 + bx^2}$

Sol^n: Here,

$$\frac{dy}{dx} = 3ax^2 + 2bx$$

or, $\frac{d^2y}{dx^2} = 6ax + 2b$

or, $x^2\frac{d^2y}{dx^2} = 6ax^3 + 2bx^2$

or, $4x\frac{dy}{dx} = 12ax^3 + 8bx^2$

or, $6y = 6ax^3 + 6bx^2$

so, $x^2\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + 6y$

$= 6ax^3 + 2bx^2 - 12ax^3 - 8bx^2 + 6ax^3 + 6bx^2 = 0$

Thus,

$x^2\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + 6y = 0$ is the required solution.

v) $\mathbf{y = a\cos(\log x) + b\sin(\log x)}$

Sol^n: Here,

$$\frac{dy}{dx} = -a\sin(\log x)\frac{1}{x} + b\cos(\log x)\frac{1}{x}$$

or, $x\frac{dy}{dx} = -a\sin(\log x) + b\cos(\log x)$

Again, differentiating both sides, we have,

$$x\frac{d^2y}{dx^2} + \frac{dy}{dx} = -\frac{a\cos(\log x)}{x} - \frac{b\sin(\log x)}{x}$$

$$= -\frac{1}{x}[\cos(\log x) + \sin(\log x)] = -\frac{1}{x}y$$

or, $x\frac{d^2y}{dx^2} + \frac{dy}{dx} + \frac{1}{x}y = 0$

or, $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$ is the required solution.

3. **Obtain the differential equation of all circles of radius a and centre (h, k).**

Sol^n: The equation of circle is;

$$(x - h)^2 + (y - k)^2 = a^2$$

or, $x^2 - 2xh + h^2 + y^2 - 2yk + k^2 = a^2$

or, $2x - 2h + 2y\frac{dy}{dx} - 2k\frac{dy}{dx} = 0$

or, $\frac{dy}{dx}(2y - 2k) = (2h - 2x)$

or, $\frac{dy}{dx} = \frac{h-x}{y-k}$ ... (i)

$$\frac{d^2y}{dx^2} = \frac{(y-k)\times -1-(h-x)\frac{dy}{dx}}{(y-k)^2}$$

or, $\frac{d^2y}{dx^2} = \frac{(k-y)^2+(h-x)^2}{(y-k)^3}$

or, $\left[1 + \left(\frac{dy}{dx}\right)^2\right] = \left[1 + \left(\frac{h-x}{y-k}\right)^2\right]$

$$= \left[\frac{y^2-2yk+k^2+h^2-2hx+x^2}{(y-k)^2}\right]$$

or, $\left(\frac{d^2y}{dx^2}\right)^2 = \left[\frac{(h-x)^2+(y-k)^2}{(y-k)^3}\right]^3$

Thus,

$$\left[1 + \left(\frac{dy}{dx}\right)^2\right]^3 = \left[\frac{(h-x)^2+(y-k)^2}{(y-k)^2}\right]^3$$

$$= [(x-h)^2 + (y-k)^2]\left[\frac{(h-x)^2+(y-k)^2}{(y-k)^2}\right]^2$$

$$= a^2\left(\frac{d^2y}{dx^2}\right)^2$$

Thus,

$$\left[1 + \left(\frac{dy}{dx}\right)^2\right]^3 = a^2\left(\frac{d^2y}{dx^2}\right)^2$$

4. **From a differential equation of simple harmonic motion given by $x = A\cos(nt + \alpha)$**

Sol^n: Here,

$$x = A\cos(nt + \alpha)$$

Differentiating both sides, we have,

$$\frac{dx}{dt} = -A\sin(nt + \alpha) . n$$

or, $\frac{d^2x}{dt^2} = -A\cos(nt + \alpha) . n^2 = -n^2x$

or, $\frac{d^2x}{dt^2} + n^2x = 0$ is the required solution

Exercise 19

Solve the following differential equations:

1. $x\sqrt{1 + y^2}dx + y\sqrt{1 + x^2}dy = 0$

Sol^n: Here,

$$x\sqrt{1 + y^2}dx + y\sqrt{1 + x^2}dy = 0$$

Separating variables, we have,

$$\frac{x\,dx}{\sqrt{1+x^2}} + \frac{y\,dy}{\sqrt{1+y^2}} = 0$$

or, $\frac{2x\,dx}{\sqrt{1+x^2}} + \frac{2y\,dy}{\sqrt{1+y^2}} = 0$

or, $2\sqrt{1 + x^2} + 2\sqrt{1 + y^2} = 2c$

Thus,

$$\sqrt{1 + x^2} + \sqrt{1 + y^2} = c$$

2. $(x^2 + 1)\frac{dy}{dx} = 1$

Sol^n: Here,

$$(x^2 + 1)\frac{dy}{dx} = 1$$

or, $dy = \frac{dx}{x^2+1}$

or, $y = \tan^{-1}x + c$ is the required solution

3. $y\,dx = (e^x + 1)dy$

Sol^n: Here,

$$y\,dx = (e^x + 1)dy$$

or, $\frac{dx}{e^x+1} = \frac{1}{y}\,dy$

or, $-\frac{e^{-x}\,dx}{1+e^{-x}} = \frac{1}{y}\,dy$

or, $-\log(e^{-x} + 1) = \log y + \log c$

or, $\log y + \log(e^{-x} + 1) + \log c = 0$

or, $y(e^{-x} + 1)c = 0$

4. $\mathbf{(xy^2 + x)dx + (yx^2 + y)dy = 0}$

Sol^n: Here,

$(xy^2 + x)dx + (yx^2 + y)dy = 0$

or, $x(1 + y^2)dx + y(1 + x^2)dy = 0$

or, $\frac{x\,dx}{1+x^2} + \frac{y\,dy}{1+y^2} = 0$

or, $\frac{2x\,dx}{1+x^2} + \frac{2y\,dy}{1+y^2} = 0$

or, $\log(1 + x^2) + \log(1 + y^2) = \log C$

or, $(1 + x^2)(y^2 + 1) = C$

5. $\mathbf{\tan y\ dx + \tan x\ dy = 0}$

Sol^n: Here,

$\cot x\,dx + \cot y\,dy = 0$

or, $\frac{\cos x\,dx}{\sin x} + \frac{\cos y\,dy}{\sin y} = 0$

or, $\log \sin x + \log \sin y = \log C$

or, $\log(\sin x\,.\,\sin y) = \log C$

or, $C = \sin x \sin y$

6. $\mathbf{y - x\frac{dy}{dx} = a\left(y^2 + \frac{dy}{dx}\right)}$

Sol^n: Here,

$y - x\frac{dy}{dx} = ay^2 + a\frac{dy}{dx}$

or, $y - ay^2 = (a + x)\frac{dy}{dx}$

or, $\frac{dx}{a+x} = \frac{dy}{y-ay^2}$

or, $\frac{dx}{a+x} = \frac{dy}{y(1-ay)} = \left(\frac{1}{y} + \frac{a}{1-ay}\right) dy$

or, $\frac{dx}{a+x} = \frac{dy}{y} - \frac{-a\,dy}{1-ay}$

or, $\log(a + x) = \log y - \log(1 - ay) + \log C$

or, $\log(a + x) + \log(1 - ay) = \log yC$

or, $(a + x)(1 - ay) = yC$ is the required solution

7. $\mathbf{(1 + x)(1 + y^2)dx + (1 + y)(1 + x^2)dy = 0}$

Sol^n: Here,

$\frac{(1+x)dx}{1+x^2} + \frac{(1+y)dy}{y^2+1} = 0$

or, $\frac{dx}{1+x^2} + \frac{x\,dx}{1+x^2} + \frac{dy}{y^2+1} + \frac{y\,dy}{y^2+1} = 0$

or, $\tan^{-1} x + \frac{1}{2}\log(1 + x^2) + \tan^{-1} y + \frac{1}{2}\log(1 + y^2) = \log C$

or, $\tan^{-1} x + \tan^{-1} y + \log\sqrt{1 + x^2}\sqrt{1 + y^2} = \log C$

which is the required solution

8. $\mathbf{(e^x + 1)y\,dy = (y + 1)e^x\,dx}$

Sol^n: Here,

$\frac{y}{1+y}dy = \frac{e^x}{e^x+1}dx$

or, $\left(1 - \frac{y}{1+y}\right) dy = \frac{e^x}{e^x+1}dx$

or, $y - \log(1 + y) = \log(e^x + 1) + \log C$

or, $y = \log(1 + y) + \log(e^x + 1) + \log C$

or, $y = \log(1 + y)\,(e^x + 1)C$

9. $\mathbf{3e^x \tan y\ dx + (1 - e^x)\sec^2 y\,dy = 0}$

Sol^n: Here,

$\frac{3e^x}{1-e^x}dx + \frac{\sec^2 y}{\tan y}dy = 0$

or, $-3\left[\frac{-e^x\,dx}{1-e^x}\right] + \frac{\sec^2 y\,dy}{\tan y} = 0$

or, $-3\log(1 - e^x) + \log \tan y = \log C$

or, $\log \tan y = \log C + \log(1 - e^x)^3$

or, $\tan y = C(1 - e^x)^3$

10. $\mathbf{e^{x-y}\,dx + e^{y-x}\,dy = 0}$

Sol^n: Here,

$\frac{e^x}{e^y}dx + \frac{e^y}{e^x}dy = 0$

or, $e^{2x}\,dx + e^{2y}\,dy = 0$

or, $\frac{e^{2x}}{2} + \frac{e^{2y}}{2} = \frac{C}{2}$

or, $e^{2x} + e^{2y} = C$

11. $(a^2 + y^2)x\,dx + y(x^2 - a^2)dy = 0$

Sol^n: Here,

$$\frac{x\,dx}{x^2 - a^2} + \frac{y\,dy}{y^2 + a^2} = 0$$

or, $\frac{2x\,dx}{x^2 - a^2} + \frac{2y\,dy}{y^2 + a^2} = 0$

or, $\log(x^2 - a^2) + \log(y^2 - a^2) = \log C$

or, $\log(x^2 - a^2)(y^2 - a^2) = \log C$

or, $(x^2 - a^2)(y^2 - a^2) = C$ is the required solution.

12. $(e^y + 1)\cos x\,dx + e^y \sin x\,dy = 0$

Sol^n: Here,

$$\frac{\cos x\,dx}{\sin x} + \frac{e^y}{e^y + 1} = 0$$

or, $\log(\sin x) + \log(e^y + 1) = \log C$

or, $\log \sin(e^y + 1) = \log C$

or, $\sin x\,(e^y + 1) = C$ is the required solution.

13. Find the particular solution of $2xy' = 3y$ given that $y = 4$ when $x = 1$.

Sol^n: Here,

$$2x\frac{dy}{dx} = 3y$$

or, $2x\frac{dy}{y} = 3\frac{dx}{x}$

or, $\frac{2\,dy}{y} = 3\frac{dx}{x}$

Integrating both sides, we have,

$$2\log y = 3\log x + \log C$$

or, $\log y^2 = \log x^3 + \log C$

or, $\frac{y^2}{x^3} = C$

When $y = 4, x = 1$

Then,

$$16.1 = C$$

or, $\frac{y^2}{x^3} = 16$

or, $y^2 = 16x^3$

or, $y = 4(x)^{\frac{3}{2}}$

14. Find the particular solution $y^1 = \sec y$, given that $y = 4$ when $x = 1$.

Sol^n: Here,

$$y^1 = \sec y$$

or, $\frac{dy}{dx} = \frac{1}{\cos y}$

or, $\cos y\;dy = dx$

or, $\sin y = x + C$

When $y = 0, x = 0$

Then,

$$\sin 0 = 0 + C$$

or, $C = 0$

$\therefore$ $\sin y = x$ is the required solution.

15. Find the particular solution of $\frac{dy}{dx} = e^{x+y}$ and it is given that for $x = 1, y = 1$ find y as $x = -1$.

Sol^n: Here,

$$\frac{dy}{dx} = e^{x+y}$$

or, $\frac{dy}{dx} = e^x.e^y$

or, $e^{-y}dy = e^x dx$

or, $-e^{-y} = e^x + C$

When $x = 1, y = 1$

Then,

$$-e^{-1} = e^1 + C$$

or, $C = -e^{-1} - e$

Thus,

$-e^{-y} = e^x - e - e^{-1}$ is the required solution.

Also,

When $x = -1$

Then,

$$-e^{-y} = e^{-1} - e - e^{-1} = -e$$

or, $e^{-y} = e^{1}$

or, $y = -1$

16. Find the equation of the curve which passes through the point $(1, 2)$ and has at every point, $\frac{dy}{dx} = \frac{-2xy}{x^2+1}$.

Sol^n: Here,

$$\frac{dy}{dx} = \frac{-2xy}{x^2+1}$$

or, $\frac{dy}{y} + \frac{2x\,dx}{x^2+1} = 0$

or, $\log y + \log(x^2 + 1) = \log C$

or, $y(x^2 + 1) = C$

When $x = 1, y = 2$

Then,

$$2(2) = C$$

or, $C = 4$

Thus,

$y(x^2 + 1) = 4$ is the required solution.

17. Find the particular solution of $y(1-x^2)\frac{dy}{dx} + x(1-y^2) = 0$ given that $y = 1$ when $x = 0$.

Sol^n: Here,

$$y(1-x^2)\frac{dy}{dx} + x(1-y^2) = 0$$

or, $\frac{y\,dy}{1-y^2} + \frac{x\,dx}{1-x^2} = 0$

or, $\frac{-2y\,dy}{1-y^2} - \frac{2x\,dx}{1-x^2} = 0$

or, $\log(1-y^2) + \log(1-x^2) = \log C$

or, $(1-y^2)(1-x^2) = C$

When $x = 0, y = 1$

Then,

$$C = 0$$

Thus,

$(1-y^2)(1-x^2) = 0$ is the required solution.

18. Find the equation of the curve represented by $y(1-x^2)\frac{dy}{dx} + x(1-y^2) = 0$ and passes the point $(1, 1)$.

Sol^n: Here,

$$y(1-x^2)\frac{dy}{dx} + x(1-y^2) = 0$$

or, $\frac{y\,dy}{1-y^2} + \frac{x\,dx}{1-x^2} = 0$

or, $y(1-x)dx + x(1+y)dy = 0$

or, $\left(\frac{1-x}{x}\right)dx + \left(\frac{1+y}{y}\right)dy = 0$

or, $\left(\frac{1}{x} - 1\right)dx + \left(\frac{1}{y} - 1\right)dy = 0$

or, $\frac{dx}{x} - dx + \frac{dy}{y} + dy = 0$

or, $\log x - x + \log y + y = C$

or, $\log xy = x - y + C$

When $x = 1, y = 1$

Then,

$$\log 1 = 1 - 1 + C$$

or, $0 = C$

Thus,

$$C = 0$$

so, $\log xy = x - y$ is the required solution.

Exercise 20

Solve the following differential equations:

1. $(x+y)^2 \frac{dy}{dx} = a^2$

Solⁿ: The given equation will be reduced to;

$$\frac{dy}{dx} + \frac{a^2}{(x+y)^2} = 0 \ldots \text{(i)}$$

Put $x + y = v$

or, $1 + \frac{dy}{dx} = \frac{dv}{dx}$

$\therefore \frac{dy}{dx} = \frac{dv}{dx} - 1$

Then, equation (i) will be;

$$\frac{dv}{dx} - 1 = \frac{a^2}{v^2}$$

or, $\frac{dv}{dx} = \frac{a^2}{v^2} + 1$

or, $\frac{dv}{dx} = \frac{a^2+v^2}{v^2}$

or, $\frac{v^2\,dv}{a^2+v^2} = dx$

or, $\left(\frac{a^2+v^2-a^2}{a^2+v^2}\right) dv = dx$

or, $1\,dv - \frac{a^2}{a^2+v^2} dv = dx$

or, $v - a^2 \frac{1}{a} \tan^{-1}\frac{v}{a} = x + C$

Putting the value in equation (ii); we get,

$$x + y - a\tan^{-1}\frac{x+y}{a} = x + C$$

or, $y = a\tan^{-1}\frac{x+y}{a} + C$

2. $\cos(x+y)\,dy = dx$

Solⁿ: The given equation will be reduced to;

$$\frac{dy}{dx} = \sec(x+y) \ldots \text{(i)}$$

Put $v = x + y$

or, $\frac{dy}{dx} = 1 + \frac{dy}{dx}$

Then equation (i) will be;

$$\frac{dv}{dx} - 1 = \sec v$$

or, $dv = (1 + \sec v)dx$

or, $\frac{dv}{\sec v + 1} = dx$

or, $\frac{(\sec v - 1)dv}{\tan^2 v} = dx$

or, $\left(\frac{\sec v}{\tan^2 v} - \cot^2 v\right) dv = x$

or, $\left(\frac{1}{\cos v} \times \cot^2 v - \cot^2 v\right) dv = x$

or, $\frac{\cos^2 v\,dv}{\sin^2 v \cos v} - (\operatorname{cosec}^2 v - 1) = x$

or, $\frac{\cos v\,dv}{\sin^2 v} - \operatorname{cosec}^2 v + 1dv = x$

Put $t = \sin v, dt = \cos v\,dv$

$$\frac{dt}{t^2} + \cot v + v = x$$

or, $-\frac{1}{t} + \cot v + v = x$

or, $-\frac{1}{\sin v} + \cot v + v = x + C$

or, $-\frac{1}{\sin(x+y)} + \cot(x+y) + x + y = x + C$

or, $y - C = \cot x + y - \frac{1}{\sin(x+y)}$

or, $y - C = \frac{\cos(x+y)}{\sin(x+y)} - \frac{1}{\sin(x+y)}$

or, $y - C = -\left[\frac{1-\cos(x+y)}{\sin(x+y)}\right] = -\frac{2\sin^2\left(\frac{x+y}{2}\right)}{2\sin\left(\frac{x+y}{2}\right).\cos\left(\frac{x+y}{2}\right)}$

or, $y - C = -\tan\left(\frac{x+y}{2}\right)$

3. $\sin^{-1}\left(\frac{dy}{dx}\right) = x + y$

Sol^n: Here,

$\frac{dy}{dx} = \sin(x+y)$

Put $x + y = v$

$\frac{dv}{dx} = 1 + \frac{dy}{dx}$

or, $\frac{dv}{dx} - 1 = \sin v$

or, $\frac{dv}{dx} = (1 + \sin v)$

or, $\frac{dv}{1+\sin v} = dx$

or, $\frac{(1-\sin v)dv}{\cos^2 v} = dx$

or, $\sec^2 v - \tan v \sec v = dx$

or, $\tan v - \sec v = x + C$

or, $\tan(x+y) - \sec(x+y) = x + C$

4. $\frac{dy}{dx} + 1 = e^{x+y}$

Sol^n: Here,

$\frac{dy}{dx} + 1 = e^{x+y}$

Put $x + y = v$

$\frac{dv}{dx} = 1 + \frac{dy}{dx}$

or, $\frac{dv}{dx} = e^v$

or, $e^{-v}\,dv = dx$

or, $-e^{-v} = x + C$

or, $-e^{-(x+y)} = x + C$

or, $-1 = e^{x+y}(x + C)$

or, $-1 = e^x(x + C)e^y$

or, $-e^{-y} = e^x(x + C)$

or, $(x + C)e^x + e^{-y} = 0$

5. $\frac{dy}{dx} + 1 = e^{x-y}$

Sol^n: Here,

$\frac{dy}{dx} + 1 = e^{x-y}$

Put $x - y = v$

$\frac{dv}{dx} = 1 - \frac{dy}{dx}$

or, $1 - \frac{dv}{dx} + 1 = e^v$

or, $2 - \frac{dv}{dx} = e^v$

or, $2 - e^v = \frac{dv}{dx}$

or, $(2 - e^v)dx = dv$

or, $\frac{dv}{2-e^v} = dx$

or, $\frac{e^{-v}\,dv}{2e^{-v}-1} = dx$

or, $-\frac{1}{2}\log(e^{-v} - 1) = x + C$

or, $-\frac{1}{2}\log(e^{y-x} - 1) = x + C$

or, $\log(2e^{y-x} - 1) = -2(x + k)$

or, $(2e^{y-x} - 1) = e^{-2x-2k}$

or, $2e^{-x}e^y - 1 = e^{-2x}e^{-2k}$

or, $2e^{-x}e^y = e^{-2x}e^{-2k} + 1$

or, $e^y = \frac{1}{2}e^{-x}e^{-2k} + \frac{1}{2}e^x$

or, $e^y = \left(\frac{e^{-2k}}{2}\right)e^{-x} + \frac{1}{2}e^x$

or, $e^y = Ce^{-x} + \frac{1}{2}e^x$

6. $\frac{dy}{dx} - x\tan(y-x) = 1$

Sol^n: This equation is;

$$\frac{dy}{dx} - 1 = x\tan(y-x) \ldots \text{(i)}$$

Put $y - x = v$

so, $\frac{dy}{dx} - 1 = \frac{dv}{dx}$

or, $\frac{dv}{dx} = x\tan v$

or, $\frac{1}{\tan v}dv = x\,dx$

or, $\cot v\ dv = x\,dx$

Integrating both sides, we have,

$$\log\sin v = \frac{x^2}{2} + \log C$$

or, $\log\frac{\sin v}{2} = \frac{x^2}{2}$

or, $\frac{\sin v}{2} = e^{\frac{x^2}{2}}$

or, $\sin v = Ce^{\frac{x^2}{2}}$

Putting the value of v, we get,

$\sin(y-x) = Ce^{\frac{x^2}{2}}$ is the required solution.

7. $\frac{dy}{dx} = (4x + y + 1)^2$

Sol^n: Here,

$$\frac{dy}{dx} = (4x + y + 1)^2$$

Put $V = y + 1$

or, $\frac{dV}{dx} = (4x + V)^2$

Put $4x + V = Z$

or, $4 + \frac{dV}{dx} = \frac{dZ}{dx}$

or, $\frac{dZ}{dx} - 4 = Z^2$

or, $\frac{dZ}{dx} = Z^2 + 4$

or, $\frac{dZ}{Z^2+4} = dx$

Integrating both sides, we have,

$$\frac{1}{2}\tan^{-1}\frac{Z}{2} = x + C$$

or, $\tan^{-1}\frac{Z}{2} = (2x + C)$

or, $\frac{Z}{2} = \tan(2x + C)$

or, $4x + V = 2\tan(2x + C)$

or, $4x + y + 1 = 2\tan(2x + C)$ is the required solution

8. $(x^2 + y^2 + 2xy + 1)\frac{dy}{dx} = (x + y)dx$

Sol^n: The given equation will be;

$$[(x+y)^2 + 1]\frac{dy}{dx} = (x+y)dx$$

Put $x + y = v$

$$1 + \frac{dy}{dx} = \frac{dv}{dx}$$

so, $\frac{dy}{dx} = \frac{x+y}{(x+y)^2+1}$

or, $\frac{dv}{dx} - 1 = \frac{v}{v^2+1}$

or, $\frac{dv}{dx} = 1 + \frac{v}{v^2+1} = \frac{v^2+1+v}{v^2+1}$

or, $\frac{v^2+1}{v^2+1+v}dv = dx$

or, $\frac{v^2+1+v-v}{v^2+1+v}dv = dx$

or, $\left(1 - \frac{v}{v^2+1+v}\right)dv = dx$

or, $dv - \frac{\frac{1}{2}(2v+1)dv - \frac{1}{2}dv}{v^2+1+v} = dx$

or, $dv - \frac{1}{2}\frac{d(v^2+v+1)}{(v^2+v+1)} + \frac{1}{2}\frac{dv}{\left(v+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = dx$

Integrating both sides, we have,

$$v - \frac{1}{2}\log(v^2 + v + 1) + \frac{1}{2}\frac{2}{\sqrt{3}}\tan^{-1}\frac{v + \frac{1}{2}}{\frac{\sqrt{3}}{2}} = x + C$$

or, $y + x - \frac{1}{2}\log[(x + y)^2 + (x + y) + 1] + \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x+2y+1}{\sqrt{3}}\right) = x + C$

or, $y = -\frac{1}{2}\log[(x + y)^2 + (x + y) + 1] + \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x+2y+1}{\sqrt{3}}\right) = C$

which is the required solution

9. $\mathbf{(x + y + 1)\frac{dy}{dx} = 1}$

Sol^n: Here,

$$(x + y + 1)\frac{dy}{dx} = 1$$

Put $x + y = v$

$$\frac{dv}{dx} = 1 + \frac{dy}{dx}$$

or, $(v + 1)\left(\frac{dv}{dx} - 1\right) = 1$

or, $\frac{dv}{dx} - 1 = \frac{1}{v+1}$

or, $\frac{dv}{dx} = 1 + \frac{1}{v+1} = \frac{v+2}{v+1}$

or, $\left(\frac{v+1}{v+2}\right) dv = dx$

or, $\left(\frac{v+2}{v+2} - \frac{1}{v+2}\right) dv = dx$

or, $\left(1 - \frac{1}{v+2}\right) dv = dx$

or, $v - \log(v + 2) = x + C$

Thus,

$$x + y - \log(x + y + 2) = x + C$$

or, $y - \log(x + y + 2) = C$ is the required solution

10. $\mathbf{\frac{dv}{dx} = \sqrt{y - x}}$

Sol^n: Here,

$$\frac{dv}{dx} = \sqrt{y - x}$$

Put $\sqrt{y - x} = v$

$$v^2 = y - x$$

or, $\frac{dy}{dx} - 1 = 2v\frac{dv}{dx}$

or, $\frac{dy}{dx} = 1 + 2v\frac{dv}{dx}$

Thus,

$$1 + 2v\frac{dv}{dx} = v$$

or, $2v\frac{dv}{dx} = v - 1$

or, $\frac{dv}{dx} = \frac{v-1}{2v}$

or, $\frac{2v}{v-1} dv = dx$

or, $\left[\frac{2(v-1)+2}{v-1}\right] dv = dx$

or, $\left(2 + \frac{2}{v-1}\right) dv = dx$

or, $2v + 2\log(v - 1) = x + C$

or, $v + \log(v - 1) = \frac{x}{2} + d$

Thus,

$\sqrt{y - x} + \log(\sqrt{y - x} - 1) = \frac{x}{2} + d$ is the required solution.

11. $\mathbf{x^2(x\,dx + y\,dy) + 2y(x\,dy - y\,dx) = 0}$

Sol^n: Here,

$$\frac{x\,dx + y\,dy}{x\,dy - y\,dx} = \frac{-2y}{x^2} \quad \ldots \text{(i)}$$

Changing into polar form, we have,

$$x = r\cos\theta$$
$$y = r\sin\theta$$

so,

$$x^2 + y^2 = r^2 \quad \ldots \text{(ii)}$$

$$\frac{y}{x} = \tan\theta \quad \ldots \text{(iii)}$$

Thus,

$$2x\,dx + 2y\,dy = 2r\,dr$$

or, $x\,dx + y\,dy = r\,dr \ldots$ (iv)

$$x\,dx + y\,dy = x^2\sec^2\theta\,d\theta$$

or, $x\,dx + y\,dy = \sec^2\theta\,(r\cos\theta)^2 d\theta = r^2\,d\theta$

Thus,

$$\frac{r\,dr}{r^2\,d\theta} = \frac{-2\times r\sin\theta}{r^2\cos^2\theta}$$

or, $\frac{1}{r}dr = -2\frac{\sin\theta}{\cos^2\theta}d\theta$

Put $\cos\theta = Z$

$$\frac{1}{r}dr = -2\frac{dZ}{Z^2}$$

Exercise 21

Solve the following differential equations.

1. $\mathbf{x + y\frac{dy}{dx} = 2y}$

Sol[n]: Here,

$$x + y\frac{dy}{dx} = 2y$$

or, $y\frac{dy}{dx} = 2y - x$

or, $\frac{dy}{dx} = \frac{2y-x}{y}$

Let, $y = vx$, diff. both sides with respect to x,

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

or, $v + \frac{dV}{dx} = \frac{2Vx-x}{Vx}$

or, $v = x\frac{dV}{dx} = \frac{2V-1}{V}$

or, $x\frac{dV}{dx} = \frac{2V-1}{V} - V$

or, $x\frac{dv}{dx} = \frac{2v-v^2-1}{v} = -\frac{(v+1)^2}{v}$

or, $\frac{vdv}{(v+1)^2} + \frac{dx}{x} = 0$

or, $\frac{vdv}{v^2+2v+1} + \frac{dx}{x} = 0$

or, $\frac{(v+1-1)dv}{(v+1)^2} + \frac{dx}{x} = 0$

or, $\frac{(v+1)dv}{(v+1)^2} + \frac{dv}{(v+1)^2} + \frac{dx}{x} = 0$

$$\frac{1dv}{v+1} + \frac{dv}{(v+1)^2} + \frac{dx}{x} = 0$$

Integrating yields,

$$\log(v-1) + \frac{1}{v-1} + \log x = c$$

or, $\log(v-1)x = c + \frac{1}{v+1}$

or, $\log(y-x) = c + \left(\frac{x}{y-x}\right)$

2. $\mathbf{(x^2 - y^2)dx + 2xydy = 0}$

Solⁿ: Here,

$$(x^2 - y^2)dx + 2xydy = 0$$

or, $\frac{dy}{dx} = \frac{y^2-x^2}{2xy}$ which is homogeneous equation

Let $y = vx$, differentiating both sides with respect to x, we get,

$$\frac{dy}{dx} = v + \frac{dv}{dx}$$

so, $v + x\frac{dv}{dx} = \frac{v^2x^2-x^2}{2x\,vx} = \frac{(v^2-1)x^2}{2vx^2}$

or, $v + x\frac{dv}{dx} = \frac{v^2-1}{2v}$

or, $x\frac{dv}{dx} = \frac{v^2-1}{2v} - v = \frac{v^2-1-2v^2}{2v}$

or, $x\frac{dv}{dx} = -\frac{1-v^2}{2v}$

Separating variables yields,

$$\frac{2v\,dv}{1+v^2} + \frac{dx}{x} = 0$$

Integrating we get,

or, $\log(1+v^2) + \log x = \log c$

or, $\log(1+v^2)x = \log c$

or, $(1+v^2)x = c$

Thus,

$$\left(1 + \frac{y^2}{x^2}\right)x = c$$

or, $x^2 + y^2 = xc$ is the required solution.

3. $\frac{dy}{dx} + \frac{x}{y} = \frac{y^2}{x^2}$

Solⁿ: Here,

$$\frac{dy}{dx} + \frac{x}{y} = \frac{y^2}{x^2}$$

Putting $y = vx$

so, $\frac{dy}{dx} = v + x\frac{dv}{dx}$

or, $v + x\frac{dv}{dx} + \frac{vx}{x} \pm \frac{v^2x^2}{x^2}$

or, $v + x\frac{dv}{dx} + v = v^2$

or, $x\frac{dv}{dx} = v^2 - 2v$

or, $\frac{dv}{v(v-2)} = \frac{dx}{x}$

or, $-\frac{1}{2}\left[\frac{1}{v} - \frac{1}{v-2}\right]dv = \frac{dx}{x}$

or, $-\frac{1}{2v}dv + \frac{1}{2(v-2)}dv = \frac{dx}{x}$

Integrating both sides, we have,

$$-\frac{1}{2}\log v + \frac{1}{2}\log(v-2) = \log x + \log c$$

or, $\log\left(\frac{v-2}{v}\right)^2 = 2\log xc$

or, $\left(\frac{v-2}{v}\right) = (xc)^2$

or, $\left(\frac{\frac{y}{x}-2}{\frac{x}{y}}\right) = (xc)^2 = x^2c^2$

or, $\frac{y-2x}{x} \times \frac{x}{y} = x^2c^2$

or, $y - 2x = x^2c^2y$ is the required solution.

4. $\mathbf{(x^2 + y^2)dx = (xy + x^2)dy}$

Solⁿ: The equation given is;

$$\frac{dy}{dx} = \frac{x^2+y^2}{xy+x^2}$$

Which is homogeneous equation, so let,

$$y = vx$$

Then,

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

or, $v + x\frac{dv}{dx} = \frac{x^2+v^2x^2}{vx^2+x^2} = \frac{v^2+1}{v+1}$

or, $x\frac{dv}{dx} = \frac{v^2+1}{v+1} - v$

or, $x\frac{dv}{dx} = \frac{v^2+1-v^2-v}{v+1} = \frac{1-v}{v+1}$

or, $\left(\frac{1+v}{1-v}\right) dv = \frac{dx}{x}$

or, $\frac{1+v}{v-1} dv + \frac{dx}{x} = 0$

or, $\left(\frac{2}{v-1} + 1\right) dv + \frac{dx}{x} = 0$

Integrating yields,

$$2\log(v-1) + v + \log x = \log c$$

or, $\log(v-1)^2 x = \log c - v$

or, $\log\left(\frac{y-x}{x}\right)^2 x = \log c - \frac{y}{x}$

or, $\log\frac{(y-x)^2}{cx} = -\frac{y}{x}$

or, $\frac{(y-x)^2}{cx} = e^{-\frac{y}{x}}$

or, $(y-x)^2 = cx\, e^{-\frac{y}{x}}$

5. $\mathbf{x(x-y)dy = y(x+y)dx}$

Sol^n: The given equation;

$$x(x-y)dy = y(x+y)dx$$

or, $\frac{dy}{dx} = \frac{xy+y^2}{x^2-xy}$

Put $y = vx$,

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

or, $v + x\frac{dv}{dx} = \frac{vx^2+v^2x^2}{x^2-vx^2} = \frac{v+v^2}{1-v}$

or, $x\frac{dv}{dx} = \frac{v+v^2}{1-v} - v$

or, $x\frac{dv}{dx} = \frac{v+v^2-v+v^2}{1-v}$

or, $x\frac{dv}{dx} = \frac{2v^2}{1-v}$

or, $\frac{1-v}{2v^2} dv = \frac{dx}{x}$

or, $\left(\frac{1}{v^2} - \frac{1}{v}\right) dv = 2\frac{dx}{x}$

Integrating yields,

$$-\frac{1}{v} - \log v = 2\log x + \log c$$

or, $-\frac{x}{y} - \log\frac{y}{x} = \log x^2 + \log c$

or, $-\frac{x}{y} = \log\frac{y}{x} + \log x^2 + \log c$

or, $-\frac{x}{y} = \log\frac{y}{x}\, x^2\, c$

or, $-\frac{x}{y} = \log xcy$

$\therefore$ $cxy = e^{-\frac{x}{y}}$

6. $\mathbf{(x^2+y^2)dy = xydx}$

Sol^n: Here,

$$(x^2+y^2)dy = xydx$$

or, $\frac{dy}{dx} = \frac{xy}{x^2+y^2}$

Put $y = vx$

Then,

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

or, $v + x\frac{dv}{dx} = \frac{vx^2}{x^2+v^2x^2}$

or, $v + x\frac{dv}{dx} = \frac{v}{1+v^2}$

or, $x\frac{dv}{dx} = \frac{v}{1+v^2} - v$

or, $x\frac{dv}{dx} = \frac{v-v-v^3}{1+v^2} = -\frac{v^3}{1+v^2}$

Separating variables, we have,

$$\left(\frac{1+v^2}{v^3}\right) dv + \frac{dx}{x} = 0$$

or, $\frac{1}{v^3} dv + \frac{1}{v} dv + \frac{dx}{x} = 0$

Integrating yields,

$$-\frac{1}{2v^2} + \log v + \log x = \log c$$

or, $-\frac{1}{2v^2} + \log vx = \log c$

or, $-\frac{1}{2v^2} = \log\frac{c}{vx}$

or, $-\frac{x^2}{2y^2} = \log\frac{c}{y}$

or, $\log\frac{y}{c} = \frac{x^2}{2y^2}$

or, $\frac{y}{c} = e^{\frac{x^2}{2y^2}}$

or, $y = ce^{\frac{x^2}{2y^2}}$

7. $\mathbf{\frac{dy}{dx} = \frac{y}{x} + \tan\frac{y}{x}}$

Sol^n: Here,

$$\frac{dy}{dx} = \frac{y}{x} + \tan\frac{y}{x}$$

Let, $y = vx$

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

or, $v + x\frac{dv}{dx} = v + \tan v$

or, $x\frac{dv}{dx} = \tan v$

or, $\cot v\, dv = \frac{dx}{x}$

or, $\frac{\cos v}{\sin v}dv = \frac{dx}{x}$

or, $\log\sin v = \log x + \log c$

or, $\log\sin\frac{y}{x} - \log x = \log c$

or, $\log\sin\frac{y}{x} \times x = \log c$

or, $\sin\frac{y}{x} = \frac{c}{x}$ is required solution.

8. $\mathbf{x^2dy + y(x + y)dx = 0}$

Sol^n: Here,

$$x^2dy + y(x + y)dx = 0$$

or, $\frac{dy}{dx} = -\frac{y(x+y)}{x^2}$

Put $y = vx$

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

or, $v + x\frac{dv}{dx} = -\frac{vx(x+vx)}{x^2}$

or, $v + x\frac{dv}{dx} = -\frac{vx^2-v^2x^2}{x^2}$

or, $v + x\frac{dv}{dx} = -v - v^2$

or, $x\frac{dv}{dx} = -2v - v^2$

or, $\frac{dv}{2v+v^2} + \frac{dx}{x} = 0$

or, $\frac{dv}{2v} + \frac{dv}{v^2} + \frac{dx}{x} = 0$

Integrating yields,

$$\log v + 2\log x - \frac{2}{v} = 2\log c = \log c$$

or, $\log v + \log x^2 - \log c = \frac{2}{v}$

or, $\log\frac{vx^2}{c} = \frac{2}{v}$

or, $\log\frac{\frac{y}{x} \times x^2}{c} = \frac{2x}{y}$

or, $\log\frac{xy}{c} = \frac{2x}{y}$

or, $\log\frac{xy}{c^2} = \frac{x}{y}$

or, $\log\frac{xy}{c^2} = \frac{2x}{y}$

9. $\mathbf{\frac{dy}{dx} = \frac{y}{x} - \frac{\sqrt{x^2+y^2}}{x}}$

Sol^n: Here,

$$\frac{dy}{dx} = \frac{y}{x} - \frac{\sqrt{x^2+y^2}}{x}$$

Put $y = vx$

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

or, $v + x\frac{dv}{dx} = v - \sqrt{v^2+1}$

or, $x\frac{dv}{dx} + \sqrt{v^2+1} = 0$

Separating variables,

$$\frac{dv}{\sqrt{v^2+1}} + \frac{dx}{x} = 0$$

Integrating yields,

$$\log\left(v + \sqrt{v^2+1}\right) + \log x = \log c$$

or, $\log\left(\frac{x}{y} + \frac{\sqrt{v^2+1}}{x}\right) + \log x = \log c$

or, $\log\left(\frac{y+\sqrt{y^2+x^2}}{x}\right)x = \log c$

or, $y + \sqrt{y^2+x^2} = c$ is the required solution.

10. $\mathbf{x\sin\frac{y}{x}dy = \left(y\sin\frac{y}{x} - x\right)dx}$

Sol^n: Here,

$$x\sin\frac{y}{x}dy = \left(y\sin\frac{y}{x} - x\right)dx$$

or, $\frac{dy}{dx} = \frac{y\sin\frac{y}{x} - x}{x\sin\frac{y}{x}}$

Which is homogeneous equation so put $y = vx$

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

or, $v + x\frac{dv}{dx} = \frac{vx\sin v - x}{x\sin v}$

or, $v + x\frac{dv}{dx} = \frac{v\sin v - 1}{\sin v}$

Separating variables,

$$x\frac{dv}{dx} = \frac{v\sin v - 1}{\sin v} - v$$

or, $x\frac{dv}{dx} = \frac{v\sin v - 1 - v\sin v}{\sin v}$

or, $x\frac{dv}{dx} = -\frac{1}{\sin v}$

or, $\sin v\,dv + \frac{dx}{x} = 0$

Integrating yields;

$$-\log v + \log x = 0$$
$$\log x = c + \log v$$

$\log x = c + \log\frac{y}{x}$ is the required solution.

11. $\mathbf{xdy - ydx = \sqrt{x^2+y^2}dx}$

Sol^n: Here,

$$xdy - ydx = \sqrt{x^2+y^2}dx$$

or, $x\frac{dy}{dx} - y = \sqrt{x^2+y^2}$

or, $x\frac{dy}{dx} = y + \sqrt{x^2+y^2}$

or, $\frac{dy}{dx} = \frac{y+\sqrt{x^2+y^2}}{x}$

Put $y = vx$

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

or, $v + x\frac{dv}{dx} = \frac{vx + x\sqrt{v^2+1}}{x}$

or, $v + x\frac{dv}{dx} = v + \sqrt{v^2+1}$

or, $x\frac{dv}{dx} = v + \sqrt{v^2+1} - v$

or, $x\frac{dv}{dx} = \sqrt{v^2+1}$

Separating variables,

$$\frac{dv}{\sqrt{v^2+1}} = \frac{dx}{x}$$

Integrating yields,

$$\log\left(v + \sqrt{v^2+1}\right) = \log x + \log c$$

or, $\log\left(\frac{y}{x} + \frac{\sqrt{y^2+x^2}}{x}\right) = \log xc$

or, $\frac{y+\sqrt{y^2+x^2}}{x} = xc$

or, $y + \sqrt{y^2 + x^2} = x^2c$

or, $(y - x^2c) = -\sqrt{y^2 + x^2}$

Thus,

$$x^2c - y = \sqrt{y^2 + x^2}$$

Thus the required equation is $(x^2c - y)^2 = y^2 + x^2$.

12. $\left(1 + e^{\frac{x}{y}}\right) \mathbf{dx} + e^{\frac{x}{y}}\left(1 - \frac{x}{y}\right) \mathbf{dy} = 0$

Sol^n: Here,

$$\left(1 + e^{\frac{x}{y}}\right) dx + e^{\frac{x}{y}}\left(1 - \frac{x}{y}\right) dy = 0$$

or, $\frac{dy}{dx} + \frac{\left(1+e^{\frac{y}{x}}\right)}{e^{\frac{x}{y}}\left(1-\frac{x}{y}\right)} = 0$

Put $y = vx$

or, $\frac{dy}{dx} = v + x\frac{dv}{dx}$

or, $v + x\frac{dv}{dx} = \frac{1+e^v}{\frac{1}{e^{v\left(\frac{v-1}{v}\right)}}}$

13. $\frac{\mathbf{dy}}{\mathbf{dx}} = \frac{\mathbf{3xy+y^2}}{\mathbf{3x^2}}$

Sol^n: This is homogeneous equation,

Put $y = vx$

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

or, $v + x\frac{dv}{dx} = \frac{3x\,vx+v^2x^2}{3x^2}$

or, $v + x\frac{dv}{dx} = \frac{3v+v^2}{3}$

or, $x\frac{dv}{dx} = \frac{3v+v^2}{3} - v$

or, $x\frac{dv}{dx} = \frac{3v+v^2-3v}{3}$

or, $x\frac{dv}{dx} = \frac{v^2}{3}$

or, $\frac{3dv}{v} - \frac{dx}{x} = 0$

Integrating yields,

or, $-\frac{3}{v} - \log x = c$

or, $-\frac{3x}{y} = \log x + c$

or, $3x = y\log x + cy$

or, $3x + y\log x = -cy = cy$

or, $cy = 3x + y\log x$ is the required equation.

14. $(\mathbf{x^2 + 2xy^2})\mathbf{dx} + (\mathbf{2x^2y + y^2})\mathbf{dy} = \mathbf{0}$

Sol^n: The given equation is:

$$(x^2 + 2xy^2)dx + (2x^2y + y^2)dy = 0$$

or, $\frac{dy}{dx} = -\frac{(x^2+2xy^2)}{2x^2y+y^2}$

or, $x^2dx + y^2dy + 2xy^2dx + 2x^2ydy = 0$

or, $x^2dx + y^2dy + (2xy^2dx + 2x^2ydy) = 0$

or, $\left(\frac{x^3}{3}\right) + d\left(\frac{y^3}{3}\right) + d(x^2y^2) = d(c)$

Integrating yields,

$$\frac{x^3}{3} + \frac{y^3}{3} + x^2y^2 = c$$

or, $x^3 + y^3 + 3x^2y^2 = 3c$

Thus, required equation is $x^3 + y^3 + 3x^2y^2 = d$

Exercise 22

Solve the differential equations.

1. $\mathbf{\frac{dy}{dx} = \frac{x+y+1}{x+y-1}}$

Sol^n: Given differential equation is;

$$\frac{dy}{dx} = \frac{x+y+1}{x+y-1} \ldots \text{(i)}$$

Comparing equation (i) with $\frac{dy}{dx} = \frac{ax+by+c}{Ax+By+c}$

$\therefore \quad \frac{a}{A} = \frac{b}{B} = 1$

i.e. the case of $\frac{a}{A} = \frac{b}{B}$

so, put

$$x + y = v \ldots \text{(i)}$$

Differentiating equation (ii) with respect to x, we get,

$$1 + \frac{dy}{dx} = \frac{dv}{dx}$$

or, $\frac{dy}{dx} = \frac{dv}{dx} - 1$

Then equation (i) becomes;

$$\frac{dv}{dx} - 1 = \frac{v+1}{v-1}$$

or, $\frac{dv}{dx} = \frac{v+1}{v-1} + 1$

or, $\frac{dv}{dx} = \frac{v+1+v-1}{v-1}$

or, $\frac{dv}{dx} = \frac{2v}{v-1}$

or, $\frac{v-1}{2v} dv = dx$

or, $\left(\frac{v}{2v} - \frac{1}{2v}\right) dv = dx$

or, $\left(\frac{1}{2} - \frac{1}{2v}\right) dv = dx$

or, $\frac{1}{2} dv - \frac{1}{2}\frac{1}{v} dv = dx$

Integrating both sides, we get,

$$\frac{1}{2}\int 1\,dv - \frac{1}{2}\int \frac{1}{v} dv = \int 1\,dx$$

or, $\int 1\,dv - \int \frac{1}{v} dv = 2\int 1\,dx$

or, $v - \log v = 2x + c$

or, $x + y - \log(x + y) = 2x + c$ $\quad$ ($\because$ using ii $v = x + y$)

or, $y - x - c = \log(x + y)$

2. $\mathbf{(x + y + 1)dx - (2x + 2y + 1)dy = 0}$

Sol^n: Given differential equation is;

$$(x + y + 1)dx - (2x + 2y + 1)dy = 0$$

or, $(x + y + 1)dx = (2x + 2y + 1)dy$

or, $\frac{dy}{dx} = \frac{x+y+1}{2x+2y+1}$

or, $\frac{dy}{dx} = \frac{x+y+1}{2(x+y)+1} \ldots \text{(i)}$

Put,

$$V = x + y \ldots \text{(ii)}$$

Differentiating equation (ii) with respect to x, we have,

$$\frac{dV}{dx} = 1 + \frac{dy}{dx}$$

or, $\frac{dy}{dx} = \frac{dV}{dx} - 1$

Then equation (i) can be written as;

$$\frac{dV}{dx} - 1 = \frac{V+1}{2V+1}$$

or, $\frac{dV}{dx} = \frac{V+1}{2V+1} + 1$

or, $\frac{dV}{dx} = \frac{V+1+2V+1}{2V+1}$

or, $\frac{dV}{dx} = \frac{3V+2}{2V+1}$

or, $\left(\frac{2V+1}{3V+2}\right) dV = dx$

or, $\frac{2\left(V+\frac{1}{2}\right)}{3V+2} dV = dx$

or, $\frac{2}{3}\frac{\left(3V+\frac{3}{2}\right)}{3V+2} dV = dx$

or, $\frac{2}{3}\left[\frac{3V+2-2+\frac{3}{2}}{3V+2}\right] dV = dx$

or, $\frac{2}{3}\left[\frac{3V+2}{3V+2} + \frac{-2+\frac{3}{2}}{3V+2}\right] dV = dx$

or, $\frac{2}{3}\left[1 - \frac{1}{2} + \frac{1}{3V+2}\right] dV = dx$

or, $\left[1 - \frac{1}{2} + \frac{1}{3V+2}\right] dV = \frac{3}{2} dx$

or, $1\, dV - \frac{1}{2}\frac{1}{3V+2} dV = \frac{3}{2} dx$

Integrating both sides, we have,

$$\int 1\, dV - \frac{1}{2}\int \frac{1}{3V+2} dV = \frac{3}{2}\int 1\, dx$$

or, $2\int 1\, dV - \int \frac{1}{3V+2} dV = 3\int 1\, dx$

or, $2V - \frac{1}{3}\log(3V+2) = 3x + C$

or, $2(x+y) - \frac{1}{3}\log\{3(x+y)+2\} = 3x + C \quad (\because V = x+y)$

or, $2x + 2y - \frac{1}{3}\log(3x+3y+2) = 3x + C$

or, $6x + 6y - \log(3x+3y+2) = 9x + 3C$

or, $6x - 3x = \log(3x+3y+2) + C_1 \quad (\because C_1 = 3C)$

3. $(\mathbf{4x + 6y + 5)dy = (3y + 2x + 4)dx}$

Sol^n: Given differential equation is;

$$(4x + 6y + 5)dy = (3y + 2x + 4)dx$$

or, $\frac{dy}{dx} = \frac{3y+2x+4}{4x+6y+5}$

or, $\frac{dy}{dx} = \frac{2x+3y+4}{4x+6y+5}$

or, $\frac{dy}{dx} = \frac{2x+3y+4}{4(2x+3y)+5} \cdots$ (i)

Put,

$$V = 2x + 3y \ldots \text{(ii)}$$

Differentiating equation (ii) with respect to x, we have,

$\frac{dV}{dx} = 2 + 3\frac{dy}{dx}$

or, $\frac{3dy}{dx} = \frac{dV}{dx} - 2$

or, $\frac{dy}{dx} = \frac{1}{3}\frac{dV}{dx} - \frac{2}{3}$

Then equation (i) can be written as;

$$\frac{1}{3}\frac{dV}{dx} - \frac{2}{3} = \frac{V+4}{2V+5}$$

or, $\frac{1}{3}\frac{dV}{dx} = \frac{V+4}{2V+5} + \frac{2}{3}$

or, $\frac{1}{3}\frac{dV}{dx} = \frac{3V+12+2(2V+5)}{3(2V+5)}$

or, $\frac{dV}{dx} = \frac{7V+22}{2V+5}$

or, $\left(\frac{2V+5}{7V+22}\right) dV = dx$

or, $2\left(\frac{V+\frac{5}{2}}{7V+22}\right) dV = dx$

or, $\frac{2}{7}\left(\frac{7V+\frac{35}{2}}{7V+22}\right) dV = dx$

or, $\frac{2}{7}\left(\frac{7V+22-22+\frac{35}{2}}{7V+22}\right) dV = dx$

or, $\left[1 - \frac{9}{2}\ \frac{1}{7V+22}\right] dV = \frac{7}{2} dx$

or, $1dV - \frac{9}{2}\ \frac{1}{7V+22} dV = \frac{7}{2} dx$

Integrating both sides, we get,

$$\int 1dV - \frac{9}{2}\int \frac{1}{7V+22} dV = \frac{7}{2}\int 1\, dx$$

or, $2\int 1dV - 9\int \frac{1}{7V+22} dV = 7\int 1\, dx$

or, $2V - 9\frac{1}{7}\log(7V+22) = 7x + C$

or, $14V - 9\log(7V+22) = 49x + 7C$

or, $14(2x+3y) - 9\log\{7(2x+3y)+22\} = 49x + C_1 (\because C_1 = 7C)$

or, $28x + 42y - 9\log(14x + 21y + 22) = 49x + C_1$

or, $42y - 21x - 9\log(14x + 21y + 22) = C_1$

or, $7(6y - 3x) - 9\log(14x + 21y + 22) = C_1$

4. $\mathbf{(2x + 2y + 3)dy - (x + y + 1)dx = 0}$

Sol^n: Given differential equation is;

$$(2x + 2y + 3)dy - (x + y + 1)dx = 0$$

or, $(2x + 2y + 3)dy = (x + y + 1)dx$

or, $\frac{dy}{dx} = \frac{x+y+1}{2x+2y+3} \cdots$ (i)

Put,

$$V = x + y \ldots \text{(ii)}$$

Differentiating equation (ii) with respect to x, we have,

$$\frac{dV}{dx} = 1 + \frac{dy}{dx}$$

or, $\frac{dy}{dx} = \frac{dV}{dx} - 1$

Then equation (i) becomes;

$$\frac{dV}{dx} - 1 = \frac{V+1}{2V+3}$$

or, $\frac{dV}{dx} = \frac{V+1}{2V+3} + 1$

or, $\frac{dV}{dx} = \frac{V+1+2V+3}{2V+3}$

or, $\frac{dV}{dx} = \frac{3V+4}{2V+3}$

or, $\left(\frac{2V+3}{3V+4}\right) dV = dx$

or, $2\left(\frac{V+\frac{3}{2}}{3V+4}\right) dV = dx$

or, $\frac{2}{3}\left(\frac{V+\frac{9}{2}}{3V+4}\right) dV = dx$

or, $\left(\frac{3V+4-4+\frac{9}{2}}{3V+4}\right) dV = \frac{3}{2}dx$

or, $\left(1 + \frac{-4+\frac{9}{2}}{3V+4}\right) dV = \frac{3}{2}dx$

or, $\left(1 + \frac{1}{2}\frac{1}{3V+4}\right) dV = \frac{3}{2}dx$

Integrating both sides, we get,

$$\int 1\, dV + \frac{1}{2}\int \frac{1}{3V+4} dV = \frac{3}{2}\int 1\, dx$$

or, $V + \frac{1}{2}\frac{1}{3}\log(3V + 4) = \frac{3}{2}x + C$

or, $6V + \log(3V + 4) = 9x + 6C$

or, $6(x + y) + \log\{3(x + y) + 4\} = 9x + C_1$ $\quad (\because C_1 = 6C)$

or, $6x + 6y + \log(3x + 3y + 4) = 9x + C_1$

or, $6y - 3x + \log(3x + 3y + 4) = C_1$

5. $\mathbf{\frac{dy}{dx} = \frac{x+y}{x+y-2}}$

Sol^n: Given differential equation is;

$$\frac{dy}{dx} = \frac{x+y}{x+y-2} \cdots \text{(i)}$$

Put,

$$V = x + y \ldots \text{(ii)}$$

Differentiating equation (ii) with respect to x, we have,

$$\frac{dV}{dx} = 1 + \frac{dy}{dx}$$

or, $\frac{dy}{dx} = \frac{dV}{dx} - 1$

Then equation (i) becomes;

$$\frac{dV}{dx} - 1 = \frac{V}{V-2}$$

or, $\frac{dV}{dx} = \frac{V}{V-2} + 1$

or, $\frac{dV}{dx} = \frac{V+V-2}{V-2}$

or, $\frac{dV}{dx} = \frac{2V-2}{V-2}$

or, $\left(\frac{V-2}{2V-2}\right) dV = dx$

or, $\frac{1}{2}\left(\frac{2V-4}{2V-2}\right) dV = dx$

or, $\frac{1}{2}\left(\frac{2V-2}{2V-2} - \frac{2}{2V-2}\right) dV = dx$

or, $\frac{1}{2}\left(1-2\frac{1}{2V-2}\right)dV = dx$

or, $\left(1-2\frac{1}{2V-2}\right)dV = 2dx$

or, $1\,dV - 2\frac{1}{2V-2}dV = 2dx$

Integrating both sides, we have,

$$\int 1\,dV - 2\int \frac{1}{2V-2}dV = 2\int 1\,dx$$

or, $V - 2\frac{1}{2}\log(2V-2) = 2x + C$

or, $V - \log(2V-2) = 2x + C$

or, $x + y - \log\{2(x+y) - 2\} = 2x + C$

or, $x + y - \log(2x + 2y - 2) = 2x + C$

or, $y - x - \log(2x + 2y - 2) = C$

6. $\mathbf{(6x - 5y + 4)dy + (y - 2x - 1)dx = 0}$

Sol^n: Here,

$$(6x - 5y + 4)dy + (y - 2x - 1)dx = 0$$

or, $\frac{dy}{dx} = \frac{y-2x-1}{6x-5y+4}$

Put $x = X = h$ and $y = Y + k$

Where, h and k are constants

$$\frac{dy}{dx} = \frac{dY}{dX}$$

Then the given equation becomes;

$$\frac{dY}{dX} = -\frac{Y-2X+(k-2h-1)}{6X-5Y+(6h-5k+4)}$$

Choose h, k such that $k - 2h - 1 = 0$ and $6h - 5k + 4 = 0$

Solving these, we get,

$$h = -\frac{1}{4}, k = \frac{1}{2}$$

So, $\frac{dY}{dX} = -\frac{Y-2X}{6X-5Y}$

which is homogeneous, so, put $Y = Vx$,

$$\frac{dY}{dX} = V + X\frac{dV}{dX}$$

$\therefore$ $V + X\frac{dV}{dX} = \frac{-Vx-2X}{6X-5VX}$

or, $X\frac{dV}{dX} = \frac{-V-2}{6-5V} - V$

or, $X\frac{dV}{dX} = \frac{5V^2-7V+2}{6-5V}$

Separating the variables, we have,

$$\frac{dV}{dx} + \frac{5V-6}{5V^2-7V+2}dV$$

or, $2\frac{dX}{X} + \frac{10V-12}{5V^2-7V+2}dV = 0$

or, $2\frac{dX}{X} + \frac{10V-7}{5V^2-7V+2}dV - \frac{5V}{5\left(V^2-\frac{7}{5}V+\frac{2}{5}\right)} = 0$

or, $2\frac{dX}{X} + \frac{10V-7}{5V^2-7V+2}dV - \frac{dV}{\left(V-\frac{7}{10}\right)^2-\left(\frac{3}{10}\right)^2} = 0$

Integrating both sides, we have,

$$2\log x + \log(5V^2 - 7V + 2) - \frac{1}{2\times\frac{3}{10}}\log\frac{\left(V-\frac{7}{10}-\frac{3}{10}\right)}{V-\frac{7}{10}+\frac{3}{10}} = \log C$$

or, $\log\{x^2(5V^2 - 7V + 2)\} - \frac{5}{3}\log\frac{V-1}{5V-2} = \log C$

or, $\frac{x^2\left(5V^2-7V+2\right)(5V-2)^{\frac{5}{3}}}{(V-1)^{\frac{5}{3}}} = C$

or, $\frac{x^2\,(5V-2)^{\frac{8}{3}}}{(V-1)^{\frac{2}{3}}} = C$

or, $(5Y - 2X)^8 = C^3(5 - X)^2$

or, $\left[5\left(y-\frac{1}{2}\right) - 2\left(x+\frac{1}{4}\right)\right]^4 = C^{\frac{3}{2}}\left[\left(y-\frac{1}{2}\right)-\left(x+\frac{1}{4}\right)\right]$

or, $(5y - 2x - 3)^4 = C(4y - 4x - 3)$ is the required solution

7. $\mathbf{\frac{dy}{dx} = \frac{x+2y-3}{2x+y-3}}$

Sol^n: Here,

$$\frac{dy}{dx} = \frac{x+2y-3}{2x+y-3}$$

Put $x = X = h$ and $y = Y + k$

Where, h and k are constants

$$\frac{dy}{dx} = \frac{dY}{dX}$$

Then the given equation becomes;

$$\frac{dY}{dX} = -\frac{X+2Y+(h+2K-3)}{2X+Y+(2h+k-3)}$$

Choose h, k such that $h + 2k - 3 = 0$ and $2h + k - 3 = 0$

Solving these, we get,

$$h = 1, k = 1$$

So, $\frac{dY}{dX} = -\frac{X+2Y}{2X+Y}$

which is homogeneous, so, put $Y = Vx$,

$$\frac{dY}{dX} = V + X\frac{dV}{dX}$$

$\therefore \quad V + X\frac{dV}{dX} = \frac{X+2VX}{2X+VX} = \frac{1+2V}{2+V}$

or, $X\frac{dV}{dX} = \frac{1+2V}{2+V} - V = \frac{1+2V-2V-V^2}{2+V} = \frac{1-V^2}{2+V}$

Separating the variables, we have,

$$\frac{dX}{X} = \frac{2+V}{1-V^2}dV$$

or, $\frac{dX}{X} = \left(\frac{2}{1-V^2} + \frac{V}{1-V^2}\right)dV$

Integrating both sides, we have,

$$\log x = 2\frac{1}{2}\log\frac{1+V}{1-V} - \frac{1}{2}\log(1 - V^2) - \log C$$

or, $X = \frac{1}{C}\frac{1+V}{1-V}\frac{1}{\sqrt{1-V^2}} = \frac{1}{C}\frac{\sqrt{1+V}}{\left(1-V^2\right)^{\frac{3}{2}}}$

or, $X^2 = \frac{C^2(1+V)}{(1-V)^3}$

or, $X^2\left(1-\frac{Y}{X}\right)^3 = \frac{1}{C^2}\left(1+\frac{Y}{X}\right)$

or, $C^2(X-Y)^3 = X + Y$

or, $(x - 1 - y + 1)^3 = C^2(x - 1 + y - 1)$

or, $C^2(x - y)^3 = (x + y - 2)$ is the required solution

8\. $\frac{dy}{dx} = \frac{2x+3y+1}{3x+5y-1}$

Sol^n: Here,

$$\frac{dy}{dx} = \frac{2x+3y+1}{3x+5y-1} \dots \text{(i)}$$

Here,

$$\frac{a}{A} = \frac{b}{B}$$

Put $x = X = h$ and $y = Y + k$

Where, h and k are constants

$$dx = dX, dy = dY$$

Then the given equation (i) becomes;

$$\frac{dY}{dX} = \frac{-2(X+h)+3(Y+k)+1}{3(X+h)+5(Y+k)-1}$$

$$= -\frac{2X+3Y+2H+3k+1}{3X+5Y+3h+5k-1}$$

Choose h, k such that $2h + 3k + 1 = 0$ and $3h + 5k - 1 = 0$

Solving these, we get,

$$h = -8, k = 5$$

So, $\frac{dY}{dX} = -\frac{2X+3Y}{3X+5Y} \dots$ (ii)

which is homogeneous, so, put $Y = Vx$,

$$\frac{dY}{dX} = V + X\frac{dV}{dX}$$

$\therefore \quad V + X\frac{dV}{dX} = -\frac{2X+3VX}{3X+5VX} = -\frac{2+3V}{3+5V}$

or, $X\frac{dV}{dX} = -\frac{2+3V}{3+5V} - V$

$$= \frac{-2-3V-3V-5V^2}{3+5V}$$

$$= \frac{-\left(5V^2+6V+2\right)}{3+5V}$$

Integrating both sides, we have,

$$\int\frac{3+5V}{5V^2+6V+2}dV = -\int\frac{1}{X}dX \dots \text{(iii)}$$

Put $5V^2 + 6V + 2 = t$

$$(10V + 6)dV = dt$$

$$2(5V + 3)dV = dt$$

$$(5V+3)dV = \frac{1}{2}dt$$

Then equation (iii) becomes;

$$\int \frac{1}{t}dt = -2\int \frac{1}{X}\, dX$$

or, $\log t = -2\log X + \log C$

or, $\log t = \log \frac{C}{X^2}$

or, $tX^2 = C$

or, $(5V^2 + 6V + 2)X^2 = C$

Since,

$Y = VX$

or, $V = \frac{Y}{X}$

or, $\left(5\frac{Y^2}{X^2} + 6\frac{Y}{X} + 2\right)X^2 = C$

or, $(5Y^2 + 6V + 2X^2) = C$

or, $5(y-5)^2 + 6(x+8)(y-5) + 2(x+8)^2 = C$

or, $5(y^2 - 10y + 25) + 6(xy - 5x + 8y - 40) + 2(x^2 + 16x + 64) = C$

or, $5y^2 - 50y + 125 + 6xy - 30x + 48y - 240 + 2x^2 + 32x + 128 = C$

or, $5y^2 + 2x^2 - 2y + 6xy + 2x + 12 = C$

or, $2x^2 + 5y^2 + 6xy - 2y + 2x = C$ is the required solution

9. $\mathbf{(x - 3y + 4)dy + (7y - 5x)dx = 0}$

Solⁿ: Here,

$(x - 3y + 4)dy + (7y - 5x)dx = 0$

or, $(x - 3y + 4)dy = -(5x - 7y)dx$

$\frac{dy}{dx} = \frac{5x-7y}{x-3y+4}$

Here,

$\frac{a}{A} \neq \frac{b}{B}$

Put $x = X = h$ and $y = Y + k$

Where, h and k are constants

$dx = dX, dy = dY$

Then the given equation (i) becomes;

$$\frac{dY}{dX} = \frac{5(X+4)-7(Y+k)}{(X+4)-3(Y+k)+4}$$

$$= \frac{(5X-7Y)+5h-7k}{(X-3Y)+h-3k+4}$$

Choose h, k such that $5h - 7k = 0$ and $h - 3k + 4 = 0$

Solving these, we get,

$h = \frac{7}{2}, k = \frac{5}{2}$

So, $\frac{dY}{dX} = -\frac{5X-7Y}{X-3Y}$

which is homogeneous, so, put $Y = vx$,

$\frac{dY}{dX} = v + X\frac{dv}{dX}$

$\therefore\ v + X\frac{dv}{dX} = -\frac{X(5-7v)}{X(1-3v)}$

or, $\frac{dv}{dX} = -\frac{5-7v}{1-3v} - v = \frac{5-7v-v+3v^2}{1-3v} = \frac{5-8v+3v^2}{1-3v}$

Integrating both sides, we have,

$\int \frac{1-3v}{3v^2-8v+5}dv = \int \frac{1}{X}\, dX$

or, $-\int \frac{3v-1}{3v^2-5v-3v+5}dv = \int \frac{1}{X}\, dX$

or, $-\int \frac{3v-1}{v(3v-5)-1(3v-5)}dv = -\int \frac{1}{X}\, dX$

or, $\int \frac{3v-1}{(v-1)(3v-5)}dv = -\int \frac{1}{X}\, dX$

or, $\int \left(-\frac{1}{v-1} + \frac{6}{3v-5}\right)dv = -\int \frac{1}{X}\, dX$

or, $-\int \frac{1}{v-1}dv + 6\int \frac{6}{3v-5}dv = -\int \frac{1}{X}\, dX$

or, $-\log(v-1) + \frac{6}{3}\log(3v-5) = -\log X + \log C$

or, $2\log(3v-5) - \log(v-1) + \log X = \log C$

or, $\frac{(3v-5)^2 X}{(v-1)} = C$

or, $(3v-5)^2 X = (v-1)C$

or, $\left(\frac{3Y}{X} - 5\right)^2 X = \left(\frac{Y}{X} - 1\right)C$

or, $\frac{(3Y-5X)^2}{X^2}=\frac{(Y-X)}{X}C$

or, $(3Y-5X)^2=(Y-X)C$

or, $\left[3\left(y-\frac{5}{2}\right)-5\left(x-\frac{7}{2}\right)\right]^2=\left[\left(y-\frac{5}{2}\right)-\left(x-\frac{7}{2}\right)\right]C$

or, $\left[\frac{3(2y-5)-5(2x-7)}{2}\right]^2=\left[\frac{2y-5-(2x-7)}{2}\right]C$

or, $\left(\frac{6y-10x-15+35}{2}\right)^2=\left(\frac{2y-2x+2}{2}\right)C$

or, $(3y-5x+10)^2=(y-x+1)C$ is the required solution

10. $\mathbf{(x-y)dy-(x+y+1)dx=0}$

Sol^n: Here,

$$(x-y)dy-(x+y+1)dx=0$$

$$\frac{dy}{dx}=\frac{x+y+1}{x-y} \ldots \text{(i)}$$

Put $x=X=h$ and $y=Y+k$

Where, h and k are constants

$$\frac{dy}{dx}=\frac{dY}{dX}$$

Then the given equation (i) becomes;

$$\frac{dY}{dX}=\frac{X+Y+(h+k+1)}{X-Y+h-k}$$

Choose h, k such that $h+k+1=0$ and $h-k=0$

Solving these, we get,

$$h=-\frac{1}{2}, k=-\frac{1}{2}$$

So, $\frac{dY}{dX}=-\frac{X+Y}{X-Y} \cdots \text{(ii)}$

which is homogeneous, so, put $Y=vx$,

$$\frac{dY}{dX}=v+X\frac{dv}{dX}$$

Thus, the equation (ii) becomes;

$\therefore \quad v+X\frac{dv}{dX}=\frac{X+vX}{X-vX}=\frac{1+v}{1-v}$

or, $X\frac{dv}{dX}=-\frac{1+v}{1-v}-v=\frac{1+v-v+v^2}{1-v}$

or, $X\frac{dv}{dX}=\frac{1+v^2}{1-v}$

Separating the variables, we have,

$$\frac{dX}{X}=\frac{1-v}{1+v^2}dv$$

or, $2\frac{dX}{X}=\frac{2}{1+v^2}dv-\frac{2v}{1+v^2}dv$

Integrating both sides, we get,

$$2\log X=2\tan^{-1}v-\log(1+v^2)-\log C$$

or, $\log X^2(1+v^2)C=2\tan^{-1}v$

or, $\frac{1}{2}\log\left\{X^2\left(\frac{X^2+y^2}{X^2}\right)C\right\}=\tan^{-1}\left(\frac{Y}{X}\right)$

or, $\tan^{-1}\left(\frac{2y+1}{2x+1}\right)=\frac{1}{2}\log\left[C\left\{\left(x+\frac{1}{2}\right)^2+\left(y+\frac{1}{2}\right)^2\right\}\right]$

which is the required solution

Exercise 23

Solve the following differential equations.

1. $\dfrac{dy}{dx}+\dfrac{2x}{1+x^2}y=\dfrac{1}{(1+x^2)^2}$

Sol^n: This is linear equation. So,

$$p=\frac{2x}{1+x^2}$$

or, $\int p\,dx=\int\frac{2x}{1+x^2}=\log(1+x^2)$

Hence

$$\text{I.F.}=e^{\int p\,dx}=e^{\log(1+x^2)}=(1+x^2)$$

Multiplying both sides by $(1+x^2)$

$$(1+x^2)\frac{dy}{dx}+2xy=\frac{1}{1+x^2}$$

or, $\dfrac{d}{dx}[(1+x^2)y]=\dfrac{1}{1+x^2}$

Integrating both sides, we have,

or, $y(1+x^2)=\displaystyle\int\frac{dx}{(1+x^2)}$

so, $y(1+x^2)=\tan^{-1}x+c$

2. $\dfrac{dy}{dx}+\dfrac{y}{x^2}=\dfrac{1}{x^2}$

Sol^n: Here,

$\dfrac{dy}{dx}+\dfrac{y}{x^2}=\dfrac{1}{x^2}$ which is of the form $\dfrac{dy}{dx}+py=Q(x)$

So, $p=\dfrac{1}{x^2}$

$$\int p\,dx=-\frac{1}{x}$$

So, $\text{I.F}=e^{-\frac{1}{x}}$

Multiplying both sides by I.F

Thus,

$$e^{-\frac{1}{x}}\frac{dy}{dx}+e^{-\frac{1}{x}}\frac{y}{x^2}=e^{-\frac{1}{x}}\left(\frac{1}{x^2}\right)$$

or, $\dfrac{dy}{dx}\left(e^{-\frac{1}{x}}y\right)=e^{-\frac{1}{x}}\left(\dfrac{1}{x^2}\right)$

Integrating both sides, we have,

$$ye^{-\frac{1}{x}}=\int e^{-\frac{1}{x}}\frac{1}{x^2}\,dx$$

or, $ye^{-\frac{1}{x}}=e^{-\frac{1}{x}}+C$?

$\therefore$ $y=1+Ce^{+\frac{1}{x}}$

3. $(1-x^2)\dfrac{dy}{dx}-xy=1$

Sol^n: Here,

$$(1-x^2)\frac{dy}{dx}-xy=1$$

or, $\dfrac{dy}{dx}-\dfrac{x}{1-x^2}y=\dfrac{1}{1-x^2}$ which is linear

So, $\text{I.F.}=e^{\int p\,dx}=e^{\int-\frac{x}{1-x^2}}=e^{\frac{1}{2}\int-\frac{2x}{1-x^2}}$

$$=e^{\frac{1}{2}\log(1-x^2)}=e^{\log(1-x^2)^{\frac{1}{2}}}$$

Multiplying by $\text{I.F.}=\sqrt{1-x^2}$

$$\frac{dy}{dx}\sqrt{1-x^2}-\frac{xy}{\sqrt{1-x^2}}=\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\left(\sqrt{1-x^2}\,y\right)=\int\frac{1}{\sqrt{1-x^2}}$$

Integrating both sides, we have,

$$y\sqrt{1-x^2}=\sin^{-1}x+c$$

$\therefore$ $y\sqrt{1-x^2}=\sin^{-1}x+c$

4. $\dfrac{dy}{dx}+2y=4x$

Sol^n: Here,

$\dfrac{dy}{dx}+2y=4x$ which is linear

So, $p=2$

$\therefore \quad e^{\int p dx} = e^{2x}$

or, $e^{2x}\frac{dy}{dx} + e^{2x}y = 4xe^{2x}$

or, $\frac{d}{dx}(e^{2x}y) = 4xe^{2x}$

Integrating both sides, we have,

$$e^{2x}y = 4\int x e^{2x} = 4\left[\frac{x \cdot e^{2x}}{2} - \int e^{2x}\, dx\right] + C$$

or, $ye^{2x} = 4\left[\frac{x}{2}e^{2x} - \frac{e^{2x}}{2}\right] + C$

$ye^{2x} = 2xe^{2x} - 2e^{2x} + C$

$\therefore \quad y = 2x - 1 + Ce^{-2x}$

5. $\mathbf{(1+x)\frac{dy}{dx} - xy = (1-x)}$

Sol^n: Here,

$(1+x)\frac{dy}{dx} - xy = (1-x)$

or, $\frac{dy}{dx} - \frac{x}{1+x} \cdot y = \left(\frac{1-x}{1+x}\right)$

or, $p = -\frac{x}{1+x} = -\left(\frac{x+1-1}{x+1}\right) = \frac{1}{x+1} - 1$

$\int pdx = \log(x+1) - x$

$\therefore \quad$ I. F. $= e^{-x} \cdot e^{\log(x+1)} = (x+1)e^{-x}$

Multiplying by I. F.

$\frac{dy}{dx} - \frac{xy}{1+x} = \frac{1-x}{1+x}$

or, $(x+1)e^{-x}\frac{dy}{dx} - \frac{x(x+1)}{(x+1)}e^{-x}y = (1-x)e^{-x}$

or, $\frac{d}{dx}[y(x+1)e^{-x}] = (1-x)e^{-x}$

Integrating both sides, we have,

$y(x+1)e^{-x} = \int(e^{-x} - xe^{-x})\, dx$

or, $y(x+1)e^{-x} = \int e^{-x}\, dx - \int xe^{-x}\, dx$

$= -e^{-x} - [e - xe^{-x} - \int -e^{-x}dx + C$

$= -e^{-x} + xe^{-x} + e^{-x} + C$

or, $y(x+1)e^{-x} = xe^{-x} + C$

$\therefore \quad y(x+1) = x + Ce^{x}$ is required solution.

6. $\mathbf{(1+x^2)\frac{dy}{dx} + y = e^{\tan^{-1}x}}$

Sol^n: Here,

$(1+x^2)\frac{dy}{dx} + y = e^{\tan^{-1}x}$

or, $\frac{dy}{dx} + \frac{1y}{1+x^2} = \frac{e^{\tan^{-1}x}}{1+x^2}$

let $p = \frac{1}{1+x^2} \int p\, dx = \int \frac{1}{1+x^2} dx = \tan^{-1}x$

$\therefore \quad$ I. F. $= e^{\int pdx} = e^{\tan^{-1}x}$

Multiplying by I. F.

$$e^{\tan^{-1}x}\frac{dy}{dx} + e^{\tan^{-1}x}\frac{y}{1+x^2} = \frac{\left(e^{\tan^{-1}x}\right)^2}{(1+x^2)}$$

$\therefore \quad \frac{d}{dx}\left(ye^{\tan^{-1}x}\right) = \int \frac{e^{2\tan^{-1}x}}{(1+x^2)} dx$

Put $\tan^{-1}x = z, dz = \frac{1}{1+x^2}dx$

Integrating both sides, we have,

$ye^{\tan^{-1}x} = \int e^{2z}\, dz = \frac{e^{2z}}{2} + C$

$\therefore \quad ye^{\tan^{-1}x} = \frac{e^{2\tan^{-1}x}}{2} + C$

7. $\mathbf{x\cos x\frac{dy}{dx} + y(x\sin x + \cos x) = 1}$

Sol^n: Here,

$x\cos x\frac{dy}{dx} + y(x\sin x + \cos x) = 1$

or, $\frac{dy}{dx} + y\left(\frac{x\sin x + \cos x}{x\cos x}\right) = \frac{1}{x\cos x}$

It is linear equation

$p = \frac{x\sin x + \cos x}{x\cos x} = \left(\tan x + \frac{1}{x}\right)$

$= \log\sec x + \log x = \log(x\sec x)$

I. F. $= e^{\int pdx} = e^{\log x\sec x} = x\sec x$

Multiplying by I. F. yields

$$x\sec x\frac{dy}{dx} + y\sec x\left(\frac{x\sin x + \cos x}{x\cos x}\right) = x\sin x\frac{1}{x\cos x}$$

$$I.F. = e^{\int \frac{1}{x\log x}} = e^{\log(\log x)} = \log x$$

$$\left(\frac{dy}{dx} + \frac{1}{x\log x}y\right)\log x = \frac{2}{x}\log x$$

or, $\frac{d}{dx}(y\log x) = 2\int \frac{1}{x}\log x\,dx$

Integrating both sides, we have,

$$y\log x = \frac{2(\log x)^2}{2} + C$$

$\therefore \quad y\log x = (\log x)^2 + C$

13. $\frac{dy}{dx} + y = \cos x$

Sol^n: Here,

$$\frac{dy}{dx} + y = \cos x$$

$$p = 1$$

$\therefore \quad \int p\,dx = x$

$\therefore \quad I.F. = e^{\int p\,dx} = e^x$

$\therefore \quad e^x\frac{dy}{dx} + ye^x = e^x\cos x$

$$\frac{d}{dx}(e^x y) = e^x\cos x$$

Integrating both sides, we have,

$$ye^x = \int e^x\cos x\,dx \ \ldots \text{(i)}$$

$$I = \int e^x\cos x\,dx = \cos x\,e^x + \int[\sin x\,e^x dx]$$

$$= \cos x\,e^x + \int e^x\sin x\,dx$$

$$= e^x\cos x + [\sin x\,e^x - \int \cos x\cdot e^x dx]$$

$\therefore \quad 2I = e^x\cos x + e^x\sin x$

$$I = \frac{e^x}{2}(\sin x + \cos x)$$

$\therefore \quad ye^x = \frac{e^x}{2}(\sin x + \cos x) + C$

or, $2y = \sin x + \cos x + Ce^{-x}$

14. $(1+y^2)dx = (\tan^{-1}y - x)dy$

Sol^n: Here,

$$(1+y^2)dx = (\tan^{-1}y - x)dy$$

or, $\frac{dx}{dy} = \frac{\tan^{-1}y}{1+y^2} - \frac{x}{1+y^2}$

or, $\frac{dx}{dy} + \frac{x}{1+y^2} = \frac{\tan^{-1}y}{1+y^2}$

Put $p = 1$

Then,

$$\int p\,dy = \tan^{-1}y$$

so, $I.F. = e^{\tan^{-1}y}$

$$x.e^{\tan^{-1}y} = \int \frac{\tan^{-1}y.\,e^{\tan^{-1}y}}{1+y^2}dy$$

Let, $z = \tan^{-1}y$

so, $dz = \frac{dy}{1+y^2}$

so, $xe^{\tan^{-1}y} = \int ze^z zd = ze^z - e^z + C$

$\therefore \quad xe^{\tan^{-1}y} = \tan^{-1}y\,e^{\tan^{-1}y} - e^{\tan^{-1}y} + C$

15. $(1+y^2) + \left(x - e^{\tan^{-1}y}\right)\frac{dy}{dx} = 0$

Sol^n: Here,

$$(1+y^2) + \left(x - e^{\tan^{-1}y}\right)\frac{dy}{dx} = 0$$

or, $\frac{dx}{dy} + \left(\frac{1}{1+y^2}\right)x = \frac{e^{\tan^{-1}y}}{1+y^2}$ which is linear

So, $I.F. = e^{\int \frac{1}{1+y^2}dy} = e^{\tan^{-1}y}$

$\therefore \quad \frac{d}{dy}\left(x\cdot e^{\tan^{-1}y}\right) = \frac{\left(e^{\tan^{-1}y}\right)^2}{1+y^2}$

Integrating both sides, we have,

$$x\cdot e^{\tan^{-1}y} = \int \frac{e^{2\tan^{-1}y}}{1+y^2}dy + k = \frac{1}{2}e^{2t} + k$$

$$= \frac{1}{2}e^{2\tan^{-1}y} + k$$

$e^{2t}/2$

$\therefore \quad x\cdot e^{\tan^{-1}y} = \frac{1}{2}e^{2\tan^{-1}y} + k$

$$2xe^{\tan^{-1}y} = e^{2\tan^{-1}y} + 2k$$

$$2xe^{\tan^{-1}y} + C = e^{2\tan^{-1}y}$$

which is required solution

16. Solve: $\frac{dy}{dx}+\frac{y}{x}=x^2$ if $y = 1$ when $x = 1$

Solⁿ: Here,

$$\frac{dy}{dx}+\frac{y}{x}=x^2$$

$$p=\frac{1}{x}$$

$$\therefore \quad I.F. = e^{\int \frac{1}{x}dx} = e^{\log x} = x$$

$$\therefore \quad \frac{d}{dx}(xy) = \int x^3\,dx$$

Integrating both sides, we have,

$$xy = \frac{x^4}{4}+C$$

$$4xy = x^4 + C \ldots \text{(i)}$$

When,

$$x = 1, y = 1$$

$$4 = 1 + 4C$$

$$\therefore \quad C = \frac{3}{4}$$

$\therefore$ $4xy = x^4 + 4$ is required equation

Exercise 24

Solve the following differential equations.

1. $\frac{dy}{dx}+\frac{y}{x}=\frac{y^2}{x^2}$

Solⁿ: Here,

$$\frac{dy}{dx}+\frac{y}{x}=\frac{y^2}{x^2}$$

Dividing both sides by y^2, we have,

$$y^{-2}\frac{dy}{dx}+\frac{1}{xy}=\frac{1}{x^2}$$

Put $v=\frac{1}{y}, \frac{dv}{dx}=-\frac{1}{y^2}\frac{dy}{dx}$

$$-\frac{dv}{dx}+\frac{v}{x}=\frac{1}{x^2}$$

$$\frac{dv}{dx}-\frac{v}{x}=-\frac{1}{x^2}$$

$$p=-\frac{1}{x}$$

$$\int p\,dx = -\log x = \log(x)^{-1}$$

$$\therefore \quad I.F. = e^{\log(x)^{-1}} = \frac{1}{x}$$

$$\frac{d}{dx}\left(\frac{v}{x}\right) = x^{-3}$$

Integrating both sides, we have,

$$\frac{v}{x} = -\int x^{-3}\,dx$$

$$\frac{v}{x} = \frac{-x^{-3+1}}{-3+1}+C$$

$$\frac{v}{x} = +\frac{1}{2}x^{-2}+C$$

or, $\frac{v}{x}+\frac{1}{2x^2}=C$

or, $\frac{1}{xy}-\frac{1}{2x^2}=C$

$$2x-y+2Cx^2y$$

2. $(1-x^2)\dfrac{dy}{dx}+xy=xy^2$

Solⁿ: Here,

$$(1-x^2)\frac{dy}{dx}+xy=xy^2$$

or, $\dfrac{dy}{dx}+\dfrac{x}{(1-x^2)}=\dfrac{xy^2}{1-x^2}$

or, $y^{-2}\dfrac{dy}{dx}+\dfrac{x}{y(1-x^2)}=x$

Put $v=\dfrac{1}{y}, \dfrac{dv}{dx}=-\dfrac{1}{y^2}\dfrac{dy}{dx}$

$$-\frac{dv}{dx}+\frac{vx}{1-x^2}=\frac{x}{1-x^2}$$

or, $\dfrac{dv}{dx}-\dfrac{vx}{1-x^2}=\dfrac{-x}{1-x^2}$

Put $p=\dfrac{-x}{(1-x^2)}, \int pdx=\frac{1}{2}\log(1-x^2)$

$$=\log(1-x^2)^{\frac{1}{2}}$$

Hence,

$$\text{I.F.}=e^{\log\sqrt{1-x^2}}=\sqrt{1-x^2}$$

$\therefore \quad \sqrt{1-x^2}\dfrac{dv}{dx}-\dfrac{vx}{(1-x^2)}=\dfrac{-x\sqrt{1-x^2}}{(1-x^2)}$

$\therefore \quad \dfrac{d}{dx}\left(v\sqrt{1-x^2}\right)=\dfrac{-x}{\sqrt{1-x^2}}$

or, $v\sqrt{1-x^2}=\sqrt{(1-x^2)}+C$

or, $\dfrac{1}{y}\sqrt{1-x^2}=\sqrt{(1-x^2)}+C$

or, $\sqrt{1-x^2}=y\sqrt{1-x^2}+cy$

or, $\sqrt{1-x^2}(1-y)=cy$

3. $\dfrac{dy}{dx}+\dfrac{1}{x}=\dfrac{e^y}{x^2}$

Solⁿ: Here,

$$\frac{dy}{dx}+\frac{1}{x}=\frac{e^y}{x^2}$$

or, $e^{-y}\dfrac{dy}{dx}+\dfrac{e^{-y}}{x}=\dfrac{1}{x^2}$

Put $v=e^{-y}$

$$\frac{dv}{dx}=-e^{-y}\frac{dy}{dx}$$

$\therefore \quad -\dfrac{dv}{dx}+\dfrac{v}{x}=\dfrac{1}{x^2}$

$$\frac{dv}{dx}-\frac{v}{x}=-\frac{1}{x^2}$$

Put $p=-\dfrac{1}{x}, \int pdx=-\log x=\log(x)^{-1}$

$$\text{I.F.}=e^{\int pdx}=\frac{1}{x}$$

$\therefore \quad \dfrac{d}{dx}\left(v\dfrac{1}{x}\right)=\dfrac{1}{x^3}$

Integrating both sides, we have,

$$\frac{v}{x}=-\int x^3\,dx=\frac{x^{-2}}{2}+C$$

or, $\dfrac{e^{-y}}{x}=\dfrac{1+2Cx^2}{2x^2}$

or, $e^{-y}2x=(1+Cx^2)$

$\therefore \quad 2x=e^y(1+Cx^2)$

4. $\dfrac{dy}{dx}+\dfrac{y}{x}\log y=\dfrac{y}{x^2}(\log y)^2$

Solⁿ: Here,

$$\frac{dy}{dx}+\frac{y}{x}\log y=\frac{y}{x^2}(\log y)^2$$

or, $\dfrac{(\log y)^{-2}}{y}+\dfrac{y}{y\log y}\times\dfrac{1}{x}=\dfrac{1}{x^2}$

Put $v=\dfrac{1}{\log y}$

$$\frac{dv}{dx}=-\frac{1}{y}(\log y)^{-2}$$

or, $-\dfrac{dv}{dx}+\dfrac{v}{x}=\dfrac{1}{x^2}$

or, $\dfrac{dv}{dx}-\dfrac{v}{x}=-\dfrac{1}{x^2}$

Put $p=-\dfrac{1}{x}$

$$\text{I.F.}=e^{\int-\frac{1}{x}dx}=e^{\log(x)^{-1}}=\frac{1}{x}$$

$\therefore \quad \frac{d}{dx}\left(v \cdot \frac{1}{x}\right) = -\frac{1}{x^3}$

Integrating both sides, we have,

$v \times \frac{1}{x} = -\int x^{-3}\,dx$

or, $\frac{v}{x} = \frac{-x^{-2}}{-2} = \frac{1}{2}x^{-2} + C$

or, $\frac{1}{\log y \times x} = \frac{1+Cx^2}{2x^2}$

or, $2x^2 = \log y + \frac{Cx^2}{2}\log y$

or, $2x = \log y\,(1 + Cx^2)$

5. $\mathbf{\frac{dy}{dx} + x\sin 2y = x^3\cos^2 y}$

Sol^n: Here,

$\frac{dy}{dx} + x\sin 2y = x^3\cos^2 y$

or, $\sec^2 y\frac{dy}{dx} + \frac{x\sin 2y}{\cos^2 y} = x^3$

Put $v = \frac{\sin 2y}{\cos^2 y} = \frac{2\sin y\cos y}{\cos^2 y} = 2\tan y$

$\frac{dv}{dx} = 2\sec^2 y\frac{dy}{dx}$

or, $\frac{1}{2}\frac{dv}{dx} + vx = x^3$

or, $\frac{dv}{dx} + v \cdot 2x = 2x^3$

Put $p = 2x$

$\therefore \quad$ I.F. $= e^{x^2}$

$\therefore \quad \frac{d}{dx}\left(v \cdot e^{x^2}\right) = 2x^3e^{x^2}$

Put $x^2 = t$

or, $2xdx = dt$

Integrating both sides, we have,

$ve^{x^2} = \int te^t\,dt = te^t - \int e^t\,dt = te^t - e^t + C$

$\therefore \quad ve^{x^2} = te^t - e^t + C$

or, $2\tan y \cdot e^{x^2} = x^2e^{x^2} - e^{x^2} + C$

$\therefore \quad \tan y \cdot e^{x^2} = \frac{e^{x^2}}{2}(x^2 - 1) + C$ is required solution

6. $\mathbf{\cos x\frac{dy}{dx} = y(\sin x - y)}$

Sol^n: Here,

$\cos x\frac{dy}{dx} = y(\sin x - y)$

or, $\frac{dy}{dx} = \tan x\, y - y\sec x$

or, $\frac{dy}{dx} - y\tan x = -y^2\sec x$

or, $\frac{1}{y^2}\frac{dy}{dx} - \frac{1}{y}\tan x = -\sec x$

Put $v = -\frac{1}{y}, \frac{dv}{dx} = \frac{1}{y^2}\frac{dy}{dx}$

$\frac{dv}{dx} + v\tan x = \sec x$

Put $p = \tan x$

$\therefore \quad$ I.F. $= e^{\log\sec x} = \sec x$

$\therefore \quad \frac{d}{dx}(v \cdot \sec x) = \sec^2 x$

Integrating both sides, we have,

$v\sec x = \int \sec^2 x\,dx$

or, $v\sec x = \tan x + C$

or, $\frac{1}{y}\sec x = (\tan x + C)$

or, $\sec x = y(\tan x + C)$ which is required solution

7. $\mathbf{x\frac{dy}{dx} + \frac{y}{x} = y^2\log x}$

Sol^n: Here,

$x\frac{dy}{dx} + \frac{y}{x} = y^2\log x$

or, $y^{-2}\frac{dy}{dx} + \frac{1}{x^2y} = \frac{\log x}{x}$

Put $v = \frac{1}{y}, \frac{dv}{dx} = -\frac{1}{y^2}\frac{dy}{dx}$

$-\frac{dv}{dx} + \frac{v}{x} = \log x$

or, $\frac{dv}{dx} - \frac{v}{x} = -\log x$

Let $p = -\frac{1}{x}, \int p\,dx = -\log x$

$\therefore$ $\quad I.F. = e^{-\log x} = \frac{1}{x}$

$$\frac{d}{dx}\left(\frac{1}{x} \times \frac{v}{a}\right) = -\frac{\log x}{x^2}$$

Integrating both sides, we have,

$$\frac{v}{x} = -\int \frac{\log x}{x^2} dx$$

or, $\frac{v}{x} = -\left[\log x \frac{x^{-2+1}}{-2+1} - \int \frac{x^{-1}}{-1} \times \frac{1}{x} dx\right] + C$

$$= -\left[-\frac{\log x}{x} + \int x^{-2} dx\right] + C = \frac{\log x}{x} + \frac{1}{x} + C$$

or, $\frac{1}{y} \times \frac{1}{x} = \frac{\log x}{x} + \frac{1}{x} + C$

or, $1 = y \log x + y + Cxy$ is required equation

8. $\frac{dy}{dx} = y \tan x - y^2 \sec x$

Sol^n: Here,

$$\frac{dy}{dx} - y \tan x = -y^2 \sec x$$

or, $y^{-2}\frac{dy}{dx} - y^{-1} \tan x = -\sec x$

let $v = \frac{1}{y}$

$\therefore$ $\frac{dv}{dx} = -\frac{1}{y^2}\frac{dy}{dx}$

$$-\frac{dv}{dx} - v \tan x = -\sec x$$

or, $\frac{dv}{dx} + v \tan x = \sec x$

$p = \tan x$

$\therefore$ $I.F. = e^{\log \sec x} = \sec x$

$\therefore$ $\frac{d}{dx}(v \times \sec x) = \sec^2 x$

Integrating both sides, we have,

$$v \times \sec x = \int \sec^2 x = \tan x + C$$

or, $v \times \sec x = \tan x + C$

or, $\frac{1}{y}\sec x = \tan x + C$

or, $\sec x = y \tan x + cy$ is required equation

9. $\frac{dy}{dx} - \frac{\tan y}{1+x} = (1 + x)e^x \sec y$

Sol^n: Here,

$$\frac{dy}{dx} - \frac{\tan y}{1+x} = (1 + x)e^x \sec y$$

or, $\cos y \frac{dy}{dx} - \frac{1}{1+x} \sin y = (1 + x)e^x$

Put $v = \sin y$

$$\cos y \frac{dy}{dx} = \frac{dv}{dx}$$

$\therefore$ $\frac{dv}{dx} - \frac{v}{1+x} = (1 + x)e^x$

It is linear, so,

$$p = -\frac{1}{1+x} = -\log(1 + x)$$

$\therefore$ $I.F. = e^{-\log(1+x)} = \frac{1}{1+x}$

$\therefore$ $\frac{d}{dx}\left(v \times \frac{1}{1+x}\right) = e^x$

Integrating both sides, we have,

$$\frac{v}{1+x} = e^x + C$$

or, $\sin y = e^x(1 + x) + C(1 + x)$

or, $\sin y = (1 + x)(e^x + C)$ is required solution

10. $\frac{dy}{dx} + x \sin 2y = x^3 \cos^2 y$

Sol^n: Here,

$$\frac{dy}{dx} + x \sin 2y = x^3 \cos^2 y$$

or, $\sec^2 y \frac{dy}{dx} + x \sec^2 y \sin 2y = x^3$

or, $\sec^2 y \frac{dy}{dx} + 2 \tan y - x = x^3$

Put $v = \tan y, \frac{dv}{dx} = \sec^2 y \frac{dy}{dx}$

$\therefore$ $\frac{dv}{dx} + 2vx = x^3$

Put $p = 2x$

$\therefore$ $I.F. = e^{x^2}$

$\therefore \quad \frac{d}{dx}(e^{x^2} \cdot v) = e^{x^2} \cdot x^3$

Integrating both sides, we have,

$$ve^{x^2} = \int e^{x^2} x^3 \, dx$$

Put $x^2 = t \Rightarrow e^{x^2} x^3$

Integrating both sides, we have,

$$ve^{x^2} = \frac{1}{2}\int e^t \cdot t \, dt = te^t - \int e^t \, dt = te^t - e^t + C$$

$\therefore \quad 2 \tan y \, e^{x^2} = x^2 e^{x^2} - e^{x^2} + C$

$\therefore \quad e^{x^2} \tan y = \frac{e^{x^2}}{2}(x^2 - 1) + C$ which is required solution

Exercise 25

Solve the following equations:

1. $(2ax + by)y \, dx + (ax + 2by)x \, dy = 0$

Sol^n: Given that;

$$(2ax + by)y \, dx + (ax + 2by)x \, dy = 0$$

or, $2axy \, dx + ax^2 \, dy + by^2 \, dx + 2bxy \, dy = 0$

or, $\int d(ax^2y) + b\int d(y^2x) = 0$

or, $ax^2y + bxy^2 = 0$

or, $ax^2y + bxy^2 = C$

2. $(x^2 - ay)dx - (ax - y^2)dy = 0$

Sol^n: Given that;

$$(x^2 - ay)dx - (ax - y^2)dy = 0$$

or, $x^2dx - ay \, dx - ax \, dy - y^2dy = 0$

or, $x^2dx - a(y \, dx + x \, dy) + y^2dy = 0$

Integrating yields,

$$\frac{x^3}{3} - axy + \frac{y^3}{3} = C$$

or, $x^3 + y^3 - 3axy = 3C$

3. $\frac{dy}{dx} = \frac{2x-y+1}{x+2y-3}$

Sol^n: Given that;

$$\frac{dy}{dx} = \frac{2x-y+1}{x+2y-3}$$

or, $(x + 2y - 3)dy = (2x - y + 1)dx$

or, $x \, dy + y \, dx + 2y \, dy - 3 \, dy + 2x \, dx - 1 \, dx = 0$

or, $d(xy) + 2y \, dy - 3 \, dy - 1 \, dx - 2x \, dx = 0$

Integrating yields,

$$-x^2 + xy + y^2 - 3y - x = C$$

or, $y^2 - x^2 - 3y + xy - x = C$

4. $\mathbf{(x+y)dy + (y-x)dx = 0}$

Sol^n: Given that;

$$(x+y)dy + (y-x)dx = 0$$

or, $x\,dy + y\,dy + y\,dx - x\,dx = 0$

or, $d(xy) + y\,dy - x\,dx = 0$

Integrating,

$$xy + \frac{y^2}{2} - \frac{x^2}{2} = C$$

or, $y^2 - x^2 + 2xy = 2C$

5. $\mathbf{2xy\,dx - (x^2 - y^2)dy = 0}$

Sol^n: Given that;

$$2xy\,dx - (x^2 - y^2)dy = 0$$

or, $2xy\,dx - x^2dy + y^2\,dy = 0$

or, $\dfrac{2xy\,dx - x^2dy}{y^2} + 1\,dy = 0$

or, $d\left(\dfrac{x^2}{y}\right) + 1\,dy = 0$

Integrating yields,

$$\frac{x^2}{y} + y = C$$

or, $x^2 + y^2 - Cy = 0$

6. $\mathbf{(x^2 + y^2 + 2x)dx + xy\,dy = 0}$

Sol^n: Given that;

$$(x^2 + y^2 + 2x)dx + xy\,dy = 0$$

or, $x^2\,dx + y^2dx + 2x\,dx + xy\,dy = 0$

or, $x^2\,dx + 2x\,dx + y^2dx + xy\,dy = 0$

or, $x^3\,dx + 2x^2\,dx + y^2x\,dx + x^2y\,dy = 0$

or, $x^3\,dx + 2x^2\,dx + \dfrac{2y^2x\,dx + 2x^2y\,dy}{2} = 0$

or, $2x^3\,dx + 4x^2\,dx + 2y^2x\,dx + 2x^2y\,dy = 0$

or, $d\left(\dfrac{2x^4}{4}\right) + d\left(\dfrac{4x^3}{3}\right) + d(x^2y^2) = C$

Integrating yields,

$$\frac{x^4}{2} + \frac{4x^3}{3} + x^2y^2 = C$$

or, $3x^4 + 8x^3 + 6x^2y^2 = 6C$

7. $\mathbf{x\dfrac{dy}{dx} + y = y^2 \log x}$

Sol^n: Given that;

$$x\frac{dy}{dx} + y = y^2 \log x$$

or, $x\,dy + y\,dx = y^2 \log x\,dx$

or, $y^2 \log x\,dx - x\,dy - y\,dx = 0$

Dividing by x^2y^2, we get,

$$\frac{\log x}{x^2}dx - \frac{1}{xy^2}dy - \frac{1}{x^2}dx = 0$$

or, $\dfrac{\log x}{x^2}dx + \left(\dfrac{1}{x}\right)\left(-\dfrac{1}{y^2}\right)dy + \dfrac{1}{y}\left(-\dfrac{1}{x^2}\right)dx = 0$

or, $\dfrac{\log x}{x^2}dx + d\left(\dfrac{1}{x} \times \dfrac{1}{y}\right) = 0$

Integrating both sides, we get,

$$\int \frac{\log x}{x^2}dx = \log x\frac{x^{-2+1}}{-2+1} - \int \frac{x^{-2+1}}{-2+1} \times \frac{1}{x}\,dx$$

$$= -\frac{1}{x}\log x + \int \frac{1}{x^2}dx = -\frac{1}{x}\log x - \frac{1}{x}$$

Thus,

$$-\frac{1}{x}\log x - \frac{1}{x} + \frac{1}{xy} + c = 0$$

or, $-y\log x - y + cxy + 1 = 0$

or, $1 + cxy = y(1 + \log x)$ is the required solution

8. $\mathbf{x\,dy - y\,dx + a(x^2 + y^2)\,dx = 0}$

Sol^n: Given that;

$$x\,dy - y\,dx + a(x^2 + y^2)\,dx = 0$$

Dividing by $x^2 + y^2$, we get,

or, $\dfrac{x\,dy - y\,dx}{x^2 + y^2} + a\,dx = 0$

or, $\dfrac{d\left(\frac{y}{x}\right)}{1 + \left(\frac{y}{x}\right)^2} + a\,dx = 0$

or, $d\left(\tan^{-1}\dfrac{y}{x}\right) + a\,dx = 0$

Integrating yields,

$$\tan^{-1}\frac{y}{x} + ax = C$$ which is the required solution

9. $\mathbf{x\,dx + y\,dy + (x^2 + y^2)dy = 0}$

Solⁿ: Given that;

$$x\,dx + y\,dy + (x^2+y^2)dy = 0$$

or, $\frac{x\,dx + y\,dy}{x^2+y^2} + dy = 0$

or, $\frac{1}{2}d[\log(x^2+y^2)] + dy = 0$

Integrating yields,

$$\frac{1}{2}\log(x^2+y^2) + y = \frac{1}{2}\log C$$

or, $\frac{1}{2}\log\left(\frac{x^2+y^2}{2}\right) = -y$

or, $\log\left(\frac{x^2+y^2}{x}\right) = -2y$

or, $\frac{x^2+y^2}{C} = e^{-2y}$

$\therefore$ $x^2+y^2 = Ce^{-2y}$

10. $\mathbf{(1+xy)y\,dx + (1-xy)x\,dy = 0}$

Solⁿ: Given that;

$$(1+xy)y\,dx + (1-xy)x\,dy = 0$$

or, $ydx + xy^2dx + xdy + x^2y\,dy = 0$

or, $ydx + xdx + y^2xdx - x^2y\,dy = 0$

Dividing by x^2y^2, we get,

$$\frac{1}{x^2y}dx + \frac{1}{xy^2}dy + \frac{dx}{x} - \frac{dy}{y} = 0$$

or, $-d\left(\frac{1}{x}\times\frac{1}{y}\right) + \frac{dx}{x} - \frac{dy}{y} = 0$

Integrating, we get,

$$-\frac{1}{xy} + \log x - \log y = c$$

or, $\frac{1}{xy} + \log\frac{x}{y} - \log y = c$ which is the required solution

11. $\mathbf{\sin x\ dy - y\cos x\ dx + y^2\,dx = 0}$

Solⁿ: Given that;

$$\sin x\ dy - y\cos x\ dx + y^2\,dx = 0$$

or, $-\frac{\sin x\,dy + y\cos x}{y^2} = dx$

or, $d\left(\frac{\sin x}{y}\right) = dx$

Integrating yields,

$$\frac{\sin x}{y} = C + x$$

or, $\frac{\sin x}{y} = (C+x)$

or, $\sin x = y(C+x)$

12. $\mathbf{x\frac{dy}{dx} = y + x^2\log x}$

Solⁿ: Given that;

$$x\frac{dy}{dx} = y + x^2\log x$$

or, $x\,dy = y\,dx + x^2\log x\ dx$

or, $\frac{x\,dy - y\,dx}{x^2} = \log x\ dx$

or, $d\left(\frac{y}{x}\right) = d(x\log x - x)$ $\quad\because \int\log x = (x\log x - x)$

Integrating yields,

$$\frac{y}{x} = (x\log x - x) + C$$

or, $y = x^2\log x - x^2 + C$

13. $\mathbf{x\cos\left(\frac{y}{x}\right)(y\,dx + x\,dy) = y\sin\frac{y}{x}(x\,dy - y\,dx)}$

Solⁿ: Given that;

$$x\cos\frac{y}{x}(y\,dx + x\,dy) = y\sin\frac{y}{x}(x\,dy - y\,dx)$$

Dividing by x, we get,

$$\cos\frac{y}{x}(y\,dx + x\,dy) = y\sin\frac{y}{x}\left(\frac{x\,dy - y\,dx}{x}\right)$$

or, $\cos\frac{y}{x}d(xy) - xy\sin\frac{y}{x}\left(\frac{x\,dy - y\,dx}{x^2}\right) = 0$

or, $\cos\frac{y}{x}d(xy) - xy\sin\frac{y}{x}d\left(\frac{xy}{x}\right) = 0$

or, $\cos\frac{y}{x}d(xy) - xy\,d\left(\cos\frac{y}{x}\right) = 0$

or, $d\left(xy\times\cos\frac{y}{x}\right) = 0$

Integrating, we get,

$$xy\cos\frac{y}{x} = c$$ which is the required equation

14. $\mathbf{\cos x(\cos x - \sin y)dx + \cos y(\cos y - \sin x)dy = 0}$

Sol^n: Given that;

$$\cos x(\cos x - \sin y)dx + \cos y(\cos y - \sin x)dy = 0$$

or, $\cos^2 x\, dx - \cos x \sin y\, dx + \cos^2 y\, dy - \sin x \cos y\, dy = 0$

or, $\cos^2 x\, dx + \cos^2 y\, dy - [d(\sin y \sin x)] = 0$

or, $\left(\frac{1+\cos 2x}{2}\right) dx + \left(\frac{1+\cos 2y}{2}\right) dy - d(\sin y \sin x) = 0$

or, $\frac{1}{2}\, dx + \frac{\cos 2x}{2} dx + \frac{1}{2}\, dy + \frac{\cos 2y}{2} dy - d(\sin x \sin y) = 0$

Integrating yields,

$$\frac{x}{2} + \frac{y}{2} + \frac{\sin 2x}{4} + \frac{\sin 2y}{4} - \sin x \sin y = C$$

or, $2x + 2y + \sin 2x + \sin 2y - 4 \sin x \sin y = d$

15. $\mathbf{(x + 2y^3)dy = y\, dx}$

Sol^n: Given that;

$$(x + 2y^3)dy = y\, dx$$

or, $x\, dy + 2y^3\, dy = y\, dx$

or, $x\, dy - y\, dx = -2y^3\, dy$

or, $\frac{y\, dx - x\, dy}{y^2} = 2y\, dy$

or, $d\left(\frac{x}{y}\right) = 2y\, dy$

Integrating yields,

$$\frac{x}{y} = \frac{2y^2}{2} + C$$

or, $\frac{x}{y} = y^2 + C$

or, $x = y^3 + Cy$

16. $\mathbf{x^2y^3\, dx + 3x^2y\, dy + 2y\, dx = 0}$

Sol^n: Given that;

$$x^2y^3\, dx + 3x^2y\, dy + 2y\, dx = 0$$

or, $y(x^2y^2dx + 2dx) + 3xy\, x\, dy = 0$

or, $y(x^2y^2 + 2)dx = -3xy\,.\,x\, dy = 0$

or, $\frac{dy}{dx} = \frac{y(x^2y^2+2)}{3xy\,.\,x}$

Let, $v = xy$

Then,

$$y + x\frac{dy}{dx} = \frac{dv}{dx}$$

or, $\frac{dy}{dx} = \left(\frac{dv}{dx} - y\right)\frac{1}{x}$

or, $\left(\frac{dv}{dx} - y\right)\frac{1}{x} = -\frac{y(x^2y^2+2)}{3xy.\, x}$

Using $xy = v$ and $y = \frac{v}{x}$

$$\left(\frac{dv}{dx} - \frac{v}{x}\right)\frac{1}{x} = -\frac{v(v^2+2)}{x.3vx}$$

or, $\frac{dv}{dx} - \frac{v}{x} = \frac{-(v^2+2)}{3x^2}$

or, $\frac{dv}{dx} = \frac{v}{x} - \frac{1}{3x}(v^2 + 2) = \frac{1}{x}\left(v - \frac{v^2+2}{3}\right)$

or, $\frac{dv}{dx} = \frac{1}{x}\left[\frac{3v-v^2-2}{3}\right]$

or, $\frac{3dv}{3v-v^2-2} = \frac{dx}{x}$

or, $\frac{3dv}{3v-v^2-2} + \frac{dx}{x} = 0$

or, $\frac{3dv}{v^2 - 2 \times \frac{3}{2}v + \left(\frac{3}{2}\right)^2 - \frac{1}{4}} + \frac{dx}{x} = 0$

or, $\frac{3dv}{\left(v-\frac{3}{2}\right)^2 - \left(\frac{1}{2}\right)^2} + \frac{dx}{x} = 0$

Integrating both sides, we get,

$$3\log\left[\frac{\left(v-\frac{3}{2}\right) \times \frac{1}{2}}{\left(v-\frac{3}{2}\right) + \frac{1}{2}}\right] + \log x = C$$

or, $3\log\left(\frac{v-2}{v-1}\right) + \log x = \log C$

or, $\log\left(\frac{v-2}{v-1}\right)^3 x = \log C$

or, $\left(\frac{v-2}{v-1}\right)^3 x = \log C$

or, $x(v - 2)^3 = C(v - 1)^3$

But $xy = v$

so, $x(xy - 2)^3 = C(xy - 1)^3$ is the required solution

17. $\dfrac{\mathbf{x\,dx + y\,dy}}{\mathbf{x\,dy - y\,dx}} = \dfrac{\sqrt{\mathbf{a^2-x^2-y^2}}}{\mathbf{x^2+y^2}}$

Sol^n: Given that;

$$\frac{x\,dx+y\,dy}{x\,dy-y\,dx} = \sqrt{\frac{a^2-x^2-y^2}{x^2+y^2}}$$

or, $a\sqrt{x^2+y^2}\dfrac{x\,dx+y\,dy}{\sqrt{a^2-x^2-y^2}} = \dfrac{x\,dx+y\,dy}{x^2+y^2}$

or, $a\sqrt{x^2+y^2}\dfrac{x\,dx+y\,dy}{\sqrt{1-\left(\frac{\sqrt{x^2+y^2}}{a}\right)^2}} = d\left(\tan^{-1}\dfrac{y}{x}\right)$

Integrating both sides, we get,

or, $\sin^{-1}\dfrac{\sqrt{x^2+y^2}}{a} = \tan^{-1}\dfrac{y}{x} + c$

or, $\dfrac{\sqrt{x^2+y^2}}{a} = \sin\left(\tan^{-1}\dfrac{y}{x} + c\right)$

$\therefore$ $\sqrt{x^2+y^2} = a\sin\left(\tan^{-1}\dfrac{y}{x} + c\right)$ is the required solution

18. $\mathbf{(x^2+y^2+2x)dx + 2y\,dy = 0}$

Sol^n: Given that;

$(x^2+y^2+2x)dx + 2y\,dy = 0$

or, $(x^2+y^2)dx + 2x\,dx + 2y\,dy = 0$

or, $(x^2+y^2)dx + d(x^2+y^2) = 0$

Dividing by x^2+y^2, we have,

$dx + \dfrac{d(x^2+y^2)}{x^2+y^2} = 0$

Integrating yields,

$x + \log(x^2+y^2) = C$

19. $\mathbf{x^2\,dy + xy\,dy + 2\sqrt{1-x^2y^2}\,dx = 0}$

Sol^n: Given that;

$x^2\,dy + xy\,dy + 2\sqrt{1-x^2y^2}\,dx = 0$

Dividing by $\sqrt{1-(xy)^2}$, we have,

$\dfrac{x^2\,dy+xy\,dy}{\sqrt{1-(xy)^2}} + 2\,dx = 0$

or, $\dfrac{x\,dy+y\,dy}{\sqrt{1-(xy)^2}} + \dfrac{2\,dx}{x} = 0$

or, $\dfrac{d(xy)}{\sqrt{1-(xy)^2}} + \dfrac{2\,dx}{x} = 0$

Integrating yields,

$\sin^{-1}xy + 2\log x = \log C$ which is the required solution

20. $\mathbf{x\,dx + y\,dy = a^2\dfrac{(x\,dy+y\,dx)}{x^2+y^2}}$

Sol^n: Given that;

$x\,dx + y\,dy = a^2\dfrac{(x\,dy+y\,dx)}{x^2+y^2}$

or, $x\,dx + y\,dy = a^2\left(\dfrac{\frac{x\,dy+y\,dx}{x^2}}{1+\frac{y^2}{x^2}}\right)$

or, $x\,dx + y\,dy = \left(\dfrac{a^2\,d\left(\frac{y}{x}\right)}{1+\left(\frac{y}{x}\right)^2}\right)$

Integration yields,

$\dfrac{x^2}{2} + \dfrac{y^2}{2} = a^2\tan^{-1}\dfrac{y}{x} + C$

$\therefore$ $x^2+y^2 = 2a^2\tan^{-1}\dfrac{y}{x} + C$

21. $\mathbf{(x^2+y^2)dx - 2xy\,dy = 0}$

Sol^n: Given that;

$(x^2+y^2)dx - 2xy\,dy = 0$

or, $y^2\,dx - 2xy\,dy = -x^2\,dx$

or, $\dfrac{2xy\,dy - y^2\,dx}{x^2} = dx$

or, $d\left(\dfrac{y^2}{x}\right) = dx$

Integration yields,

$\dfrac{y^2}{x} = x + C$

$\therefore$ $y^2 = x(x+C)$ is the required solution

Exercise 26

Solve the following equations.

1. $\mathbf{P^2 + P - 6 = 0}$

Solⁿ: Given,

$P^2 + P - 6 = 0$

or, $P^2 + 3P - 2P - 6 = 0$

or, $P(P+3) - 2(P+3) = 0$

or, $(P-2)(P+3) = 0$

Either,	Or,
$P - 2 = 0$	$P + 3 = 0$
or, $\frac{dy}{dx} - 2 = 0$	or, $\frac{dy}{dx} + 3 = 0$
or, $dy = 2\,dx$	or, $dy + 3\,dx = 0$
or, $y = 2x + C$	or, $y + 3x = C$

Hence, general solution is;

$$(y + 3x - C)(y - 2x - C) = 0$$

2. $\mathbf{P^2 + 2Px - 3x^2 = 0}$

Solⁿ: Given,

$P^2 + 2Px - 3x^2 = 0$

or, $P^2 + 3Px - Px - 3x^2 = 0$

or, $P(P + 3x) - x(P + 3x) = 0$

or, $(P - x)(P + 3x) = 0$

Either,	Or,
$P - x = 0$	$P + 3x = 0$
or, $\frac{dy}{dx} = x$	or, $\frac{dy}{dx} + 3x = 0$
or, $dy = x\,dx$	or, $dy + 3x\,dx = 0$
or, $y = \frac{x^2}{2} + C$	or, $y + \frac{3x^2}{2} = C$
or, $(2y - x^2 - d) = 0$	or, $2y + 3x^2 - d = 0$

Thus, general solution is;

$$(2y + 3x^2 - d)(2y - x^2 - d) = 0$$

3. $\mathbf{P^2 - P(e^x + e^{-x}) + 1 = 0}$

Solⁿ: Given,

$P^2 - P(e^x + e^{-x}) + 1 = 0$

or, $P^2 - Pe^x + Pe^{-x} + 1 = 0$

or, $P(P - e^x) - e^x(P - e^x) = 0$

or, $(P - e^x)(P - e^x) = 0$

Either,	Or,
$P - e^x = 0$	$P - e^x = 0$
or, $\frac{dy}{dx} - e^x = 0$	or, $dy = e^x\,dx$
or, $dy - e^x\,dx = 0$	or, $y = e^x + C$
or, $y - e^x dx = 0$	or, $y + e^x - C = 0$
or, $y - e^x - C = 0$	

Thus, general solution is;

$$(y - e^x - C)(y + e^x - C) = 0$$

4. $\mathbf{P(P^2 + xy) = P^2(x + y)}$

Solⁿ: Given,

$P^3 + Pxy = P^2x + P^2y$

or, $P^3 + Pxy - P^2x - P^2y = 0$

or, $P^3 - P^2x - P^2y + Pxy = 0$

or, $P^2(P - x) - Py(P - x) = 0$

or, $(P^2 - Py)(P - x) = 0$

or, $P(P - y)(P - x) = 0$

Either,	Or,	Or,
$P = 0$	$P - y = 0$	$P - x = 0$
or, $\frac{dy}{dx} = 0$	or, $\frac{dy}{dx} - y = 0$	or, $\frac{dy}{dx} - x = 0$
or, $y = C$	or, $\frac{dy}{y} - dx = 0$	or, $dy = x\,dx$
	or, $\log_e \frac{y}{C} = x$	or, $y = \frac{x^2}{2} + c$
	or, $\frac{y}{C} = e^x$	or, $\frac{y}{C} = e^x$
	$\because y = \log_e x = x = a^y$	or, $y = Ce^x$
	or, $y = Ce^x$	

or, $2y - x^2 - d = 0$

Thus, general solution is $(y - C)(2y - x^2 - d)(y - Ce^x) = 0$

5. $yP^2 + (x - y)P - x = 0$

Sol^n: Given,

$yP^2 + (x-y)P - x = 0$

or, $yP^2 + xP - yP - x = 0$

or, $yP(P-1) + x(P-1) = 0$

or, $(yP + x)(P - 1) = 0$

Either,

$yP + x = 0$

or, $\frac{ydy}{dx} + x = 0$

or, $ydy + x\,dx = 0$

or, $\frac{y^2}{2} + \frac{x^2}{2} = \frac{C}{2}$

or, $y^2 + x^2 - C = 0$

Or,

$P - 1 = 0$

or, $\frac{dy}{dx} - 1 = 0$

or, $dy = dx$

or, $y = x + C$

or, $y - x - C = 0$

Thus, general solution is $(y^2 + x^2 - C)(y - x - C) = 0$

6. $P^2 + 2Px + Py + 2xy = 0$

Sol^n: Given,

$P^2 + 2Px + Py + 2xy = 0$

or, $P^2 + Py + 2Px + 2xy = 0$

or, $P(P + y) + 2x(P + y) = 0$

or, $(P + 2x)(P + y) = 0$

Either,

$P + 2x = 0$

or, $\frac{dy}{dx} + 2x = 0$

or, $dy + 2x\,dx = 0$

or, $y + x^2 = C$

or, $y + x^2 - C = 0$

Or,

$P + y = 0$

or, $\frac{dy}{dx} + y = 0$

or, $\frac{dy}{y} + dx = 0$

or, $\log y + x = \log C$

or, $\log\frac{y}{C} = -x$

or, $\frac{y}{C} = e^{-x}$

or, $y = Ce^{-x}$

Thus, general solution is $(y + x^2 - C)(y - Ce^{-x}) = 0$

7. $P^3 + 2xP^2 - y^2P^2 - 2xy^2P = 0$

Sol^n: Given,

$P^3 + 2xP^2 - y^2P^2 - 2xy^2P = 0$

or, $P^2(P - y^2) + 2xP(P - y^2) = 0$

or, $(P^2 + 2xP)(P - y^2) = 0$

or, $P(P + 2x)(P - y^2) = 0$

Either,

$P = 0$

or, $\frac{dy}{dx} + 2x = 0$

or, $dy = 0$

or, $y = C$

or, $y - C = 0$

Or,

$P + 2x = 0$

or, $\frac{dy}{dx} + 2x = 0$

or, $dy + 2xdx = 0$

or, $y + x^2 = C$

or, $y + x^2 - C = 0$

Or,

$P - y^2 = 0$

or, $\frac{dy}{dx} - y^2 = 0$

or, $\frac{dy}{y^2} - dx = 0$

or, $y^{-2}dy - dx = 0$

or, $\frac{y^{-2+1}}{-2+1} - x = C$

or, $-y^{-1} - x = C$

or, $-\frac{1}{y} - x = C$

or, $-1 - xy = Cy$

or, $xy + Cy + 1 = 0$

Thus, general solution is $(y - C)(y + x^2 - C)(xy + Cy + 1) = 0$

8. $P^3 - P(x^2 + xy + y^2) + x^2y + xy^2 = 0$

Sol^n: Given,

$P^3 - Px^2 - Pxy - Py^2 + x^2y + xy^2 = 0$

or, $P(P^2 - x^2) - Pxy + x^2y - Py^2 + xy^2 = 0$

or, $P(P^2 - x^2) - xy(P - x) - y^2(P - x) = 0$

or, $P(P - x)(P + x) - xy(P - x) - y^2(P - x) = 0$

or, $(P - x)(P^2 + Px - xy - y^2) = 0$

or, $(P - x)[P^2 - y^2 + x(P - y)] = 0$

or, $(P - x)[(P - y)(P + y) + x(P - y)] = 0$

or, $(P - x)(P - y)(P + x + y) = 0$

Either,

$P - x = 0$

or, $\frac{dy}{dx} - x = 0$

Or,

$P - y = 0$

or, $\frac{dy}{dx} - y = 0$

or, $dy - xdx = 0$

or, $y - \frac{x^2}{2} = d$

or, $2y - x^2 - 2d = 0$

or, $2y - x^2 - C = 0$

or, $\frac{dy}{y} - dx = 0$

or, $\log y - x = \log C$

or, $\log \frac{y}{C} = x$

or, $\frac{y}{C} = e^x$

or, $y - Ce^x = 0$

Or,

$$P + x + y = 0$$

or, $\frac{dy}{dx} + x + y = 0$

or, $\frac{dy}{dx} + y = -x$

Here,

$$IF = e^{\int dx} = e^x$$

$$e^x\left(\frac{dy}{dx} + y\right) = xe^x$$

Integrating yields,

or, $ye^x = -\int xe^x\,dx$

or, $ye^x = -xe^x - \int e^x\,dx$

or, $ye^x = -(xe^x + \int e^x\,dx)$

or, $ye^x = xe^x + e^x + C$

or, $y = -x + 1 + Ce^x$

or, $y + x - 1 - Ce^x = 0$

Thus, general solution is;

$(y - Ce^x)(2y - x^2 - C)(y + x - 1 - Ce^x) = 0$

9. $\mathbf{x^2\left(\frac{dy}{dx}\right)^2 + xy\frac{dy}{dx} - 6y^2 = 0}$

Sol^n: Given,

$$x^2\left(\frac{dy}{dx}\right)^2 + xy\frac{dy}{dx} - 6y^2 = 0$$

or, $x^2P^2 + xyP - 6y^2 = 0$

or, $x^2P^2 + 3xyP - 2xyP - 6y^2 = 0$

or, $xP(xP + 3y) - 2y(xP + 3y) = 0$

or, $(xP - 2y)(xP + 3y) = 0$

Either,

$$xP - 2y = 0$$

or, $x\frac{dy}{dx} - 2y = 0$

or, $\frac{dy}{y} - \frac{2dx}{x} = 0$

or, $\log y - 2\log x = \log C$

or, $\log y - \log x^2 = \log C$

or, $\log \frac{y}{x^2} = \log C$

or, $\frac{y}{x^2} = C$

or, $y = Cx^2$

or, $y - Cx^2 = 0$

Or,

$$xP + 3y = 0$$

or, $x\frac{dy}{dx} + 3y = 0$

or, $x\,dy + 3y\,dx = 0$

or, $\frac{dy}{y} + \frac{3dx}{x} = 0$

or, $\log y + 3\log x = \log C$

or, $\log y + \log x^3 = \log C$

or, $\log yx^3 = \log C$

or, $yx^3 = C$

or, $yx^3 - C = 0$

Thus, general solution is $(yx^3 - C)(y - Cx^2) = 0$

10. $\mathbf{\left(\frac{dy}{dx}\right)^2 + 2y\cot x\frac{dy}{dx} = y^2}$

Sol^n: The given equation is;

$$\left(\frac{dy}{dx}\right)^2 + 2y\cot x\frac{dy}{dx} = y^2$$

The given equation is;

$$P^2 + 2y\cot x\,P - y^2 = 0$$

The equation is quadratic in P so,

$$P = \frac{-2y\cot x \pm \sqrt{4y^2\cot^2 x - 4\times(y^2)}}{2\times 1}$$

or, $P = -y\cot x \pm \sqrt{y^2\cot^2 x + y^2}$

$= -y\cot x \pm \sqrt{y^2\operatorname{cosec}^2 x}$

or, $P = -y\cot x \pm y\operatorname{cosec} x$

or, $P = \frac{-y\cos x}{\sin x} \pm \frac{y}{\sin x}$

or, $\frac{dy}{y} = \left(\frac{-\cos x}{\sin x} \pm \frac{y}{\sin x}\right)dx$

or, $\frac{dy}{y} = -\left(\frac{\cos x}{\sin x} \pm \frac{y}{\sin x}\right)dx$

Taking +ve sign

$$\frac{dy}{y} = -\frac{(\cos x+1)(1-\cos x)}{\sin x(1-\cos x)}dx$$

or, $\frac{dy}{y} = -\frac{\sin x}{(1-\cos x)}$

Integrating, we get,

$\log y = -\log(1-\cos x) + \log c$

or, $\log y \times (1-\cos x) = \log c$

or, $y - y\cos x = c \ldots$ (i)

Taking –ve sign

$$\frac{dy}{y} = -\left(\frac{\cos x-1}{\sin x}\right) = \frac{1-\cos x}{\sin x}$$

or, $\frac{dy}{y} = \frac{(1-\cos x)(1+\cos x)dx}{\sin x(1+\cos x)}$

or, $\frac{dy}{y} = \frac{\sin x}{1+\cos x}dx$

or, $\log y = -\log(1+\cos x) + \log C$

or, $\log y + \log(1+\cos x) = \log C$

or, $\log y(1+\cos x) = \log C$

or, $y(1+\cos x) = C \ldots$ (ii)

From equation (i) and (ii)

$y(1 \pm \cos x) = C$ is the required solution

Exercise 27

Find the general solution of the following differential equations.

1. $\mathbf{P^3x - P^2y - 1 = 0}$

Sol[n]: Given,

$P^3x - P^2y - 1 = 0$

or, $P^3x = P^2y + 1$

or, $x = \frac{y}{P} + \frac{1}{P^3}$ which is solvable for x,

So, differentiating both sides with respect to y, we get,

$$\frac{dx}{dy} = \frac{P\frac{dy}{dy} - y\frac{dP}{dy}}{P^2} + 3P^{-2}\frac{dP}{dy}$$

or, $\frac{dx}{dy} = \frac{P - y\frac{dP}{dy}}{P^2} - \frac{3}{P^4}\frac{dP}{dy}$

or, $\frac{1}{P} = \frac{P}{P^2} - \frac{y}{P^2}\frac{dP}{dy} - \frac{3}{P^2}\frac{dP}{dy}$

or, $\frac{y}{P^2}\frac{dP}{dy} = -\frac{3}{P^2}\frac{dP}{dy}$

or, $\frac{dP}{dy}\left(\frac{y}{P^2} + \frac{3}{P^2}\right) = 0$

or, $\frac{dP}{dy}\left(\frac{3+y}{P^2}\right) = 0$

or, $\frac{dP}{dy} = 0$

or, $dP = 0$

or, $P = C \ldots$ (i)

Solving equation (i) and $P^3x - P^2y - 1 = 0$ yields;

$C^3x - C^2y - 1 = 0$

2. $\mathbf{y = 2Px + P^3y^2}$

Sol[n]: Given,

$y = 2Px + P^3y^2$

or, $2Px = y - P^3y^2$

or, $x = \frac{y}{2P} - y^2P^2$ which is solvable for x,

So, differentiating both sides with respect to y, we get,

$$\frac{dx}{dy} = \frac{2P - 2y\frac{dP}{dy}}{4P^2} - 2yP^2 - y^2 \times 2P\frac{dP}{dy}$$

or, $\frac{1}{P} = \frac{2P}{4P^2} - \frac{2y}{4P^2}\frac{dP}{dy} - 2yP^2 - y^2 2P\frac{dP}{dy}$

or, $\frac{1}{P} - \frac{2P}{4P^2} + 2yP^2 = -\frac{dP}{dy}\left(\frac{2y}{4P^2} + y^2 2P\right)$

or, $\frac{1}{P} - \frac{2P}{4P^2} + 2yP^2 = -\frac{dP}{dy}\left(\frac{y}{2P^2} + 2Py^2\right)$

or, $\left(2yP^2 + \frac{1}{2P}\right) = -\frac{dP}{dy}\left(\frac{y + 4P^3y^2}{2P^2}\right)$

or, $\left(\frac{4yP^3 + 1}{2P}\right) = -\frac{dP}{dy}\left[\frac{y(1 + 4P^3y)}{2P^2}\right]$

or, $\left(\frac{2P^2}{2P}\right) = -\frac{dP}{dy}.y$

or, $P = -\frac{dP}{dy}.y$

or, $Pdy + y\,dP = 0$

or, $\frac{dy}{y} + \frac{dP}{P} = 0$

or, $yP = C \ldots$ (i)

Thus,

$$y = 2 \times \frac{C}{y}x + \frac{C^3}{y^3}y^2$$

or, $y^2 = 2Cx + C^3$

3. $x = 4P + 4P^3$

Sol^n: Given,

$x = 4P + 4P^3$ which is solvable for x,

So, differentiating both sides with respect to y, we get,

$$\frac{dx}{dy} = 4\frac{dP}{dy} + 12P^2\frac{dP}{dy}$$

or, $\frac{1}{P} = 4\frac{dP}{dy} + 12P^2\frac{dP}{dy}$

or, $\frac{1}{P} = (4 + 12P^2)\frac{dP}{dy}$

or, $dy = (4P + 12P^3)dP$

or, $y = 2P^2 + 3P^4 + C$

Hence,

$x = 4P + 4P^3$ and $y = 2P^2 + 3P^4 + C$ is the required solution

4. $\sin y \cos Px - \cos y \sin Px = P$

Sol^n: Given,

$\sin y \cos Px - \cos y \sin Px = P$

or, $\sin(y - Px) = P$ —(i)

or, $y - Px = \sin^{-1} P$

or, $y = Px + \sin^{-1} P$ which is solvable for y,

So, differentiating both sides with respect to x, we get,

$$\frac{dy}{dx} = P + x\frac{dP}{dx} + \frac{1}{\sqrt{1-P^2}}\frac{dP}{dx}$$

or, $P = P + \left(x + \frac{1}{\sqrt{1-P^2}}\right)\frac{dP}{dx}$

or, $0 = \frac{dP}{dx}\left(x + \frac{1}{\sqrt{1-P^2}}\right)$

or, $(x\sqrt{1 - P^2} + 1)dP = 0$

or, $dP = 0$

or, $P = C$

Thus, eq^n in (i)

$\sin(y - Cx) = C$ is the required solution

5. $y + Px = x^4P^2$

Sol^n: Given,

$y + Px = x^4P^2$

or, $y = -Px - x^4P^2$ which is solvable for y,

so, $\frac{dy}{dx} = -P - x\frac{dP}{dx} - x^4\,2P\frac{dP}{dx} - P^2 \times 4x^3$

or, $P = -P - 4x^3P^2 - \frac{dP}{dx}(x + 2Px^4)$

or, $2P + 4x^3P^2 = -\frac{dP}{dx}(1 + 2Px^3)x$

or, $2P(1 + 2Px^3) = -\frac{dP}{dx}(1 + 2Px^3)x$

or, $2P = -\frac{dP}{dx}x$

or, $2\frac{dx}{x} = -\frac{dP}{P}$

or, $2\log x + \log P = \log C$

or, $x^2P = C$

or, $P = \frac{C}{x^2}$

Thus, from the question,

so, $y + \frac{Cx}{x^2} = x^4\left(\frac{C}{x^2}\right)^2$

or, $y + \frac{C}{x} = x^4 \times \frac{C^2}{x^4} = C^2$

$xy + C = C^2$ is the required solution

6. $\mathbf{P^2 - Py + x = 0}$

Sol^n: Given,

$P^2 - Py + x = 0$

or, $yP = x + P^2$

or, $y = \frac{x}{P} + P$

Differentiating both sides with respect to x, we get,

$P = \frac{d}{dx}\left(\frac{x}{P} + P\right)$

or, $P = \left[P^{-1} + x(-P^{-2})\frac{dP}{dx}\right] + \frac{dP}{dx}$

or, $P = \frac{1}{P} - \frac{x}{P^2}\frac{dP}{dx} + \frac{dP}{dx}$

or, $\left(P - \frac{1}{P}\right) = \frac{dP}{dx}\left(1 - \frac{x}{P^2}\right)$

or, $\frac{dx}{dP}\left(\frac{P^2-1}{P}\right) = \left(1 - \frac{x}{P^2}\right)$

or, $-\frac{dx}{dP} + \frac{x}{P^2} \times \frac{P}{P^2-1} = \frac{P}{P^2-1}$

or, $\frac{dx}{dP} - \frac{x}{P(P^2-1)} = -\frac{P}{P^2-1}$

which is linear in x and P

$$I.F. = e^{\int -\frac{1}{P(P^2-1)}dP} = e^{\int\left(-\frac{1}{P} + \frac{1}{P^2-1}\right)dP}$$

$$= e^{-\log P + \frac{1}{2}\log(P^2-1)}$$

$$= e^{\log\frac{\sqrt{P^2-1}}{P}}$$

or, $I.F. = \frac{\sqrt{P^2-1}}{P}$

Multiplying equation (ii) by I.F. and integrating both sides, we have,

$$\frac{\sqrt{P^2-1}}{P}.x = \int \frac{\sqrt{P^2-1}}{P} \times \frac{P}{P^2-1}dP$$

or, $\sqrt{P^2-1}\,\frac{x}{P} = \int \frac{1}{P^2-1}dP$

or, $\sqrt{P^2-1}\,\frac{x}{P} = \log\left(P + \sqrt{P^2-1}\right) + C$

or, $x\sqrt{P^2-1} = P\left[\log\left(P + \sqrt{P^2-1}\right)\right] + C$

$$\therefore \left.\begin{array}{l} x = \dfrac{P\left[\log\left(P+\sqrt{P^2-1}\right)\right]+C}{\sqrt{P^2-1}} \\ x = Py - P^2 \end{array}\right\} \text{is the required solution}$$

7. $\mathbf{e^y - P^3 - P = 0}$

Sol^n: Given,

$e^y - P^3 - P = 0$

or, $e^y = P^3 + P$

or, $y = \log(P^3 + P)$ which is solvable for y,

So, differentiating both sides with respect to x, we get,

$$\frac{dy}{dx} = \frac{1}{P^3+P}\left(3P^2\frac{dP}{dx} + \frac{dP}{dx}\right)$$

or, $P = \frac{1}{P^3+P}\frac{dP}{dx}(3P^2+1)$

or, $\frac{P^4+P^2}{3P^2+1} = \frac{dP}{dx}$

or, $\frac{(3P^2+1)dP}{P^4+P^2} = dx$

or, $\frac{(3P^2+1)}{P^2(1+P^2)}dP = dx$

or, $\frac{1+P^2+2P^2}{P^2(1+P^2)}dP = dx$

or, $\left(\frac{2}{1+P^2} + \frac{1}{P^2}\right)dP = dx$

or, $\frac{2}{1+P^2}dP + \frac{1}{P^2}dP = dx$

or, $C + 2\tan^{-1}P - \frac{1}{P} = x$

Thus,

$$x = 2\tan^{-1}P - \frac{1}{P} + C$$

and $y = \log(P^3 + P)$ is the required solution

8. $\mathbf{4(xP^2 + yP) = y^4}$

Sol^n: Given,

$$4(xP^2 + yP) = y^4$$

or, $4xP^2 = y^4 - 4yP$

or, $4x = \frac{y^4}{P^2} - \frac{4yP}{P^2}$

or, $4x = \frac{y^4}{P^2} - \frac{4y}{P}$ which is solvable for x,

So, differentiating both sides with respect to y, we get,

$$4\frac{dx}{dy} = \frac{P^2 \times 4y^3 - y^4 \times 2P\frac{dy}{dy}}{P^4} - \frac{4P - 4y\frac{dP}{dy}}{P^2}$$

or, $\frac{4}{P} = \frac{4P^2y^3}{P^4} - \frac{2Py^4}{P^4}\frac{dP}{dy} - \frac{4}{P} - \frac{4y}{P^2}\frac{dP}{dy}$

or, $\frac{8}{P} - \frac{4P^2y^3}{P^4} = \frac{dP}{dy}\left(\frac{4y}{P^2} - \frac{2Py^4}{P^4}\right)$

or, $\left(\frac{8P^3 - 4P^2y^3}{P^4}\right) = \frac{dP}{dy}\left(\frac{4yP^2 - 2Py^4}{P^4}\right)$

or, $4P^2(2P - y^3) = \frac{dP}{dy}(2P - y^3)2Py$

or, $4P^2\,dy = 2Py\,dP$

or, $2P\,dy = y\,dP$

or, $2\frac{dy}{y} = \frac{dP}{P}$

or, $2\log y = \log P + \log C$

or, $y^2 = PC$

or, $P = \frac{y^2}{C}$

Thus, from the given equation, we have,

$$4x = \frac{y^4}{y^4} \times C^2 - \frac{4y}{y^2}.C$$

or, $4x = C^2 - \frac{4}{y}C$

9. $\mathbf{x = \frac{P}{\sqrt{1-P^2}} = a}$

Sol^n: Given,

$$x + \frac{P}{\sqrt{1-P^2}} = a$$

Differentiating both sides with respect to y, we get,

$$\frac{dx}{dy} + \frac{d}{dy}\left(\frac{P}{\sqrt{1-P^2}}\right) = 0$$

or, $\frac{1}{P} + \frac{dP}{dy}(1-P^2)^{-\frac{1}{2}} + P\left(-\frac{1}{2}\right)(1-P^2)^{-\frac{3}{2}}(-2P)\frac{dP}{dy} = 0$

or, $\frac{1}{P} + \frac{dP}{dy}\frac{1}{\sqrt{1-P^2}} + \frac{P^2}{(1-P^2)^{-\frac{3}{2}}}\frac{dP}{dy} = 0$

or, $\frac{1}{P}+\frac{dP}{dy}\left[\frac{1}{\sqrt{1-P^2}}+\frac{P^2}{\left(1-P^2\right)^{-\frac{3}{2}}}\right]=0$

or, $\frac{1}{P}+\frac{dP}{dy}\left[\frac{1-P^2+P^2}{\left(1-P^2\right)^{-\frac{3}{2}}}\right]=0$

or, $\frac{1}{P}+\frac{dP}{dy}\left[\frac{1}{\left(1-P^2\right)^{-\frac{3}{2}}}\right]=0$

or, $\frac{1}{P}=\frac{dP}{dy}\frac{1}{\left(1-P^2\right)^{-\frac{3}{2}}}$

or, $dy=-\frac{P\,dP}{\left(1-P^2\right)^{\frac{3}{2}}}$

Integrating both sides, we get,

$$y+C=-\int\frac{P}{\left(1-P^2\right)^{\frac{3}{2}}}\,dP$$

Put,

$1-P^2=t$

or, $-2P\,dP=dt$

or, $P\,dP=-\frac{dt}{2}$

So,

$$\int\frac{-P\,dP}{\left(1-P^2\right)^{\frac{3}{2}}}=\frac{1}{2}\int\frac{dt}{t^{\frac{3}{2}}}+C=\frac{t^{-\frac{3}{2}+1}}{-\frac{3}{2}+1}+C=\frac{-1}{\sqrt{t}}=\frac{-1}{\sqrt{1-P^2}}$$

$\therefore\quad y+C=\frac{-1}{\sqrt{1-P^2}}$

Squaring both sides, we get,

$(y+C)^2=\frac{1}{\left(1-P^2\right)}$

or, $1-P^2=\frac{1}{(y+1)^2}$

or, $P^2=1-\frac{1}{(y+1)^2}$

So, $(x-a)^2=\frac{P^2}{1-P^2}$ reduces to

$$(x-a)^2=\frac{1-\frac{1}{(y+1)^2}}{\frac{1}{(y+1)^2}}$$

or, $(x-a)^2=(y+1)^2-1$

or, $(x-a)^2-(y+1)^2=-1$

or, $(y+1)^2-(x-a)^2=1$ is the required solution.

10. $\mathbf{xP^3=a+bp}$

Solⁿ: Given,

$xP^3=a+bp$

or, $x=\frac{a}{P^3}+\frac{b}{P^2}$

or, $\frac{dx}{dy}=\frac{-3a}{P^4}-\frac{2b}{P^3}$

or, $\frac{1}{P}=\frac{-3a}{P^4}\frac{dP}{dy}-\frac{2b}{P^3}\frac{dP}{dy}$

or, $1=\left(\frac{-3a}{P^3}-\frac{2b}{P^2}\right)\frac{dP}{dy}$

or, $P^3dy=(-3a-2bP)dP$

or, $dy=-\frac{3a}{P^3}dP-\frac{2b}{P^2}dP$

Thus,

$y=\frac{3a}{2P^2}+\frac{2b}{P}$

and $x=\frac{a}{P^3}+\frac{b}{P^2}$ is the required solution

11. $\mathbf{y = 2Px + P^2}$

Sol^n: Given,

$$y = 2Px + P^2 \ldots \text{(i)}$$

Differentiating both sides with respect to x, we get,

$$P = 2P + 2x\frac{dP}{dx} + 2P\frac{dP}{dx}$$

or, $-P = 2\frac{dP}{dx}(x + P)$

or, $\frac{dx}{dP} = \frac{2(x+P)}{-P}$

or, $\frac{dx}{dP} + \frac{2}{P}x = -2 \ldots \text{(ii)}$

This is linear so,

$$\text{I.F.} = e^{\int \frac{2}{P} dP} = e^{2\log P} = P^2$$

Now, multiplying both sides by P^2, we get,

$$P^2\frac{dx}{dP} + 2Px = -2P^2$$

or, $\frac{d}{dP}(xP^2) = -2P^2$

Integrating both sides, we get,

$$xP^2 = \int -2P^2\, dP + k$$

or, $xP^2 = -\frac{2P^3}{3} + k$

or, $x = -\frac{2P}{3} + \frac{k}{P^2} \ldots \text{(iii)}$

Eliminating P from equation (i) and (iii), we have,

$$\left.\begin{array}{l} x = kP^2 - \frac{2P}{3} \\ y = 2Px + P^2 \end{array}\right\} \text{which is the required solution}$$

We can expand it as;

Since $y = 2Px + P^2$

$$P^2 + 2xP - y = 0 \text{ which is quadratic in P}$$

$\therefore \quad P = \frac{-2x \pm \sqrt{4x^2 - 4\times1(-y)}}{2\times1}$

or, $P = -x \pm \sqrt{x^2 + y}$

Hence,

Taking positive sign, we have,

$$P = -x + \sqrt{x^2 + y}$$

Putting the value of P in equation (iii);

$$xP^2 = -\frac{2}{3}P^3 + k$$

or, $xP^2 = -\frac{2}{3}P^3 + k$

or, $x\left(-x + \sqrt{x^2 + y}\right)^2 = -\frac{2}{3}\left(-x + \sqrt{x^2 + y}\right)^3 + k$

or, $x\left(x^2 + x^2 + y - 2x\sqrt{x^2 + y}\right) + \frac{2}{3}\left[-x^3 + (x^2 + y)^{\frac{3}{2}} + 3x^2\sqrt{x^2 + y} - 3x(x^2 + y)\right] = k$

or, $x\left(2x^2 + y - 2x\sqrt{x^2 + y}\right) + \frac{2}{3}\left[-4x^3 - 3xy + (x^2 + y)^{\frac{3}{2}} + 3x^2\sqrt{x^2 + y}\right] = k$

or, $2x^3 + xy - 2x^2\sqrt{x^2 + y} - \frac{8x^3}{3} - 2xy + \frac{2}{3}(x^2 + y)^{\frac{3}{2}} + 2x^2\sqrt{x^2 + y} = k$

or, $-\frac{2x^3}{3} - xy + \frac{2}{3}(x^2 + y)^{\frac{3}{2}} = k$

or, $-2x^3 - 3xy + 2(x^2 + y)^{\frac{3}{2}} = k$

or, $2(x^2 + y)^{\frac{3}{2}} = 2x^3 + 3xy + 3k$

Squaring gives;

$$(2x^2 + 3xy + 3k)^2 - 4(x^2 + y)^3 = 0$$

Let, $C = 3k$

Thus,

$$(2x^3 + 3xy + C)^2 - 4(x^2 + y)^3 = 0 \text{ is the required solution}$$

12. $\mathbf{P^3 - P(y + 3) + x = 0}$

Sol^n: Given,

$$x = -P^3 + P(y + 3) \ldots \text{(i)}$$

Differentiating both sides of equation (i) with respect to y, we get,

$$\frac{dx}{dy} = -3P^2\frac{dP}{dy} + P(1) + (y + 3)\frac{dP}{dy}$$

or, $\frac{1}{P} - P = \frac{dP}{dy}(-3P^2 + y + 3)$

or, $\frac{1-P^2}{P} = \frac{dP}{dy}(-3P^2 + y + 3)$

or, $\frac{dy}{dP} = \frac{P}{1-P^2}(-3P^2 + y + 3)$

or, $\frac{dy}{dP} - \frac{Py}{1-P^2} = \frac{P}{1-P^2}(-3P^2 + 3)$

or, $\frac{dy}{dP} - \frac{Py}{1-P^2} = 3P \ldots$ (ii)

This is linear, so,

$$\text{I. F.} = e^{\int \frac{-P}{1-P^2} dP} = e^{-\frac{1}{2}\log(1-P^2)} = e^{\log\sqrt{1-P^2}} = \sqrt{1-P^2}$$

Multiplying equation (ii) by I. F. $= \sqrt{1-P^2}$, we get,

$$\frac{d}{dP}\left(y\sqrt{1-P^2}\right) = 3P\sqrt{1-P^2}\, dP$$

Integrating both sides, we have,

$$y\sqrt{1-P^2} = \int 3P\sqrt{1-P^2}\, dP$$

or, $y\sqrt{1-P^2} = -\int 3\sqrt{t}\frac{dt}{2} + C$

Where,

$$t = 1 - P^2$$

$$y\sqrt{1-P^2} = -\frac{3}{2}\frac{t^{\frac{3}{2}}}{\frac{3}{2}} + C = -(1-P^2)^{\frac{3}{2}} + C$$

$\therefore$ $y\sqrt{1-P^2} + (1-P^2)^{\frac{3}{2}} = C \ldots$ (iii)

Eliminating P between equation (i) and (iii), we have,

$$\left.\begin{array}{l} x = p(y+3) - P^3 \\ y\sqrt{1-P^2} + (1-P^2)^{\frac{3}{2}} = C \end{array}\right\} \text{which is the required solution}$$

13. $\mathbf{x + yP = aP^2}$

Sol^n: Given,

$$yP = -x + aP^2 \ldots \text{(i)}$$

Differentiating both sides of equation (i) with respect to x, we get,

$$P = -\frac{d}{dx}(xP^{-1}) + a\frac{dP}{dx}$$

or, $P = -\left[P^{-1} + x(-P^{-2})\frac{dP}{dx}\right] + \frac{dP}{dx}$

or, $P = -\frac{1}{P} - \frac{x}{P^2}\left(\frac{dP}{dx}\right) + a\frac{dP}{dx}$

or, $\left(P + \frac{1}{P}\right) = \frac{dP}{dx}\left(\frac{x}{P^2} + a\right)$

or, $\frac{dx}{dP}\left(\frac{P^2+1}{P}\right) = \frac{x}{P^2} + a$

or, $\frac{dx}{dP} = \frac{x}{P(P^2+1)} + \frac{aP}{P^2+1}$

or, $\frac{dx}{dP} - \left[\frac{1}{P(P^2+1)}\right]x = \frac{aP}{P^2+1}$

This is linear so,

$$\text{I. F.} = e^{\int \frac{-1}{P(P^2+1)} dP} = e^{-\frac{1}{2} + \frac{P}{P^2+1}} = e^{-\log P + \frac{1}{2}\log(P^2+1)}$$

$$= P^{-1}\sqrt{P^2+1} = \frac{\sqrt{P^2+1}}{P}$$

$\therefore$ $\frac{\sqrt{P^2+1}}{P}\frac{dx}{dP} = \frac{\sqrt{P^2+1}}{P}\frac{1}{P^2+1}x = \frac{\sqrt{P^2+1}}{P} \times \frac{aP}{P^2+1}$

or, $\frac{d}{dP}\left(\frac{x\sqrt{P^2+1}}{P}\right) = \frac{a}{\sqrt{P^2+1}}$

Integrating both sides, we get,

$$\frac{x\sqrt{P^2+1}}{P} = \int \frac{a}{\sqrt{P^2+1}} dP = a\log\left(P + \sqrt{1+P^2}\right) + C$$

or, $(1+P^2)^{\frac{3}{2}} = P\left[a\log\left(P + \sqrt{1+P^2}\right)\right] + C$

Thus,

$$x + yP = aP^2$$

or, $x(1+P^2)^{\frac{1}{2}} = P\left[a\log\left(P + \sqrt{1+P^2}\right)\right] + C$

which is the required solution

14. $\mathbf{y = \sin P - P\cos P}$

Solⁿ: Given,

$$y = \sin P - P\cos P \ldots \text{(i)}$$

Differentiating both sides of equation (i) with respect to x, we get,

$$\frac{dy}{dx} = \frac{d}{dx}\sin P - \frac{d}{dx}(P\cos P)$$

or, $P = \cos P\frac{dP}{dx} + P\sin P\frac{dP}{dx} - \cos P\frac{dP}{dx}$

or, $P = P\sin P\frac{dP}{dx}$

or, $\text{cosec}\, P = \frac{dP}{dx}$

or, $dx = \sin P\ dP$

Integrating both sides, we have,

$$x = -\cos P + C$$

or, $-(x - C) = \cos P$

or, $\cos P = (C - x)$

Hence,

$$\sin P = \sqrt{1 - \cos^2 P} = \sqrt{1 - (C - x)^2}$$

or, $P\cos P = \sin P - y$

or, $P = \frac{\sin P - y}{\cos P}$

or, $P = \frac{\sqrt{1 - \cos^2 P} - y}{C - x}$

or, $P = \left[\frac{\sqrt{1 - (C - x)^2} - y}{C - x}\right]$

Thus,

$C - x = \cos P$ reduced to

$$C - x = \cos\left[\frac{\sqrt{1 - (C - x)^2} - y}{C - x}\right] \text{ is the required solution}$$

15. $\mathbf{P^2 - 2Px + 1 = 0}$

Solⁿ: Given,

$$P^2 - 2Px + 1 = 0$$

or, $2Px = P^2 + 1$

or, $2x = P + \frac{1}{P}$

Differentiating both sides with respect to y, we get,

$$\frac{2}{P} = \frac{dP}{dy} + \left(-\frac{1}{P^2}\right)\frac{dP}{dy}$$

or, $\frac{2}{P} = \left(1 - \frac{1}{P^2}\right)\frac{dP}{dy}$

or, $\frac{dy}{dP} = \frac{1}{2}\left(P - \frac{1}{P}\right)$

or, $dy = \frac{1}{2}\left(P - \frac{1}{P}\right)dP$

Integrating both sides, we have,

$$y = \frac{P^2}{4} - \frac{1}{2}\log P + C$$

Thus,

$$\left.\begin{aligned} x &= \frac{1}{2}\left(P + \frac{1}{P}\right) \\ y &= \frac{P^2}{4} - \frac{1}{2}\log P + C \end{aligned}\right\} \text{is the required solution}$$

16. $\mathbf{y = (1 + P)x + aP^2}$

Solⁿ: Given,

$$y = (1 + P)x + aP^2 \ldots \text{(i)}$$

Differentiating both sides of equation (i) with respect to x, we get,

$$P = 1 + P + x\left(\frac{dP}{dx}\right) + a \times 2P\frac{dP}{dx}$$

or, $0 = 1 + \frac{dP}{dx} + (x + 2aP)$

or, $-1 = \frac{dP}{dx} + (x + 2aP)$

or, $\frac{dx}{dP} + x = -2aP \ldots \text{(ii)}$

This is linear so, differentiating equation, we have,

$$\text{I.F.} = e^{\int 1\, dP} = e^P$$

$$\frac{d}{dP}(e^P \times x) = -2aP \times e^P$$

Integrating both sides, we get,

$$e^P \times x = -2a\int P \,.\, e^P\, dP$$

$$= -2a(Pe^P - e^P) + C$$
$$= -2ae^P(P + 1) + C$$

or, $x = 2a(1 - P) + Ce^{-P}$

Thus, from equation (i), we get,

$$y = (1 + P)[2a(1 - P) + Ce^{-P}] + aP^2$$
$$= 2a - aP^2 + (1 + P)Ce^{-P} \dots \text{(iv)}$$

Eliminating P between equation (iii) and (iv), we get,

$$\left.\begin{aligned} x &= 2a(1 - P) + Ce^{-P} \\ y &= 2a - aP^2 + (1 + P)Ce^{-P} \end{aligned}\right\} \text{which is the required solution}$$

Exercise 28

1. $y = Px + P - P^2$

Soln: Given that;

$$y = Px + P - P^2$$
$$= Px + P(1 - P)$$

This is Clairaut's equation

So, differentiating both sides with respect to x, we get,

$$\frac{dy}{dx} = P + x\frac{dP}{dx} + \frac{dP}{dx}(1 - P) + Px - \frac{dP}{dx}$$

or, $P = P + x\frac{dP}{dx} + \frac{dP}{dx}(1 - P) + Px - \frac{dP}{dx}$

or, $0 = \frac{dP}{dx}[x + 1 - P - P]$

Either,

$\frac{dP}{dx} = 0$ or, $P = C$

or, $x + 1 - 2P = 0$

or, $x + 1 = 2P$

or, $P = \frac{x+1}{2} \dots$ (i)

From $y = Px + P(1 - P)$ and equation (i), we have,

$$y = \left(\frac{x+1}{2}\right)x + \left(\frac{x+1}{2}\right)\left(1 - \frac{x+1}{2}\right)$$

or, $y = \left(\frac{x+1}{2}\right)\left(x + 1 - \frac{x+1}{2}\right) = \left(\frac{x+1}{2}\right)\left(\frac{x+1}{2}\right) = \frac{(x+1)^2}{4}$

$\therefore$ $4y = (x + 1)^2$ is the singular solution.

and from $P = C$ and given equation;

$y = Cx + C(1 - C)$ is the complete solution

2. $y = Px + \frac{a}{P}$

Soln: Given that;

$$y = Px + \frac{a}{P}$$

This is clairaut's equation, so, differentiating both sides with respect to x.

$$\frac{dy}{dx} = P + x\frac{dP}{dx} + ax - \frac{1}{P^2}\frac{dP}{dx}$$

$$P = P + x\frac{dP}{dx} - \frac{a}{P^2}\frac{dP}{dx}$$

or, $\frac{dP}{dx}\left(x - \frac{a}{P^2}\right) = 0$

Either,

$\frac{dP}{dx} = 0$ or, $P = C$

or, $x - \frac{a}{P^2} = 0$

or, $P^2 = \frac{a}{x}$

or, $P = \sqrt{\frac{a}{x}}$

Using it in $y = Px + \frac{a}{P}$

$$y = \sqrt{\frac{a}{x}}x + \frac{a\sqrt{x}}{\sqrt{x}}$$

or, $y = \sqrt{ax} + \sqrt{ax} = 2\sqrt{ax}$

i.e. $y^2 = 4ax$ is the singular solution

and using $P = C$ in $y = Px + \frac{a}{P}$, we get,

$y = Cx + \frac{a}{C}$ is the complete solution

3. $\mathbf{Py = P^2(x - b) + a}$

Sol^n: The given equation is;

$$y = P(x - b) + \frac{a}{P}$$

or, $y = Px + \frac{a}{P} - Pb$

This equation is of the form $y = Px + f(P)$ so is of the clairaut's form.

Differentiating both sides with respect to x, we get,

$$\frac{dy}{dx} = P + x\frac{dP}{dx} - \frac{a}{P^2}\frac{dP}{dx} - b\frac{dP}{dx}$$

or, $P = P + x\frac{dP}{dx} - \frac{a}{P^2}\frac{dP}{dx} - b\frac{dP}{dx}$

or, $\frac{dP}{dx}\left[x - \frac{a}{P^2} - b\right] = 0$

Either,

$\frac{dP}{dx} = 0$

or, $dP = 0$

or, $P = C$

So, complete solution is;

$$Cy = C^2(x - b) + a$$

Again,

if $x - \frac{a}{P^2} - b = 0$

or, $\frac{a}{P^2} = x - b$

or, $P^2 = \frac{a}{x-b}$

so, $P = \sqrt{\frac{a}{x-b}}$ … (i)

Eliminating P from equation (i) and $Py = P^2(x - b) + a$

$$\sqrt{\frac{a}{x-b}}\cdot y = \frac{a}{(x-b)}(x - b) + a$$

or, $y\frac{\sqrt{a}}{\sqrt{x-b}} = a + a = 2a$

or, $y^2a = 4a^2(x - b)$

or, $y^2 = 4a(x - b)$ is the required singular solution

4. $\mathbf{y = Px + aP - aP^2}$

Sol^n: Given,

$$y = Px + aP - aP^2$$
$$= Px + aP(1 - P) \dots \text{(i)}$$

or, $P = P + x\frac{dP}{dx} + a\frac{dP}{dx}(1 - P) + aP\left(-\frac{dP}{dx}\right)$

or, $\frac{dP}{dx}(x + a - aP - aP) = 0$

or, $\frac{dP}{dx}(x + a - 2aP) = 0$

If $\frac{dP}{dx} = 0$

Then,

$P = C$ … (ii)

Thus,

$$y = Cx + ac - aC^2 \qquad \text{OR}$$

$$x + a - 2aP = 0$$

or, $x + a = 2aP$

$\therefore \quad P = \frac{x+a}{2a} \ldots \text{(iii)}$

Eliminating P from equation (i) and (iii), we have,

$$y = \left(\frac{x+a}{2a}\right)x + a\left(\frac{x+a}{2a}\right)\left(1 - \frac{x+a}{2a}\right)$$

or, $y = \frac{x+a}{2a}\left[x + a\left(1 - \frac{x+a}{2a}\right)\right]$

or, $y = \frac{x+a}{2a}\left[x + a - \frac{(x+a)}{2}\right]$

or, $y = \left(\frac{x+a}{2a}\right)\left[\frac{2(x+a)-(x+a)}{2}\right]$

or, $y = \frac{x+a}{2a} \times \frac{x+a}{2}$

or, $4ay = (x + a)^2$

Hence,

$$(x + a)^2 = 4ay$$

and $y = Cx + aC - aC^2$ is the required solution

5. $\mathbf{(y - Px)^2(1 + P^2) = a^2P^2}$

Sol^n: Given,

$$(y - Px)^2(1 + P^2) = a^2P^2$$

or, $(y - Px)\sqrt{1 + P^2} = \sqrt{a^2P^2} = ap$

or, $y - Px = \frac{ap}{\sqrt{1+P^2}}$

or, $y = Px + \frac{ap}{\sqrt{1+P^2}}$ which is the Clairaut's equation

Differentiating both sides with respect to x, we get,

$$\frac{dy}{dx} = P + x\frac{dP}{dx} + \left[\frac{\sqrt{1+P^2} \times a\frac{dP}{dx} - ap \times \frac{2P}{2\sqrt{1+P^2}}\frac{dP}{dx}}{\left(1+P^2\right)}\right]$$

or, $P = P + x\frac{dP}{dx} + \frac{dP}{dx}\left[\frac{a\sqrt{1+P^2}}{\left(1+P^2\right)} - \frac{ap^2}{\left(1+P^2\right)\sqrt{\left(1+P^2\right)}}\right]$

or, $0 = \frac{dP}{dx}\left[x + a\left\{\frac{1+P^2 - P^2}{\left(1+P^2\right)\sqrt{\left(1+P^2\right)}}\right\}\right]$

or, $0 = \frac{dP}{dx}\left[x + \frac{a}{\left(1+P^2\right)\sqrt{\left(1+P^2\right)}}\right]$

Either,

$$\frac{dP}{dx} = 0$$

or, $P = C$

so, complete solution is;

$$(y - Cx)^2(1 + C^2) = a^2C^2$$

Thus,

$$x = -\frac{a}{\left(1+P^2\right)^{\frac{3}{2}}}$$

or, $(x)^{\frac{2}{3}} = \frac{a^{\frac{2}{3}}}{\left(1+P^2\right)}$

Also, from $y = Px + \frac{ap}{\sqrt{1+P^2}}$

$$y = \frac{-Pa}{\left(1+P^2\right)^{\frac{3}{2}}} + \frac{ap}{\sqrt{1+P^2}} = \frac{aP\left[-1+1+P^2\right]}{\left(1+P^2\right)^{\frac{3}{2}}}$$

Thus,

$$y = \frac{aP^3}{\left(1+P^2\right)^{\frac{3}{2}}}$$

or, $(y)^{\frac{2}{3}} = \frac{a^{\frac{2}{3}}P^2}{\left(1+P^2\right)}$

$$\therefore \quad (x)^{\frac{2}{3}} + (y)^{\frac{2}{3}} = \frac{a^{\frac{2}{3}}}{(1+P^2)} + \frac{a^{\frac{2}{3}}.1}{(1+P^2)} = \frac{a^{\frac{2}{3}}(1+P^2)}{(1+P^2)} = a^{\frac{2}{3}}$$

Thus,

$(x)^{\frac{2}{3}} + (y)^{\frac{2}{3}} = a^{\frac{2}{3}}$ is the singular solution

6. **$P = \log(Px - y)$**

Sol^n: Given,

$$P = \log(Px - y)$$

or, $\log(Px - y) = P$

or, $Px - y = e^P$

or, $y = Px - e^P$ is the clairaut's equation, so,

Differentiating both sides with respect to x, we get,

$$\frac{dy}{dx} = P + x\frac{dP}{dx} - e^P \frac{dP}{dx}$$

or, $P = P + \frac{dP}{dx}(x - e^P)$

or, $\frac{dP}{dx}(x - e^P) = 0$

Either,

$$\frac{dP}{dx} = 0$$

or, $P = C$

So, complete solution is;

$$y = Cx - e^C$$

or, $x = e^P$

or, $P = \log x$

Thus, singular solution is;

$$y = x\log x - e^{\log x} = x\log x - x$$

7. **$(y + 1)P - xP^2 + 2 = 0$**

Sol^n: Given,

$$(y + 1)P - xP^2 + 2 = 0$$

Thus it can be written as;

$$yP = xP^2 - P - 2$$

or, $y = xP - 1 - \frac{2}{P}$ which is the Clairaut's equation, so,

Differentiating both sides with respect to x, we get,

$$\frac{dy}{dx} = P + x\frac{dP}{dx} - \frac{2}{P^2}\frac{dP}{dx}$$

or, $P = P + \frac{dP}{dx}\left(x + \frac{2}{P^2}\right)$

or, $\frac{dP}{dx}\left(x + \frac{2}{P^2}\right) = 0$

Either,

$$\frac{dP}{dx} = 0$$

or, $P = C$

so, it's complete solution is;

$$(y + 1)C - xC^2 + 2 = 0$$

and if $x + \frac{2}{P^2} = 0$

$$P^2 = -\frac{2}{x}$$

Thus, using it in equation;

$$(y + 1)P = xC^2 - 2$$

or, $(y + 1)^2P^2 = (xP^2 - 2)^2$

Hence,

$$(y + 1)^2\left(-\frac{2}{x}\right) = \left[x\left(-\frac{2}{x}\right) - 2\right]^2$$

or, $-\frac{2}{x}(y + 1)^2 = (-4)^2$

or, $-\frac{2}{x}(y + 1)^2 = 16$

or, $(y + 1)^2 = -8x$

i.e. $(y + 1)^2 + 8x = 0$ is the required singular solution

8. **$(xP - y)^2 = P^2 - 1$**

Sol^n: Given,

$$(xP - y)^2 = P^2 - 1$$

The given equation can be written as;

$$xP - y = \sqrt{P^2 - 1}$$

or, $y = xP + \sqrt{P^2 - 1}$ which is in Clairaut's form, so,

Differentiating both sides with respect to x, we get,

$$\frac{dy}{dx} = P + x\frac{dP}{dx} + \frac{2}{\sqrt{P^2-1}}\frac{dP}{dx}$$

or, $P = P + x\frac{dP}{dx} + \frac{2}{\sqrt{P^2-1}}\frac{dP}{dx}$

or, $0 = \frac{dP}{dx}\left(x + \frac{2}{\sqrt{P^2-1}}\right)$

Either,

$\frac{dP}{dx} = 0$

or, $P = C$

so, it's complete solution is;

$y = Cx + \sqrt{C^2-1}$

Again,

$x - \frac{P}{\sqrt{P^2-1}} = 0$

or, $x = \frac{P}{\sqrt{P^2-1}}$ ... (i)

From equation (i) and given equation;

$y = \frac{P}{\sqrt{P^2-1}} - \sqrt{P^2-1}$

or, $y = \frac{P^2-P^2+1}{\sqrt{P^2-1}}$

or, $y = \frac{1}{\sqrt{P^2-1}}$

or, $y^2 = \frac{1}{P^2-1}$

or, $P^2 - 1 = \frac{1}{y^2}$

or, $P^2 = 1 + \frac{1}{y^2} = \frac{y^2+1}{y^2}$ ... (ii)

Using equation (i) and (ii)

$x^2 = \frac{P^2}{P^2-1}$

or, $x^2 = \frac{\frac{1+y^2}{y^2}}{\frac{1+y^2}{y^2}-1} = \frac{1+y^2}{y^2} \times \frac{y^2}{1}$

$\therefore$ $x^2 = 1 + y^2$ is the required general solution

Exercise 29

1. $\mathbf{\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 0}$

Sol^n: Here,

$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 0$

or, $(D^2 + 3D + 2)y = 0$

So, it's auxiliary equation is;

$m^2 + 3m + 2 = 0$

or, $m^2 + 2m + m + 2 = 0$

or, $m(m + 2) + 1(m + 2) = 0$

or, $(m + 1)(m + 2) = 0$

Either $m = -1$ or $m = -2$

Thus, $y = C_1e^{-x} + C_2e^{-2x}$ is the required general solution.

2. $\mathbf{\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 4y = 0}$

Sol^n: Here,

$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 4y = 0$

or, $(D^2 + 5D + 4)y = 0$

So, it's auxiliary equation is;

$m^2 + 5m + 4 = 0$

or, $m^2 + 4m + m + 4 = 0$

or, $m(m + 4) + 1(m + 4) = 0$

or, $(m + 1)(m + 4) = 0$

Either $m = -1$ or $m = -4$

Thus, $y = C_1e^{-x} + C_2e^{-4x}$ is the required general solution.

3. $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 0$

Sol^n: Here,

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 0$$

or, $(D^2 + 2D + 1)y = 0$

So, it's auxiliary equation is;

$$m^2 + 2m + 1 = 0$$

or, $(m + 1)(m + 1) = 0$

Either $m = -1$ or $m = -1$

Since roots are equal;

$y = (C_1 + C_{2x})e^{-x}$ is the required general solution.

4. $\frac{d^2y}{dx^2} + \frac{dy}{dx} = 0$

Sol^n: Here,

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} = 0$$

or, $(D^2 + D)y = 0$

So, it's auxiliary equation is;

$$m^2 + m = 0$$

or, $m(m + 1) = 0$

Either $m = 0$ or $m = -1$

so, $y = C_1e^{0x} + C_2e^{-x}$

$= C_1 + C_2e^{-x}$ is the required general solution.

5. $\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 25y = 0$

Sol^n: Here,

$$\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 25y = 0$$

or, $(D^2 + 6D + 25)y = 0$

So, it's auxiliary equation is;

$$m^2 + 6m + 25 = 0$$

or, $m = \frac{-6 \pm \sqrt{36-100}}{2}$

or, $m = \frac{-6 \pm 8i}{2}$

or, $m = -3 \pm 4i$

Thus, the required general solution is;

$$y = e^{-3x}(A\cos 4x + B\sin 4x)$$

6. $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 13y = 0$

Sol^n: Here,

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 13y = 0$$

or, $(D^2 + 4D + 13)y = 0$

So, it's auxiliary equation is;

$$m^2 + 4m + 13 = 0$$

or, $m = \frac{-4 \pm \sqrt{16-52}}{2}$

or, $m = \frac{-4 \pm \sqrt{36i^2}}{2}$

or, $m = \frac{-4 \pm 6i}{2} = -2 \pm 3i$

Thus, the required general solution is;

$$y = e^{-2x}(A\cos 3x + B\sin 3x)$$

7. $16\frac{d^2y}{dx^2} + 24\frac{dy}{dx} + 9y = 0$

Sol^n: Here,

$$16\frac{d^2y}{dx^2} + 24\frac{dy}{dx} + 9y = 0$$

or, $(16D^2 + 24D + 9)y = 0$

So, it's auxiliary equation is;

$$16m^2 + 24m + 9 = 0$$

or, $16m^2 + 12m + 12m + 9 = 0$

or, $4m(m + 3) + 3(4m + 3) = 0$

or, $(4m + 3)(4m + 3) = 0$

Either $m = -\frac{3}{4}$ or $m = -\frac{3}{4}$

Since roots are equal so the required general solution is;

$$y = (C_1 + C_2x)e^{-\frac{3}{4}x}$$

8. $(D+3)^2 = 0$

Solⁿ: Here,

$$(D+3)^2 = 0$$

So, it's auxiliary equation is;

$$(m+3)^2 = 0$$

so, $m = -3$ or $m = -3$

Since roots are equal so the required general solution is;

$$y = (C_1 + C_2x)e^{-3x}$$

9. $(D^2 + 3aD - 4a^2)y = 0$

Solⁿ: Here,

$$(D^2 + 3aD - 4a^2)y = 0$$

So, it's auxiliary equation is;

$$m^2 + 3am - 4a^2 = 0$$

or, $m^2 + 4am - am - 4a^2 = 0$

or, $m(m+4a) - a(m+4a) = 0$

or, $(m-a)(m+4a) = 0$

Either $m = a$ or $m = -4a$

Thus, the required general solution is;

$$y = C_1e^{ax} + C_2e^{-4ax}$$

10. Solve $\frac{d^2x}{dt^2} + \mu x = 0, \mu > 0$ given that $x = a$, and $\frac{dx}{dt} = 0$ when $t = \frac{\mu}{2\sqrt{\mu}}$.

Solⁿ: Here,

$$\frac{d^2x}{dt^2} + \mu x = 0$$

or, $(D^2 + \mu x) = 0$

So, it's auxiliary equation is;

$$m^2 + \mu = 0$$

or, $m^2 = -\mu$

or, $m = \pm\sqrt{\mu}i$

Thus, the required general solution is;

$$x = A\cos\sqrt{\mu}t + B\sin\sqrt{\mu}t$$

When,

$$t = \frac{\pi}{2\sqrt{\mu}}; x = a$$

$$a = A\cos\frac{\pi}{2} + B\sin\frac{\pi}{2}$$

or, $a = B \times 1$

$\therefore$ $B = a$

and $\frac{dx}{dt} = -A\sin\sqrt{\mu}t \times \sqrt{\mu} + A\cos\sqrt{\mu}t \times \sqrt{\mu}$

or, $0 = -A\sqrt{\mu} \times \frac{\sin\pi}{2\sqrt{\mu}} + A\cos\sqrt{\mu} \times \frac{\sin\pi}{2\sqrt{\mu}} \times \sqrt{\mu}$

or, $0 = -A\sqrt{\mu} \times 1$

$\therefore$ $A = 0$

Thus, the required general solution is;

$$x = a\sin\sqrt{\mu}t$$

11. $\frac{d^2x}{dt^2} - 3\frac{dx}{dt} + 2x = 0$ given that $x = 1$ when $t = 0$ and $\frac{dx}{dt} = 0$ when $t = 0$.

Solⁿ: Here,

$$\frac{d^2x}{dt^2} - 3\frac{dx}{dt} + 2x = 0$$

or, $D^2 - 3D + 2 = 0$

So, it's auxiliary equation is;

$$m^2 - 3m + 2 = 0$$

or, $m^2 - m - 2m + 2 = 0$

or, $m(m-1) - 2(m-1) = 0$

or, $(m-1)(m-2) = 0$

Either $m = 1$ or $m = 2$

Thus, the required general solution is;

$$x = C_1e^t + C_2e^{2t}$$

$$x = C_1e^t + C_2e^{2t}$$

When,

$$t = 0, x = 1$$

so, $1 = C_1e^0 + C_2e^0$

or, $1 = C_1 + C_2 \ldots$ (i)

When,

$$t = 0, \frac{dx}{dt} = 0$$

so, $\frac{dx}{dt} = C_1e^t + 2C_2e^{2t}$

or, $0 = C_1e^0 + 2C_2e^0$

or, $0 = C_1 + 2C_2 \ldots$ (ii)

From equation (i) and (ii), we have,

$$C_1 + 2C_2 = 0$$
$$\underline{\underset{-}{C_1} + \underset{-}{C_2} = \underset{-}{1}}$$
$$\therefore \quad C_2 = -1$$

Again, from equation (i), we have,

$$C_1 = 2$$

Thus, the required general solution is;

$$x = 2e^t - e^{2t}$$

12. $\frac{d^2y}{dx^2} + y = 0$ **given that** $y = 4$ **when** $x = 0$ **and** $\frac{dy}{dx} = 0$ **when** $x = 0$.

Solⁿ: Here,

$$\frac{d^2y}{dx^2} + y = 0$$

or, $(D^2 + 1) = 0$

So, it's auxiliary equation is;

$$m^2 + 1 = 0$$

or, $m^2 = -1$

or, $m = \pm i$

Thus, the required general solution is;

$y = A\cos x + B\sin x$

When,

$x = 0, y = 4$

$4 = A\cos 0° + B\sin 0°$

or, $A = 4$

Again,

$$\frac{dy}{dx} = 0$$

When,

$x = 0$

so, $\frac{dy}{dx} = -A\sin x + B\cos x$

or, $0 = -A\sin 0° + B\cos 0°$

$\therefore$ $B = 0$

Thus, the required general solution is;

$$y = 4\cos x$$

Exercise 30

Solve the following equations.

1. $(D^2 - 1)y = 5e^{2x}$

Solⁿ: Here,

$$(D^2 - 1)y = 5e^{2x}$$

It's auxiliary equation is;

$$m^2 - 1 = 0$$

or, $m = \pm 1$

so, C.F. $= C_1e^x + C_2e^{-x}$

For particular integral;

$$\text{P.I.} = \frac{5e^{2x}}{(D^2 - 1)} = \frac{5e^{2x}}{3} = \frac{5}{3}e^{2x}$$

Thus, the required general solution is;

$$y = \text{C.F.} + \text{P.I.} = C_1e^x + C_2e^{-x} + \frac{5}{3}e^{2x}$$

2. $(D - 2)^2y = e^{4x}$

Solⁿ: Here,

$$(D - 2)^2y = e^{4x}$$

It's auxiliary equation is;

$$(m - 2)^2 = 0$$

or, $m = 2, 2$

so, C.F. $= (C_1 + C_2x)e^{2x}$

For particular integral;

$$\text{P.I.} = \frac{e^{4x}}{(D-2)^2} = \frac{e^{4x}}{(4-2)^2} = \frac{e^{4x}}{4}$$

Thus, the required general solution is;

$$y = (C_1 + C_2x)e^{2x} + \frac{e^{4x}}{4}$$

3. $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 3y = 2e^{3x}$

Solⁿ: Here,

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 3y = 2e^{3x}$$

or, $(D^2 - 4D + 3)y = 2e^{3x}$

It's auxiliary equation is;

$m^2 - 4m + 3 = 0$

or, $m^2 - m - 3m + 3 = 0$

or, $m(m-1) - 3(m-1) = 0$

Either $m = 1$ or $m = 3$

Thus,

C.F. $= C_1e^x + C_2e^{3x}$

For particular integral;

$$\text{P.I.} = \frac{2e^{3x}}{(D^2-4D+3)} = \frac{x.2e^{3x}}{(2D-4)} = \frac{2xe^{3x}}{6-4} = xe^{3x}$$

Thus, the required general solution is;

$y = C_1e^x + C_2e^{3x} + xe^{3x}$

4. $\frac{d^2y}{dx^2} + 4y = \sin 2x$

Solⁿ: Here,

$\frac{d^2y}{dx^2} + 4y = \sin 2x$

or, $(D^2 + 4D)y = \sin 2x$

It's auxiliary equation is;

$m^2 - 4 = 0$

or, $m = \pm 2i$

Thus,

C.F. $= (A\cos 2x + B\sin 2x)$

For particular integral;

$$\text{P.I.} = \frac{\sin 2x}{(D^2+4)} = x\frac{1}{2D}\sin 2x = \frac{x(-\cos 2x)}{2\times 2} = -\frac{x}{4}\cos 2x$$

Thus, the required general solution is;

$y = A\cos 2x + B\sin 2x - \frac{x}{4}\cos 2x$

5. $(D^2 + 16)y = \cos 4x$

Solⁿ: Here,

$(D^2 + 16)y = \cos 4x$

It's auxiliary equation is;

$m^2 + 16 = 0$

or, $m^2 = -16$

or, $m = \pm 4i$

Thus,

C.F. $= (A\cos 4x + B\sin 4x)$

For particular integral;

$$\text{P.I.} = \frac{\cos 4x}{(D^2+16)} = \frac{x\cos 4x}{2D} = \frac{x\sin 4x}{2\times 4} = \frac{x}{8}\sin 4x$$

Thus, the required general solution is;

$y = A\cos 4x + B\sin 4x + \frac{x}{8}\sin 4x$

6. $\frac{d^2y}{dx^2} + y = \cos^2 x$

Solⁿ: Here,

$\frac{d^2y}{dx^2} + y = \cos^2 x$

or, $(D^2 + 1)y = \cos^2 x$

It's auxiliary equation is;

$m^2 + 1 = 0$

or, $m = \pm i$

Thus,

C.F. $= (A\cos x + B\sin x)$

For particular integral;

$$\text{P.I.} = \frac{\cos^2 x}{(D^2+1)}$$

$$= \frac{1+\cos 2x}{2(D^2+1)} = \frac{1}{2}\left[\frac{e^{0x}}{D^2+1} + \frac{\cos 2x}{D^2+1}\right] = \frac{1}{2}\left[\frac{e^{0x}}{1} + \frac{\cos 2x}{-4+1}\right]$$

$$= \frac{1}{2}\left[1 - \frac{1}{3}\cos 2x\right]$$

Thus, the required general solution is;

$y = A\cos x + B\sin x + \frac{1}{2} - \frac{1}{6}\cos 2x$

7. $(D^2 - D - 2)y = \sin 2x + e^x$

Solⁿ: Here,

$(D^2 - D - 2)y = \sin 2x + e^x$

It's auxiliary equation is;

$m^2 - m - 2 = 0$

or, $m = \frac{1\pm\sqrt{1+4}}{2} = \frac{1\pm 3}{2}$

or, $m = -1, 2$

Thus,

$$C.F. = C_1e^{-x} + C_2e^{2x}$$

For particular integral;

$$P.I. = \frac{\sin 2x + e^x}{D^2 - D - 2}$$

$$= \frac{e^x}{D^2-D-2} + \frac{\sin 2x}{D^2-D-2}$$

$$= \frac{e^x}{1-1-2} + \frac{\sin 2x}{-4-D-2}$$

$$= \frac{e^x}{-2} - \frac{\sin 2x}{D+6}$$

$$= -\frac{1}{2}e^x - \frac{(D-6)\sin 2x}{(D+6)(D-6)}$$

$$= -\frac{e^x}{2} - \frac{(D-6)\sin 2x}{D^2-36}$$

$$= -\frac{e^x}{2} - \frac{(D-6)\sin 2x}{-4-36}$$

$$= -\frac{e^x}{2} + \frac{4}{40}[2\cos 2x - 6\sin 2x]$$

$$= -\frac{e^x}{2} + \frac{1}{20}[\cos 2x - 3\sin 2x]$$

Thus, the required general solution is;

$$y = C_1e^{-x} + C_2e^{2x} + \frac{1}{20}(\cos 2x - 3\sin 2x)$$

8. $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = e^{2x}\sin x$

Sol^n: Here,

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = e^{2x}\sin x$$

or, $(D^2 + 3D + 2)y = e^{2x}\sin x$

It's auxiliary equation is;

$m^2 + 3m + 2 = 0$

or, $m^2 + m + 2m + 2 = 0$

or, $m(m+1) + 2(m+1) = 0$

or, $(m+2)(m+1) = 0$

Either $m = -2$ or -1

Thus,

$$C.F. = C_1e^{-2x} + C_2e^{-x}$$

For particular integral;

$$P.I. = \frac{e^{2x}\sin x}{D^2+3D+2}$$

$$= e^{2x}\left[\frac{\sin x}{(D+2)^2+3(D+2)+2}\right] = e^{2x}\left[\frac{\sin x}{D^2+4D+4+3D+6+2}\right]$$

$$= e^{2x}\left[\frac{\sin x}{D^2+7D+12}\right] = e^{2x}\left[\frac{\sin x}{-1+12+7D}\right]$$

$$= e^{2x}\left[\frac{\sin x}{7D+11}\right] = e^{2x}\frac{(7D-11)}{49D^2-121}\sin x$$

$$= e^{2x}\frac{(7D\sin x - 11\sin x)}{-49-121} = e^{2x}\frac{(7\cos x - 11\sin x)}{-170}$$

$$= e^{2x}\frac{(11\sin x - 7\cos x)}{170}$$

Thus, the required general solution is;

$$y = C.F. + P.I.$$

$$= C_1e^{-x} + C_2e^{-2x} + \frac{e^{2x}}{170}(11\sin x - 7\cos x)$$

9. $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = x^2e^{3x}$

Sol^n: Here,

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = x^2e^{3x}$$

or, $(D^2 - 2D + 1)y = x^2e^{3x}$

It's auxiliary equation is;

$m^2 - 2m + 1 = 0$

or, $m^2 - m - m + 1 = 0$

or, $m(m-1) - 1(m-1) = 0$

or, $(m-1)(m-1) = 0$

Either $m = 1$ or 1

Thus,

$$C.F. = (A + Bx)e^x$$

For particular integral;

$$P.I. = \frac{x^2e^{3x}}{D^2-2D+1}$$

$$= e^{3x}\left[\frac{x^2}{(D+3)^2-2(D+3)+1}\right]$$

$$= e^{3x}\left[\frac{x^2}{D^2+6D+9-2D-6+1}\right]$$

$$= e^{3x}\left[\frac{x^2}{D^2+4D+4}\right] = e^{3x}\left[\frac{x^2}{4\left(1+\frac{D^2+4D}{4}\right)}\right]$$

$$= \frac{e^{3x}}{4}\left[\left(1+\frac{D^2+4D}{4}\right)^{-1} x^2\right]$$

$$= \frac{e^{3x}}{4}\left[1-\left(\frac{D^2+4D}{4}\right)+D^2\right]x^2$$

$$= \frac{e^{3x}}{4}\left(1+\frac{3D^2}{4}-D\right)x^2 = \frac{e^{3x}}{4}\left(x^2+\frac{6}{4}-2x\right)$$

$$= \frac{e^{3x}}{4}\left(x^2+\frac{3}{2}-2x\right)$$

$$= \frac{e^{3x}}{8}(2x^2-4x+3)$$

Thus, the required general solution is;

$$y = C.F. + P.I.$$

$$= (C_1 + C_2x)e^x + \frac{e^{3x}}{8}(2x^2-4x+3)$$

10. $(D^2+2)y = \sin\sqrt{2}x$

Sol^n: Given:

$$(D^2+2)y = \sin\sqrt{2}\,x$$

Here the auxiliary equation is;

$$m^2 + 2 = 0$$

or, $m = \pm\sqrt{2}i$

$$C.F. = A\cos\sqrt{2}x + B\sin\sqrt{2}x$$

$$P.I. = \frac{\sin\sqrt{2}x}{D^2+2} = x \times \frac{1}{20}\sin\sqrt{2}x$$

$$= \frac{x}{2}\left(\frac{-\cos\sqrt{2}x}{\sqrt{2}}\right) = -\frac{x}{2\sqrt{2}}\cos\sqrt{2}x$$

Hence the general solution is;

$$y = C.F. + P.I.$$

$$= A\cos\sqrt{2}x + B\sin\sqrt{2}x - \frac{x}{2\sqrt{2}}\cos\sqrt{2}x$$

11. $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 4y = e^x\cos x$

Sol^n: Here,

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 4y = e^x\cos x$$

or, $(D^2 - 2D + 4)y = e^x\cos x$

It's auxiliary equation is;

$$m^2 - 2m + 4 = 0$$

or, $m = \frac{2\pm\sqrt{4-16}}{2} = \frac{2\pm2\sqrt{3}i}{2} = 1\pm\sqrt{3}i$

Thus,

$$C.F. = e^x(C_1\cos\sqrt{3}x + C_2\sin\sqrt{3}x)$$

For particular integral;

$$P.I. = \frac{e^x\cos x}{D^2-2D+4}$$

$$= e^x\left[\frac{\cos x}{(D+1)^2-2(D+1)+4}\right] = e^x\left(\frac{\cos x}{D^2+2D+1-2D+2+4}\right)$$

$$= e^x\left(\frac{\cos x}{D^2+3}\right) = e^x\left(\frac{\cos x}{-1+3}\right) = \frac{e^x\cos x}{2}$$

Thus, the required general solution is;

$$y = C.F. + P.I.$$

$$= e^x(C_1\cos\sqrt{3}x + C_2\sin\sqrt{3}x) + \frac{e^x\cos x}{2}$$

12. $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y = x + \sin x$

Sol^n: Here,

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y = x + \sin x$$

or, $(D^2 + D - 2)y = x + \sin x$

It's auxiliary equation is;

$$m^2 + m - 2 = 0$$

or, $m = \frac{-1\pm\sqrt{1+8}}{2} = \frac{-1\pm3}{2}$

Either $m = -2$ or 1

Thus,

$$C.F. = C_1e^x + C_2e^{-2x}$$

For particular integral;

$$\text{P.I.} = \frac{x+\sin x}{D^2+D-2}$$
$$= \frac{x}{D^2+D-2} + \frac{\sin x}{D^2+D-2}$$
$$= \frac{x}{-2\left(1-\frac{D^2+D}{2}\right)} + \frac{\sin x}{-1+D-2}$$
$$= -\frac{1}{2}\left(1-\frac{D^2+D}{2}\right)^{-1} x + \frac{\sin x}{D-3}$$
$$= -\frac{1}{2}\left(1+\frac{D^2}{2}+\frac{D}{2}\right)x + \frac{(D+3)\sin x}{D^2-9}$$
$$= -\frac{1}{2}\left(X+\frac{1}{2}\right) + \frac{\cos x + 3\sin x}{-1-9}$$
$$= -\frac{x}{2} - \frac{1}{4} - \frac{\cos x}{10} - \frac{3\sin x}{10}$$
$$= -\frac{1}{4}(2x+1) - \frac{1}{10}(\cos x + 3\sin x)$$

Thus, the required general solution is;

$$y = \text{C.F.} + \text{P.I.}$$
$$= C_1 e^{2x} + C_2 e^{x} - \frac{1}{10}(\cos x + 3\sin x) - \frac{1}{4}(2x+1)$$

13. $(D^2 - 4D + 4)y = x^2 + e^{2x}$

Sol^n: Here,

$$(D^2 - 4D + 4)y = x^2 + e^{2x}$$

It's auxiliary equation is;

$$m^2 - 4m + 4 = 0$$

or, $(m-2)(m-2) = 0$

Either $m = 2$ or 2

Thus,

$$\text{C.F.} = (C_1 + C_2 x)e^{2x}$$

For particular integral;

$$\text{P.I.} = \frac{x^2 + e^{2x}}{D^2-4D+4} = \frac{x^2}{D^2-4D+4} + \frac{e^{2x}}{D^2-4D+4}$$
$$= \frac{1}{4}\left(\frac{x^2}{1+\frac{D^2-4D}{4}}\right)x^2 + \frac{x\,e^{2x}}{2D-4}$$

$$= \frac{1}{4}\left(1-\frac{D^2}{4}+\frac{4D}{4}+D^2\right)x^2 + \frac{x^2 e^{2x}}{2}$$
$$= \frac{1}{4}\left(x^2 - \frac{2}{4} + 2x + 2\right) + \frac{x^2 e^{2x}}{2}$$
$$= \frac{1}{4}\left(x^2 + 2x + \frac{3}{2}\right) + \frac{x^2 e^{2x}}{2}$$

Thus, the required general solution is;

$$y = \text{C.F.} + \text{P.I.}$$
$$= (C_1 + C_2 x)e^{2x} + \frac{1}{4}\left(x^2 + 2x + \frac{3}{2}\right) + \frac{x^2 e^{2x}}{2}$$

14. $(D-2)^2 y = x^2 e^{2x}$

Sol^n: Here,

$$(D-2)^2 y = x^2 e^{2x}$$

It's auxiliary equation is;

$$(m-2)^2 = 0$$

Either $m = 2$ or 2

Thus,

$$\text{C.F.} = (C_1 + C_2 x)e^{2x}$$

For particular integral;

$$\text{P.I.} = \frac{x^2 e^{2x}}{(D-2)^2}$$
$$= e^{2x}\left[\frac{x^2}{(D+2-2)^2}\right] = e^{2x}\left(\frac{x^2}{D^2}\right) = e^{2x}\left(\frac{x^3}{3D}\right) = e^{2x}\left(\frac{x^4}{12}\right)$$
$$= \frac{x^4 e^{2x}}{12}$$

Thus, the required general solution is;

$$y = \text{C.F.} + \text{P.I.}$$
$$= (C_1 + C_2 x)e^{2x} + \frac{x^4 e^{2x}}{12}$$

15. $(D^2 - 3D + 2)y = \cos hx$

Sol^n: Here,

$$(D^2 - 3D + 2)y = \cos hx$$

It's auxiliary equation is;

$$m^2 - 3m + 2 = 0$$

or, $m^2 - m - 2m + 2 = 0$

or, $m(m-1) - 2(m-1) = 0$

or, $(m-2)(m-1)=0$

Either $m = 2$ or 1

Thus,

$$C.F. = C_1e^{2x} + C_2e^{x}$$

For particular integral;

$$P.I. = \frac{\cos hx}{D^2-3D+2}$$

$$= \left[\frac{e^x+e^{-x}}{2(D^2-3D+2)}\right] = \frac{e^{-x}}{2(D^2-3D+2)} + \frac{e^x}{2(D^2-3D+2)}$$

$$= \frac{e^{-x}}{2(1+3+2)} + \frac{xe^x}{4D-6}$$

$$= \frac{e^{-x}}{12} - \frac{xe^x}{2}$$

Thus, the required general solution is;

$$y = C.F. + P.I. = C_1e^{2x} + C_2e^x + \frac{e^{-x}}{12} - \frac{xe^x}{2}$$

16. **$(D^2-1)y = \sin hx$**

Sol^n: Here,

$$(D^2-1)y = \sin hx$$

It's auxiliary equation is;

$$m^2 - 1 = 0$$

or, $m = \pm 1$

Thus,

$$C.F. = C_1e^x + C_2e^{-x}$$

For particular integral;

$$P.I. = \frac{\sin hx}{D^2-1}$$

$$= \left[\frac{e^x-e^{-x}}{2(D^2-1)}\right] = \frac{x}{2}\left(\frac{e^x-e^{-x}}{2D}\right) = \frac{x}{2}\left(\frac{e^x}{2}+\frac{e^{-x}}{2}\right)$$

$$= \frac{x}{2}\left(\frac{e^x+e^{-x}}{2}\right)$$

$$= \frac{x}{2}\cos hx$$

Thus, the required general solution is;

$$y = C.F. + P.I.$$

$$= C_1e^x + C_2e^{-x} + \frac{x}{2}\cos hx$$

17. **$(D^2+4)y = x\sin^2 x$**

Sol^n: Here,

$$(D^2+4)y = x\sin^2 x$$

It's auxiliary equation is;

$$m^2 + 4 = 0$$

or, $m = \pm 2i$

Thus,

$$C.F. = A\cos 2x + B\sin 2x$$

For particular integral;

$$P.I. = \frac{x\sin^2 x}{D^2+4}$$

$$= \frac{1}{2}\left[\frac{x(1-\cos 2x)}{D^2+4}\right] = \frac{1}{2}\left[\frac{x}{D^2+4} - \frac{x\cos 2x}{D^2+4}\right]$$

$$= \frac{1}{2}\left[\frac{\left(1+\frac{D^2}{4}\right)^{-1}x}{4} - \frac{\text{Real port of } xe^{i2x}}{D^2+4}\right]$$

$$= \frac{1}{8}\left[\left(1-\frac{D^2}{4}\right)x - \frac{1}{2}\,\frac{\text{Real port of } xe^{i2x}}{D^2+4}\right]$$

$$= \frac{1}{8}x - \frac{1}{2}R.\,e^{i2x}\frac{x}{(D+2i)^2+4}$$

$$= \frac{1}{8}x - \frac{1}{2}\text{Real port of } e^{i2x}\frac{x}{D^2+4iD}$$

$$= \frac{x}{8} - \frac{1}{2}\text{Real port of }\left[\frac{e^{i2x}.1}{4iD\left(1+\frac{D}{4i}\right)x}\right]$$

$$= \frac{x}{8} - \frac{1}{2}\text{Real port of }\left[e^{i2x}\frac{1}{4iD}\left(1+\frac{D}{4i}\right)^{-1}x\right]$$

$$= \frac{x}{8} - \frac{1}{2}\text{Real port of }\left[e^{i2x}\frac{1}{4iD}\left(1-\frac{D}{4i}\right)x\right]$$

$$= \frac{x}{8} - \frac{1}{2}\text{Real port of }\left[e^{i2x}\frac{1}{4iD}\left(x+\frac{1}{4i}\right)\right]$$

$$= \frac{x}{8} - \frac{1}{2}\text{Real port of }\left[(\cos 2x + i\sin 2x)\left(\frac{x^2}{8i}+\frac{x}{16}\right)\right]$$

$$= \frac{x}{8} - \frac{1}{2}\text{Real port of }\left[(\cos 2x + i\sin 2x)\left(\frac{-ix^2}{8}-\frac{x}{16}\right)\right]$$

$$= \frac{x}{8} + \frac{1}{32}\text{Real port of }\left[(\cos 2x + i\sin 2x)(2ix^2 - x)\right]$$

$$= \frac{x}{8} + \frac{1}{32}[-2x^2 \sin 2x - x\cos 2x]$$

$$= \frac{x}{8} - \frac{x}{32}\cos 2x - \frac{1}{16}x^2 \sin 2x$$

Thus, the required general solution is;

$$y = \text{C.F.} + \text{P.I.}$$

$$= A\cos 2x + B\sin 2x + \frac{x}{8} - \frac{1}{32}x\cos 2x - \frac{1}{16}x^2 \sin 2x$$

18. $(D^2 - 4)y = x\sin hx$

Sol^n: Here,

$$(D^2 - 4)y = x\sin hx$$

It's auxiliary equation is;

$$m^2 - 4 = 0$$

or, $m = \pm 2$

Thus,

$$\text{C.F.} = C_1e^{2x} + C_2e^{-2x}$$

For particular integral;

$$\text{P.I.} = \frac{x\sin hx}{D^2-4}$$

$$= \frac{x(e^x - e^{-x})}{2(D^2-4)} = \frac{xe^x}{2(D^2-4)} - \frac{xe^{-x}}{2(D^2-4)}$$

$$= \frac{e^x}{2}\left[\frac{x}{(D+1)^2-4}\right] - \frac{e^{-x}}{2}\left[\frac{x}{(D-1)^2-4}\right]$$

$$= \frac{e^x}{2}\left[\frac{x}{D^2+2D+1-4}\right] - \frac{e^{-x}}{2}\left[\frac{x}{D^2-2D+1-4}\right]$$

$$= \frac{e^x}{2}\left[\frac{x}{D^2+2D-3}\right] - \frac{e^{-x}}{2}\left[\frac{x}{D^2-2D-3}\right]$$

$$= \frac{e^x}{2}\left[\frac{x}{-3\left(1-\frac{D^2+2D}{3}\right)}\right] - \frac{e^{-x}}{2}\left[\frac{x}{-3\left(1-\frac{D^2-2D}{3}\right)}\right]$$

$$= \frac{e^x}{-6}\left(1 + \frac{D^2}{3} + \frac{2D}{3}\right)x + \frac{e^{-x}}{6}\left(1 + \frac{D^2}{3} - \frac{2D}{3}\right)x$$

$$= \frac{e^x}{-6}\left(x + \frac{2}{3}\right) + \frac{e^{-x}}{6}\left(x - \frac{2}{3}\right)$$

$$= \frac{-x}{3}\left(\frac{e^x - e^{-x}}{2}\right) - \frac{2}{18}\left(\frac{e^x + e^{-x}}{2}\right)$$

$$= \frac{-x}{3}\sin hx - \frac{2}{9}\left(\frac{e^x + e^{-x}}{2}\right)$$

$$= \frac{-x}{3}\sin hx - \frac{2}{9}\cos hx$$

Thus, the required general solution is;

$$y = \text{C.F.} + \text{P.I.}$$

$$= C_1e^{2x} + C_2e^{-2x} - \frac{x}{3}\sin hx - \frac{2}{9}\cos hx$$

19. $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = xe^x \sin x$

Sol^n: Here,

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = xe^x \sin x$$

or, $(D^2 - 2D + 1)y = xe^x \sin x$

It's auxiliary equation is;

$$m^2 - 2m + 1 = 0$$

or, $(m-1)^2 = 0$

or, $m = 1, 1$

Thus,

$$\text{C.F.} = (C_1 + C_2x)e^x$$

For particular integral;

$$\text{P.I.} = \frac{xe^x \sin x}{D^2-2D+1}$$

$$= e^x\left[\frac{x\sin x}{(D+1)^2-2(D+1)+1}\right]$$

$$= e^x\left[\frac{x\sin x}{D^2+2D+1-2D-2+1}\right]$$

$$= e^x\left[\frac{x\sin x}{D^2}\right]$$

$$= e^x\,.\,\text{Imaginary port of } e^{ix}\,.\left[\frac{x}{D^2}\right]$$

$$= e^x\,.\,\text{Imaginary port of } e^{ix}\,.\left[\frac{x}{(D+1)^2}\right]$$

$$= e^x\,.\,\text{I.P. of } e^{ix}\,.\left[\frac{x}{D^2+2Di-1}\right]$$

$$= e^x\,.\,\text{I.P. of } e^{ix}[-1\{1-(D^2+2iD)\}^{-1}x]$$

$$= -e^x \times \text{I.P. of } e^{ix}[1 + D^2 + 2iD]x$$

$$= -e^x \times \text{I. P. of } (\cos x + i\sin x)(x + 2i)$$

$$= -e^x \times \text{I. P. of } (x\cos x + 2i\cos x + ix\sin x - 2\sin x)$$

$$= -e^x(2\cos x + x\sin x)$$

Thus, the required general solution is;

$$y = \text{C. F.} + \text{P. I.}$$
$$= (C_1 + C_2x)e^x - e^x(2\cos x + x\sin x).$$

20. $(D^2 + 2D + 1)y = x\cos x$

Sol^n: Here,

$$(D^2 + 2D + 1)y = x\cos x$$

It's auxiliary equation is;

$$m^2 + 2m + 1 = 0$$

or, $m = -1, -1$

Thus,

$$\text{C. F.} = (C_1 + C_2x)e^{-x}$$

For particular integral;

$$\text{P. I.} = \frac{x\cos x}{D^2+2D+1}$$

$$= \left(\frac{\text{Real port of } xe^{ix}}{D^2+2D+1}\right)$$

$$= \text{Real port of } e^{ix}\left[\frac{x}{(D+1)^2+2(D+1)+1}\right]$$

$$= \text{R. P. of } e^{ix}\left(\frac{x}{D^2+2iD-1+2D+2i+1}\right)$$

$$= \text{R. P. of } e^{ix}\left(\frac{x}{20+D^2+2iD+2i}\right)$$

$$= \text{R. P. of } \frac{e^{ix}}{2i}\left(1+\frac{D^2+2iD+2i}{2i}\right)^{-1} x$$

$$= \text{R. P. of } \frac{e^{ix}}{2i}\left(1-\frac{D^2}{2i}-D-\frac{D}{i}\right)x$$

$$= \text{R. P. of } \frac{e^{ix}}{2i}\left(x-1-\frac{1}{i}\right)$$

$$= \text{R. P. of } \frac{(\cos x+i\sin x)}{2i}(x-1-i)$$

$$= \text{R. P. of} \frac{1}{2}(\cos x + i\sin x)(x-1+i)(-i)$$

$$= \frac{1}{2}[(x-1)\sin x + \cos x]\frac{1}{2}x\sin x - \frac{1}{2}\sin x + \frac{1}{2}\cos x$$

Thus, the required general solution is;

$$y = \text{C. F.} + \text{P. I.}$$
$$= (C_1 + C_2x)e^{-x} + \frac{x}{2}\sin x - \frac{1}{2}(\sin x - \cos x)$$

Exercise 31

Solve the following differential equations.

1. $x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = \frac{1}{x}$

$D = \frac{d}{dx}$

Solⁿ: Given,

$$x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = \frac{1}{x}$$

which is in the form;

$$(x^2D^2 - 2xD + 2)y = \frac{1}{x}$$

This is homogeneous equation so,

Put $x = e^z; z = \log x$

Thus,

$xDy = (\delta^2 - \delta)y$ δy

$x^2D^2y = (\delta^2 - \delta)y$

Where,

$$\delta = \frac{d}{dz}$$

Thus equation is reduced into;

$(\delta^2 - \delta)y - 2\delta y + 2y = e^{-z}$

or, $(\delta^2 - 3\delta + 2)y = e^{-z}$

It's auxiliary equation is;

$m^2 - 3m + 2 = 0$

or, $m^2 - m - 2m + 2 = 0$

or, $m(m-1) - 2(m-1) = 0$

or, $(m-2)(m-1) = 0$

Either m = 2 or 1

Hence,

$$C.F. = C_1e^{2z} + C_1e^{z} = C_1x^2 + C_2x$$

$$\therefore \quad P.I. = \frac{e^{-z}}{\delta^2 - 3\delta + 2} = \frac{e^{-z}}{\underbrace{(1-3)(-1+2)}_{1+3+2}} = \frac{e^{-z}}{6} = \frac{1}{6x}$$

Hence,

$$y = C.F. + P.I. = C_1x^2 + C_2x + \frac{1}{6x}$$

2. $(x^2D^2 + xD - 1)y = x^2$

Solⁿ: Given,

$(x^2D^2 + xD - 1)y = x^2$

This is homogeneous equation so,

Put $xDy = \delta y$ and $x^2D^2y = (\delta^2 - \delta)y$

So that,

$x = e^z$

Thus equation is reduced into;

$\delta(\delta - 1)y + \delta y - y = e^{2z}$

or, $(\delta^2 - \delta + \delta - 1)y = e^{2z}$

or, $(\delta^2 - 1)y = e^{2z}$

It's auxiliary equation is;

$m^2 - 1 = 0$ implies $m = \pm 1$

Hence,

$$C.F. = C_1e^{z} + C_1e^{-z} = C_1x + \frac{C_2}{x}$$

$$\therefore \quad P.I. = \frac{e^{2z}}{\delta^2 - 1} = \frac{e^{2z}}{4-1} = \frac{e^{2z}}{3} = \frac{x^2}{3}$$

Hence,

$$y = C.F. + P.I. = C_1x + \frac{C_2}{x} + \frac{x^2}{3}$$

3. $x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = 4x^3$

Solⁿ: Given,

$$x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = 4x^3$$

which is in the form;

$$(x^2D^2 - 2xD + 2)y = 4x^3$$

This is homogeneous equation so,

Put $x = e^z$

so, $x^2D^2y = \delta(\delta - 1)y$

$xDy = \delta y$

Thus equation is reduced into;

$[\delta(\delta - 1) - 2\delta + 2]y = 4e^{3z}$

or, $(\delta^2 - \delta - 2\delta + 2)y = 4e^{3z}$

or, $(\delta^2 - 3\delta + 2)y = 4e^{3z}$

It's auxiliary equation is;

$m^2 - 3m + 2 = 0$

or, $m^2 - m - 2m + 2 = 0$

or, $m(m-1) - 2(m-1) = 0$

or, $(m-1)(m-2) = 0$

Either $m = 1$ or 2

Hence,

$$C.F. = C_1e^z + C_2e^{2z} = C_1x + C_2x^2$$

$$\therefore \quad P.I. = \frac{4e^{3z}}{D^2-3D+2} = \frac{4e^{3z}}{9-9+2} = 2e^{3z} = 2x^3$$

Hence,

$$y = C.F. + P.I. = C_1x + C_2x^2 + 2x^3$$

4. $\mathbf{x^2\frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + 4y = x^6}$

Sol^n: Given,

$$x^2\frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + 4y = x^6$$

which is in the form;

$$x^2D^2y - 3xDy + 4y = x^6$$

This is homogeneous equation so,

Put $x = e^z$

so, $z = \log x$

i.e. $x^2D^2y = \delta(\delta-1)y$

$xDy = \delta y$

Thus equation is reduced into;

$$[\delta(\delta-1) - 3\delta + 4]y = e^{6z}$$

or, $(\delta^2 - \delta - 3\delta + 4)y = e^{6z}$

or, $(\delta^2 - 4\delta + 4)y = e^{6z}$

It's auxiliary equation is;

$$m^2 - 4m + 4 = 0$$

or, $(m-2)^2 = 0$

Either $m = 2$ or 2

Hence,

$$C.F. = (C_1 + C_2z)e^{2x} = (C_1 + C_2\log x)x^2$$

$$\therefore \quad P.I. = \frac{e^{6z}}{\delta^2-4\delta+4} = \frac{e^{6z}}{36-24+4} = \frac{e^{6z}}{16} = \frac{x^6}{16}$$

Hence,

$$y = C.F. + P.I. = (C_1 + C_2\log x)x^2 + \frac{x^6}{16}$$

$\mathbf{x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = 6x}$

Given,

$$x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = 6x$$

which is in the form;

$$x^2D^2y - 2xDy = 6x^2$$

This is homogeneous equation so,

Put $x = e^z$

so, $z = \log x$

i.e. $xDy = \delta y$

$x^2D^2y = \delta(\delta-1)y$

Thus equation is reduced into;

$$[\delta(\delta-1) + 2\delta]y = 6e^{2z}$$

or, $(\delta^2 + \delta)y = 6e^{2z}$

It's auxiliary equation is;

$$m^2 + m = 0$$

Either $m = 0$ or -1

Hence,

$$C.F. = C_1 + C_2e^{-z} = C_1 + \frac{C_2}{x}$$

$$\therefore \quad P.I. = \frac{6e^{2z}}{\delta^2+\delta} = \frac{6e^{2z}}{4+2} = e^{2z} = x^2$$

Hence,

$$y = C.F. + P.I. = C_1 + \frac{C_2}{x} + x^2$$

$\mathbf{x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} - 4y = x^4}$

Given,

$$x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} - 4y = x^4$$

which is in the form;

$$x^2D^2y - 2xDy - 4y = x^4$$

This is homogeneous equation so,

Put $x = e^z$

so, $xDy = \delta y$

$x^2D^2y = \delta(\delta-1)y$

Thus equation is reduced into;

$$\delta(\delta-1)y - 2\delta y - 4y = e^{4z}$$

or, $(\delta^2 - \delta - 2\delta - 4)y = e^{4z}$

or, $(\delta^2 - 3\delta - 4)y = e^{4z}$

It's auxiliary equation is;

$m^2 - 3m - 4 = 0$

or, $m^2 - 4m + m - 4 = 0$

or, $m(m-4) + 1(m-4) = 0$

or, $(m+1)(m-4) = 0$

Either $m = -1$ or 4

Hence,

$$C.F. = C_1e^{-z} + C_2e^{4z} = \frac{C_1}{x} + C_2x^4$$

$$\therefore \quad P.I. = \frac{e^{4z}}{\delta^2-3\delta-4} = \frac{ze^{4z}}{2\delta-3} = \frac{ze^{4z}}{8-3} = \frac{ze^{4z}}{5} = \frac{x^4}{5}\log x$$

Hence, the required general solution is;

$$y = C.F. + P.I. = \frac{C_1}{x} + C_2x^4 + \frac{x^4}{5}\log x$$

7. $\mathbf{x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - 4y = x^2}$

Sol^n: Given,

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - 4y = x^2$$

which is in the form;

$$x^2D^2y + xDy - 4y = x^2$$

This is homogeneous equation so,

Put $x = e^z$

or, $z = \log x$

i.e. $x^2D^2y = \delta(\delta - 1)y$

$xDy = \delta y$

Thus equation is reduced into;

$\delta(\delta - 1)y + \delta y - 4y = e^{2z}$

or, $(\delta^2 - \delta + \delta - 4)y = e^{2z}$

or, $(\delta^2 - 4)y = e^{2z}$

It's auxiliary equation is;

$m^2 - 4 = 0$

or, $m = \pm 2$

Hence,

$$C.F. = C_1e^{2z} + C_2e^{-2z} = C_1x^2 + \frac{C_2}{x^2}$$

$$\therefore \quad P.I. = \frac{e^{2z}}{\delta^2-4} = \frac{ze^{2z}}{2\delta} = \frac{ze^{2z}}{4} = \frac{\log x . x^2}{4}$$

Hence, the required general solution is;

$$y = C.F. + P.I. = C_1x^2 + \frac{C_2}{x^2} + \frac{\log x . x^2}{4}$$

$\mathbf{x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = \log x}$

Sol^n: Given,

$$x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = \log x$$

which is in the form;

$$(x^2D^2 - xD + 1)y = \log x$$

This is homogeneous equation so,

Put $x = e^z$ $z = \log x$

so, $xDy = \delta y$

$x^2D^2y = \delta(\delta - 1)y$

Thus equation is reduced into;

$\delta(\delta - 1)y - \delta y + y = z$

or, $(\delta^2 - \delta - \delta + 1)y = z$

or, $(\delta^2 - 2\delta + 1)y = z$

It's auxiliary equation is;

$m^2 - 2m + 1 = 0$

or, $m = 1, 1$

Hence,

$$C.F. = (C_1 + C_2z)e^z = (C_1 + C_2\log x)x$$

$$\therefore \quad P.I. = \frac{z}{1-2\delta+\delta^2} = [1 - (2\delta + \delta^2)]^{-1}z$$

$$= [1 + 2\delta + \delta^2]z = (z + 2) = \log x + 2$$

Hence, the required general solution is;

$$y = C.F. + P.I. = (C_1 + C_2\log x)x + \log x + 2$$

9. $\mathbf{x^2\frac{d^2y}{dx^2} + 4x\frac{dy}{dx} + 2y = e^x}$

Sol^n: Given,

$$x^2\frac{d^2y}{dx^2} + 4x\frac{dy}{dx} + 2y = e^x$$

which is in the form;

$$(x^2D^2 + 4xD + 2)y = e^x$$

This is homogeneous equation so,

Put $x = e^z$

or, $z = \log x$

i.e. $x^2D^2y = \delta(\delta - 1)y$

$xDy = \delta y$

Thus equation is reduced into;

$\delta(\delta - 1)y + 4\delta y + 2y = e^{e^z}$

or, $(\delta^2 - \delta + 4\delta + 2)y = e^{e^z}$

or, $(\delta^2 + 3\delta + 2)y = e^{e^z}$

It's auxiliary equation is;

$m^2 + 3m + 2 = 0$

or, $m^2 + m + 2m + 2 = 0$

or, $m = -1$ or -2

Hence,

$$\text{C.F.} = C_1e^{-z} + C_2e^{-2z} = \frac{C_1}{x} + \frac{C_2}{x^2}$$

$$\therefore \quad \text{P.I.} = \frac{e^{e^z}}{\delta^2+3\delta+2}$$

$$= \frac{e^{e^z}}{(\delta+1)(\delta+2)} = \left(\frac{1}{\delta+1} - \frac{1}{\delta+2}\right)e^{e^z} = \frac{e^{e^z}}{\delta+1} - \frac{e^{e^z}}{\delta+2}$$

We know,

$$\frac{x}{D-a} = e^{ax}\int e^{-ax}\,dx$$

$$\therefore \quad \text{P.I.} = \frac{e^{e^z}}{\delta+1} - \frac{e^{e^z}}{\delta+2}$$

$$= e^{-z}\int e^z e^{e^z}\,dz - e^{-2z}\int e^{2z}e^{e^z}\,dz$$

Now,

$$\int e^z e^{e^z}\,dz = \int e^t\,dt = e^t + c = e^{e^z}\,dz$$

Put $t = e^z$

and $\int e^{2z}e^{e^z}\,dz = \int e^z e^{e^z} e^z\,dz$ $te^t dt$

Put $t = e^z$

$e^z\,dz = dt$

$= \int te^t\,dt$ (by parts)

$= te^t - \int e^t\,dt$

$= te^t - e^t$

$= e^z e^{e^z} - e^{e^z}$

Thus,

$$\text{P.I.} = e^{-z}e^{e^z} - e^{-2z}\left(e^z e^{e^z} - e^{e^z}\right) = e^{-2z}e^{e^z}$$

Hence, the required general solution is;

$y = \text{C.F.} + \text{P.I.}$

$= C_1e^{-z} + C_2e^{-2z} + e^{-2z}e^{e^z}$

$= C_1x^{-1} + C_2x^{-2} + x^{-2}e^x$

$(x^2D^2 + xD + 1)y = \sin(\log x^2)$

Given,

$$(x^2D^2 + xD + 1)y = \sin(\log x^2)$$

This is homogeneous equation so,

Put $x = e^z$

so, $z = \log x$

i.e. $xDy = \delta y$

$x^2D^2y = \delta(\delta - 1)y$

Thus equation is reduced into;

$\delta(\delta - 1)y + \delta y + y = \sin(2z)$

or, $(\delta^2 - \delta + \delta + 1)y = \sin(2z)$

or, $(\delta^2 + 1)y = \sin(2z)$

It's auxiliary equation is;

$m^2 + 1 = 0$

or, $m = \pm 1$

Hence,

$$\text{C.F.} = (A\cos x + B\sin x)$$

$$\therefore \quad \text{P.I.} = \frac{\sin 2z}{\delta^2+1} = \frac{\sin 2z}{-4+1} = -\frac{1}{3}\sin 2z = -\frac{1}{3}\sin(2\log x)$$

$$= -\frac{1}{3}\sin(\log x^2)$$

Hence, the required general solution is;

$$y = \text{C.F.} + \text{P.I.} = (A\cos z + B\sin z) - \frac{1}{3}\sin(\log x^2)$$

$$= A\cos(\log x) + B\sin(\log x) - \frac{1}{3}\sin(\log x^2)$$

11. **$(x^2D^2 - 2)y = x^2 + \frac{1}{x}$**

Sol^n: Given,

$$(x^2D^2 - 2)y = x^2 + \frac{1}{x}$$

This is homogeneous equation so,

Put $x = e^z$

so, $x^2D^2y = \delta(\delta - 1)y$

$xDy = \delta y$

Thus equation is reduced into;

$$\delta(\delta - 1)y - 2y = x^2 + \frac{1}{x}$$

or, $(\delta^2 - \delta - 2)y = e^{2z} + e^{-z}$

It's auxiliary equation is;

$m^2 - m - 2 = 0$

or, $m^2 - 2m + m - 2 = 0$

or, $m(m - 2) + 1(m - 2) = 0$

or, $(m + 1)(m - 2) = 0$

or, $m = -1$ or 2

Hence,

$$\text{C.F.} = C_1e^{-z} + C_2e^{2z} = \frac{C_1}{x} + C_2x^2$$

$$\therefore \quad \text{P.I.} = \frac{e^{2z}+e^{-z}}{\delta^2-\delta-2} = \frac{e^{2z}}{\delta^2-\delta-2} + \frac{e^{-z}}{\delta^2-\delta-2}$$

$$= \frac{ze^{2z}}{2\delta-1} + \frac{ze^{-z}}{2\delta-1}$$

$$= \frac{zx^2}{3} - \frac{1}{3}\frac{\log x}{x}$$

$$= \frac{x^2\log x}{3} - \frac{\log x}{3x}$$

Hence, the required general solution is;

$$y = \text{C.F.} + \text{P.I.} = \frac{C_1}{x} + C_2x^2 + \frac{x^2}{3}\log x - \frac{1}{3x}\log x$$

12. $\mathbf{(x + P)^2 \frac{d^2y}{dx^2} - 4(x + P)\frac{dy}{dx} + 6y = x}$

Sol^n: Given,

$$(x + P)^2 \frac{d^2y}{dx^2} - 4(x + P)\frac{dy}{dx} + 6y = x$$

This is homogeneous equation so,

Put $x + P = e^z$

or, $z = \log(x + P)$

or, $\frac{dz}{dx} = \frac{1}{x+P}$

or, $\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} = \frac{dy}{dz}\frac{1}{x+P}$

and $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{1}{x+P}\frac{dy}{dz}\right)$

$$= \frac{1}{x+P}\frac{d^2y}{dz^2} \times \frac{dz}{dx} + \frac{dy}{dz}\left[-\frac{1}{(x+P)^2}\right]$$

$$= \frac{1}{(x+P)^2}\frac{d^2y}{dz^2} - \frac{1}{(x+P)^2}\frac{dy}{dz}$$

$$= \frac{1}{(x+P)^2}\left[\frac{d^2y}{dz^2} - \frac{dy}{dz}\right]$$

$$\therefore \quad (x + P)^2\frac{d^2y}{dx^2} = \frac{d^2y}{dz^2} - \frac{dy}{dz}$$

So, equation becomes;

$$\frac{d^2y}{dz^2} - \frac{dy}{dz} - 4\frac{dy}{dz} + 6y = e^z - 2$$

or, $\frac{d^2y}{dz^2} - 5\frac{dy}{dz} + 6y = e^z - 2$

or, $(\delta^2 - 5\delta + 6)y = e^z - 2$

It's auxiliary equation is;

$m^2 - 5m + 6 = 0$

or, $m^2 - 3m - 2m + 6 = 0$

or, $m(m - 3) - 2(m - 3) = 0$

or, $(m - 3)(m - 2) = 0$

Either $m = 3$ or 2

Hence,

$$\text{C.F.} = C_1e^{3z} + C_2e^{2z} = C_1(x + P)^3 + C_2(x + P)^2$$

$$\therefore \quad \text{P.I.} = \frac{e^z-2}{\delta^2-5\delta+6}$$

$$= \frac{e^z}{\delta^2-5\delta+6} - \frac{2e^{0.z}}{\delta^2-5\delta+6}$$

$$= \frac{e^z}{1-5+6} - \frac{2}{6}$$

$$= \frac{e^z}{2} - \frac{2}{6}$$

$$= \frac{x+P}{2} - \frac{P}{3} = \frac{3x+3P-P}{6} = \frac{3x+2P}{6}$$

Hence, the required general solution is;

$$y = \text{C.F.} + \text{P.I.} = C_1(x + P)^3 + C_2(x + P)^2 + \frac{3x+2P}{6}$$

13. $(x+1)^2\dfrac{d^2y}{dx^2}+(x+1)\dfrac{dy}{dx}=(2x+3)(2x+4)$

Sol^n: Given,

$$(x+1)^2\frac{d^2y}{dx^2}+(x+1)\frac{dy}{dx}=(2x+3)(2x+4) \quad \text{(i)}$$

Put $x+1=e^z$

or, $z=\log(x+1)$

Now,

$$(x+1)^2\frac{d^2y}{dx^2}=\frac{d^2y}{dz^2}-\frac{dy}{dz}$$

or, $(x+1)\dfrac{dy}{dx}=\dfrac{dy}{dz}$

Putting in (i)

so, $\left(\dfrac{d^2y}{dz^2}-\dfrac{dy}{dz}+\dfrac{dy}{dz}\right)=4x^2+14x+12$

or, $\left(\dfrac{d^2y}{dz^2}-\dfrac{dy}{dz}+\dfrac{dy}{dz}\right)=4(e^z-1)^2+14(e^z-1)+12$

or, $\dfrac{d^2y}{dz^2}=4(e^{2z}-2e^z+1)+14e^z-2$

or, $\delta^2y=4e^{2z}-8e^z+4+14e^z-2$

or, $\delta^2y=4e^{2z}+6e^z+2$

It's auxiliary equation is;

$m^2=0$

Either $m=0$ or 0

Hence,

$$C.F.=(C_1+C_2z)e^{0z}=C_1x+C_2z=C_1+C_2\log(x+1)$$

$\therefore$ $P.I.=\dfrac{4e^{2z}+6e^z+2}{\delta^2}$

$=\dfrac{4e^{2z}}{\delta^2}+\dfrac{6e^z}{\delta^2}+\dfrac{2e^{0z}}{\delta^2}$

$=\dfrac{4e^{2z}}{4}+\dfrac{6e^z}{1}+\dfrac{2z^2}{2}$

$=e^{2z}+6e^z$

$=(x+1)^2+6(x+1)+[\log(x+1)]^2$

Hence, the required general solution is;

$y=C.F.+P.I.$

$=C_1+C_2\log(x+1)+(x+1)^2+6(x+1)+[\log(x+1)]^2$

14. $(2x+3)^2\dfrac{d^2y}{dx^2}+2(2x+3)\dfrac{dy}{dx}-4y=8x$

Sol^n: Given,

$$(2x+3)^2\frac{d^2y}{dx^2}+2(2x+3)\frac{dy}{dx}-4y=8x$$

Put $2x+3=e^z$

or, $z=\log(2x+3)$

so, $\dfrac{dz}{dx}=\dfrac{2}{2x+3}$

or, $\dfrac{dy}{dx}=\dfrac{dy}{dz}-\dfrac{dz}{dx}=\dfrac{dy}{dz}-\dfrac{2}{2x+3}$

or, $(2x+3)\dfrac{dy}{dz}=\dfrac{2dy}{dz}$

Also,

$$\frac{d^2y}{dx^2}=\frac{4}{(2x+3)^2}\left(\frac{d^2y}{dz^2}-\frac{dy}{dz}\right)$$

i.e. $(2x+3)^2\dfrac{d^2y}{dx^2}=4\left(\dfrac{d^2y}{dz^2}-\dfrac{dy}{dz}\right)$

Thus,

$$4\left(\frac{d^2y}{dz^2}-\frac{dy}{dz}\right)+4\frac{dy}{dz}-4y=8\left(\frac{e^z-3}{2}\right)$$

or, $(4\delta^2-4\delta+4\delta-4)y=8\left(\dfrac{e^z-3}{2}\right)$

or, $(4\delta^2-4)y=8\left(\dfrac{e^z-3}{2}\right)$

It's auxiliary equation is;

$4m^2-4=0$

or, $m^2=1$

or, $m=\pm1$

Hence,

$$C.F.=C_1e^z+C_2e^{-z}=\frac{C_2}{e^z}+C_1e^z=\frac{C_2}{2x+3}+C_1(2x+3)$$

$\therefore$ $P.I.=\dfrac{4(e^z-3e^{0z})}{4\delta^2-4}=\dfrac{e^z-3e^{0z}}{\delta^2-1}=\dfrac{e^z}{\delta^2-1}-\dfrac{3e^{0z}}{\delta^2-1}$

$=\dfrac{ze^z}{2\delta}+1\times3=\dfrac{\log(2x+3)(2x+3)}{2}+3$

Hence, the required general solution is;

$$y=C.F.+P.I.=\frac{C_1}{2x+3}+C_2(2x+3)+\left(\frac{2x+3}{2}\right)\log(2x+3)+3$$

15. $(2x+1)^2\dfrac{d^2y}{dx^2} - 6(2x+1)\dfrac{dy}{dx} + 16y = 8(2x+1)^2$

Solⁿ: Given,

$$(2x+1)^2\frac{d^2y}{dx^2} - 6(2x+1)\frac{dy}{dx} + 16y = 8(2x+1)^2$$

Put $2x+1 = e^z$

or, $z = \log(2x+1)$

so, $\dfrac{dz}{dx} = \dfrac{2}{2x+1}$

or, $\dfrac{dy}{dx} = \dfrac{dy}{dz}\times\dfrac{dz}{dx} = \dfrac{2}{2x+1}\dfrac{dy}{dz}$ ... (i)

Similarly,

$$\frac{d^2y}{dx^2} = \frac{4}{(2x+1)^2}\left(\frac{d^2y}{dz^2} - \frac{dy}{dz}\right)$$

Thus,

$$4\left(\frac{d^2y}{dz^2} - \frac{dy}{dz}\right) - 2.6\frac{dy}{dz} + 16y = 8e^{2z}$$

or, $\left(4\dfrac{d^2y}{dz^2} - 4\dfrac{dy}{dz} - 12\dfrac{dy}{dz} + 16\right)y = 8e^{2z}$

or, $(4\delta^2 - 16\delta + 16)y = 8e^{2z}$

It's auxiliary equation is;

$4m^2 - 16m + 16 = 0$

or, $2m^2 - 8m + 8 = 0$

or, $m^2 - 4m + 4 = 0$

or, $(m-2)^2 = 0$

Either $m = 2, 2$

Hence,

$$C.F. = (C_1 + C_2 z)e^{2z} = [C_1 + C_2\log(2x+1)](2x+1)^2$$

$$\therefore \quad P.I. = \frac{8e^{2z}}{4\delta^2 - 16\delta + 16} = \frac{z.8e^{2z}}{8\delta - 16} = \frac{z^2.8e^{2z}}{8} = z^2e^{2z}$$

$$= [\log(2x+1)]^2(2x+1)^2$$

Hence, the required general solution is;

$$y = C.F. + P.I.$$

$$= [C_1 + C_2\log(2x+1)](2x+1)^2 + [\log(2x+1)]^2(2x+1)^2$$

$$= [C_1 + C_2\log(2x+1) + \{\log(2x+1)\}^2](2x+1)^2$$

Application of differential equations

Important problems

1. **Resisted motion:**

A particle of mass 'm' is projected vertically upward underground the resistance of air being mk times the velocity. Show that the greatest height attained by the particle is $\frac{V^2}{g}[\lambda - \log(1+\lambda)]$ where V is the greatest velocity which the above mass well attain when it falls freely and λV is the initial velocity.

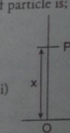
Solⁿ: Let 'u' be the velocity of the particle at time 't'. The forces acting on the particle are:

i) its weight 'mg' acting vertically downwards

ii) the resistance 'mk u' of the air acting vertically downward

Accelerating force on the particle $= -mg - mku$

$\therefore$ By Newton's second law, the equation of motion of particle is;



$$mu\frac{du}{dx} = -mg - mku$$

or, $u\dfrac{du}{dx} = -g - ku$ ... (i)

When the particle falls freely under gravity equation (i) reduces to;

$u\dfrac{du}{dx} = g - ku$ ... (ii) (replacing g by $-g$)

When the particle attains the greatest velocity 'V', its acceleration is zero. Hence, from equation (ii);

$g - kV = 0$

or, $k = \dfrac{g}{V}$

$\therefore$ From equation (i);

$u\dfrac{du}{dx} = -g - \dfrac{g}{V} = -\dfrac{g}{V}(u+V)$

or, $\dfrac{u}{u+V}du = -\dfrac{g}{V}dx$

Integrating;

$\int\dfrac{u}{u+V}du = \int -\dfrac{g}{V}dx$

or, $\int\left(1 - \dfrac{u}{u+V}\right)du = -\dfrac{g}{V}x + C$

or, $u - V\log(u+V) = -\frac{g}{V}x + C \ldots$ (iii)

Initially when $x = 0, u = \lambda V$ (the given initial velocity)

Hence, from equation (iii);

$$\lambda V - V\log(u+V) = C$$

Putting the value of C in equation (iii);

$$u - V\log(u+V) = -\frac{g}{V}x + V[\lambda - \log V\ (1+\lambda)] \ldots \text{(iv)}$$

Let,

h = the greatest height attained by particle

Then,

$x = h$

When,

$u = 0$

Hence, from equation (iv);

$$-V\log V = -\frac{g}{V}h + V[\lambda - \log V(1+\lambda)]$$

or, $\frac{g}{V}h = V\lambda - V[\log V(1+\lambda) - \log V]$

$= V\lambda - V\log(1+\lambda)$

or, $h = \frac{V^2}{g}[\lambda - \log(1+\lambda)]$ Proved

2. a) Resisted motion:

A body of mass 'm', falling from rest, is subject to the force of gravity and an air resistance proportional to the square of the velocity (i.e. kV^2). If it falls through a distance x and possesses a velocity V at that instant, prove that:

$$\frac{2kx}{m} = \log\frac{a^2}{a^2 - v^2};$$

Where,

$mg = ka^2$

Sol^n:

The forces acting on the body are:

i) its weight 'mg' vertically downwards

ii) the air resistance kV^2 vertically upwards

Accelerating force on the body $= mg - kV^2 = ka^2 - kV^2$

$= k(a^2 - V^2)$ $(\because mg = ka^2)$

By Newton's second law, equation of motion is;

$$mV\frac{dv}{dx} = k(a^2 - V^2)$$

or, $\frac{v}{a^2 - v^2} = \frac{k}{m}dx$

Integrating,

$$-\frac{1}{2}\log(a^2 - V^2) = \frac{k}{m}x + C \ldots \text{(i)}$$

Initially, when $x = 0, V = 0$

Hence from equation (i);

$$-\frac{1}{2}\log a^2 = C$$

$\therefore$ From equation (i);

$$-\frac{1}{2}\log(a^2 - V^2) = \frac{k}{m}x - \frac{1}{2}\log a^2$$

or, $\frac{1}{2}\log\frac{a^2}{a^2 - v^2} = \frac{kx}{m}$

or, $\frac{2kx}{m} = \log\frac{a^2}{a^2 - v^2}$ Proved

2. b) **The equations of emf in terms of current i for an electrical circuit having resistant 'R' and a condenser of capacity 'C', in series, is $E = Ri + \int \frac{i}{C}dt = E_0 \sin\omega t$. Find the current I at any time 't', when $E = E_0 \sin\omega t$.**

Sol^n:

Given equation can be written as;

$$Ri + \int \frac{i}{C}dt = E_0 \sin\omega t$$

Differentiating both sides with respect to t, we get,

$$R\frac{di}{dt} + \frac{i}{C} = \frac{\omega E_0}{R}\cos\omega t$$

or, $\frac{di}{dt} + \frac{i}{RC} = \frac{\omega E_0}{R}\cos\omega t \ldots$ (i)

which is a linear equation

$$I_0F_0 = e^{\int \frac{1}{RC}dt} = e^{\frac{t}{RC}}$$

or, $\frac{di}{dt} + \frac{i}{RC} = e^{\frac{t}{RC}}$

$\therefore$ Solution of equation (i) is;

$$i \cdot e^{\frac{t}{RC}} = \int \frac{\omega E_0}{R}\cos\omega t \cdot e^{\frac{t}{RC}}dt$$

$$= \frac{\omega E_0}{R}\int e^{\frac{t}{RC}}\cos\omega t\, dt$$

$$= \frac{\omega E_0}{R}\left[\frac{\frac{1}{RC}e^{\frac{t}{RC}}.\cos\omega t - e^{\frac{t}{RC}}(-\omega\sin\omega t)}{\frac{1}{R^2C^2} + \omega^2}\right] + C$$

$$= \frac{\omega E_0 e^{\frac{t}{RC}}}{R}\left[\frac{\cos\omega t + RC\omega \sin\omega t}{1+R^2C^2\omega^2}\right] . RC + C'$$

$$\text{or,} \quad i . e^{\frac{t}{RC}} = \omega E_0 C e^{\frac{t}{RC}} \left(\frac{\cos\omega t + RC\omega \sin\omega t}{1+R^2C^2\omega^2}\right) + C'$$

$$\therefore \quad i = \frac{\omega E_0 C}{1+R^2C^2\omega^2}(\cos\omega t + RC\omega \sin\omega t) + C' e^{-\frac{t}{RC}}$$

3. Electric Circuits:

a) A constant e.m.f. E volts is applied to a circuit containing a constant resistance R ohms in series and a constant inductance L henries. If the initial current is zero, show that the current builds up to half its theoretical maximum in $\frac{L(\log 2)}{R}$ seconds.

Sol^n:

Let,

i = current at any time 't'

By Kirchhoff's law,

$$L\frac{di}{dt} + Ri = E$$

$$\text{or,} \quad \frac{di}{dt} + \frac{R}{L} i = \frac{E}{L} \ldots \text{(i)}$$

It is a linear differential equation of first order and first degree.

So,

$$I.F. = e^{\int \frac{R}{L} dt} = e^{\frac{Rt}{L}}$$

Hence, solution of (i) is;

$$i . \left(e^{\frac{Rt}{L}}\right) = \int \frac{E}{L} e^{\frac{Rt}{L}} dt$$

$$= \frac{E}{L}\frac{1}{\frac{R}{L}} e^{\frac{Rt}{L}} + C$$

$$= \frac{E}{R} e^{\frac{Rt}{L}} + C$$

$$\therefore \quad i = \frac{E}{R} + Ce^{-\frac{Rt}{L}} \ldots \text{(ii)}$$

Given,

Initial current is zero i.e. when $t = 0, i = 0$

Hence, from equation (ii);

$$0 = \frac{E}{R} + C$$

$$\text{or,} \quad C = -\frac{E}{R}$$

Then, equation (ii) reduces to;

$$i = \frac{E}{R}\left(1 - e^{-\frac{Rt}{L}}\right) \ldots \text{(iii)}$$

Clearly i increases with 't' and attains the maximum value $\frac{E}{R}$. Let the current i in the circuit be half its theoretical maximum after a time 'T'. Then, from equation (iii);

$$\frac{1}{2}\frac{E}{R} = \frac{E}{R}\left(1 - e^{-\frac{Rt}{L}}\right)$$

$$\text{or,} \quad e^{-\frac{Rt}{L}} = \frac{1}{2}$$

$$\text{or,} \quad -\frac{RT}{L} = \log\frac{1}{2}$$

Hence, the heat lost per min through 20 meter length of pipe,

$$= 60 \times 2000\, Q$$
$$= 120000 \times 163$$
$$= 19560000$$

Now, let,

$T = t$

When

$x = 7.5$

From equation (i);

$$I = -\frac{Q}{2\pi k}\log_e 7.5$$

$$\text{or,} \quad t = -\frac{Q}{2\pi k}\log(7.5) + C \ldots \text{(iv)}$$

Subtracting equation (ii) from (iv);

$$t - 200 = -\frac{Q}{2\pi k}[\log(7.5) - \log_e(5)]$$

$$\text{or,} \quad t - 200 = -\frac{Q}{2\pi k}\log_e 1.5 \ldots \text{(v)}$$

Dividing equation (v) by (iv), we get,

$$\frac{t-200}{150} = \frac{\log(1.5)}{\log(2)}$$

or, $t = 113$

When,

$x = 7.5$ m

Then,

$T = 113°C$

Exercise 32

1. **Transform the equation $x^2 - 3y^2 + 4x + 6y = 0$ by transferring the origin to the point $(-2, 1)$ and the co–ordinate axes remaining parallel.**

Sol^n: Here, the given equation is $x^2 - 3y^2 + 4x + 6y = 0$ and the origin is transformed to $(-2,1)$ the axis remains parallel, so Put $x = x - 2$ and $y = y + 1$ in given equation,

So, we get,

$$(x-2)^2 - 3(y+1)^2 + 4(x-2) + 6(y+1) = 0$$

or, $x^2 - 4x + 4 - 3(y^2 + 2y + 1) + 4x - 8 + 6y + 6 = 0$

or, $x^2 - 4x + 4 - 3y^2 + 6y + 3 + 4x - 8 + 6y + 6 = 0$

or, $x^2 - 3y^2 = 1$ is required equation referred to new ones.

2. **Transform the equation $x^2 + 3y^2 + 3x - 40 = 0$ to parallel axes through $(4, -1)$.**

Sol^n: Here, the given equation is $x^2 + 3y^2 + 3x - 40 = 0$ and the origin is transformed to $(4, -1)$ having the axis parallel, so put $x = x + 4$ and $y = y - 1$.

Then,

$$(x+4)^2 + 3(y-1)^2 + 3(x+4) - 40 = 0$$

or, $x^2 + 8x + 16 + 3y^2 - 6y + 3 + 3x + 12 - 40 = 0$

or, $x^2 + 3y^2 + 11x - 6y - 9 = 0$

Hence, the equation referred to new axis is;

$$x^2 + 3y^2 + 11x - 6y - 9 = 0$$

3. **Transform the equation $x^2 + y^2 - 5xy + 4 = 0$ to parallel axes through the point $(1, 3)$.**

Sol^n: The given equation is $x^2 + y^2 - 5xy + 4 = 0$ and the axis are parallel whereas the origin is transformed to $(1,3)$.

For this put $x = x + 1$ and $y = y + 2$ in given equation, so that,

$$(x+1)^2 + (y+3)^2 - 5(x+1)(y+3) + 4 = 0$$

or, $x^2 + 2x + 1 + y^2 + 6y + 9 - 5(xy + 3x + y + 3) + 4 = 0$

or, $x^2 + 2x + 1 + y^2 + 6y + 9 - 5xy + 15x + 5y + 15 + 4 = 0$

or, $x^2 + y^2 - 5xy - 13x + y - 1 = 0$

Hence, the equation referred to new axis is;

$$x^2 + y^2 - 5xy - 13x + y = 1$$

4. **Transform the equation $2x^2 + 4xy + 5y^2 - 4x - 22y + 7 = 0$ to parallel axes through $(-2, 3)$.**

Sol^n: The given equation is $2x^2 + 4xy + 5y^2 - 4x - 22y + 7 = 0$ and the origin is transformed to $(-2,3)$ so substitute $x = x - 2$ and $y = y + 3$ in a given equation.

Thus,

$$2(x-2)^2 + 4(x-2)(y+3) + 5(y+3)^2 - 4(x-2) - 2(y+3) + 7 = 0$$

or, $2(x^2 - 4x + 4) + 4(xy + 3x - 2y - 6) + 5(y^2 + 6y + 9) - 4x + 8 - 22y - 66 + 7 = 0$

or, $2x^2 - 8x + 8 + 4xy + 12x - 8y - 24 + 5y^2 + 30y + 45 - 4x + 8 - 22y - 66 + 7 = 0$

or, $2x^2 + 5y^2 + 4xy - 22 = 0$

Hence the required equation referred to new axis is;

$$2x^2 + 5y^2 + 4xy - 22 = 0$$

5. **What does the equation $x^2 + 2\sqrt{3}xy - y^2 = 2a^2$ become when the axes are turned through an angle 30° to the original axes.**

Sol^n: Here, the given equation is $x^2 + 2\sqrt{3}xy - y^2 = 2a^2$ and it is turned through one angle $\theta = 30^\circ$ to the original axis, so,

Put $x = x\cos\theta - y\sin\theta$

$y = y\cos\theta + x\sin\theta$

Hence,

$$x = x\cos 30 - y\sin 30$$

or, $x = \frac{\sqrt{3}x}{2} - \frac{y}{2} = \frac{\sqrt{3}x - y}{2}$

and $y = y\cos 30 + x\sin 30 = \frac{\sqrt{3y}}{2} + \frac{x}{2} = \frac{x + \sqrt{3}y}{2}$

Thus, substituting the values of x and y in given equation, yields;

or, $\left(\frac{\sqrt{3}x - y}{2}\right)^2 + 2\sqrt{3}\left(\frac{\sqrt{3}x - y}{2}\right)\left(\frac{x + \sqrt{3}y}{2}\right) - \left(\frac{x + \sqrt{3}y}{2}\right)^2 = 2a^2$

or, $\frac{3x^2 - 2\sqrt{3}xy + y^2}{4} + 2\sqrt{3}\left(\frac{x^2\sqrt{3} + 3xy - xy - \sqrt{3}y^2}{4}\right) - \left(\frac{x^2 + 2\sqrt{3}xy + 3y^2}{4}\right) = 2a^2$

or, $3x^2 - 2\sqrt{3}xy + y^2 + 6x^2 + 6\sqrt{3}xy - 2\sqrt{3}xy - 6y^2 - x^2 - 2\sqrt{3}xy - 3y^2 = 8a^2$

or, $8x^2 - 8y^2 = 8a^2$

or, $x^2 - y^2 = a^2$

Hence, $x^2 - y^2 = a^2$ is the required equation referred to new axis.

6. **What does the equation $2x + \sqrt{3}y = \sqrt{2}$ become when the axes are turned through an angle $45°$ to the original axes.**

Sol^n: The given equation is $2x + \sqrt{3}y = \sqrt{2}$ and the axis are turned through an angle 45° with axis, so for the new required equation, we substitute,

$$x = x\cos\theta - y\sin\theta$$
$$= x\cos 45 - y\sin 45$$
$$= \frac{x-y}{\sqrt{2}}$$

And $y = 1 = y\cos\theta + x\sin\theta$

$$= \frac{y+x}{\sqrt{2}}$$

Hence, the equation $2x + 3y = \sqrt{2}$ is reduced into

$$2\frac{(x-y)}{\sqrt{2}} + 3\frac{(y+x)}{\sqrt{2}} = \sqrt{2}$$

or, $2(x-y) + 3(y+x) = 2$

or, $2x - 2y + 3y + 3x = 2$

or, $y + 5x = 2$

Hence, $y + 5x = 2$ is required turned equation.

7. **What does the equation $3x^2 + 3y^2 + 2xy = 2$ become when the axes are turned through an angle $45°$ to the original axes.**

Sol^n: The given equation is $3x^2 + 3y^2 + 2xy = 2$ and it is turned through an angle $\theta = 45°$ with original axis, so for this substitute

$$x = x\cos\theta - y\sin\theta$$
$$= x\cos 45 - y\sin 45 = \frac{1}{\sqrt{2}}x - \frac{1}{\sqrt{2}}y = \frac{x-y}{\sqrt{2}}$$

And $y = y\cos\theta + x\sin\theta$

$$= y\cos 45 + x\sin 45 = \frac{x+y}{\sqrt{2}}$$

Thus, the equation $3x^2 + 3y^2 + 2xy = 2$ is changed into;

$$3\left(\frac{x-y}{\sqrt{2}}\right)^2 + 3\left(\frac{x+y}{\sqrt{2}}\right)^2 + 2\left(\frac{x-y}{\sqrt{2}}\right)\left(\frac{x+y}{\sqrt{2}}\right) = 2$$

or, $3\frac{(x^2-2xy+y^2)}{2} + 3\frac{(x^2+2xy+y^2)}{2} + 2\frac{(x^2-y^2)}{2} = 2$

or, $3x^2 - 6xy + 3y^2 + 3x^2 + 6xy + 3y^2 + 2x^2 - 2y^2 = 4$

or, $8x^2 + 4y^2 = 4$

or, $2x^2 + y^2 = 1$

Hence, $2x^2 + y^2 = 1$ is the required equation turned under given angle $\theta = 45°$.

Through what angle should the axes be rotated so that equation $9x^2 - 2\sqrt{3}xy + 7y^2 = 10$ may be changed to $3x^2 + 5y^2 = 5$.

Sol^n: The given equation is $9x^2 - 2\sqrt{3}xy + 7y^2 = 10$ and is required to change into $3x^2 + 5y^2 = 5$ for this by question rotation is allowed.

Let θ be the required rotation;

So, $x = x\cos\theta - y\sin\theta$

$y = y\cos\theta + x\sin\theta$

Then,

$9x^2 + 7y^2 - 2\sqrt{3}xy = 10$ is reduced into

or, $9(x\cos\theta - y\sin\theta)^2 + 7(y\cos\theta + x\sin\theta)^2$
$-2\sqrt{3}(x\cos\theta - y\sin\theta)(y\cos\theta + x\sin\theta) = 10$

or, $9[x^2\cos^2\theta - 2xy\sin\theta\cos\theta + y^2\sin\theta]$
$+7[y^2\cos^2\theta + 2xy\cos\theta\sin\theta + x^2\sin^2\theta]$
$-2\sqrt{3}[xy\cos^2\theta + x^2\sin\theta\cos\theta - y^2\sin\theta\cos\theta - xy\sin^2\theta] = 10$

or, $9x^2\cos^2\theta - 18xy\sin\theta\cos\theta + 9y^2\sin\theta + 7y^2\cos^2\theta$
$+14xy\cos\theta\sin\theta + 7x^2\sin^2\theta - 2\sqrt{3}xy\cos^2\theta$
$-2\sqrt{3}x^2\sin\theta\cos\theta + 2\sqrt{3}y^2\sin\theta\cos\theta + 2\sqrt{3}xy\sin^2\theta = 10$

or, $x^2[9\cos^2 + 7\sin^2\theta - 2\sqrt{3}\sin\theta\cos\theta] + y^2[9\cos^2\theta + 7\cos^2\theta$
$+2\sqrt{3}\sin\theta\cos\theta] + xy[-18\sin\theta\cos\theta + 14\sin\theta\cos\theta$
$+2\sqrt{3}\sin^2\theta - 2\sqrt{3}\cos^2\theta] = 10$

For coeff: $x^2 = 3$

$9\cos^2\theta + 7\sin^2\theta - 2\sqrt{3}\sin\theta\cos\theta = 3$ ... (I)

For coeff: $y^2 = 5$

$9\sin^2\theta + 7\cos^2\theta + 2\sqrt{3}\sin\theta\cos\theta = 5$ ... (II)

and coeff: $xy = 0$

$14\sin\theta\cos\theta - 18\sin\theta\cos\theta + 2\sqrt{3}\sin^2\theta - 2\sqrt{3}\cos^2\theta = 0$

or, $-4\sin\theta\cos\theta = -2\sqrt{3}\sin^2\theta + 2\sqrt{3}\cos^2\theta$

or, $-2\sin\theta\cos\theta \times 2 = 2\sqrt{3}(\cos^2\theta - \sin^2\theta)$

or, $-\sin 2\theta = 2\sqrt{3}\cos 2\theta$

or, $\tan 2\theta = -\sqrt{3} = \tan 120$

Thus, $2\theta = 120°$

$$\theta = \frac{120}{2} = 60$$

Hence, $\theta = 60°$

9. Find the angle through which the axes may be turned so that the equation $x + 2y + 5 = 0$ may be reduced to the form $x = c$ and determine the value of c.

Solⁿ: Let θ be the required angle of rotation, so that $x + 2y + 5 = 0$ is reduced into the form $x = c$ so substitute

$$x = x\cos\theta - y\sin\theta$$
$$y = y\cos\theta + x\sin\theta$$

Thus,

$$(x\cos\theta - y\sin\theta) + 2(y\cos\theta + x\sin\theta) + 5 = 0$$
$$x[\cos\theta + 2\sin\theta] + y[2\cos\theta - \sin\theta] + 5 = 0$$

For coeff: $y = 0$

$$2\cos\theta - \sin\theta = 0$$
$$\sin\theta = 2\cos\theta$$
$$\tan\theta = 2$$

Thus, $\theta = \tan^{-1} 2$

Hence, $x(\cos\theta + 2\sin\theta) + y(2\cos\theta - \sin\theta) + 5 = 0$ is reduced into $x(\cos\tan^{-1}2 + 2\sin\tan^{-1}2) + 5 = 0$

Since $\tan\theta = \frac{2}{1} = \frac{p}{b}$

$$h = \sqrt{2^2 + 1^2} \rightarrow h = \sqrt{5}$$

So, $\cos\tan^{-1}2 = \frac{1}{\sqrt{5}}$ $\sin\tan^{-1}2 = \frac{2}{\sqrt{5}}$

cos θ   sin θ

or, $x\left(\frac{1}{\sqrt{5}} + 2 \times \frac{2}{\sqrt{5}}\right) + 5 = 0$

or, $\frac{5x}{\sqrt{5}} = -5$

or, $x = -\sqrt{5}$

Hence, $c = -\sqrt{5}$

Thus, $\theta = \tan^{-1}2$ and $c = -\sqrt{5}$

10. What does the equation of the straight lines $7x^2 + 4xy + 4y^2 = 0$ become when the axes are the bisectors of the angles between them.

Solⁿ: The equation of straight lines is $7x^2 + 4xy + 4y^2 = 0$ and of the axis are so rotated that they will bisect the angle between the original axis so $\theta = \frac{\pi}{4}$ is given.

how?

Hence
$$x = x\cos\theta - y\sin\theta$$
$$= x\cos 45 - y\sin 45$$
$$= \frac{x-y}{\sqrt{2}}$$

and
$$y = y\cos\theta + x\sin\theta$$
$$= y\cos 45 - x\sin 45$$
$$= \frac{y}{\sqrt{2}} - \frac{x}{\sqrt{2}}$$
$$= \frac{y-x}{\sqrt{2}}$$

Thus, substituting the values of x and y by $\frac{x-y}{\sqrt{2}}$ and $\frac{x+y}{\sqrt{2}}$ in given equation, yields;

$$7\left(\frac{x-y}{\sqrt{2}}\right)^2 + 4\left(\frac{x-y}{\sqrt{2}}\right)\left(\frac{x+y}{\sqrt{2}}\right) + 4\left(\frac{x+y}{\sqrt{2}}\right)^2 = 0$$

or, $7\left(\frac{x^2-2xy+y^2}{2}\right) + 4\left(\frac{x^2-y^2}{2}\right) + 4\left(\frac{x^2+2xy+y^2}{2}\right) = 0$

or, $\frac{7(x^2-2xy+y^2)+4(x^2-y^2)+4(x^2+y^2+2xy)}{2} = 0$

or, $7x^2 - 14xy + 7y^2 + 4x^2 - 4y^2 + 4x^2 + 4y^2 + 8xy = 0$

or, $15x^2 - 6xy + 7y^2 = 0$

11. Through what angle must the axes be rotated to remove the term containing xy in $3x^2 + 2xy + 3y^2 - \sqrt{2}x = 0$. Also find the transformed equation.

Solⁿ: Here, given equation is $3x^2 + 2xy + 3y^2 - \sqrt{2}x = 0$ let θ be the required angle of rotation so,

Put
$$x = x\cos\theta - y\sin\theta$$
$$y = y\cos\theta + x\sin\theta$$

Then, $3x^2 + 2xy + 3y^2 - \sqrt{2}x = 0$ becomes;

$$3(x\cos\theta - y\sin\theta)^2 + 2(x\cos\theta - y\sin\theta)(y\cos\theta + x\sin\theta) + 3(y\cos\theta + x\sin\theta)^2 - \sqrt{2}(x\cos\theta - y\sin\theta) = 0$$

or, $3[x^2\cos^2\theta - 2xy\sin\theta\cos\theta + y^2\sin^2\theta] + 2[xy\cos^2\theta - y^2\sin\theta\cos\theta + x^2\sin\theta\cos\theta - xy\sin^2\theta] + 3[y^2\cos^2\theta + xy\sin\theta\cos\theta + x^2\sin^2\theta] - \sqrt{2}[x\cos\theta - y\sin\theta] = 0$

Here coefficient xy is;

$$-6\sin\theta\cos\theta + 2\cos^2\theta - 2\sin^2\theta + 6\sin\theta\cos\theta$$

By question to remove the coefficient of xy it must be zero;

$$3\sin\theta\cos\theta = 2\cos^2\theta - 2\sin^2\theta - 3\sin\theta\cos\theta$$

or, $3\sin\theta\cos\theta = \cos 2\theta - 3\sin\theta\cos\theta$

or, $\cos 2\theta = 0 = \cos 90$

or, $2\theta = 90$

or, $\theta = \frac{\pi}{4} = 45^\circ$

Hence, the rotation must be 45° to remove xy term. Then the equation will be when it is rotated by 45° as;

$$x = x\cos 45 - y\sin 45 = \frac{x-y}{\sqrt{2}}$$

$$y = x\cos 45 + x\sin 45 = \frac{x+y}{\sqrt{2}}$$

Thus, equation is;

$$3\left(\frac{x-y}{\sqrt{2}}\right)^2 + 3\left(\frac{x+y}{\sqrt{2}}\right)^2 + 2\left(\frac{x-y}{\sqrt{2}}\right)\left(\frac{x+y}{\sqrt{2}}\right) - \sqrt{2}\left(\frac{x-y}{\sqrt{2}}\right) = 0$$

or, $3\frac{(x^2-2xy+y^2)}{2} + 3\frac{(x^2+2xy+y^2)}{2} + 2\frac{(x^2-y^2)}{2} - (x-y) = 0$

or, $3x^2 - 6xy + 3y^2 + 3x^2 + 6xy + 3y^2 + 2x^2 - 2y^2 - x + y = 0$

or, $6x^2 + 6y^2 - x + y = 0$ is the required equation obtained by rotating given axis with 45°.

12. **Transform the equation $12x^2 - 10xy + 2y^2 + 11x - 5y + 2 = 0$ by translating the axes into an equation with linear term missing.**

Sol^n: The given equation is $12x^2 - 10xy + 2y^2 + 11x - 5y + 2 = 0$ which is to be transferred in such a way that the linear terms are missing. Let the required transform is to the point (h, k) so;

$$x = x + h$$
$$y = y + k$$

Then,

$$12(x+h)^2 - 10(x+h)(y+k) + 2(y+k)^2 + 11(x+h) - 5(y+k) + 2 = 0$$

i.e. $12(x^2 + 2xh + h^2) - 10(xy + xk + hy + hk) + 2(y^2 + 2yk + k^2) + 11x + 11h - 5y - 5k + 2 = 0$

$$12x^2 + 2y^2 + x(24h - 10k + 11) + y(-10h + 4k - 5) + xy(-10) + 12h^2 - 10hk + 2k^2 + 11h - 5k + 2 = 0$$

For coeff. x = 0 = coeff. y

We have,

$$24h - 10k + 11 = 0 \ldots \text{(I)}$$
$$10h - 4k + 5 = 0 \ldots \text{(II)}$$

So,

$$48h - 20k + 22 = 0$$
$$\underline{50h - 20k + 25 = 0}$$

From I $(24 \times \frac{-3}{2} + 11 = 10k)$

$\therefore \quad k = -\frac{25}{10} = -\frac{5}{2}$

Hence,

$$(h, k) = \left(-\frac{3}{2}, -\frac{5}{2}\right)$$

Thus, equation reduces to;

$$12x^2 + 2y^2 + 12\left(-\frac{3}{2}\right)^2 - 10 \times -\frac{3}{2} \times -\frac{5}{2} - 10xy + 2\left(-\frac{5}{2}\right)^2 + 11 \times -\frac{3}{2} - 5 \times -\frac{5}{2} + 2 = 0$$

or, $12x^2 + 2y^2 + 12 \times \frac{9}{4} - \frac{150}{4} + \frac{50}{4} - \frac{33}{2} + \frac{25}{2} + 2 = 0$

or, $12x^2 + 2y^2 - 10xy + \frac{108}{4} - \frac{150}{4} + \frac{50}{4} - \frac{66}{4} + \frac{50}{4} + \frac{8}{4} = 0$

or, $12x^2 + 2y^2 - 10xy = 0$

or, $6x^2 - 5xy + 2y^2 = 0$ is the required equation after transforming the equation in such a way that the linear terms are missing.

13. **The equation $x^2 - y^2 = a^2$ is transferred to $xy = c^2$ by a change of rectangular axes. Find the inclination of new axes to the original axes and the value of c^2.**

Sol^n: Since $x^2 - y^2 = a^2$ is transformed so, let the axis is rotated by θ, so

$$x = x\cos\theta - y\sin\theta$$
$$y = y\cos\theta + x\sin\theta$$
$$(x\cos\theta - y\sin\theta)^2 - (y\cos\theta + x\sin\theta)^2 = a^2$$

or, $x^2\cos^2\theta - 2xy\sin\theta + y^2\sin^2\theta - y^2\cos^2\theta - 2xy\sin\theta\cos\theta - x^2\sin^2\theta = a^2$

or, $x^2(\cos^2\theta - \sin^2\theta) + y^2(\sin^2\theta - \cos^2\theta) + xy[-2\sin\theta\cos\theta - 2\sin\theta\cos\theta] = a^2$

Since $x^2 - y^2$ terms are missing

So, $\cos^2\theta - \sin^2\theta = 0$

$\sin^2\theta - \cos^2\theta = 0$

Hence,

$$\cos^2\theta - \sin^2\theta = 0$$

or, $\cos 2\theta = 0 = \cos 90$

or, $2\theta = 90$

or, $\theta = 45°$

Hence,

$$xy[-4\sin\theta\cos\theta] = a^2$$

or, $xy[-2\sin 2\theta] = a^2$

or, $xy[-2\sin 90] = a^2$

or, $xy(-2) = a^2$

or, $xy = -\frac{a^2}{2}$

Comparing it with $xy = c^2$ yields;

$$c^2 = -\frac{a^2}{2} \text{ and the rotation is } \theta = 45^\circ$$

14. **If (x, y) and (x_1, y_1) be the co–ordinates of the same point referred to two sets of rectangular axes with same origin and if $ux + vy$ where u and v are independent x, y, become $u_1x_1 + v_1y_1$. Show that $u^2 + v^2 = u_1{}^2 + v_1{}^2$.**

Sol^n: By question;

Two axes have same origin so let Q be the rotation. Then,

$$x = x_1 \cos\theta - y_1 \sin\theta \text{ and}$$
$$y = y_1 \cos\theta + x_1 \sin\theta$$

Then, by question,

$$ux + vy = u_1x_1 + v_1y_1$$

i.e. $u(x_1 \cos\theta - y_1 \sin\theta) + v(y_1 \cos\theta + x_1 \sin\theta) = u_1x_1 + v_1y_1$

i.e. $x_1[u\cos\theta + v\sin\theta] + y_1[v\cos\theta - u\sin\theta] = u_1x_1 + v_1y_1$

Thus,

$$u_1 = u\cos\theta + v\sin\theta$$
$$v_1 = v\cos\theta - u\sin\theta$$

Thus,

$$\begin{aligned} u_1{}^2 + v_1{}^2 &= (u\cos\theta + v\sin\theta)^2 + (v\cos\theta - u\sin\theta)^2 \\ &= u^2\cos^2\theta + v^2\sin^2\theta + v^2\cos^2\theta + u^2\sin^2\theta \\ &\qquad -2uv\sin\theta\cos\theta + 2uv\sin\theta\cos\theta \\ &= u^2(\cos^2\theta + \sin^2\theta) + v^2(\sin^2\theta + \cos^2\theta) \\ &= u^2 + v^2 \text{ Proved.} \end{aligned}$$

15. **Transform the equation $x^2 - 2xy + y^2 + x - 3y = 0$ to axes through the point $(-1, 0)$ parallel to the lines bisecting the angles between original axes.**

Sol^n: The given equation is $x^2 - 2xy + y^2 + x - 3y = 0$ so, the angle between the axis is 90, so 17 bisects is $\theta = 45°$. Hence,

$$x = x\cos\theta - y\sin\theta + h = x\cos\frac{\pi}{4} - y\sin\frac{\pi}{4} - 1 = \frac{x-y-\sqrt{2}}{\sqrt{2}}$$

$$y = y\cos\theta + x\sin\theta + 0 = \frac{y+x}{\sqrt{2}}$$

Thus, substituting yields;

$$\left(\frac{x-y}{\sqrt{2}} - 1\right)^2 - 2\left(\frac{x-y}{\sqrt{2}} - 1\right)\left(\frac{y+x}{\sqrt{2}}\right) + \left(\frac{y+x}{\sqrt{2}}\right)^2 + \frac{x-y-\sqrt{2}}{\sqrt{2}} - 3\left(\frac{y+x}{\sqrt{2}}\right) = 0$$

or, $\left(\frac{x-y}{\sqrt{2}}\right)^2 - 2\left(\frac{x-y}{\sqrt{2}}\right) + 1 - 2\left(\frac{x-y}{\sqrt{2}}\right)\frac{(x+y)}{\sqrt{2}} + 2\frac{(x+y)}{\sqrt{2}} + \left(\frac{x+y}{\sqrt{2}}\right)^2 + \frac{x-y}{\sqrt{2}} - 1 - 3\frac{(x+y)}{\sqrt{2}} = 0$

or, $\frac{x^2-2xy+y^2}{2} - 2\left(\frac{x-y}{\sqrt{2}}\right) + 1 - 2\left(\frac{x^2-y^2}{2}\right) + 2\left(\frac{x+y}{\sqrt{2}}\right) + \frac{x^2+2xy+y^2}{2} + \frac{x-y}{\sqrt{2}} - 1 - 3\left(\frac{x+y}{\sqrt{2}}\right) = 0$

or, $\frac{x^2}{2} - \frac{2xy}{2} - \frac{y^2}{2} - 2\left(\frac{x-y}{\sqrt{2}}\right) + 1 - 2\left(\frac{x^2}{2} - \frac{y^2}{2}\right) + 2\left(\frac{x+y}{\sqrt{2}}\right) + \frac{x^2}{2} + \frac{y^2}{2} + \frac{2xy}{2} + \frac{x}{\sqrt{2}} - \frac{y}{\sqrt{2}} - 1 - \frac{3x}{\sqrt{2}} - \frac{3y}{\sqrt{2}} = 0$

or, $x^2 + y^2 - x^2 + y^2 + \frac{2x}{\sqrt{2}} + \frac{2y}{\sqrt{2}} + \frac{x}{\sqrt{2}} - \frac{y}{\sqrt{2}} - 1 - \frac{3x}{\sqrt{2}} - \frac{3y}{\sqrt{2}} = 0$

or, $2y^2 + 2\left(\frac{x+y}{\sqrt{2}}\right) - \frac{2x}{\sqrt{2}} - \frac{4y}{\sqrt{2}} = 1$

or, $2y^2 + \sqrt{2}(x+y) - \frac{2x-4y}{\sqrt{2}} = 1$

or, $2\sqrt{2}y^2 + 2(x+y) - (2x-4y) = \sqrt{2}$

or, $2\sqrt{2}y^2 + 6y - \sqrt{2} = 0$ is the required solution

Exercise 33

1. Find the equation of ellipse

i) whose foci $(\pm 2, 0)$ and $e = \frac{1}{2}$

Sol^n: Here,

foci $(\pm 2, 0)$ and $e = \frac{1}{2}$

Here, the major axis is along x–axis

Given that: $ae = 2, e = \frac{1}{2}$

So,

$a \times \frac{1}{2} = 2$

$\therefore \quad a = 4$

Now,

$b^2 = a^2 - a^2e^2, a > b$

or, $b^2 = 16 - 4$

or, $b^2 = 12$

or, $b = \sqrt{12}$

Hence, the equation of ellipse is;

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

or, $\frac{x^2}{16} + \frac{y^2}{12} = 1$

or, $12x^2 + 16y^2 = 192$

or, $3x^2 + 4y^2 = 48$ is the required equation under given condition.

ii) whose vertices $(\pm 5, 0)$ and foci $= (\pm 4, 0)$

Sol^n: Here,

Vertices $(\pm 5, 0)$ and foci $= (\pm 4, 0)$

Here, the major axis is along x–axis hence vertices $a = 5$ and the foci $ae = 4$ thus $e = \frac{4}{5}$

Since, the major axis is along x–axis;

$b^2 = a^2(1 - e^2)$ for $a > b$

or, $b^2 = 25\left(1 - \frac{16}{25}\right)$

or, $b^2 = 25\left(\frac{9}{25}\right) = 9$

or, $b = \sqrt{9}$

Hence, the equation of eclipse is;

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

or, $\frac{x^2}{25} + \frac{y^2}{9} = 1$

or, $9x^2 + 25y^2 = 225$ is the required equation.

iii) focus $(2, 4)$, eccentricity $= \frac{1}{\sqrt{3}}$ and directrix is $x - y + 13 = 0$

Sol^n: Here,

focus $(2, 4)$, eccentricity $= \frac{1}{\sqrt{3}}$ and directrix is $x - y + 13 = 0$

We know that, if (x, y) be any point on ellipse then the e times perpendicular distance from (x, y) to $x - y + 13 = 0$ must equal the distance of (x, y) from focus, so,

$(x - 2)^2 + (y - 4)^2 = e^2\left(\frac{x-y+13}{\sqrt{2}}\right)^2$

or, $x^2 - 4x + 4 + y^2 - 8y + 16 = \frac{1}{13}\frac{(x-y+13)^2}{2}$

or, $26x^2 + 26y^2 - 104x - 208y + 520 = x^2 + y^2 - 2xy + 26x - 26y + 169$

or, $25x^2 + 25y^2 + 2xy - 130x - 182y + 451 = 0$ is the required equation of eclipse under given condition.

iv) focus $(-1, 1)$, $e = \frac{1}{2}$ and directrix is $x - y + 3 = 0$

Sol^n: Here,

focus $(-1, 1)$, $e = \frac{1}{2}$ and directrix is $x - y + 3 = 0$

By definition, we know that the distance from any point (x, y) of the ellipse to the focus $(-1, 1)$ is equal to e times the perpendicular distance from the point to the directrix. So,

$(x + 1)^2 + (y - 1)^2 = \frac{1}{4}\left[\frac{x-y+3}{\sqrt{2}}\right]^2$

or, $x^2 + 2x + 1 + y^2 - 2y + 1 = \frac{1}{8}[x^2 - 2xy + y^2 + 6x - 6y + 9]$

or, $8x^2 + 8y^2 + 16x - 16y = 16 = x^2 - 2xy + y^2 + 6x - 6y + 9$

or, $7x^2 + 7y^2 + 2xy + 10x - 10y + 7 = 0$ is the required equation of ellipse.

2. **Find the equation of the ellipse whose centre is the origin, whose axes are the axes of co–ordinates and which passes through**

i) **the point $(1, 4)$ and $(-3, 2)$**

Sol^n: Here,

The equation of ellipse having centre as origin and axis the axis of coordinate is;

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \ldots \text{(i)}$$

By question (i) passes through $(1, 4)$

So, $\frac{1}{a^2} + \frac{16}{b^2} = 1 \ldots$ (ii)

and since it passes through $(-3, 2)$ so is;

$$\frac{9}{a^2} + \frac{4}{b^2} = 1 \ldots \text{(iii)}$$

From (ii) and (iii) solving yields;

$$\begin{array}{r} \frac{9}{a^2} + \frac{144}{b^2} = 9 \\ \underline{-\frac{9}{a^2} \mathbin{\underset{-}{+}} \frac{4}{b^2} = \underset{-}{1}} \\ \frac{140}{b^2} = 8 \end{array}$$

or, $8b^2 = 140$

or, $b^2 = \frac{70}{4} = \frac{35}{2}$

Similarly,

$$\begin{array}{r} \frac{1}{a^2} + \frac{16}{b^2} = 1 \\ \underline{-\frac{36}{a^2} \mathbin{\underset{-}{+}} \frac{16}{b^2} = \underset{-}{4}} \\ -\frac{35}{a^2} = -3 \end{array}$$

or, $a^2 = \frac{35}{3}$

Hence, equation is $\frac{3x^2}{35} + \frac{2y^2}{35} = 1$

Thus, $3x^2 + 2y^2 = 35$ is the required equation of ellipse.

ii) **the point $(2, 1)$ and $(1, -3)$**

Sol^n: Here,

Let the equation of ellipse is;

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \ldots \text{(i)}$$

Since it passes through (2, 1) and (1, –3), these coordinate mist satisfy (i) so;

$$\frac{4}{a^2} + \frac{1}{b^2} = 1 \ldots \text{(ii)}$$

or, $\frac{1}{a^2} + \frac{9}{b^2} = 1 \ldots$ (iii)

Solving yields,

$$\begin{array}{r} \frac{4}{a^2} + \frac{1}{b^2} = 1 \\ \underline{-\frac{4}{a^2} \mathbin{\underset{-}{+}} \frac{36}{b^2} = \underset{-}{4}} \\ -\frac{35}{b^2} = -3 \end{array}$$

$$b^2 = \frac{35}{3}$$

Similarly,

$$a^2 = \frac{35}{8}$$

Hence, the required equation of ellipse is $\frac{x^2}{\frac{35}{8}} + \frac{y^2}{\frac{35}{3}} = 1$

or, $8x^2 + 3y^2 = 35$ is the required equation of ellipse.

3. **Find the equation of ellipse whose latus rectum 5 eccentricity $\frac{2}{3}$ and its axes as the axes of co–ordinates.**

Sol^n: Here,

The equation of ellipse is;

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \ldots \text{(i)}$$

By the question,

Eccentricity $e = \frac{2}{3}$ and

Latus rectum $\frac{2b^2}{a} = 5 \ldots$ (ii)

Since,

$b^2 = a^2(1 - e^2)$

or, $b^2 = a^2\left(1 - \frac{4}{9}\right)$

or, $\frac{5a}{2} = a^2 \times \frac{5}{9}$

or, $9a = 2a^2$

or, $a = \frac{9}{2}$

Thus,

$b^2 = a^2(1 - e^2) = \frac{9}{2} \times \frac{9}{2}\left(1 - \frac{4}{9}\right) = \frac{81}{4} \times \frac{5}{9} = \frac{45}{4}$

$\therefore$ $b^2 = \frac{45}{4}$.

Hence, required equation of ellipse is;

$\frac{x^2}{\frac{81}{4}} + \frac{y^2}{\frac{45}{4}} = 1$

or, $\frac{4x^2}{81} + \frac{4y^2}{45} = 1$

or, $\frac{20x^2+36y^2}{405} = 1$

i.e. $20x^2 + 36y^2 = 405$ is required ellipse.

4. Find the eccentricity, latus rectum, directrix and foci of the ellipse.

i) $\mathbf{3x^2 + 4y^2 = 12}$

Sol^n: Here,

or, $\frac{x^2}{4} + \frac{y^2}{3} = 1$

Here,

$a > b$

Thus, eccentricity is obtained from

$b^2 = a^2(1 - e^2)$

or, $3 = 4(1 - e^2)$

or, $\frac{3}{4} = 1 - e^2$

or, $e^2 = 1 - \frac{3}{4} = \frac{1}{4}$

or, $e = \frac{1}{2}$

Latus rectum $= \frac{2b^2}{a} = \frac{2\times3}{2} = 3$

Directrix is $x = \pm\frac{a}{e} = \pm\frac{2}{\frac{1}{2}} = \pm4$

Foci $= (\pm ae, 0) = (\pm1, 0)$

ii) $\mathbf{\frac{x^2}{9} + \frac{y^2}{16} = 1}$

Sol^n: Here,

$a < b$, thus for eccentricity,

$a^2 = b^2(1 - e^2)$

or, $9 = 16(1 - e^2)$

or, $\frac{9}{16} = 1 - e^2$

i.e. $e^2 = 1 - \frac{9}{16} = \frac{7}{16}$

so, $e = \frac{\sqrt{7}}{4}$

We have,

Latus rectum $= \frac{2a^2}{b} = \frac{2\times9}{4} = \frac{9}{2}$

The equation of directrix $y = \pm\frac{b}{e}$

$= \pm\frac{4}{\frac{\sqrt{7}}{4}} = \pm\frac{16}{\sqrt{7}}$

The coordinate of foci $= (0, \pm be) = \left(0, \pm4 \times \frac{\sqrt{7}}{4}\right) = (0, \pm\sqrt{7})$

iii) $\mathbf{4x^2 + y^2 - 8x + 2y + 1 = 0}$

Sol^n: Here, $4x^2 + y^2 - 8x + 2y + 1 = 0$ can be written as;

$4x^2 + y^2 - 8x + 2y + 1 = 0$

$4(x^2 - 2x) + y^2 + 2y + 1 = 0$

or, $4(x^2 - 2x + 1) + y^2 + 2y + 1 - 4 = 0$

or, $4(x - 1)^2 + (y + 1)^2 = 4$

or, $\frac{(x-1)^2}{1} + \frac{(y+1)^2}{4} = 1$

Here,

$b > a$

Thus, for eccentricity

$a^2 = b^2(1 - e^2)$

or, $1 = 4(1 - e^2)$

or, $\frac{1}{4} = 1 - e^2$

or, $e^2 = 1 - \frac{1}{4} = \frac{3}{4}$

Hence,

$$e = \frac{\sqrt{3}}{2}$$

The length of latus rectum is $\frac{2a^2}{b} = \frac{2 \times 1}{2} = 1$

The equation of directrix is;

$$y = k \pm \frac{b}{c} = -1 \pm \frac{2}{\frac{\sqrt{3}}{2}} = -1 \pm \frac{4}{\sqrt{3}}$$

The coordinate of foci $= (h, k \pm be)$

$$= \left(1, -1 \pm 2 \times \frac{\sqrt{3}}{2}\right)$$

$$= \left(1, -1 \pm \sqrt{3}\right)$$

iv) $\mathbf{7x^2 + 6y^2 - 42x - 24y + 86 = 0}$

Sol^n: Here,

$7x^2 + 6y^2 - 42x - 24y + 86 = 0$ can be written as;

$7(x^2 - 6x) + 6(y^2 - 4y) + 86 = 0$

or, $7(x^2 - 6x + 9) + 6(y^2 - 4y + 4)86 = 63 + 24$

or, $7(x - 3)^2 + 6(y - 2)^2 = 1$

or, $\frac{(x-7)^2}{\frac{1}{7}} + \frac{(y-2)^2}{\frac{1}{6}} = 1$

Since $b > a$, so for eccentricity

$a^2 = b^2(1 - e^2)$

or, $\frac{1}{7} = \frac{1}{6}(1 - e^2)$

or, $\frac{6}{7} = 1 - e^2$

or, $e^2 1 - \frac{6}{7} = \frac{1}{7}$

So, $e = \frac{1}{\sqrt{7}}$

The latus rectum $= \frac{2a^2}{b}$

or, $\frac{2 \times \frac{1}{7}}{\frac{1}{\sqrt{6}}} = \frac{2\sqrt{6}}{7}$

The equation of directrix is;

$$y = k \pm \frac{b}{e} = 2 \pm \frac{\sqrt{7}}{\sqrt{6}}$$

The coordinate of foci $= (h, k \pm be) = \left(3, 2 \pm \frac{1}{\sqrt{42}}\right)$

v) $\mathbf{8(x - 1)^2 + 6(y + 1)^2 = 1}$

Sol^n: Here,

The equation of ellipse given is;

$8(x - 1)^2 + 6(y + 1)^2 = 1$ which can be written as;

$$\frac{(x-1)^2}{\frac{1}{8}} + \frac{(y+1)^2}{\frac{1}{6}} = 1$$

Here,

$b > a$, so for eccentricity we use

$a^2 = b^2(1 - e^2)$

or, $\frac{1}{8} = \frac{1}{6}(1 - e^2)$

or, $\frac{3}{4} = 1 - e^2$

or, $e^2 = \frac{1}{2}$

The latus rectum $= \frac{2a^2}{b} = \frac{2 \times \frac{1}{8}}{\sqrt{\frac{1}{6}}} = \frac{2}{8} \times \frac{\sqrt{6}}{1} = \frac{\sqrt{6}}{4}$

The equation of latus rectum is;

$$y = k \pm \frac{b}{e} = -1 \pm \frac{1}{\sqrt{6}} \times \frac{2}{1} = -1 \pm \frac{2}{\sqrt{6}}$$

The coordinate of foci are $h, k \pm be$

$$= \left(1, -1 \pm \frac{1}{\sqrt{6}} \times \frac{1}{2}\right)$$

$$= \left(1, -1 \pm \frac{1}{2\sqrt{6}}\right)$$

5. **Find the centre, length of axes, eccentricity and directrix of the ellipse.**

i) $\mathbf{2x^2 + 3y^2 - 4x + 5y + 4 = 0}$

Sol^n: Here,

The given equation of ellipse is;

$2x^2 + 3y^2 - 4x + 5y + 4 = 0$ which can be written as;

or, $2(x^2 - 2x) + 3\left(y^2 + \frac{5}{3}y\right) + 4 = 0$

or, $(x^2-2x+1)+3\left(y^2+2y\frac{5}{6}y+\frac{25}{36}\right)-2-3\times\frac{25}{36}+4=0$

or, $2(x-1)^2+3\left(y+\frac{5}{6}\right)^2=\frac{75}{36}-2=\frac{3}{36}=\frac{1}{12}$

or, $\frac{(x-1)^2}{\frac{1}{24}}+\frac{\left(y+\frac{5}{6}\right)^2}{\frac{1}{36}}=1$

Thus centre is $(h,k)=\left(1,-\frac{5}{6}\right)$

Length of axis are;

Major axis $=2\times\frac{1}{\sqrt{24}}=\frac{2}{2\sqrt{6}}=\frac{1}{\sqrt{6}}$

Minor axis $2\times\frac{1}{\sqrt{36}}=\frac{2}{6}=\frac{1}{3}$

For eccentricity, since $a>b$

$b^2=a^2(1-e^2)$

or, $\frac{1}{36}=\frac{1}{24}(1-e^2)$

or, $\frac{2}{3}=1-e^2$

or, $e^2=1-\frac{2}{3}=\frac{1}{3}$

or, $e=\frac{1}{\sqrt{3}}$

The equation of directrix is;

$$x=h\pm\frac{a}{e}=1\pm\frac{1}{\sqrt{24}}\times\frac{\sqrt{3}}{1}=1\pm\frac{1}{\sqrt{8}}=1\pm\frac{1}{2\sqrt{2}}$$

ii) $\mathbf{2x^2+3y^2-4x-12y+13=0}$

Sol^n: Here,

The given equation of ellipse is;

$2x^2+3y^2-4x-12y+13=0$ which can be written as

or, $2(x^2-2x+1)+3(y^2-4y+4)-2-12+13=0$

or, $\frac{(x-1)^2}{\frac{1}{2}}+\frac{(y-2)^2}{\frac{1}{3}}=1$

Since $a>b$, so,

For eccentricity we use

$b^2=a^2(1-e^2)$

or, $\frac{1}{3}=\frac{1}{2}(1-e^2)$

or, $\frac{2}{3}=1-e^2$

or, $e^2=1-\frac{2}{3}=\frac{1}{3}$

or, $e=\frac{1}{\sqrt{3}}$

Centre is $(h,k)=(1,2)$

Length of major axis $=2a=2\times\frac{2}{\sqrt{2}}=\sqrt{2}$

The length of minor axis $=2\times\frac{1}{\sqrt{3}}=\frac{2}{\sqrt{3}}$

The equation of directrix is;

$$x=h\pm\frac{a}{e}=1\pm\sqrt{\frac{3}{2}}$$

6. **Find the co–ordinates of the second focus and the equation of second directrix of the ellipse whose one focus is $S(1,2)$ and the corresponding directrix is the line $x-y=5$ and $e=\frac{1}{2}$.**

Sol^n: The coordinate of focus and the equation of directrix is required to find. One focus is $S(1,2)$ and the corresponding directrix is $x-y=5$ and eccentricity $e=\frac{1}{2}$.

Since the equation of directrix is $x-y=5\rightarrow$ (i) whose slope is 1 so the major axis has slop -1. The equation of major axis passes through focus $(1,2)$ so ……. Equation with slope -1 is;

$y-2=-1(x-1)$

$y-2=-x+1$

$y+x-3=0\longrightarrow$ (ii)

Solving (i) and (ii) gives

$x=4$

$y=-1$

Thus, we have,

$\frac{SA}{AZ}=\frac{1}{2}$

Hence A divides SZ internally;

So, $A=\left(\frac{m_1x_2+m_2x_1}{x_1+m_2},\frac{m_1y_2+m_2y_1}{m_1+m_2}\right)$

$=\left(\frac{1\times4+2\times1}{1+2},\frac{1\times-1+2\times2}{3}\right)$

$=(2,1)$

and $A^1=(-2,5)$

Since, C is centre so $(0,3)$ C is mid–point of ZZ^1 as well

So,

Let, $Z = (h,k)$, then

or, $\frac{h+4}{2} = 0$

$\therefore$ $\boxed{h = -4}$

and $\frac{k-1}{2} = 3$

or, $k - 1 = 6$

$\therefore$ $\boxed{k = 7}$

Hence,

$Z^1 = (-4,7)$

The directrix is parallel to $z - y - 5 = 0$ is $x - y + k = 0$ which passes through $(-4,7)$ is;

$-4 - 7 + k = 0$

or, $k = 11$

So the required directrix is $x - y + 11 = 0$.

To find focus let (h^1, k^1) be focus:

So, $\left(\frac{h^1+1}{2}, \frac{k^1+2}{2}\right) = (0,3)$

or, $\frac{h^1+1}{2} = 0$

or, $h^1 = -1$

and $\frac{k^1+2}{2} = 3$

or, $k^1 = 4$

Thus, focus is $(-1,4)$

7. **By transferring the origin to the point $(2,3)$ and turning the axes through an angle $\frac{\pi}{4}$ prove that $3x^2 + 2xy + 3y^2 - 18x - 22y + 50 = 0$ represents an ellipse.**

Sol^n: If the axis is turned through $\frac{\pi}{4}$ and origin is transformed to $(2,3)$, then the equation is changed and is obtained by substituting

$$x = x\cos\theta - y\sin\theta + h$$
$$y = y\cos\theta + x\sin\theta + k$$

i.e. $x = x\cos\frac{\pi}{4} - y\sin\frac{\pi}{4} + 2 = \frac{x}{\sqrt{2}} - \frac{y}{\sqrt{2}} + 2 = \frac{x-y}{\sqrt{2}} + 2$

$$y = \frac{y}{\sqrt{2}} + \frac{x}{\sqrt{2}} + k = \frac{x+y}{\sqrt{2}} + 3$$

or, $3\left(\frac{x-y}{\sqrt{2}} + 2\right)^2 + 2\left(\frac{x+y}{\sqrt{2}} + 3\right)\left(\frac{x-y}{\sqrt{2}} + 2\right) + 3\left(\frac{x+y}{\sqrt{2}} + 3\right)^2$
$-18\left(\frac{x-y}{\sqrt{2}} + 2\right) - 22\left(\frac{x+y}{\sqrt{2}} + 3\right) + 50 = 0$

or, $\frac{3}{2}(x-y)^2 + \frac{12}{\sqrt{2}}(x-y) + 12 + x^2 - y^2 + \frac{6}{\sqrt{2}}(x-y) + 12$
$+ \frac{4}{\sqrt{2}}(x+y) + \frac{3}{2}(x+y)^2 + \frac{18}{\sqrt{2}}(x+y) + 27 - \frac{18}{\sqrt{2}}(x-y) - 36$
$- \frac{22}{\sqrt{2}}(x+y) - 66 + 50 = 0$

or, $\frac{3}{2}(x-y)^2 + \frac{3}{2}(x+y)^2 + x^2 - y^2 - 1 = 0$

or, $3[x^2 - 2xy + y^2 + x^2 + 2xy + y^2] + 2x^2 - 2y^2 - 2 = 0$

$\therefore$ $8x^2 + 4y^2 - 2 = 0$

$\therefore$ $4x^2 + 2y^2 = 1$

Hence, transformed equation is $4x^2 + 2y^2 = 1$

i.e. $\frac{x^2}{\frac{1}{4}} + \frac{y^2}{\frac{1}{2}} = 1$ which is the equation of ellipse.

8. **Show that the point $\frac{1-t^2}{1+t^2}$ and $\frac{2t}{1+t^2}$ where t is a variable parameter, lies on an ellipse.**

Sol^n: Here,

$$x = a\frac{(1-t^2)}{1+t^2}$$
$$y = b\left(\frac{2t}{1+t^2}\right)$$

We have to show that, these points lies on the equation of ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Substituting values of x and y yields;

$$\frac{a^2}{a^2}\left(\frac{1-t^2}{1+t^2}\right)^2 + b^2\left(\frac{2t}{1+t^2}\right)^2$$
$$= \frac{(1-t^2)^2}{(1+t^2)^2} + \frac{4t^2}{(1+t^2)^2} = \frac{(1-t^2)^2 + 4t^2}{(1+t^2)^2} = \frac{(1+t^2)^2}{(1+t^2)^2} = 1$$

Hence, proved

9. **A point P moves in such a way that the sum of its distances from S and S_1 is 10 and $SS_1 = 8$. Show that the locus of P is an ellipse.**

Sol^n: Let P(x, y) be any point on locus and S and S' be considered in such a way that $SS' = 8$. So let $S(-4, 0)$ and $S'(4, 0)$ be standard co–ordinates on x–axis. By question;

$PS + PS' = 10$

or, $\sqrt{(x+4)^2+y^2} + \sqrt{(x-4)^2+y^2} = 10$

or, $\sqrt{(x+4)^2+y^2} - 10 = -\sqrt{(x-4)^2+y^2}$

Squaring both sides, we get,

$(x+4)^2 + y^2 - 20\sqrt{(x+4)^2+y^2} + 100 = x^2 - 8x + 16 + y^2$

or, $x^2 + 8x + 16 + y^2 - 20\sqrt{(x+4)^2+y^2} + 100 = x^2 - 8x + 16 + y^2$

or, $16x - 100 = 20\sqrt{(x+4)^2+y^2}$

Again, squaring both sides, we get,

$(16x - 100)^2 = 400(x^2 - 8x + 16 + y^2)$

or, $16(4x - 25)^2 = 400(x^2 - 8x + 16 + y^2)$

or, $(4x - 25)^2 = 25(x^2 - 8x + 16 + y^2)$

or, $16x^2 - 200x + 625 = 25x^2 - 200x + 400 + 25y^2$

or, $9x^2 + 25y^2 = 225$

or, $\frac{x^2}{25} + \frac{y^2}{9} = 1$ which is the required equation of ellipse.

Note:

You can prove for the co–ordinate $S(-4, 0)$ and $S'(4, 0)$ as well.

10. **Find the equation of the ellipse whose foci are at (−2, 4) and (4, 4) and major axis is 10. Also find the eccentricity of the ellipse.**

Sol^n: The foci of ellipse are $(-2, 4)$ and $(4, 4)$ and major axis is 10.

We have,

$(h - ae, k) = (-2, 4)$

$(h + ae, k) = (4, 4)$

and major axis;

$2a = 10$

$a = 5$

Thus,

$h - ae = -2$

$\underline{h + ae = 4}$

$2h = 2$

$h = 1$

so, $h - ae = -2$

or, $1 - 5 \times e = -2$

or, $-5e = -2 - 1$

$\therefore$ $\boxed{e = \frac{3}{5}}$

We know that,

$b^2 = a^2(1 - e^2)$

or, $b^2 = 5^2\left(1 - \frac{9}{25}\right)$

or, $b^2 = 25\left(\frac{16}{25}\right)$

$\therefore$ $\boxed{b^2 = 16}$

Again $h = 1$ and $k = 4$ so centre is $(1, 4)$

So equation is;

$$\frac{(x-1)^2}{25} + \frac{(y-4)^2}{16} = 1$$

or, $16(x^2 - 2x + 1) + 25(y^2 - 8y + 16) = 400$

or, $16x^2 - 32x + 16 + 25y^2 - 200y + 400 = 400$

or, $16x^2 + 25y^2 - 32x - 200y + 16 = 0$ is the required equation of ellipse under given condition.

11. **Find the eccentric angle of a point on the ellipse $\frac{x^2}{4} + \frac{y^2}{5} = 2$ whose distance from the centre is $\frac{\sqrt{34}}{2}$.**

Sol^n: The equation of ellipse is;

$\frac{x^2}{4} + \frac{y^2}{5} = 2$

i.e. $\frac{x^2}{8} + \frac{y^2}{10} = 1$

Let, ϕ be the eccentric angle of point P on ellipse $\frac{x^2}{8} + \frac{y^2}{10} = 1$

The point P is;

$P(\sqrt{8}\cos\phi, \sqrt{10}\sin\phi)$

The centre of $\frac{x^2}{8} + \frac{y^2}{10} = 1$ is O(0, 0)

By question;

The distance $OP = \frac{\sqrt{34}}{2}$

so, $8\cos^2\phi + 10\sin^2\phi = \frac{34}{4}$

or, $16\cos^2\phi + 20\sin^2\phi - 17 =$

or, $16\cos^2\phi + 20 - 20\cos^2\phi - 17 = 0$

or, $-4\cos^2\phi + 3 = 0$

or, $4\cos^2\phi = 3$

or, $\cos\phi = \pm\frac{\sqrt{3}}{2}$

Thus,

$\phi = \pm\frac{\pi}{3}$ or $\pm\frac{4\pi}{3}$ is the required eccentric angles.

Exercise 34

1. **Find the equations of tangent and normal.**

i) **at the point (3, −2) of the ellipse $4x^2 + 9y^2 = 36$**

Sol[n]: The given equation of ellipse is $4x^2 + 9y^2 = 36$ which is of the form $\frac{x^2}{9} + \frac{y^2}{4} = 1$ and we have to find the equation of tangent to it at (3, −2) so by formula, equation of tangent is;

$$\frac{x.3}{9} + \frac{y(-2)}{4} = 1$$

i.e. $\frac{3x}{9} - \frac{2y}{4} = 1$

or, $12x - 18y = 36$

or, $2x - 3y = 6$ is required equation of tangent under given condition.

The given equation of ellipse is $\frac{x^2}{9} + \frac{y^2}{4} = 1$ and at (3, −2) we have to find the equation of normal.

The equation of normal at (3, −2) to $\frac{x^2}{9} + \frac{y^2}{4} = 1$ is;

$$\frac{x-3}{\frac{3}{9}} = \frac{y+2}{-\frac{2}{4}}$$

or, $3(x-3) = -2(y+2)$

or, $3x - 9 = -2y - 4$

So, $3x + 2y - 5 = 0$ is required equation of normal.

ii) **at the point $\left(1, \frac{4}{3}\right)$ of the ellipse $4x^2 + 9y^2 = 20$**

Sol[n]: The given equation of ellipse is $4x^2 + 9y^2 = 20$ which is of the form $\frac{x^2}{5} + \frac{y^2}{\frac{20}{9}} = 1$ and we have to find the equation of tangent at $\left(1, \frac{4}{3}\right)$. So by formula the equation of tangent is;

$$\frac{x.1}{5} + \frac{y.\frac{4}{3}}{\frac{20}{9}} = 1$$

or, $\frac{x}{5} + \frac{12y}{20} = 1$

or, $4x + 12y = 20$

or, $x + 3y = 5$ is required equation of tangent

The equation of normal is always perpendicular to tangent so is $3x - y + k = 0$. But it passes through $\left(1, \frac{4}{3}\right)$ so it must satisfy;

$$3x - y + k = 0$$

or, $3 \times 1 - \frac{4}{3} + k = 0$

or, $\frac{9-4}{3} = -k$

or, $k = -\frac{5}{3}$

or, $3x - y - \frac{5}{3} = 0$

or, $9x - 3y = 5$ is required equation of normal.

iii) at the end of the latus rectum of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Sol^n: The equation of ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ so the end of latus rectum are $\left(-ae, \frac{b^2}{a}\right)$ and $\left(-ae, \frac{-b^2}{a}\right)$.

Thus, the equation of tangent at $\left(-ae, \frac{b^2}{a}\right)$ to ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is;

$$\frac{x(-ae)}{a^2} + \frac{y\left(\frac{b^2}{a}\right)}{b^2} = 1$$

or, $\frac{-xe}{a} + \frac{y}{a} = 1$

or, $-xe + y = a$

Hence, tangent at $\left(-ae, \frac{b^2}{a}\right)$ is $-xe + y = a$. Similarly at $\left(-ae, \frac{-b^2}{a}\right)$ is $-xe - y = a$ i.e. $y + xe = -a$. For normal at $\left(-ae, \frac{b^2}{a}\right)$ normal is perpendicular to tangent so $-x - ey + k = 0$ since it passes through $\left(-ae, \frac{b^2}{a}\right)$ so,

$$ae - e \times \frac{b^2}{a} + k = 0$$

or, $a^2e - b^2e = -ka$

or, $e(a^2 - b^2) = -ka \ldots$ (i)

Since,

$$b^2 = a^2(1 - e^2)$$

or, $b^2 - a^2 = -a^2e^2$

Hence,

$$-ka = a^2e^2 \times e$$

or, $k = -ae^3$

Thus, normal at $\left(-ae, \frac{b^2}{a}\right)$ is;

$$-x - ey - ae^3 = 0$$

or, $x + ey = ae^3$

Similarly for another

2. **Find the equation of the tangents to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and passes through the point $(2, 3)$.**

Sol^n: We know that the equation of tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is;

$$y = mx \pm \sqrt{a^2m^2 + b^2}$$

So the equation of tangent to

$$\frac{x^2}{16} + \frac{y^2}{9} = 1 \text{ is}$$

or, $y = mx \pm \sqrt{16m^2 + 9} \ldots$ (i)

Since it passes through $(2, 3)$, so it satisfies (i)

$$3 = 2m \pm \sqrt{16m^2 + 9}$$

or, $3 - 2m = \pm\sqrt{16m^2 + 9}$

Squaring both sides, yields;

$$(3 - 2m)^2 = 16m^2 + 9$$

or, $9 - 12m + 4m^2 = 16m^2 + 9$

or, $-12m + 4m^2 - 16m^2 = 0$

or, $-12m^2 - 12m = 0$

So, $m = -1$ or $m = 0$

Thus, equations of tangents are

$$y - 3 = -1(x - 2)$$

or, $y - 3 = -x + 2$

or, $\boxed{y + x = 5}$ and

$$y - 3 = 0(x - 2)$$

or, $\boxed{y = 3}$

3. **Show that $4x + y + 7 = 0$ is a tangent in the ellipse $x^2 + 3y^2 = 3$ also find the point of contact.**

Solⁿ: The equation of ellipse is $x^2 + 3y^2 = 3$ which is of the form $\frac{x^2}{3} + \frac{y^2}{1} = 1$ and the given line is $4x + y + 7 = 0$

i.e. $y = -4x - 7$

Now the point of intersection of $y = -4x - 7$ and ellipse $x^2 + 3y^2 = 3$ is;

$x^2 + 3(-4x - 7)^2 = 3$

i.e. $x^2 + 3(16x^2 + 56x + 49) = 3$

or, $x^2 + 48x^2 + 168x + 147 = 3$

or, $49x^2 + 168x + 147 = 3$

or, $49x^2 + 168x + 144 = 0$

or, $(7x)^2 + 2 \times 7x \times 12x + (12)^2 = 0$

or, $(7x + 12)^2 = 0$

or, $7x + 12 = 0$

or, $x = -\frac{12}{7}$

When,

$x = -\frac{12}{7}$

$y = +4 \times \frac{12}{7} - 7 = -\frac{1}{7}$

Since it is perfect square so $y + 4x + 7 = 0$ is tangent and the point of contact is $\left(-\frac{12}{7}, -\frac{1}{7}\right)$

4. **Show that $x + y = \sqrt{a^2 + b^2}$ is a tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and find the co–ordinates of the point of contact.**

Solⁿ: The equation of ellipse is;

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

and line is;

$x + y = \sqrt{a^2 + b^2}$

i.e. $y = \sqrt{a^2 + b^2} - x$ ... (i)

Thus, their point of contact is;

$b^2x^2 + a^2\left(\sqrt{a^2 + b^2} - x\right)^2 = a^2b^2$

i.e. $b^2x^2 + a^2\left(a^2 + b^2 - 2x\sqrt{a^2 + b^2} + x^2\right) = a^2b^2$

or, $b^2x^2 + a^4 + a^2b^2 - 2a^2\sqrt{a^2 + b^2}x + a^2x^2 = a^2b^2$

or, $(b^2 + a^2)x^2 - 2a^2\sqrt{a^2 + b^2}x + a^4 = 0$

or, $\left(\sqrt{a^2 + b^2}x\right)^2 - 2 \times a \times \sqrt{a^2 + b^2}x + (a^2)^2 = 0$

i.e. $\left(x\sqrt{a^2 + b^2} - a^2\right)^2 = 0$ which is perfect square so the given line is tangent.

To find point of contact:

$x\sqrt{a^2 + b^2} - a^2 = 0$

i.e. $x = \frac{a^2}{\sqrt{a^2+b^2}}$

From (i)

$y = \sqrt{a^2 + b^2} - \frac{a^2}{\sqrt{a^2+b^2}} = \frac{\left(a^2+b^2\right)-a^2}{\sqrt{a^2+b^2}}$

Thus, point of contact is $\left(\frac{a^2}{\sqrt{a^2+b^2}}, \frac{b^2}{\sqrt{a^2+b^2}}\right)$

5. **Find the equation of the tangents to the ellipse $3x^2 + 2y^2 = 1$ perpendicular to the line $x + 3y = 1$.**

Solⁿ: The given equation of ellipse is $3x^2 + 2y^2 = 1$ which can be written as;

$\frac{x^2}{\frac{1}{3}} + \frac{y^2}{\frac{1}{2}} = 1$

and the line is;

$x + 3y = 1$

Hence, the tangent perpendicular to it is;

$3x - y + k = 0$ ... (i)

If it is tangent to $3x^2 + 2y^2 = 1$ then their point of intersection is;

$3x^2 + 2(3x + k)^2 = 1$

or, $3x^2 + 2(9x^2 + 6kx + k^2) = 1$

or, $3x^2 + 18x^2 + 12kx + 2k^2 - 1 = 0$

or, $21x^2 + 12kx + 2k^2 - 1 = 0$

If the (i) is tangent so;

$B^2 - 44c = 0$

or, $144k^2 - 84(2k^2 - 1) = 0$

or, $144k^2 - 168k^2 + 84 = 0$

or, $-24k^2 = -84$

or, $2k^2 = 7$

or, $k^2 = \frac{7}{2}$

Thus,

$$k = \pm\frac{7}{2}$$

Thus, required equation of tangents perpendicular to $3y + x = 1$ are $3x - y \pm \frac{7}{2} = 0$.

6. **Find the equation of tangent to the ellipse $x^2 + 3y^2 = 3$ which are perpendicular to the straight line $x - 4y - 5 = 0$.**

Solⁿ: The equation of lines which are perpendicular to $x - 4y - 5 = 0$ are $-4x - y + k = 0$ if it is tangent to $x^2 + 3y^2 = 3$ so their point of intersection is;

$$x^2 + 3(k - 4x)^2 = 3$$

or, $x^2 + 3(k^2 - 8kx + 16x^2) = 3$

or, $x^2 + 3k^2 - 24kx + 48x^2 = 3$

or, $49x^2 - 24kx + 3k^2 - 3 = 0$

Thus, by condition of tangency $B^2 - 4AC = 0$ gives;

$$(-24k)^2 - 4 \times 49 \times (3k^2 - 3) = 0$$

or, $576k^2 - 588k^2 + 588 = 0$

or, $-12k^2 = -588$

or, $k^2 = 49$

or, $k = \pm 7$

Thus required equation of tangents is;

$$-4x - y \pm 7 = 0$$

i.e. $\boxed{4x + y \pm 7 = 0}$

7. **Find the equation is to the tangents to the ellipse $4x^2 + 3y^2 = 5$ which are parallel to the straight line $3x - y + 7 = 0$, also find the point of contact.**

Solⁿ: The given equation of line is $3x - y + 7 = 0$ and the lines parallel to it are $3x - y + k = 0$ and if it is tangent to $4x^2 + 3y^2 = 5$ then their point of intersection is obtained by;

$$4x^2 + 3(3x + k)^2 = 5$$

or, $4x^2 + 3(9x^2 + 6kx + k^2) = 5$

or, $4x^2 + 27x^2 + 18kx + 3k^2 - 5 = 0$

or, $31x^2 + 18kx + 3k^2 - 5 = 0$

If the line is tangent so $B^2 - 4AC = 0$ yields;

$$(18k^2) - 4 \times 31 \times (3k^2 - 5) = 0$$

or, $324k^2 - 372k^2 + 620 = 0$

or, $-48k^2 = -620$

or, $k^2 = \frac{620}{48} = \frac{310}{24} = \frac{155}{12}$

or, $k = \pm\sqrt{\frac{155}{12}} = \pm\frac{1}{2}\sqrt{\frac{155}{3}}$

Hence tangents are $y = 3x \pm \frac{1}{2}\sqrt{\frac{155}{3}}$ ... (i)

To find the point of contact we solve (1) and ellipse.

Thus,

$$4x^2 + 3\left(3x \pm \frac{1}{2}\sqrt{\frac{155}{3}}\right)^2 = 5$$

or, $4x^2 + 3\left(9x^2 \pm 3x\sqrt{\frac{155}{3}} + \frac{155}{6}\right) = 5$

or, $4x^2 + 27x^2 \pm 9x\sqrt{\frac{155}{3}} + \frac{155}{2} = 5$

or, $31x^2 \pm 9x\sqrt{\frac{155}{3}} + \frac{145}{2} = 0$

Thus,

$$x = \frac{9\sqrt{\frac{155}{3}} \pm \sqrt{9^2 \times \frac{155}{3} - 4 \times 31 \times \frac{145}{2}}}{2 \times 31}$$

$$\therefore \quad x = \frac{\sqrt{155 \times 27} \pm \sqrt{27 \times 155 - 31 \times 145}}{62}$$

8. **Find the equations of the tangents to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ which make equal intercepts on the both axes.**

Solⁿ: Let, the equation of tangents which makes equal intercept k on both axis is;

$$\frac{x}{k} + \frac{y}{k} = 1$$

i.e. $\boxed{x + y = k}$

If it is tangent to $b^2x^2 + a^2y^2 = a^2b^2$ (ellipse) is their point of intersection

So, $b^2x^2 + a^2(k - x)^2 = a^2b^2$

or, $b^2x^2 + a^2(k^2 - 2kx + x^2) = a^2b^2$

or, $b^2x^2 + a^2(k^2 - 2kx + x^2) - a^2b^2 = 0$

or, $(b^2 + a^2)x^2 - 2ka^2x + a^2k^2 - a^2b^2 = 0$

If (i) is tangent to ellipse given then,

$$B^2 - 4AC = 0$$

or, $4k^2a^4 = 4(a^2 + b^2)(a^2k^2 - a^2b^2)$

or, $4k^2a^4 = 4k^2a^4 - 4a^4b^2 + 4a^2b^2k^2 - 4a^2b^4$

or, $4a^4b^2 + 4a^2b^4 = 4a^2b^2k^2$

or, $4a^2b^2(a^2 + b^2) = 4a^2b^2k^2$

$\therefore$ $k^2 = a^2 + b^2$

or, $k = \pm\sqrt{a^2 + b^2}$

Thus, required equation of tangent is; $x + y = \pm\sqrt{a^2 + b^2}$

9. **Find the points on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ such that the tangents at each of them makes equal angles with the axes. Show that the length of the perpendicular from the centre on either of these tangents is $\sqrt{\frac{a^2+b^2}{2}}$.**

Sol^n: Any point on ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is $(a\cos\theta, b\sin\theta)$. The equation of line at this point makes the angle equal to both axis so its slope is 1 or –1.

The equation of tangent at $(a\cos\theta, b\sin\theta)$ to $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is;

$$\frac{x\,a\cos\theta}{a^2} + \frac{y\,b\sin\theta}{b^2} = 1$$

or, $\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1$

It's slope $= -\frac{\text{coeff } x}{\text{coeff } y} = -\frac{\cos\theta}{a} \times \frac{b}{\sin\theta} = \frac{b}{a} \times -\cot\theta = \frac{a}{b}$

or, $\tan\theta = \frac{b}{a}$

or, $\frac{p}{b} = \frac{b}{a}$

So, $h = \sqrt{a^2 + b^2}$

Thus,

$$\sin\theta = \frac{b}{\sqrt{a^2+b^2}}$$

$$\cos\theta = \frac{a}{\sqrt{a^2+b^2}}$$

Thus coordinate is;

$$(a\cos\theta, b\sin\theta) = \frac{a^2}{\sqrt{a^2+b^2}}, \frac{b^2}{\sqrt{a^2+b^2}}$$

We have to prove that the length of perpendicular from centre (0,0) upon $\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta - 1 = 0$ is equal to the distance form (0,0) upon;

$$\frac{x}{a} \times \frac{a}{\sqrt{a^2+b^2}} + \frac{y}{b}\frac{b}{\sqrt{a^2+b^2}} - 1 = 0$$

i.e. $\frac{x}{\sqrt{a^2+b^2}} + \frac{y}{\sqrt{a^2+b^2}} - 1 = 0$

So, $\frac{1}{\sqrt{\frac{1}{a^2+b^2} + \frac{1}{a^2+b^2}}} = \frac{1}{\sqrt{\frac{2}{a^2+b^2}}} = \sqrt{\frac{a^2+b^2}{2}}$ proved

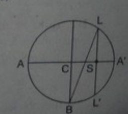
If the normal at the end of a latus rectum of an ellipse passes through one extremity of the minor axis, then show that the eccentricity of the curve is given by the equation $e^4 + e^2 - 1 = 0$.

Sol^n: Let the equation of ellipse is;

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Suppose, LSL' be a latus rectum with co–ordinate of L and L' are respectively $\left(ae, \frac{b^2}{a}\right)$ and $\left(ae, \frac{-b^2}{a}\right)$.

Firstly, we take the normal at L $\left(ae, \frac{b^2}{a}\right)$ so,



$$\frac{x-x_1}{\frac{x_1}{a^2}} = \frac{y-y_1}{\frac{y_1}{b^2}}$$

or, $\frac{x-ae}{\frac{ae}{a^2}} = \frac{y-\frac{b^2}{a}}{\frac{\frac{b^2}{a}}{b^2}}$

or, $\frac{a}{e} = a\left(y - \frac{b^2}{a}\right)$

or, $\frac{a}{e}(x - ae) = ay - b^2$ ... (i)

But, if this normal passes through the minor axis $(0, -b)$ of ellipse, then equation (i) will be;

$$\frac{a}{e}(x - ae) = ay - b^2$$

or, $-a^2 = -(ab + b^2)$

or, $a^2 = ab + b^2$

or, $a^2 = ab + a^2 - a^2e^2$

or, $a^2e^2 = ab$

Squaring both sides, we get,

or, $a^4e^4 = a^2b^2$

or, $a^4e^4 = a^4(1-e^2)$

or, $e^4 = 1 - e^2$

or, $e^4 + e^2 - 1 = 0$ Proved

11. Prove that locus of the foot of the perpendicular drawn from the centre on any tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is $r^2 = a^2\cos^2\theta + b^2\sin^2\theta$.

Sol^n: Let the given equation of ellipse be;

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \ldots \text{(i)}$$

Any tangent to equation (i) is;

$$y = mx + \sqrt{a^2m^2 + b^2} \ldots \text{(ii)}$$

Whose slope is m. Hence slope of normal is $-\frac{1}{m}$. So any line from centre (0, 0) and perpendicular to equation (ii) is;

$$y - 0 = -\frac{1}{m}(x - 0)$$

or, $y = -\frac{x}{m} \ldots$ (iii)

Let, (h, k) be the foot of perpendicular drawn from centre to equation (ii). So (h, k) is satisfied by equation (ii) and (iii).

$$k = mh + \sqrt{a^2m^2 + b^2} \ldots \text{(iv)}$$

$$k = -\frac{h}{m} \ldots \text{(v)}$$

To find the locus of (h, k) we have to eliminate the variable m between equations (iv) and (v),

Using $m = -\frac{h}{k}$ from equation (v) on (iv) then,

$$k = -\left(\frac{h}{k}\right)k + \sqrt{a^2\left(\frac{-h}{k}\right)^2 + b^2}$$

or, $k^2 = -h^2 + \sqrt{a^2h^2 + b^2k^2}$

Locus of (h, k) is;

$$(x^2 + y^2)^2 = a^2x^2 + b^2y^2$$

Changing into polar form;

$$x = r\cos\theta ; y = r\sin\theta$$

So, $x^2 + y^2 = r^2$

$\therefore$ $(r^2)^2 = a^2r^2\cos^2\theta + b^2r^2\sin^2\theta$

or, $r^4 = r^2(a^2\cos^2\theta + b^2\sin^2\theta)$

$\therefore$ $r^2 = a^2\cos^2\theta + b^2\sin^2\theta$ is the required locus.

12. Find the locus of the feet of the perpendicular drawn upon any tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ from either focus.

Sol^n: The equation of ellipse is;

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \ldots \text{(i)}$$

Any tangent to equation (i) is;

$$y = mx + \sqrt{a^2m^2 + b^2} \ldots \text{(ii)}$$

Whose slope is m. Hence slope of normal is $-\frac{1}{m}$. Hence any line through locue (ae, 0) is;

$$y - 0 = -\frac{1}{m}(x - ae)$$

or, $y = -\frac{1}{m}(x - ae) \ldots$ (iii)

Let (h, k) be the feet of perpendicular drawn from locus to the tangent (ii). So (h, k) satisfies equation (ii) and (iii).

$$k = mh + \sqrt{a^2m^2 + b^2} \ldots \text{(iv)}$$

$$k = -\frac{h}{m} + \frac{ae}{m} \ldots \text{(v)}$$

To find the locus of (h, k) we have to eliminate the variable m between equations (iv) and (v),

So, $k = \frac{-h+ae}{m}$

or, $m = \frac{-h+ae}{k}$

or, $k = -\left(\frac{h+ae}{m}\right)h + \sqrt{a^2\left(\frac{h+ae}{m}\right)^2 + b^2}$

or, $k^2 = -(h + ae)h + \sqrt{a^2(h - ae)^2 + b^2k^2}$

or, $k^2 = -h^2 - aeh + \sqrt{a^2(h - ae)^2 + b^2k^2}$

or, $(h^2 + k^2 + aeh)^2 = a^2(h - ae)^2 + b^2k^2$

or, $(h^2+k^2)^2 + 2(h^2+k^2)aeh + a^2e^2h^2 = a^2(h^2 - 2aeh + a^2e^2) + b^2k^2$

Thus, solving yields;

$$(h^2+k^2)^2 = a^4$$

So, $h^2+k^2 = a^2$

Thus, locus is $x^2+y^2=a^2$

13. If Q be the point on the auxiliary circle corresponding to a point P on an ellipse, then show that the normals at P and Q meet on a fixed circle $x^2+y^2=(a+b)^2$.

Sol^n: Let, Q be any point $Q(a\cos\theta, a\sin\theta)$ on auxiliary circle and $P(a\cos\theta, b\sin\theta)$ be the point on ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

The equation of normal at $P(a\cos\theta, a\sin\theta)$ is y has slope;

$$\frac{a\sin\theta}{a\cos\theta - 0} = \tan\theta$$

So, its equation is;

$$y - a\sin\theta = \tan\theta\,(x - a\cos\theta)$$

or, $y - a\sin\theta = x\tan\theta - a\sin\theta$

So,

$$y = x\tan\theta \ldots \text{(i)}$$

Similarly the equation of normal at $(a\cos\theta, a\sin\theta)$ is;

$$\frac{a^2(x-a\cos\theta)}{a\cos\theta} = \frac{b^2(y-b\sin\theta)}{b\sin\theta}$$

or, $\frac{ax-a^2\cos\theta}{\cos\theta} = \frac{by-b^2\sin\theta}{\sin\theta}$

or, $ax\sec\theta - a^2 = by\,\text{cosec}\,\theta - b^2$

or, $ax\sec\theta - by\,\text{cosec}\,\theta = b^2 - a^2 \ldots \text{(ii)}$

Solving equation (i) and (ii) to eliminate $\sec\theta$ and $\text{cosec}\,\theta$, we get,

$$x^2+y^2=(a+b)^2$$

Hence, Proved

14. Find the locus of a point from which two tangents can be drawn to an ellipse making angles θ_1 and θ_2 with the major axis such that;

i) $\tan\theta_1 - \tan\theta_2 = d$ (constant)

i) $\cot\theta_1 - \cot\theta_2 = d$ (constant)

Sol^n: Let, the equation of ellipse is;

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \ldots \text{(i)}$$

Any tangent to ellipse (i) is;

$$y = mx + \sqrt{a^2m^2+b^2} \ldots \text{(ii)}$$

Let (h,k) be a point from which two tangents are drawn. Then (h,k) lies on equation (ii).

$$k = mh + \sqrt{a^2m^2+b^2}$$

$$k - mh = \sqrt{a^2m^2+b^2}$$

Squaring both sides, we get,

$$(k-mh)^2 = a^2m^2+b^2$$

or, $k^2 + 2mkh + m^2h^2 = a^2m^2 + b^2$

or, $m^2(h^2-a^2) - 2kmh + k^2 - b^2 = 0 \ldots \text{(iii)}$

This is quadratic in m, so it gives two values of m say m_1 and m_2. Corresponding to two slopes m_1, m_2 we get to tangents through $P(h,k)$.

Let, θ_1, θ_2 be the angles of inclination of tangents with x–axis.

$$\tan\theta_1 = m_1 \text{ and } \tan\theta_2 = m_2$$

Then by equation (iii), we have,

$$m_1 + m_1 = \frac{2hk}{h^2-a^2}$$

or, $\tan\theta_1 + \tan\theta_2 = \frac{2hk}{h^2-a^2} \ldots \text{(iv)}$

Product of $m_1 . m_1 = \frac{k^2-b^2}{h^2-a^2}$

or, $\tan\theta_1 . \tan\theta_2 = \frac{k^2-b^2}{h^2-a^2} \ldots \text{(v)}$

or, $(\tan\theta_1 - \tan\theta_2)^2 = (\tan\theta_1 + \tan\theta_2)^2 - 4\tan\theta_1\tan\theta_2$

or, $d^2 = \frac{4h^2k^2}{(h^2-a^2)^2} - 4\frac{k^2-b^2}{h^2-a^2}$

or, $d^2(h^2-a^2)^2 = 4h^2k^2 - 4(k^2-h^2)(h^2-a^2)$

or, $d^2(x^2-a^2)^2 = 4x^2y^2 - 4(y^2-x^2)(x^2-a^2)$

which is the required locus

Also,

$$\cot\theta_1 . \cot\theta_2 = 2d$$

or, $\tan\theta_2 - \tan\theta_1 = 2d$ [illegible]

or, $(\tan\theta_1 - \tan\theta_2)^2 =$ [illegible]

or, $(\tan\theta_1 + \tan\theta_2)^2$ [illegible] or, $\frac{4h^2k^2}{\left(h^2-a^2\right)^2} - 4\frac{k^2-b^2}{h^2-a^2} = 4d^2\left(\frac{k^2-b^2}{h^2-a^2}\right)^2$

or, $4h^2k^2 - 4(k^2 - b^2)(h^2 - a^2) = 4d^2(k^2 - b^2)^2$

So, the required locus is;

$$4x^2y^2 - 4(y^2 - b^2)(x^2 - a^2) = 4d^2(y^2 - b^2)^2$$

Exercise 35

Find the centre, eccentricity, vertices, foci, directrices, length of axes and latus rectum of the hyperbola.

i) $4x^2 - 9y^2 = 36$

Soln: The given equation of hyperbola is $4x^2 - 9y^2 = 36$ which can be written as;

$$\frac{x^2}{9} - \frac{y^2}{4} = 1$$

Thus,

Centre = $(h, k) = (0,0)$

Since, $b^2 = a^2(e^2 - 1)$

So, $4 = 9(e^2 - 1)$

or, $\frac{4}{9} = e^2 - 1$

or, $e^2 = 1 + \frac{4}{9} = \frac{13}{9}$

$\therefore$ $e = \frac{\sqrt{13}}{3}$

Vertices = $(\pm a, 0) = (\pm 3,10)$

Foci = $(\pm ae, 0) = \left(\pm 3 \times \frac{\sqrt{3}}{3}, 0\right) = (\pm\sqrt{3}, 0)$

Directrix is;

$x = \pm\frac{a}{e}$

or, $x = \pm\frac{3}{\frac{\sqrt{13}}{3}}$

or, $x = \pm\frac{9}{\sqrt{13}}$

Length of axes,

$2a = 2 \times 3 = 6$

$2b = 2 \times 2 = 4$

The length of latus rectum = $\frac{2b^2}{a}$ = [illegible]

ii) $\mathbf{16x^2 - 9y^2 = -144}$

Sol^n^: The given equation of ellipse is $16x^2 - 9y^2 = -144$ which is of the form;

$$\frac{x^2}{9} - \frac{y^2}{16} = -1$$

Thus,

Centre = (0,0)

Eccentricity

$$a^2 = b^2(e^2 - 1)$$

or, $9 = 16(e^2 - 1)$

or, $\frac{9}{16} + 1 = e^2$

or, $e = \frac{5}{4}$

Foci $= (0, \pm be) = \left(0, \pm 4 \times \frac{5}{4}\right) = (0, \pm 5)$

Directrix

$$y = \pm \frac{b}{e}$$

or, $y = \pm \frac{4}{\frac{5}{4}} = \pm \frac{16}{5}$

Length of axis $= 2b = 2 \times 4 = 8$

$= 2a = 2 \times 3 = 6$

The length of latus rectum $= \frac{2a^2}{b} = \frac{2 \times 3^2}{4} = \frac{9}{2}$

iii) $\mathbf{16x^2 - 25y^2 = 400}$

Sol^n^: The given equation of ellipse is $16x^2 - 25y^2 = 400$ which can be written as;

$$\frac{x^2}{25} - \frac{y^2}{16} = 1$$

Thus,

Centre = (h, k) = (0,0)

For eccentricity

$$b^2 = a^2(e^2 - 1)$$

or, $16 = 25(e^2 - 1)$

or, $\frac{16}{25} = e^2 - 1$

or, $\frac{16}{25} + 1 = e^2$

or, $e = \frac{\sqrt{41}}{5}$

Foci $= (\pm a, 0) = (\pm 5, 0)$

Equation of directrix is;

$$x = \pm \frac{a}{e}$$

or, $x = \pm \frac{5}{\frac{\sqrt{41}}{5}}$

or, $x = \pm \frac{25}{\sqrt{41}}$

The length of axis;

$2a = 2 \times 5 = 10$

$2b = 2 \times 4 = 8$

The length of latus rectum $= \frac{2b^2}{a} = \frac{2 \times 16}{5} = \frac{32}{5}$

iv) $\mathbf{9x^2 - 16y^2 + 72x - 32y - 16 = 0}$

Sol^n^: The given equation of hyperhola is:

$9x^2 - 16y^2 + 72x - 32y - 16 = 0$

or, $9(x^2 + 8x) - 16(y^2 + 2y) = 16$

or, $9(x^2 + 2.x.4 + 4^2 - 16) - 16(y^2 + 2y + 1 - 1) = 16$

or, $9(x + 4)^2 - 144 - 16(y + 1)^2 + 16 = 16$

or, $9(x + 4)^2 - 16(y + 1)^2 = 144$

or, $\frac{(x+4)^2}{16} - \frac{(y+1)^2}{9} = 1$

Thus,

Centre (h, k) = (−4,1)

For eccentricity;

$$b^2 = a^2(e^2 - 1)$$

or, $9 = 16(e^2 - 1)$

or, $\frac{9}{16} + 1 = e^2$

or, $e = \frac{5}{4}$

The foci $= (h \pm ae, k) = \left(-4 \pm 4 \times \frac{5}{4}, -1\right)$

$= (-4 \pm 5, -1)$

$(-9, -1)$ and $(1, -1)$

Vertices $= (h \pm a, k) = (-4 \pm 4, -1)$

$(0, -1)$ and $(-8, -1)$

Length of axis;

$2a = 2 \times 4 = 8$

$2b = 2 \times 3 = 6$

The equation of directrix;

$$x = h \pm \frac{a}{e}$$

or, $x = -4 \pm \frac{4\times4}{5}$

or, $x = -4 \pm \frac{16}{5}$

$\therefore$ $x = -\frac{4}{5}$ and $x = \frac{36}{5}$

2. **Find the equation to the hyperbola referred to its axes as axes of co–ordinates.**

i) **with distances between the foci is 9, eccentricity is $\sqrt{3}$.**

Solⁿ: The distance between foci is 9 that is $(\pm ae, 0)$ and $(-ae, 0)$ have distance 9

So, $\boxed{2ae = 9}$

Also $e = \sqrt{3}$ is given;

So, $a = \frac{9}{2\sqrt{3}} = \frac{3\sqrt{3}}{2}$

By using;

$$b^2 = a^2(e^2 - 1)$$

or, $b^2 = \frac{27}{4}(3 - 1)$

or, $b^2 = \frac{27}{2}$

Thus, required equation of hyperbola is;

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

or, $\frac{x^2}{\frac{27}{4}} - \frac{y^2}{\frac{27}{2}} = 1$

or, $4x^2 - 2y^2 = 27$ is required equation of hyperbola.

ii) **with distances between the foci is 16, eccentricity is $\sqrt{2}$.**

Solⁿ: By the question;

The distance between foci;

i.e. $\boxed{2ae = 16}$

Also,

$e = \sqrt{2}$ is given;

Thus,

$$a = \frac{16}{2e} = \frac{8}{e} = \frac{8}{\sqrt{2}} = 4\sqrt{2}$$

We know that,

$$b^2 = a^2(e^2 - 1)$$

So, $b^2 = 32(2 - 1)$

$b^2 = 32$

Thus, $a^2 = 32$ and $b^2 = 32$ so the required equation of hyperbola is;

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

or, $\frac{x^2}{32} - \frac{y^2}{32} = 1$

So, $x^2 - y^2 = 32$ is required equation.

iii) **with length of transverse and conjugate axes are 3 and 4 respectively.**

Solⁿ: The length of transverse axis $2a = 3$ and conjugate axis $2b = 4$

so, $a = \frac{3}{2}$

and $b = 2$

Thus, required equation of hyperbola is;

$$\frac{x^2}{\frac{9}{4}} - \frac{y^2}{4} = 1$$

or, $\frac{4x^2}{9} - \frac{y^2}{4} = 1$

i.e. $16x^2 - 9y^2 = 36$ is required equation.

iv) **with length of conjugate axis is 10, passes through the point $(2, -1)$.**

Solⁿ: The standard equation of hyperbola is;

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \ldots (i)$$

By question, the length of conjugate axis;

$2b = 10$

$\therefore$ $b = 5$ also (i) passes through $(2, -1)$ thus,

$$\frac{2^2}{a^2} - \frac{(-1)^2}{b^2} = 1$$

or, $\frac{4}{a^2} - \frac{1}{b^2} = 1$

Since $b = 5$ is given;

$$\frac{4}{a^2} = 1 + \frac{1}{25} = \frac{26}{25}$$

or, $a^2 = \frac{50}{13}$

Thus from (i);

$$\frac{x^2}{\frac{50}{13}} - \frac{y^2}{25} = 1$$

or, $\frac{13x^2}{50} - \frac{y^2}{25} = 1$

or, $13x^2 - 2y^2 = 50$ is required of hyperbola under the given condition.

v) **with length of conjugate axis 7 and passes through the $(3, -2)$.**

Sol^n: The equation of hyperbola is;

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \ldots \text{(i)}$$

By question the length of conjugate axis $2b = 7$ is given so $b = \frac{7}{2}$ we are given that (i) passes through $(3, -2)$ so

$$\frac{9}{a^2} - \frac{4}{b^2} = 1$$

or, $\frac{9}{a^2} - \frac{4}{\frac{49}{4}} = 1$

or, $\frac{9}{a^2} - \frac{16}{49} = 1$

or, $\frac{9}{a^2} = \frac{16}{49} + 1 = \frac{65}{49}$

or, $a^2 = \frac{49 \times 9}{65} = \frac{441}{65}$

Thus, equation of hyperbola is;

$$\frac{x^2}{\frac{441}{65}} - \frac{y^2}{\frac{49}{4}} = 1$$

or, $65x^2 - 36y^2 = 441$ is the required equation of hyperbola.

3. **Find the equation of the hyperbola whose directrix is $2x + y - 1 = 0$ and focus $(1, 2)$ and eccentricity $\sqrt{3}$.**

Sol^n: The equation of the hyperbola is to be determined.

Focus is $(1,2)$ and $2x + y - 1 = 0$ is directrix

Thus by definition of hyperbola;

$$e = \frac{sp}{pm}$$

Where, $p(x, y)$ be any point of hyperbola so

$$\sqrt{3} = \frac{\sqrt{(x-1)^2 + (y-2)^2}}{\left(\frac{2x+y-1}{\sqrt{5}}\right)}$$

Squaring both sides, we get,

$$3 = \frac{(x-1)^2 + (y-2)^2 \times 5}{(2x+y-1)^2}$$

or, $3[4x^2 + 4xy + y^2 - 4x - 2y + 1] = (x^2 - 2x + 1 + y^2 - 4y + 4)5$

or, $12x^2 + 12xy + 3y^2 - 12x - 6y + 3 = 5x^2 10x + 5 + 5y^2 - 20y + 20$

i.e. $7x^2 - 2y^2 + 12xy + 14y - 22 = 0$ is required equation of hyperbola.

4. **Find the equation of the hyperbola with vertices $(\pm 5, 0)$ and foci and $(\pm 7, 0)$.**

Sol^n: The vertex of hyperbola is;

$$(\pm a, 0) = (\pm 5, 0)$$

so, $a = 5$

Foci $= (\pm ae, 0) = (\pm 7, 0)$

so, $ae = 7$

Thus,

$$e = \frac{7}{5}$$

We know that,

$$b^2 = a^2(e^2 - 1)$$

or, $b^2 = 25\left(\frac{49}{25} - 1\right) = 24$

or, $b^2 = 29$

Thus, required equation of hyperbola is,

$$\frac{x^2}{15} - \frac{y^2}{24} = 1$$

i.e. $24x^2 - 25y^2 = 600$

Hence required equation is $24x^2 - 25y^2 = 600$

5. **Prove that the locus of a point which moves in such a way that the difference of its distances from the points $(5, 0)$ and $(-5, 0)$ is 2 is a hyperbola.**

Sol^n: Let, $P(x, y)$ be the locus of point

By question; the difference of its distance form $A(5,0)$ and $B(-5,0)$ is 2.

i.e. $PA - PB = 2$

So, $\sqrt{(x-5)^2 + (y-0)^2} - \sqrt{(x+5)^2 + (y-0)^2} = 2$

Squaring yields;

or, $\sqrt{x^2 - 10x + 25 + y^2} - \sqrt{x^2 + 10x + 25 + y^2} = 2$

i.e. $\sqrt{x^2 - 10x + 25 + y^2} = 2 + \sqrt{x^2 + 10x + 25 + y^2}$

Squaring both sides, we get,

$$x^2 - 10x + 25 + y^2 = x^2 + 10x + 25 + y^2 + 4 + 2\sqrt{x^2 + 10x + 25 + y^2} \times 2$$

or, $-20x - 4 = 4\sqrt{x^2 + 10x + 25 + y^2}$

or, $-5x - 1 = \sqrt{x^2 + 10x + 25 + y^2}$

Again squaring both sides, we get,

$$25x^2 + 10x + 1 = x^2 + 10x + y^2 + 25$$

or, $24x^2 - y^2 = 24$

or, $\frac{x^2}{1} - \frac{y^2}{24} = 1$ which is required equation of hyperbola. Proved

6. **Prove that the lines $\frac{x}{a} + \frac{y}{b} = k$ and $\frac{x}{a} - \frac{y}{b} = \frac{1}{k}$ where k is variable parameter always intersect on a hyperbola.**

Sol^n: The given lines are;

$$\frac{x}{a} + \frac{y}{b} = k \text{ and } \frac{x}{a} - \frac{y}{b} = \frac{1}{k}$$

To find their point of intersection;

$$\frac{bx+ay}{ab} = k \dots \text{(i)}$$

or, $\frac{bx-ay}{ab} = \frac{1}{k}$

i.e. $k = \frac{ab}{bx-ay} \cdots$ (ii)

So from (i) and (ii)

$$\frac{bx+ay}{ab} = \frac{ab}{bx-ay}$$

or, $b^2x^2 - a^2y^2 = a^2b^2$

or, $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

Since their point of intersection is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is hyperbola.

Proved

Show that the eccentricity of a hyperbola with transverse axis is 2a and axes are the axes of co–ordinates which passes through the point (h, k) is given by $\left(\frac{k^2+h^2-a^2}{h^2-a^2}\right)^{\frac{1}{2}}$.

Sol^n: The equation of hyperbola in its standard form is;

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Since it passes through (h, k) so;

$$\frac{h^2}{a^2} - \frac{k^2}{b^2} = 1$$

or, $h^2b^2 - a^2k^2 = a^2b^2$

or, $h^2b^2 - a^2b^2 = a^2k^2$

or, $k^2 = \frac{b^2\left(h^2-a^2\right)}{a^2} \dots$ (i)

We know that;

$$b^2 = a^2(e^2 - 1) \dots \text{(ii)}$$

If transverse axis is 2a

So, from equation (i), we have,

$$\frac{a^2k^2}{h^2-a^2} = b^2 \text{ (using in equation (ii))}$$

or, $b^2 = a^2(e^2 - 1)$

or, $\frac{a^2k^2}{h^2-a^2} = a^2(e^2 - 1)$

or, $k^2 = (e^2 - 1)(h^2 - a^2)$

or, $k^2 = e^2(h^2 - a^2) - h^2 + a^2$

or, $k^2 + h^2 - a^2 = e^2(h^2 - a^2)$

or, $e^2 = \frac{k^2+h^2-a^2}{h^2-a^2}$

Thus,

$$e = \sqrt{\frac{k^2+h^2-a^2}{h^2-a^2}} \quad \text{Proved}$$

Exercise 36

1. **Find the equation of tangent, and normal at the point $(5\sqrt{2}, 4)$ to the hyperbola $16x^2 - 25y^2 = 400$.**

Sol^n: The equation of hyperbola is;

$$16x^2 - 25y^2 = 400$$

i.e. $\frac{x^2}{25} - \frac{y^2}{16} = 1$

We have to find the equation of tangent at $(5\sqrt{2}, 4)$ so by formula

$$\frac{x.5\sqrt{2}}{25} - \frac{y.4}{16} = 1$$

or, $\frac{x\sqrt{2}}{5} - \frac{y}{4} = 1$

or, $4\sqrt{2}x - 5y = 20$ is required equation of tangent.

The equation of normal is perpendicular to tangent, so it is perpendicular to tangent, so is

$-5x - 4\sqrt{2}y + k = 0$ but it passes through $(5\sqrt{2}, 4)$ so,

or, $-25\sqrt{2} - 16\sqrt{2} + k = 0$

or, $k = 41\sqrt{2}$

Hence, equation of normal is $5x + 4\sqrt{2}y = 41\sqrt{2}$ is the required equation of normal.

2. **Find the equation of the tangent and normal to the hyperbola $2x^2 - 3y^2 = 6$ at the end of its latus rectum in the first quadrant.**

Sol^n: The equation of hyperbola is $2x^2 - 3y^2 = 6$ which can be written as;

$$\frac{x^2}{3} - \frac{y^2}{2} = 1$$

We know that, the end of lactus rectum in first quadrant is;

$$\left(ae, \frac{b^2}{a}\right)$$

Here,

$$a = \sqrt{3}, b = \sqrt{2}$$

So, $b^2 = a^2(e^2 - 1)$

or, $2 = 3(e^2 - 1)$

or, $\frac{2}{3} + 1 = e^2$

or, $e^2 = \frac{5}{3}$

or, $e = \sqrt{\frac{5}{3}}$

Hence coordinate is $\left(\sqrt{3} \times \frac{\sqrt{5}}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right) = \left(\sqrt{5}, \frac{2}{\sqrt{3}}\right)$

Hence equation of tangent is;

$$\frac{x.\sqrt{5}}{3} - \frac{y.\frac{2}{\sqrt{3}}}{2} = 1$$

or, $\frac{x\sqrt{5}}{3} - \frac{y}{\sqrt{3}} = 1$

or, $x\sqrt{15} - 3y = 3\sqrt{3}$

or, $x\sqrt{5} - \sqrt{3}y = 3$ is required equation of tangent. To find the equation of normal which is always perpendicular to tangent is $-\sqrt{3}x - \sqrt{5}y + k = 0$ which passes through $\left(\sqrt{5}, \frac{2}{\sqrt{3}}\right)$ so,

or, $-\sqrt{15} - \sqrt{5} \times \frac{2}{\sqrt{3}} + k = 0$

or, $\frac{-\sqrt{45} - \sqrt{20}}{\sqrt{3}} = -k$

or, $k = \frac{\sqrt{65}}{\sqrt{3}}$

Thus,

$$-\sqrt{3}x - \sqrt{5}y + \frac{\sqrt{65}}{\sqrt{3}} = 0$$

or, $-3x - \sqrt{15}y = -\sqrt{65}$

$3x + \sqrt{15}y = \sqrt{65}$ is required equation of normal.

3. **Prove that the straight line $21x + 5y = 116$ is a tangent to the hyperbola $7x^2 - 5y^2 = 232$ and find its point of contact.**

Sol^n: The equation of straight line given is $21x + 5y = 116$ and given equation of hyperbola is $7x^2 - 5y^2 = 232$.

Solving these equations yields;

$$7x^2 - 5\left(\frac{116 - 21x}{5}\right)^2 = 232$$

or, $7x^2 - \left(\frac{13456-7872x+441x^2}{5}\right) = 232$

or, $\frac{35x^2-(13456-4872x+441x^2)}{5} = 232$

or, $35x^2 - 13456 + 4872x - 441x^2 = 1160$

or, $-406x^2 + 4872x - 14616 = 0$

or, $406x^2 - 4872x + 14616 = 0$

or, $406x(x-6) - 2436(x-6) = 0$

or, $(406x - 2436)(x-6) = 0$

or, $x = 6$

When,

$x = 6$

Then,

$21 \times 6 + 5y = 116$

or, $5y = 116 - 126 = -10$

or, $y = -2$

Thus point of contact is only one so it is tangent and point of contact is $(6, -2)$.

4. **Find the equation of tangent to hyperbola $3x^2 - 4y^2 = 1$ which is parallel to the line $2x - y + 4 = 0$.**

Sol^n: The equation of line parallel to $2x - y + 4 = 0$ is $2x - y + k = 0$ if $y = 2x + k$ is tangent to the line $3x^2 - 4y^2 = 1$ so their point of intersection is obtained.

$3x^2 - 4(2x+k)^2 = 1$

or, $3x^2 - 4[4x^2 + 4xk + k^2] = 1$

or, $3x^2 - 16x^2 - 16xk - 4k^2 - 1 = 0$

or, $-13x^2 - 16xk - 4k^2 - 1 = 0$

or, $13x^2 + 16xk + 4k^2 + 1 = 0$

Since for tangent $B^2 - 4AC = 0$

$(16k)^2 - 4 \times 13 \times (4k^2 + 1) = 0$

or, $256k^2 - 208k^2 - 52 = 0$

or, $48k^2 - 52 = 0$

or, $k^2 = \frac{52}{48} = \frac{13}{12}$

or, $k^2 = \pm\sqrt{\frac{13}{12}} = \pm\frac{\sqrt{13}}{2\sqrt{3}}$

Thus equation of tangent is;

$2x - y \pm \pm\frac{\sqrt{13}}{2\sqrt{3}} = 0$

or, $4\sqrt{3}x - 2\sqrt{3}y \pm \sqrt{13} = 0$

5. **Find the equation of tangent to the hyperbola $3x^2 - 4y^2 - 1 = 0$ and perpendicular to the line $2x + 3y - 5 = 0$.**

Sol^n: The equation of line which is perpendicular to line $2x + 3y - 5 = 0$ is $3x - 2y + k = 0$ if it is tangent to the hyperbola $3x^2 - 4y^2 - 1 = 0$, so their point of intersection is,

$3x^2 - 4\left(\frac{3x+k}{2}\right)^2 - 1 = 0$

or, $3x^2 - 4\frac{(9x^2+6xk+k^2)}{4} - 1 = 0$

or, $3x^2 - 9x^2 - 6xk - k^2 - 1 = 0$

or, $-6x^2 - 6xk - k^2 - 1 = 0$

or, $6x^2 + 6xk + k^2 + 1 = 0$

Since $B^2 - 4AC$ must be zero

$(6k)^2 - 4 \times 6(k^2 + 1) = 0$

or, $36k^2 - 24k^2 - 24 = 0$

or, $12k^2 = 24$

or, $k^2 = 2$

or, $k = \pm\sqrt{2}$

Thus, equation of tangent is $3x - 2y \pm \sqrt{2} = 0$.

6. **Find the point of intersection of the tangents at t_1 and t_2 on the hyperbola $xy = c^2$.**

Sol^n: Any point t_1 is $\left(ct_1, \frac{c}{t_1}\right)$ and t_2 is $\left(ct_2, \frac{c}{t_2}\right)$

The equation of tangent at $\left(ct_1, \frac{c}{t_1}\right)$ to $xy = c^2$ is;

$\frac{x}{ct_1} + \frac{t_1 y}{c} = 2$

or, $x + yt_1^2 = 2ct_1 \dots$ (i)

Similarly, the equation of tangent at $\left(ct_2, \frac{c}{t_2}\right)$ is;

$x + yt_2^2 = 2ct_2 \dots$ (ii)

Solving (i) and (ii),

$$2ct_1 - yt_1^2 + yt_2^2 = 2ct_2$$

or, $2ct_1 - 2ct_2 = yt_1^2 - yt_2^2$

or, $2c(t_1 - t_2) = y(t_1 - t_2)(t_1 + t_2)$

$$\therefore \quad y = \left(\frac{2c}{t_1+t_2}\right)$$

Also from (i),

$$x = 2ct_2 - yt_2^2$$

$$= 2ct_2 - t_2^2 \times \frac{2c}{t_1+t_2}$$

$$= 2ct_2\left[1 - \frac{t_2}{t_1+t_2}\right]$$

$$= 2ct_2\left[\frac{t_1+t_2-t_2}{t_1+t_2}\right] = \left[\frac{2ct_1t_2}{t_1+t_2}\right]$$

Hence, the point of intersection of tangents at t_1 and t_2 is;

$$\left(\frac{2ct_1t_2}{t_1+t_2}, \frac{2c}{t_1+t_2}\right)$$

7. **Find the equation of a hyperbola whose asymptotes are $2x - y - 3 = 0$ and $3x + y - 7 = 0$ and passes through the point $(1, 1)$.**

Solⁿ: Here,

The equations of asymptotes are;

$$2x - y - 3 = 0$$

and $3x + y - 7 = 0$

We know that, the combined equation of asymptotes and the equation of hyperbola is differ by constant on y so let it be λ.

So equation of hyperbola is;

$$(2x - y - 3)(3x + y - 7) + \lambda = 0$$

or, $(-2)(-3) + \lambda = 0$

or, $\lambda + 6 = 0$

or, $\lambda = -6$

Hence, required equation of hyperbola is;

$$(3x + y - 7)(2x - y - 3) - 6 = 0$$

or, $6x^2 - 3xy - 9x + 2xy - y^2 - 3y - 14x + 7y + 21 - 6 = 0$

or, $6x^2 - y^2 - xy - 23x + 4y + 15 = 0$ is the required equation of hyperbola.

8. **Prove that the normal at the point 't' of the rectangular hyperbola $xy = c^2$ meets the curve again at a point 't_1' such that $t^3t_1 = -1$.**

Solⁿ: The equation of normal at t with the hyperbola passes $xy = c^2$ is;

$$xt^2 - y = ct^3 - \frac{c}{t}$$

By the question, it passes through $\left(ct_1, \frac{c}{t_1}\right)$ so,

$$ct_1 \times t^2 - \frac{c}{t_1} = ct^3 - \frac{c}{t}$$

or, $t^2t_1 - \frac{1}{t_1} = t^3 - \frac{1}{t}$

or, $\frac{t^2t_1^2 - 1}{t_1} = \frac{t^4 - 1}{t}$

or, $t^3t_1^2 - t = t^4t_1 - t_1$

or, $t^3t_1^2 - t^4t_1 = t - t_1$

or, $t^3t_1(t - t_1) = -(t - t_1)$

$\therefore \quad t^3t_1 = -1$

Exercise 37

1. **Show that the conic $9x^2 - 24xy + 16y^2 - 50x - 100y + 225$ represents a parabola. Find the latus rectum, vertex, focus and directrix of the parabola.**

Sol^n: The given equation of conic is;

$$9x^2 - 24xy + 16y^2 - 50x - 100y + 225 = 0$$

Comparing it with $ax^2 + 2hxy + by^2 + 29x + 2fy + c = 0$ yields;

$$b = 16, a = 9, h = -12, g = -25, f = -50, \; c = 225$$

Here,

$$\begin{aligned} & abc + 2fgh - af^2 - bg^2 - ch^2 \\ &= 9 \times 16 \times 225 + 2 \times -50 \times -25 \times -12 - 9 \times -(50)^2 \\ &\qquad -16 \times (-25)^2 - 225 \times (-12)^2 \\ &= 32400 - 30000 - 9 \times 2500 - 16 \times 625 - 225 \times 144 \\ &= 2400 - 22500 - 10000 - 32400 \neq 0 \end{aligned}$$

and $ab - h^2 = 9 \times 16 - (-12)^2 = 144 - 144 = 0$

So it is parabola,

Also,

$$9x^2 - 24xy + 16y^2 - 50x - 100y + 225 = 0$$

Yields;

$$(3x - 4y + \lambda)^2 - 50x - 100y + 225 - 6\lambda x + 8\lambda y + \lambda^2 = 0$$

or, $(3x - 4y + \lambda)^2 - x(50 + 6\lambda) - y(100 - 8\lambda) - \lambda^2 + 225 = 0$

or, $(3x - 4y + \lambda)^2 = x(50 + 6\lambda) + y(100 - 8\lambda) + \lambda^2 - 225$

Let us choose λ in such a way that $3x - 4y + \lambda = 0$ and $x(50 + 6\lambda) + y(100 - 8\lambda) + \lambda^2 - 225 = 0$ are perpendicular to each other.

So, $\frac{3}{4} \times \frac{-50-6\lambda}{100-8\lambda} = -1$

or, $-150 - 18\lambda = -400 + 32\lambda$

or, $250 = 50\lambda$

or, $\lambda = 5$

Hence,

$$(3x - 4y + 5)^2 = 80x + 60y - 200$$

or, $\left(\frac{3x-4y+5}{\sqrt{3^2+4^2}}\right)^2 (25) = \frac{80x+60y-200}{\sqrt{80^2+(60)^2}} \times 100$

or, $\left(\frac{3x-4y+5}{5}\right)^2 = \frac{100}{25}\left[\frac{80x+60y-200}{100}\right]$

which is of the form $Y^2 = 4AX$

Length of latus rectum $= 4A = \frac{100}{25} = 4$

For vertex:

$$\begin{array}{ccccc} 3 & -4 & 5 & 3 & -4 \\ 8 & 6 & -2 & 8 & 6 \end{array}$$

$\therefore \quad \frac{x}{80-30} = \frac{y}{40+60} = \frac{1}{18+32}$

So, $x = \frac{50}{50} = 1$

$y = \frac{100}{50} = 2$

Thus, vertex is $= (1,2)$

Focus:

We know that focus is the intersection of equation of latus rectum and axis of parabola. We first find the equation of latus rectum, so

$X = A$

or, $\frac{80x+60y-200}{100} = 1$

or, $80x + 60y - 200 = 100$

or, $80x + 60y - 300 = 0$

or, $8x + 6y - 30 = 0$

or, $4x + 3y - 15 = 0 \ldots$ (i)

Also, the equation of axis is;

$Y = 0$

or, $\frac{3x-4y+5}{5} = 0$

or, $3x - 4y + 5 = 0 \ldots$ (ii)

Solving (i) and (ii);

$$\begin{array}{ccccc} x & y & 1 & & \\ 4 & 3 & -15 & 4 & 3 \\ 3 & -4 & 5 & 3 & -4 \end{array}$$

$\therefore \quad \frac{x}{-45} = \frac{y}{-65} = \frac{1}{-25}$

$x = \frac{-45}{-25} = \frac{9}{5}$

$y = \frac{-65}{-25} = \frac{13}{5}$

Hence $(x, y) = \left(\frac{9}{5}, \frac{13}{5}\right)$ is focus.

For directrix:

The equation of directrix is $X = -A$ for $Y^2 = 4AX$

So, $\frac{80x+60y-200}{100} = -1$

or, $80x + 60y - 200 = -100$

or, $80x + 60y - 100 = 0$

or, $8x + 6y - 10 = 0$

or, $\boxed{4x + 3y - 5 = 0}$ is required equation of directrix.

2. **Show that the conic $x^2 - 2xy + y^2 - 2x - 2y + 3$ represents a parabola. Find latus rectum, vertex, focus and directrix of the parabola.**

Sol^n: The equation given is;

$$x^2 - 2xy + y^2 - 2x - 2y + 3 = 0$$

Comparing it with $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ gives;

$$a = 1, h = -1, b = 1, g = -1, f = -1, c = 3$$

and $\Delta = abc + 2fgh - af^2 - bg^2 - ch^2$

$$= 1 \times 1 \times 3 + 2 \times -1 \times -1 \times -1 \times -1 \times (-1)^2 - 1 \times (-1)^2 - 3 \times (-1)^2$$

$$= 3 - 2 - 1 - 1 - 3$$

$$= -4 \neq 0$$

and $ab - h^2 = 1 \times 1 - (-1)^2$

$$= 1 - 1 = 0$$

Since, $\Delta \neq 0$ and $ab - h^2 = 0$ so given equation represents the equation of parabola.

Hence,

$$x^2 - 2xy + y^2 - 2x - 2y + 3 = 0$$

or, $(x - y)^2 - 2(x) - 2y + 3 = 0$

or, $(x - y + \lambda)^2 - 2x - 2y + 3 + 2\lambda y - 2\lambda x - \lambda^2 + 3 = 0$

or, $(x - y + \lambda)^2 - x(2\lambda + 2) - y(2 - 2\lambda) + 3 - \lambda^2 = 0$

We choose λ in such a way that,

$x - y + \lambda = 0$ and

or, $x(2\lambda + 2) + y(2 - 2\lambda) + \lambda^2 - 3 = 0$ becomes perpendicular to each other.

So, $1 \times -\frac{(2\lambda+2)}{2-2\lambda} = -1$

i.e. $2\lambda + 2 = 2 - 2\lambda$

$4\lambda = 0$

or, $\lambda = 0$

Hence,

$$(x - y)^2 = 2x + 2y - 3$$

or, $\left(\frac{x-y}{\sqrt{2}}\right)^2 \times 2 = \frac{2x+2y-3}{\sqrt{4+4}} \times \sqrt{8}$

or, $\left(\frac{x-y}{\sqrt{2}}\right)^2 = \frac{\sqrt{8}}{2}\left[\frac{2x+2y-3}{\sqrt{8}}\right]$

Which is of the form $y^2 = 4AX$

Hence length of latus rectum $= \frac{\sqrt{8}}{2} = \frac{2\sqrt{2}}{2} = \sqrt{2}$

For vertex:

x	y	1		
1	−1	0	1	−1
2	2	−3	2	2

or, $\frac{x}{3} = \frac{y}{3} = \frac{1}{4}$

Hence $x = \frac{3}{4}$ and $y = \frac{3}{4}$

Hence vertex $= \left(\frac{3}{4}, \frac{3}{4}\right)$

For focus:

We know that focus is the point of intersection of equation of latus rectum and axis of parabola. So we find equation of directrix and the equation of axis and then solve them.

We know that, $X = A$ is directrix;

$$\frac{2x+2y-3}{\sqrt{8}} = \frac{\sqrt{8}}{8}$$

or, $2x + 2y - 3 = 1$

or, $2x + 2y - 4 = 0$

or, $x + y = 2$

Also, the equation of axis is;

$Y = 0$

or, $x - y = 0$

Thus solving,

$$\begin{array}{l} x - y = 0 \\ \underline{x + y = 2} \\ 2x = 2 \end{array}$$

or, $x = 1$ and $y = 1$

Thus focus is (1,1).

For directrix:

The equation of directrix is $Y = -A$ for $Y^2 = 4AX$

So, $\frac{2x+2y-3}{\sqrt{8}} = -\frac{\sqrt{8}}{8}$

or, $2x + 2y - 3 = -1$

or, $2x + 2y - 2 = 0$

or, $x + y = 1$

Hence, the equation of directrix is $x + y = 1$.

3. **Find the positions and lengths of the axes of the conic $x^2 - 4xy - 2y^2 + 10x + 4y$.**

Sol^n: The given conic is;

$$x^2 - 4xy - 2y^2 + 10x + 4y = 0$$

Comparing it with $ax^2 + 2hxy + hy^2 + 2gx + 2fy + c = 0$

We get,

$$a = 1, h = -2, b = -2, g = 5, f = 2, c = 0$$

and $\Delta = abc + 2fgh - af^2 - bg^2 - ch^2$

$$= 2 \times 2 \times 5 \times -2 - 1 \times (2)^2 + 2 \times (5)^2$$

$$= -40 - 4 + 50$$

$$= 6 \neq 0$$

and $ab - h^2 = 1 \times -2 - (-2)^2$

$$= -2 - 4$$

$$= -6 < 0$$

Since,

$$\Delta \neq 0$$

and $ab - h^2 = 1 \times (-2) - (-2)^2 - 2 - 4 = -6$

Hence, given equation is hyperbola.

For the length of axis, we have the equation;

$$\frac{1}{\gamma^4} - \frac{1}{\gamma^2}(A + B) + (AB - H^2) = 0$$

For this we have to find centre;

so, $\emptyset = x^2 - 4xy - 2y^2 + 10x + 4y$

or, $\frac{\delta\emptyset}{dx} = 2x - 4y + 10$

or, $\frac{\delta\emptyset}{dy} = -4x - 4y + 4$

For coordinate of centre

$\frac{\delta\emptyset}{dx} = 0$ and $\frac{\delta\emptyset}{dy} = 0$ is solved

or, $2x - 4y + 10 = 0$

or, $-4x - 4y + 4 = 0$

$$\begin{array}{l} x - 2y = -5 \\ \underline{-x - y = -1} \\ -3y = -6 \end{array}$$

or, $y = 3$

and $x = 2 \times 3 - 5 = 1$

Thus centre is (1,3)

Standard form referred to centre is;

$$\frac{a(h^2-ab)}{\Delta}x^2 + \frac{2h(h^2-ab)}{\Delta}xy + \frac{b(b^2-ab)}{\Delta}y^2 = 1$$

or, $x^2 - 4xy - 2y^2 = 1$

or, $x^2 - 4xy - 2y^2 - 1 = 0$

so comparing it with;

$$Ax^2 + 2Hxy + By^2 = 1$$

$$A = 1, H = -2, B = -2$$

Thus, length of axis;

$$\frac{1}{\gamma^4} - \frac{1}{\gamma^2}(1 - 2) + (-1 \times 2 - 4) = 0$$

or, $\frac{1}{\gamma^4} + \frac{1}{\gamma^2} - 6 = 0$

or, $1 + \gamma^2 - 6\gamma^4 = 0$

or, $6\gamma^4 - \gamma^2 - 1 = 0$

or, $6\gamma^4 - 3\gamma^2 + 2\gamma^2 - 1 = 0$

or, $3\gamma^2(2\gamma^2 - 1) - 1 = 0$

or, $2\gamma^4 + 2\gamma^2 - \gamma^2 - 1 = 0$

or, $2\gamma^2(\gamma^2 + 1) - 1(\gamma^2 + 1) = 0$

or, $(2\gamma^2 - 1)(\gamma^2 + 1) = 0$

or, $2\gamma^2 = 1$

$\therefore \quad \gamma = \frac{1}{\sqrt{2}}$ and $\gamma = 1$

Hence, to find the equation of asymptotes we have that the equation of asymptotes differs by constant compared to hyperbola so asymptote is;

$$x^2 - 4xy - 2y^2 + 10x + 4y - \frac{\Delta}{ab-b^2} = 0$$

or, $x^2 - 4xy - 2y^2 + 10x + 4y - \frac{6}{-6} = 0$

or, $x^2 - 4xy - 2y^2 + 10x + 4y + 1 = 0$

4. **Find the product of semi–axis of the conic $x^2 - 4xy + 5y^2 = 2$.**

Sol^n: The equation of conic is;

$$x^2 - 4xy + 5y^2 = 2$$

Comparing it with $ax^2 + 2hxy + hy^2 + 2gx + 2fy + c = 0$

We get,

$$a = 1, h = -2, b = 5, g = 0, f = 0, c = -2$$

Now,

$$\begin{aligned}\Delta &= abc + 2fgh - af^2 - bg^2 - ch^2 \\ &= 1 \times 5(-2) + 2 \times 0 \times 0(-2) - 1 \times 1^2 - 5 \times 0^2 + 2(-2)^2 \\ &= -10 + 8 \\ &= -2 \neq 0\end{aligned}$$

and $ab - h^2 = 1 \times 5 - (-2)^2$

$= 5 - 4 = 1$

$\therefore \quad h^2 - ab = -1$

Thus, the equation of conic referred to its centre is;

$$\frac{a\left(h^2-ab\right)}{\Delta}X^2 + 2h\frac{\left(h^2-ab\right)}{\Delta}XY + \frac{b\left(h^2-ab\right)}{\Delta}Y^2 = 1$$

or, $1\left(\frac{-1}{-2}\right)X^2 - 4\left(\frac{-1}{-2}\right)XY + 5\left(\frac{-1}{-2}\right)Y^2 = 1$

or, $\frac{X^2}{2} - \frac{4XY}{2} + \frac{5Y^2}{2} = 1$

or, $\frac{X^2}{2} - 2XY + \frac{5}{2}Y^2 = 1$

Comparing it with $AX^2 + 2HXY + BY^2 = 1$

We get,

$$A = \frac{1}{2}, H = -1, B = \frac{5}{2}$$

So,

$$\frac{1}{r^4} - \frac{1}{r^2}(A + B) + (AB - H^2) = 0$$

or, $\frac{1}{r^4} - \left(\frac{1}{2} + \frac{5}{2}\right)\frac{1}{r^2} + \left(\frac{5}{4} - 1\right) = 0$

or, $\frac{1}{r^4} - \frac{2}{r^2} + \frac{1}{4} = 0$

or, $r^4 - 12r^2 + 4 = 0$

It is quadratic in r^2 so it has two values. So, the product of $r_1{}^2$ and $r_1{}^2 = \frac{c}{a}$ in quadratic equation.

$$r_1{}^2 r_2{}^2 = 4$$

or, $r_1 \times r_2 = 2$

The product of axes = 2

5. **Find the centre, lengths of the axes and eccentricity of the conic.**

i) $\mathbf{9x^2 + 4xy + 6y^2 - 22x - 16y + 9 = 0}$

Sol^n: The equation of conic is;

$$9x^2 + 4xy + 6y^2 - 22x - 16y + 9 = 0$$

Comparing it with $ax^2 + 2hxy + hy^2 + 2gx + 2fy + c = 0$

We get,

$$a = 9, h = 2, b = 6, g = -11, f = -8, c = 9$$

Now,

$$\begin{aligned}\Delta &= abc + 2fgh - af^2 - bg^2 - ch^2 \\ &= 9 \times 6 \times 9 + 2(-8)(-11) \times 2 - 9(-8)^2 - 6(-11)^2 - 9(2)^2 \\ &= 486 + 352 - 576 - 726 - 36 \\ &= -500 \neq 0\end{aligned}$$

and $ab - h^2 = 9 \times 6 - (2)^2$

$= 54 - 4$

$= 50 > 0$

So, it represents ellipse.

For centre:

Let,

$$\phi = 9x^2 + 4xy + 6y^2 - 22x - 16y + 9$$

$$\frac{\partial\phi}{\partial x} = 18x + 4y - 22$$

$$\frac{\partial\phi}{\partial y} = 4x + 12y - 16$$

For centre:

We solve $\frac{\partial\phi}{\partial x} = 0$ and $\frac{\partial\phi}{\partial y} = 0$

So,

$$9x + 2y - 11 = 0$$

or, $2x + 6y - 8 = 0$

or, $\frac{x}{-16+66} = \frac{y}{-22+72} = \frac{1}{54-4}$

or, $\frac{x}{50} = \frac{y}{50} = \frac{1}{50}$

$\therefore \quad x = 1, y = 1$

So, centre is (1, 1)

For length of axes:

The equation referred to centre is;

$$\frac{a(h^2-ab)}{\Delta}x^2 + 2h\frac{(h^2-ab)}{\Delta}xy + \frac{b(h^2-ab)}{\Delta}y^2 = 1$$

or, $9\left(\frac{50}{500}\right)x^2 + 2 \times 2\left(\frac{50}{500}\right)xy + 6\left(\frac{50}{500}\right)y^2 = 1$

or, $\frac{9}{10}x^2 + \frac{4}{10}xy + \frac{6}{10}y^2 = 1$

So,

$$A = \frac{9}{10}, H = \frac{2}{10}, B = \frac{6}{10}$$

So,

$$\frac{1}{r^4} - \frac{1}{r^2}(A + B) + (AB - H^2) = 0$$

or, $\frac{1}{r^4} - \frac{1}{r^2}\left(\frac{15}{10}\right) + \left(\frac{54}{100} - \frac{4}{100}\right) = 0$

or, $\frac{1}{r^4} - \frac{3}{3r^2} + \frac{50}{100} = 0$

or, $50r^4 - 150r^2 + 100 = 0$

or, $r^4 - 3r^2 + 2 = 0$

or, $r^4 - r^2 - 2r^2 + 2 = 0$

or, $r^2(r^2 - 1) - 2(r^2 - 1) = 0$

or, $(r^2 - 1)(r^2 - 2) = 0$

Either $r^2 = 2$ or, $r^2 = 1$

$\therefore \quad r = \sqrt{2}$ or, $r = 1$

Major axis $= \sqrt{2}$

Minor axis $= 1$

For eccentricity:

$$r_1 = 1, r_2 = \sqrt{2}$$

Now,

$$r_1^2 = r_2^2 = (1 - e^2)$$

or, $1 = 2(1 - e^2)$

or, $\frac{1}{2} = 1 - e^2$

or, $e^2 = 1 - \frac{1}{2} = \frac{1}{2}$

or, $e = \frac{1}{\sqrt{2}}$

ii) $\mathbf{2x^2 + 3y^2 - 4x - 12y + 13 = 0}$

Soln: The equation of conic is;

$$2x^2 + 3y^2 - 4x - 12y + 13 = 0$$

Comparing it with $ax^2 + 2hxy + hy^2 + 2gx + 2fy + c = 0$

We get,

$$a = 2, h = 0, b = 3, g = -2, f = -6, c = 13$$

Now,

$$\begin{aligned}\Delta &= abc + 2fgh - af^2 - bg^2 - ch^2 \\ &= 78 + 2(-6)(-2) \times 0 - 2 \times 36 - 3 \times 4 - 13(0)^2 \\ &= 78 - 72 - 12 \\ &= -6 \neq 0\end{aligned}$$

and $ab - h^2 = 2 \times 3 - 0$

$= 6 > 0$

So, it represents ellipse.

For centre:

Let,

$$\phi = 2x^2 + 3y^2 - 4x - 12y + 13$$

$$\frac{\partial\phi}{\partial x} = 4x - 4 = 0$$

$$\frac{\partial\phi}{\partial y} = 6y - 12 = 0$$

$$x = 1, y = 2$$

Thus,

Centre $= (1, 2)$

The equation referred to centre is;

$$\frac{a(h^2-ab)}{\Delta}x^2 + 2h\frac{(h^2-ab)}{\Delta}xy + \frac{b(h^2-ab)}{\Delta}y^2 = 1$$

or, $2\left(\frac{-6}{-6}\right)x^2 + 3\left(\frac{-6}{-6}\right)y^2 = 1$

or, $2x^2 + 3y^2 = 1$

So,

$$A = 2, H = 0, B = 3$$

We have equation:

$$\frac{1}{r^4} - \frac{1}{r^2}(A + B) + (AB - H^2) = 0$$

or, $\frac{1}{r^4} - \frac{1}{r^2}(2 + 3) + (6) = 0$

or, $\frac{1}{r^4} - \frac{5}{r^2} + 6 = 0$

or, $6r^4 - 5r^2 + 1 = 0$

or, $6r^4 - 3r^2 - 2r^2 + 1 = 0$

or, $3r^2(2r^2 - 1) - 1(2r^2 - 1) = 0$

or, $(3r^2 - 1)(2r^2 - 1) = 0$

Either $3r^2 = 1$ or, $2r^2 = 1$

$\therefore$ $r = \frac{1}{\sqrt{3}}$ or, $r = \frac{1}{\sqrt{2}}$

Thus,

$$\left(\frac{1}{\sqrt{3}}\right)^2 = \left(\frac{1}{\sqrt{2}}\right)^2 (1 - e^2)$$

or, $\frac{1}{3} = \frac{1}{2}(1 - e^2)$

or, $\frac{2}{3} = 1 - e^2$

or, $e^2 = 1 - \frac{2}{3} = \frac{1}{3}$

$\therefore$ $e = \frac{1}{\sqrt{3}}$

6. **Find the asymptotes of the hyperbola $2x^2 + 5xy + 2y^2 + 4x + 5y = 0$.**

Sol^n: The hyperbola is;

$$2x^2 + 5xy + 2y^2 + 4x + 5y = 0$$

Thus, the equation of asymptotes is;

$$2x^2 + 5xy + 2y^2 + 4x + 5y - \frac{\Delta}{ab - h^2} = 0$$

Where,

$$\begin{aligned}\Delta &= abc + 2fgh - af^2 - bg^2 - ch^2 \\ &= 0 + 2 \times \frac{5}{2} \times 2 \times \frac{5}{2} - 2\left(\frac{5}{2}\right)^2 - 2(2)^2 - 0 \\ &= 25 - \frac{25}{2} - 8 \\ &= \frac{25}{2} - 8 = \frac{9}{2}\end{aligned}$$

and $ab - h^2 = 2 \times 2 - \left(\frac{5}{2}\right)^2 = 4 - \frac{25}{4} = -\frac{9}{4}$

So, $2x^2 + 5xy + 2y^2 + 4x + 5y - \frac{\frac{9}{2}}{-\frac{9}{4}} = 0$

or, $2x^2 + 5xy + 2y^2 + 4x + 5y + 2 = 0$

or, $(2x + y + 2)(x + 2y + 1) = 0$

Thus, asymptotes are;

$$2x + y + 2 = 0$$

and $x + 2y + 1 = 0$

7. **Find the foci and eccentricity of the conic $x^2 + 4xy + y^2 - 2x + 2y - 6 = 0$.**

Sol^n: The equation of conic is;

$$x^2 + 4xy + y^2 - 2x + 2y - 6 = 0$$

Comparing it with $ax^2 + 2hxy + hy^2 + 2gx + 2fy + c = 0$

We get,

$$a = 1, h = 2, b = 1, g = -1, f = 1, c = -6$$

Now,

$$\begin{aligned}\Delta &= abc + 2fgh - af^2 - bg^2 - ch^2 \\ &= -6 + 2 \times 1(-1) \times 2 - 1(1)^2 - 1(-1)^2 + 6 \times 4 \\ &= -6 - 4 - 1 - 1 + 24 \\ &= 12 \neq 0\end{aligned}$$

and $ab - h^2 = \frac{1}{4} = -3$

So, $\frac{a(h^2 - ab)}{\Delta}x^2 + 2h\frac{(h^2 - ab)}{\Delta}xy + \frac{b(h^2 - ab)}{\Delta}y^2 = 1$

or, $1\left(\frac{3}{12}\right)x^2 + 4\left(\frac{3}{12}\right)xy + 1\left(\frac{3}{12}\right)y^2 = 1$

or, $\frac{x^2}{4} + \frac{4xy}{4} + \frac{1}{4}y^2 = 1$

So,

$$A = \frac{1}{4}, H = \frac{1}{2}, B = \frac{1}{4}$$

We have equation;

$$\frac{1}{r^4} - \frac{1}{r^2}(A + B) + (AB - H^2) = 0$$

or, $\frac{1}{r^4} - \frac{2}{4}\frac{1}{r^2} + \frac{1}{16} - \frac{1}{4} = 0$

or, $\frac{1}{r^4} - \frac{1}{2r^2} + \frac{3}{16} = 0$

or, $-3r^4 - 8r^2 + 16 = 0$

or, $3r^4 + 8r^2 - 16 = 0$

or, $3r^4 + 12r^2 - 4r^2 - 16 = 0$

or, $3r^2(r^2 + 4) - 4(r^2 + 4) = 0$

or, $(3r^2 - 4)(r^2 + 4) = 0$

Either $r^2 = \frac{4}{3}$ or, $r^2 = -4$

For centre (α, β):

$$\phi = x^2 + 4xy + y^2 - 2x + 2y - 6$$

$$\frac{\partial\phi}{\partial x} = 2x + 4y - 2$$

$$\frac{\partial\phi}{\partial y} = 2y + 2 + 4x$$

So,

$$2x + 4y - 2 = 0$$

i.e. $x + 2y - 1 = 0$

and $4x + 2y + 2 = 0$

i.e. $2x + y + 1 = 0$

Now,

$$\frac{x}{2+1} = \frac{y}{-2-1} = \frac{1}{1-4}$$

or, $x = -1, y = 1$

Thus,

$$\text{Centre} = (-1, 1)$$

The equation of transverse axis $= A - \left(\frac{1}{r_1^2}\right)X + HY = 1$

Thus, we can find $\tan\theta$ = slope of transverse axis as 1 as $x - y = 0$

For e:

$$4 = \frac{4}{3}(e^2 - 1)$$

or, $3 = e^2 - 1$

or, $e = 2$

or, $r_1 = \frac{2}{\sqrt{3}}$

Thus,

$$\text{Foci} = (\alpha \pm ae \cos\theta, \beta \pm ae \sin\theta)$$

$$= \left(-1 \pm 2 \times \frac{2}{\sqrt{3}} \times \frac{1}{\sqrt{2}}, 1 \pm 2 \times \frac{2}{\sqrt{3}} \times \frac{1}{\sqrt{2}}\right)$$

$$= \left(-1 \pm \frac{2\sqrt{2}}{\sqrt{3}}, 1 \pm \frac{2\sqrt{2}}{\sqrt{3}}\right)$$

and eccentricity = 2

Exercise 38

Describe the sketch the graph of the following polar equation of conic section.

1. $r = \frac{2}{1-\cos\theta}$

Sol^n: Here,

$$r = \frac{2}{1-\cos\theta}$$

Comparing it with $r = \frac{de}{1-e\cos\theta}$

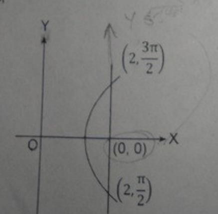
We find

$$d = 2, e = 1$$

Therefore, the conic represents parabola with focus as pole. The expression $\cos\theta$ tells us its axis in on the polar axis. The co-ordinates of $P(r, \theta)$ on the curve are shown on the table.

θ	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$
$r = \frac{2}{1-\cos\theta}$	2	1	2
(r, θ)	$\left(2, \frac{\pi}{2}\right)$	$(1, \pi)$	$\left(2, \frac{3\pi}{2}\right)$

The graph is;



focus = (0,0)
vertex (1, π)
eqn of directrix: d cos θ
x = 2 cos π
x = -2

2. $r = \frac{10}{3-2\cos\theta}$

Sol^n: Here, the equation of conic is;

$$r = \frac{10}{3-2\cos\theta}$$

or, $r = \frac{\frac{10}{3}}{3-\frac{2}{3}\cos\theta} = \frac{5\times\frac{3}{2}}{1-\frac{2}{3}\cos\theta}$

Comparing it with $r = \frac{de}{1-e\cos\theta}$

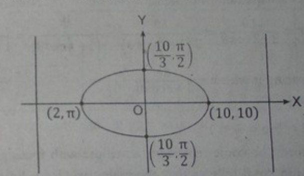
We find

$$d = 5, e = \frac{2}{3} < 1$$

Therefore, the conic represents an ellipse with focus as pole. The expression $\cos\theta$ tells us that the major axis is on polar axis. The co–ordinate $P(r, \theta)$ on the curve is shown on the table below.

θ	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$
$r = \frac{10}{3-2\cos\theta}$	$\frac{10}{1}$	$\frac{10}{3}$	$\frac{10}{5} = 2$	$\frac{10}{3}$
(r, θ)	$(10, 0)$	$\left(\frac{10}{3}, \frac{\pi}{2}\right)$	$(2, \pi)$	$\left(\frac{10}{3}, \frac{3\pi}{2}\right)$

The vertex are $V(10, 0)$ and $V^1(2, \pi)$. The figure is;



3. $r = \frac{12}{3+\cos\theta}$

Sol^n: Here, the equation of conic is;

$$r = \frac{12}{3+\cos\theta} = \frac{12}{3\left(1+\frac{1}{3}\cos\theta\right)} = \frac{4}{1+\frac{1}{3}\cos\theta} = \frac{12\times\frac{1}{3}}{1+\frac{1}{3}\cos\theta}$$

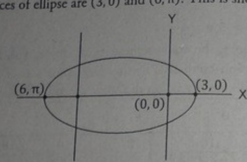
Comparing it with $r = \frac{de}{1+e\cos\theta}$

We find; $d = 12, e = \frac{1}{3} < 1$

Therefore, the conic represents an ellipse with focus as pole. The expression $\cos\theta$ tells us that its axis is on the polar axis. The co-ordinates of $P(r,\theta)$ on the curve is shown on table below.

θ	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$
$r=\frac{12}{3+\cos\theta}$	3	4	6	4
(r,θ)	$(3,0)$	$\left(4,\frac{\pi}{2}\right)$	$(6,\pi)$	$\left(4,\frac{3\pi}{2}\right)$

The vertices of ellipse are $(3,0)$ and $(6,\pi)$. This is shown as;



4. $r=\frac{10}{3+2\cos\theta}$

Sol^n: Here, the equation of conic is;

$$r=\frac{10}{3+2\cos\theta}=\frac{10}{3\left(1+\frac{2}{3}\cos\theta\right)}=\frac{\frac{10}{3}}{\left(1+\frac{2}{3}\cos\theta\right)}=\frac{5\times\frac{2}{3}}{1+\frac{2}{3}\cos\theta}$$

Comparing it with $r=\frac{de}{1+e\cos\theta}$

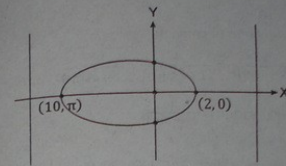
We find

$$d=5, e=\frac{2}{3}<1$$

Therefore, the conic represents an ellipse with focus as pole. The expression $\cos\theta$ tells us that the major axis is on polar axis. The co–ordinate $P(r,\theta)$ on the curve is shown on the table below.

θ	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$
$r=\frac{10}{3+2\cos\theta}$	2	$\frac{10}{3}$	10	$\frac{10}{3}$
(r,θ)	$(2,0)$	$\left(\frac{10}{3},\frac{\pi}{2}\right)$	$(10,\pi)$	$\left(\frac{10}{3},\frac{3\pi}{2}\right)$

The vertex of ellipse are $V(2,0)$ and $V^1(10,\pi)$. The figure is shown as;



5. $r=\frac{10}{3+2\sin\theta}$

Sol^n: Here, the equation of conic is;

$$r=\frac{10}{3+2\sin\theta}=\frac{\frac{10}{3}}{\left(1+\frac{2}{3}\sin\theta\right)}=\frac{5\times\frac{2}{3}}{1+\frac{2}{3}\sin\theta}$$

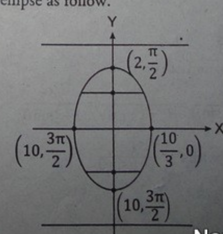
Comparing it with $r=\frac{de}{1+e\sin\theta}$

We find; $d=5, e=\frac{2}{3}<1$

Therefore, the conic represents an ellipse with focus as pole. The expression $\sin\theta$ tells us that the major axis is on polar axis. The co–ordinate $P(r,\theta)$ on the curve is shown on the table below.

θ	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$
$r=\frac{10}{3+2\sin\theta}$	$\frac{10}{3}$	2	$\frac{10}{3}$	10
(r,θ)	$\left(\frac{10}{3},0\right)$	$\left(2,\frac{\pi}{2}\right)$	$\left(\frac{10}{3},\pi\right)$	$\left(10,\frac{3\pi}{2}\right)$

The vertex of the major axis of ellipse are $V\left(2,\frac{\pi}{2}\right)$ and $V^1\left(10,\frac{3\pi}{2}\right)$. These points together type of conic leads to sketch the graph of ellipse as follow.



6. $\mathbf{r = \dfrac{12}{3-2\sin\theta}}$

Sol^n: Here, the equation of conic is;

$$r = \frac{12}{3-2\sin\theta} = \frac{4}{\left(1-\frac{2}{3}\sin\theta\right)} = \frac{6\times\frac{2}{3}}{1-\frac{2}{3}\sin\theta}$$

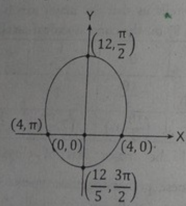
Comparing it with $r = \dfrac{de}{1-e\sin\theta}$

We find; $d = 6, e = \frac{2}{3} < 1$

So, it represents ellipse with focus as pole. The expression $\sin\theta$ tells us that the major axis is perpendicular to the polar axis. The co–ordinate $P(r,\theta)$ on the curve is shown on the table below.

θ	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$
$r = \frac{12}{3-2\sin\theta}$	4	·12	4	$\frac{12}{5}$
(r,θ)	$(4,0)$	$\left(12,\frac{\pi}{2}\right)$	$(4,\pi)$	$\left(\frac{12}{5},\frac{3\pi}{2}\right)$

The vertex of the major axis of ellipse are $V\left(12,\frac{\pi}{2}\right)$ and $V^1\left(\frac{12}{5},\frac{3\pi}{2}\right)$. The figure is;



7. $\mathbf{r = \dfrac{6}{2+3\sin\theta}}$

Sol^n: Here, the equation of conic is;

$$r = \frac{6}{2+3\sin\theta} = \frac{3}{\left(1+\frac{3}{2}\sin\theta\right)} = \frac{\frac{1}{2}\times\frac{3}{2}}{1+\frac{3}{2}\sin\theta}$$

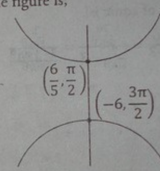
Comparing it with $r = \dfrac{de}{1+e\sin\theta}$

We find; $d = \frac{1}{2}, e = \frac{3}{2} > 1$

Therefore the conic represents a hyperbola with focus as the pole. The expression $\sin\theta$ tells us that the transverse axis of the hyperbola is perpendicular to polar axis. The co–ordinate $P(r,\theta)$ on the curve is shown on the table below.

θ	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$
$r = \frac{6}{2+3\sin\theta}$	3	$\frac{6}{5}$	3	-6
(r,θ)	$(3,10)$	$\left(\frac{6}{5},\frac{\pi}{2}\right)$	$(3,\pi)$	$\left(-6,\frac{3\pi}{2}\right)$

The vertex of the major axis of ellipse are $V\left(\frac{6}{5},\frac{\pi}{2}\right)$ and $V^1\left(-6,\frac{3\pi}{2}\right)$. The figure is;



8. $\mathbf{r = \dfrac{4}{1+3\cos\theta}}$

Sol^n: Here, the equation of conic is;

$$r = \frac{4}{1+3\cos\theta} = \frac{3\times\frac{4}{3}}{1+3\cos\theta}$$

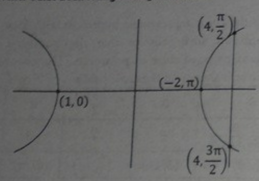
Comparing it with $r = \dfrac{de}{1+e\cos\theta}$

We find; $d = \frac{4}{3}, e = 3 > 1$

So, it represents hyperbola. The expression $\cos\theta$ tells us that its transverse axis is on the polar axis. The co–ordinate $P(r,\theta)$ on the curve is shown on the table below.

θ	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$
$r = \frac{4}{1+3\cos\theta}$	1	4	-2	4
(r,θ)	$(1,0)$	$\left(4,\frac{\pi}{2}\right)$	$(-2,\pi)$	$\left(4,\frac{3\pi}{2}\right)$

The vertex of hyperbola are $(1,0)$ and $(-2,\pi)$. The graph is;



9. $r = \dfrac{\mathbf{10\ cosec\,\theta}}{\mathbf{2\,cosec\,\theta + 3}}$

Sol^n: Here, the equation of conic is;

$$r = \frac{10\ \text{cosec}\,\theta}{2\,\text{cosec}\,\theta + 3} = \frac{\frac{10}{\sin\theta}}{\frac{2}{\sin\theta} + 3} = \frac{\frac{10}{\sin\theta}}{\frac{2+3\sin\theta}{\sin\theta}}$$

so, $r = \dfrac{10}{2+3\sin\theta}$

Proceed as Q. no. 7

10. $r = \dfrac{\mathbf{7}}{\mathbf{2+5\cos\theta}}$

Sol^n: Here, the equation of conic is;

$$r = \frac{7}{2+5\cos\theta}$$

or, $r = \dfrac{\frac{5}{2}\times\frac{14}{5}}{1+\frac{5}{2}\cos\theta}$

Comparing it with $r = \dfrac{de}{1+e\sin\theta}$

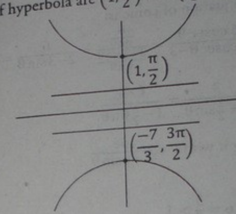
We get,

$$d = \frac{14}{5}, e = \frac{5}{2} > 1$$

So, it represents hyperbola. The expression $\sin\theta$ tells us that the transverse axis of hyperbola is perpendicular to polar axis. The co–ordinate $P(r, \theta)$ on the curve is shown on the table below.

θ	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$
$r = \frac{7}{2+5\cos\theta}$	$\frac{7}{2}$	1	$\frac{7}{2}$	$\frac{-7}{2}$
(r, θ)	$\left(\frac{7}{2}, 0\right)$	$\left(1, \frac{\pi}{2}\right)$	$\left(\frac{7}{2}, \pi\right)$	$\left(\frac{-7}{3}, \frac{3\pi}{2}\right)$

The vertex of hyperbola are $\left(1, \frac{\pi}{2}\right)$ and $\left(\frac{-7}{3}, \frac{3\pi}{2}\right)$. The graph is;



11. $r = \dfrac{\mathbf{12}}{\mathbf{2+4\sin\theta}}$

Sol^n: Here, the equation of conic is;

$$r = \frac{12}{2+4\sin\theta} = \frac{6}{1+2\sin\theta}$$

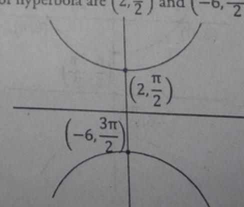
Comparing it with $r = \dfrac{de}{1+e\cos\theta}$

We get, $d = 3, e = 2 > 1$

So, it represents hyperbola. The expression $\cos\theta$ tells us that the transverse axis of hyperbola is perpendicular to polar axis. The co–ordinate $P(r, \theta)$ on the curve is shown on the table below.

θ	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$
$r = \frac{12}{2+4\sin\theta}$	6	2	6	-6
(r, θ)	$(6, 0)$	$\left(2, \frac{\pi}{2}\right)$	$(6, \pi)$	$\left(-6, \frac{3\pi}{2}\right)$

The vertex of hyperbola are $\left(2, \frac{\pi}{2}\right)$ and $\left(-6, \frac{3\pi}{2}\right)$. The graph is;



12. $r = \dfrac{6 \operatorname{cosec}\theta}{2\operatorname{cosec}\theta - 3}$

Solⁿ: Here, the equation of conic is;

$$r = \frac{6 \operatorname{cosec}\theta}{2\operatorname{cosec}\theta-3} = \frac{\frac{6}{\sin\theta}}{\frac{2-3\sin\theta}{\sin\theta}} = \frac{6}{2-3\sin\theta} = \frac{6}{2\left(1-\frac{3}{2}\sin\theta\right)}$$

$$= \frac{3}{1-\frac{3}{2}\sin\theta} = \frac{6\times\frac{3}{2}}{1-\frac{3}{2}\sin\theta}$$

Comparing it with $r = \dfrac{de}{1-e\sin\theta}$

We get,

$$d = 6, e = \frac{3}{2} > 1$$

Therefore conic represents a hyperbola with focus as pole. The expression $\sin\theta$ tells us that the transverse axis of the hyperbola is perpendicular to the polar axis. The co–ordinates $P(r,\theta)$ on the curve is shown on the table.

θ	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$
$r = \frac{7}{2+5\cos\theta}$	3	-6	3	$\frac{6}{5}$
(r,θ)	$(3,0)$	$\left(-6,\frac{\pi}{2}\right)$	$(3,\pi)$	$\left(\frac{6}{5},\frac{3\pi}{2}\right)$

The points $(3,0)$ and $(3,\pi)$ on the graph helps us to sketch the lower branch of hyperbola. The vertex of hyperbola are $V\left(-6,\frac{\pi}{2}\right)$ and $V'\left(\frac{6}{5},\frac{3\pi}{2}\right)$. The graph is shown as;