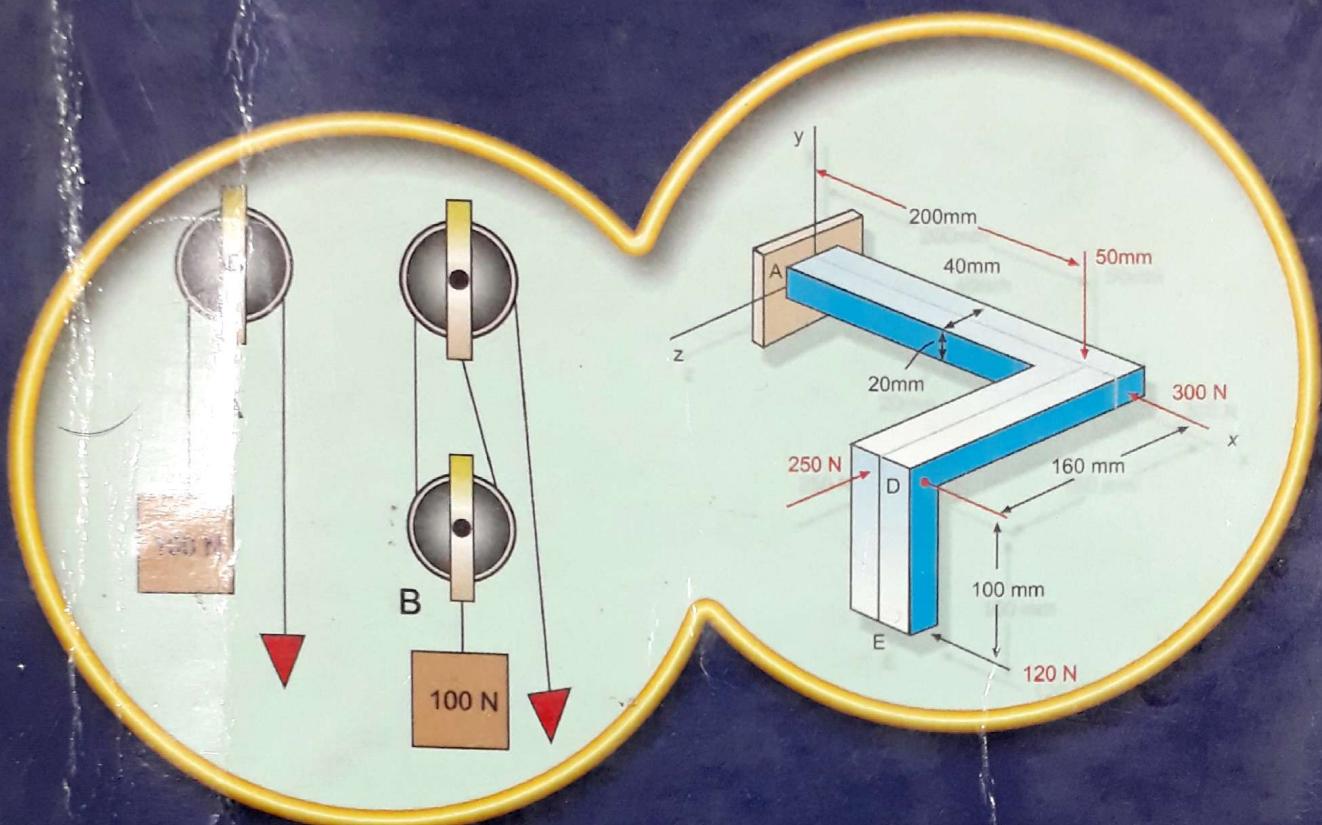


For first year students of BCE, BIE, BME, BAME, BCT, BEX, BEL, BGE, B. Agri & B.Arch



# APPLIED MECHANICS

FOR ENGINEERS

MK

Dr. Hari Ram Parajuli  
Er. Sudan Neupane

# **Applied Mechanics for Engineers**

**Dr. Hari Ram Parajuli  
Er. Sudan Neupane**

**M.K. Publishers and Distributors  
Bhotahity, Kathmandu**

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## *Preface*

This book is an attempt to present the fundamentals and basic concepts of applied mechanics. The book is the outcome of class delivered to various groups of students at bachelor level courses. The approach is to ensure conceptual understanding of theory of applied mechanics and their application to solve engineering problems. The text presents numerous ideas and influences which is the outcome of the distillation of the available text book written by distinguished authors. An attempt has been made to provide a solution to a wide variety of standard problem set for engineering degree examination in a simple, lucid and understandable language. The Authors believe that this book will comfortably meet the requirements of students for which it has been written. Our effort in writing this book will be amply awarded if readers send their constructive suggestions and objective criticism with a view to enhance the usefulness of the book.

We would like to pay our sincere gratitude and respect to Professor Narendra Man Shakya and Associate Professors Dr. Rajendra Shrestha for advising us on teaching and learning which encouraged us to write this book. A special mention in this respect is reserved for our family members. We also thank Mr. Miller Shrestha, Kiran Shakya and the publisher M.K. Publisher and Distributor for bringing out the book.

December, 2015

**Dr. Hari Ram Parajuli**  
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## *foreword*

We have great pleasure in writing this foreword to the book entitled "Applied Mechanics for Engineers" by Dr. Hari Ram Parajuli and Er. Sudan Neupane. This book is the first level text book for applied mechanics courses of engineering degree in bachelor level for all faculties. This book is a new effort to elaborate the subject matters and it provides a very concise and clear presentation of applied mechanics. Each chapter is presented with self-explanatory sketches which curtails long and wordy explanations. Number of solved examples and practice questions at the end of each chapter serve as a good source to understand the basic principle. The style of this book reflects experienced teaching culture since the vast topic has been covered in a comprehensive manner.

The young dynamic authors are to be commended on their efforts in authoring such a comprehensive textbook. With great enthusiasm, we recommend this book for the professionals, teachers and students who are learning applied mechanics.

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Engineering

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Civil Engineering

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# **Chapter 1**

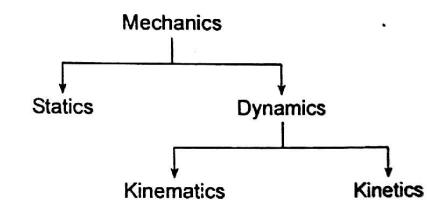
## **Applied Mechanics : Introduction**

### **1.1 Definition**

Mechanics is a basic subject that deals with the response of particles and rigid bodies to mechanical disturbance. Mechanical disturbances in our context means forces, moments, displacements, couple. It essentially deals with the study of bodies at rest or in motion when subjected to external mechanical disturbances. This subject can be subdivided into three branches:

- a) Rigid body mechanics
- b) Deformable body mechanics
- c) Fluid mechanics

In this course we will study rigid body mechanics as it is a basics for the study of fluid mechanics and mechanics of deformable body. Rigid body mechanics is further divided into statics and dynamics; the former dealing with bodies in rest, and later with bodies in motion. Dynamics is further divided into kinematics and kinetics. Kinematics is used to relate displacement, velocity, acceleration and time without reference to the cause of motion. In kinetics, both the motion and its causes are considered. Hence kinetics is science of relation between force mass and motion of body.



#### **1.1.1 Scope of applied mechanics**

Mechanics is the foundation of most engineering sciences and is prerequisite to their study. The purpose of mechanics is to explain and predict natural and physical phenomena and thus forms the foundations for engineering application. It is essential to have a knowledge of applied

mechanics for the design and analysis of many types of structural members, mechanical components and electrical devices encountered in engineering.

## 1.2 Concept of Rigid and Deformed bodies

### 1.2.1 Concept of rigid body

A body has a definite shape and consists of numbers of particles. A particle is an object that has infinitely small volume (size that can be neglected) but has a mass. Hence the rigid body can be considered as a combination of large number of particles in which the distance between any two particles remains constant (i.e. size and shape of body do not change) when it is acted upon by a force system. This model is important because the material properties of any body that is assumed to be rigid will not have to be considered when studying the effects of forces acting on body. In most of the case, the actual deformations occurring in structures, machines and mechanism and the like are usually small, and the rigid body assumption is suitable for analysis.

### 1.2.2 Deformable body

If a body changes its shape and size when it is acted upon by an external force, then it is called deformable body. An elastic body undergoes deformation but regains its original shape after the removal of external forces.

## 1.3 Fundamental concept and principle of mechanics: Newtonian Mechanics

### 1.3.1 Fundamental concept

Although the study of mechanics goes back to the time of Aristotle (384–322 B.C.) and Archimedes (287–212 B.C.), one has to wait Newton (1642–1727 A.D.) to find a satisfactory formulation of its fundamental principles. While its limitation have now been recognized, Newtonian mechanics still remains the basis of today's engineering science. After the Newton's laws were postulated, important techniques for their application were developed by Euler, D'Alembert, Lagrange and others. Their validity remains unchallenged until Einstein formulated his theory of relativity.

### Introduction

The basic concepts used in mechanics are space, time, mass and force. These concept cannot be truly defined rather they should be accepted on the basis of our intuition and experience. The concept of space is associated with the notation of the position of point P. The position of P can be defined by three length measured from certain reference called origin, in the three direction. These length are called coordinates of P. Time is a measure of succession of events. The concept of mass is used to characterize and compare bodies on the basis of experiment. A force represents the action of one body on another. Force is characterized by its point of application, its magnitude, and its direction. In Newtonian mechanics, space, time and mass are absolute concepts, independent of each other. (This is not true in relativistic mechanics, by Einstein, where the time of an event depends upon its position and where the mass of a body varies with its velocity.)

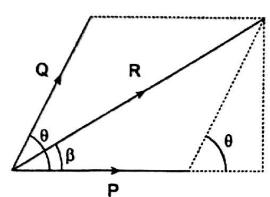
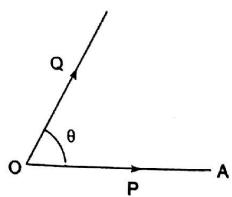
### 1.3.2 Principle of Mechanics: Newtonian

The study of elementary mechanics rests on six fundamental principles based on experimental evidence.

1. The parallelogram law of forces
2. The principle of transmissibility of force
3. Newton's first law of motion
4. Newton's Second law of motion
5. Newton's third law of motion
6. Newton's law of gravitation

### 1. Parallelogram law of forces

The parallelogram law is used to determine the resultant of two forces acting at a point in a plane and inclined to each other at an angle. It states that "If two forces, acting at a point be represented in magnitude and direction by the two adjacent sides of a parallelogram, then their resultant is represented in magnitude and direction by the diagonal of the parallelogram passing through that point."



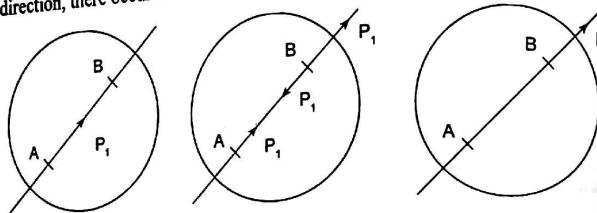
$$\therefore \vec{R} = \vec{P} + \vec{Q}$$

$$R^2 = P^2 + Q^2 + 2PQ \cos \theta$$

$$\tan \beta = \frac{Q \sin \theta}{P + Q \cos \theta}$$

## 2. Principle of transmissibility of force

When the point of application of force acting on a body is shifted to any other point on the line of action of the force without changing its direction, there occurs no change in the equilibrium of the body.



## 3. Newton's first law

"Every body continues in its state of rest or of uniform motion in a straight line if there is no unbalanced force acting upon it."

## 4. Newton's second law

"The rate of change of linear momentum is directly proportional to the applied force and it takes place in the direction of the applied force."

## 5. Newton's third law

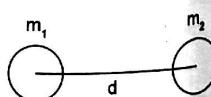
"To every action, there is equal and opposite reaction."

## 6. Newton's law of gravitation

"Everybody in the universe attracts every other body with a force directly proportional to the product of their mass and inversely proportional to the square of the distance separating them."

$$F = \frac{G m_1 m_2}{r^2}; G = 6.67 \times 10^{-11} \text{ Nm}^2/\text{kg}^2$$

where G is constant.



## 1.4 System of unit

Based on the units used to represent the three fundamental quantities namely mass (m), length (l) and time (t) the commonly used system of units are

- a) FPS system — Foot, pound, second
- b) CGS system — centimeter, gram, second
- c) MKS system — meter, kilogram, second

### 1.4.1 SI prefixes

When a numerical quantity is either very large or very small, the units used to define its size may be modified by using a prefix.

Exponential form	Prefix	Symbol
$10^{12}$	tera	T
$10^9$	giga	G
$10^6$	mega	M
$10^3$	kilo	K
$10^{-3}$	milli	m
$10^{-6}$	micro	$\mu$
$10^{-9}$	nano	n
$10^{-12}$	pico	p

## 1.5 Review of vector algebra

### 1.5.1 Dot product

If  $\vec{A}$  and  $\vec{B}$  are two vectors and  $\theta$  is the angle between them, then the dot product is defined as

$$\vec{A} \cdot \vec{B} = |\vec{A}| |\vec{B}| \cos \theta = AB \cos \theta$$

The result of the dot product of two vectors is scalar. So it is also called scalar product.

### 1.5.1.1 Properties of scalar product

- a)  $\vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A}$
- b)  $\vec{A} \cdot (\vec{B} + \vec{C}) = \vec{A} \cdot \vec{B} + \vec{A} \cdot \vec{C}$
- c)  $\vec{i} \cdot \vec{i} = \vec{j} \cdot \vec{j} = \vec{k} \cdot \vec{k} = 1$
- d)  $\vec{i} \cdot \vec{j} = \vec{j} \cdot \vec{k} = \vec{k} \cdot \vec{i} = 0$

If  $\vec{A} \cdot \vec{B} = 0$  and  $A$  and  $B \neq 0$ ; then  $A$  and  $B$  are orthogonal (perpendicular) vector.

### 1.5.2 Cross product

If  $\vec{A}$  and  $\vec{B}$  are two vectors and  $\theta$  is the angle between them, then the cross product is defined as

$$\vec{A} \times \vec{B} = |\vec{A}| |\vec{B}| \sin \theta \hat{n}$$

The result of the cross product of two vector is again a vector, so it is also called vector product.

### 1.5.2.1 Properties of vector product

$$\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$$

$$\vec{A} \times (\vec{B} + \vec{C}) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C}$$

$$\vec{i} \times \vec{i} = \vec{j} \times \vec{j} = \vec{k} \times \vec{k} = 0$$

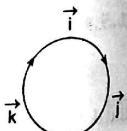
$$\vec{i} \times \vec{j} = \vec{k}$$

$$\vec{j} \times \vec{k} = \vec{i}$$

$$\vec{k} \times \vec{i} = \vec{j}$$

If  $\vec{A} = A_1 \vec{i} + A_2 \vec{j} + A_3 \vec{k}$

$$\vec{B} = B_1 \vec{i} + B_2 \vec{j} + B_3 \vec{k}$$



### Introduction

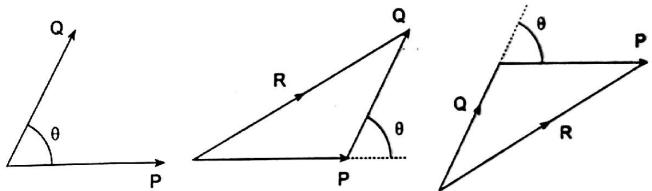
then,

$$\vec{A} \times \vec{B} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix}$$

### 1.5.3 Triangle law of forces

"If two forces acting simultaneously on a body are represented by the sides of a triangle taken in order, their resultant is represented by the third side of triangle taken in the opposite order."

$$\vec{R} = \vec{P} + \vec{Q}$$



The following trigonometric relations can be applied while working out solution by the triangle law.

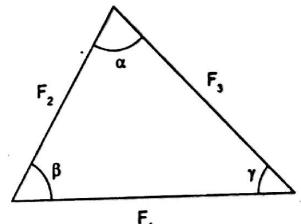
$$\frac{F_1}{\sin \alpha} = \frac{F_2}{\sin \gamma} = \frac{F_3}{\sin \beta}$$

and

$$F_1^2 = F_2^2 + F_3^2 - 2F_2F_3 \cos \alpha$$

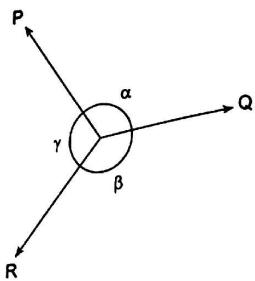
$$F_2^2 = F_1^2 + F_3^2 - 2F_1F_3 \cos \gamma$$

$$F_3^2 = F_1^2 + F_2^2 - 2F_1F_2 \cos \beta$$



### 1.5.4 Lami's theorem

"If a body is in equilibrium under the action of three forces, then each force is proportional to the sine of the angle between the other two forces."



Mathematically,

$$\frac{P}{\sin \beta} = \frac{Q}{\sin \gamma} = \frac{R}{\sin \alpha}$$

All the forces must be outgoing from common point.

#### 1.5.5 Resolution of forces

Finding out the component of a given force in two given directions (normally perpendicular) is called resolution.

Consider angle  $\alpha$ :

$$R_x = R \cos \alpha$$

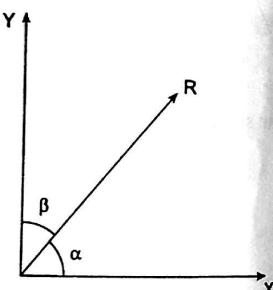
$$R_y = R \sin \alpha$$

Consider angle  $\beta$ :

$$R_x = R \sin \beta$$

$$R_y = R \cos \beta$$

Note: If X and Y are not perpendicular, then the case will be different. It will be discussed in chapter 3.



## Chapter 2

### Basic Concept in Statics and Static Equilibrium

#### 2.1 Concept of particles and free body diagram

##### 2.1.1 Concept of particles

A particle is an object that has infinitely small volume (occupies negligible space) but has a mass which can be considered to be concentrated at a point. A rigid body is a combination of large number of particles occupying fixed positions with respect to each other. The study of mechanics of particle is obviously a prerequisite to that of rigid body. Besides, the results obtained for a particle can be used directly in a large number of problems dealing with the conditions of rest or motion of actual bodies.

When the distances travelled by the body are much more larger than the size of body itself, then we can approximate that body as particle. Sometimes heavenly bodies like earth is treated as particle, although size is very large, why? It is because the perimeter of earth is very small in comparison to the distance travelled by earth in revolving around the sun.

If the size is comparable to the distance travelled, then we treat them as finite body.

##### 2.1.2 Free body Diagram (FBD)

The force analysis of a structure is made in a simplified way by considering the equilibrium of a portion of the structure. For that, the portion is drawn separately showing applied forces, self weight and reactions at the point of contact with other bodies. The resulting diagram is known as Free Body Diagram (FBD). So, FBD is a sketch of body (space diagram) drawn in such a way that it shows all the reaction forces, applied forces and moments on the body.

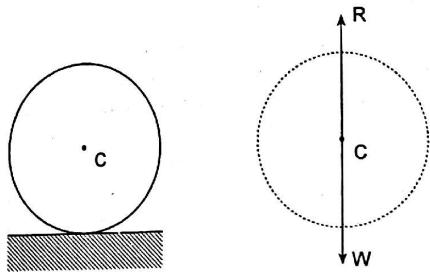
##### 2.1.2.1 Guidelines for drawing FBD

- i) FBD should have no external supports or connections.

- ii) The self weight ( $W = mg$ ) should be indicated with vertical downward arrow.
- iii) The reactions from the supports and connections should be indicated.
- iv) The uncut member force should not be shown in FBD.
- v) Tension in rope or string is directed towards support.
- vi) The adopted coordinate system and sense of unknown force should be shown in FBD.

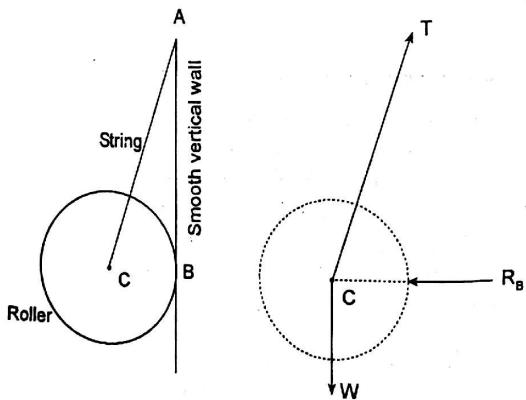
### 2.1.2.2 Few Examples of FBD

a) A sphere resting on a frictionless plane surface



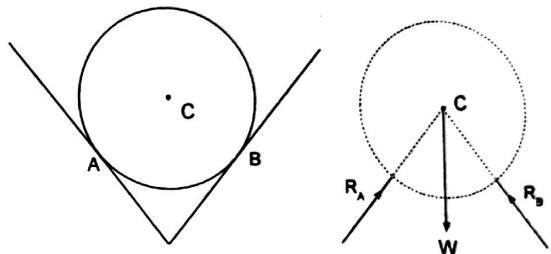
- Force  $W = mg$  acting downward through centroid.
- Reaction  $R$  at the point of contact with surface. This  $R$  acts upward normal to the surface.

b) A circular roller (of weight  $W$ ) hangs by a string and rest against a smooth (frictionless) vertical wall



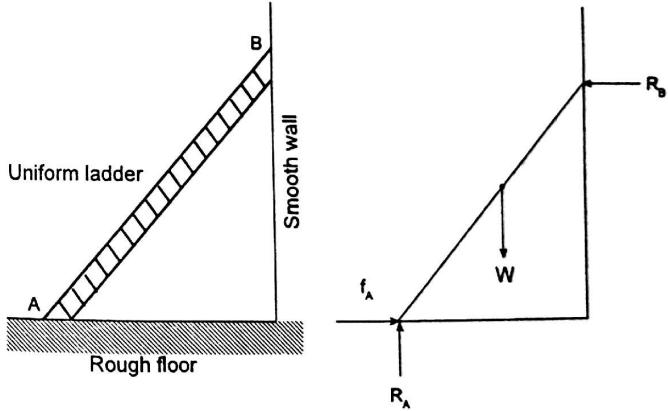
### Basic Concept in Statics and Static Equilibrium

- Force  $W = mg$  acting downward through centroid.
- Wall reaction  $R_B$  at point of contact, normal to wall along BC.
- Tension  $T$  towards support A along CA.
- c) A sphere resting in a V-shaped groove.



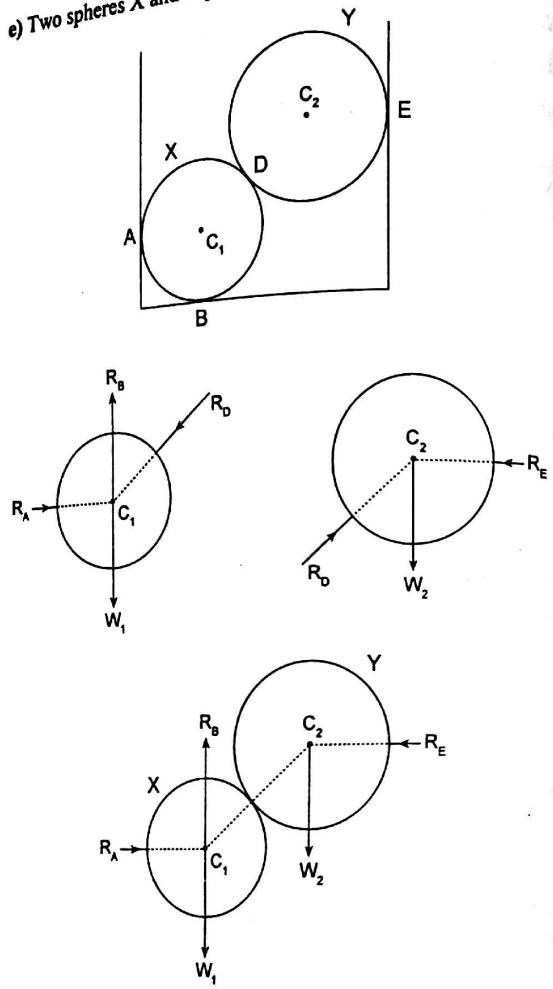
- Weight  $W = mg$  acting downward through centroid.
- Reaction forces  $R_A$  and  $R_B$ , normal to contact points A and B respectively.

d) A uniform ladder leans against a smooth wall and rest on rough floor.



- Weight  $W = mg$  acting downward through centroid
- Reaction force  $R_A$  and  $R_B$  normal to contact point A and B respectively.
- Frictional force  $f_A$  in opposite to the motion of slippage.

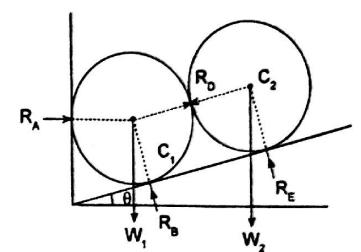
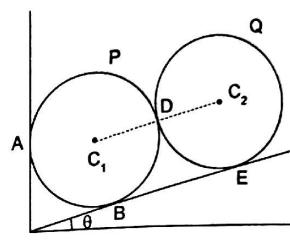
e) Two spheres X and Y placed in a vessel.



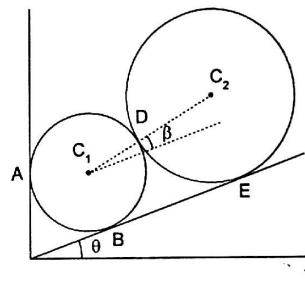
- Weight w<sub>1</sub> and w<sub>2</sub> vertically downward through e.g.
- R<sub>A</sub>, R<sub>B</sub> and R<sub>E</sub> normal to contact surface.
- R<sub>D</sub> on both sphere at contact point O along C<sub>1</sub>C<sub>2</sub> in opposite direction.

Note: Reaction of sphere X on sphere Y and vice-versa at contact point D are equal in magnitude, opposite in direction and are collinear. So, they do not appear in the FBD.

f) Two rollers of the same diameter are supported by an inclined plane and a vertical wall.



g) Two rollers of the unequal diameter are supported by an inclined plane and a vertical wall.



Everything will be similar to previous case (case f) but the line C<sub>1</sub>C<sub>2</sub> will make angle β with inclined plane.

$$\text{So, } \sin \beta = \frac{r_2 - r_1}{r_2 + r_1}$$

Here W<sub>1</sub> makes angle θ and W<sub>2</sub> makes angle θ with R<sub>B</sub> and R<sub>E</sub> respectively. Here θ is the angle of inclined plane.

## 2.2 Physical meaning of equilibrium and its essence in structural application

When the numbers of forces act on a rigid body, we have various methods of finding the resultant force. This resultant force will produce same effect as produced by number of forces. In some cases, it is quite possible for a resultant to be zero. In such case, the net effect of the given force is zero, and the particle is said to be in equilibrium. Such a set of forces, where resultant is zero, are called equilibrium forces. Hence the resultant refers to the single force which produces the same effect as is done by the combined effect of several forces. The force, which brings the set of forces in equilibrium is called an equilibrant. As a fact, the equilibrant is equal to the resultant force in magnitude, but opposite in direction.

In any structure, there will be number of applied force, moments and couple. For a structure to be in equilibrium, all these applied forces and moment is to be balanced.

## 2.3 Equation of equilibrium in two dimensions

Consider a body is acted upon by number of coplanar and non-concurrent forces. Because of these forces, the body may have one of the following states:

- a) A body may move in any one direction
- b) The body may rotate about itself without moving
- c) The body may move in any one direction and rotate about itself at same time
- d) The body may be completely at rest

**Case a:**

The equilibrium equations will be

- i)  $\Sigma F_x = 0$  i.e.  $\Sigma H = 0$
- ii)  $\Sigma F_y = 0$  i.e.  $\Sigma V = 0$

**Case b:**

The equilibrium equation will be

$$\Sigma M = 0$$

**Case c:**

The equilibrium equations will be

i)  $\Sigma F_x = 0$

ii)  $\Sigma F_y = 0$

iii)  $\Sigma M = 0$

**Case d:**

It is in equilibrium condition and following three equations of equilibrium are already satisfied.

i)  $\Sigma F_x = 0$

ii)  $\Sigma F_y = 0$

iii)  $\Sigma M = 0$

So, it can be concluded that for a system to be in equilibrium

$$\vec{\Sigma F} = \vec{R} = 0$$

$$\Sigma M = 0$$

The equilibrium equations for 3-D can be

$$\vec{\Sigma F} = 0$$

$$\Sigma M = 0$$

which can be further expanded as

$$\Sigma F_x = 0 \quad \Sigma F_y = 0 \quad \Sigma F_z = 0$$

$$\Sigma M_x = 0 \quad \Sigma M_y = 0 \quad \Sigma M_z = 0$$

where

$\Sigma F_x$  = algebraic sum of forces in x-direction

$\Sigma F_y$  = algebraic sum of forces in y-direction

$\Sigma F_z$  = algebraic sum of forces in z-direction

$\Sigma M_x$  = algebraic sum of moment about y-z plane

$\Sigma M_y$  = algebraic sum of moment about x-z plane

$\Sigma M_z$  = algebraic sum of moment about x-y plane

### Worked Out Examples

1. The resultant of two forces is 10 N and it is inclined at  $60^\circ$  to one of the forces whose magnitude is 5 N. Determine the magnitude and direction of the other force.

**Solution:**

$$\text{Let } P = 5$$

$$R = 10 \text{ N}$$

$Q$  = unknown force

Let the angle between  $R$  and  $Q$  be  $\beta$ .

From the figure;  $\Delta OAC$

$$\frac{P}{\sin \beta} = \frac{Q}{\sin 60^\circ} = \frac{R}{\sin (180^\circ - (60^\circ + \beta))}$$

$$\therefore \frac{Q}{\sin 60^\circ} = \frac{R}{\sin (60^\circ + \beta)} = \frac{P}{\sin \beta}$$

$$\frac{R}{\sin (60^\circ + \beta)} = \frac{P}{\sin \beta}$$

$$\Rightarrow \frac{10}{\sin (60^\circ + \beta)} = \frac{5}{\sin \beta}$$

solving,

$$\beta = 30^\circ$$

$$\therefore Q = \frac{P}{\sin 30^\circ} \times \sin 60^\circ = 5\sqrt{3} \text{ N} = 8.66 \text{ N}$$

2. Determine the magnitude and direction of the resultant

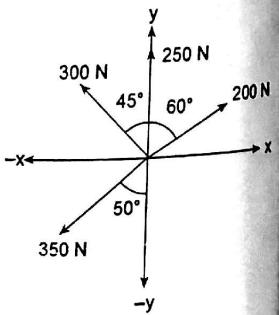
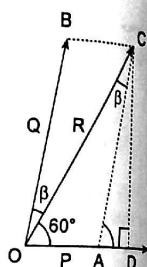
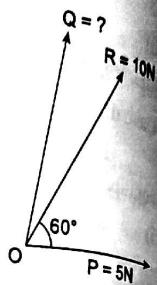
$$\vec{R} = \vec{\Sigma F_x} + \vec{\Sigma F_y}$$

$$\Sigma F_x (\rightarrow +ve)$$

$$= 200 \sin 60^\circ - 300 \sin 45^\circ - 350 \sin 50^\circ$$

$$= -307.04 \text{ N}$$

$$= 307.04 \text{ N} (\leftarrow)$$



$$\Sigma F_y (\uparrow +ve)$$

$$= 200 \cos 60^\circ + 250 + 300 \cos 45^\circ - 350 \cos 50^\circ$$

$$= 337.156 \text{ N} (\uparrow)$$

$$\therefore \vec{R} = \vec{\Sigma F_x} + \vec{\Sigma F_y}$$

$$R = \sqrt{(\Sigma F_x)^2 + (\Sigma F_y)^2}$$

$$= \sqrt{(307.04)^2 + (337.15)^2}$$

$$= 456.01 \text{ N}$$

$$\tan \alpha = \frac{\Sigma F_y}{\Sigma F_x} = \frac{337.15}{307.04}$$

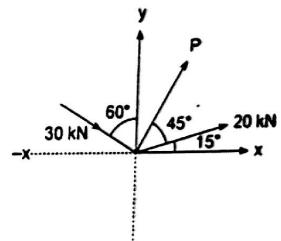
$$\therefore \alpha = 47.67^\circ$$

$$R = 456.01 \quad \alpha = 47.67^\circ$$

3. Determine the value of  $P$  such that the resultant is horizontal. Also find the value of equilibrant.

**Solution:**

Here 30 kN is directed towards the origin. To resolve the force it must be directed away from origin. So, produce 30 kN on the same line of action.



$$\Sigma F_x (\rightarrow +ve)$$

$$= 20 \cos 15^\circ + P \cos 60^\circ + 30 \sin 60^\circ$$

$$= 45.3 + 0.5P$$

$$\Sigma F_y (\uparrow +ve)$$

$$= 20 \sin 15^\circ + P \sin 60^\circ - 30 \sin 60^\circ$$

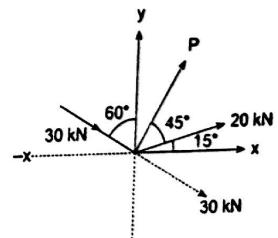
$$= -9.823 + \frac{\sqrt{3}}{2} P$$

Since the resultant is horizontal

$$\Sigma F_y = 0$$

$$\therefore \frac{\sqrt{3}}{2} P = 9.823$$

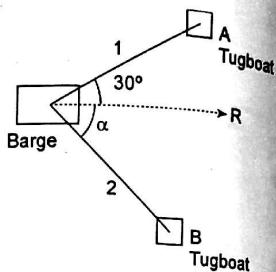
$$\therefore P = 11.34$$



$$\begin{aligned}\vec{R} &= \sum \vec{F}_x \\ &= 45.3 + 0.5 \times 11.34 \\ R &= 50.97 (\rightarrow) \\ \therefore \text{value of equilibrant} &= -R = 50.97 (\leftarrow)\end{aligned}$$

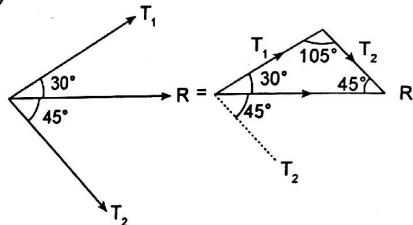
4. A barge is pulled by two tugboats. If the resultant of the forces exerted by the tugboat is a 5000 N force directed along the axis of barge, determine

- the tension in each of the rope knowing  $\alpha = 45^\circ$
- the value of  $\alpha$  for which the tension in rope 2 is minimum and minimum  $T_2$ .



**Solution:**

Case a)



$$\alpha = 45^\circ$$

Draw FBD and triangle for forces

Using sine law

$$\frac{T_1}{\sin 45^\circ} = \frac{T_2}{\sin 30^\circ} = \frac{R}{\sin 105^\circ}$$

$$T_1 = \frac{R}{\sin 105^\circ} \times \sin 45^\circ = \frac{5000}{\sin 105^\circ} \times \sin 45^\circ = 3660.25 \text{ N}$$

$$T_2 = \frac{R}{\sin 105^\circ} \times \sin 30^\circ = \frac{5000}{\sin 105^\circ} \times \sin 30^\circ = 2588.19 \text{ N}$$

### Basic Concept in Statics and Static Equilibrium

#### Case b:

We know that the perpendicular distance is the shortest distance. So, we draw a perpendicular line on  $T_1$  to find minimum  $T_2$ .

So, from figure

$$\alpha + 90^\circ + 30^\circ = 180^\circ$$

$$\therefore \alpha = 60^\circ$$

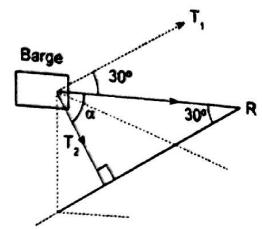
$$\frac{R}{\sin 90^\circ} = \frac{T_2}{\sin 30^\circ} = \frac{T_1}{\sin 60^\circ}$$

$$T_2 = R \sin 30^\circ$$

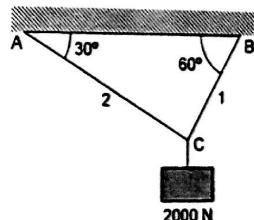
$$= 5000 \times \frac{1}{2} = 2500 \text{ N (minimum)}$$

$$T_1 = R \sin 60^\circ$$

$$= 5000 \times \frac{\sqrt{3}}{2} = 4330.12 \text{ N}$$



5. A weight of 2000 N is supported by two chains AC and BC as shown in figure. Determine the tension in each chord.



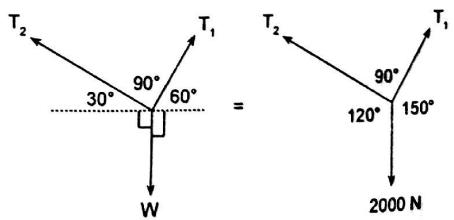
**Solution:**

Draw FBD.

Hint: Condition to use Lami's theorem

a) At most 3 forces

b) All forces directed away from centre.



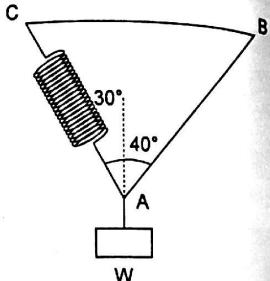
Given FBD satisfy the condition to use Lami's theorem.

$$\text{So, } \frac{T_1}{\sin 120} = \frac{T_2}{\sin 150} = \frac{2000}{\sin 90}$$

$$\therefore T_1 = 2000 \sin 120 = 1732 \text{ N}$$

$$T_2 = 200 \sin 150 = 1000 \text{ N}$$

- (6) A container of weight  $W$  is subjected from ring A to which cable AB and spring AC are attached. The constant of spring is  $100 \text{ N/m}$  and its unstretched length is  $3 \text{ m}$ . Determine the tension in the cable when a)  $W = 120 \text{ N}$  b)  $W = 160 \text{ N}$



**Solution:**

Case a)

$$W = 120 \text{ N}$$

Using Lami's theorem

$$\frac{T_{AC}}{\sin 140} = \frac{T_{AB}}{\sin 150} = \frac{W}{\sin 70}$$

$$\therefore T_{AC} = \frac{120}{\sin 70} \times \sin 140 = 82.08 \text{ N}$$

$$T_{AB} = \frac{120}{\sin 70} \times \sin 150 = 63.85 \text{ N}$$

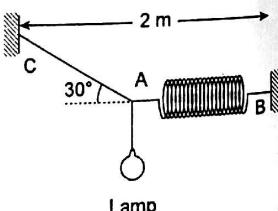
Similarly change  $W = 160 \text{ N}$  and get new value of  $T_{AB} = 85.13 \text{ N}$ .

Note: Spring constant and unstretched length are superfluous data (here). Students can be asked to calculate the extension on the spring.

7. Determine the required length of cord AC so that  $8 \text{ kg}$  lamp can be suspended in the position shown. The underformed length of spring AB is  $0.4 \text{ m}$  and the spring has stiffness of  $300 \text{ N/m}$ .

**Solution:**

Draw FBD



Using Lami's theorem

$$\frac{T_{AC}}{\sin 90} = \frac{T_{AB}}{\sin 120} = \frac{W}{\sin 150}$$

$$T_{AC} = \frac{78.48}{\sin 150} \times \sin 90$$

$$= 156.96 \text{ N}$$

$$T_{AB} = \frac{78.48}{\sin 150} \times \sin 120$$

$$= 135.93 \text{ N}$$

Again, let  $x_{AB}$  be extension of spring AB

$$T_{AB} = k_{AB} \times x_{AB} \quad [F = kx]$$

$$135.93 = 300 \times x_{AB}$$

$$\therefore \text{stretched length of AB} = l_{AB} = 0.453 + 0.4 = 0.853 \text{ m}$$

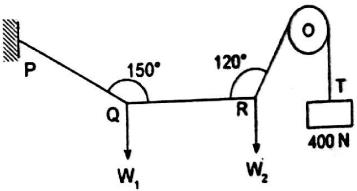
To satisfy given condition

$$l_{AC} \cos 30 + l_{AB} = 2$$

$$l_{AC} = \frac{2 - l_{AB}}{\cos 30} = \frac{2 - 0.853}{\cos 30} = 1.324 \text{ m}$$

(8)

- A light string PQRST whose extremity P is fixed, has weight  $W_1$  and  $W_2$  attached to it at Q and R. It passes round a small smooth peg at S carrying a weight of  $400 \text{ N}$  at the free end T as shown in figure. If QR is horizontal, find the magnitude of  $W_1$  and  $W_2$  and tension in the portion PQ, QR, RS and ST of string.



**Solution:**

For a smooth frictionless peg, pulley and continuous string, the tension induced in the string is same throughout.

$$\therefore \text{tension at ST} = T_{ST} = 400 \text{ N}$$

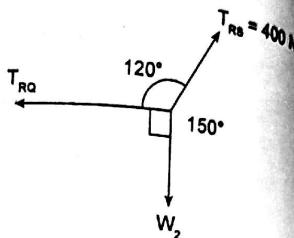
$$\text{tension at RS} = T_{RS} = 400 \text{ N}$$

At point R:

$$\frac{W_2}{\sin 120} = \frac{T_{RQ}}{\sin 90} = \frac{T_{RO}}{\sin 150}$$

$$\therefore W_2 = \frac{400}{\sin 90} \times \sin 120 \\ = 461.88 \text{ N}$$

$$T_{RQ} = \frac{400}{\sin 90} \times \sin 150 \\ = 200 \text{ N}$$

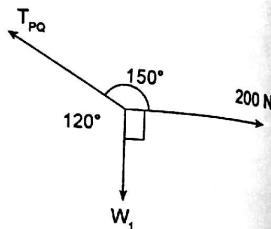


At point Q:

$$\frac{W_1}{\sin 150} = \frac{200}{\sin 120} = \frac{T_{PQ}}{\sin 90}$$

$$\therefore W_1 = \frac{200}{\sin 120} \times \sin 150 \\ = 115.47 \text{ N}$$

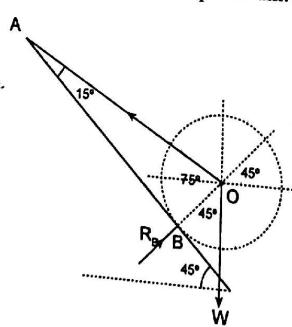
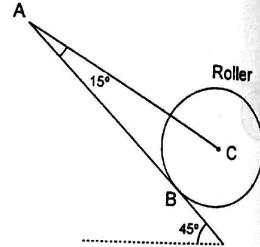
$$T_{PQ} = 230.94 \text{ N}$$



9. A roller of weight 500 N rests on a smooth inclined plane and is kept free from rolling down by a string as shown in figure. Work out the tension in the string and reaction at contact point B.

**Solution:**

Draw the FBD and use the trigonometry and equation of equilibrium.



Using Lami's theorem

$$\frac{W}{\sin 105} = \frac{T}{\sin 135} = \frac{R_B}{\sin 120}$$

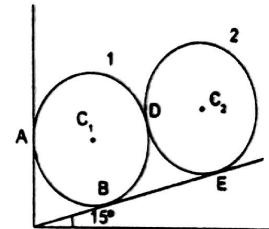
$$T = \frac{W}{\sin 105} \times \sin 135$$

$$= \frac{500}{\sin 105} \times \sin 135 = 366.02 \text{ N}$$

$$R_B = \frac{W}{\sin 105} \times \sin 120$$

$$= \frac{500}{\sin 105} \times \sin 120 = 448.28 \text{ N}$$

10. Two rollers of the same diameter are supported by an inclined plane and a vertical wall. The upper and the lower rollers are respectively 200 N and 250 N in weight. Assuming smooth surface, find the reactions induced at the points of support A, B, C and D.



**Solution:**

Since the two rollers are of same diameter, line joining the centre  $C_1C_2$  will be parallel to inclined plane.

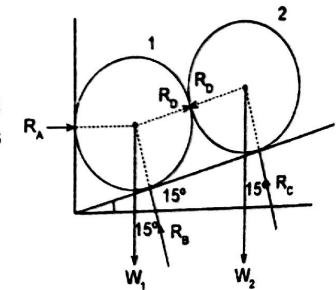
$$W_1 = 250 \text{ N}$$

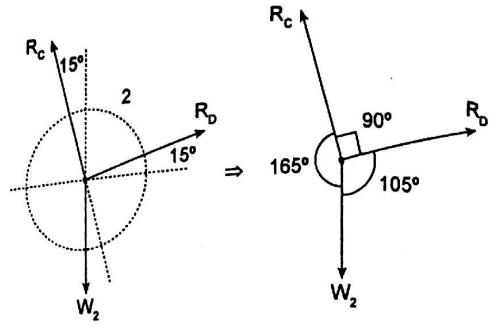
$$W_2 = 200 \text{ N}$$

Since roller 2 has only 3 forces, so we will first solve this roller using Lami's theorem.

Using Lami's theorem

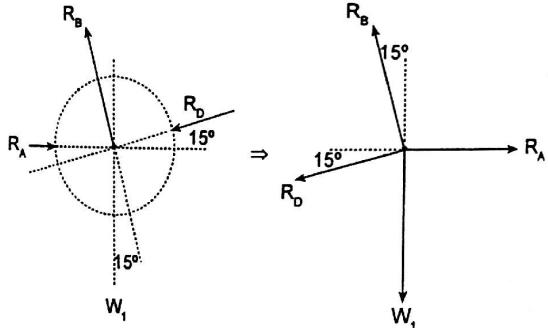
$$\frac{R_C}{\sin 105} = \frac{R_D}{\sin 165} = \frac{W_2}{\sin 90}$$





$$\begin{aligned} R_C &= \frac{W_2}{\sin 90} \times \sin 105 \\ &= \frac{200}{\sin 90} \times \sin 105 = 193.18 \text{ N} \\ R_D &= \frac{W_2}{\sin 90} \times \sin 165 \\ &= \frac{200}{\sin 90} \times \sin 165 = 51.76 \text{ N} \end{aligned}$$

Now for further calculation, draw FBD of roller 1.



Here in the FBD of roller 2, the number of force system is greater than 3 (i.e. 4), so, Lami's theorem cannot be used. So, we go for resolution of force and use equation of equilibrium  
i.e.  $\sum F_x = 0$     $\sum F_y = 0$

Now, from FBD  
i)  $\sum F_y = 0$  ( $\uparrow +ve$ )

$$R_B \cos 15 - R_D \sin 15 - W_1 = 0$$

$$\begin{aligned} \text{or, } R_B \cos 15 - 51.76 \sin 15 - 250 &= 0 \\ \therefore R_B &= 272.688 \text{ N} \\ \text{ii) } \sum F_x &= 0 (\rightarrow +ve) \\ R_A - R_B \sin 15 - R_D \cos 15 &= 0 \\ \text{or, } R_A - 272.688 \sin 15 - 51.76 \cos 15 &= 0 \\ \therefore R_A &= 120.57 \text{ N} \\ \therefore R_A &= 120.57 \text{ N}, R_B = 272.688 \text{ N}, R_C = 193.18 \text{ N}, R_D = 51.76 \text{ N} \end{aligned}$$

- (11) The cylinder A and B rest in an inclined plane which makes an angle of  $25^\circ$  with horizontal as shown in figure below. Determine reaction at contact points. Take

weight of cylinder A ( $W_A$ )

$$= 100 \text{ N}$$

weight of cylinder B ( $W_B$ )

$$= 200 \text{ N}$$

$$\text{diameter of cylinder A } (D_A) = 60 \text{ mm}$$

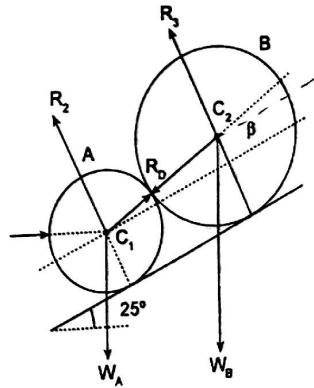
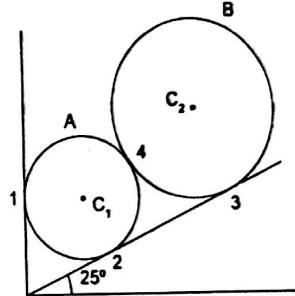
$$\text{diameter of cylinder B } (D_B) = 90 \text{ mm}$$

**Solution:**

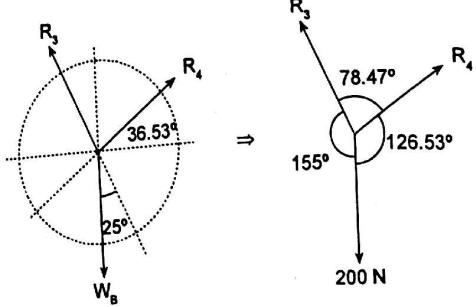
Let the four contact points be 1, 2, 3 and 4. Here the diameter of two roller is different. So, the line joining  $C_1 C_2$  will make an angle with inclined plane (say  $\beta$ ).

Calculation of angle  $\beta$

$$\begin{aligned} \sin \beta &= \frac{r_B - r_A}{r_B + r_A} \\ &= \frac{45 - 30}{45 + 30} = \frac{15}{75} = \frac{1}{5} \\ \therefore \beta &= 11.53^\circ \end{aligned}$$



First we will solve cylinder B as it includes 3 forces; using Lami's theorem

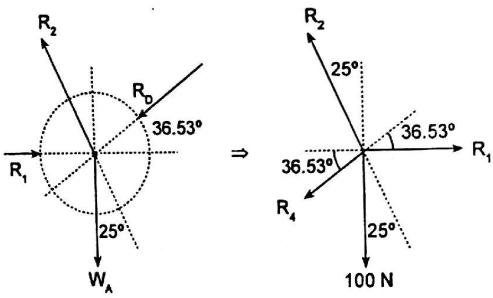


$$\frac{R_3}{\sin 126.53^\circ} = \frac{R_4}{\sin 155^\circ} = \frac{200}{\sin 78.47^\circ}$$

$$\therefore R_3 = \frac{200}{\sin 78.47^\circ} \times \sin 126.53^\circ = 164.01 \text{ N}$$

$$R_4 = \frac{200}{\sin 78.47^\circ} \times \sin 155^\circ = 86.1972 \text{ N}$$

For further calculation, draw FBD of sphere A.



$$\Sigma F_y = 0 (\uparrow +ve)$$

$$R_2 \cos 25^\circ - R_4 \sin 36.53^\circ - W_A = 0$$

$$R_2 \cos 25^\circ - 86.19 \sin 36.53^\circ - 100 = 0$$

$$\therefore R_2 = 166.94 \text{ N}$$

$$\Sigma F_x = 0 (\rightarrow +ve)$$

$$R_1 - R_2 \sin 25^\circ - R_4 \cos 36.53^\circ = 0$$

$$R_1 - 166.94 \sin 25^\circ - 86.19 \cos 36.53^\circ = 0$$

$$\therefore R_1 = 139.81 \text{ N}$$

$$\therefore R_1 = 139.81 \text{ N}, \quad R_2 = 166.94 \text{ N},$$

$$R_3 = 164.01 \text{ N}, \quad R_4 = 86.197 \text{ N}$$

- (12) A vertical pole is guyed by three cables PA, PB and PC tied at a common point P 10 m above the ground. The base points of the cables are A(-3, 0, -4), B(1, -1, 5), C(5, 0, -1). If the tensile force in the cables are adjusted to be 15, 18 and 20 kN respectively. Make calculation for the force on the pole at P.

**Solution:**

The steps in the solution of this kind is to find out the direction of different vector say  $\vec{PA}$ ,  $\vec{PB}$  and  $\vec{PC}$ . For this we will first calculate the unit vector and multiply it by magnitude.

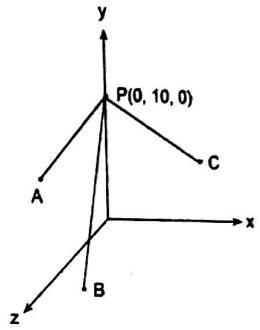
Calculation of unit vector along PA

$$\begin{aligned} n_1 &= \frac{\vec{PA}}{|\vec{PA}|} = \frac{\vec{A} - \vec{P}}{|\vec{PA}|} = \frac{(-3, 0, -4) - (0, 10, 0)}{|\vec{PA}|} \\ &= \frac{-3 \vec{i} - 10 \vec{j} - 4 \vec{k}}{\sqrt{3^2 + 10^2 + 4^2}} \\ &= -0.268 \vec{i} - 0.894 \vec{j} - 0.3577 \vec{k} \end{aligned}$$

$$\therefore \text{Force along } \vec{PA} = \vec{F}_1 = n_1 \cdot 15$$

$$= 15(-0.268 \vec{i} - 0.894 \vec{j} - 0.3577 \vec{k})$$

$$\vec{F}_1 = -4.02 \vec{i} - 13.41 \vec{j} - 5.36 \vec{k}$$



along  $\vec{PB}$ ,

$$\vec{n}_2 = \frac{\vec{PB}}{|\vec{PB}|} = \frac{\vec{B} - \vec{P}}{|\vec{PB}|} = \frac{(1, -1, 5) - (0, 10, 0)}{|\vec{PB}|}$$

$$= \frac{\vec{i} - 11\vec{j} + 5\vec{k}}{\sqrt{1^2 + 11^2 + 5^2}}$$

$$= 0.0824\vec{i} - 0.9072\vec{j} + 0.4123\vec{k}$$

$$\text{Force along } \vec{PB} = \vec{F}_2 = n_2 \cdot 18$$

$$= 18(0.0824\vec{i} - 0.9072\vec{j} + 0.4123\vec{k})$$

$$\vec{F}_2 = 1.48\vec{i} - 16.34\vec{j} + 7.42\vec{k}$$

along  $\vec{PC}$ ,

$$\vec{n}_3 = \frac{\vec{PC}}{|\vec{PC}|} = \frac{\vec{C} - \vec{P}}{|\vec{PC}|} = \frac{(5, 0, -1) - (0, 10, 0)}{|\vec{PC}|}$$

$$= \frac{5\vec{i} - 10\vec{j} - \vec{k}}{\sqrt{5^2 + 10^2 + 1^2}}$$

$$\vec{n}_3 = 0.4454\vec{i} - 0.8908\vec{j} - 0.08908\vec{k}$$

$$\text{Force along } \vec{PC} = \vec{F}_3 = n_3 \cdot 20$$

$$= 20(0.4454\vec{i} - 0.8908\vec{j} - 0.08908\vec{k})$$

$$\vec{F}_3 = 8.91\vec{i} - 17.8\vec{j} - 1.78\vec{k}$$

$\therefore$  Force on pole P is

$$\vec{R} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3$$

$$= -4.02\vec{i} - 13.41\vec{j} - 5.36\vec{k} + 1.48\vec{i} - 16.34\vec{j} + 7.42\vec{k}$$

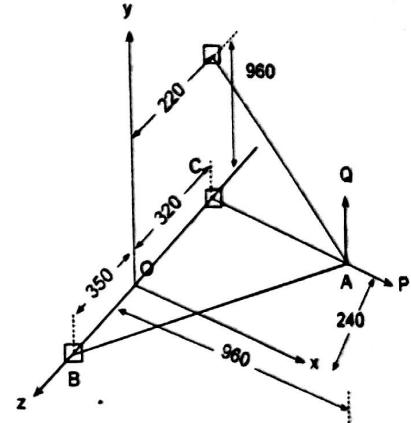
$$+ 8.91\vec{i} - 17.8\vec{j} - 1.78\vec{k}$$

$$\therefore \vec{R} = 6.37\vec{i} - 47.54\vec{j} + 0.28\vec{k}$$

- (13) Three cables are connected at A, where the force P and Q are applied as shown. Knowing that  $P = 1200$  N, determine the range of values of Q for which cable AD is taut.

**Solution:**

First, find the number of forces  $\vec{AB}, \vec{AC}, \vec{AD}, \vec{P}, \vec{Q}$ .



Second is to find 3D coordinates of A, B, C and D.

$$A (0.96, 0.24, 0) \quad [\text{look value parallel to axes}]$$

$$B (0, 0, 0.35)$$

$$C (0, 0, -0.32)$$

$$D (0, 0.96, -0.22)$$

$\vec{P}$  is parallel to x-axis;  $\vec{P} \vec{i}$

$\vec{Q}$  is parallel to y-axis;  $\vec{Q} \vec{j}$

Now, find the force  $\vec{AB}, \vec{AC}, \vec{AD}, \vec{P}$  and  $\vec{Q}$

Unit vector along  $\vec{AB}$

$$\hat{\vec{AB}} = \vec{n}_1 = \frac{\vec{AB}}{|\vec{AB}|} = \frac{\vec{B} - \vec{A}}{|\vec{AB}|}$$

$$= -0.914\vec{i} - 0.25\vec{j} + 0.33\vec{k}$$

Unit vector along AC

$$\hat{AC} = n_2 = \frac{\vec{AC}}{|AC|} = \frac{\vec{C} - \vec{A}}{|AC|}$$

$$= -0.923 \vec{i} - 0.23 \vec{j} - 0.3 \vec{k}$$

Unit vector along AD

$$\hat{AD} = n_B = \frac{\vec{AD}}{|AD|} = \frac{\vec{D} - \vec{A}}{|AD|}$$

$$= -0.78 \vec{i} + 0.59 \vec{j} - 0.18 \vec{k}$$

$$\vec{P} = P \vec{i}$$

$$\vec{Q} = Q \vec{j}$$

Although  $Q = 0$ , AD cable remains taut. This is the minimum range of Q. To find the maximum range, we first assume the condition of unstretching AD i.e.  $T_{AD} = 0$ .

For equilibrium at A;

$$\vec{T}_{AB} + \vec{T}_{AC} + \vec{T}_{AD} + \vec{P} + \vec{Q} = 0$$

$$\text{if } \vec{T}_{AD} = 0$$

$$\vec{T}_{AB} + \vec{T}_{AC} + \vec{P} + \vec{Q} = 0$$

$$T_{AB}(\hat{AB}) + T_{AC}(\hat{AC}) + 1200 \vec{i} + Q \vec{j} = 0$$

$$T_{AB}(-0.914 \vec{i} - 0.25 \vec{j} + 0.33 \vec{k}) + T_{AC}(-0.923 \vec{i} - 0.23 \vec{j} - 0.3 \vec{k}) + 1200 \vec{i} + Q \vec{j} = 0$$

Equating the coefficient

$$-0.914T_{AB} - 0.923T_{AC} + 1200 = 0 \quad \dots \dots \text{(i)}$$

$$-0.25T_{AB} - 0.23T_{AC} + Q = 0 \quad \dots \dots \text{(ii)}$$

$$0.33T_{AB} - 0.3T_{AC} = 0 \quad \dots \dots \text{(iii)}$$

Solving we get,

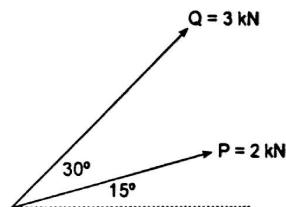
$$T_{AB} = 621.98 \text{ N} \quad T_{AC} = 684.185 \text{ N}$$

$$Q = 312.859$$

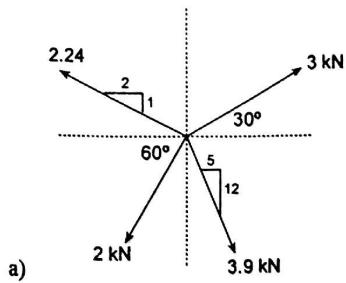
$\therefore$  Range of Q is  $0 < Q < 312 \text{ N.s}$

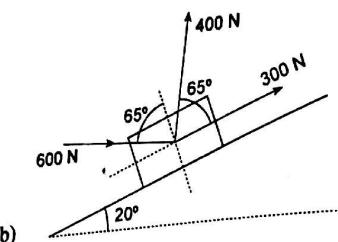
### Practice Questions

- The magnitude of two forces is such that when acting at right angle produce a resultant force of  $\sqrt{20}$  and when acting at  $60^\circ$  produce a resultant equal to  $\sqrt{28}$ . Workout the two forces.
- Use triangle law to find the resultant. Also verify using parallelogram law.

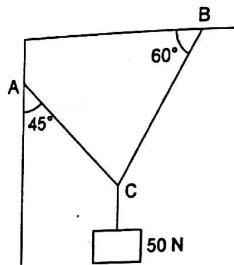


- The resultant of two forces P and Q acting at a point is R. The resultant R gets doubled when Q is either doubled or its direction is reversed. Show that  $P : Q : R = \sqrt{2} : \sqrt{3} : \sqrt{2}$ .
- Determine the resultant





5. Determine the tension in string.



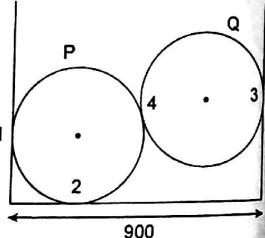
6. Determine the reaction at all contact points.

a) Radius of P = 25 cm

Radius of Q = 25 cm

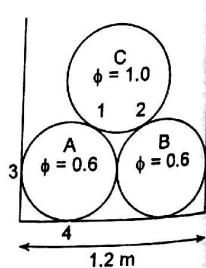
Wt. of P = 500 N

Wt. of Q = 500 N



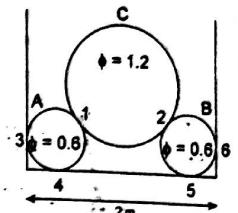
b) Wt. of A = Wt. of B = 1000 N

Wt. of C = 2000 N

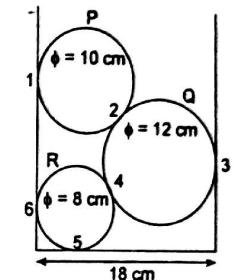


Basic Concept in Statics and Static Equilibrium

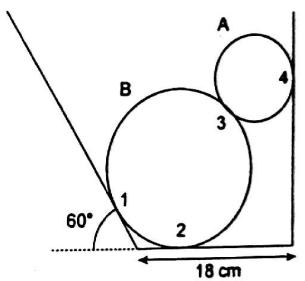
- c) Wt. of A = Wt. of B = 1000 N  
Wt. of C = 2000 N



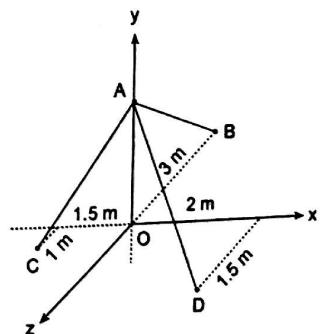
- d) Wt. of P = 2 kN  
Wt. of Q = 4 kN  
Wt. of R = 1.5 kN



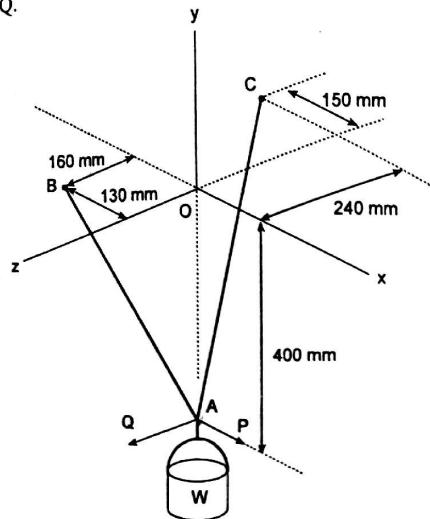
7. Two cylinders A and B rest in a channel as shown in figure. The cylinder A has diameter of 10 cm and weighs 200 N whereas the cylinder B has diameter of 18 cm and weighs 500 N. Determine the contact forces.



8. In a system shown a 5 m long beam is held in vertical position AO by three guy wire AB, AC and AD. If a tension equivalent to 600 N is induced in AD and the resultant force at A is to be vertical. Calculate tension in wire AC and AB.



- (9) A container of weight  $W$  is suspended from ring A. Cable BAC passes through the ring and is attached to fixed supports at B and C. Two forces  $P = P_i$  and  $Q = Q_k$  are applied to the ring to maintain the container in the position shown. Knowing that  $W = 376 \text{ N}$ , determine  $P$  and  $Q$ .



(Hint: The tension is the same in both portions of cable BAC.)

#### Answers

1. (2, 4 or 4, 2 unit      2.  $R = 4.836, \alpha = 33.06$
4. a)  $R = 3.03, \alpha = -68.86$       b)  $R = 1019 \text{ N}, \alpha = 26.01$
5.  $T_{AC} = 25.88 \text{ N}, T_{BC} = 36.59 \text{ N}$
6. a)  $R_1 = 666.67 \text{ N}, R_2 = 1000 \text{ N}, R_3 = 666.67 \text{ N}, R_4 = 833.3 \text{ N}$   
b),  
c)  $R_1 = R_2 = 1590.87 \text{ N}, R_3 = R_6 = 1237.38 \text{ N}, R_4 = R_5 = 1999.87 \text{ N}$   
d)  $R_2 = 2.593 \text{ kN}, R_4 = 10 \text{ kN}, R_6 = 8 \text{ kN}$
7.  $R_1 = 154.893 \text{ N}, R_2 = 622.55 \text{ N}, R_3 = 240.82 \text{ N}, R_4 = 134.14 \text{ N}$
8.  $T_{AB} = 591.66 \text{ N}, T_{AC} = 761.24 \text{ N}$
9.  $P = 131.2 \text{ N}, Q = 29.6 \text{ N}$

## Chapter 3

### Force acting on particles and rigid body

#### 3.1 Different types of force

Force is defined as an external agent which change or tends to change the speed, direction or shape of system. Different types of forces are concentrated load (point force), surface traction (pressure load) and body force (weight) are applied in different shapes structure.

##### Characteristics of force

- Force have magnitude and direction, so it is a vector quantity.
- Force have point of application.
- Force is transmissible vector i.e. it can be moved along its line of action.

##### 3.1.1 Point force

A finite force which is assumed to act through point.

##### 3.1.2 Body force

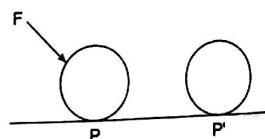
A force which acts on each element of body is called body force. e.g. gravitational force, inertia force, electromagnetic force etc.

##### 3.1.3 Surface force

The force which acts on the surface or area elements of the body is called surface force. When the area considered lies on the actual boundary of the body, the surface force distribution is termed as surface traction.

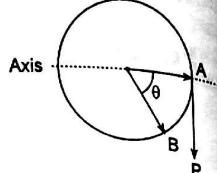
##### 3.1.4 Translation force

The force which moves or tends to move a body from one point to another point is called translation force.



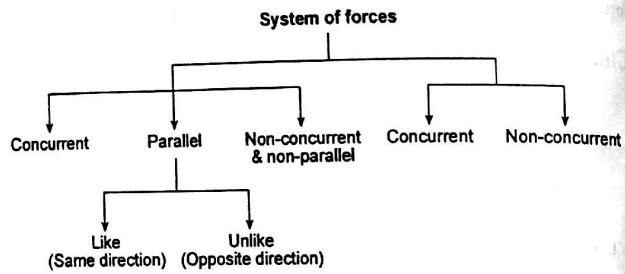
### 3.1.5 Rotational force

The force which rotates or tends to rotate a body around a central axis is called rotational force. e.g. turning of bolt by wrench.



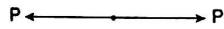
## 3.2 System of forces

When several forces of different magnitude and direction act upon a body, they constitute a force system considering the plane in which forces are applied and depending upon the position of line of action, force may be classified as follows:



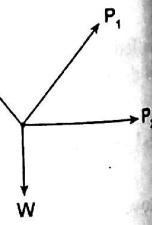
### • Collinear force

The line of action of all forces lie along the same straight line. e.g. force on rope.



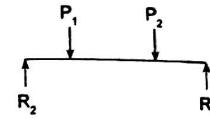
### • Coplanar concurrent force

All forces lie in the same plane having different directions but their lines of action act at a point, called point of concurrency. e.g. forces on rod resting against wall.



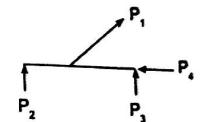
### • Coplanar parallel force

The line of action of all forces are parallel to each other and lie in single plane. e.g. system of vertical load on beam.



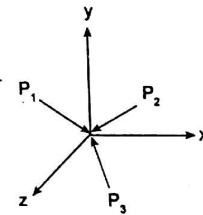
### • Coplanar non concurrent and non parallel forces

All the forces lie in the same plane but their line of action do not pass through a single point and force are not parallel. e.g. forces on a ladder resting against wall and person climbing.



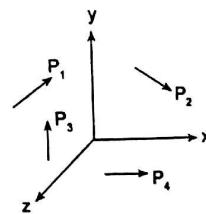
### • Non-coplanar concurrent forces

All forces do not lie in the same plane but their line of action passes through single point. e.g. forces on a tripod carrying a camera.



### • Non-coplanar non concurrent forces

All forces do not lie in the same plane, and their lines of action do not meet at a single point. e.g. forces acting on a moving bus.



### 3.3 Resolution and composition of force

#### 3.3.1 Resolution of forces

The process to find the components of given force in two given directions is called resolution. These component forces will produce same effect on the body as the given single force.

Let us find the component of force  $R$  in the direction making angles  $\alpha$  and  $\beta$  with its line of action. Let the resolved force be  $P$  and  $Q$  as shown in figure.

From trigonometry

$$\angle OCB = \angle AOC = \alpha$$

$$\angle OBC = 180 - (\alpha + \beta)$$

Applying sin rule to  $\triangle OBC$

$$\frac{OB}{\sin \alpha} = \frac{BC}{\sin \beta} = \frac{OC}{\sin (180 - \alpha + \beta)}$$

$$\frac{Q}{\sin \alpha} = \frac{P}{\sin \beta} = \frac{R}{\sin (\alpha + \beta)}$$

$$\therefore P = R \frac{\sin \beta}{\sin (\alpha + \beta)}$$

$$Q = R \frac{\sin \alpha}{\sin (\alpha + \beta)}$$

When the force  $R$  is to be resolved along perpendicular direction, then

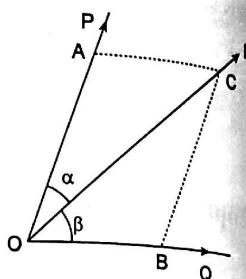
$$\alpha + \beta = 90^\circ$$

$$\therefore \alpha = 90^\circ - \beta$$

$$\therefore P = R \frac{\sin \beta}{\sin (\alpha + \beta)}$$

$$= R \frac{\sin \beta}{\sin 90}$$

$$\therefore P = r \sin \beta$$



$$\text{Also, } Q = R \frac{\sin \alpha}{\sin (\alpha + \beta)}$$

$$= R \frac{\sin \alpha}{\sin 90}$$

$$= R \sin \alpha$$

$$= R \sin (90 - \beta)$$

$$= R \cos \beta$$

$\therefore$  for perpendicular axes

$$Q = R \cos \beta \quad \text{and} \quad P = R \sin \beta$$

$$\text{or} \quad Q = R \sin \alpha \quad \text{and} \quad P = R \cos \alpha$$

(Note the difference for different reference angle.)

#### 3.3.2 Composition of force

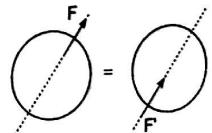
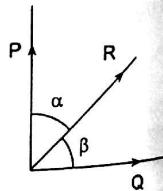
The process of finding out the resultant force, out of number of given forces, is called composition of forces or sometimes compounding of forces. The resultant of forces can be calculated using following methods

1. Parallelogram law
2. Triangle law
3. Method of resolution

(The numerical method and solve example has been discussed in chapter two.)

### 3.4 Principle of transmissibility and equivalent force

The principle of transmissibility states that the conditions of equilibrium or motion of rigid body will remain unchanged if a force  $F$  acting at a given point of the rigid body is replaced by a force  $F'$  of same magnitude and same direction, but acting at different point, provided that the two forces have the same line of action as in figure. The two forces  $F$  and  $F'$  have the same effect on the rigid body and are said to be equivalent.



### 3.5 Moments and couples

#### 3.5.1 Moments of force

A coplanar non concurrent force system consists of a set of forces that lie in the same plane but the line of action of all the forces do not meet at a single point.

Moment of force about a point is defined as the turning tendency of a force about that point. It is measured by the product of force and the perpendicular distance of the line of action of force from that point.

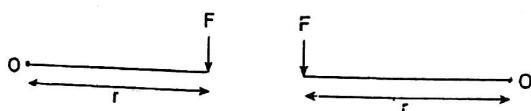
Moment of force  $F$  about  $O$

$$\vec{M}_O = \vec{r}_{F/O} \times \vec{F}$$

Here  $\vec{r}_{F/G}$  is the distance of point of application of force with respect to point  $O$ , about which moment is to be calculated.

Scalarly,  $M_O = r \times F$

The point  $O$  is called moment centre and the distance  $r$  is called moment arm. The tendency of turning of the body due to the moment of force may be clockwise or anticlockwise. The corresponding moments are referred as clockwise moment and anticlockwise moment.



For a 3D case:

$$M_O = \vec{r} \times \vec{F}$$

$$M_O = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x & y & z \\ F_x & F_y & F_z \end{vmatrix}$$

Notes:

- Moment of force about a point

$$\vec{M}_O = \vec{r} \times \vec{F}$$

(Be careful to select  $\vec{r}$ )

Moment of force about an axis  $\vec{i}, \vec{j}$  and  $\vec{k}$  is

$$\vec{M} \cdot \vec{i}, \vec{M} \cdot \vec{j}, \vec{M} \cdot \vec{k}$$

In general the moment of  $F$  about any axis  $AB$  will be  $\vec{M} \cdot \hat{n}$  where

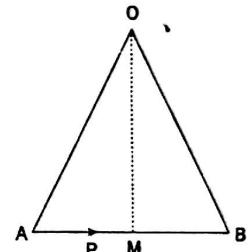
$M$  is the moment of force  $F$  about any point on the axis  $AB$  and  $\hat{n}$  is the unit vector along  $AB$ .

#### 3.5.1.1 Graphical representation of moment

Let a force  $P$  be represented by vector  $AB$  and  $O$  be the point about which the moment is to be determined.

Moment of force  $P$  about  $O$

$$\begin{aligned} M_P &= P \times OM \\ &= AB \times OM \\ &= 2 \times \frac{1}{2} \times AB \times OM \\ &= 2 \times \left( \frac{1}{2} \times AB \times OM \right) \end{aligned}$$



$$M_P = 2 \times \text{Area of triangle } AOB$$

Hence the moment of force about any point is geometrically equal to twice the area of the triangle whose base is the line that represent the force and whose vertex is the point about which the moment is required to calculate.

#### 3.5.1.2 Principle of moments

A body acted upon by a number of coplanar forces will be in equilibrium, if the algebraic sum of the moments of all forces about a point in the same plane is zero.

$$\text{i.e. } \Sigma \text{moment} = 0$$

$\Sigma M = 0$   
i.e. clockwise moment = anticlockwise moment  
Note: For a body to be in equilibrium

$$\Sigma F_x = 0$$

$$\Sigma F_y = 0$$

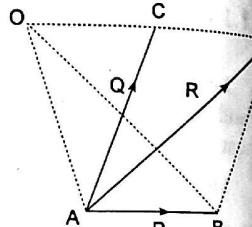
$$\Sigma F_z = 0$$

### 3.5.1.3 Varignon's Theorem

It states that "Moment of a resultant of two forces, about a point lying in the plane of the forces, is equal to the algebraic sum of moments of these forces about the same point."

Consider two concurrent forces P and Q as shown. Let O be the point about which moment is to be calculated. Through O draw a line parallel to the direction of force P and let this line meet the line of action of force Q at C. Suppose R be the resultant force.

$$\text{Moment of force } P \text{ about } O = 2 \times \Delta AOB$$



$$\text{Moment of force } Q \text{ about } O = 2 \times \Delta AOC$$

$$\text{Moment of force } R \text{ about } O = 2 \times \Delta AOD$$

From geometry:

$$\Delta AOD = \Delta AOC + \Delta ACD$$

$$\Delta AOD = \Delta AOC + \Delta ABD \quad (\text{why?})$$

$$\Delta AOD = \Delta AOC + \Delta AOB \quad (\text{why?})$$

$$\therefore \Delta AOD = \Delta AOC + \Delta AOB$$

$$\frac{M_R}{2} = \frac{M_Q}{2} = \frac{M_P}{2}$$

$$\therefore M_R = M_Q + M_P$$

This principle can be extended for number of forces "moment of resultant of number of forces about a point lying in the plane of force is equal to the algebraic sum of the moment of these forces about the same point."

$$\begin{aligned}\vec{r} \times (\vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \dots) &= \vec{r} \times \vec{F}_1 + \vec{r} \times \vec{F}_2 + \vec{r} \times \vec{F}_3 + \dots \\ \vec{r} \times \vec{R} &= \vec{r} \times \vec{F}_1 + \vec{r} \times \vec{F}_2 + \vec{r} \times \vec{F}_3 + \dots\end{aligned}$$

### 3.5.2 Moment of a couple

Two parallel forces equal in magnitude but opposite in direction separated by a finite distance are said to form a couple.

$$\text{Moment of couple} = P \times d$$

Sign convention

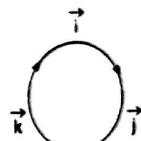
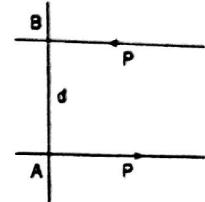
Clockwise moment is negative

Anticlockwise moment is positive.

If we go for vector method

$$\begin{aligned}\text{Moment of couple} &= \vec{r}_{A/B} \times \vec{P} i \\ &= -d j \times \vec{P} i \\ &= -dP(-k) \\ &= dP k\end{aligned}$$

(why  $\vec{r}_{A/B}$  but not  $\vec{r}_{B/A}$ ? Discuss in class)



#### 3.5.2.1 Characteristic of couple

- a) algebraic sum of the forces constituting the couple is zero.
- b) couple cannot be balanced by single force.
- c) couple is a free vector.

#### 3.5.2.2 Couple as a free vector

The only effect of a couple is to produce a moment. It is possible to represent a couple with a vector, the couple vector, which is equal to the moment of the couple. The couple vector is a free vector and will be represented by a special symbol  $\bigcirclearrowright$  to distinguish it from force vector.

Proof:

To prove couple is a free vector, we need to show moment of couple about any point will be same.

Taking moment about O:

$$\begin{aligned} M_O &= \vec{r} \times \vec{F} \\ &= \vec{r}_{C/O} \times -\vec{F} i \\ &= [(0, b) - (0, 0)] \times -\vec{F} i \\ &= b \vec{j} \times -\vec{F} i \\ &= -bF(-\vec{k}) \end{aligned}$$

$$M_O = bF \vec{k}$$

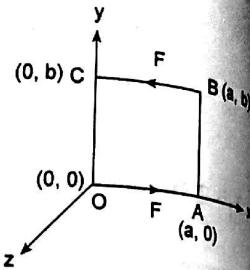
Taking moment about A

$$\begin{aligned} M_A &= \vec{r} \times \vec{F} \\ &= \vec{r}_{C/A} \times -\vec{F} i \\ &= [(0, b) - (a, 0)] \times -\vec{F} i \\ &= (-a \vec{i} + b \vec{j}) \times -\vec{F} i \\ &= aF \vec{i} \times \vec{i} - bF(-\vec{k}) \\ &= 0 + bF \vec{k} \end{aligned}$$

$$M_A = bF \vec{k}$$

Taking moment about B

$$\begin{aligned} M_B &= \vec{r} \times \vec{F} \\ &= \vec{r}_{A/B} \times \vec{F} i \\ &= [(a, 0) - (a, b)] \times \vec{F} i \end{aligned}$$



$$\begin{aligned} &= -b \vec{j} \times \vec{F} \vec{i} \\ &= -bF(-\vec{k}) \\ M_B &= bF \vec{k} \end{aligned}$$

Since  $M_O = M_A = M_B$ , it is proved that couple is a free vector.

### 3.5.3 Resultant of coplanar, non concurrent force system

Different steps which are to be followed to determine the resultant of coplanar, non-concurrent force system in magnitude, direction and position are listed below:

Steps:

1. Resolve the given forces horizontally ( $\Sigma F_x$ ) and vertically ( $\Sigma F_y$ ).

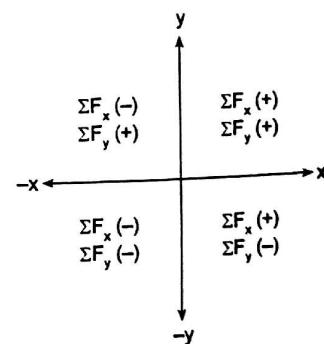
2. Determine magnitude of resultant

$$R = \sqrt{(\Sigma F_x)^2 + (\Sigma F_y)^2}$$

3. Find the direction of resultant force with respect to x-axis using

$$\tan \alpha = \frac{\Sigma F_y}{\Sigma F_x} \Rightarrow \alpha = \tan^{-1} \frac{\Sigma F_y}{\Sigma F_x}$$

Depending upon the sign of  $\Sigma F_x$  and  $\Sigma F_y$  decide in which quadrant the resultant lies.



4. Obtain the algebraic sum of moments of all forces about any point say O.

5. Mark the position of resultant such that it produces the same direction of moment about point O.  
 6. Apply Varignon's theorem of moment to find the exact position of resultant.

$$\Sigma M_O = R \times d$$

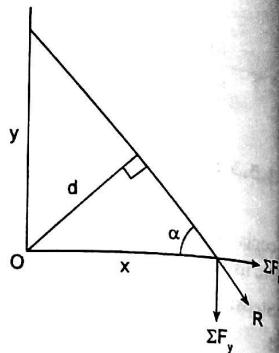
$$\text{also, } \Sigma M_O = \Sigma F_y \times x$$

$$\text{also, } \sin \alpha = \frac{d}{x}$$

$$\therefore d = x \sin \alpha$$

$$\text{also, } \tan \alpha = \frac{y}{x}$$

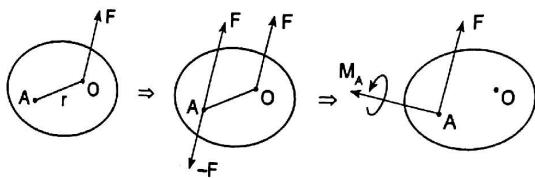
$$y = x \tan \alpha$$



### 3.6 Resolution of force into forces and a couple

#### 3.6.1 Reducing a force system to a force and a couple at a given point A

The force is the resultant  $R$  of the system and is obtained by adding the various forces; the moment of the couple is the moment resultant of the system and is obtained by adding the moments about A of the various forces.



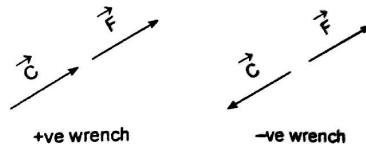
$$R = \sum F$$

$$M_A^R = \sum r \times F$$

where the position vector  $r$  is drawn from A to any point on the line of action of  $F$ .

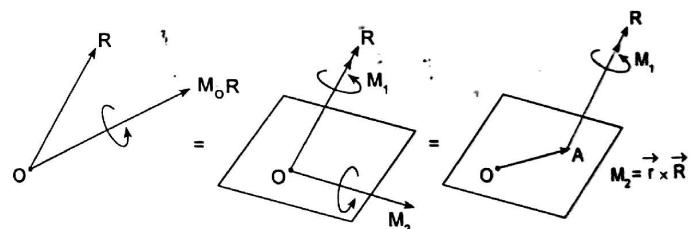
#### 3.6.2 Reduction of a system of forces to a wrench

A wrench consists of a force and a couple with the same line of action. If the directions are same, it is called positive wrench and if the directions are opposite, it is called negative wrench.



A zero wrench is a condition in which both forces and couple are zero.

If a given system is comprised of forces which are not concurrent, coplanar or parallel, the equivalent force-couple system at a point O will consist of a force  $R$  and a couple vector  $M_O^R$  which are not mutually perpendicular [to check use dot product]. In this case the system cannot be reduced to a single force but can be reduced to a wrench – the combination of a force  $R$  and a couple vector  $M_1$  directed along a common line of action called axis of wrench. The ratio  $p = \frac{M_1}{R}$  is called pitch of wrench.



Steps:

1. Reduce the given system to an equivalent force couple system  $[R, M_O^R]$  typically located at origin O.
2. Determine the pitch  $p$

$$p = \frac{M_1}{R} = \frac{\vec{R} \cdot \vec{M_O^R}}{R^2}$$

$$\text{Couple vector } \vec{M}_1 = p \cdot \vec{R}$$

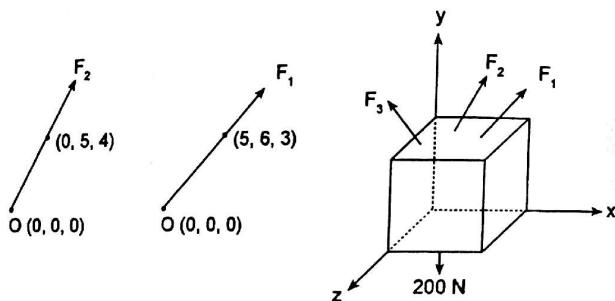
3. Express that the moment about O of the wrench is equal to the moment resultant  $M_O^R$  of the force-couple system at O.

$$\vec{M}_l + \vec{r} \times \vec{R} = \vec{M}_O^R$$

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$$

### Worked Out Examples

1. Three forces  $F_1$ ,  $F_2$  and  $F_3$  supports a 200 N block as shown in figure. If the magnitude of  $F_1$  and  $F_2$  are 60 N and 50 N respectively. Find the magnitude and direction of  $F_3$  for equilibrium.



**Solution:**

$$W = 200 \text{ N}$$

$$\vec{W} = -200\vec{j}$$

$$\text{Unit vector along } F_1 = \hat{n}_1 = \frac{\vec{F}_1}{|\vec{F}_1|}$$

$$= \frac{5\vec{i} + 6\vec{j} + 3\vec{k}}{\sqrt{5^2 + 6^2 + 3^2}} = \frac{5\vec{i} + 6\vec{j} + 3\vec{k}}{8.3667}$$

$$\hat{F}_1 = n_1 = 0.597\vec{i} + 0.717\vec{j} + 0.358\vec{k}$$

$$\text{Unit vector along } F_2 = \hat{n}_2 = \frac{\vec{F}_2}{|\vec{F}_2|}$$

$$= \frac{0\vec{i} + 5\vec{j} + 4\vec{k}}{\sqrt{0^2 + 5^2 + 4^2}} = \frac{5\vec{j} + 4\vec{k}}{6.403}$$

$$\hat{F}_2 = n_2 = 0.78\vec{j} + 0.624\vec{k}$$

$$\vec{F}_1 = 60\hat{F}_1$$

$$= 60(0.597\vec{i} + 0.717\vec{j} + 0.358\vec{k})$$

$$\vec{F}_1 = 35.82\vec{i} + 43.02\vec{j} + 21.48\vec{k}$$

$$= 50\hat{F}_2$$

$$= 50(0.78\vec{j} + 0.624\vec{k})$$

$$= 0\vec{i} + 39\vec{j} + 31.2\vec{k}$$

For equilibrium,

$$\sum \vec{F} = 0$$

$$\vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \vec{W} = 0$$

$$35.82\vec{i} + 43.02\vec{j} + 21.48\vec{k} + 39\vec{j} + 31.2\vec{k} + \vec{F}_3 - 200\vec{j} = 0$$

$$\text{or, } 35.82\vec{i} - 117.98\vec{j} + 52.68\vec{k} + \vec{F}_3 = 0$$

$$\therefore \vec{F}_3 = -35.82\vec{i} + 117.98\vec{j} - 52.68\vec{k}$$

$$|\vec{F}_3| = \sqrt{(-35.82)^2 + (117.98)^2 + (-52.68)^2}$$

$$= 134 \text{ N}$$

2. A homogeneous plate of mass 30 kg is supported by three vertical wires as shown in figure. The plate is 60 cm in diameter. Determine the tension in wire.

**Solution:**

The plate rests on XZ plane.

$$A(-0.3, 0, 0)$$

$$B(r \sin 30, 0, r \cos 30)$$

$$\text{or, } B(0.3 \times 0.5, 0, 0.3 \times 0.866)$$

$$\text{or, } B(0.15, 0, 0.26)$$

$$C(0, 0, -0.3)$$

$$\therefore A(-0.3, 0, 0), B(0.15, 0, 0.26), C(0, 0, -0.3)$$

For equilibrium

$$\Sigma F = 0 \text{ and } \Sigma M = 0$$

$$\Sigma \vec{F} = 0$$

$$\vec{T}_1 j + \vec{T}_2 j + \vec{T}_3 j - w j = 0$$

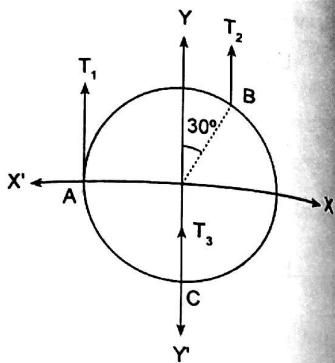
$$T_1 + T_2 + T_3 = 30 \times 9.81$$

$$T_1 + T_2 + T_3 = 294.3 \dots\dots(i)$$

$$\Sigma \vec{M}_O = 0$$

$$\vec{r}_{AO} \times \vec{T}_1 + \vec{r}_{BO} \times \vec{T}_2 + \vec{r}_{CO} \times \vec{T}_3 = 0$$

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -0.3 & 0 & 0 \\ 0 & T_1 & 0 \end{vmatrix} + \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0.15 & 0 & 0.26 \\ 0 & T_2 & 0 \end{vmatrix} + \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 0 & -0.3 \\ 0 & T_3 & 0 \end{vmatrix} = 0$$



$$-0.3T_1 \vec{k} - 0.26T_2 \vec{i} + 0.3T_3 \vec{i} + 0.15T_2 \vec{k} = 0$$

$$(0.3T_3 - 0.26T_2) \vec{i} + (0.15T_2 - 0.3T_1) \vec{k} = 0$$

Equating coefficient of like vectors

$$0.3T_3 - 0.26T_2 = 0 \Rightarrow T_3 = 0.8667T_2 \dots\dots(ii)$$

$$0.15T_2 - 0.3T_1 = 0 \Rightarrow T_1 = 0.5T_2 \dots\dots(iii)$$

From eq. (i), (ii) and (iii)

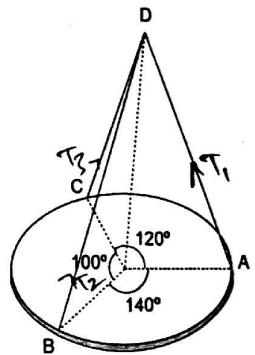
$$0.5T_2 + T_2 + 0.8667T_2 = 294.3$$

$$\therefore T_2 = \frac{294.3}{2.3667} = 124.35 \text{ N}$$

$$T_1 = 62.175 \text{ N}$$

$$T_3 = 107.77 \text{ N}$$

3. A homogeneous circular plate of mass 50 kg is supported by three wires. The angular distance between the points of attachment on the circumference of the plate with respect to centre of the plate makes an angle of  $100^\circ$  degree while other two angular distances are  $120^\circ$  and  $140^\circ$  as shown in figure. The three wires are attached to a single point on the ceiling which is 5 m vertically above the centroid of the plate. The plate has diameter of 1 m. Calculate the force developed in each wire.



**Solution:**

$$A(0.5, 0, 0)$$

$$B(-0.5 \sin 50, 0, -0.5 \cos 50) = (-0.383, 0, -0.321)$$

$$C(-0.5 \cos 60, 0, 0.5 \sin 60) = (-0.25, 0, 0.433)$$

$$D(0, 5, 0)$$

$$\therefore A(0.5, 0, 0), B(-0.383, 0, -0.321),$$

$$C(-0.25, 0, 0.433), D(0, 5, 0)$$

Unit vector along AD

$$\hat{AD} = \hat{n}_1 = \frac{\vec{AD}}{|\vec{AD}|} = \frac{\vec{D} - \vec{A}}{|\vec{AD}|}$$

$$= \frac{(0, 5, 0) - (0.5, 0, 0)}{\sqrt{(-0.5)^2 + 5^2}}$$

$$= \frac{-0.5 \vec{i} + 5 \vec{j}}{5.024}$$

$$= -0.1 \vec{i} + 0.99 \vec{j}$$

$$\vec{T}_{AD} = T_{AD}(-0.1 \vec{i} + 0.99 \vec{j})$$

Unit vector along BD

$$\hat{BD} = \hat{n}_2 = \frac{\vec{BD}}{|\vec{BD}|}$$

$$= \frac{(0, 5, 0) - (-0.38, 0, -0.32)}{|BD|}$$

$$= \frac{0.38 \vec{i} + 5 \vec{j} + 0.32 \vec{k}}{\sqrt{0.38^2 + 5^2 + 0.32^2}} = \frac{0.38 \vec{i} + 5 \vec{j} + 0.32 \vec{k}}{5.024}$$

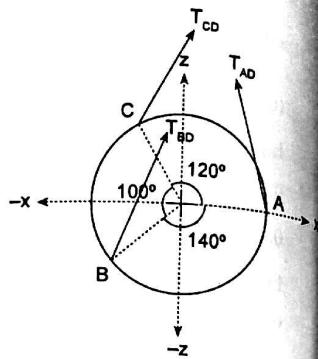
$$= 0.076 \vec{i} + 0.99 \vec{j} + 0.064 \vec{k}$$

$$\therefore \vec{T}_{BD} = T_{BD}(0.076 \vec{i} + 0.99 \vec{j} + 0.064 \vec{k})$$

Unit vector along CD

$$\hat{CD} = \hat{n}_3 = \frac{\vec{CD}}{|\vec{CD}|}$$

$$= \frac{(0, 5, 0) - (-0.25, 0, 0.433)}{|\vec{CD}|}$$



$$= \frac{0.25 \vec{i} + 5 \vec{j} - 0.433 \vec{k}}{\sqrt{0.25^2 + 5^2 + 0.433^2}} = \frac{0.25 \vec{i} + 5 \vec{j} - 0.433 \vec{k}}{5.024}$$

$$= 0.05 \vec{i} + 0.99 \vec{j} - 0.086 \vec{k}$$

$$\therefore \vec{T}_{CD} = T_{CD}(0.05 \vec{i} + 0.99 \vec{j} - 0.086 \vec{k})$$

Now for equilibrium

$$\sum \vec{F} = 0$$

$$\vec{T}_{AD} + \vec{T}_{BD} + \vec{T}_{CD} + \vec{W} = 0$$

$$T_{AD}(-0.1 \vec{i} + \vec{j}) + T_{BD}(0.076 \vec{i} + \vec{j} + 0.064 \vec{k})$$

$$+ T_{CD}(0.05 \vec{i} + \vec{j} - 0.086 \vec{k}) - 490.5 \vec{j} = 0$$

Equating the coefficient of like vectors

$$-0.1 T_{AD} + 0.076 T_{BD} + 0.05 T_{CD} = 0 \quad \dots \dots \text{(i)}$$

$$T_{AD} + T_{BD} + T_{CD} = 490.5 \quad \dots \dots \text{(ii)}$$

$$0.064 T_{BD} - 0.086 T_{CD} = 0 \quad \dots \dots \text{(iii)}$$

Solving,

$$T_{AD} = 193 \text{ N}, T_{BD} = 170.5 \text{ N}, T_{CD} = 126.9 \text{ N}$$

4. A force of 800 N acts on a bracket as shown. Determine the moment of the force about B.

**Solution:**

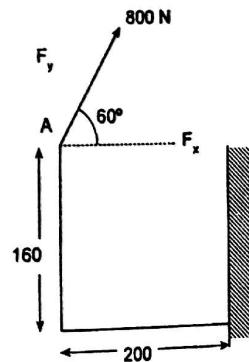
You can use both scalar and vector method.

#### Method 1: Scalar method

Resolve the force 800 N to x and y axes.

$$\therefore F_x = 800 \cos 60 = 400 \text{ N}$$

$$F_y = 800 \sin 60 = 692.82 \text{ N}$$



$$\begin{aligned}\Sigma M_B +ve &= -F_x 160 - F_y 200 \\ &= -400 \times 0.1 - 692.82 \times 0.2 \\ &= -202.56 \text{ Nm}\end{aligned}$$

∴ moment about B is 202.56 Nm clockwise.

**Method 2: Vector method**

A(-0.2, 0.16), B(0, 0)

$$\begin{aligned}\vec{M}_B &= \vec{r} \times \vec{F} \\ &= \vec{r}_{A/B} \times (800 \cos 60 \vec{i} + 800 \sin 60 \vec{j}) \\ &= (-0.2 \vec{i} + 0.16 \vec{j}) \times 400 \vec{i} + 692.82 \vec{j} \\ &= (-0.2 \times 692.82) \vec{k} - (0.16 \times 400) \vec{k} \\ &= -202.56 \text{ Nm } \vec{k}\end{aligned}$$

[Here -ve sign indicates clockwise moment.]

55

Determine the tension in the cable AB which holds a post BC of 4 m length from sliding. The post has a mass of 10 kg. Assume all surfaces are smooth.

**Solution:**

Draw a FBD, calculate length BD and use the equilibrium equation.

$$\text{Length } BD = \frac{3}{\sin 60} = 2\sqrt{3}$$

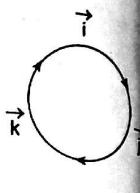
Here weight ( $w = mg$ ) lies downward from centroid and reaction  $R_{BV}$  and  $R_D$  perpendicular to contact point, tension  $T_{BA}$  towards support.

Length BG = 2 m (why?); where G is the point c.g.

$$\Sigma M_B = 0 +ve$$

(It is better to use moment equation at a point where maximum force acts.)

$$mg \times 2 \cos 60 - R_D 2\sqrt{3} = 0$$



$$98.1 - R_D 2\sqrt{3} = 0$$

$$\therefore R_D = 28.31 \text{ N}$$

$$\Sigma F_y = 0 +ve$$

$$R_{BV} - mg + R_D \sin 30 = 0$$

(not useful for this question.)

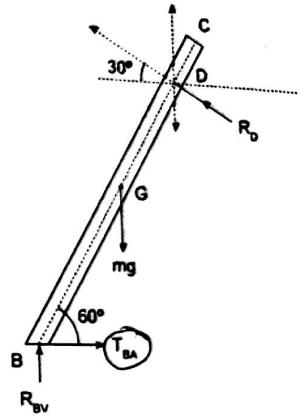
$$\Sigma F_x = 0 +ve$$

$$T_{BA} - R_D \cos 30 = 0$$

$$T_{BA} = R_D \cos 30$$

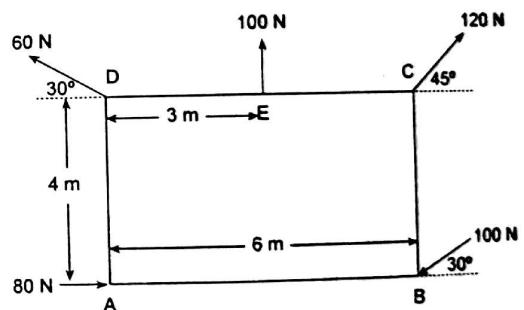
$$= 28.31 \cdot \frac{\sqrt{3}}{2}$$

$$= 24.51 \text{ N}$$



6.

A plate measuring 6 m × 5 m is acted upon by a set of forces in its plane as shown in figure. Determine the magnitude direction and position of resultant force.

**Solution:**

$$\Sigma F_x = 120 \cos 45 - 60 \cos 30 + 80 - 100 \cos 30$$

$$= 84.85 - 51.96 + 80 - 86.6$$

$$= 26.29 \text{ N}$$

$$\Sigma F_y = 100 + 60 \sin 30 + 120 \sin 45 - 100 \sin 30$$

$$= 100 + 30 + 84.85 - 50$$

$$= 164.85 \text{ N}$$

$$\begin{aligned}\vec{R} &= \vec{\sum F_x} + \vec{\sum F_y} \\ &= \sqrt{(\sum F_x)^2 + (\sum F_y)^2} \\ &= \sqrt{(26.29)^2 + (164.85)^2} \\ &= 166.93 \text{ N}\end{aligned}$$

$$\begin{aligned}\tan \alpha &= \frac{\sum F_y}{\sum F_x} \\ &= \frac{164.85}{26.29} = 6.27\end{aligned}$$

$$\therefore \alpha = 80.94^\circ$$

Since both  $\sum F_x$  and  $\sum F_y$  are positive, resultant lies in first quadrant.

To find the position, we have to calculate moment at any point (say A).

$$\therefore \sum M_A +ve = -100 \sin 30 \times 6 - 120 \cos 45 \times 4 + 120 \sin 45 \times 6 + 100 \times 3 + 60 \cos 30 \times 4$$

$$\therefore \sum M_A = 377.5 \text{ Nm}$$

The resultant R should lie so that it can produce anticlockwise moment of 377.5 Nm about point A.

$$\therefore R \times d = 377.5$$

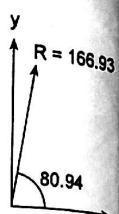
$$\therefore d = \frac{377.5}{166.93} = 2.261 \text{ m}$$

Again for small  $\Delta APQ$

$$\sin 80.94 = \frac{d}{x}$$

$$\therefore x = \frac{d}{\sin 80.94}$$

$$= \frac{2.261}{\sin 80.94} = 2.29 \text{ m}$$

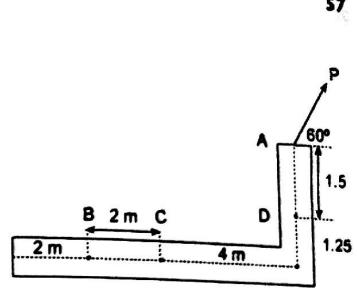


### Force acting on particles and rigid body

- 7 A 160 N force P is applied at point A of the structural member. Replace P with

a) Equivalent force couple at C

b) Equivalent system consisting of a vertical force at B and second force at D.



**Solution:**

Equivalent force at C = P

$$= 160 \text{ N}$$

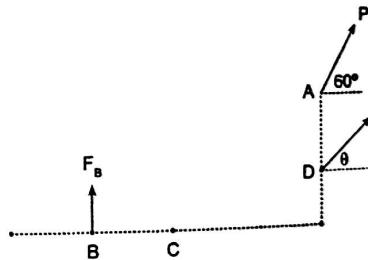
Equivalent couple is moment of this force P about C

$$M_C = -P \cos 60 \times (1.5 + 1.25) + P \sin 60 \times 4$$

$$= -160 \cos 60 \times 2.75 + 160 \sin 60 \times 4$$

$$= 334.25 \text{ Nm}$$

- b) Let  $F_B$  be vertical force at B and  $F_D$  be force at D at angle  $\theta$  with x-axis.



Using Varignon's theorem

Moment of P about D = moment of  $F_B$  about D + moment of  $F_D$  about D

$$\therefore P \cos 60 \times 1.5 = F_B \times 6 + 0$$

$$\frac{160 \cos 60 \times 1.5}{6} = F_B$$

$$\therefore F_B = 20 \text{ N} (\uparrow)$$

Also, equating vertical component  
or,  $P \sin 60 = F_B + F_D \sin \theta$   
or,  $160 \sin 60 = 20 + F_D \sin \theta$   
 $\therefore F_D \sin \theta = 118.56 \quad \dots \text{(i)}$

Also, equating horizontal component  
or,  $P \cos 60 = F_D \cos \theta$   
or,  $160 \times \cos 60 = F_D \cos \theta$   
 $\therefore F_D \cos \theta = 80 \quad \dots \text{(ii)}$

From (i) and (ii)  
 $\tan \theta = 1.482$   
 $\therefore \theta = 56^\circ$   
 $\therefore F_D = \frac{80}{\cos 56} = 143.03 \text{ N}$

- (8) Replace the couple and force shown in figure by an equivalent single force applied to lever.

**Solution:**

Here 200 N forces forms a couple.

So, couple  $C_O = 200 \times 0.12$   
 $= 24 \text{ Nm}$

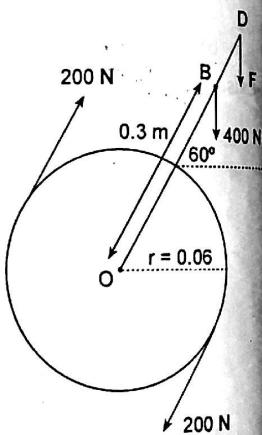
$\therefore C_O = -24 \text{ Nm}$

Note this  $C_O$  is a free vector.

Moment of force 400 N at O

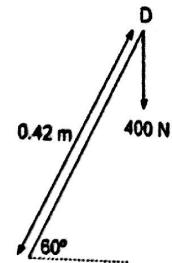
$$\begin{aligned} M_O &= 400 \times OB \cos 60 \\ &= 400 \times 0.15 \\ &= -60 \text{ Nm} \end{aligned}$$

$$\begin{aligned} \therefore \text{Total moment} &= (-60 - 24) \text{ Nm} \\ &= -84 \text{ Nm about O} \end{aligned}$$



### Force acting on particles and rigid body

Now the equivalent single force also produce the same moment about O. Let it be vertical through D at distance d from O. Then,

$$\begin{aligned} F \times d \cos 60 &= 84 \\ 400 \times d \cos 60 &= 84 \\ \therefore d &= 0.42 \text{ m} \\ \text{So, the equivalent force is } F &= 400 \text{ N at } d = 0.42 \end{aligned}$$


9. Three pairs of couples are acted on the triangular block as shown in figure below. Determine the single resultant couple.

**Solution:**

We know couple is a free vector. So, to calculate the resultant couple, we will calculate the moment at any point. Here, the maximum force passes through E. So, we will calculate moment at E. Let

$$C_1 = \text{moment of couple at E due to } 70 \text{ N force}$$

$$C_2 = \text{moment of couple at E due to } 80 \text{ N force}$$

$$C_3 = \text{moment of couple at E due to } 100 \text{ N force}$$

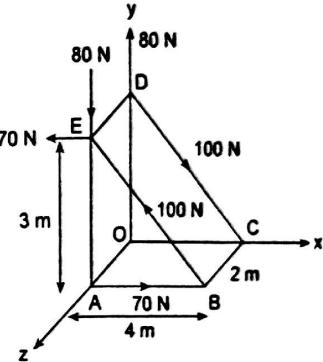
Secondly, we will write the coordinates

$$A(0, 0, 2), \quad B(4, 0, 2), \quad C(4, 0, 0)$$

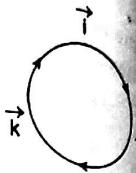
$$D(0, 3, 0), \quad E(0, 3, 2)$$

Here the direction of 100 N is unknown. So, find the unit vector along DC.

$$\begin{aligned} \text{Unit vector of } DC &= \hat{DC} = \frac{\vec{DC}}{|\vec{DC}|} = \frac{\vec{C} - \vec{D}}{|\vec{DC}|} \\ &= \frac{(4, 0, 0) - (0, 3, 0)}{|\vec{DC}|} \end{aligned}$$



$$\begin{aligned} &= \frac{4 \vec{i} - 3 \vec{j}}{\sqrt{4^2 + 3^2}} \\ &= \frac{4}{5} \vec{i} - \frac{3}{5} \vec{j} = 0.8 \vec{i} - 0.6 \vec{j} \end{aligned}$$



$$\begin{aligned} \therefore \vec{C}_1 &= \vec{r} \times \vec{F} \\ &= \vec{r}_{A/E} \times 70 \vec{i} \\ &= [(0, 0, 2) - (0, 3, 2)] \times 70 \vec{i} \\ &= -3 \vec{j} \times 70 \vec{i} = 210 \vec{k} \\ \vec{C}_2 &= \vec{r}_{D/E} \times \vec{F} \\ &= (0, 3, 0) - (0, 3, 2) \times 80 \vec{j} \\ &= -2 \vec{k} \times 80 \vec{j} \\ &= 160 \vec{i} \\ \vec{C}_3 &= \vec{r}_{D/E} \times \vec{F} \\ &= -2 \vec{k} \times 100(0.8 \vec{i} - 0.6 \vec{j}) \\ &= -2 \vec{k} \times (80 \vec{i} - 60 \vec{j}) \\ &= -160 \vec{j} - 120 \vec{i} \end{aligned}$$

∴ Single resultant couple

$$\begin{aligned} \vec{C} &= \vec{C}_1 + \vec{C}_2 + \vec{C}_3 \\ &= 210 \vec{k} + 160 \vec{i} - 160 \vec{j} - 120 \vec{i} \\ &= 40 \vec{i} - 160 \vec{j} + 210 \vec{k} \text{ Nm} \end{aligned}$$

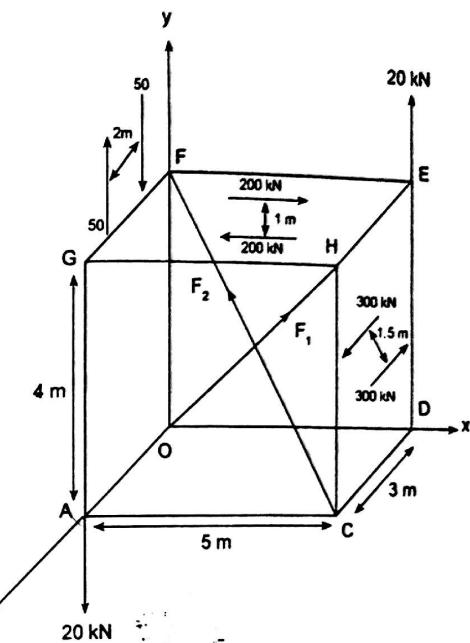
magnitude = 267.02 Nm

- 10 Find the resultant of force couple system at point A as shown in figure below.  $F_1 = 100 \text{ kN}$ ,  $F_2 = 300 \text{ kN}$ .

**Solution:**

Coordinates

O	(0, 0, 0)
A	(0, 0, 3)
C	(5, 0, 3)
D	(5, 0, 0)
E	(5, 4, 0)
F	(0, 40, 0)
G	(0, 4, 3)
H	(5, 4, 3)



Let

$C_1$  = moment of couple at A due to 50 N forces

$C_2$  = moment of couple at A due to 200 kN force

$C_3$  = moment of couple at A due to 300 kN force

$$\text{Unit vector along } F_1 = \hat{F}_1 = \frac{\vec{OH}}{|\vec{OH}|}$$

$$\begin{aligned} &= \frac{5 \vec{i} + 4 \vec{j} + 3 \vec{k}}{\sqrt{5^2 + 4^2 + 3^2}} \\ &= 0.707 \vec{i} + 0.565 \vec{j} + 0.424 \vec{k} \end{aligned}$$

$$\begin{aligned}\therefore \vec{F}_1 &= 100 \hat{F}_1 \\ &= 70.7 \vec{i} + 56.5 \vec{j} + 42.4 \vec{k} \\ \text{Unit vector along } F_2 &= \hat{F}_2 = \frac{\vec{CF}}{|\vec{CF}|} = \frac{\vec{F} - \vec{C}}{|\vec{CF}|} \\ &= \frac{(0, 4, 0) - (5, 0, 3)}{|\vec{CF}|} \\ &= \frac{-5 \vec{i} + 4 \vec{j} - 3 \vec{k}}{\sqrt{5^2 + 4^2 + 3^2}} \\ &= -0.707 \vec{i} + 0.565 \vec{j} - 0.424 \vec{k}\end{aligned}$$

$$\therefore \vec{F}_2 = 300 \hat{F}_2$$

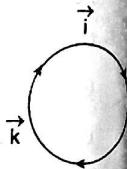
$$\begin{aligned}\therefore \vec{F}_2 &= 300(-0.707 \vec{i} + 0.565 \vec{j} - 0.424 \vec{k}) \\ &= -212.1 \vec{i} + 169.68 \vec{j} - 127.27 \vec{k}\end{aligned}$$

Moment due to  $F_1$  about A is

$$\begin{aligned}\vec{M}_1 &= \vec{r} \times \vec{F} \\ &= \vec{r}_{O/A} \times \vec{F}_1 \\ &= -3 \vec{k} \times (70.7 \vec{i} + 56.5 \vec{j} + 42.4 \vec{k}) \\ &= -212.1 \vec{j} + 169.68 \vec{i} \\ &= 169.68 \vec{i} - 212.1 \vec{j}\end{aligned}$$

Moment due to  $F_2$  about A is

$$\begin{aligned}\vec{M}_2 &= \vec{r} \times \vec{F} \\ &= \vec{r}_{C/A} \times \vec{F}_2\end{aligned}$$



$$\begin{aligned}&= 5 \vec{i} \times (-212.1 \vec{i} + 169.68 \vec{j} - 127.27 \vec{k}) \\ &= 636.39 \vec{j} + 848.4 \vec{k}\end{aligned}$$

Moment due to force 20 kN at E about A

$$\begin{aligned}\vec{M}_3 &= \vec{r} \times \vec{F} \\ &= \vec{r}_{E/A} \times 20 \vec{j} \\ &= [(5, 4, 0) - (0, 0, 3)] \times 20 \vec{j} \\ &= (5 \vec{i} + 4 \vec{j} - 3 \vec{k}) \times 20 \vec{j} \\ &= 100 \vec{k} + 60 \vec{i} \\ &= 60 \vec{i} + 100 \vec{k}\end{aligned}$$

$$\vec{C}_1 = -(50 \times 2) \vec{i} = -100 \vec{i}$$

$$\vec{C}_2 = -(200 \times 1) \vec{k} = -200 \vec{k}$$

$$\vec{C}_3 = -(300 \times 1.5) \vec{i} = -450 \vec{i}$$

$$\therefore \text{Resultant force} = \vec{R} = \vec{F}_1 + \vec{F}_2$$

$$\begin{aligned}\therefore \vec{R} &= 70.7 \vec{i} + 56.5 \vec{j} + 42.4 \vec{k} - 212.1 \vec{i} + 160.68 \vec{j} - 127.27 \vec{k} \\ &= -141.4 \vec{i} + 226.1 \vec{j} - 84.87 \vec{k}\end{aligned}$$

$$\vec{M}_R = \vec{M}_1 + \vec{M}_2 + \vec{M}_3 + \vec{C}_1 + \vec{C}_2 + \vec{C}_3$$

$$\begin{aligned}&= 169.68 \vec{i} - 212.1 \vec{j} + 636.39 \vec{j} + 848.4 \vec{k} + 60 \vec{i} + 100 \vec{k} \\ &\quad - 100 \vec{i} - 200 \vec{k} - 450 \vec{i}\end{aligned}$$

$$\therefore \vec{M}_R = -320.32 \vec{i} + 424.29 \vec{j} + 748.4 \vec{k} \text{ about A.}$$

11. Replace the two wrenches as shown in figure by a single equivalent wrench. Indicate its line of action in xz plane.

**Solution:**

First find resultant of force and moment at origin.

$$\vec{R} = 50\vec{i} + 40\vec{j}$$

$$M_O^R$$

$$= -100\vec{j} + 80\vec{i} + (6\vec{k} \times 50\vec{i})$$

$$= -100\vec{j} + 80\vec{i} + 300\vec{j}$$

$$= 80\vec{i} + 200\vec{j}$$

$$\text{pitch } P = \frac{\vec{R} \cdot \vec{M}_O^R}{\vec{R}^2}, \therefore$$

$$= \frac{(50\vec{i} + 40\vec{j}) \cdot (80\vec{i} + 200\vec{j})}{(\sqrt{(50^2 + 40^2)})^2}$$

$$= \frac{4000 + 8000}{4100}$$

$$= \frac{12000}{4100}$$

$$\therefore P = 2.926$$

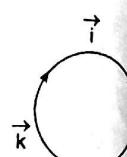
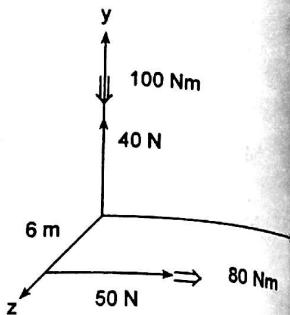
$$\therefore \text{couple vector } \vec{M}_1 = P \cdot \vec{R}$$

$$= 2.926(50\vec{i} + 40\vec{j})$$

$$= 146.34\vec{i} + 117.04\vec{j}$$

To find line of action

$$\vec{M}_1 + \vec{r} \times \vec{R} = \vec{M}_O^R$$



$$146.34\vec{i} + 117.04\vec{j} + (\vec{x}\vec{i} + \vec{y}\vec{j} + \vec{z}\vec{k}) \times (50\vec{i} + 40\vec{j}) \\ = 80\vec{i} + 200\vec{j}$$

for xz plane  $y = 0$

$$\therefore 146.34\vec{i} + 117.04\vec{j} + (\vec{x}\vec{i} + \vec{z}\vec{k}) \times (50\vec{i} + 40\vec{j}) \\ = 80\vec{i} + 200\vec{j}$$

$$40x\vec{k} + 50z\vec{j} - 40z\vec{i} = -66.34\vec{i} + 82.96\vec{j}$$

Equating the coefficient

$$40x = 0$$

$$\therefore x = 0$$

$$z = 1.65$$

$$\therefore \vec{r} = 1.65\vec{k}$$

$$\vec{M}_1 = 146.34\vec{i} + 117.04\vec{j}$$

12. Compute a wrench out of a force  $\vec{F} = 5\vec{i} + 7\vec{j} + 18\vec{k}$  acting at  $(4, 4, 3)$  and a couple  $\vec{C} = 18\vec{i} + 7\vec{j} + 5\vec{k}$

**Solution:**

$$\vec{R} = 5\vec{i} + 7\vec{j} + 18\vec{k}$$

$$\vec{M}_O^R = \vec{r} \times \vec{F} + \vec{C}$$

$$= (4\vec{i} + 4\vec{j} + 3\vec{k}) \times (5\vec{i} + 4\vec{j} + 18\vec{k}) \\ + (18\vec{i} + 7\vec{j} + 5\vec{k})$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 4 & 4 & 3 \\ 5 & 7 & 18 \end{vmatrix} + (18\vec{i} + 7\vec{j} + 5\vec{k})$$

$$\begin{aligned}
 &= (72 - 21) \vec{i} - (72 - 15) \vec{j} + (28 - 20) \vec{k} + 18 \vec{i} + 7 \vec{j} + 5 \vec{k} \\
 &= 51 \vec{i} - 57 \vec{j} + 8 \vec{k} + 18 \vec{i} + 7 \vec{j} + 5 \vec{k} \\
 &= 69 \vec{i} - 50 \vec{j} + 13 \vec{k}
 \end{aligned}$$

$$\therefore \text{pitch } p = \frac{\vec{R} \cdot M_0^R}{R^2}$$

$$\begin{aligned}
 &= \frac{(5 \vec{i} + 7 \vec{j} + 18 \vec{k}) \cdot (69 \vec{i} - 50 \vec{j} + 13 \vec{k})}{(5^2 + 7^2 + 18^2)} \\
 &= \frac{345 + (-350) + 234}{398} \\
 &= 0.575
 \end{aligned}$$

$$\begin{aligned}
 \therefore \vec{M}_1 &= p \cdot \vec{R} = 0.575(5 \vec{i} + 7 \vec{j} + 18 \vec{k}) \\
 &= 2.875 \vec{i} + 4.02 \vec{j} + 10.35 \vec{k}
 \end{aligned}$$

13. Force  $\vec{F} = (3 \vec{i} - 6 \vec{j} + 4 \vec{k}) \text{ N}$  passes through point  $(6, 3, 2) \text{ m}$

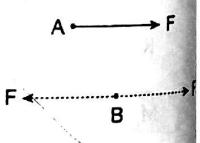
Replace this force with an equivalent system, where the force  $\vec{F}$  passes through  $(2, 5, 10) \text{ m}$ .

**Solution:**

Let A(6, 3, 2) and B(2, 5, 10)

Introduce two forces  $\vec{F}$  at B as shown.

New value of force at B =  $\vec{F}$



$$= 3 \vec{i} - 6 \vec{j} + 4 \vec{k}$$

You can calculate couple at A or B.

$$\text{Couple } C = \vec{r}_{AB} \times \vec{F}$$

$$= (\vec{A} - \vec{B}) \times (3 \vec{i} - 6 \vec{j} + 4 \vec{k})$$

$$= (4 \vec{i} - 2 \vec{j} - 3 \vec{k}) \times (3 \vec{i} - 6 \vec{j} + 4 \vec{k})$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 4 & -2 & -8 \\ 3 & -6 & 4 \end{vmatrix}$$

$$= (-8 - 48) \vec{i} - (16 + 24) \vec{j} + (-24 + 6) \vec{k}$$

$$\text{Couple } C = -56 \vec{i} - 40 \vec{j} - 18 \vec{k}$$

$\therefore$  Equivalent system at B is

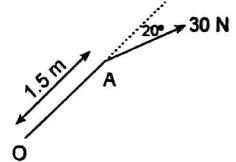
$$\vec{F} = 3 \vec{i} - 6 \vec{j} + 4 \vec{k}$$

$$\vec{C} = -56 \vec{i} - 40 \vec{j} - 18 \vec{k}$$

Check: couple  $C = \vec{r}_{B/A} \times -\vec{F}$  (why -ve for  $\vec{F}$ ?)

### Practice Questions

1. A 30 N force acts on the end of the 1.5 m lever as shown. Determine the moment of force about O

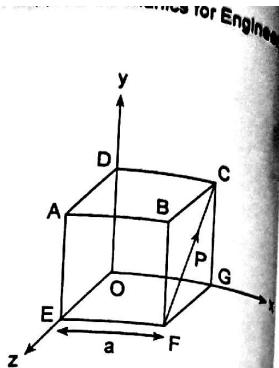


[Ans. 15.4 Nm]

- 2: Forces  $\vec{F} = (3 \vec{i} - 6 \vec{j} + 4 \vec{k}) \text{ N}$  goes through point  $(6, 3, 2) \text{ m}$ . Replace this force with an equivalent system, where the force  $\vec{F}$  goes through point  $(2, -5, 10) \text{ m}$ .

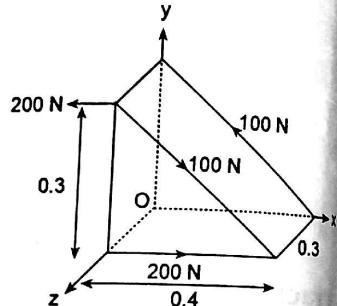
[Ans.  $F = 3 \vec{i} - 6 \vec{j} + 4 \vec{k}$ ;  $C = -16 \vec{i} - 40 \vec{j} - 48 \vec{k}$ ]

3. A cube of side  $a$  is acted upon by a force  $P$  as shown. Determine the moment of  $P$
- about A
  - about edge AB
  - about diagonal AG of cube



$$[\text{Ans. a)} \frac{aP}{\sqrt{2}} (\vec{i} + \vec{j} + \vec{k}); \text{ b)} \frac{aP}{\sqrt{2}}; \text{ c)} -\frac{aP}{\sqrt{6}}]$$

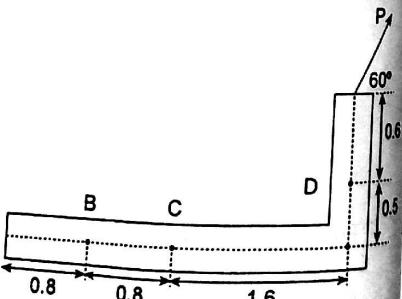
4. Replace the two couples of a triangular block as shown in figure by a single resultant couple.



$$[\text{Ans. } (18 \vec{i} + 24 \vec{j} + 60 \vec{k}) \text{ Nm}]$$

5. A 700 N force  $P$  is applied at point A of structural member. Replace  $P$  with

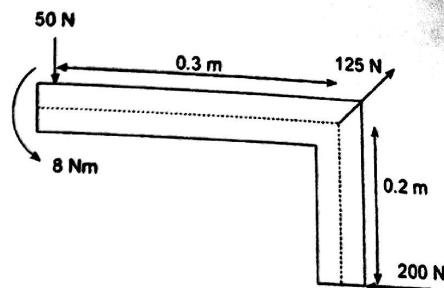
- An equivalent force couple at point C.
- An equivalent system consisting of a vertical force at B and second force at D.



$$[\text{Ans. a)} 700 \text{ N, } 585 \text{ Nm; b)} 87.5 \text{ N, } 626 \text{ N}]$$

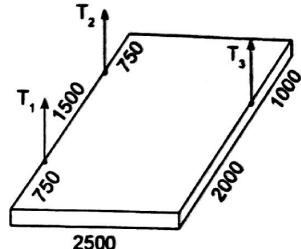
### Force acting on particles and rigid body

6. An angle bracket has been subjected to three forces and a couple as shown in figure. Determine the resultant of this system of forces. Locate the point of position of resultant.



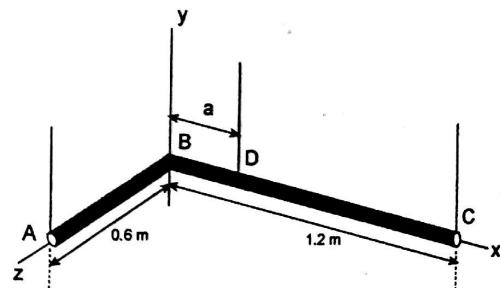
$$[\text{Ans. } R = 149.33, \alpha = 157.44, x = 0.292 \text{ m, } y = 0.124 \text{ m, } d = 0.1138]$$

7. Three vertical wires as shown in figure support a plate of 50 kg. Determine the tension in each wire.



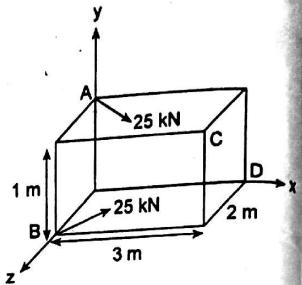
$$[\text{Ans. } T_1 = 20.83 \text{ kg, } T_2 = 4.17 \text{ kg, } T_3 = 25 \text{ kg}]$$

8. Two steel pipe AB and BC each having a weight per unit length of 30 N/m are welded together at B and supported by three wires. Knowing  $a = 0.4$  m, determine the tension in each wire.



$$[\text{Ans. } T_A = 9 \text{ N, } T_D = 40.5 \text{ N, } T_C = 4.5 \text{ N}]$$

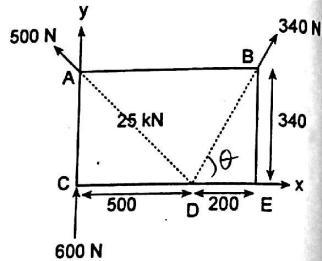
9. If two forces of same magnitude 25 kN acts at point A and B as shown in figure. Force A passes through C and B through D.
- Determine equivalent force couple system at C.
  - Find equivalent wrench, pitch and axis of wrench.



[Ans. a)  $41.5 \vec{i} + 13.75 \vec{j} + 41.5 \vec{k}$ ; b)  $13.75 \vec{i}, 0.33 \text{ m}, x = 0, y = 0.5, z = 1$ ]

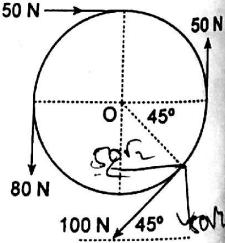
10. Four forces act on a  $700 \times 375$  mm plate as shown

- Find the resultant of the force
- Find the position of resultant.



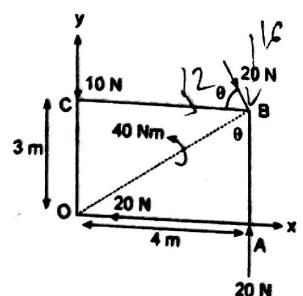
[Ans.  $(519.81 \vec{i} + 1199.2 \vec{j}) \text{ N}$ ; 250 mm from point C of edge CE and 412.76 mm from A of edge AB]

11. Determine the resultant of the four forces acting tangentially to a circle of radius 3 m as shown. What will be the location of the resultant with respect to centre of the circle.



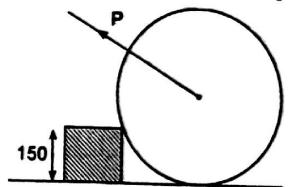
[Ans.  $R = 128.18 \text{ N}, \alpha = 51.8^\circ, d = 2.795 \text{ m}$ ]

12. For the force system acting on a body OABC as shown. Determine the magnitude direction and position of resultant force.



[Ans.  $R = 10 \text{ N}, \alpha = 216.87^\circ, d = 2 \text{ m}, x = 3.33 \text{ m}, y = 2.5 \text{ m}$ ]

13. A uniform wheel weighing 30 kN and of 600 mm diameter rests against 150 mm thick rigid block. Calculate the least pull ( $P_{\min}$ ) through centre to just turn over the corner of block. Also calculate reaction (assume no friction).



[Ans.  $P_{\min} = 25.981, R_{\min} = 15 \text{ kN}$ ]

## Chapter 4

# Centre of Gravity, Centroid and Moment of Inertia

### 4.1 Concept of centre of gravity and centroid

A body comprises of several parts and its every part possess weight. So weight is the force of attraction between a body and the earth. The weight is proportional to mass of the body. The weight of all parts of a body can be considered as parallel forces directed towards the centre of earth. Thus centre of gravity is defined as the point through which the whole weight of the body is assumed to act. It is denoted by c.g. or G.

Centre of mass is the point where the whole mass of body is assumed to act. It differs from centre of gravity only when the gravitational field is not uniform and non-parallel.

The centroid or centre of area is defined as the point where the whole area of the figure is assumed to be concentrated. It is analogous to centre of gravity when a body has area but not weight.

#### 4.1.1 Centroid of Lines, areas and volume

##### a) Centroid of lines

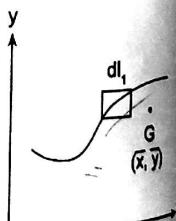
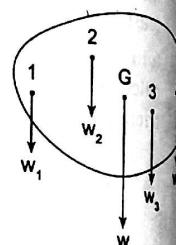
Let us consider a homogeneous slender rod or curved wire of length L in the xy plane. Let the cross-sectional area be A and  $dl$  be the differential length.

Then, differential volume  $dv = Adl$

Total volume  $V = AL$

Using principle of moments

$$\bar{X} \cdot V = \int x dv$$



#### Centre of Gravity, Centroid and Moment of Inertia

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$$\bar{x} = \frac{1}{V} \int x dv$$

$$= \frac{1}{AL} \int x Adl$$

$$= \frac{1}{A} \int x dl$$

$$\bar{Y} v = \int y dv$$

$$\bar{Y} = \frac{1}{V} \int y dv$$

$$= \frac{1}{AL} \int y Adl$$

$$\bar{Y} = \frac{1}{A} \int y dl$$

For composite line

$$\bar{X} = \frac{x_1 l_1 + x_2 l_2 + x_3 l_3 + \dots}{l_1 + l_2 + l_3 + \dots}$$

$$\bar{Y} = \frac{y_1 l_1 + y_2 l_2 + y_3 l_3 + \dots}{l_1 + l_2 + l_3 + \dots}$$

##### b) Centroid of area

Consider a homogeneous thin plate of uniform thickness. Let  $dA$  be the differential area of the plate in the  $xy$  plane so that the differential volume is given by

$$dV = t dA$$

$$V = t \cdot A$$

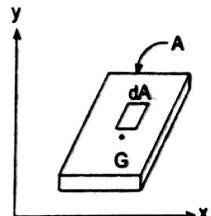
Using the principle of moment

$$\bar{X} V = \int x dv$$

$$= \frac{1}{V} \int x dv$$

$$= \frac{1}{t \cdot A} \int x t dA$$

$$= \frac{1}{A} \int x dA$$



$$\begin{aligned}\bar{Y}_V &= \frac{1}{V} \int y \, dv \\ \bar{Y} &= \frac{1}{V} \int y \, dv \\ &= \frac{1}{t \cdot A} \int y \cdot t \, dA \\ &= \frac{1}{A} \int y \, dA\end{aligned}$$

For composite area

$$\begin{aligned}\bar{X} &= \frac{A_1 X_1 + A_2 X_2 + A_3 X_3 + \dots}{A_1 + A_2 + A_3 + \dots} \\ \bar{Y} &= \frac{A_1 Y_1 + A_2 Y_2 + A_3 Y_3 + \dots}{A_1 + A_2 + A_3 + \dots}\end{aligned}$$

### c) Centroid of volume

$$\begin{aligned}\bar{X} &= \frac{1}{V} \int x \, dv \\ \bar{Y} &= \frac{1}{V} \int y \, dv \\ \bar{Z} &= \frac{1}{V} \int z \, dv\end{aligned}$$

### 4.1.2 First moment of an area

The first moment of element of an area about any axis is the product of area of element and the perpendicular distance between the element and axis.

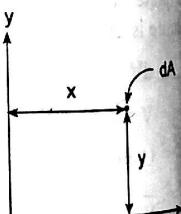
about x-axis:  $dM_x = y \, dA$

about y-axis:  $dM_y = x \, dA$

for a finite area

$$M_x = \bar{Y} \cdot A \text{ or } \int y \, dA$$

$$M_y = \bar{X} \cdot A \text{ or } \int x \, dA$$



### Centre of Gravity, Centroid and Moment of Inertia

### 4.1.3 Determination of centroid of area of rectangle

Consider a rectangle ABCD of breadth b and height h. Consider a strip of thickness dy located at a distance of y as shown.

Elemental area  $dA = b \, dy$

Moment about x-axis  $dM_x = y \, dA = by \, dy$

$dy$

Area of rectangle  $A = b \times h$

Let  $\bar{Y}$  be the distance of centroid from x-axis.

Then from moment principle:

$$A \times \bar{Y} = \int dM_x$$

$$bh \times \bar{Y} = \int_0^h by \, dy$$

$$= b \left[ \frac{y^2}{2} \right]_0^h$$

$$\bar{Y} = \frac{b}{2} \cdot h^2 \frac{1}{bh}$$

$$\therefore \bar{Y} = \frac{h}{2}$$

$$\text{Similarly, } \bar{x} = \frac{b}{2}$$

$$\therefore (\bar{x}, \bar{y}) = \left( \frac{b}{2}, \frac{h}{2} \right)$$

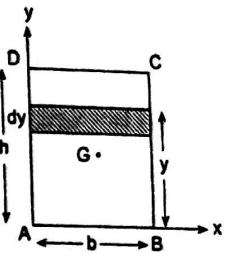
### 4.1.4 Determination of centroid of area of triangle

Consider a triangle ABC with width b and height h. Consider an elemental strip of width l thickness dy and located at distance y from the base BC as shown.

Elemental area  $dA = l \, dy$

Moment about x-axis  $dM_x = y \, dA$

Area of triangle  $A = \frac{1}{2} \times b \times h$



Let  $\bar{y}$  be the distance of centroidal from x-axis; then by moment principle

$$A \times \bar{y} = \int dMx$$

$$\frac{1}{2}bh \times \bar{y} = \int_0^h y dA$$

$$\frac{1}{2} \times bh \times \bar{y} = \int_0^h y \cdot 1 dy$$

$$= \int_0^h b \left(1 - \frac{y}{h}\right) \cdot y dy$$

$$= b \left[ \int_0^h y dy - \int_0^h \frac{y^2}{h} dy \right]$$

$$= b \left[ \frac{h^2}{2} - \frac{h^3}{3h} \right]$$

$$\therefore \bar{y} = \frac{h}{3}$$

Centroid of triangle is  $\frac{1}{3}$  of height from base and  $\frac{2}{3}$  of height from vertex.

B

### Centroid of common shapes of lines

#### a) Quarter-circular arc

$$\bar{x} = \frac{2r}{\pi} \quad \bar{y} = \frac{2r}{\pi}$$

$$\text{Length} = \frac{\pi r}{2}$$

#### b) Semi-circular arc

$$\bar{x} = 0 \quad \bar{y} = \frac{2r}{\pi}$$

$$\text{Length} = \pi r$$

#### c) Arc of circle

$$\bar{x} = \frac{r \sin \alpha}{\alpha} \quad \bar{y} = 0$$

$$\text{Length} = 2ar$$

### Centroid of common shapes of area

#### a) Triangular area

$$\bar{y} = \frac{h}{3} \quad \text{Area} = \frac{bh}{2}$$

#### b) Quarter-circular area

$$\bar{x} = \frac{4r}{3\pi} \quad \bar{y} = \frac{4r}{3\pi}$$

$$\text{Area} = \frac{\pi r^2}{4}$$

### Centre of Gravity, Centroid and Moment of Inertia

#### c) Semicircular area

$$\bar{x} = 0 \quad \bar{y} = \frac{4r}{3\pi}$$

$$\text{Area} = \frac{\pi r^2}{2}$$

#### d) Quarter elliptical area

$$\bar{x} = \frac{4a}{3\pi} \quad \bar{y} = \frac{4b}{3\pi}$$

$$\text{Area} = \frac{\pi ab}{4}$$

#### e) Semi elliptical area

$$\bar{x} = 0 \quad \bar{y} = \frac{4b}{3\pi}$$

$$\text{Area} = \frac{\pi ab}{4}$$

#### f) Semiparabolic area

$$\bar{x} = \frac{3a}{8} \quad \bar{y} = \frac{3h}{5}$$

$$\text{Area} = \frac{2ah}{3}$$

#### g) Parabolic area

$$\bar{x} = 0 \quad \bar{y} = \frac{3h}{5}$$

$$\text{Area} = \frac{4ah}{3}$$

#### h) Parabolic spandrel

$$\bar{x} = \frac{3a}{4} \quad \bar{y} = \frac{3h}{10}$$

$$\text{Area} = \frac{ah}{3}$$

#### i) General spandrel

$$\bar{x} = \frac{n+1}{n+2} a \quad \bar{y} = \frac{n+1}{4n+2} h$$

$$\text{Area} = \frac{ah}{n+1}$$

#### j) Circular sector

$$\bar{x} = \frac{2r \sin \alpha}{3\alpha} \quad \bar{y} = 0$$

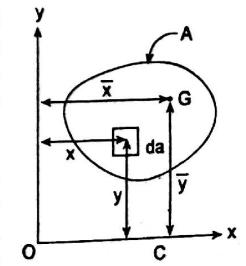
$$\text{Area} = \alpha r^2$$

## 4.2 Second Moment of area/Moment of inertia and Radius of Gyration

### 4.2.1 Moment of Inertia/Second Moment of area

If the first moment of area is again multiplied by the perpendicular distance between axes and centroid, the product so obtained is called second moment of area or more broadly moment of inertia (MOI).

Consider the area A with centroid at G. The moment of inertia of any plane area A is the second moment of all the small area  $dA$ . Comprising the area A about any axis in the plane of area A.



### Second moment of area about y-axis

$$I_{yy} = \sum (dA \cdot x) x$$

where  $(dA \cdot x)$  is the first moment of area  $dA$  about y-axis and  $(dA \cdot x)x$  is the moment of the first moment of area  $A$ , called second moment, about same y-axis.

$$\text{Similarly, } I_{xx} = \sum (dA \cdot y) y$$

In integration form:

$$I_{xx} = \int y^2 dA \quad I_{yy} = \int x^2 dA$$

Inertia refers to the property of a body by virtue of which body resists any change in its state of rest or of uniform motion. Moment of area is essentially a measure of resistance to bending and is applied while dealing with the deflection or deformation of number in bending. Its unit is fourth power of length  $\text{mm}^4, \text{cm}^4, \text{m}^4$  etc.

#### 4.2.1.1 Moment of inertia of rectangular lamina

Consider a rectangular lamina of width  $b$  and height  $h$  as shown. Here  $G$  is the centroid. Consider an elemental area  $dA$  at a distance of  $y$  from base OA.

$$\text{Elemental area } dA = b dy$$

$$\text{MOI about OA} = I_{xx}$$

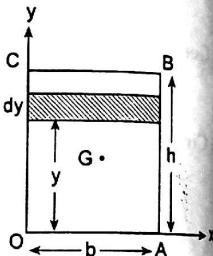
$$I_{xx} = \int y^2 dA$$

$$\therefore = \int_0^h y^2 b dy$$

$$= b \left[ \frac{y^3}{3} \right]_0^h$$

$$\therefore I_{xx} = \frac{bh^3}{3}$$

$$\text{Similarly, } I_{yy} = \frac{hb^3}{3} \quad [\text{simply change the orientation.}]$$



### Centre of Gravity, Centroid and Moment of Inertia

elemental component parallel to the base BC. Let one such elemental component at a distance  $y$  from base BC and thickness  $dy$ .

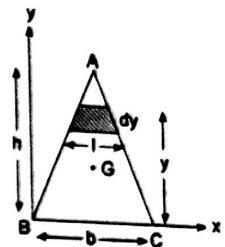
Let  $l$  be the length of element

$$\frac{l}{b} = \frac{h-y}{h}$$

$$\therefore l = b \left( 1 - \frac{y}{h} \right)$$

$$\text{elemental area } dA = l dy$$

$$= b \left( 1 - \frac{y}{h} \right) dy$$



Moment of inertia (MOI) about BC =  $I_{xx}$

$$I_{xx} = \int y^2 dA$$

$$= \int_0^h y^2 b \left( 1 - \frac{y}{h} \right) dy$$

$$= b \int_0^h y^2 dy - \frac{b}{h} \int_0^h y^3 dy$$

$$= b \frac{h^3}{3} - \frac{bh^4}{4h}$$

$$= \frac{bh^3}{3} - \frac{bh^3}{4}$$

$$= \frac{bh^3}{12}$$

#### 4.2.1.3 Moment of inertia of circular lamina

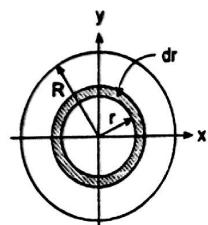
The circular lamina may be considered as consisting of elemental concentric rings. Consider one such elemental ring at a radius  $r$  and having thickness  $dr$ .

$$\text{Polar MOI of elemental ring} = dA \times r^2$$

$$= 2\pi r dr \times r^2$$

$$dI_p = 2\pi r^3 dr$$

$$\therefore I_p = \int_0^R 2\pi r^3 dr$$



#### 4.2.1.2 Moment of inertia of triangular lamina

Consider a triangular lamina of base  $b$  and height  $h$  as shown. The triangle may be considered to consist of a number of infinitely small

$$= 2\pi \frac{R^4}{4}$$

$$= \frac{\pi R^4}{2}$$

if diameter  $D = 2R$ ,  $I_p = \frac{\pi D^4}{32}$

$$\therefore I_{xx} = I_{yy} = \frac{I_p}{2} = \frac{\pi D^4}{64} = \frac{\pi R^4}{4}$$

Note polar MOI =  $I_{xx} + I_{yy}$

#### 4.2.2 Radius of gyration

If the entire area (or mass) of a lamina is concentrated at a point such that there is no change in the moment of inertia about a given axis, then distance of that point from the given axis is called radius of gyration.

Mathematically,

$$I = AK^2$$

$I$  = Moment of inertia

$A$  = Area of body

$K$  = Radius of gyration

$$\therefore K = \sqrt{\frac{I}{A}}$$

$$k_{xx} = \sqrt{\frac{I_{xx}}{A}}; k_{yy} = \sqrt{\frac{I_{yy}}{A}}$$

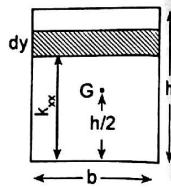
$$\text{polar radius of gyration} = k_{zz} = \sqrt{\frac{I_{xx} + I_{yy}}{A}}$$

It should not be confused with centroid of area, ordinate  $\bar{y} = \frac{h}{2}$ . Here;

$$I_{xx} = \frac{bh^3}{3}, A = bh$$

$$\therefore k_{xx} = \sqrt{\frac{I_{xx}}{A}} = \sqrt{\frac{bh^3/3}{bh}}$$

$$\therefore k_{xx} = \frac{h}{\sqrt{3}}$$



Centroid  $\bar{y}$  depends upon the first moment about radius of gyration  $k_{xx}$  depends upon second moment of area.

#### 4.3 Use of parallel axis theorem and perpendicular axis theorem

##### 4.3.1 parallel axis theorem

This is also called transfer theorem. It is used to transfer the MOI about one axis to other axis parallel to each other. It states "If the moment of inertia of a plane about an centroidal axis is represented by  $I_G$ ; the moment of inertia about any other axis AB, parallel to centroidal axis, and at a distance of  $h$  from centroid is given by

$$I_{AB} = I_G + Ah^2$$

$A$  is area of lamina.

**Proof:**

Consider a strip with elemental area  $dA$  at a distance of  $y$  from centroidal axis. Then,

MOI of strip about an axis through C.G. is  $y^2 dA$

MOI of whole area about C.G. is

$$I_G = \int y^2 dA$$

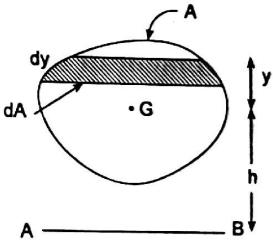
MOI of whole area about AB is

$$\begin{aligned} I_{AB} &= \int (y + h)^2 dA \\ &= \int (y^2 dA + 2hy dA + h^2 dA) \\ \therefore I_{AB} &= \int y^2 dA + 2h \int y dA + h^2 \int dA \\ &= I_G + 2h \int y dA + Ah^2 \end{aligned}$$

Here  $\int y dA$  is the first moment of area about c.g. and is equal to  $\bar{y} \cdot A$ , where  $\bar{y}$  is the distance between the section and the axis passing through c.g. when whole area is considered, this  $\bar{y} = 0$

$$\therefore \int y dA = 0$$

$$\therefore I_{AB} = I_G + Ah^2$$



#### 4.3.2 Perpendicular axis theorem

If  $I_{xx}$  and  $I_{yy}$  be the moment of inertia of a plane section about two perpendicular axes meeting at O, the moment of inertia  $I_{zz}$  about the axis z, perpendicular to both previous axes is given by

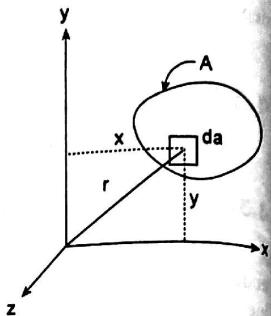
$$I_{zz} = I_{xx} + I_{yy}$$

Consider a small lamina of area  $dA$  at  $x$  and  $y$  distance from perpendicular axes  $y$  and  $x$  respectively.

$$\text{From geometry; } r^2 = x^2 + y^2$$

$$\begin{aligned} I_{zz} &= \int r^2 dA \\ &= \int (x^2 + y^2) dA \\ &= \int x^2 dA + \int y^2 dA \end{aligned}$$

$$\therefore I_{zz} = I_{xx} + I_{yy}$$



#### 4.4 Moment of inertia of composite section

##### Steps:

- Divide the given section into simple area i.e. rectangle, triangle, circle etc.
- Find out the MOI about their own centre of gravity.
- Find the centroid of composite section.
- Now use parallel axes theorem to transfer to new centroidal axes (of composite section) and about new required axes (if any).

##### Moments of inertia of common geometric shapes

###### a) Rectangle

$$\begin{aligned} I_x &= \frac{1}{12}bh^3; & I_y &= \frac{1}{12}b^3h; \\ I_x &= \frac{1}{3}bh^3; & I_y &= \frac{1}{3}b^3h; & I_c &= \frac{1}{12}bh(b^2 + h^2) \end{aligned}$$

###### b) Triangle

$$I_x = \frac{1}{36}bh^3; \quad I_y = \frac{1}{12}bh^3$$

###### c) Circle

$$I_x = I_y = \frac{1}{4}\pi r^4; \quad I_O = \frac{1}{2}\pi r^4$$

###### Semicircle

$$I_x = I_y = \frac{1}{8}\pi r^4; \quad I_O = \frac{1}{4}\pi r^4$$

###### Quarter circle

$$I_x = I_y = \frac{1}{16}\pi r^4; \quad I_O = \frac{1}{8}\pi r^4$$

###### Ellipse

$$I_x = \frac{1}{4}\pi ab^3; \quad I_y = \frac{1}{4}\pi a^3b; \quad I_O = \frac{1}{4}\pi ab(a^2 + b^2)$$

#### Worked Out Examples

- Locate the centroid of line of semi-circular arc.

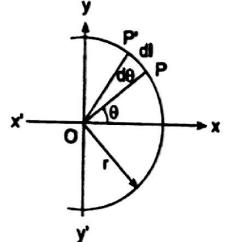
##### Solution:

Locate the small arc  $PP'$  which subtends angle  $d\theta$  at origin. Let P is at an angle  $\theta$  with OB.

Length of arc  $PP' = dl = rd\theta$

coordinates of P is  $(r \cos \theta, r \sin \theta)$

$$\begin{aligned} \bar{x} &= \frac{1}{l} \int x dl \\ &= \frac{1}{\int dl} \int x dl \\ &= \frac{1}{\int_{-\pi/2}^{\pi/2} r d\theta} \int_{-\pi/2}^{\pi/2} (r \cos \theta) \cdot r d\theta \\ &= \frac{1}{\pi r} r^2 \sin \theta \Big|_{-\pi/2}^{\pi/2} \\ &= \frac{1}{\pi r} 2r^2 \\ &= \frac{2r}{\pi} \end{aligned}$$



Since figure is symmetrical about x-axis,  $\bar{Y} = 0$   
 $\therefore (\bar{X}, \bar{Y}) = \left(\frac{2r}{\pi}, 0\right)$

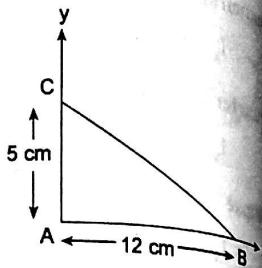
2. Locate the centroid of the composite line. Also find the first moment of area about X and Y axes.

**Solution:**

$$\text{Here, } AB = 12 \text{ cm}$$

$$AC = 5 \text{ cm}$$

$$BC = \sqrt{5^2 + 12^2} = 13 \text{ cm}$$



Segment	Length $L_i$ (cm)	$\bar{X}_i$ cm	$\bar{Y}_i$ cm	$\bar{X}_i L_i$	$\bar{Y}_i L_i$
AB	12	6	0	72	0
AC	5	0	2.5	0	12.5
BC	13	6	2.5	78	32.5
	30			150	45

$$\therefore \bar{X} = \frac{\sum X_i L_i}{\sum L_i}$$

$$= \frac{150}{30} = 5 \text{ cm}$$

$$\bar{Y} = \frac{\sum Y_i L_i}{\sum L_i}$$

$$= \frac{45}{30} = 1.5 \text{ cm}$$

$$\text{First moment about x-axis } M_x = \sum Y_i L_i = 45 \text{ cm}^2$$

$$\text{First moment about y-axis } M_y = \sum X_i L_i = 150 \text{ cm}^2$$

3. Determine by direct integration the location of the centroid of shaded area.

**Solution:**

Steps:

Determine constant k.

- i) Calculate small elemental rectangle and integrate it.

Given curve

$$y = kx^2$$

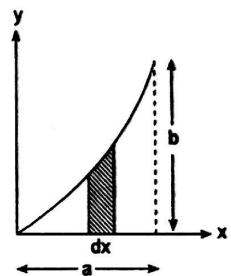
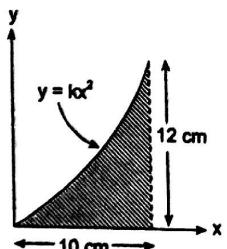
At origin, 0 = k.0<sup>2</sup> not useful.

Again, b = ka<sup>2</sup>

$$\therefore k = \frac{b}{a^2}$$

$$\text{So, } y = \frac{b}{a^2} x^2$$

$$x = \frac{a}{\sqrt{b}} \cdot \sqrt{y}$$



Draw a elemental rectangle parallel to y-axis having width of dx. The height of elemental rectangle will be y.

Elemental area  $dA = y dx$

$$\therefore A = \int dA = \int_0^a y dx$$

$$= \int_0^a \frac{b}{a^2} \cdot x^2 dx$$

$$= \frac{b}{a^2} \cdot \frac{a^3}{3}$$

$$\therefore A = \frac{ab}{3}$$

$$\text{Now, } \bar{X} = \frac{1}{A} x_{el} dA$$

where  $x_{el}$  is centroid of elemental rectangle.

$$\bar{x} = \frac{1}{ab} \int x \cdot y \, dx$$

$$= \frac{3}{ab} \int_0^a x \cdot \frac{b}{a^2} x^2 \, dx$$

$$= \frac{3}{ab} \frac{b}{a^2} \int_0^a x^3 \, dx$$

$$= \frac{3}{a^3} \cdot \frac{a^4}{4}$$

$$\therefore \bar{x} = \frac{3a}{4}$$

Again,

$$\bar{y} = \frac{1}{A} \int y_{el} \, dA$$

$y_{el}$  is the centroid of elemental rectangle.

$$\bar{y} = \frac{1}{ab} \int \frac{y}{2} \cdot y \, dx \quad (\text{Here } y_{el} = \frac{y}{2} \text{ why?})$$

$$= \frac{3 \times 1}{ab} \int_0^a y^2 \, dx$$

$$= \frac{3}{2ab} \int_0^a \frac{b^2}{a^4} \cdot x^4 \, dx$$

$$= \frac{3}{2ab} \cdot \frac{b^2}{a^4} \cdot \frac{a^5}{5}$$

$$\therefore \bar{y} = \frac{3b}{10}$$

$$\therefore \bar{x} = \frac{3a}{4}, \bar{y} = \frac{3b}{10}$$

if  $a = 10 \text{ cm}$ ,  $b = 12 \text{ cm}$

$$\bar{x} = 7.5 \text{ cm} \quad \bar{y} = 3.6 \text{ cm}$$

Alternative method:

Till now we take vertical elemental rectangle, but we can take horizontal rectangle also.

(here  $x_{el} = x$ , why?)

Be careful while calculating the length of elemental rectangle.

Elemental area  $dA$

$$\therefore dA = (a - x) dy \quad (\text{but not } x dy, \text{ why?})$$

$$\therefore A = \int dA = \int_0^b (a - x) dy$$

$$= \frac{ab}{3}$$

$$\bar{x} = \frac{1}{A} \int x_{el} \, dA$$

$$= \frac{1}{A} \int_0^b \left( x + \frac{a-x}{2} \right) \cdot (a-x) dy \quad (\text{why?})$$

$$= \frac{3a}{4}$$

Again,

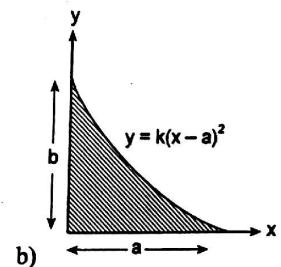
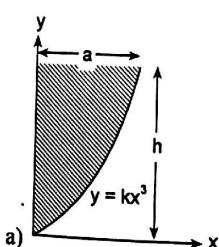
$$\bar{y} = \frac{1}{A} \int y_{el} \, dA$$

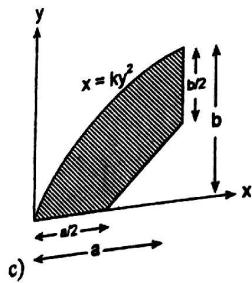
$$= \frac{1}{A} \int_0^b y \cdot (a-x) dy \quad (\text{why?})$$

$$= \frac{3b}{10}$$

So, proper selection of elemental rectangle is required for fast and easy solution.

#### 4. Locate the centroid of shaded area:





**Solution:**

a) Take a horizontal strip of thickness  $dy$ .

$$y = kx^3$$

$$h = ka^3$$

$$\therefore k = \frac{h}{a^3}$$

$$\therefore y = \frac{h}{a^3} \cdot x^3$$

$$x = \frac{ay^{1/3}}{h^{1/3}}$$

$$\text{Small area } dA = x \, dy$$

$$\therefore A = \int dA$$

$$= \int_0^h x \, dy$$

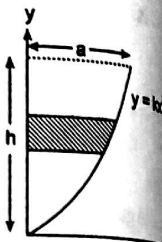
$$= \int_0^h \frac{a \cdot y^{1/3}}{h^{1/3}} \, dy = \frac{a}{h^{1/3}} \cdot \frac{y^{4/3}}{4/3} \Big|_0^h$$

$$= \frac{a}{h^{1/3}} \cdot \frac{3}{4} h^{4/3}$$

$$= \frac{3ah}{4}$$

$$\bar{x} = \frac{1}{A} \int x_{el} dA$$

$$= \frac{4}{3ah} \int_0^h \frac{x}{2} \cdot x \, dy$$



(b) Take vertical strip

Centre of Gravity, Centroid and Moment of Inertia

$$\begin{aligned} &= \frac{2}{3ah} \int_0^h x^2 \, dy \\ &= \frac{2}{3ah} \int_0^h \frac{a^2 y^{2/3}}{h^{2/3}} \, dy \\ &= \frac{2a}{3h^{5/3}} \int_0^h y^{2/3} \, dy \\ &= \frac{2a}{3h^{5/3}} \cdot \frac{y^{5/3}}{5/3} \Big|_0^h \\ &= \frac{2a}{3} \times \frac{3}{5} \end{aligned}$$

$$\therefore \bar{x} = \frac{2a}{5}$$

$$\begin{aligned} \bar{y} &= \frac{1}{A} \int y_{el} dA \\ &= \frac{4}{3ah} \int_0^h y \cdot x \, dy \\ &= \frac{4}{3ah} \int_0^h y \cdot \frac{a}{h^{1/3}} \cdot y^{1/3} \, dy \\ &= \frac{4}{3h^{4/3}} \cdot \int_0^h y^{4/3} \, dy \\ &= \frac{4}{3} \cdot \frac{1}{h^{4/3}} \cdot \frac{y^{7/3}}{7/3} \Big|_0^h \\ &= \frac{4}{7} \cdot h \end{aligned}$$

$$\therefore \bar{y} = \frac{4h}{7}$$

b) Take vertical strip

$$y = k(x - a)^2$$

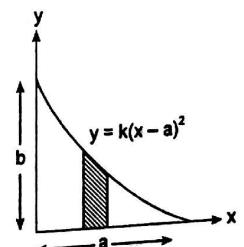
$$x \rightarrow a, y \rightarrow 0 \text{ at point A.}$$

$$O = k \cdot 0 \text{ not useful.}$$

$$x \rightarrow 0, y \rightarrow b \text{ at point B}$$

$$b = k(0 - a)^2$$

$$b = k \cdot a^2$$



$$k = \frac{b}{a^2}$$

$$\therefore y = \frac{b}{a^2} \cdot (x - a)^2$$

$$dA = y dx$$

$$\therefore \text{Area } A = \int_0^a y dx$$

$$= \frac{b}{a^2} \int_0^a (x - a)^2 dx$$

$$= \frac{b}{a^2} \int_0^a (x^2 - 2ax + a^2) dx$$

$$= \frac{b}{a^2} \cdot \left[ \frac{a^3}{3} - 2a \cdot \frac{a^2}{2} + a^3 \right]$$

$$= \frac{b}{a^2} \cdot \frac{a^3}{3}$$

$$\therefore A = \frac{ab}{3} \text{ unit}$$

$$\bar{x} = \frac{1}{A} \int x_{el} dA$$

$$= \frac{3}{ab} \int_0^a xy dx$$

$$= \frac{3}{ab} \int_0^a x \cdot \frac{b}{a^2} (x - a)^2 dx$$

$$= \frac{3}{ab} \int_0^a \frac{b}{a^2} \cdot x(x^2 - 2ax + a^2) dx$$

$$= \frac{3}{ab} \int_0^a \left( x^3 - 2ax^2 + a^2x \right) dx$$

$$= \frac{3}{a^3} \cdot \left[ \frac{a^4}{4} - 2a \cdot \frac{a^3}{3} + \frac{a^4}{2} \right]$$

$$= \frac{3}{a^3} \cdot \left[ \frac{3a^4 - 8a^4 + 6a^4}{12} \right]$$

$$= \frac{3}{a^3} \cdot \frac{a^4}{12}$$

$$\therefore \bar{x} = \frac{a}{4}$$

$$\begin{aligned}\bar{y} &= \frac{1}{A} \int y_{el} dA \\ &= \frac{3}{ab} \int_0^a \frac{y}{2} \cdot y dx \\ &= \frac{3}{2ab} \int_0^a y^2 dx \\ &= \frac{3}{2ab} \int_0^a k^2(x - a)^4 dx \\ &= \frac{3}{2ab} \cdot \frac{b^2}{a^4} \int_0^a (x - a)^4 dx \\ &= \frac{3b}{2a^4} \int_0^a (x - a)^4 dx\end{aligned}$$

Let  $x - a = z$

$$dx = dz$$

$$x \rightarrow 0; \quad z \rightarrow -a$$

$$x \rightarrow a; \quad z \rightarrow 0$$

$$\begin{aligned}\therefore \bar{y} &= \frac{3b}{2a^4} \int_{-a}^0 z^4 dz \\ &= \frac{3b}{2a^4} \cdot \frac{z^5}{5} \Big|_{-a}^0 \\ &= \frac{3b}{10a^4} \cdot a^4 \\ &= \frac{3b}{10}\end{aligned}$$

c) Required area = Area of curve - Area of triangle

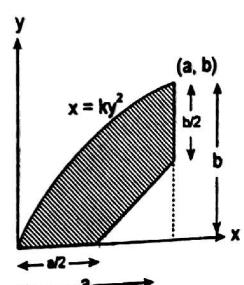
Let area of curve =  $A_1$

area of triangle =  $A_2$

Centroid of curve  $(\bar{X}_1, \bar{Y}_1)$

Centroid of triangle  $= (\bar{X}_2, \bar{Y}_2)$

Then,



$$\bar{X} = \frac{A_1 \bar{X}_1 - A_2 \bar{X}_2}{A_1 - A_2}, \quad \bar{Y} = \frac{A_1 \bar{Y}_1 - A_2 \bar{Y}_2}{A_1 - A_2}$$

Now consider the curve only

$$x = ky^2$$

$$a = kb^2$$

$$\therefore k = \frac{a}{b^2}$$

$$\therefore x = \frac{a}{b^2} \cdot y^2$$

$$y = \sqrt{\frac{b}{a}} x$$

$$dA = \int y \, dx$$

$$A_1 = \int_0^a \frac{b}{\sqrt{a}} \cdot \sqrt{x} \, dx$$

$$= \frac{b}{\sqrt{a}} \cdot \int_0^a x^{1/2} \, dx \Rightarrow A_1 = \frac{b}{\sqrt{a}} \cdot \frac{a^{3/2}}{3/2} = \frac{2ab}{3} \text{ unit}$$

$$\therefore A_1 = \frac{2ab}{3} \text{ unit}$$

$$\bar{X}_1 = \frac{1}{A} \int x_{el} \cdot dA$$

$$= \frac{3}{2ab} \int_0^a x \cdot y \, dx$$

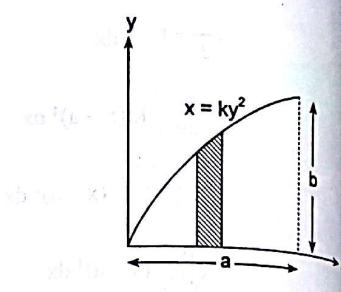
$$= \frac{3}{2ab} \cdot \frac{b}{\sqrt{a}} \int_0^a x \cdot x^{1/2} \, dx$$

$$= \frac{3}{2a^{3/2}} \cdot \int_0^a x^{3/2} \, dx$$

$$= \frac{3}{2a^{3/2}} \cdot \frac{a^{5/2}}{5/2}$$

$$\therefore \bar{X}_1 = \frac{3a}{5}$$

$$\bar{Y}_1 = \frac{1}{A} \int y_{el} \, dA$$



$$\begin{aligned} &= \frac{3}{2ab} \int_0^a \frac{y}{2} \cdot y \, dx \\ &= \frac{3}{4ab} \int_0^a \frac{b^2}{a} \cdot x \, dx \\ &= \frac{3b}{4a^2} \cdot \frac{x^2}{2} \Big|_0^a \end{aligned}$$

$$\therefore \bar{Y}_1 = \frac{3b}{8}$$

Consider triangle only.

$$\text{Area } A_2 = \frac{1}{2} \times \frac{a}{2} \times \frac{b}{2} = \frac{ab}{8}$$

$$\bar{X}_2 = \frac{a}{2} + \frac{2}{3} \cdot \frac{a}{2} = \frac{a}{3} + \frac{a}{2}$$

$$\bar{Y}_2 = \frac{1}{3} \cdot \frac{b}{2} = \frac{b}{6}$$

$$\therefore \text{Area required} = A_1 - A_2$$

$$= \frac{2ab}{3} - \frac{ab}{8} = \frac{13ab}{24}$$

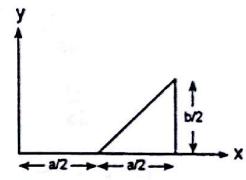
$$\begin{aligned} \bar{X} &= \frac{A_1 \bar{X}_1 - A_2 \bar{X}_2}{A_1 - A_2} \\ &= \frac{\frac{2ab}{3} \times \frac{3a}{5} - \frac{ab}{8} \times \left(\frac{a}{3} + \frac{a}{2}\right)}{\frac{13ab}{24}} \end{aligned}$$

$$= \frac{\frac{2a^2b}{5} - \frac{ab}{8} \times \frac{5a}{6}}{\frac{13ab}{24}}$$

$$= \frac{96a^2b - 25a^2b}{48 \times 5} \times \frac{24}{13ab}$$

$$= \frac{71a^2b}{130ab}$$

$$\therefore \bar{X} = \frac{71a}{130}$$



$$\bar{Y} = \frac{A_1 Y_1 - A_2 Y_2}{A_1 - A_2}$$

$$= \frac{\frac{2ab}{3} \times \frac{3b}{8} - \frac{ab}{8} \times \frac{b}{6}}{\frac{13ab}{24}}$$

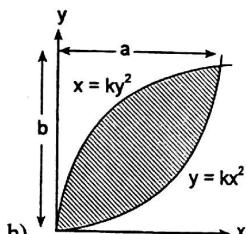
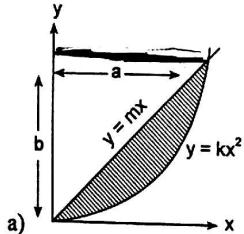
$$= \frac{\frac{ab^2}{4} - \frac{ab^2}{48}}{\frac{13ab}{24}}$$

$$= \frac{11ab^2}{48} \times \frac{24}{13ab}$$

$$\bar{Y} = \frac{11b}{26}$$

$$\therefore (\bar{X}, \bar{Y}) = \left( \frac{71a}{130}, \frac{11b}{26} \right)$$

5. Locate the centroid of shaded area



**Solution:**

a) For the intersection of curve, the solving method is easier when we draw both horizontal and vertical strip.

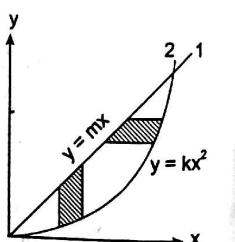
To calculate  $\bar{X}$ , take vertical strip.

To calculate  $\bar{Y}$ , take horizontal strip.

$$\text{Let } y_1 = mx_1, \quad y_2 = kx_2^2$$

$$b = mab = k.a^2$$

$$\therefore m = \frac{b}{a} \quad k = \frac{b}{a^2}$$



$$\text{or, } y_1 = \frac{b}{a} x_1 \quad x_1 = \frac{a}{b} y_1$$

$$y_2 = \frac{b}{a^2} x_2^2 \quad x_2 = \frac{a}{\sqrt{b}} \cdot \sqrt{y_2}$$

For the calculation of area you can take either strip.

Taking vertical strip

$$dA = (y_1 - y_2) dx$$

$$\therefore A = \int dA$$

$$= \int_0^a (y_1 - y_2) dx$$

$$= \int_0^a \left( \frac{b}{a} x_1 - \frac{b}{a^2} x_2^2 \right) dx$$

For vertical strip  $x_1 = x_2 = x$  (say)

$$\therefore A = \int_0^a \left( \frac{b}{a} x - \frac{b}{a^2} x^2 \right) dx$$

$$= \int_0^a \frac{b}{a} x dx - \int_0^a \frac{b}{a^2} x^2 dx$$

$$= \frac{b}{a} \cdot \frac{a^2}{2} - \frac{b}{a^2} \cdot \frac{a^3}{3}$$

$$= \frac{ab}{2} - \frac{ab}{3}$$

$$\therefore A = \frac{ab}{6} \text{ unit}$$

Calculation of  $\bar{X}$ ; consider vertical strip

$$\bar{X} = \frac{1}{A} \int x_{el} dA$$

$$= \frac{6}{ab} \int_0^a x \cdot (y_1 - y_2) dx$$

$$= \frac{6}{ab} \int_0^a x \left[ \frac{b}{a} x_1 - \frac{b}{a^2} x_2^2 \right] dx$$

$$= \frac{6}{ab} \int_0^a \left( \frac{b}{a} x^2 - \frac{b}{a^2} x^3 \right) dx$$

$$\begin{aligned} &= \frac{6}{ab} \left[ \frac{b}{a} \cdot \frac{a^3}{3} - \frac{b}{a^2} \cdot \frac{a^4}{4} \right] \\ &= \frac{6}{ab} \cdot \left( \frac{a^2 b}{3} - \frac{a^2 b}{4} \right) \end{aligned}$$

$$\therefore \bar{x} = \frac{a}{2}$$

Calculation of  $\bar{Y}$ , consider horizontal strip

$$\bar{Y} = \frac{1}{A} \int y_{el} dA$$

$$\begin{aligned} &= \frac{6}{ab} \int_0^a y(x_2 - x_1) dy \\ &= \frac{6}{ab} \int_0^a y \left( \sqrt{b} \cdot \sqrt{y_2} - \frac{a}{b} y_1 \right) dy \\ &= \frac{6}{ab} \int_0^b y \left( \frac{a}{\sqrt{b}} \cdot y^{3/2} - \frac{a}{b} y^2 \right) dy \\ &= \frac{6}{ab} \left[ \frac{a}{\sqrt{b}} \cdot \frac{b^{5/2}}{5/2} - \frac{a}{b} \cdot \frac{b^3}{3} \right] \\ &= \frac{6}{ab} \left[ \frac{2a}{5} \cdot b^2 - \frac{ab^2}{3} \right] \end{aligned}$$

$$\therefore \bar{Y} = \frac{2b}{5}$$

$$\therefore (\bar{x}, \bar{Y}) = \left( \frac{a}{2}, \frac{2b}{5} \right)$$

b) Here

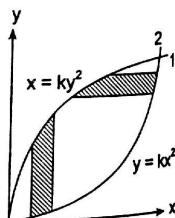
$$x_1 = ky_1^2 \quad y_2 = kx_2^2$$

$$a = k \cdot b^2 \quad b = k \cdot a^2$$

$$\therefore k = \frac{a}{b^2} \quad k = \frac{b}{a^2}$$

$$x_1 = \frac{a}{b^2} \cdot y_1^2 \quad y_2 = \frac{b}{a^2} \cdot x_2^2$$

$$y_1 = \frac{b}{\sqrt{a}} \cdot \sqrt{x_1} \quad x_2 = \frac{a}{\sqrt{b}} \cdot \sqrt{y_2}$$



$$\begin{aligned} A &= \int dA = \int_0^a (y_1 - y_2) dx \\ &= \int_0^a \frac{b}{\sqrt{a}} \sqrt{x_1} - \frac{b}{a^2} x_2^2 dx \end{aligned}$$

Here for vertical strip,  $x_1 = x_2 = x$

$$\begin{aligned} A &= \int_0^a \left( \frac{b}{\sqrt{a}} \sqrt{x} - \frac{b}{a^2} x^2 \right) dx \\ &= \frac{b}{\sqrt{a}} \cdot \frac{a^{3/2}}{3/2} - \frac{b}{a^2} \cdot \frac{a^3}{3} \\ &= \frac{2b}{3} \cdot a - \frac{ab}{3} \\ &= \frac{ab}{3} \text{ unit} \end{aligned}$$

Calculation of  $\bar{X}$ , consider vertical strip

$$\bar{X} = \frac{1}{A} \int x_{el} dA$$

$$\begin{aligned} &= \frac{1}{A} \int_0^a x(y_1 - y_2) dx \\ &= \frac{1}{3} \int_0^a x \left[ \frac{b}{\sqrt{a}} \sqrt{x} - \frac{b}{a^2} x^2 \right] dx \end{aligned}$$

$$\begin{aligned} &= \frac{1}{3} \int_0^a \left( \frac{b}{\sqrt{a}} x^{3/2} - \frac{b}{a^2} x^3 \right) dx \\ &= \frac{1}{3} \left[ \frac{b}{\sqrt{a}} \cdot \frac{a^{5/2}}{5/2} - \frac{b}{a^2} \cdot \frac{a^4}{4} \right] \\ &= \frac{3}{ab} \left[ \frac{2b}{5} \cdot a^2 - \frac{a^2 b}{4} \right] \\ &= \frac{3}{ab} \cdot \frac{8a^2 b - 5a^2 b}{20} \end{aligned}$$

$$\therefore \bar{X} = \frac{9a}{20}$$

Calculation of  $\bar{Y}$ , consider horizontal strip

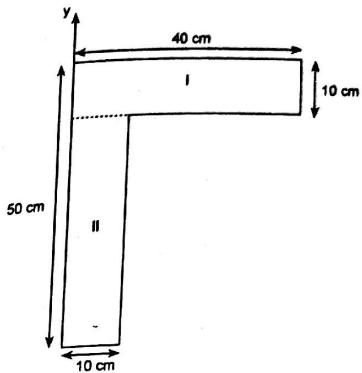
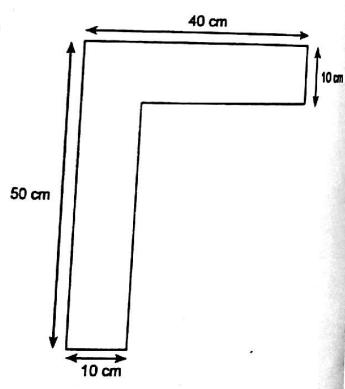
$$\begin{aligned}\bar{Y} &= \frac{1}{A} \int y_{el} dA \\ &= \frac{3}{ab} \int_0^b y (x_2 - x_1) dy \\ &= \frac{3}{ab} \int_0^b y \left( \frac{a}{\sqrt{b}} \sqrt{y_2} - \frac{a}{b^2} y_1^2 \right) dy \\ &= \frac{3}{ab} \int_0^b \left( \frac{a}{\sqrt{b}} y^{3/2} - \frac{a}{b^2} y^3 \right) dy \\ &= \frac{3}{ab} \left[ \frac{a}{\sqrt{b}} \cdot \frac{b^{5/2}}{5/2} - \frac{a}{b^2} \cdot \frac{b^4}{4} \right] \\ &= \frac{3}{ab} \left[ \frac{2a}{b} b^2 - \frac{ab^2}{4} \right] \\ &= \frac{3}{ab} \cdot \frac{3ab^2}{20} \\ \therefore \bar{Y} &= \frac{9b}{20}\end{aligned}$$

$$\therefore (\bar{X}, \bar{Y}) = \left( \frac{9a}{20}, \frac{9b}{20} \right)$$

6. Locate the centroid of composite area.

**Solution:**

Here the axis is not given, so we have to assume axes. Also, the required section is sum of two rectangles.



segment	Area ( $A_i$ ) cm $^2$	$X_i$ cm	$Y_i$ cm	$A_i X_i$	$A_i Y_i$
Rect (i)	400	20	45	8000	18000
Rect (ii)	400	5	20	2000	8000
	800			1000	26000

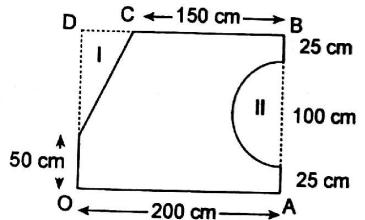
$$\bar{X} = \frac{\sum A_i X_i}{\sum A_i} = \frac{10000}{800} = 12.5 \text{ cm}$$

$$\bar{Y} = \frac{\sum A_i Y_i}{\sum A_i} = \frac{26000}{800} = 32.5 \text{ cm}$$

7. Locate the centroid of the given composite area.

**Solution:**

$$\begin{aligned}\text{Required area} &= \text{Rect. OABD} \\ &- \text{Fig (i)} - \text{fig (ii)}\end{aligned}$$



Segment	Area ( $A_i$ )	$X_i$	$Y_i$	$A_i X_i$	$A_i Y_i$
Rect OABD	$200 \times 150 = 30000$	100	75	3000000	2250000

Fig. (i)	$-\frac{1}{2} \times 50$ $\times 100 = -2500$	$\frac{50}{3}$	$50 + \frac{100}{3} \times 2 = 116.67$	(-) 41666.67	(-) 29167 $\frac{1}{3}$
Fig. (ii)	$-\pi \times \frac{50^2}{2} = -3925$	$200 - \frac{4r}{3\pi} = 178.77$	75	(-) 701672.25	(-) 29437 $\frac{1}{3}$
	$\Sigma = 23575$			$\Sigma = 2256661$	$\Sigma = 1663950$

(-) area for not required area

$$\bar{X} = \frac{\sum A_i X_i}{\sum A_i} = \frac{225661}{23575} = 95.72$$

$$\bar{Y} = \frac{\sum A_i Y_i}{\sum A_i} = \frac{1663950}{23575} = 70.58$$

8. Calculate the MOI of the shaded area by direct integration about axes.

#### Solution:

To find the moment of inertia, we draw two rectangular strip parallel to x-axis and y-axis.

Take vertical strip for  $I_{yy}$ . Take horizontal strip for  $I_{xx}$ .

(Note sometimes it is easier to take vertical strip to calculate  $I_{xx}$ . In such case we use MOI of rectangle about base is  $\frac{bh^3}{3}$  and integrate it.)

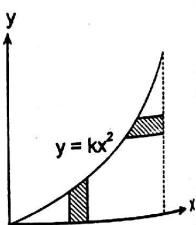
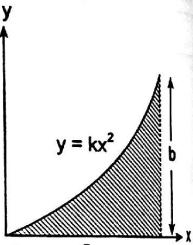
Calculation of  $I_{xx}$ :

Take horizontal strip

$$I_{xx} = \int y^2 dA$$

$$= \int y^2 (a-x) dy$$

$$= \int_0^b y^2 a dy - \int_0^b y^2 x dy$$



$$\begin{aligned}
 &= \frac{ab^3}{3} - \int_0^b y^2 \frac{a}{\sqrt{b}} \sqrt{y} dy \\
 &= \frac{ab^3}{3} - \frac{a}{\sqrt{b}} \cdot \frac{b^{7/2}}{7/2} \\
 &= \frac{ab^3}{3} - \frac{2}{7} ab^3 \\
 &= \frac{ab^3}{21} \text{ unit}
 \end{aligned}$$

Alternative:  
Take vertical strip

$$\begin{aligned}
 I_{xx} &= \int_0^a \frac{y^3}{3} dx \\
 &= \frac{1}{3} \int_0^a \frac{b^3}{a^6} x^6 dx \\
 &= \frac{1}{3} \frac{b^3}{a^6} \frac{a^7}{7} \\
 I_{xx} &= \frac{ab^3}{27} \text{ unit}
 \end{aligned}$$

Calculation of  $I_{yy}$ :

Take vertical strip

$$I_{yy} = \int x^2 dA$$

$$= \int_0^a x^2 \cdot y dx$$

$$= \int_0^a x^2 \frac{b}{a^2} x^2 dx$$

$$= \frac{b}{a^2} \int_0^a x^4 dx$$

$$= \frac{b}{a^2} \frac{a^5}{5}$$

$$I_{yy} = \frac{a^3 b}{5} \text{ unit}$$

9. Calculate the MOI of given area.

**Solution:**

Suppose  $y = mx$  be eq (ii)

$y = kx^2$  be eq (i)

Then,

$$y_2 = mx_2 \quad y_1 = kx_1^2$$

$$b = m.a \quad b = ka^2$$

$$\therefore m = \frac{b}{a} \quad \therefore k = \frac{b}{a^2}$$

$$y_2 = \frac{b}{a}x_2 \quad y_1 = \frac{b}{a^2}x_1^2$$

$$x_2 = \frac{a}{b}y_2 \quad y_2 = \frac{a}{\sqrt{b}}\sqrt{y_1}$$

Calculation of  $I_{xx}$ :

Take horizontal strip

$$I_{xx} = \int y^2 dA$$

$$= \int_0^b y^2 (x_1 - x_2) dy$$

$$= \int_0^b y^2 \left( \frac{a}{\sqrt{b}}\sqrt{y_1} - \frac{a}{b}y_2 \right) dy$$

$$= \int_0^b \left( \frac{a}{\sqrt{b}}y^{5/2} - \frac{a}{b}y^3 \right) dy$$

$$= \frac{a}{\sqrt{b}} \cdot \frac{b^{7/2}}{7/2} - \frac{a}{b} \cdot \frac{b^4}{4}$$

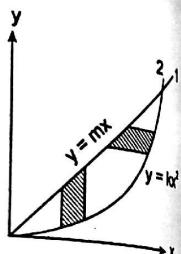
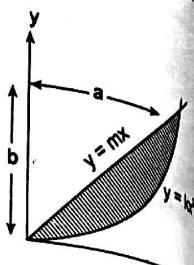
$$= \frac{2a}{7} \cdot b^3 - \frac{ab^3}{4}$$

$$\therefore I_{xx} = \frac{ab^3}{28} \text{ unit}$$

(It is not recommended to use vertical strip to calculate  $I_{xx}$  here, why?)

Calculation of  $I_{yy}$ :

Take vertical strip



$$\begin{aligned} I_{yy} &= \int x^2 dA \\ &= \int_0^a x^2 (y_2 - y_1) dx \\ &= \int_0^a x^2 \left[ \frac{b}{a}x_2 - \frac{b}{a^2}x_1^2 \right] dx \\ &= \int_0^a \left( \frac{b}{a}x^3 - \frac{b}{a^2}x^4 \right) dx \\ &= \frac{b}{a} \cdot \frac{a^4}{4} - \frac{b}{a^2} \cdot \frac{a^5}{5} \\ &= \frac{a^3b}{4} - \frac{a^3b}{5} \\ &\therefore I_{yy} = \frac{a^3b}{20} \text{ unit} \end{aligned}$$

10. Calculate the MOI about x-axis.

**Solution:**

Here it is recommended to use vertical strip to calculate  $I_{xx}$ .

$$I_{xx} = \frac{bh^3}{3}$$

$$\therefore I_{xx} = \int_0^2 \frac{1}{3}y^3 dx$$

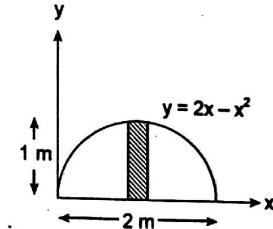
$$= \frac{1}{3} \int_0^2 (2x - x^2)^3 dx$$

$$= \frac{1}{3} \int_0^2 (8x^3 - x^6 - 3(2x)^2x^2 + 3.2x \cdot x^4) dx$$

$$= \frac{1}{3} \int_0^2 (8x^3 - x^6 - 12x^4 + 6x^5) dx$$

$$= \frac{1}{3} \left[ \frac{8x^4}{4} - \frac{x^7}{7} - 12 \cdot \frac{x^5}{5} + \frac{6x^6}{6} \right]_0^2$$

$$= \frac{1}{3} \left[ 2 \times 16 - \frac{128}{7} - \frac{384}{5} + 64 \right]$$



$$= \frac{1}{3}[0.914]$$

$$I_{xx} = 0.3047 \text{ m}^4$$

11. Calculate the moment of inertia of rectangle and triangle through centroid.

**Solution:**

a) for rectangle:

$$\text{We know, } I_{AB} = \frac{bh^3}{3}$$

Using parallel axis theorem;

$$I_{AB} = I_{GXGX} + A(\bar{y})^2$$

$$\frac{bh^3}{3} = I_{GXGX} + bh\left(\frac{h}{2}\right)^2$$

$$I_{GXGX} = \frac{bh^3}{3} - bh\frac{h^2}{4}$$

$$= \frac{bh^3}{3} - \frac{bh^3}{4}$$

$$\therefore I_{GXGX} = \frac{bh^3}{12}$$

b) For triangle:

$$\text{We know, } I_{BC} = \frac{bh^3}{12}$$

Using parallel axis

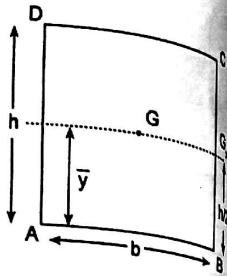
$$I_{BC} = I_{GXGX} + A(\bar{y})^2$$

$$\frac{bh^3}{12} = I_{GXGX} + \frac{1}{2} \times b \times h \left(\frac{h}{3}\right)^2$$

$$\therefore I_{GXGX} = \frac{bh^3}{12} - \frac{1}{2} \cdot \frac{bh^3}{9}$$

$$= \frac{bh^3}{12} - \frac{bh^3}{18}$$

$$= \frac{bh^3}{36} \text{ unit}$$

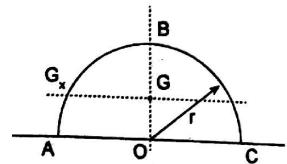


12. Calculate the MOI of semicircular section about centroidal axes.

**Solution:**

Here, MOI of semicircular about the base AC is equal to the half of MOI of circular section about AC.

$$\therefore \text{MOI}_{AC} = \frac{1}{2} \times \frac{\pi r^4}{4} = \frac{\pi r^4}{8}$$



$$\text{The } \bar{y} = \frac{4r}{3\pi} \text{ [Distance of G from AC]}$$

Using parallel axes theorem

$$I_{AC} = I_{GXGX} + A.h^2$$

$$\frac{\pi r^4}{8} = I_{GXGX} + \frac{\pi r^2}{2} \cdot \left(\frac{4r}{3\pi}\right)^2$$

$$\frac{\pi r^4}{8} = I_{GXGX} + \frac{\pi r^2}{2} \cdot \frac{16r^2}{9\pi^2}$$

$$I_{GXGX} = \frac{\pi r^4}{8} - \frac{8r^4}{9\pi}$$

$$\therefore I_{GXGX} = 0.11r^4$$

$$\text{MOI about centroidal y-axis is } I_{GYGY} = \frac{\pi r^4}{8} = 0.393r^4$$

13. Find the MOI about axes.

**Solution:**

Here the MOI about y axis can be directly calculated using vertical strip. To calculate MOI about x axis, we first calculate MOI of curve about x axis and subtract the MOI of triangle about same x axis.

$$\text{Let } y = kx^2 + c \quad \dots \dots \text{(i)}$$

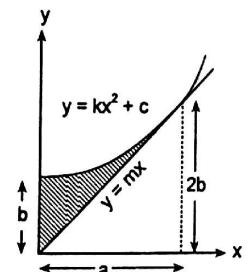
$$y = mx \quad \dots \dots \text{(ii)}$$

$$\therefore y_1 = kx_1^2 + c$$

This equation passes through (0, b) and (a, 2b)

$$b = k \cdot 0 + c$$

$$\therefore c = b$$



Again,  $2b = k \cdot a^2 + b$

$$b = ka^2$$

$$\therefore k = \frac{b}{a^2}$$

$$\therefore y_1 = \frac{b}{a^2} x_1^2 + b \quad \text{and}$$

$$x_1 = \sqrt{(y_1 - b)} \cdot \frac{a}{\sqrt{b}}$$

Again,

$$y_2 = mx_2$$

$$2b = m \cdot a$$

$$\therefore m = \frac{2b}{a}$$

$$\therefore y_2 = \frac{2b}{a} x_2 \quad \text{and} \quad x_2 = \frac{a}{2b} y_2$$

Calculation of  $I_{yy}$

Consider a vertical strip

$$dA = (y_1 - y_2) dx$$

$$= \left\{ \left( \frac{b}{a^2} x_1^2 + b \right) - \left( \frac{2b}{a} x_2 \right) \right\} dx$$

$$= \left( \frac{b}{a^2} x^2 + b - \frac{2b}{a} x \right) dx$$

$$I_{yy} = \int x^2 dA$$

$$= \int_0^a x^2 \left( \frac{b}{a^2} x^2 + b - \frac{2b}{a} x \right) dx$$

$$= \int_0^a \left( \frac{b}{a^2} x^4 + bx^2 - \frac{2b}{a} x^3 \right) dx$$

$$= \frac{b}{a^2} \cdot \frac{a^5}{5} + b \cdot \frac{a^3}{3} - \frac{2b}{a} \cdot \frac{a^4}{4}$$

$$= \frac{a^3 b}{5} + \frac{a^3 b}{3} - \frac{a^3 b}{2}$$

$$= \frac{6a^3 b + 10a^3 b - 15a^3 b}{30}$$

$$\therefore I_{yy} = \frac{a^3 b}{30} \text{ unit}$$

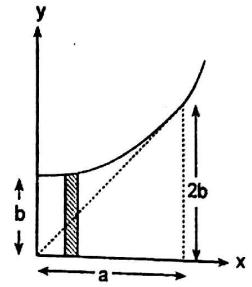
Calculation of  $I_{xx}$

Required MOI = MOI of curve - MOI of triangle

$$I_{xx} = \int \frac{1}{3} y^3 dx - \text{MOI of } \Delta$$

$$= \int_0^a \frac{1}{3} \left( \frac{b}{a^2} x^2 + b \right)^3 dx - \frac{a(2b)^3}{12}$$

$$[\Delta \text{MOI} = \frac{bh^3}{12}]$$



$$= \frac{1}{3} \int_0^a \left( \frac{b^3}{a^6} x^6 + b^3 + \frac{3b^2 x^4}{a^4} \cdot b + 3 \frac{b}{a^2} x^2 b^2 \right) dx - \frac{8ab^3}{12}$$

$$= \frac{1}{3} \left[ \frac{b^3 a^7}{a^6} + b^3 a + \frac{3b^2 a^5}{a^4} b + \frac{3b^3 a^3}{a^2} b \right] - \frac{2}{3} ab^3$$

$$= \frac{1}{3} \left[ \frac{ab^3}{7} + ab^3 + \frac{3ab^3}{5} + ab^3 \right] - \frac{2}{3} ab^3$$

$$= \frac{15ab^3 + 70ab^3 + 21ab^3}{35} - \frac{2}{3} ab^3$$

$$= \frac{96ab^3}{105} - \frac{2ab^3}{3}$$

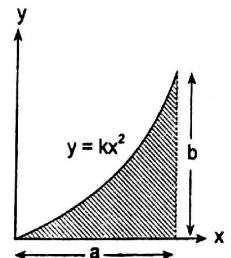
$$\therefore I_{xx} = \frac{26ab^3}{105} \text{ unit}$$

14. Find the moment of inertia about centroidal axis of the shaded plane area by using integration method.

*Solution:*

Steps:

1. Calculate the centroid of area.
2. Calculate MOI about axes
3. Transfer the MOI about axes to this centroidal axes.



In previous questions, we have calculated

$$A = \frac{ab}{3}; \quad \bar{X} = \frac{3a}{4}; \quad \bar{Y} = \frac{3b}{10}$$

$$I_{xx} = \frac{ab^3}{21}; \quad I_{yy} = \frac{a^3b}{5}$$

Now, MOI about centroidal x axis

Let  $I_{GXGX}$  = MOI about centroidal x axis

Using parallel axis theorem:

$$I_{xx} = I_{GXGX} + A(\bar{y})^2$$

$$\frac{ab^3}{21} = I_{GXGX} + \frac{ab}{3} \left( \frac{3b}{10} \right)^2$$

$$\therefore I_{GXGX} = \frac{ab^3}{21} - \frac{3ab^3}{100} = \frac{37ab^3}{100} \text{ unit}$$

MOI about centroidal y axis:

Let  $I_{GYGY}$  be the MOI about centroidal y-axis.

Then,

$$I_{yy} = I_{GYGY} + A(\bar{x})^2$$

$$\frac{a^3b}{5} = I_{GXGX} + \frac{ab}{3} \cdot \left( \frac{3a}{4} \right)^2$$

$$I_{GYGY} = \frac{a^3b}{5} - \frac{3a^3b}{16}$$

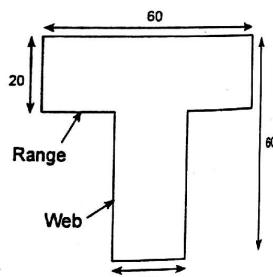
$$= \frac{16a^3b - 15a^3b}{80} = \frac{a^3b}{80} \text{ unit}$$

15. Calculate the MOI about centroidal axes. All dimension in cm.

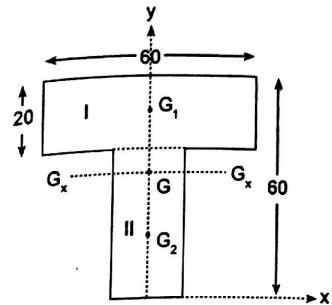
**Solution:**

[Horizontal rectangle is called flange and vertical rectangle is called web]

Here the axis is not given. So, we should first give the axes. If possible draw a axis to divide the figure symmetrically.



Required figure = Rect. I + Rect. II



Since the figure is symmetric about y-axis;  $\bar{X} = 0$ .

Calculation of  $\bar{Y}$

Segment	Area ( $A_i$ )	$\bar{y}_i$	$A_i \bar{y}_i$
Rectangle I	$60 \times 20 = 1200$	$40 + 10 = 50$	60000
Rectangle II	$40 \times 20 = 800$	20	16000
	2000		76000

$$\therefore \bar{Y} = \frac{\sum A_i \bar{y}_i}{\sum A_i}$$

$$= \frac{76000}{2000} = 38$$

$$\therefore G (0, 38)$$

MOI about centroidal x-axis

$$I_{GXGX} = \left( \frac{bh^3}{12} + Ah^2 \right)_I + \left( \frac{bh^3}{12} + Ah^2 \right)_{II}$$

$$= \frac{60 \times 20^3}{12} + 60 \times 20 \times 12^2 + \frac{20 \times 40^3}{12} + 20 \times 40 \times 18^2 \\ = 212800 + 365866$$

$$\therefore I_{GXGX} = 578666 \text{ cm}^4$$

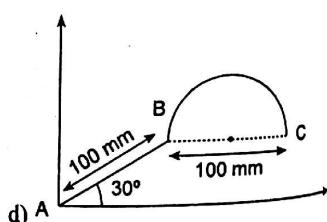
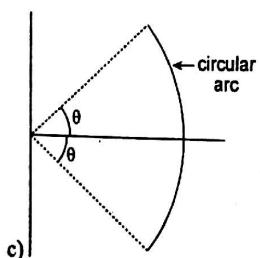
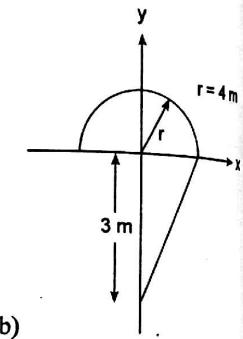
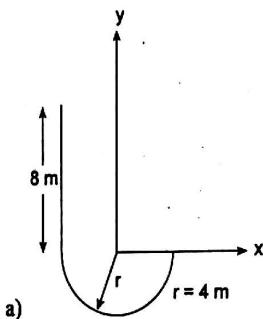
MOI about centroidal y-axis

Since the area is symmetrical about y-axis

$$\begin{aligned}
 I_{GXY} &= \left(\frac{bh^3}{12}\right) + \left(\frac{bh^3}{12}\right) \\
 &= \frac{20 \times 60^3}{12} + \frac{40 \times 20^3}{12} \\
 &= 360000 + 26666 \\
 \therefore I_{GXY} &= 386666 \text{ cm}^4
 \end{aligned}$$

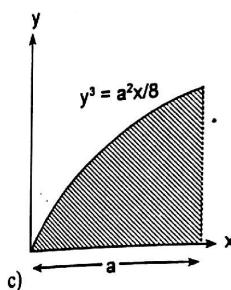
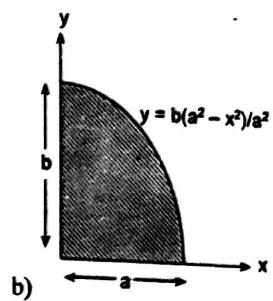
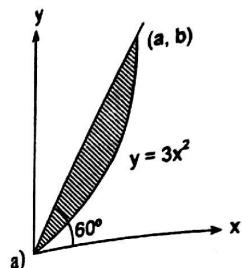
**Practice Questions**

1. Locate the centroid of composite lines



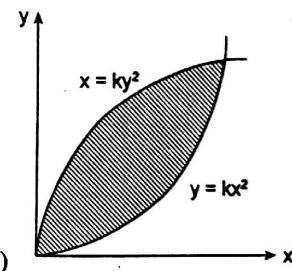
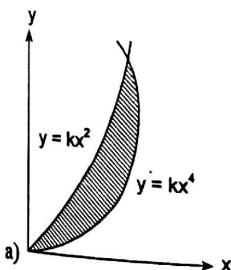
[Ans. a)  $\bar{X}, \bar{Y} = -1.57, 0$ ; b)  $\bar{X}, \bar{Y} = 0.569, 1.395$   
c)  $\bar{X} = \frac{r \sin \theta}{\theta}$ ; d)  $\bar{X} = 100.3 \text{ mm}, \bar{Y} = 59.72 \text{ mm}]$

2. Locate the centroid of the shaded area



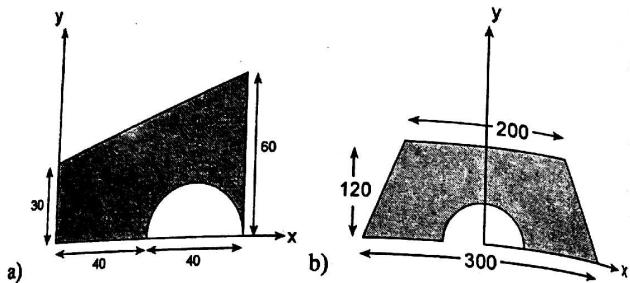
[a)  $\bar{X} = 0.289, \bar{Y} = 0.4$ ; b)  $\bar{X} = \frac{3a}{8}, \bar{Y} = \frac{2b}{5}$ ; c)  $\bar{X} = \frac{4a}{7}, \bar{Y} = \frac{a}{5}$ ]

3. Locate the centroid of shaded area:



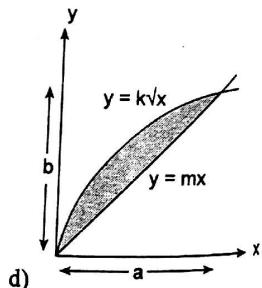
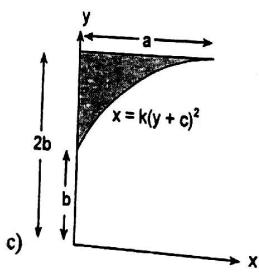
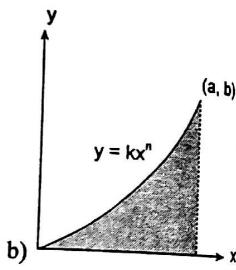
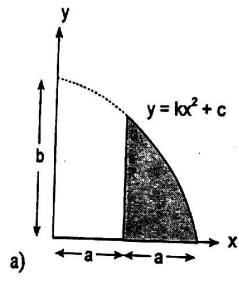
[a)  $\bar{X} = \frac{5a}{8}, \bar{Y} = \frac{b}{3}$ ; b)  $\bar{X} = 9 \text{ cm}, \bar{Y} = 9 \text{ cm}]$

4. Locate the centroid of composite area:



$$[a] \bar{X} = 41.1 \text{ cm}, \bar{Y} = 26.5 \text{ cm}; b) \bar{X} = 0, \bar{Y} = 69.1 \text{ cm}]$$

5. Find the moment of inertia about x-y axes.

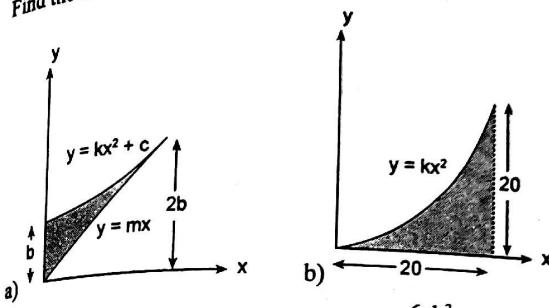


$$[Ans.: a) I_{xx} = 0.0430ab^3 \text{ unit}, I_{yy} = \frac{47}{60}a^3b \text{ unit};$$

$$b) I_{xx} = \frac{ab^3}{3(3n+1)} \text{ unit}, I_{yy} = \frac{a^3b}{n+3} \text{ unit};$$

$$c) I_{xx} = \frac{31}{30}ab^3 \text{ unit}, I_{yy} = \frac{1}{21}a^3b \text{ unit}; d) I_{xx} = \frac{ab^3}{20} \text{ unit}, I_{yy} = \frac{a^3b}{28} \text{ unit}]$$

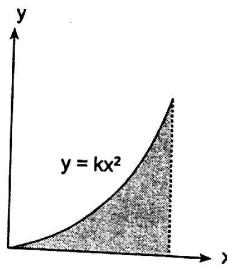
6. Find the moment of inertia about centroidal axes:



$$[Ans.: a) I_{GXGX} = \frac{6ab^3}{175} \text{ unit}, I_{GYGY} = \frac{a^3b}{80} \text{ unit};$$

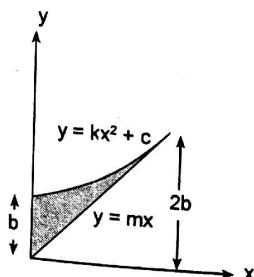
$$b) I_{GXGX} = 59200 \text{ cm}^4, I_{GYGY} = 2000 \text{ cm}^4]$$

7. Determine the radius of gyration of the shaded area about axes:



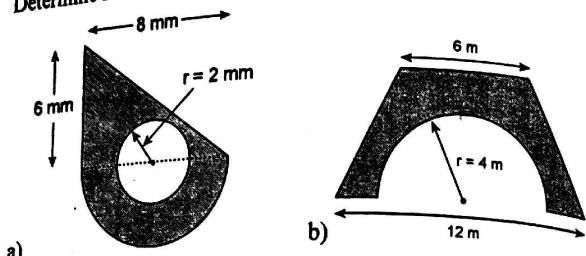
$$[Ans.: K_{xx} = \frac{b}{\sqrt{7}} \text{ unit}, K_{yy} = \frac{a\sqrt{3}}{\sqrt{5}} \text{ unit}]$$

8. Determine the polar radius of gyration about axes

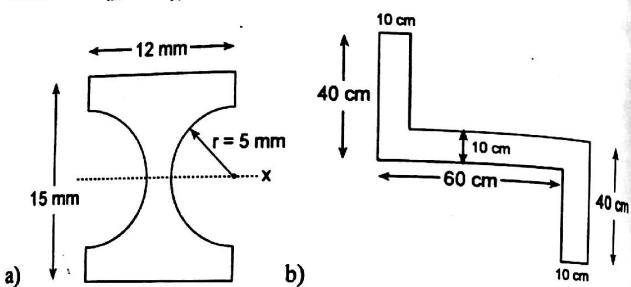


$$[Ans.: K_{zz} = \sqrt{\frac{7a^2 + 52b^2}{70}}]$$

9. Determine MOI about centroidal x-axis.

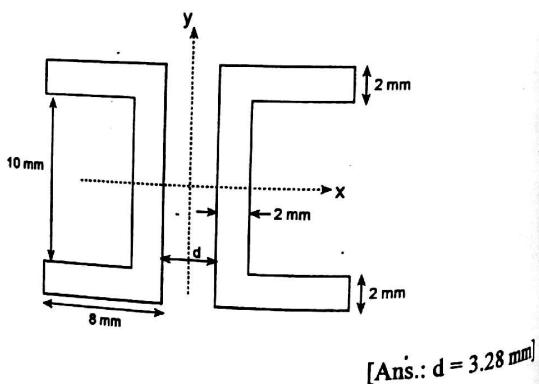


10. Determine  $I_{xx}$  and  $I_{yy}$  of centroidal axes:



$$[Ans.: a) I_{XX} = 2884.13 \text{ mm}^4, I_{YY} = 839.74 \text{ mm}^4; \\ I_{xx} = 2.9 \times 10^5 \text{ cm}^4, I_{yy} = 5.6 \times 10^5 \text{ cm}^4]$$

11. Two equal channels are kept as shown at a distance  $d$  between them. Calculate the value of distance  $d$  if the centroidal moments of inertia  $I_{xx}$  and  $I_{yy}$  are equal.



## Chapter 5

### Friction

#### 5.1 Introduction

When a body slides over another body, a force is exerted at a surface of contact by the stationary body on a moving body. This resisting force is called the force of friction and acts in the direction opposite to the direction of motion.

Friction is quite undesirable and needs to be mitigated in some machines and process like power screw, bearing, gear, fluid flow etc. where the presence of friction would cause loss of power, wearing and tearing of equipment etc. However, the working of many device such as frictional brake and clutches, belt and rope drive, holding and fastening of device depends upon friction. Friction in both a liability and a necessity and is often referred as a necessary evil.

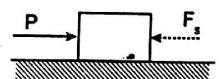
#### 5.1.1 Characteristic of friction

- It always acts in a direction opposite to that in which motion is intended.
- It exists as long as the tractive force acts. So, it is passive force.
- It is self-adjusting force i.e. only that much comes to play as is just sufficient to prevent motion.

#### 5.2 Static and Dynamic frictions

##### 5.2.1 Static friction

The friction force that develops between mating surface when subjected to external force but there is no relative motion between them is called static friction.

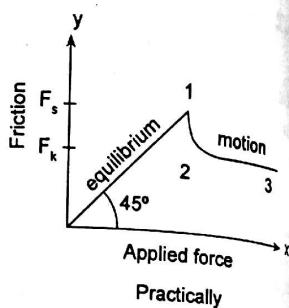
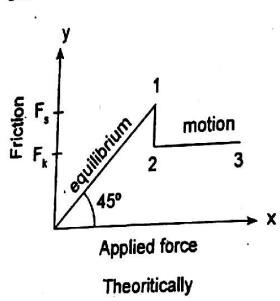


If force  $P$  acts towards right, then  $F_s$  equal to  $P$  acts in opposite direction to prevent motion. Then the force  $F_s$  is static friction.

### 5.2.2 Dynamic friction

The frictional force that develops between mating surfaces when subjected to external forces and there is relative motion between them is called dynamic friction. It is also called kinetic friction.

If force  $P$  acts towards right, then  $f_k$  less than  $P$  acts in opposite direction and the body moves with acceleration  $a$ .



### 5.3 Laws of solid friction (static or dynamic)

Based on the experiment, following laws of friction have been stated:

- Friction acts tangential to the surface in contact and is in a direction opposite to that in which motion is to take place.
- Frictional force is maximum at the instant of impending motion.
- The magnitude of limiting friction bears a constant ratio to the normal reaction between the mating surface. This ratio drops to a slightly lower value when motion starts.
- Limiting friction is independent of the area and shape of contact surface.
- Limiting friction depends upon the nature of the surface in contact.
- At low velocity, friction is independent of the velocity. But at high speed, there will be slight reduction in friction.

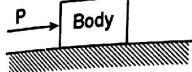
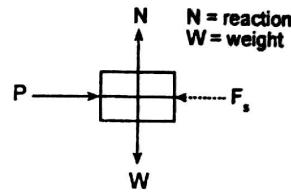
### 5.4 Terms related to friction

#### a) Coefficient of friction

The frictional force is proportional to normal reaction i.e.  $F \propto R$ . The ratio  $\frac{F}{R}$  is called coefficient of friction. So, coefficient of friction is defined as the ratio of frictional force to normal reaction.

When the system is in state of impending motion, the frictional force has maximum value  $f_s$ . If  $\mu_s$  is the coefficient of static friction, then

$$\mu_s = \frac{F_s}{R}$$



For impending motion

$$P - f_s = 0$$

$$P - \mu_s N = 0$$

$$\therefore \mu_s = \frac{P}{N} \dots\dots (i)$$

When motion starts, the maximum force of friction falls to some lower value, called kinetic friction  $F_k$ . If  $\mu_k$  is the coefficient of kinetic friction, then

$$\mu_k = \frac{f_k}{R}$$

for motion

$$P - f_k = m \cdot a \quad \text{where } m = \text{mass and } a = \text{acceleration}$$

$$P - \mu_k N = m \cdot a$$

$$\frac{P - ma}{N} = \mu_k \quad \dots\dots (ii)$$

Since  $a > 0$ , from eq. (i) and (ii)  $\mu_s > \mu_k$ . So, the coefficient of static friction is always greater than kinetic friction.

### b) Limiting friction

The maximum value of frictional force which acts on the body when it just starts to slide over another body is called limiting friction.

$$\text{Mathematically, } F_{\mu_m} = \mu_s \cdot N$$

where  $N$  is normal reaction.

### c) Impending motion

When the applied force on a body becomes equal to maximum frictional force i.e. limiting friction acting on it, the body just comes to motion. This stage of motion is called impending motion.

### d) Angle of friction

It is defined as the angle which the resultant of normal reaction and limiting frictional force makes with the normal reaction.

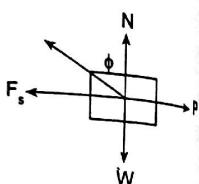
$N$  = Normal reaction

$f_s$  = Limiting friction

$R$  = Resultant of  $N$  and  $f_s$

$$= \sqrt{N^2 + f_s^2}$$

$$\therefore \tan \phi = \frac{f_s}{N} = \mu_s \text{ where } \phi \text{ is angle of friction.}$$



### e) Angle of repose

The angle  $\alpha$  of the inclined plane at which the block resting on it is about to slide down the plane is called angle of repose.

At equilibrium

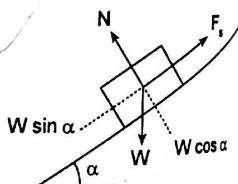
$$f_s = W \sin \alpha$$

$$\mu_s N = W \sin \alpha$$

$$\mu_s W \cos \alpha = W \sin \alpha$$

$$\mu_s = \frac{W \sin \alpha}{W \cos \alpha}$$

$$\therefore \tan \alpha = \mu_s$$



### Friction

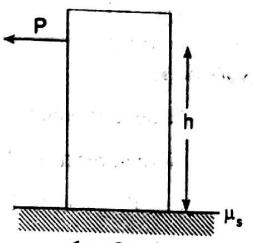
In previous, angle of friction  $\tan \phi = \mu_s$ .

So, angle of friction = angle of repose.

### 5.5 Condition of Tipping and sliding of a block

Here tipping refers to the turning of block due to the formation of couple. The condition for no tipping is  $h = \frac{a}{2\mu_s}$

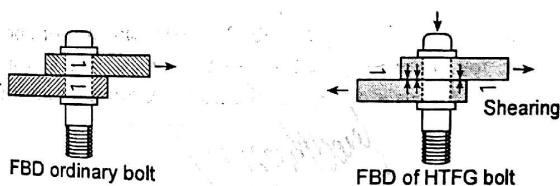
$$\text{or, } h < \frac{a}{2\mu_s} \quad \text{i.e. } h \leq \frac{a}{2\mu_s}$$



$h$  is point of application of force  $P$  above the base of block.

### 5.6 High tension friction grip bolts

High tension (strength) friction grip bolt are high strength structural bolts which have been tightened such as to induce tension in the bolt shank. Due to the tension in the bolt, the interface between the plies (steel member in joint) cannot move relative to each other because of friction resistance. The bolt act differently than normal bolts or rivets. Friction along interface takes load in case of high tension friction grip bolt subject to shear.



### Worked Out Example

- A block of mass 20 kg is rest on a horizontal floor. To slide the block a force  $P$  is applied along the horizontal. The coefficient of static friction and dynamic friction between the block and floor are stated to be 0.3 and 0.25 respectively. For the each value of  $P$  given below, workout the corresponding value of frictional force  $F$  and complete the table.

Applied force P(N)	15	30	45	<b>58.86</b>	60	75
Frictional force F (N)						

**Solution:**

$$\text{Weight of block } w = mg = 20 \times 9.81 = 196.2 \text{ N}$$

$$\text{Normal reaction } R = w = 196.2 \text{ N}$$

$$\text{Limiting friction } F_{\text{lim}} = \mu_s R = 0.3 \times 196.2 = 58.86 \text{ N}$$

The block will remain at rest till the applied force is less than 58.86 N.

When applied force is greater than  $F_{\text{lim}}$ , block will move and frictional force reduces to  $\mu_k R = 0.25 \times 196.2 \text{ N} = 49.05 \text{ N}$ .

Applied force P(N)	15	30	45	<b>58.86</b>	60	75
Frictional force F (N)	15	30	45	<b>58.86</b>	49.05	49.05

2. A wooden block of weight 100 N rests on a horizontal plane. Determine the force required to just a) pull it b) push it. Take coefficient of friction  $\mu = 0.3$  for all surface. Comment on the result.

**Solution:**

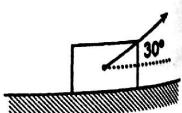
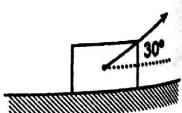
**Case a)**

Let  $F_1$  be the force required to pull.

$$\Sigma F_x = 0 \rightarrow +ve$$

$$F_1 \cos 30 - f_i = 0$$

$$F_1 \cos 30 - \mu_s R = 0 \dots\dots (i)$$



### Friction

$$\Sigma F_y = 0 \uparrow +ve$$

$$F_1 \sin 30 + R - W = 0$$

$$F_1 \sin 30 + R - 100 = 0$$

$$0.5F_1 + R = 100$$

$$\therefore R = 100 - 0.5F_1 \dots\dots (ii)$$

From (i) and (ii)

$$F_1 \frac{\sqrt{3}}{2} - 0.3(100 - 0.5F_1) = 0$$

$$0.866F_1 - 30 + 0.15F_1 = 0$$

$$1.016F_1 = 30$$

$$\therefore F_1 = 29.53 \text{ N}$$

**Case b)**

Let  $F_2$  be the force required to push.

For just motion

$$\Sigma F_y = 0 \uparrow +ve$$

$$R - W - F_2 \sin 30 = 0$$

$$R - 100 - 0.5F_2 = 0$$

$$\therefore R = 100 + 0.5F_2$$

$$\Sigma F_x = 0 \rightarrow +ve$$

$$f_i - F_2 \cos 30 = 0$$

$$\mu_s R - 0.866F_2 = 0$$

$$0.3(100 + 0.5F_2) - 0.866F_2 = 0$$

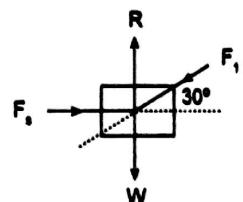
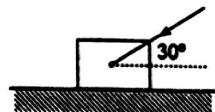
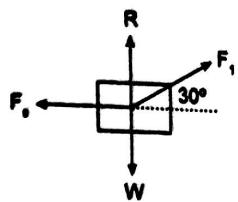
$$30 + 0.15F_2 - 0.866F_2 = 0$$

$$30 + (-)0.716F_2 = 0$$

$$0.716F_2 = 30$$

$$\therefore F_2 = 41.89 \text{ N}$$

**Comment:** Since  $F_2 > F_1$ ; it is easier to pull the block than to push it.



3. Two blocks A and B of 40 N and 20 N respectively are in equilibrium position as shown in figure. Calculate the force P required to move block A. Take  $\mu = 0.3$  for all surfaces.

**Solution:**  
First draw FBD of block A and B.

Consider block B

$$\Sigma F_y = 0 \uparrow +ve$$

$$R_B + T \sin 30 - W_B = 0$$

$$R_B + 0.5T = 20$$

$$\therefore R_B = 20 - 0.5T$$

$$\Sigma F_x = 0 \rightarrow +ve$$

$$T \cos 30 - f_B = 0$$

$$T \cos 30 - \mu_s R_B = 0$$

$$T \cos 30 - 0.3(20 - 0.5T) = 0$$

$$0.866T - 6 + 0.15T = 0$$

$$\therefore T = 5.905 \text{ N}$$

$$\therefore R_B = 20 - 0.5 \times 5.905 = 17.047 \text{ N}$$

Consider block A

$$\Sigma F_y = 0 \uparrow +ve$$

$$R_A - W_A - R_B = 0$$

$$\therefore R_A = 40 + 17.04 = 57.04 \text{ N}$$

$$\Sigma F_x = 0 \rightarrow +ve$$

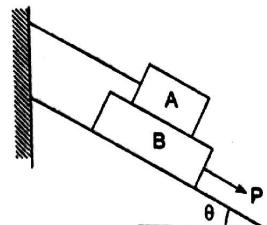
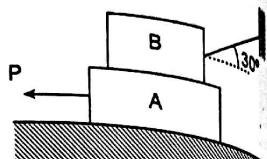
$$f_A + f_B - P = 0$$

$$\mu_s(R_A + R_B) - P = 0$$

$$0.3(57.04 + 17.047) = P$$

$$\therefore P = 22.22 \text{ N}$$

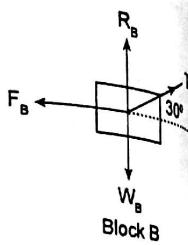
4. Block A in the figure weighs 120 N and block B weigh 200 N. The cord is parallel to inclined. If the  $\mu = 0.6$ ,  $\theta = 30^\circ$ , what force P applied to B acting down and parallel to incline will start motion. Find the value of tension in cord.



**Solution:**

Consider a block A

For a inclined plane, we use x-axis as axis parallel to incline plane and y-axis as axis perpendicular to incline plane.



$$\Sigma F_y = 0 (+ve)$$

$$R_A - W_A \cos \theta = 0$$

$$R_A - 120 \cos 30 = 0$$

$$R_A - 103.92 = 0$$

$$\therefore R_A = 103.92 \text{ N}$$

$$\Sigma F_x = 0 (+ve)$$

$$f_A - T + W_A \sin \theta = 0$$

$$\mu_s R_A - T + 120 \sin 30 = 0$$

$$0.6 \times 103.92 - T + 60 = 0$$

$$\therefore T = 122.35 \text{ N}$$

Consider a block B

Note: A body transfer reaction not the weight. So, we have to consider  $R_A$  not  $W_A$ .

$$\Sigma F_y = 0 (+ve)$$

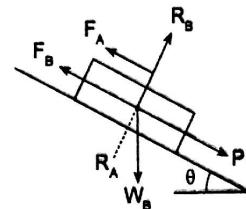
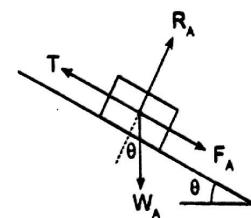
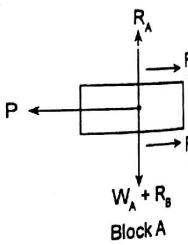
$$R_B - R_A - W_B \cos \theta = 0$$

$$R_B = 103.92 + 200 \cos 30$$

$$= 277.12 \text{ N}$$

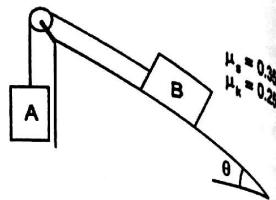
Also,  $\Sigma F_x = 0 (+ve)$

$$P - f_A - f_B + w_B \sin \theta = 0$$



$$\begin{aligned} P - \mu(R_A + R_B) + 200 \sin 30 &= 0 \\ P - 0.6(381.04) + 100 &= 0 \\ \therefore P &= 128.624 \text{ N} \end{aligned}$$

5. Knowing that  $W_A = 25 \text{ kg}$  and  $\theta = 30^\circ$ , determine  
 a) the smallest value.  
 b) the largest value of  $W_B$  for which system is in equilibrium.



**Solution:**

- a) For the smallest value of  $W_B$ , it will move upward.

$$\Sigma F_y = 0 \text{ (+ve)}$$

$$R - W_B \cos \theta = 0$$

$$R = W_B \cos 30$$

$$\Sigma F_x = 0 \text{ (+ve)}$$

$$f_B - T + W_B \sin \theta = 0$$

$$\mu_s R - 25 + W_B \sin 30 = 0 \quad [T = W_A]$$

$$0.35 \times W_B \cos 30 + W_B \sin 30 = 25$$

$$W_B (0.303 + 0.5) = 25$$

$$\therefore W_B (\min) = 31.13 \text{ kg}$$

- b) For the largest value of  $W_B$ , it will move downward.

$$\Sigma F_y = 0 \text{ (+ve)}$$

$$R - W_B \cos \theta = 0$$

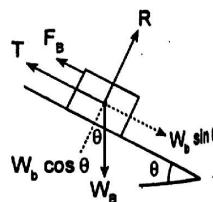
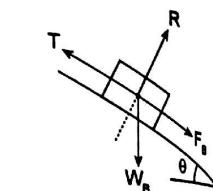
$$R = W_B \cos 30$$

$$\Sigma F_x = 0 \text{ (+ve)}$$

$$W_B \sin 30 - T - f_B = 0$$

$$W_B \times 0.5 - 25 - \mu_s R = 0$$

$$W_B \times 0.5 - 25 - 0.35(0.866 W_B) = 0$$



### Friction

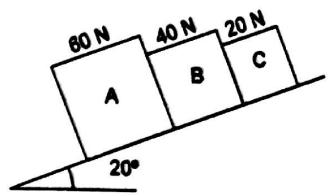
$$\therefore W_B = \frac{25}{0.1968}$$

$$\therefore W_B (\max) = 127 \text{ kg}$$

6. Three blocks are placed on a  $20^\circ$  incline in contact with each other. Determine which of the block will move and friction force on each block.

for A and C:  $\mu_s = 0.5$  and  $\mu_k = 0.4$

for B:  $\mu_s = 0.3$  and  $\mu_k = 0.2$



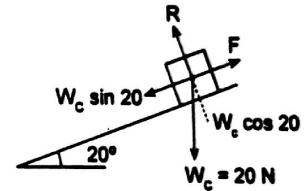
**Solution:**

Block C:

$$\begin{aligned} f_{\lim} &= \mu_s R = 0.5 \times 20 \cos 20^\circ \\ &= 0.396 \text{ N} \end{aligned}$$

$$f = 20 \sin 20 = 6.8404 \text{ N}$$

$f < f_{\lim}$ ; so block C does not move.



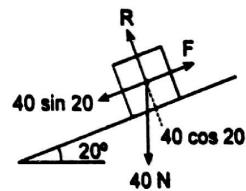
Frictional force  $= f = 6.8404 \text{ N}$

Block B:

$$\begin{aligned} f_{\lim} &= \mu_s R = 0.3 \times 40 \cos 20^\circ \\ &= 11.27 \text{ N} \end{aligned}$$

$$f = 40 \sin 20 = 13.68$$

$f > f_{\lim}$ ; so block B will move.



Frictional force  $= f = 11.27 \text{ N}$

Resulting force that pushes the block A  $= 13.68 - 11.27 = 2.4 \text{ N}$

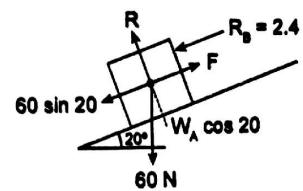
Block A:

$$\begin{aligned} f_{\lim} &= \mu_s R = 0.5 \times 60 \cos 20^\circ \\ &= 28.19 \text{ N} \end{aligned}$$

$$f - R_B - 60 \sin 20 = 0$$

$$f - 2.4 - 20.52 = 0$$

$$\therefore f = 22.92$$

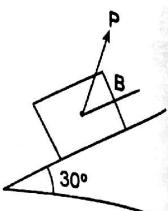


$f < f_{\text{lim}}$ ; so block A does not move.

Frictional force = 22.92 N

Since block A is in rest and it does not slide, none of the block will slide.

7. Knowing that the coefficient of friction between 25 kg block and incline is  $\mu = 0.25$ ; determine  
 a) smallest value of P required to move the block upward the inclined.  
 b) corresponding value of  $\beta$ .



**Solution:**

**Method one: Drawing method**

Replace frictional force f and normal reaction R by S where  
 $S^2 = f^2 + R^2$

For just motion:

$$\tan \phi_s = \mu_s$$

$$\tan \phi_s = 0.25$$

$$\therefore \phi_s = 14.036^\circ$$

For minimum P; P is perpendicular to S and B must equal to  $\phi_s$ .

Angle between S and W is  $30 + \phi_s = 44.036^\circ$

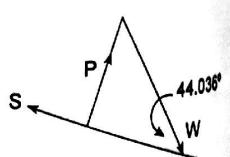
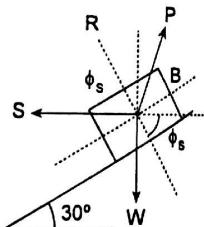
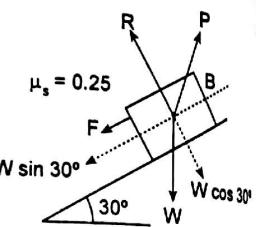
$$\therefore \sin 44.036 = \frac{P}{W}$$

$$\therefore P = P_{\min} = W \sin 44.036$$

$$= 25 \times 9.81 \times \sin 44.036$$

$$= 170.97 \text{ N}$$

$$\beta = 14.036^\circ$$



### Friction

**Alternative method:**

$$\sum F_x = 0 (+ve)$$

$$P \cos \beta - W \sin 30 - f = 0$$

$$P \cos \beta - W \sin 30 - \mu_s R = 0$$

$$P \cos \beta - 0.5 W - 0.25 R = 0 \dots\dots (i)$$

$$\sum F_x = 0 (+ve)$$

$$P \sin \beta + R - W \cos 30 = 0$$

$$P \sin \beta + R - 0.866 W = 0$$

$$\therefore R = 0.866 W - P \sin \beta \dots\dots (ii)$$

From eq. (i) and (ii)

$$\text{or, } P \cos \beta - 0.5 W - 0.25(0.866 W - P \sin \beta) = 0$$

$$\text{or, } P \cos \beta - 0.5 W - 0.2165 W + 0.25 P \sin \beta = 0$$

$$\text{or, } 0.25 P \sin \beta + P \cos \beta - 0.7165 W = 0$$

$$\text{or, } 0.25 P \sin \beta + P \cos \beta - 0.7165 \times 25 \times 9.81 = 0$$

$$\text{or, } 0.25 P \sin \beta + P \cos \beta - 175.72 = 0$$

For minimum value of P;  $\frac{dP}{d\beta} = 0$

$$\frac{d}{d\beta}(0.25 P \sin \beta + P \cos \beta - 175.72) = 0$$

$$0.25 P \cos \beta + 0.25 \sin \beta \frac{dP}{d\beta} - P \sin \beta + \cos \beta \frac{dP}{d\beta} = 0$$

$$0.25 P \cos \beta - P \sin \beta = 0$$

$$0.25 \cos \beta = \sin \beta$$

$$\therefore \tan \beta = 0.25$$

$$\therefore \beta = \tan^{-1}(0.25) = 14.036^\circ$$

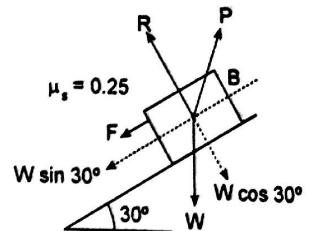
Again,

$$0.25 P_{\min} \sin \beta + P_{\min} \cos \beta - 175.72 = 0$$

$$0.25 P_{\min} \sin 14.036 + P_{\min} \cos 14.036 - 175.72 = 0$$

$$P_{\min}(0.0606 + 0.9701) = 175.72$$

$$\therefore P_{\min} = \frac{175.72}{1.0307} = 170.48 \text{ N}$$



8. Determine the minimum angle  $\theta$  at which a uniform ladder can be placed against a wall without slippage under its own weight. The coefficient of friction for all surfaces is 0.2.

**Solution:**

For equilibrium:

$$\Sigma M_A = 0 \text{ (+ve)}$$

$$\text{or, } W \frac{L}{2} \cos \theta - R_B L \sin \theta - f_B L \cos \theta = 0$$

$$\text{or } \frac{W}{2} \cos \theta - R_B \sin \theta - \mu R_B \cos \theta = 0 \dots\dots\text{(i)}$$

$$\Sigma F_y = 0 \text{ (+ve)}$$

$$R_A - W + f_B = 0$$

$$R_A - W + \mu R_B = 0$$

$$R_A + \mu R_B - W = 0 \dots\dots\text{(ii)}$$

$$\Sigma F_x = 0 \text{ (+ve)}$$

$$f_A - R_B = 0$$

$$\mu R_A = R_B \dots\dots\text{(iii)}$$

From eq. (ii) and (iii)

$$R_A + \mu R_A = W$$

$$R_A(1 + 0.04) = W$$

$$\therefore W = 1.04 R_A \dots\dots\text{(iv)}$$

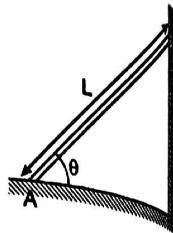
From eq. (i)

$$\frac{W}{2} \cos \theta - R_B \sin \theta - \mu R_B \cos \theta = 0$$

$$\frac{1.04 R_A}{2} \cos \theta - 0.2 R_A \sin \theta - 0.4 R_A \cos \theta = 0$$

$$0.52 \cos \theta - 0.2 \sin \theta - 0.04 \cos \theta = 0$$

$$0.48 \cos \theta = 0.2 \sin \theta$$



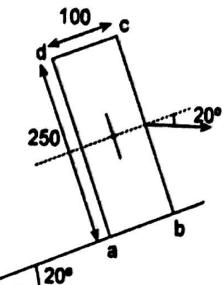
$$\therefore \tan \theta = \frac{0.48}{0.2} = 2.4$$

$$\therefore \theta = \tan^{-1} 2.4 = 67.38^\circ$$

9. The homogeneous block has a mass of 50 kg and the coefficient of friction between the block and plane is 0.4. If the force P is increased gradually until the motion ensures, will the block slide or tip and for what value of P.

**Solution:**

Let the block be represented by abcd. It is visible that the tipping of block will be about b (if tipping condition).



$$W = mg = 50 \times 9.81 = 490.5 \text{ N}$$

For equilibrium:

$$\Sigma M_b = 0 \text{ (+ve)}$$

$$P \cos 20 \times 125 - W \cos 20 \times 50 - W \sin 20 \times 125 = 0$$

$$117.47P - 23045.96 - 20970.11 = 0$$

$$117.47P = 44016.07$$

$$\therefore P = 374.7 \text{ N}$$

$$\Sigma F_y = 0 \text{ (+ve)}$$

$$R - W \cos 20 - P \sin 20 = 0$$

$$R = \sin 20 + W \cos 20$$

$$= 374.7 \sin 20 + 490.5 \cos 20$$

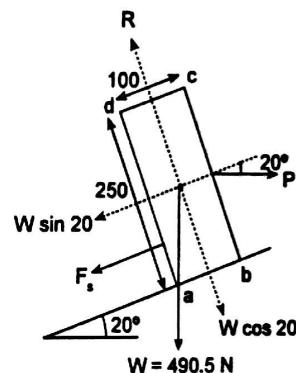
$$= 589.08 \text{ N}$$

$$\therefore \text{Limiting friction } F_{lim} = \mu R$$

$$= 0.4 \times 589.08 = 235.63 \text{ N}$$

$$\Sigma F_x = 0 \text{ (+ve)}$$

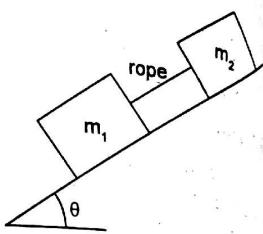
$$P \cos 20 - 2 \sin 20 - f_s = 0$$



$$\therefore f_s = 374.7 \cos 20 - 490.5 \sin 20 \\ = 352.1 - 167.7 \\ = 184.4 \text{ N}$$

Since limiting friction  $f_{lim} > f_s$ , there will not be sliding. If motion ensues, there will be tipping.

10. Two masses  $m_1 = 240 \text{ N}$  and  $m_2 = 150 \text{ N}$  are tied together by a rope parallel to the incline plane surface as shown in figure. If  $\mu_s$  for  $m_1$  is 0.3 and that for  $m_2$  is 0.55. Find:
- the value of  $\theta$  for which masses will just start slipping
  - tension in the rope.



#### Solution:

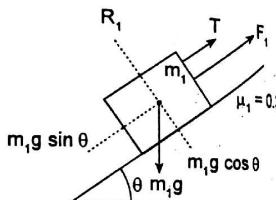
The tension in the rope will be same for both blocks. Let it be  $T$ . During just slipping, the block tends to move downward.

Consider mass  $m_1$

$$\Sigma F_y = 0 \text{ (+ve) } \uparrow$$

$$R_1 - m_1 g \cos \theta = 0$$

$$\therefore R_1 = m_1 g \cos \theta \text{ N}$$



$$\Sigma F_x = 0 \text{ (+ve) } \rightarrow$$

$$T + f_1 - m_1 g \sin \theta = 0$$

$$T + \mu_1 R_1 - m_1 g \sin \theta = 0$$

$$T + 0.3 m_1 g \cos \theta - m_1 g \sin \theta = 0$$

$$\therefore T = m_1 g (\sin \theta - 0.3 \cos \theta)$$

$$= 240(\sin \theta - 0.3 \cos \theta) \dots\dots (i)$$

#### Friction

Consider mass  $m_2$

$$\Sigma F_y = 0 \text{ (+ve) } \uparrow$$

$$R_2 - m_2 g \cos \theta = 0$$

$$R_2 = m_2 g \cos \theta = 150 \cos \theta$$

$$\Sigma F_x = 0 \text{ (+ve) } \rightarrow$$

$$f_2 - T - m_2 g \sin \theta = 0$$

$$\mu_2 R_2 - 240 (\sin \theta - 0.3 \cos \theta) - 150 \sin \theta = 0$$

$$\text{or, } 0.55(150 \cos \theta) - 140 \sin \theta + 72 \cos \theta - 150 \sin \theta = 0$$

$$\text{or, } 82.5 \cos \theta - 240 \sin \theta + 72 \cos \theta - 150 \sin \theta = 0$$

$$\text{or, } 390 \sin \theta = 154.5 \cos \theta$$

$$\text{or, } \tan \theta = \frac{154.5}{390} = 0.3961$$

$$\therefore \theta = \tan^{-1} 0.3961 = 21.62^\circ$$

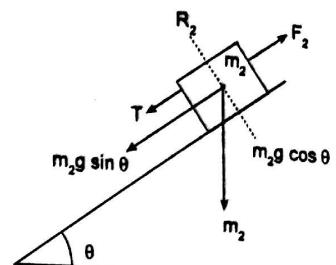
Again,

$$T = 240 (\sin \theta - 0.3 \cos \theta)$$

$$= 240 (0.368 - 0.2788)$$

$$= 240 \times 0.0891$$

$$= 21.4 \text{ N}$$

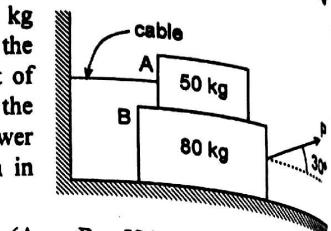


#### Practice Question

- A body of weight 100 N rests on a rough horizontal surface ( $\mu = 0.3$ ) and is acted upon by a force applied at an angle of  $30^\circ$  to the horizontal. What force is required to just cause the body to slide over the surface? Also, determine the inclination and magnitude of minimum force required to set the block into impending motion.

(Ans.: 29.53 N,  $16.7^\circ$ , 28.73 N)

2. Two blocks A and B weighing 50 kg and 80 kg are in equilibrium in the position as shown. The coefficient of friction for all surfaces = 0.3. Find the value of P required to move the lower block B and also find the tension in cable.



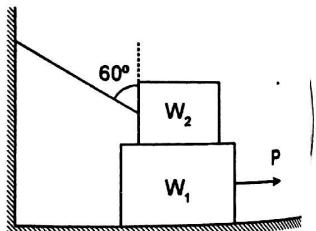
(Ans.:  $P = 521.4 \text{ N}$ ,  $T = 147.15 \text{ N}$ )

3. Two blocks of weight  $W_1$  and  $W_2$  rest on a rough inclined plane and are connected by a string. The coefficient of friction between these blocks and the plane is  $\mu_1$  and  $\mu_2$  respectively. Show that for the impending motion, the inclination is given by  $\tan \theta = \frac{\mu_1 W_1 + \mu_2 W_2}{W_1 + W_2}$

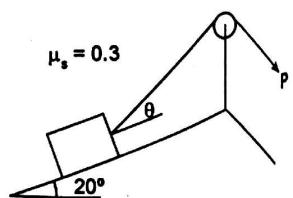
4. A uniform ladder of weight 250 N and length 5 m is placed against a vertical wall in a position where its inclination to horizontal is  $60^\circ$ . A man weighing 800 N climbs the ladder. If the coefficient of friction for all surfaces = 0.2; at what position along a ladder will he induce slippage?

(Ans.: 1.66 m from base)

5. A block of weight  $W_1 = 1800 \text{ N}$  rests on a horizontal surface and supports on the top of it another block of weight  $W_2 = 1000 \text{ N}$  as shown in figure. The coefficient of friction for all surfaces is 0.4. Find the magnitude of P to slide lower block.

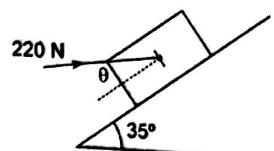


6. Knowing that  $\theta = 25^\circ$ , determine the range of values of P for which equilibrium is maintained.



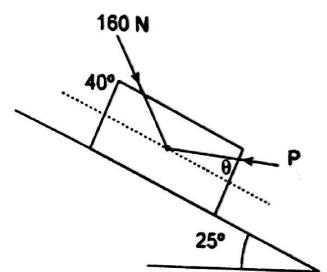
(Ans.:  $7.56 \leq P \leq 59.2 \text{ N}$ )

7. Determine the smallest value of  $\theta$  for which the block of 20 kg is in equilibrium. Take  $\mu_s = 0.3$ .



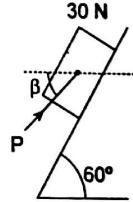
(Ans.:  $\theta = 28.9^\circ$ )

8. Neglecting the mass of block, determine  
 a) the smallest value of P for which the block is in equilibrium.  
 b) the corresponding value of  $\theta$ .  
 Take  $\mu_s = 0.35$ .



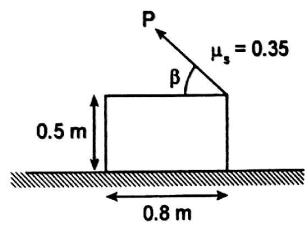
(Ans.:  $P_{\min} = 81.7 \text{ N}$ ,  $\theta = 19.29^\circ$ )

9. For the 30 N block, determine  
 a) the smallest value of P required to maintain equilibrium.  
 b) the corresponding value of  $\beta$ .



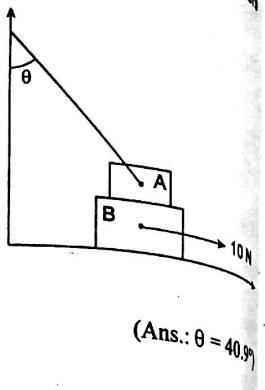
(Ans.:  $P_{\min} = 21.6 \text{ N}$ ,  $\beta = 46^\circ$ )

10. A block of mass 40 kg must be moved to the left along the floor without tipping. Determine  
 a) largest allowable value of  $\beta$ .  
 b) the corresponding value of P.



(Ans.:  $\beta = 58.1^\circ$ ,  $P = 166.4 \text{ N}$ )

11. The 16 N block A is attached to link AC and rests on the 24 N block B. Determine the value of  $\theta$  for which motion of block B is impending. Take  $\mu = 0.2$  for all.



## Chapter 6

### Analysis of Beam and Frame

#### 6.1. Introduction

A structural member designed to support loads applied at various point along with the member is known as beam. In most cases, the loads are perpendicular to the axis of beam and will cause only shear and bending in the beam. When the load are not at right angle to the beam, they will also produce axial force in the beam.

Frames are simply the combination of the network of horizontal beam and vertical column. They are designed to carry the loads and transfer it to the support.

#### 6.2 Objective of structural design

The structural design should have following main objectives:

- a) **Strength:** To resist safely the stresses induced by the loads in various section.
- b) **Stability:** To prevent overturning, sliding or buckling of structure under the action of load.
- c) **Serviceability:** To ensure satisfactory performance under load.

#### 6.3 Discrete and continuum

Various assumption are made during the application of principles of mechanics to practical problems on the basis of assumption, a structure may be:

- a) **Continuum:** In this idealization, the elements are assumed to be distributed continuously throughout the surface. The solutions of this structures are represented by partial differential equations and are more complex.
- b) **Discrete (Skeletal):** In this idealization, the elements are assumed to be composed of finite number of small element. Depending upon the

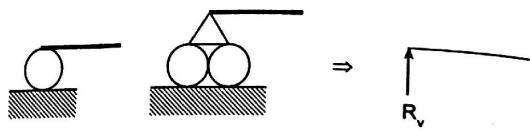
resources and time availability, we divide the structure into number of small compound i.e. in basic 1-D element (line element), 2-D element (basic triangle element) and 3-D element (cube or parallelepiped). Larger the number of discrete elements, more accurate will be the result. This assumption makes the analysis easier.

#### 6.4 Types of support

Basically there are three types of supports. These are based on natures of independent displacement called degree of freedom.

##### a) Roller support

It gives rise to one force which is perpendicular to the plane supporting the plane.



No. of reaction = 1

##### b) Hinge support

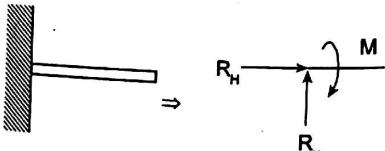
It gives rise to one force reaction whose direction is unknown. It can be resolved into two forces along x and y axes.



No. of reaction = 2

##### c) Fixed support

It gives rise to one force reaction (having two components) and one reaction moment.



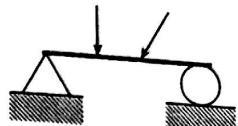
No. of reaction = 3

#### 6.5 Classification of loads

Different types of loads are:

##### a) Point load:

It is assumed to act at a point. It may be vertical, horizontal or inclined. Unit: kg, N, kN.

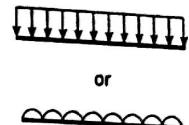


##### b) Uniformly distributed load (UDL):

Load is distributed along the structure uniformly. It is in rectangular form. Unit: kg/m, kN/m.

Total load is calculated as  

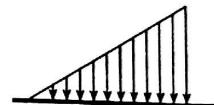
$$W = \text{base} \times \text{height} = \text{Area of rectangle}$$



##### c) Uniformly varying load (UVL):

Load which varies with uniform intensity. Intensity varies with respect to length. Unit: kg/m, kN/m

Total load =  $\frac{1}{2} \times \text{base} \times \text{height} = \text{Area of triangle}$



##### d) Couple:

It is the combination of two equal and opposite forces separated by certain distance. Unit: Nm.

##### e) Moment:

It is the product of force and distance. Unit: Nm

##### f) Static and Dynamic load:

Static load are gradually applied load. They do not give rise to vibration whereas the dynamic loads give rise to load and vibration as well.

##### g) Imposed load:

Extra load like wind, rain, seismic load etc.

##### h) Dead and live load:

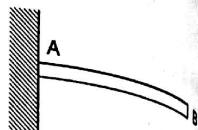
Dead load is the self-load or load of structure itself whereas the live load are movable load.

## 6.6 Classification of beams

The beams are classified as:

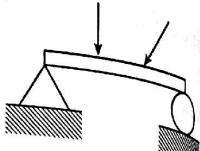
### a) Cantilever beam:

A cantilever is a beam whose one end is fixed and the other end is free. AB is the length of cantilever.



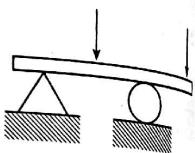
### b) Simply supported beam:

A beam which rests as one hinge support and other roller support.



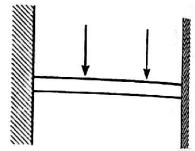
### c) Over handing beam:

An overhanging beam is one or both the ends projects beyond the support.



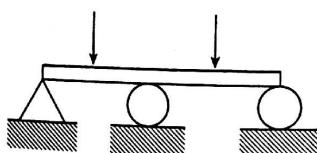
### d) Fixed beam:

A beam whose both ends are rigidly fixed or built into its supporting walls and columns.



### e) Continuous beam:

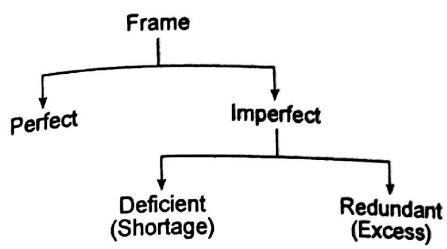
This have more than two supports.



## 6.7 Classification of frame

A frame is the network of beams and columns joined together to carry load and transfer to the support. If the joint is rigid, the frame is rigid.

jointed frame and if the joints are hinge connected, it is called pin jointed frame.



### a) Perfect frame:

There is sufficient number of support connections and structural member. It is determinate and stable both statically and geometrically.

### b) Deficient:

It may lack either structural member or support connection or both.

### c) Redundant:

There may be either excess of support connections or member or both.

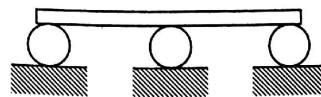
## 6.8 Concept of beams and frames

### 6.8.1 Stability

The structure is stable if and only if it is statically and geometrically stable.

a) **Statical stability:** This is the necessary condition for the stability. If unknown quantities are equal to the equations of static equilibrium, then the structure is said to be statically stable.

b) **Geometrical stability:** This is the sufficient condition for the stability. There must be at least one end fixed or combination of one hinge and one roller.



It is determinate and statically stable but geometrically unstable since it cannot resist inclined load.

### 6.8.2 Static determinacy and indeterminacy

A structure is said to be statically determinate if all the internal member forces and reactions can be determined using the equation of static equilibrium.

For beam:

$$\text{External indeterminacy} = r - (3 + c)$$

$$\text{Internal indeterminacy} = 0$$

$$\begin{aligned}\text{Total degree of static indeterminacy} &= r - (3 + c) + 0 \\ &= r - (3 + c)\end{aligned}$$

where  $r$  = support reaction and  $c$  = equation due to condition (hinge).

For frame:

$$\text{External indeterminacy} = r - (3 + c)$$

$\text{Internal indeterminacy} = 3 \times \text{total no. of cuts required to have open configuration}$

Directly, you can use

$$\text{Total degree of static indeterminacy} = (3m + r) - (3j + c)$$

To get internal indeterminacy you can subtract external indeterminacy from total.

### 6.8.3 Kinematic determinacy and indeterminacy

It is related with the joint displacements. It is the total number of freedom that are allowed for the joints in a structure.

For beam:

We can find degree of kinematic indeterminacy by counting the freedom at joints.

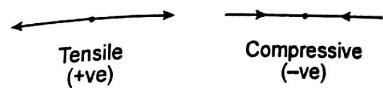
For frame:

$$\text{Degree of kinematic indeterminacy (DKI)} = 3j - (m + r)$$

### 6.9 Calculation of axial force, shear force and bending moment and drawing AFD, SFD and BMD

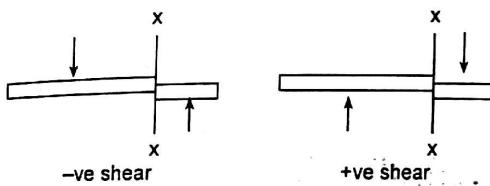
#### 6.9.1 Axial force (AF)

At any cross-section of member it is the algebraic sum of all forces acting parallel to the longitudinal axis on either side of the section. A tensile is positive while compressive is negative.



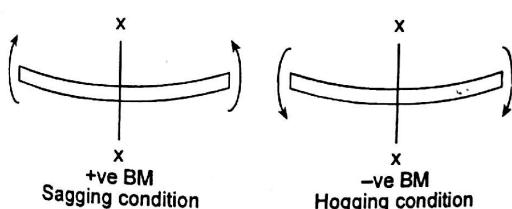
#### 6.9.2 Shear force (SF)

It is the algebraic sum of all forces acting transverse to the member on either side of the section.



#### 6.9.3 Bending moment (BM)

It is the algebraic sum of the moments of all forces on either side of the section.



The beam in hogging condition is subjected to negative bending moment and one in a sagging condition is subjected to positive bending moment.

### 6.9.4 Axial force, shear force and bending moment diagrams (AFD, SFD and BMD)

The graphical representation of axial force, shear force and bending moment which are introduced in the structural members are called AFD, SFD and BMD. Values of axial force, shear force and bending moment are plotted along y-axis corresponding to the length of the member along x-axis.

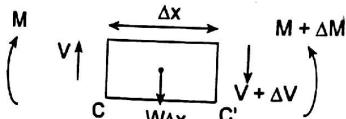
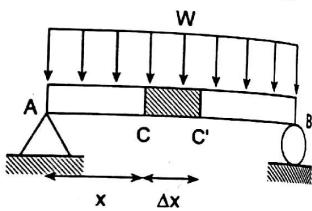
### 6.9.5 Important properties of SF and BM:

- BM is zero at simple support.
- Under UDL, SFD varies linearly but BMD varies parabolically.
- Maximum BM occurs at point of zero shear.
- The point of zero BM is known as point of contraflexure or inflection point or point of virtual hinge.
- Abrupt change in loading cause change in slope of SFD and BMD. Concentrated load produce vertical lines in SFD and BMD.

### 6.9.6 Relation among load, shear and bending moment.

Consider a simply supported beam AB carrying distributed  $W$  per unit length. Let C and C' be the two points on the beam at a distance of  $\Delta x$  from each other. The shear and BM at C will be denoted by V and M and is assumed positive. Therefore, the shear and BM at C' will be denoted by  $v + \Delta v$  and  $M + \Delta M$  respectively.

Let us detach the portion of beam CC' and redraw the free body diagram FBD.



The force exerted on the free body include a load of magnitude  $W\Delta x$  and internal forces.

### a) Relation among load and shear:

$$\sum F_y = 0 \uparrow ve$$

$$V - (V + \Delta V) - W\Delta x = 0$$

$$\Delta V = -W\Delta x$$

Dividing both side by  $\Delta x$  and let  $\Delta x \rightarrow 0$ ;

$$\lim_{\Delta x \rightarrow 0} \frac{\Delta V}{\Delta x} = -W$$

$$\frac{dv}{dx} = -W$$

indicates for a beam loaded as shown. Slope of shear curve is negative and absolute value of slope at any point equals to load per unit length at that point.

Integrating

$$\int_A^B dv = - \int_{x_A}^{x_B} W dx$$

$$V_B - V_A = -(Area \text{ under load curve})$$

### b) Relation among shear and BM

$$\sum M_c' = 0 +ve$$

$$M + \Delta M - M - V\Delta x + W\Delta x \frac{\Delta x}{2} = 0$$

$$\Delta M = V\Delta x - \frac{1}{2} W(\Delta x)^2$$

Dividing the equation by  $\Delta x$  and  $\Delta x \rightarrow 0$

$$\lim_{\Delta x \rightarrow 0} \frac{\Delta M}{\Delta x} = V$$

$$\frac{dM}{dx} = V$$

This indicates that slope of bending moment curve equals to value of shear. The equation shows that BM is maximum when shear force is zero. This property tells where the beam is likely to fall under bending.

Integrating,

$$\int_A^B dM = \int_{x_A}^{x_B} V dx$$

$M_B - M_A$  = area under shear curve

## Worked Out Examples

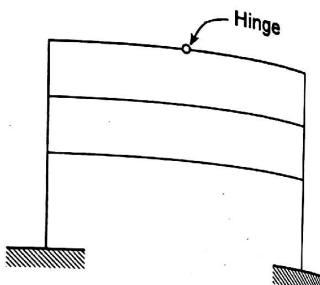
1. Calculate external, internal and total statical indeterminacy of given frame.

**Solution:**

Here,  $m = 9, j = 8, r = 6$  (fixed support),  $c = 1$  (hinge)

External indeterminacy

$$= r - (3 + c) \\ = 6 - (3 + 1) = 2$$



Total indeterminacy =  $(3m + r) - (3j + c)$

$$= (3 \times 9 + 6) - (3 \times 8 + 1) = 8$$

$\therefore$  Internal indeterminacy =  $8 - 2 = 6$

2. Calculate external indeterminacy of given beam.

**Solution:**

Here,  $r = 4$  (for hinge support)

$c = 1$  (hinge point)

External indeterminacy =  $r - (3 + c)$

$$= 4 - (3 + 1) = 4 - 4 = 0$$

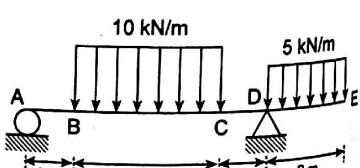
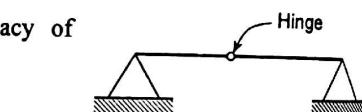
3. Draw SFD, BMD of the given beam loaded as shown.

**Solution:**

Draw FBD:

$$R_{AV} \rightarrow Rx^n \text{ at A}$$

$$R_{DH} \rightarrow Rx^n \text{ at D horizontal}$$



$R_{DV} \rightarrow Rx^n$  at D vertical

Calculation of  $Rx^n$

$\sum M_A = 0$  +ve

$$-(5 \times 2) \cdot (5 + \frac{2}{2}) + R_{DV} \times 5 - (10 \times 3) (1 + \frac{3}{2}) = 0$$

$$-60 + 5R_{DV} - 75 = 0$$

$$\therefore R_{DV} = 27 \text{ kN} (\uparrow)$$

$\sum F_y = 0$   $\uparrow$  +ve

$$R_{AV} - 30 + R_{DV} - 10 = 0$$

$$R_{AV} - 40 + 27 = 0$$

$$R_{AV} = 13 \text{ kN} (\uparrow)$$

$\sum F_x = 0$   $\rightarrow$  +ve

$$R_{DH} = 0$$

Calculation of shear force and BM

Let  $V_x$  represent shear force and  $M_x$  represent moment at section.

- (i) For span AB [ $0 \leq x \leq 1\text{m}$ ]

$$V_x = 13$$

$$x = 0; V_A = 13 \text{ kN}$$

$$x = 1; V_B = 13 \text{ kN}$$

$$M_x = 13x$$

$$x = 0; M_A = 0$$

$$x = 1; M_B = 13 \text{ kNm}$$

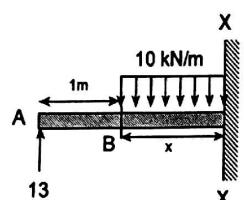
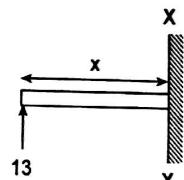
- (ii) For Span BC [ $0 \leq x \leq 3\text{m}$ ]

$$V_x = 13 - 10x$$

$$x = 0; V_B = 13 \text{ kN}$$

$$x = 3; V_C = -17 \text{ kN}$$

Here values of shear force varies from +ve to -ve; so check for zero-shear



$$V_x = 13 - 10x$$

$$0 = 13 - 10x$$

$$\therefore x = 1.3 \text{ m}$$

It is 1.3 m from B towards C.

$$M_x = 13(1+x) - (10x) \frac{x}{2}$$

$$= 13 + 13x - 5x^2$$

$$x = 0; M_B = 13 \text{ kNm}$$

$$x = 3; M_C = 13 + 13 \times 3 - 5 \times 9$$

$$= 7 \text{ kNm}$$

$$x = 1.3; M_{\max} = 21.45 \text{ kNm}$$

For span ED [0 ≤ x ≤ 2m]

For our simplicity, we have solved few span parts from left to right and will solve few span from right to left. But you can also solve either totally from left to right or right to left. The important thing is sign convention. Be sure you have used proper sign convention as stated.

(iii) For span ED [0 ≤ x ≤ 2m]

$$V_x = 5x$$

$$x = 0; V_E = 0$$

$$x = 2; V_D = 10 \text{ kN}$$

$$M_x = -5x \frac{x}{2} = -2.5x^2$$

$$x = 0; M_E = 0$$

$$x = 2; M_D = -10 \text{ kNm}$$

Here  $M_x = -2.5x^2$  represent the curve. So to plot we have to check whether it will be concave upward or concave downward. We will have two approach.

#### a) Straight line

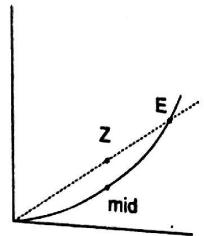
Calculate the moment  $M_x$  at middle of span.

at  $x = 1; M_{\text{mid}} = -2.5 \times 1 = -2.5 \text{ kN} = \text{point Z (say)}$

Now, If moments value three point E, Z and D forms straight line

$$M_Z = \frac{M_D + M_E}{2}$$

$$= \frac{0 - 10}{2} = -5 \text{ kNm}$$



$M_Z > M_{\text{mid}}$ ; so the curve is concave upward.

#### b) Calculus approach

Since we have taken span from right to left, -ve sign is to be neglected for equation.

$$M_x = 2.5x^2$$

$$dM_x = 5x$$

$$d^2M_x = 5$$

which is positive and gives the minimum value and is concave upward.

(iv) For span DC [0 ≤ x ≤ 1m]

$$V_x = 10 - 27 = -17$$

$$x = 0; V_D = -17 \text{ kN}$$

$$x = 1; V_C = -17 \text{ kN}$$

$$M_x = -10(1+x) + 27x$$

$$= -10 - 10x + 27x$$

$$= 17x - 10$$

$$x = 0; M_D = -10 \text{ kNm}$$

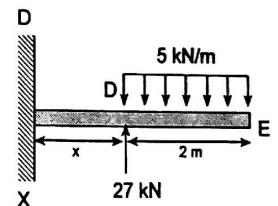
$$x = 1; M_C = 7 \text{ kNm}$$

Here values of BM varies from -ve to +ve, so check for zero BM.

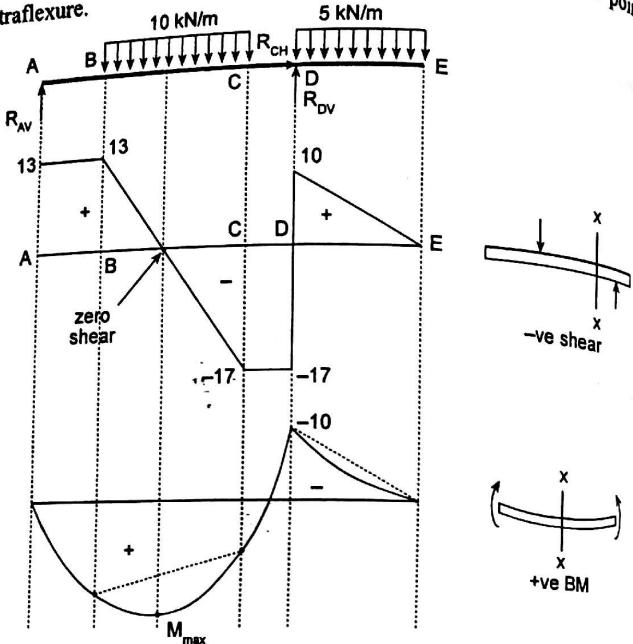
$$M_x = 17x - 10$$

$$0 = 17x - 10$$

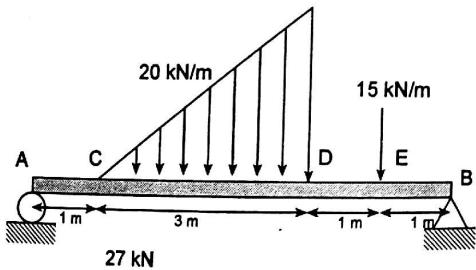
$$x = \frac{10}{17} = 0.59$$



It is 0.59 m from D towards C. This point is known as point of contraflexure.



4. Draw SFD and BMD of the beam loaded as follows:



**Solution:**

Calculation of reaction:

$$\sum M_A = 0 \text{ +ve}$$

$$\left(\frac{1}{2} \times 3 \times 20\right) \cdot \left(1 + \frac{2}{3} \cdot 3\right) + 15 \times 5 - R_{BV} \times 6 = 0$$

$$30 \times 3 + 15 \times 5 - 6 R_{BV} = 0$$

$$\therefore R_{BV} = 27.5 (\uparrow) \text{ kN}$$

$$\sum F_y = 0 \uparrow +\text{ve}$$

$$R_{AV} - \frac{1}{2} \times 3 \times 20 - 15 + R_{BV} = 0$$

$$R_{AV} - 30 - 15 + 27.5 = 0$$

$$R_{AV} = 17.5 \uparrow \text{ kN}$$

#### Calculation of shear force and BM

For span AC [0 ≤ x ≤ 1m]

$$V_x = 17.5$$

$$x = 0; V_A = 17.5 \text{ kN}$$

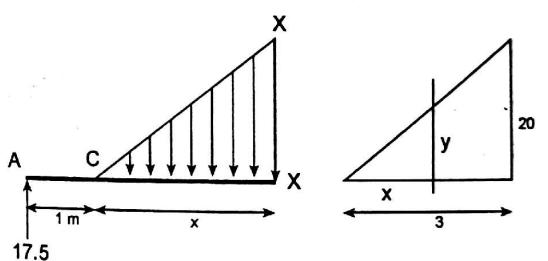
$$x = 1; V_B = 17.5 \text{ kN}$$

$$M_x = 17.5x$$

$$x = 0; M_A = 0$$

$$x = 1; M_C = 17.5 \text{ kNm}$$

(ii) For span CD [0 ≤ x ≤ 3m]



$$V_x = 17.5 - \frac{1}{2} \times 20 \frac{x}{3}$$

$$= 17.5 - \frac{10}{3} x^2$$

$$x = 0; V_C = 17.5 \text{ kN}$$

$$x = 3; V_D = -12.5 \text{ kN}$$

Values of shear varies from +ve to -ve, so check for zero shear.

$$0 = 17.5 - \frac{10}{3}x^2$$

$$\frac{10}{3}x^2 = 17.5$$

$$x = 2.29 \text{ m}$$

It is from C to D.

$$M_x = 17.5(1+x) - \frac{1}{2} \times \frac{20x}{3} \times \frac{x}{3}$$

$$M_x = 17.5 + 17.5x - \frac{10}{9}x^3$$

$$x = 0; M_C = 17.5 \text{ kNm}$$

$$x = 3; M_D = 40 \text{ kNm}$$

$$x = 2.29, M_{\max} = 44.23$$

$$x = 1.5; M_{\text{mid}} = 40 \text{ kNm}$$

$$\frac{M_C + M_D}{2} = 27.75 \text{ kNm}$$

(iii) For span BE [0 ≤ x ≤ 1m]

$$V_x = -27.5$$

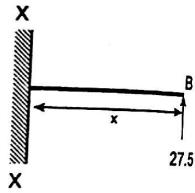
$$x = 0, V_B = -27.5 \text{ kN}$$

$$x = 1, V_C = -27.5 \text{ kN}$$

$$M_x = 27.5x$$

$$x = 0, M_B = 0$$

$$x = 1, M_E = 27.5 \text{ kNm}$$



(iv) For span ED [0 ≤ x ≤ 1m]

$$V_x = -27.5 + 15 = -12.5 \text{ kN}$$

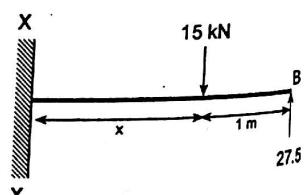
$$x = 0, V_E = -12.5 \text{ kN}$$

$$x = 1, V_D = -12.5 \text{ kN}$$

$$M_x = 27.5(1+x) - 15x \\ = 27.5 + 12.5x$$

$$x = 0, M_E = 27.5 \text{ kNm}$$

$$x = 1, M_D = 40 \text{ kNm}$$



### Alternative method of span DC

Sometimes the triangular load has to be solved by making trapezoid.

For span DC [0 ≤ x ≤ 3m]

Before shear force calculation the height of trapezium at section X, let say it is y.

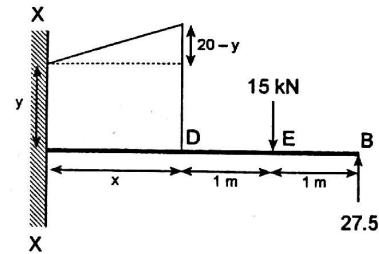
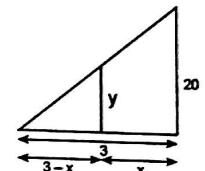
$$\frac{20}{y} = \frac{3}{3-x}$$

$$y = \left(\frac{3-x}{3}\right)20$$

Now the trapezoid can be divided into triangle and rectangle as shown.

$$V_x = -27.5 + 15 + xy + \frac{1}{2}x(20-y)$$

$$V_x = -12.5 + x\left(\frac{3-x}{3} \cdot 20\right) + \frac{1}{2}x\left(20 - \frac{3-x}{3} \cdot 20\right)$$



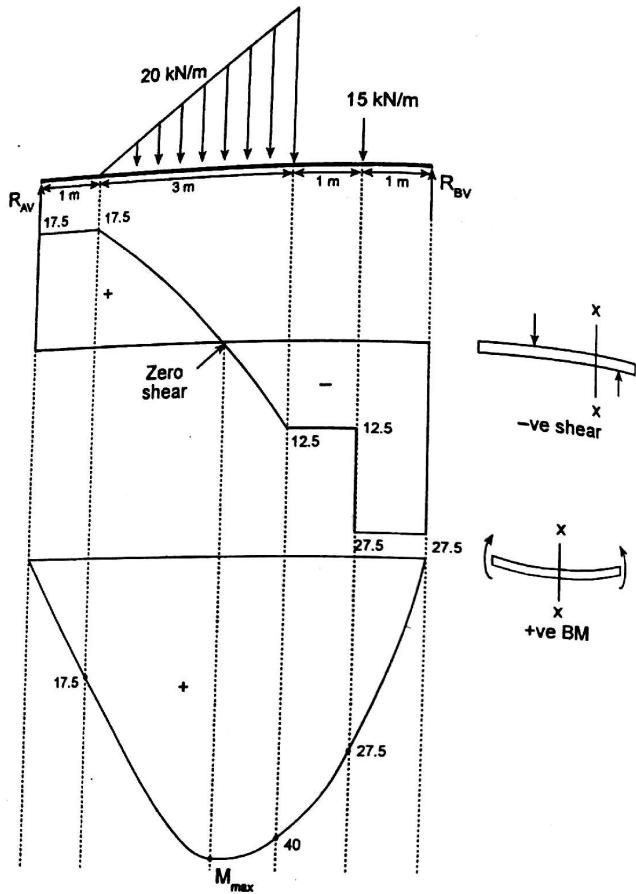
(you can solve it)

$$M_x = 27.5(2+x) - 15(1+x) - \frac{1}{2}x(20-y)\frac{2x}{3} - xy\frac{x}{2}$$

$$\text{Put } y = \frac{3-x}{3} \cdot 20$$

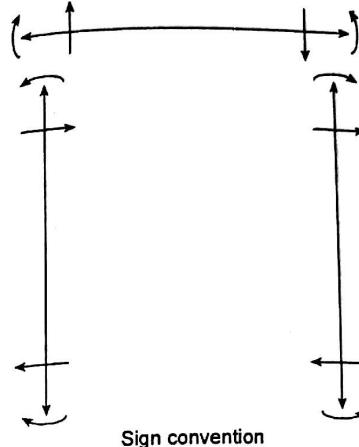
$$M_x = \frac{10}{9}x^3 - 10x^2 + 12.5x + 40$$

(you can solve now.)



### Analysis of frame

Assume the individual elements of frame should be in equilibrium if whole frame is in equilibrium.



5. Draw AFD, SFD and BMD for the given frame.

**Solution:**

Calculation of reactions:

$$\sum M_E = 0 \text{ +ve}$$

$$-R_{AV} \times 5 - 20 \times 3 + 25 \times \frac{5}{2} = 0$$

$$-5R_{AV} - 60 + 62.5 = 0$$

$$-5R_{AV} = -2.5$$

$$\therefore R_{AV} = 0.5 \text{ kN} (\uparrow)$$

$$\sum F_y = 0 \uparrow +\text{ve}$$

$$R_{AV} - 5 \times 5 + R_{EV} = 0$$

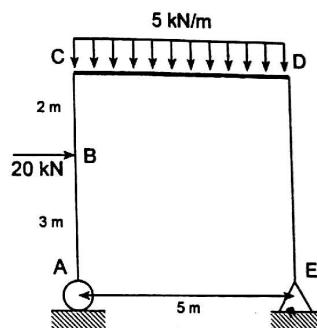
$$0.5 - 25 + R_{EV} = 0$$

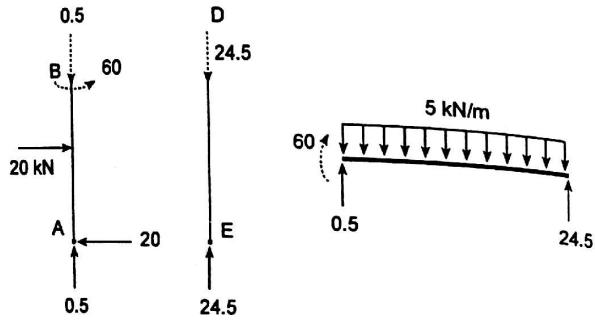
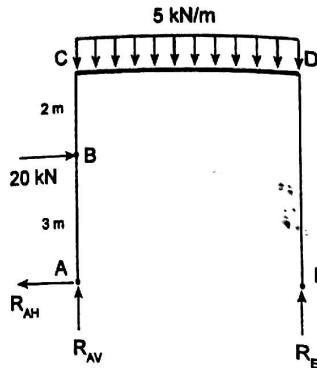
$$\therefore R_{EV} = 24.5 \text{ kN} (\uparrow)$$

$$\sum F_x = 0 \rightarrow +\text{ve}$$

$$20 - R_{AH} = 0$$

$$\therefore R_{AH} = 20 \text{ kN}$$





(i) For span AB [0 ≤ x ≤ 3m]

$$V_x = 20$$

$$x = 0, V_A = 20 \text{ kN}$$

$$x = 3; V_B = 20 \text{ kN}$$

$$M_x = 20x$$

$$x = 0; M_A = 0$$

$$x = 3; M_B = 60 \text{ kNm}$$

(ii) For span BC [0 ≤ x ≤ 2m]

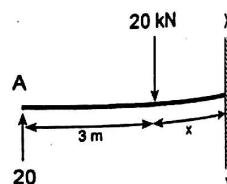
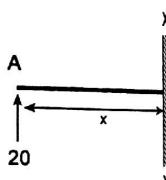
$$V_x = 20 - 20 = 0$$

$$V_B = V_C = 0$$

$$M_x = 20(3 + x) - 20x = 60$$

$$x = 0; M_B = 60 \text{ kNm}$$

$$x = 2; M_C = 60 \text{ kNm}$$



(iii) For span CD [0 ≤ x ≤ 5m]

$$V_x = 0.5 - 5x$$

$$x = 0; V_C = 0.5 \text{ kN}$$

$$x = 5; V_D = -24.5 \text{ kN}$$

Here shear force varies from +ve to -ve; so check for zero shear

$$V_x = 0.5 - 5x$$

$$0 = 0.5 - 5x$$

$$5x = 0.5$$

$$x = 0.1 \text{ m}$$

This is 0.1 m from C to D.

$$M_x = 60 + 0.5x - 5x \frac{x}{2} \quad d^2M_x = -5; \text{ so, concave downward.}$$

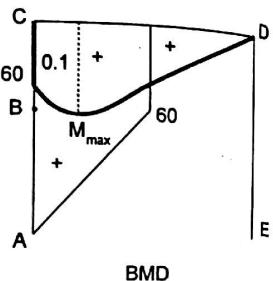
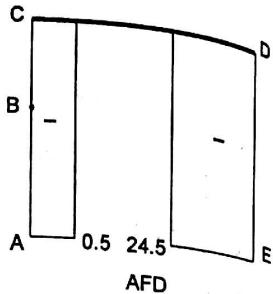
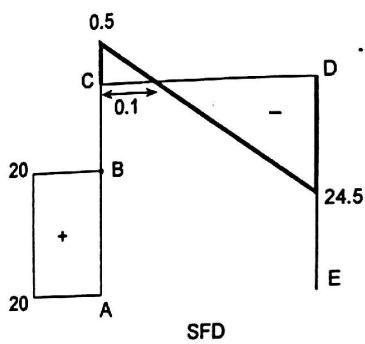
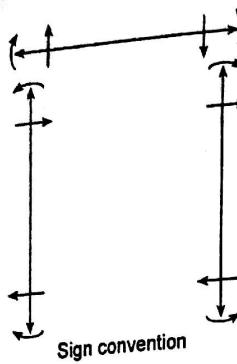
$$= 60 + 0.5x - 2.5x^2$$

$$x = 0; M_C = 60 \text{ kNm}$$

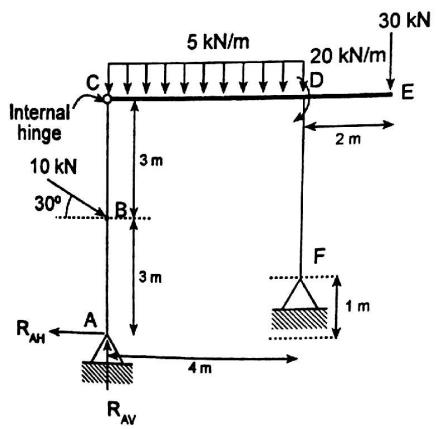
$$x = 5; M_D = 60 - 60 = 0$$

$$x = 2; M_{\text{mid}} = 51 \text{ kNm}$$

$$\frac{M_C + M_D}{2} = 30 \text{ kNm}$$



6. Draw AFD, SFD and BMD of given frame. Indicate also the salient features if any.



*Solution:*

Calculation of reaction

Taking left of internal hinge C

$$\sum M_C = 0 \text{ +ve}$$

$$-R_{AH} \times 6 + 10 \cos 30 \times 3 = 0$$

$$-6R_{AH} = 15\sqrt{3}$$

$$\therefore R_{AH} = 4.33 \text{ kN (left)}$$

Taking  $\sum M_F = 0 \text{ +ve}$

$$-R_{AV} \times 4 - 4.33 \times 1 - 10 \cos 30 \times 2 + 10 \sin 30 \times 4 + (5 \times 4) \cdot 2 - 20$$

$$-30 \times 2 = 0$$

$$-4R_{AV} = 41.65$$

$$\therefore R_{AV} = (-) 10.4125 = 10.4125 \text{ (down)}$$

$\sum F_x = 0 \rightarrow +\text{ve}$

$$-4.33 + 10 \cos 30 - R_{FH} = 0$$

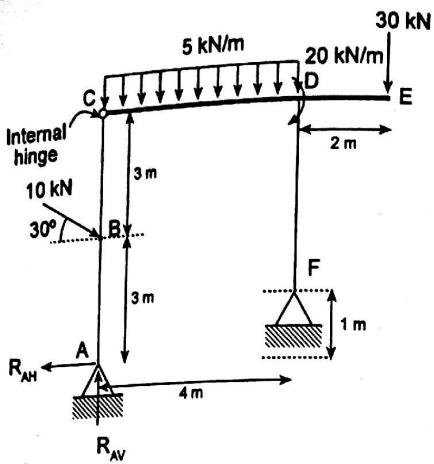
$$\therefore R_{FH} = 8.66 - 4.33 = 4.33 \text{ kN (left)}$$

$\sum F_y = 0 \uparrow +\text{ve}$

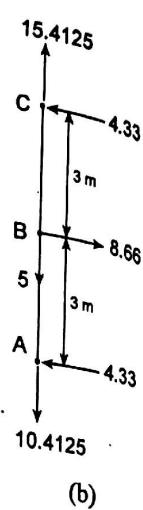
$$R_{AV} - 10 \sin 30 - 5 \times 4 - 30 + R_{FV} = 0$$

$$-10.4125 - 5 - 20 - 30 + R_{FV} = 0$$

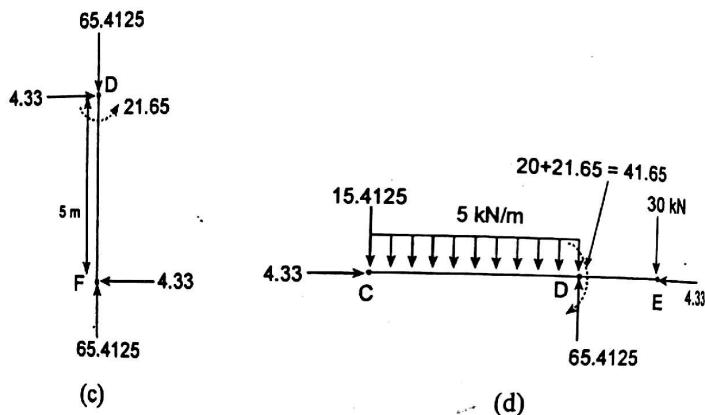
$$\therefore R_{FV} = 65.4125 \text{ (up)}$$



(a)



(b)



(c)

(d)

(i) For span AB [0 ≤ x ≤ 3m]

$$V_x = 4.33$$

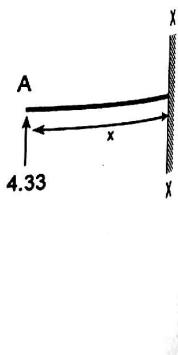
$$x = 0; \quad V_A = 4.33 \text{ kN}$$

$$x = 3; \quad V_B = 4.33 \text{ kN}$$

$$M_x = 4.33x$$

$$x = 0; \quad M_A = 0$$

$$x = 3; \quad M_B = 13$$



Analysis of Beam and Frame

(ii) For span BC [0 ≤ x ≤ 3m]

$$V_x = 4.33 - 8.66 = -4.33 \text{ kN}$$

$$x = 0;$$

$$V_B = -4.33 \text{ kN}$$

$$x = 3;$$

$$V_C = -4.33 \text{ kN}$$

$$M_x = 4.33(3 + x) - 8.66x$$

$$= 13 - 4.33x$$

$$x = 0; \quad M_B = 13 \text{ kNm}$$

$$x = 3; \quad M_C = 0$$

(iii) For span FD [0 ≤ x ≤ 5m]

$$V_x = 4.33$$

$$x = 0; \quad V_F = 4.33 \text{ kN}$$

$$x = 5; \quad V_D = 4.33 \text{ kN}$$

$$M_x = -4.33x$$

$$x = 0; \quad M_F = 0$$

$$x = 5; \quad M_D = -21.65 \text{ kNm}$$

(iv) For span ED [0 ≤ x ≤ 2m]

$$V_x = 30$$

$$x = 0; \quad V_E = 30 \text{ kN}$$

$$x = 2; \quad V_D = 30 \text{ kN}$$

$$M_x = -30x$$

$$x = 0; \quad M_E = 0$$

$$x = 2; \quad M_D = -60 \text{ kNm}$$

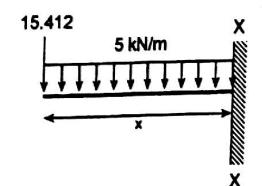
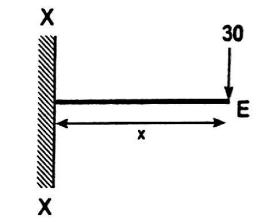
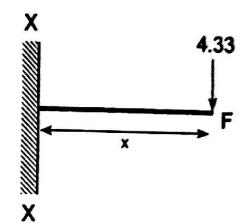
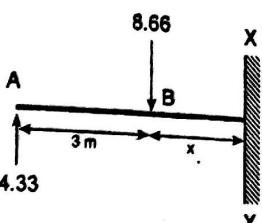
(v) For span CD [0 ≤ x ≤ 4m]

$$V_x = -15.412 - 5x$$

$$x = 0; \quad V_C = -15.412 \text{ kN}$$

$$x = 4; \quad V_D = -35.412 \text{ kN}$$

$$M_x = -15.412x - 5x \cdot \frac{x}{2}$$



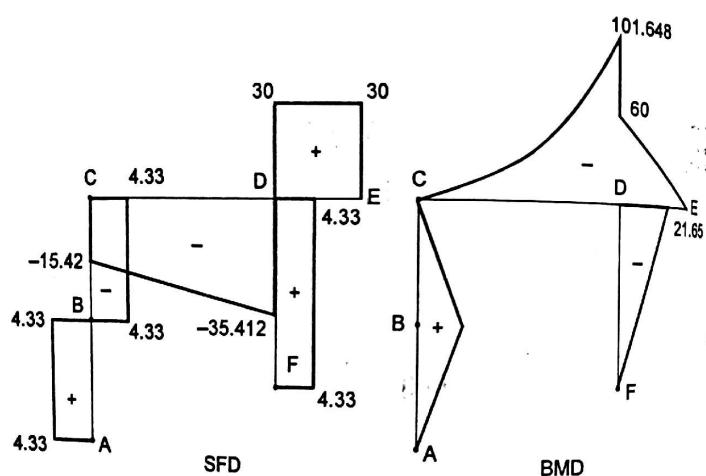
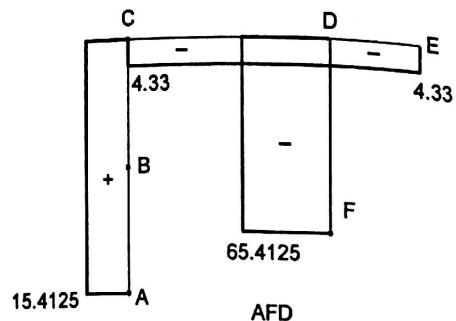
$d^2M_x = 5$ ; so, concave upward.

$$= -15.412x - 2.5x^2$$

$$x = 0; \quad M_C = 0$$

$$x = 4; \quad M_D = -101.648 \text{ kNm}$$

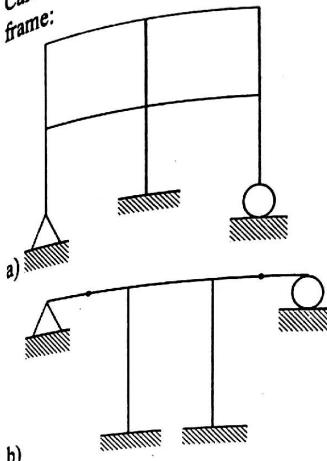
$$\frac{M_C + M_D}{2} = -50.824$$



$(3m+3) - (3j+3)$

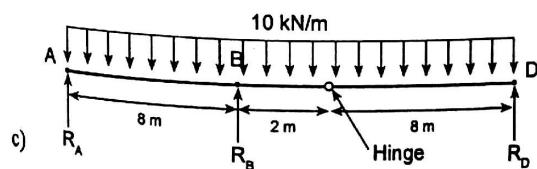
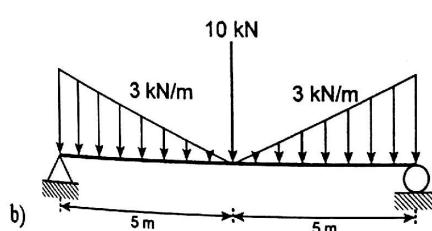
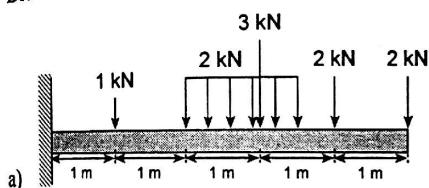
### Practice Question

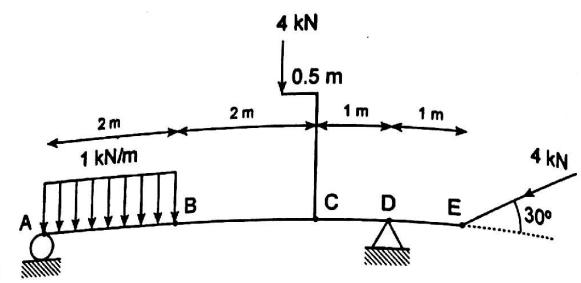
1. Calculate external, internal and total indeterminacy of following frame:



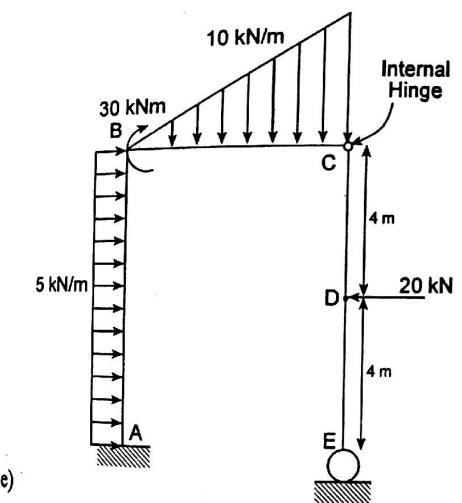
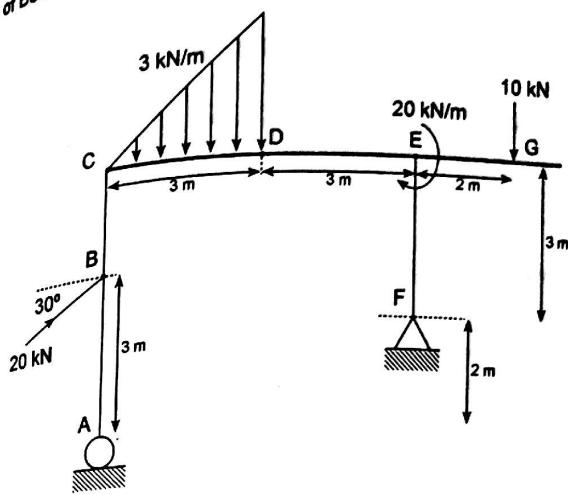
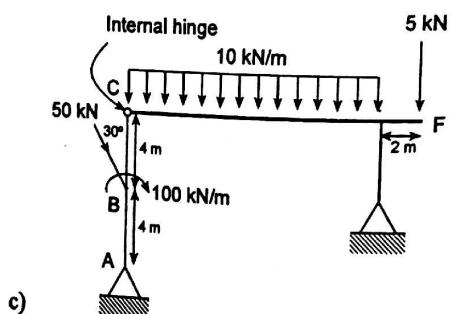
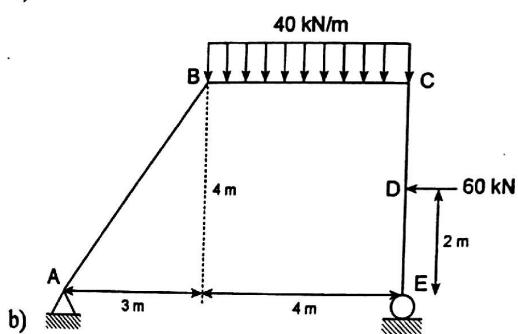
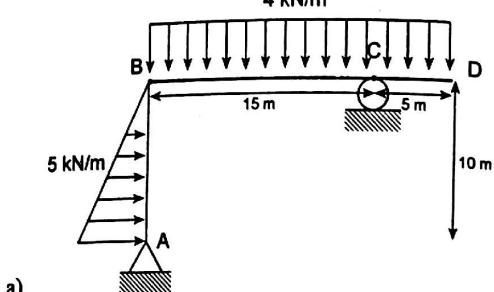
(Ans.: a) 3, 6, 9 b) 4, 0, 4)

2. Draw SFD and BMD for following beam.





3. Draw AFD, SFD and BMD and show the salient point (if any)



## Chapter 7

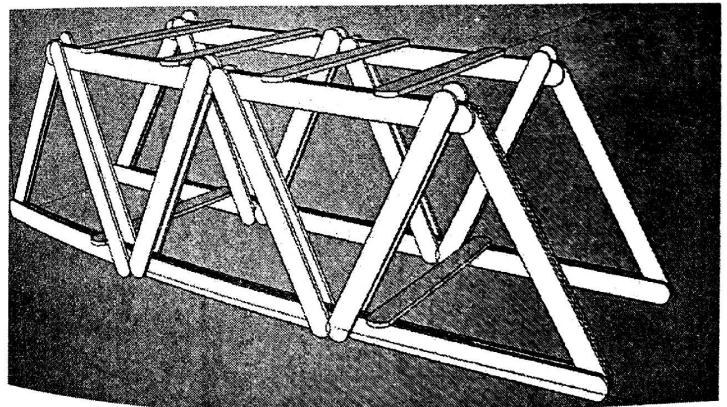
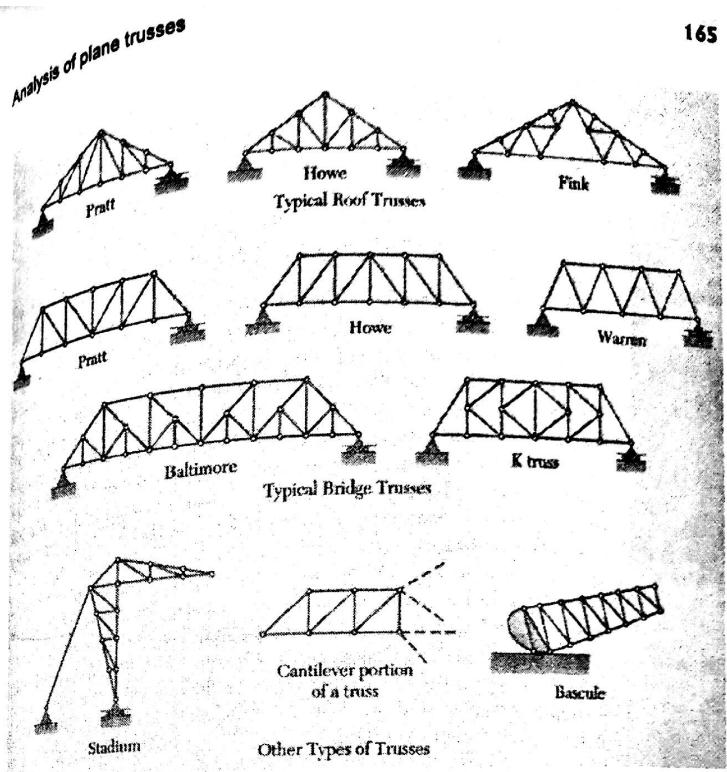
### Analysis of plane trusses

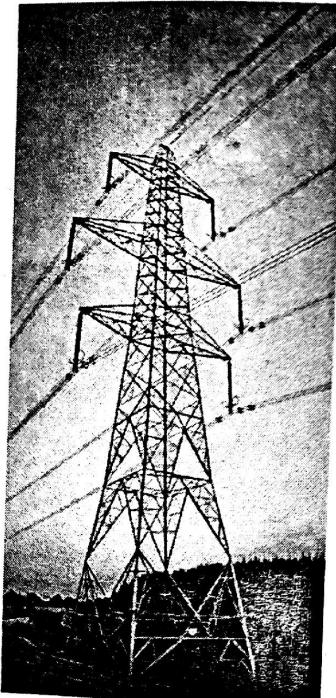
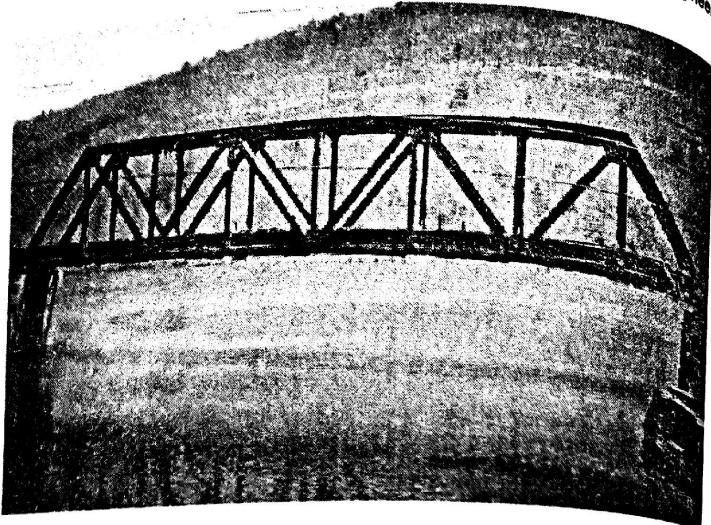
#### 7.1 Introduction and uses

A truss is an assembly of members that are fastened together to support stationary or moving loads. They may be fastened with riveted joints, welded joints or pin joints. The truss is one of the major type of engineering structure. It provides both a practical and an economical solution to many engineering situations especially in the design of bridges, long span roof and buildings. Truss member are connected at their extremities only. In the system, the loads are transmitted through joints.

#### 7.2 Classification of truss

- a) According to the analysis
  - i) Plane truss or 2-D truss
  - ii) Space truss or 3-D truss
- b) According to their construction
  - i) Simple truss
  - ii) Compound truss
  - iii) Complex truss
- c) According to support condition
  - i) Simply supported truss
  - ii) Cantilever truss
- d) According to purpose of utilization
  - i) Roof truss
  - ii) Bridge truss
  - iii) Tower truss etc.





### 7.3 Determinancy and Indeterminacy of a structure

The structure in which the unknown reaction elements can be worked out by applying the equations of static equilibrium is called statically determinate, otherwise indeterminate structure. In case of plane truss, statics provides three equations which are useful to find the support reaction.

$$\Sigma F_x = 0; \quad \Sigma F_y = 0; \quad \Sigma M = 0$$

If  $m$  be the number of members,  $r$  be the number of support reactions and  $j$  be the number of joints in the truss, then the following three conditions may arise.

a)  $m + r > 2j$

The truss is said to be redundant or over rigid. These truss can be solved with the help of static equilibrium alone. So, it is statically indeterminate.

b)  $m + r = 2j$

The truss is said to be efficient or perfect or rigid truss. Truss is statically determinate and stable.

c)  $m + r < 2j$

The truss is said to be deficient or imperfect truss. Truss is unstable and is mechanism.

#### 7.3.1 Degree of static indeterminacy

A structure is said to be statically determinate if all member forces and reactions can be evaluated using the equation of static equilibrium.

For plane truss

- a) If  $m = 2j - r$  Determinate
- b) If  $m > 2j - r$  Indeterminate
- c) If  $m < 2j - r$  Unstable
- d) Total external indeterminacy =  $r - 3$
- e) Total internal indeterminacy =  $m - (2j - 3)$
- f) Total degree of static indeterminacy =  $(r - 3) + m - (2j - 3)$   
 $= m + r - 2j$

### 7.3.2 Degree of kinematic indeterminacy (DKI)

- It is related with the joint displacement. It is the total number of freedoms that are allowed for the joints in a structure.
- Roller support can resist only vertical translation but not horizontal translation and rotation. So, degree of freedom is 2.
  - For hinge support degree of freedom is 1.
  - For fixed support degree of freedom is 0.

For truss

$$DKI = 2j - r$$

$$= 2 \times \text{no. of joints} - \text{no. of support reaction}$$

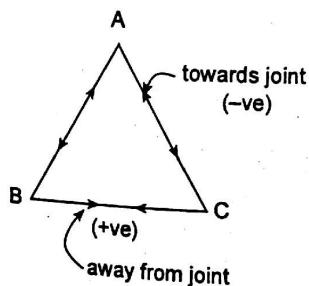
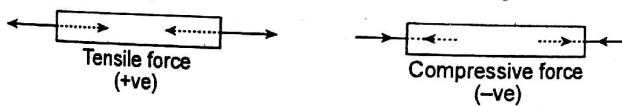
### 7.4 Idealization of a truss

Following assumptions are made while finding out the forces in the members of truss:

- The members are connected by smooth frictionless pins.
- Self weight of member is neglected.
- Loads are applied only at the joint.
- Truss is statically determinate.

### 7.5 Nature of force

A tensile force in the member is indicated by arrows pointing away from the joints while a compressive force is towards the joints as shown.



During analysis, start by assuming either tensile (T) or compressive (C). After solving for a force at a joint, if positive forces indicate correct assumption of the direction of forces. A negative forces means that the actual direction of force is opposite to the assumed.

### 7.6 Analysis of truss

Two methods are available:

- Method of joints
- Method of sections

#### 7.6.1 Method of joints

The solution proceeds from one joint to the other at every joint the forces are concurrent as the centre line of each member at a joint meet at a point. Equation of equilibrium at a joint is

$$\Sigma F_x = 0; \quad \Sigma F_y = 0$$

Thus only two unknown forces can be determined. Start with a joint with two unknown force at most.

#### 7.6.2 Method of section

This method is more economical than the method of joints, if forces in only few members of the truss are desired. Divide the truss into two imaginary section by passing a cut (straight or crooked) through the truss such that the member in which force is to be determined is cut. To solve problem more than one cut may be required. Take one cut section of truss and use

$$\Sigma M = 0; \quad \Sigma F_y = 0; \quad \Sigma F_x = 0$$

to solve the unknown forces.

## Worked Out Examples

1. Find the forces in all the members of truss by joint methods.

**Solution:**

Draw FBD of given truss and use simple trigonometry to find length and angles.

$$AC = AB \cos 60^\circ = 2.5 \text{ m}$$

Let us assume:

$$R_{AV} \rightarrow Rx^n \text{ at A, (vertical)}$$

$$R_{AH} \rightarrow Rx^n \text{ at A, (horizontal)}$$

$$R_{BV} \rightarrow Rx^n \text{ at B, (vertical)}$$

Calculation of  $Rx^n$

$$\text{Taking } \sum M_A = 0 \rightarrow +ve$$

$$\text{or, } 30 \times AD - R_{BV} \times 5 = 0$$

$$\text{or, } 30 \times 2.5 \cos 60^\circ = 5 R_{BV}$$

$$\therefore T R_{BV} = 30 \times 1.25$$

$$\therefore R_{BV} = 7.5 \text{ kN } (\uparrow)$$

$$\Sigma F_y = 0 \uparrow +ve$$

$$\text{or, } R_{AV} - 30 + R_{BV} = 0$$

$$\text{or, } R_{AV} - 30 + 7.5 = 0$$

$$\text{or, } R_{AV} = 22.5 \text{ kN } (\uparrow)$$

$$\Sigma F_x = 0 \rightarrow +ve$$

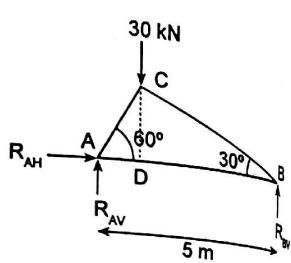
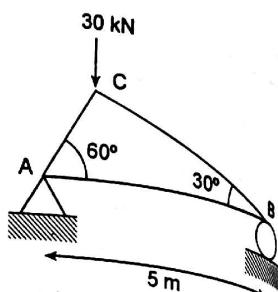
$$R_{AH} = 0$$

**Joint A:**

Unknown force is assumed to be tensile.

$$\Sigma F_y = 0 \uparrow +ve$$

$$22.5 + F_{AC} \sin 60^\circ = 0$$



$$\therefore F_{AC} = -\frac{22.5}{\sin 60^\circ}$$

$$= -25.97 \text{ kN}$$

$$= 25.97 \text{ kN (C)}$$

-ve sign for AC indicate that the actual direction of force in the member AC is opposite to assumed direction.

$$\Sigma F_x = 0 \rightarrow +ve$$

$$F_{AB} + F_{AC} \cos 60^\circ = 0$$

$$F_{AB} - 25.97 \cos 60^\circ = 0$$

$$\therefore F_{AB} = 12.99 \text{ kN (T)}$$

**Joint B:**

$$\Sigma F_y = 0 \uparrow +ve$$

$$7.5 + F_{BC} \sin 30^\circ = 0$$

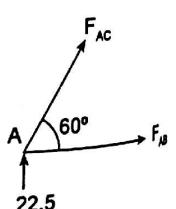
$$7.5 + F_{BC} \frac{1}{2} = 0$$

$$\therefore F_{BC} = -15 \text{ kN}$$

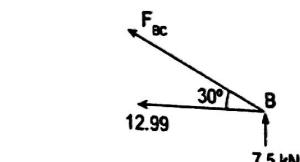
$$F_{BC} = 15 \text{ kN (C)}$$

Member	Magnitude (kN)	Nature
AB	12.99	Tensile (T)
BC	15	Compressive (C)
CA	25.97	Compressive (C)

You can solve this truss without calculating the reaction value. It is because joint C has only two unknown. You can directly start from here.



(Try yourself)



2. Find the forces in members of truss shown.

**Solution:**

Draw FBD:

**Calculation of reaction:**

$$\Sigma M_A = 0 \text{ +ve}$$

$$\text{or, } 5 \times 3 + 2.5 \times 6 - 9 R_{DV} = 0$$

$$\text{or, } 30 = 9 R_{DV}$$

$$\therefore R_{DV} = 3.33 \text{ kN } (\uparrow)$$

$$\Sigma F_y = 0 \uparrow +\text{ve}$$

$$\text{or, } R_{AV} - 5 - 2.5 + R_{DV} = 0$$

$$\text{or, } R_{AV} - 7.5 + 3.33 = 0$$

$$\therefore R_{AV} = 4.167 \text{ kN } (\uparrow)$$

$$\Sigma F_x = -0 \rightarrow +\text{ve}$$

$$R_{AH} = 0$$

**Joint A:**

From trigonometry

$$\tan \theta = \frac{4}{3}$$

$$\sin \theta = \frac{4}{5} = 0.8$$

$$\cos \theta = \frac{3}{5} = 0.6$$

$$\Sigma F_y = 0 \uparrow +\text{ve};$$

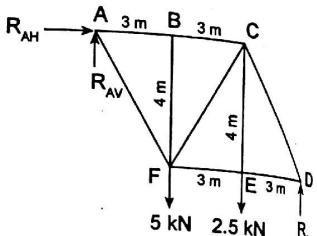
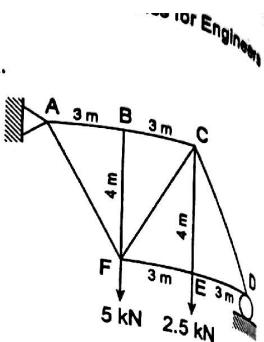
$$\text{or, } 4.167 - F_{AF} \sin \theta = 0$$

$$\therefore F_{AF} = 5.21 \text{ kN } (\text{T})$$

$$\Sigma F_x = 0 \rightarrow +\text{ve}$$

$$\text{or, } F_{AB} + F_{AF} \cos \theta = 0$$

$$\text{or, } F_{AB} + 5.21 \times 0.6 = 0$$



Analysis of plane ...

$$\text{or, } F_{AB} + 3.126 = 0$$

$$\therefore F_{AB} = -3.126 \text{ kN} = 3.126 \text{ kN } (\text{C})$$

**Joint B:**

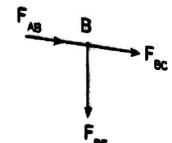
$$\Sigma F_y = 0 \uparrow +\text{ve}$$

$$F_{BF} = 0$$

$$\Sigma F_x = 0 \rightarrow +\text{ve}$$

$$\text{or, } 3.126 + F_{BC} = 0$$

$$\therefore F_{BC} = -3.126 \text{ kN} = 3.126 \text{ kN } (\text{C})$$



**Joint F**

From trigonometry,

$$\tan \alpha = \frac{3}{4}$$

$$\therefore \alpha = 36.87$$

$$\tan \theta = \frac{4}{3}$$

$$\therefore \theta = 53.13$$

$$\Sigma F_y = 0 \uparrow +\text{ve}$$

$$\text{or, } 5.21 \cos 36.87 - 5 + F_{FC} \sin 53.13 = 0$$

$$\therefore F_{FC} = 1.04 \text{ kN } (\text{T})$$

$$\Sigma F_x = 0 \rightarrow +\text{ve}$$

$$\text{or, } F_{FE} + 1/04 \cos 53.13 - 5.21 \sin 36.87 = 0$$

$$\therefore F_{FE} = 2.502 \text{ kN } (\text{T})$$

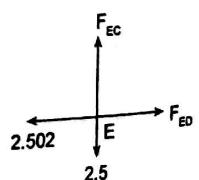
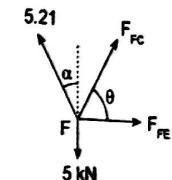
**Joint E:**

$$\Sigma F_y = 0 \uparrow +\text{ve}$$

$$F_{EC} = 2.5 \text{ kN } (\text{T})$$

$$\Sigma F_x = 0 \rightarrow +\text{ve}$$

$$F_{ED} = 2.502 \text{ kN } (\text{T})$$



Joint D:

$$\tan \theta = \frac{4}{3}$$

$$\therefore \theta = 53.13$$

$$\Sigma F_y = 0 \uparrow +ve$$

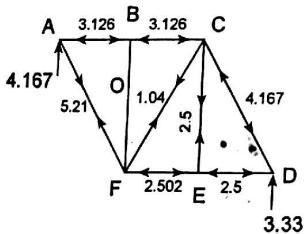
$$\text{or, } F_{DC} \sin 53.13 + 3.33 = 0$$

$$\therefore F_{DC} = -4.167 \text{ kN}$$

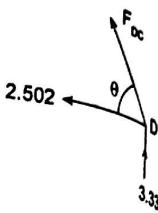
$$= 4.167 \text{ kN (C)}$$

Member	Magnitude (kN)	Nature
AB	3.126	C
BC	3.126	C
CD	4.167	C
DE	2.5	T
CE	2.5	T
EF	2.502	T
CF	1.04	T
BF	0	-
AF	5.21	T

## Graphical presentation



3. Determine the total degree of internal, external indeterminacy of the given truss. Also determine the member force in CE, BE and BD for given truss.



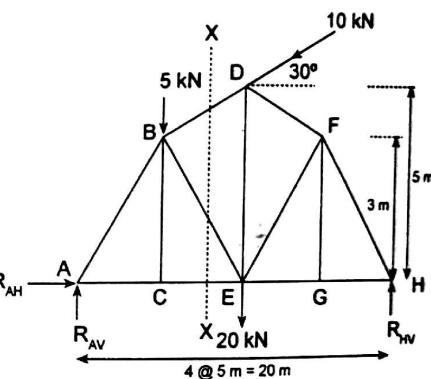
Solution:

$$\text{Total external indeterminacy} = r - 3$$

$$= 3 - 3 = 0$$

$$\text{Total internal indeterminacy} = m - (2j - 3)$$

$$= 13 - (2 \times 8 - 3) = 13 - 13 = 0$$

Calculation of  $R_{AH}$ 

$$\Sigma M_A = 0 +ve \rightarrow$$

$$\text{or, } -5 \times 5 - 20 \times 10 + R_{HV} \times 20 + 10 \cos 30 \times 5 - 10 \sin 30 \times 10 = 0$$

$$\text{or, } -25 - 200 + 20R_{HV} + 25\sqrt{3} - 50 = 0$$

$$\therefore R_{HV} = 11.5849 \text{ kN } (\uparrow)$$

$$\Sigma F_y = 0 \quad \uparrow +ve$$

$$\text{or, } R_{AV} - 5 - 20 - 10 \sin 30 + 11.5849 = 0$$

$$\therefore R_{AV} = 18.4151 \text{ kN} (\uparrow)$$

$$\Sigma F_x = 0 \rightarrow +ve$$

$$R_{AH} - 10 \cos 30 = 0$$

$$\therefore R_{AH} = 8.66 \text{ kN} (\rightarrow)$$

Draw a section X-X as shown. Take the left part of section.

By solving,

$$\alpha = 21.8^\circ$$

$$\beta = 30.9^\circ$$

$$\theta = 59.037^\circ$$

$$\Sigma M_B = 0 +ve \curvearrowright$$

$$\text{or, } 8.66 \times 3 - 18.41 \times + F_{CE} \times 3 = 0$$

$$\therefore F_{CE} = 22.03 \text{ kN}$$

$$\Sigma M_E = 0 +ve \curvearrowleft$$

$$\text{or, } -18.41 \times 10 + 5 \times 5 - F_{BD} \cos 21.8 \times 3 - F_{BD} \sin 21.8 \times 8 = 0$$

$$\text{or, } -F_{BD} \times 4.6423 = 159.15$$

$$\therefore F_{BD} = -34.282 \text{ kN}$$

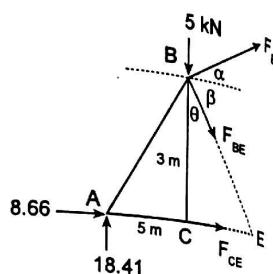
$$= 34.282 \text{ kN (C)}$$

$$\Sigma F_y = 0 \uparrow +ve$$

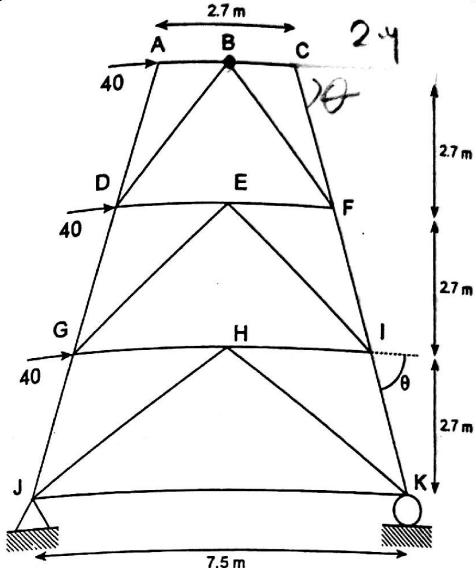
$$\text{or, } 18.415 - 5 - F_{BE} \sin 30.96 - 34.28 \sin 21.81 = 0$$

$$\text{or, } 13.415 - 0.51448 F_{BE} - 12.7317 = 0$$

$$F_{BE} = 1.32816 \text{ kN (T)}$$



4. Determine the force in the member IK for the truss shown.



**Solution:**

Draw a section a-a

$$\text{Use } \tan \theta = \frac{2.7}{0.8}$$

$$\therefore \theta = 73.5^\circ$$

Here we have select the section that cuts four unknown force (>3) because by visualization it is found that 3 unknown force passes through G.

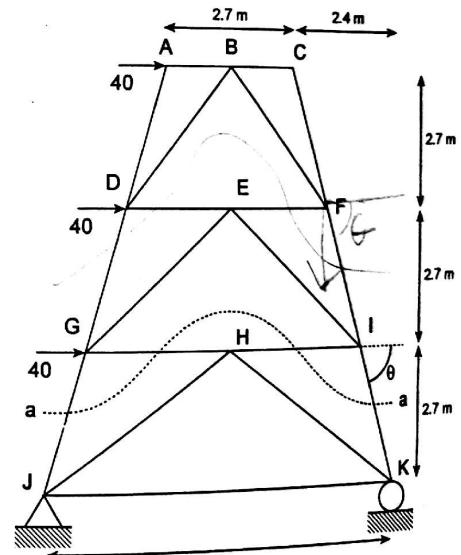
$$GI = 7.5 - 2 \times 0.8 = 5.9 \text{ m make in same line.}$$

$$\Sigma M_G = 0 +ve \curvearrowleft$$

$$\text{or, } F_{IK} \sin \theta \times 5.9$$

$$+ 40 \times 2.7$$

$$+ 40 \times 5.4 = 0$$



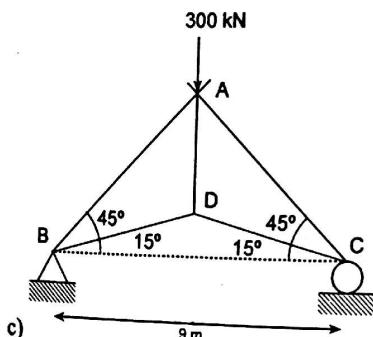
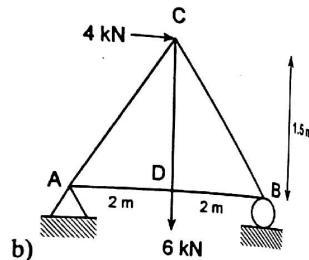
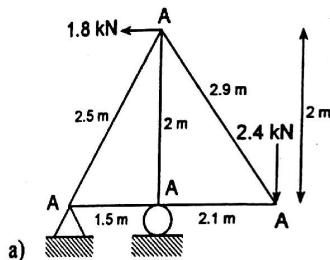
$$\therefore F_{IK} \sin 73.5 = (-)54.915$$

$$\therefore F_{IK} = 57.27 \text{ kN (C)}$$

$$F_{GJ} = 57.27 \text{ kN (T)}$$

### Practice Questions

1. Find the forces in different members of truss by methods of joints.



(Ans. a) AB = 1.2 kN (T), BC = 2.52 kN (C), AD = 3.48 kN (T),

AC = 3.36 kN (C), CD = 2.52 kN (C)

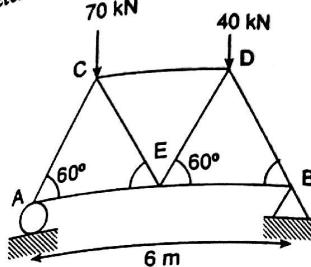
b) AC = 2.5 (C), BC = 7.5 (C), AD = 6 (T),

DB = 6 (T), CD = 6 (T);

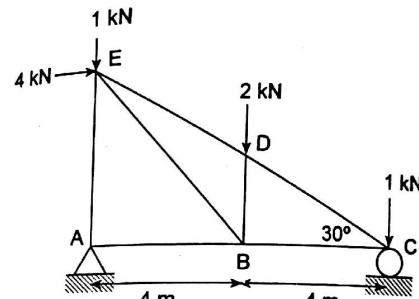
c) AB = 280 (C), AD = 100 (T), AC = 280 (C),

CD = 205 (T), DB = 205 (T)

2. Determine the member force using section methods.



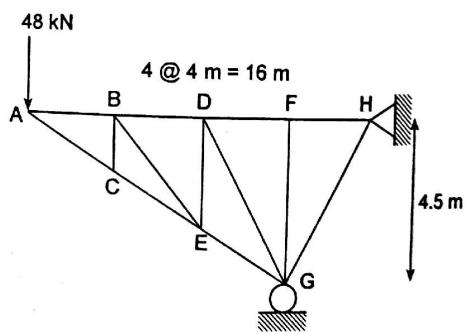
a)



b)

(Ans. a) AC = 29 (C), BD = 40 (C), CD = 17 (C), AE = 15 (T), EB = 20 (T), ED = 5.5 (C), EC = 5.5 (T))

3. Determine the forces in each member of the truss.

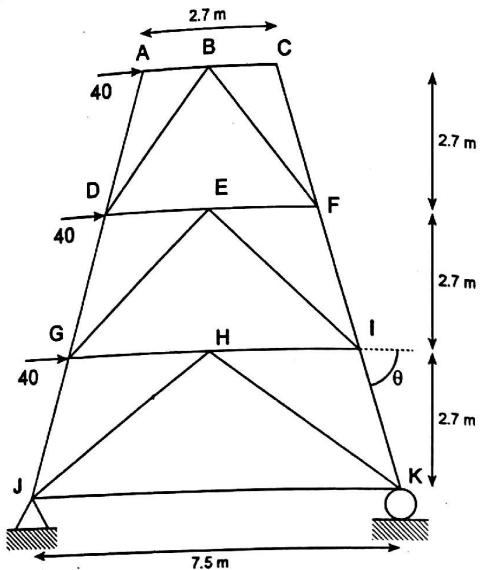


Ans. AB = 128 (T), AC = 136.7 (C), BD = DF = FH = 128 (T), CE = EG = 136.7 (C), GH = 192.7 (C)

[Hint: for this type of cantilever truss, do not calculate the reaction. Start from the free end which has at most two unknowns. The joint A has

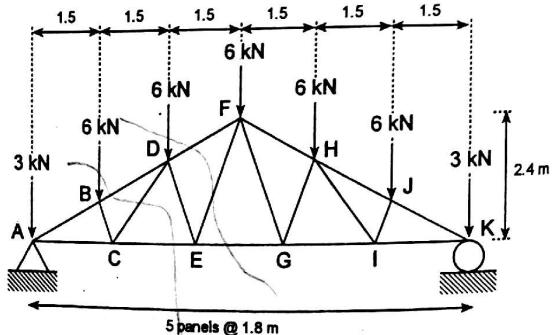
two unknowns. So, start from here. Use trigonometry to calculate angle]

4. Determine the force in member FI for the truss shown.



(Ans.: 26.2 kN(C))

5. A Fink roof truss is loaded as shown. Determine the force in members BD, CD and CE.



(Ans.: BD = 29.8 kN(C), CD = 6.25 kN(T), CE = 22.5 kN(T))

## Chapter 8

### Kinematics of particle and rigid body

Kinematics is the study of geometry of motion. It is used to relate displacement, velocity, acceleration and time, without reference to the cause of the motion.

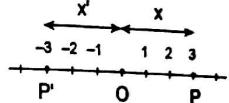
#### 8.1 Rectilinear motion of particle

When a particle moves along a straight line, the motion is said to be a rectilinear motion.

At any instant of time  $t$ , the particle will occupy a certain position on the straight line. To define the position  $P$  of the particle, we choose a fixed origin  $O$  on the straight line and a direction (say positive along  $x$  axis).

$$OP = x = +3\text{ m}$$

$$OP' = x' = -3\text{ m}$$



Here  $P$  and  $P'$  are called position coordinate and  $OP$  and  $OP'$  are called position vector.

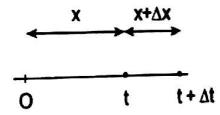
When the position coordinate  $x$  of a particle is known for every value of time  $t$ , we say that the motion of particle is known. The timetable of the motion can be given in the form of an equation in  $x$  and  $t$ .

Then,

$$\text{Average velocity} = \frac{\Delta x}{\Delta t}$$

The instantaneous velocity  $v$  of the particle at the instant  $t$  is obtained from the average velocity by choosing shorter and shorter time intervals  $\Delta t$  and displacement  $\Delta x$ .

$$\text{Instantaneous velocity } v = \lim_{\Delta t \rightarrow 0} \frac{\Delta x}{\Delta t}$$



$$\therefore v = \frac{dx}{dt}$$

$$\text{Similarly, } a = \frac{dv}{dt} = \frac{d^2x}{dt^2}$$

$$v = \frac{dx}{dt}$$

$$\therefore dt = \frac{dx}{v}$$

$$\text{Again, } a = \frac{dv}{dt} = \frac{dv}{\frac{dx}{v}} = v \frac{dv}{dx}$$

The relate the acceleration, velocity and position of particle.

## 8.2 Determination of motion of particle and rigid body

The condition of motion will be specified by the type of acceleration that the particle possess. The acceleration of the particle may be expressed as the function of one or more of the variable  $x$ ,  $v$  and  $t$ . In order to determine  $x$  in terms of  $t$ , it is necessary to perform two successive integration.

### a) When acceleration is a given function of time i.e. $a = f(t)$

$$\text{We know, } a = \frac{dv}{dt}$$

$$\text{or, } dv = a dt$$

$$\therefore dv = f(t) dt$$

Integrating both sides

$$\int dv = \int f(t) dt$$

$$\text{initially; } t = 0, v = v_0$$

$$\text{finally, } t = t, v = v$$

$$\int_{v_0}^v dv = \int_0^t f(t) dt$$

$$v - v_0 = \int_0^t f(t) dt$$

$$v = v_0 + \int_0^t f(t) dt \dots\dots (i)$$

$$\text{Again, } v = \frac{dx}{dt}$$

$$dx = v dt$$

Integrating both side:

$$\int dx = \int v dt$$

$$\text{at } t \rightarrow 0; \quad x \rightarrow x_0$$

$$\text{at } t \rightarrow t; \quad x \rightarrow x$$

$$\int_{x_0}^x dx = \int_0^t v dt$$

$$\text{or, } x - x_0 = \int_0^t [v_0 + \int_0^t f(t) dt] dt$$

$$\therefore x = x_0 + \int_0^t [v_0 + \int_0^t f(t) dt] dt$$

Hence position  $x$  is obtained.

### b) When acceleration is a given function of position i.e. $a = f(x)$

$$\text{We know, } a = \frac{dv}{dt}$$

$$\text{Also, } a = v \frac{dv}{dx}$$

$$\text{or, } v dv = a dx$$

$$\text{or, } v dv = f(x) dx$$

Integrating both side

$$\int v dv = \int f(x) dx$$

$$\text{at } t \rightarrow 0; \quad x \rightarrow x_0; \quad v \rightarrow v_0$$

$$\text{at } t \rightarrow t; \quad x \rightarrow x, \quad v \rightarrow v$$

$$\int_{v_0}^v v dv = \int_{x_0}^x f(x) dx$$

$$\text{or, } \frac{v^2}{2} \Big|_{v_0}^v = \int_{x_0}^x f(x) dx$$

$$\text{or, } \frac{v^2 - v_0^2}{2} = \int_{x_0}^x f(x) dx$$

$$\text{or, } v^2 = v_0^2 + 2 \int_{x_0}^x f(x) dx$$

$$\text{or, } v = \left[ v_0^2 + 2 \int_{x_0}^x f(x) dx \right]^{1/2}$$

$$\text{Again, } v = \frac{dx}{dt}$$

$$\text{or, } dx = v dt$$

$$\text{or, } \int dx = \int v dt$$

$$\text{or, } \int_{x_0}^x dx = \int_0^t v dt$$

$$\text{or, } x - x_0 = \int_0^t v dt$$

$$\therefore x = x_0 + \int_0^t [v_0^2 + 2 \int_{x_0}^x f(x) dx]^{1/2} dt$$

Hence position  $x$  is obtained.

**c) When acceleration is a given function of velocity  
i.e.  $a = f(v)$**

$$\text{We know, } a = \frac{dv}{dt}$$

$$\text{Also, } a = v \frac{dv}{dx}$$

$$\text{for, } a dx = v dv$$

$$\text{or, } f(v) dx = v dv$$

$$\text{or, } dx = \frac{v dv}{f(v)}$$

$$\text{at } t \rightarrow 0; \quad x \rightarrow x_0, \quad v \rightarrow v_0$$

$$\text{at } t \rightarrow t; \quad x \rightarrow x; \quad v \rightarrow v$$

$$\text{or, } \int dx = \int \frac{v dv}{f(v)}$$

$$\text{or, } \int_{x_0}^x dx = \int_{v_0}^v \frac{v dv}{f(v)}$$

$$\text{or, } x - x_0 = \int_{v_0}^v \frac{v dv}{f(v)}$$

$$\therefore x = x_0 + \int_{v_0}^v \frac{v dv}{f(v)}$$

Hence position  $x$  is obtained.

**d) Uniformly rectilinear motion**

Uniform motion means covering equal distance over equal intervals of time. In this case velocity is constant throughout the motion and hence acceleration  $a$  is zero for every value of  $t$ .

$$\therefore v = \frac{dx}{dt} = \text{constant} = u \text{ (say)}$$

$$\text{or, } \frac{dx}{dt} = u$$

$$\text{or, } dx = u dt$$

$$\text{or, } \int dx = \int u dt$$

$$\text{or, } x = ut + c_1$$

$$\text{at } t \rightarrow 0; \quad x \rightarrow x_0$$

$$\therefore c_1 = x_0$$

$$\therefore x = x_0 + ut$$

Hence the position of particle is obtained.

**e) Uniformly accelerated rectilinear motion**

In this motion acceleration  $a$  of the particle is constant.

$$\therefore \frac{dv}{dt} = \text{constant} = a \text{ (say)}$$

$$\text{or, } dv = a dt$$

$$\text{or, } \int dv = \int a dt$$

$$\text{or, } v = at + c_1$$

at  $t \rightarrow 0$ ,  $v \rightarrow v_0$

$$\therefore c_1 = v_0$$

$$\therefore v = v_0 + at \dots\dots (i)$$

$$\text{Also, } v = \frac{dx}{dt}$$

$$\text{or, } \frac{dx}{dt} = v_0 + at$$

$$\text{or, } dx = (v_0 + at) dt$$

$$\text{or, } \int dx = \int (v_0 + at) dt$$

$$\text{at } t \rightarrow 0, \quad x \rightarrow x_0$$

$$\text{or, } x = v_0 t + \frac{at^2}{2} + c_1$$

$$\therefore c_1 = x_0$$

$\therefore x = x_0 + v_0 t + \frac{1}{2} at^2$  is required position.

$$\text{Also, } a = v \frac{dv}{dx}$$

$$\text{or, } v dv = a dx$$

$$\text{or, } \int v dv = \int a dx$$

$$\text{or, } \frac{v^2}{2} = ax + c_1$$

$$\text{at } t \rightarrow 0, \quad v \rightarrow v_0, \quad x \rightarrow x_0$$

$$\text{or, } \frac{v_0^2}{2} = ax_0 + c_1$$

$$\therefore c_1 = \frac{v_0^2}{2} - ax_0$$

$$\therefore \frac{v^2}{2} = ax + \frac{v_0^2}{2} - ax_0$$

$$\text{or, } \frac{v^2}{2} - \frac{v_0^2}{2} = a(x - x_0)$$

$$v^2 = v_0^2 + 2a(x - x_0)$$

### i) Relative motion of two particles

Let position coordinate of A =  $x_A$

position coordinate of B =  $x_B$

Relative position of B with respect to A is  $x_{B/A}$  or  $x_{AB}$

$$x_{B/A} = x_B - x_A$$

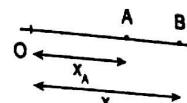
$$x_B = x_A + x_{B/A}$$

Differentiating:

$$V_B = V_A + V_{B/A}$$

Again, differentiating

$$a_B = a_A + a_{B/A}$$



### g) Dependent motion

The position of a particle sometimes depend upon the position of another or several other particles. The motion of such kind is called dependent motion.

Let the position of A be  $x_A$  and B be  $x_B$  with respect to reference XY.

From the figure; the position of block B depends upon the position of block A. Since the rope abcdef is of constant length.

$$ab + bc + cd + de + ef = \text{constant}$$

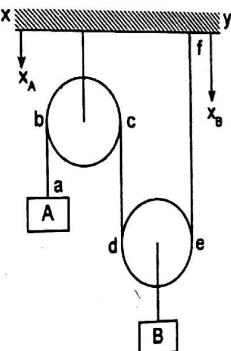
The length of portion of rope bc and de wrapped around the pulley remains fixed. So,

$$ab + cd + ef = \text{constant}$$

Since the length segment ab differ from  $x_A$  only by a constant and the length of segment cd and ef differ from  $x_B$  by a constant, we can write

$$x_A + x_B + x_B = \text{constant}$$

$$x_A + 2x_B = k \dots\dots (i)$$



Differentiating

$$v_A + 2V_B = 0 \dots\dots (ii)$$

Again differentiating

$$a_A + 2A_B = 0 \dots\dots (iii)$$

### 8.3 Curvilinear motion

The motion of a particle along a curved path, other than a straight line is known as curvilinear motion. e.g. projectile motion, motion of satellite.

Position of particle on curved path at any instant is defined as position vector  $\vec{r}$  where

$$\vec{r} = x \vec{i} + y \vec{j}$$

$$|\vec{r}| = \sqrt{x^2 + y^2}$$

$$\text{velocity vector } \vec{v} = \frac{d\vec{r}}{dt} = \frac{d}{dt}(x \vec{i} + y \vec{j})$$

$$\vec{v} = \vec{i} \frac{dx}{dt} + \vec{j} \frac{dy}{dt}$$

$$\vec{v} = v_x \vec{i} + v_y \vec{j}$$

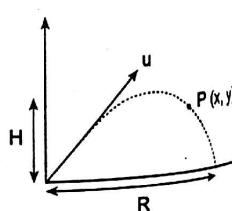
$$\vec{a} = a_x \vec{i} + a_y \vec{j}$$

#### 8.3.1 Projectile motion

Let a body is projected with initial velocity  $u$  at an angle  $\theta$  with ground as shown in figure.

Let  $H$  be the greatest height,  $R$  be the horizontal range and  $T$  be the time of flight of projectile.

Consider at instant of time  $t$  projectile reaches point  $P(x, y)$ .



Then,

$$u_y = u \sin \theta$$

$u_x = u \cos \theta$  (remains constant throughout the motion. why?)

for point P

$$x = u \cos \theta \times t$$

$$\therefore t = \frac{x}{u \cos \theta}$$

Consider vertical motion;

$$s_y = u_y t + \frac{1}{2} a_y t^2$$

for point P(x, y)

$$y = u \sin \theta \cdot t + \frac{1}{2} (-g) t^2$$

$$\text{or, } y = u \sin \theta \frac{x}{u \cos \theta} - \frac{1}{2} \cdot g \cdot \frac{x^2}{u^2 \cos^2 \theta}$$

$$\text{or, } y = x \tan \theta - \frac{1}{2} \frac{g}{u^2} x^2 \sec^2 \theta$$

$$\therefore y = x \tan \theta - \frac{1}{2} \frac{g}{u^2} x^2 (1 + \tan^2 \theta)$$

This is the equation of parabola. Hence the projectile motion is parabolic in nature.

#### a) Calculation of time of flight (T)

During the total time of flight, the net height gain by projectile will be zero.

$$s_y = v_y T + \frac{1}{2} a_y T^2$$

$$0 = u \sin \theta \cdot T + \frac{1}{2} (-g) T^2$$

$$2 u \sin \theta \cdot T = g T^2$$

$$\therefore T = \frac{2 u \sin \theta}{g}$$

**b) Calculation of greatest height**

At the maximum height of projectile the vertical component of velocity will be zero.

$$v_y^2 = u^2 + 2a_y s_y$$

$$0 = (u \sin \theta)^2 + 2(-g).H$$

$$u^2 \sin^2 \theta = 2gh$$

$$\therefore H = \frac{u^2 \sin^2 \theta}{2g}$$

For maximum height,

$$\sin^2 \theta = 1 \text{ i.e. } \theta = 90^\circ$$

$$H_{\max} = \frac{u^2}{2g} \text{ at } \theta = 90^\circ$$

**c) Calculation of horizontal range**

Since horizontal component of projectile velocity is constant.

$$\begin{aligned} R &= v \cos \theta \times T \\ &= u \cos \theta \times \frac{2u \sin \theta}{g} \\ &= \frac{u^2}{g} 2 \sin \theta \cos \theta \\ &= \frac{u^2 \sin 2\theta}{g} \end{aligned}$$

For maximum range

$$\sin 2\theta = 1 \text{ i.e. } \theta = 45^\circ$$

$$R_{\max} = \frac{u^2}{g} \text{ at } \theta = 45^\circ$$

**8.4 Tangential and normal component of acceleration**

The tangential component of acceleration deflects the change in speed of particle while its normal component deflects change in direction of motion of particle. The acceleration of particle will be zero only if both of its component is zero.

The normal component is always directed towards the centre while tangential component is tangent to the curve. If  $\hat{e}_t$  and  $\hat{e}_n$  be the unit vector along tangential and normal component respectively. Then,

$$\vec{a} = \frac{d\vec{v}}{dt} \hat{e}_t + \frac{v^2}{s} \hat{e}_n$$

$$\vec{a} = \vec{a}_t + \vec{a}_n$$

$$\vec{a}_t = \frac{d\vec{v}}{dt}; \vec{a}_n = \frac{\vec{v}^2}{\delta} \text{ where } \delta \text{ is radius of curvature.}$$

For a straight line motion i.e. rectilinear motion, radius of curvature  $\delta = \infty, a_n = 0$ .

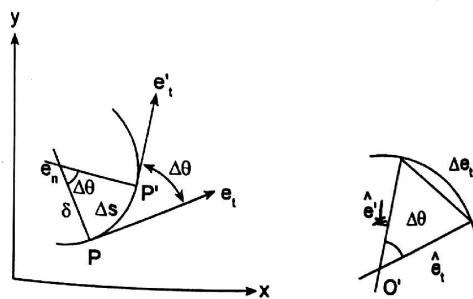
$$\therefore \frac{d\vec{v}}{dt} = \vec{a}_t = \vec{a} \quad [\text{for uniform circular motion } a_t = 0; a = a_n = \frac{v^2}{\delta}]$$

Sometime  $\delta$ , radius of curvature is calculated using

$$\delta = \frac{\left[ 1 + \left( \frac{dy}{dx} \right)^2 \right]^{3/2}}{\frac{d^2y}{dx^2}}$$

**8.4.1 Derivation of tangential and normal component**

The velocity of a particle is a vector tangent to the path of a particle but that the acceleration vector is in general, not tangent to the path. The acceleration vector may be resolved into component directed along the tangent and normal to the path of particle.



Consider  $\hat{e}_t$  and  $\hat{e}_n$  be the unit vectors along tangential and normal direction. Consider a particle  $a$  at point  $P$  having unit vector  $\hat{e}_t$  tangent to the path and  $\hat{e}_n$  perpendicular to  $\hat{e}_t$ . When  $P$  reaches to  $P'$ ,  $\hat{e}_t$  changes to  $\hat{e}'_t$  and  $\hat{e}_n$  changes to  $\hat{e}'_n$  having radius of curvature  $\delta$ .

Drawing both vectors from the same origin  $O$ , we define  $\Delta e_t = \hat{e}'_t - \hat{e}_t$ . Since  $\hat{e}_t$  and  $\hat{e}'_t$  are of unit length, their tips lie on the circle of radius  $r$ . Denoting by  $\Delta\theta$  the angle formed by  $\hat{e}_t$  and  $\hat{e}'_t$ ,

$$\text{Now, } \Delta s = PP' = \delta \cdot \Delta\theta \quad [\text{use } \theta = \frac{1}{r}]$$

$$\Delta \hat{e}_t = \hat{e}'_t - \hat{e}_t \equiv \Delta\theta \cdot \hat{e}_n$$

$$\text{Now, } \frac{d\theta}{ds} = \lim_{\Delta s \rightarrow 0} \frac{\Delta\theta}{\Delta s} = \frac{1}{\delta}$$

$$\frac{d \hat{e}_t}{d\theta} = \lim_{\Delta\theta \rightarrow 0} \frac{\Delta \hat{e}_t}{\Delta\theta} = \hat{e}_n$$

$$\text{Now } \vec{a} = \frac{d \vec{v}}{dt} = \frac{d}{dt} (v \hat{e}_t)$$

$$= \frac{dv}{dt} \hat{e}_t + v \frac{d \hat{e}_t}{dt}$$

$$= \frac{dv}{dt} \hat{e}_t + v \frac{d \hat{e}_t}{d\theta} \cdot \frac{d\theta}{ds} \cdot \frac{ds}{dt}$$

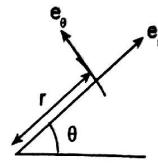
$$= \frac{dv}{dt} \hat{e}_t + v \cdot \hat{e}_n \cdot \frac{1}{\delta} v$$

$$\vec{a} = \frac{dv}{dt} \hat{e}_t + \frac{v^2}{\delta} \hat{e}_n$$

$$= \vec{a}_t + \vec{a}_n$$

### 8.5 Radial and transverse component

In certain problems of plane motion the position of the particle  $P$  is defined by its polar coordinates  $r$  and  $\theta$  as shown in figure. It is then convenient to resolve the velocity and acceleration of the particle into component parallel and perpendicular to the line  $OP$  respectively. These component are called radial and transverse component respectively.



The unit vector  $\hat{e}_r$  defines the radial direction i.e. the direction in which  $P$  would move if  $r$  were increased and  $\theta$  were kept constant.

The unit vector  $\hat{e}_\theta$  defines the transverse direction i.e. the direction in which  $P$  would move if  $\theta$  were increased and  $r$  were kept constant.

$$\vec{v}_r = \dot{r} \hat{e}_r$$

$$\therefore \vec{v} = \vec{v}_r + \vec{v}_\theta$$

$$= \dot{r} \hat{e}_r + r \dot{\theta} \hat{e}_\theta$$

$$\vec{a}_r = \ddot{r} - r \dot{\theta}^2 \quad \vec{a}_\theta = r \ddot{\theta} + 2 \dot{r} \dot{\theta}$$

$$\vec{a} = \vec{a}_r + \vec{a}_\theta$$

$$\vec{a} = (\ddot{r} - r \dot{\theta}^2) \hat{e}_r + (r \ddot{\theta} + 2 \dot{r} \dot{\theta}) \hat{e}_\theta$$

In case of particle moving along a circle of centre  $O$ , we have

$$r = \text{constant}$$

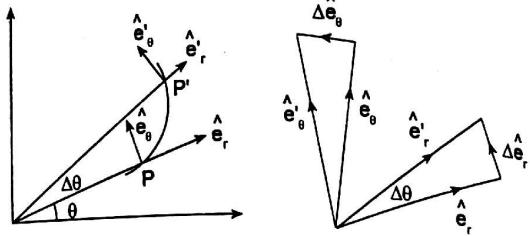
$$\dot{r} = 0, \ddot{r} = 0$$

$$\therefore \vec{v} = r \dot{\theta} \hat{e}_\theta; \vec{a} = -r \dot{\theta}^2 \hat{e}_r + r \ddot{\theta} \hat{e}_\theta$$

#### 8.5.1 Derivation of radial and transverse components

The position of the particle at  $P$  is defined by the coordinates  $r$  and  $\theta$  as shown where  $r$  is length in metre and  $\theta$  is angle in radian. The unit

vectors in radial and transverse direction are denoted by  $\hat{e}_r$  and  $\hat{e}_\theta$  respectively.



As a particle moves from P to P', the unit vector  $\hat{e}_r$  and  $\hat{e}_\theta$  changes to  $\hat{e}'_r$  and  $\hat{e}'_\theta$ .

$$\hat{e}'_r = \frac{d\hat{e}_r}{dt} = \frac{d\hat{e}_r}{d\theta} \cdot \frac{d\theta}{dt} = \hat{e}_\theta \cdot \dot{\theta}$$

Since  $d\hat{e}_r$  is directed in the direction of  $\hat{e}_\theta$

$$\hat{e}'_\theta = \frac{d\hat{e}_\theta}{dt} = \frac{d\hat{e}_\theta}{d\theta} \cdot \frac{d\theta}{dt} = -\hat{e}_r \cdot \dot{\theta}$$

Since  $d\hat{e}_\theta$  is directed in the direction of  $-\hat{e}_r$ .

$$\text{Now, } \vec{r} = r\hat{e}_r$$

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{d}{dt}(r\hat{e}_r)$$

$$\begin{aligned} &= r \frac{d\hat{e}_r}{dt} + \hat{e}_r \frac{dr}{dt} \\ &= r \dot{e}_r + \dot{e}_r r \\ &= r e_\theta \dot{\theta} + r \dot{e}_r \\ &= r \dot{\theta} \hat{e}_\theta + r \dot{e}_r \end{aligned}$$

$$\vec{v} = r \dot{e}_r + r \dot{\theta} \hat{e}_\theta$$

$$\begin{aligned} \vec{v} &= \vec{v}_r + \vec{v}_\theta \\ \vec{a} &= \frac{d\vec{v}}{dt} \\ \vec{a} &= \frac{d}{dt} [\vec{v}_r + \vec{v}_\theta] \\ &= \frac{d}{dt} [r \dot{e}_r + r \dot{\theta} \hat{e}_\theta] \\ \vec{a} &= \ddot{r} \hat{e}_r + \dot{r} \dot{e}_r + \dot{r} \dot{\theta} \hat{e}_\theta + r \ddot{\theta} \hat{e}_\theta + r \dot{\theta} \dot{\theta} \hat{e}_\theta \\ &= \ddot{r} \hat{e}_r + r \dot{\theta} \hat{e}_\theta + \dot{r} \dot{\theta} \hat{e}_\theta + r \ddot{\theta} \hat{e}_\theta + r \dot{\theta} (-\dot{\theta}) \hat{e}_r \\ &= (\ddot{r} - r \dot{\theta}^2) \hat{e}_r + (2r \dot{\theta} + r \ddot{\theta}) \hat{e}_\theta \\ &= \vec{a}_r + \vec{a}_\theta \end{aligned}$$

#### Quick Revision

$$\vec{v}_r = \dot{r} \hat{e}_r; \quad \vec{v}_\theta = r \dot{\theta} \hat{e}_\theta$$

$$\vec{v} = \vec{v}_r + \vec{v}_\theta$$

$$\vec{a}_r = (\ddot{r} - r \dot{\theta}^2) \hat{e}_r; \quad \vec{a}_\theta = (2r \dot{\theta} + r \ddot{\theta}) \hat{e}_\theta$$

$$\vec{a} = \vec{a}_r + \vec{a}_\theta$$

#### Worked Out Example

- The motion of particle is defined by the relation  $x = t^2 - 10t + 20$ , where  $x$  is expressed in meter and  $t$  in second. Determine
  - when the velocity is zero?
  - the position and the total distance travel when  $t = 8$  seconds.

*Solution:*

$$x = t^2 - 10t + 20$$

$$v = \frac{dx}{dt} = 2t - 10$$

If  $v = 0$ ;

$$2t - 10 = 0$$

$$\therefore t = 5 \text{ sec}$$

$$\text{At } t = 0; \quad x_0 = 20 \text{ m}$$

$$\text{At } t = 5; \quad x_5 = -5 \text{ m}$$

$$\text{At } t = 8; \quad x_8 = 4 \text{ m}$$

$x_8$  is the position at  $t = 8$  sec.

$$\therefore x_8 = 4 \text{ m}$$

Total distance travelled when  $t = 8$  sec is  $s_8$

$$\begin{aligned} s_8 &= |x_5 - x_0| + |x_8 - x_5| \\ &= |-5 - 20| + |4 - (-5)| \\ &= 25 + 9 = 34 \text{ m} \end{aligned}$$

2. The acceleration of a particle is directly proportional to the time  $t$ . At time  $t = 0$  sec, the velocity of the particle  $v = -16$  m/s. Knowing that both the velocity and the position coordinate are zero when  $t = 4$  sec, write the equation of motion of particle.

**Solution:**

$$a \propto t$$

$$a = kt$$

Integrating

$$\int a dt = \int kt dt$$

$$\text{or, } v = k \frac{t^2}{2} + c_1$$

$$\text{At } t = 0, \quad v = -16,$$

$$\therefore -16 = c_1$$

$$\therefore v = \frac{kt^2}{2} - 16$$

$$\text{At } t = 4, \quad v = 0$$

$$0 = k \cdot \frac{16}{2} - 16$$

$$\therefore k = 2$$

$$v = t^2 - 16$$

Integrating with respect to time;

$$x = \frac{t^3}{3} - 16t + c_2$$

$$\text{At } t = 4 \text{ sec; } x = 0$$

$$0 = \frac{64}{3} - 64 + c_2$$

$$\therefore c_2 = 42.67$$

$$\therefore x = \frac{t^3}{3} - 16t + 42.67$$

Equations of motion are

$$\text{a) } x = \frac{t^3}{3} - 16t + 42.67$$

$$\text{b) } v = t^2 - 16$$

$$\text{c) } a = 2t$$

3. The equation of motion of a particle is defined by the relation  $a = -0.4 v$  where  $a$  is the acceleration in  $\text{m/s}^2$  and  $v$  is the velocity in  $\text{m/s}$ . Find the time for which velocity will be zero. It is known that velocity is 30  $\text{m/s}$  at the instant of zero seconds.

**Solution:**

$$a = -0.4 v$$

$$\frac{dv}{dt} = -0.4 dt$$

Integrating

$$\ln v = 0.4t + c_1$$

$$\text{At } t = 0, \quad v = 30$$

$$\therefore \ln 30 = c_1$$

$$\therefore \ln v = -0.4t + \ln 30$$

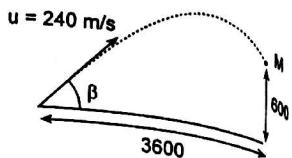
When  $v = 0$

$$\ln 0 = -0.4t + \ln 30$$

or,  $t = \infty$

$\therefore$  Time = infinity

4. A projectile is fired with an initial velocity of 240 m/s at a target M located 600 m above a gun G and at a horizontal distance of 3600 m. Neglecting air resistance, determine the value of firing angle  $\beta$ .



**Solution:**

For projectile motion

$$y = x \tan \beta - \frac{1}{2} g \frac{x^2}{u^2} (1 + \tan^2 \beta)$$

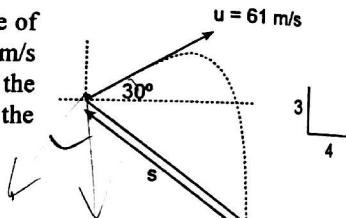
$$\text{or, } 600 = 3600 \tan \beta - 4.905 \frac{(3600)^2}{(240)^2} (1 + \tan^2 \beta)$$

$$\text{or, } 600 = 3600 \tan \beta - 1103.6 - 1103.6 \tan^2 \beta$$

$$\text{or, } 1103.6 \tan^2 \beta - 3600 \tan \beta + 1703.6 = 0$$

$$\therefore \beta = 69.58 \text{ and } 29.85$$

5. A particle is projected at an angle of  $30^\circ$  with an initial velocity of 61 m/s as shown in figure. Find, the slopping distance covered by the projectile.



**Solution:**

$$\text{Initial velocity } u = 61 \text{ m/s}$$

$$\text{Projection angle } \theta = 30^\circ$$

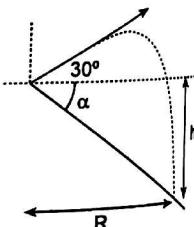
Let  $\alpha$  be the angle of inclination. Then,

$$\tan \alpha = \frac{3}{4}$$

$$\alpha = 36.87^\circ$$

Then,

$$\tan \alpha = \frac{h}{R}$$



$$\frac{3}{4} = \frac{h}{R}$$

$$R = \frac{4}{3}h \quad \text{and} \quad h = \frac{3}{4}R$$

Consider a vertical motion of projectile;

$$s_y = v_y t + \frac{1}{2} a_y t^2$$

$$-h = u \sin \theta \cdot t - \frac{1}{2} g t^2$$

$$-h = 61 \sin 30^\circ t - \frac{1}{2} \times 9.81 \times t^2$$

$$-h = 30.5 t - 4.905 t^2 \dots\dots (i)$$

Again,

$$R = u_x \times t$$

$$\frac{4h}{3} = u \cos \theta \times t$$

$$\frac{4h}{3} = 61 \cos 30 \times t$$

$$h = 39.62t \dots\dots (ii)$$

From eq. (i) and (ii)

$$-39.62t = 30.5t - 4.905t^2$$

$$4.905t^2 = 70.12t$$

$$\therefore t = 14.31 \text{ sec}$$

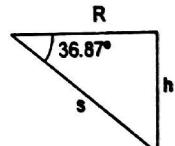
$$\therefore h = 39.62 \times 14.31 = 566.97 \text{ m}$$

To find the slopping distance s

$$\sin 36.87 = \frac{h}{s}$$

$$\therefore s = \frac{h}{\sin 36.87}$$

$$= \frac{566.97}{0.6} = 944.94 \text{ m}$$



6. It is determined that radius of curvature at A is  $\delta_A = 9$  m. Determine the velocity  $v_A$ . The radius of curvature of trajectory at its maximum height.

**Solution:**

**Case (a)**

$a_t$  will be along the  $v_A$  and  $a_n$  will be perpendicular to  $a_t$ .

Now, again draw a triangle

$$\cos 40^\circ = \frac{a_n}{g}$$

$$\therefore a_n = g \cos 40^\circ \\ = 9.81 \cos 40^\circ \\ = 7.51 \text{ m/s}^2$$

$$\text{Also, } a_n = \frac{v_A^2}{\delta_A}$$

$$7.51 = \frac{v_A^2}{9}$$

$$\therefore v_A = 8.22 \text{ m/s}$$

**Case (b)**

At maximum height

$$v_A' = v_A \cos 40^\circ = 6.2968 \text{ m/s}$$

$$a_n = g$$

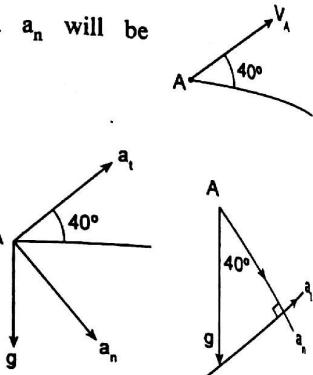
$$\therefore A_n = \frac{v_A^2}{\delta}$$

$$\delta = \frac{(6.2968)^2}{9.81} = 4.04 \text{ m}$$

This is the minimum radius of curvature.

7. The rectangular component of acceleration for a particle are  $a_x = 3t$ ,  $a_y = 30 - 10t$  where  $a$  is in  $\text{m/s}^2$  and  $t$  in seconds.

If the particle starts from rest at the origin, find the radius of curvature of the path at instant of 2 seconds.



**Solution:**

$$a_x = 3t$$

$$v_x = \frac{3t^2}{2} + c_1$$

$$\text{At } t = 0, v_x = 0, v_y = 0$$

$$\therefore c_1 = 0$$

$$\therefore v_x = \frac{3t^2}{2}$$

$$\text{Also, } a_y = 30 - 10t$$

$$v_y = 30t - 5t^2 + c_2$$

$$\text{At } t = 0, v_x = 0, v_y = 0$$

$$\therefore c_2 = 0$$

$$\therefore v_y = 30t - 5t^2$$

Again,

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{dt} \cdot \frac{dt}{dx} \\ &= \frac{dy}{dt} \times \frac{1}{\frac{dx}{dt}} \end{aligned}$$

$$\frac{dy}{dx} = \frac{v_y}{v_x}$$

$$\frac{dy}{dx} = \frac{30t - 5t^2}{\frac{3t^2}{2}}$$

$$\frac{dy}{dx} = \frac{60t - 10t^2}{3t^2}$$

$$\frac{dy}{dx} = \frac{20}{t} - \frac{10}{3}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx} \left[ \frac{20}{t} - \frac{10}{3} \right]$$

$$= \frac{d}{dt} \left( \frac{20}{t} - \frac{10}{3} \right) \cdot \frac{dt}{dx}$$

$$= \frac{-20}{t^2} \cdot \frac{1}{v_x}$$

$$= \frac{-20}{t^2} \cdot \frac{2}{3t^2}$$

$$\therefore \frac{d^2y}{dx^2} = -\frac{40}{3t^4}$$

At  $t = 2$  sec

$$\begin{aligned}\frac{dy}{dx} &= \frac{20}{t} - \frac{10}{3} \\ &= 10 - \frac{10}{3} = \frac{20}{3} \\ \frac{d^2y}{dx^2} &= -\frac{40}{3t^4} \\ &= -\frac{40}{3 \cdot 2^4} = -\frac{40}{48}\end{aligned}$$

Radius of curvature

$$\delta = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}$$

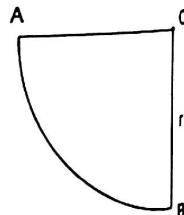
$$= \frac{\left[1 + \left(\frac{20}{3}\right)^2\right]^{3/2}}{-\frac{40}{48}} = -367.62$$

$\therefore$  radius of curvature = 368 m.

8. An automobile enters a curved road at 30 km/hr and then leaves at 48 km/hr. The curbed road is in the form of quarter of circle and has a length of 400 m. If the car travels at constant acceleration along the curve, calculate resultant acceleration at both ends of curve.

**Solution:**

$$P = \frac{2\pi r}{4}$$



$$400 = \frac{2\pi \times r}{4}$$

$$\therefore r = 254.6 \text{ m}$$

$$\text{Initially, } v_A = 30 \text{ km/hr} = 8.33 \text{ m/s}$$

$$\text{Finally, } v_B = 48 \text{ km/hr} = 13.33 \text{ m/s}$$

$$\text{Again, } v_B^2 = v_A^2 + 2 \cdot a_t \cdot s$$

$$(13.33)^2 = (8.33)^2 + 2 \times a_t \times 400$$

$$\therefore a_t = 0.1353 \text{ m/s}^2$$

At point A:

$$a_n = \frac{v_A^2}{\delta} = \frac{(8.33)^2}{254.6} = 0.6979 \text{ m/s}^2$$

At point B:

$$a_n = \frac{v_B^2}{\delta} = \frac{(13.33)^2}{254.6} = 0.6979 \text{ m/s}^2$$

Resultant acceleration at

a) Point A

$$(a_R)_A = (a_n)_A + a_t$$

$$= \sqrt{(a_n)_A^2 + a_t^2}$$

$$= \sqrt{(0.2725)^2 + (0.1353)^2}$$

$$= 0.304 \text{ m/s}$$

$$\tan \theta = \frac{a_t}{a_n}$$

$$\therefore \theta = 26.40^\circ$$

b) Point B:

$$(a_R)_B = (a_n)_B + a_t$$

$$= \sqrt{(a_n)_B^2 + a_t^2}$$

$$= \sqrt{(0.6979)^2 + (0.1353)^2}$$

$$= 0.7108 \text{ m/s}^2$$

$$\tan \beta = \frac{at}{a_n}$$

$$\therefore \beta = 10.97^\circ$$

9. For a pulley system as shown in figure, calculate the velocity and acceleration of the block C. If the velocity and accelerations of the blocks A and B are  $4 \text{ m/s}$  ( $\downarrow$ ),  $2 \text{ m/s}^2$  ( $\uparrow$ ),  $6 \text{ m/s}$  ( $\uparrow$ ) and  $4 \text{ m/s}^2$  ( $\uparrow$ ) respectively.

**Solution:**

Let the position of block A, B and C be  $x_A$ ,  $x_B$ ,  $x_C$  from the reference MN.

$$\text{Then, } x_A + x_B + x_C = k$$

$$x_A + 2x_B + x_C = k$$

Differentiating

$$v_A + 2v_B + v_C = 0$$

$$a_A + 2a_B + a_C = 0$$

$$\text{Given, } v_A = 4 \text{ m/s} (\downarrow),$$

$$v_B = 6 \text{ m/s} (\uparrow)$$

$$\therefore v_A + 2v_B + v_C = 0$$

$$\text{or, } -4 + 2 \times 6 + v_C = 0$$

$$\text{or, } -4 + 12 + v_C = 0$$

$$\therefore v_C = -8 \text{ m/s} = 8 \text{ m/s} (\downarrow)$$

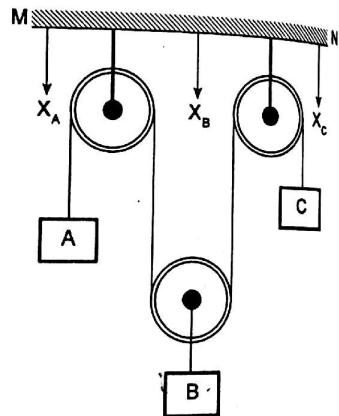
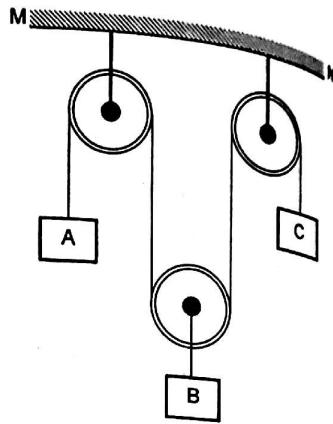
Also, given  $a_A = 2 \text{ m/s}^2$  ( $\uparrow$ ),  $a_B = 4 \text{ m/s}^2$  ( $\uparrow$ )

$$\text{or, } a_A + 2a_B + a_C = 0$$

$$\text{or, } 2 + 2 \times 4 + a_C = 0$$

$$\text{or, } 2 + 8 + a_C = 0$$

$$\therefore a_C = 10 \text{ m/s}^2$$
 ( $\uparrow$ )



10. The connection of block A, B and C with pulley as shown in figure. If the block B has velocity  $2 \text{ m/s}$  ( $\uparrow$ ) and its acceleration  $3 \text{ m/s}^2$  ( $\downarrow$ ) while block C has velocity and acceleration of  $2 \text{ m/s}$  ( $\uparrow$ ) and  $4 \text{ m/s}^2$  ( $\downarrow$ ) respectively. Find the velocity and acceleration of block A.

**Solution:**

Here are two different ropes.

$$\text{Here, } ab + bc + cd = \text{constant} = k \dots \text{(i)}$$

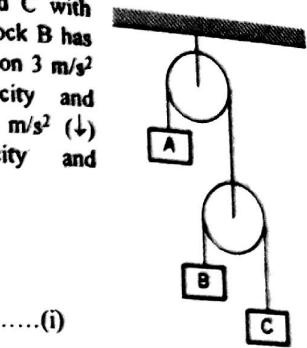
$$ef + fg + gh = \text{constant} = k \dots \text{(ii)}$$

From eq. (i)

$$ab + bc + cd = k$$

$$x_A + k + x_2 = k$$

$$x_A + x_2 = k \dots \text{(iii)}$$



From eq. (ii)

$$ef + fg + gh = k$$

$$x_B - x_2 + k + x_C - x_2 = k$$

$$x_B + x_C - 2x_2 = k \dots \text{(iv)}$$

Multiply eq. (iii) by (ii) and add to eq. (iv)

$$\text{or, } 2x_A + 2x_2 + x_B + x_C - 2x_2 = k$$

$$\text{or, } 2x_A + x_B + x_C = k$$

$$\therefore 2v_A + v_B + v_C = 0$$

$$\therefore 2a_A + a_B + a_C = 0$$

Given,  $v_B = 2 \text{ m/s}$  ( $\uparrow$ ),  $v_C = 2 \text{ m/s}$  ( $\uparrow$ )

$$\text{or, } 2v_A + v_B + v_C = 0$$

$$\text{or, } 2v_A + 2 + 2 = 0$$

$$\text{or, } 2v_A = -4$$

$$\therefore v_A = -2 \text{ m/s} = 2 \text{ m/s} (\downarrow)$$

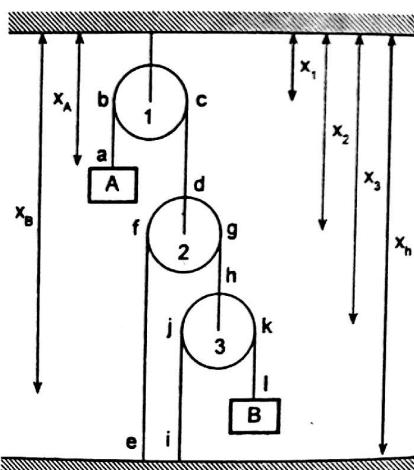
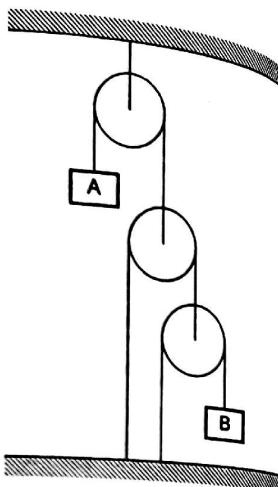
Given,  $a_B = 3 \text{ m/s}^2$  ( $\downarrow$ ),  $a_C = 4 \text{ m/s}^2$  ( $\downarrow$ )

$$\begin{aligned} \text{or, } & 2a_A + a_B + a_C = 0 \\ \text{or, } & 2a_A - 3 - 3 = 0 \\ \text{or, } & 2a_A = 6 \\ \therefore & a_A = 3 \text{ m/s}^2 (\uparrow) \end{aligned}$$

11. Find the acceleration of body B if the acceleration of A is  $4 \text{ m/s}^2 (\downarrow)$  for the following connection.

**Solution:**

$$\begin{aligned} ab + bc + cd = k \\ x_A + k + x_2 = k \\ x_A + x_2 = k \quad \dots \dots \text{(i)} \\ ef + fg + gh = k \\ \text{or, } x_h - x_2 + k + x_3 - x_2 = k \\ \text{or, } x_h - 2x_2 + x_3 = k \\ \text{or, } x_3 - 2x_2 = k \quad \dots \dots \text{(ii)} [x_n = k] \end{aligned}$$



$$ij + jk + kl = \text{constant} = k$$

$$\begin{aligned} \text{or, } & x_h - x_3 + k + x_B - x_3 = k \\ \text{or, } & x_h - 2x_3 + x_B = k \end{aligned}$$

$$\text{or, } x_B - 2x_3 = k \quad \dots \dots \text{(iii)}$$

Multiply eq. (i) by 4, eq. (ii) by 2

$$4x_A + 4x_2 = k \quad \dots \dots \text{(iv)}$$

$$2x_3 - 4x_2 = k \quad \dots \dots \text{(v)}$$

$$x_B - 2x_3 = k \quad \dots \dots \text{(vi) from eq. (iii)}$$

Adding all (iv), (v) and (vi)

$$4x_A + x_B = k$$

$$\text{or, } 4v_A + v_B = 0$$

$$\text{or, } 4a_A + a_B = 0$$

$$\therefore a_B = -4.a_A$$

$$= -4 \times 4 = -16 \text{ m/s}^2 (\downarrow) = 16 \text{ m/s}^2 (\uparrow)$$

12. The rotation of the 0.9 m arm OA about O is defined by the relation  $\theta = 0.15t^2$  where  $\theta$  is expressed in radian and  $t$  in second. Collar B slides along the arm in such a way that its distance from O is  $r = 0.9 - 0.12t^2$ ,  $r$  in meters and  $t$  in secs. After the arm OA has rotated through  $30^\circ$ , determine

- a) total velocity of the collar  
b) total acceleration of collar.

**Solution:**

$$\text{Given, } \theta = 30^\circ$$

$$\theta = 30 \times \frac{\pi}{180} = 0.524^\circ$$

$$\text{Also, } \theta = 0.15t^2 \text{ (by question)}$$

$$0.524 = 0.15 t^2$$

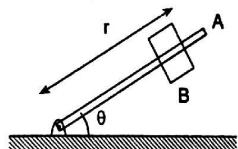
$$\text{or, } t^2 = 3.4906$$

$$\therefore t = 1.868 \text{ sec.}$$

$$\text{Now, } r = 0.9 - 0.12t^2 ; \quad \theta = 0.15 \times (1.868)^2$$

$$\text{At } t = 1.868 \text{ sec}$$

$$r = 0.9 - 0.12 \times (1.868)^2 ; \quad \theta = 0.15 \times (1.868)^2$$



$$\begin{aligned} r &= 0.481 \text{ m} \\ \dot{r} &= -2 \times 0.12 \text{ t} \\ &= -0.24 \times 1.868 \\ &= -0.448 \text{ m/s} \\ \ddot{r} &= -0.24 \text{ m/s}^2 \end{aligned} ; \quad \begin{aligned} \theta &= 0.524^\circ \\ \dot{\theta} &= 0.3t \\ &= 0.3 \times 1.868 \\ &= 0.56 \text{ rad/sec} \\ \ddot{\theta} &= 0.3 \text{ rad/sec}^2 \end{aligned}$$

$$v_r = \dot{r} = -0.448 \text{ m/s}$$

$$v_\theta = r\dot{\theta} = 0.481 \times 0.56 = 0.270 \text{ m/s}$$

$$\begin{aligned} v_B &= \sqrt{v_r^2 + v_\theta^2} \\ &= \sqrt{(0.448)^2 + (0.270)^2} = 0.523 \text{ m/s} \end{aligned}$$

$$\begin{aligned} \tan \beta &= \frac{v_\theta}{v_r} \\ &= \frac{0.270}{0.448} \end{aligned}$$

$$\therefore \beta = 31.076^\circ$$

$$\begin{aligned} a_r &= \ddot{r} - r\dot{\theta}^2 \\ &= -0.24 - 0.481 \times (0.561)^2 \\ &= -0.391 \text{ m/s}^2 \end{aligned}$$

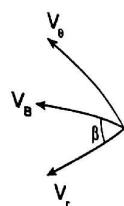
$$\begin{aligned} a_\theta &= r\ddot{\theta} + 2\dot{r}\dot{\theta} \\ &= 0.481 \times 0.3 + 2 \times (-0.449) \times 0.561 \\ &= -0.359 \text{ m/s}^2 \end{aligned}$$

$$\begin{aligned} \vec{a}_B &= \vec{a}_r + \vec{a}_\theta \\ \therefore a_B &= \sqrt{a_r^2 + a_\theta^2} \\ &= \sqrt{(0.391)^2 + (0.359)^2} = 0.531 \text{ m/s}^2 \end{aligned}$$

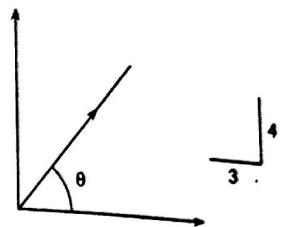
$$\begin{aligned} \tan \beta &= \frac{a_\theta}{a_r} \\ &= \frac{0.359}{0.391} \end{aligned}$$

$$\therefore \beta = 42.55^\circ$$

Acceleration of B with respect to OA is  $a_{B/OA} = \ddot{r} = -0.240 \text{ m/s}^2$



13. A particle starting from origin is subjected to acceleration, such that  $a_x = -3 \text{ m/s}^2$  and  $a_y = -11 \text{ m/s}^2$ . If the initial velocity is  $80 \text{ m/s}$  directed at a slope of  $4 : 3$  as shown. Compute the radius of curvature of path after  $4 \text{ sec}$ . Also find the position at the end of  $4 \text{ sec}$ .



*Solution:*

$$\tan \theta = \frac{4}{3} = 1.333$$

$$\text{So, } \sin \theta = \frac{4}{5} = 0.8$$

$$\cos \theta = \frac{3}{5} = 0.6$$

At  $t = 0 \text{ sec}$

$$(v_0)_x = 80 \cos \theta = 48 \text{ m/s}$$

$$(v_0)_y = 80 \sin \theta = 64 \text{ m/s}$$

At  $t = 4 \text{ sec}$ ,

$$\begin{aligned} (v_t)_x &= (v_0)_x + a_x t \\ &= 48 - 3 \times 4 = 36 \text{ m/s} \end{aligned}$$

$$\begin{aligned} (v_t)_y &= (v_0)_y + a_y t \\ &= 64 - 11 \times 4 = 20 \text{ m/s} \end{aligned}$$

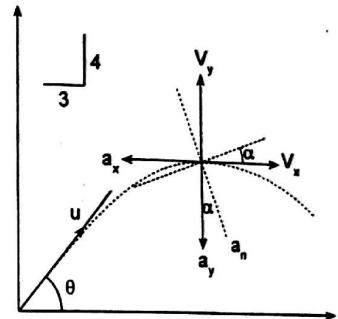
$$\begin{aligned} |\vec{v}_t| &= \sqrt{(v_t)_x^2 + (v_t)_y^2} \\ &= \sqrt{36^2 + 20^2} = 41.18 \text{ m/s} \end{aligned}$$

$$\tan \alpha = \frac{(v_t)_y}{(v_t)_x} = \frac{20}{36}$$

$$\therefore \alpha = 29.05^\circ$$

From figure  $a_n = a_y \cos \alpha - a_x \sin \alpha$

$$\begin{aligned} a_n &= 11 \cos 29.05 - 3 \sin 29.05 \\ &= 8.1594 \text{ m/s}^2 \end{aligned}$$



$$\text{Also, } a_n = \frac{v^2}{\delta}$$

$$\Rightarrow \delta = \frac{v^2}{a_n} = \frac{(41.18)^2}{8.1594} = 207.9 \text{ m}$$

$$x_4 = (v_0)_x \cdot t + \frac{1}{2} \cdot a_x \cdot t^2$$

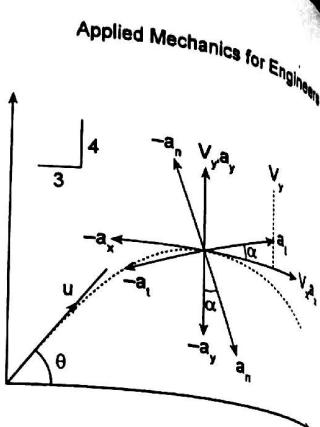
$$= 48 \times 4 + \frac{1}{2} \cdot -3 \cdot 4^2$$

$$= 168$$

$$y_4 = (v_0)_y \cdot t + \frac{1}{2} a_y t^2$$

$$= 64 \times 4 + \frac{1}{2} \cdot -11 \cdot 4^2 = 168$$

$$\therefore x_4, y_4 = 168, 168$$



### Practice Questions

1. The acceleration of a particle is defined by the relation  $a = -2 \text{ m/s}^2$ . Initially if velocity  $v_0 = 8 \text{ m/s}$  and position  $x_0 = 8 \text{ m}$ ; determine the velocity, position and the total distance travelled at the instant of 6 sec.

(Ans.  $-4 \text{ m/s}$ ;  $12 \text{ m}$ ;  $20 \text{ m}$ )

2. The acceleration of a particle is defined by the relation  $a = 12x - 28$  where  $a$  in  $\text{m/s}^2$  and  $x$  in m. knowing that  $v = 8 \text{ m/s}$  when  $x = 0$ ; determine:

a) the maximum value of  $x$ .

b) the velocity when the particle has travelled a total distance of 3 m.

(Ans. a)  $2 \text{ m}$ , b)  $-4.47 \text{ m/s}$ )

3. The acceleration of a particle is defined by the relation  $a = kt^2$ . Knowing that velocity is  $-32 \text{ m/s}$  when time is zero and again velocity is  $+32 \text{ m/s}$  when time is 4 sec. Write down the equation of motion knowing that the position of the particle is zero at the instant of 4 sec.

(Ans.  $x = \frac{t^4}{4} - 32t + 64$ ;  $v = t^3 - 32$ ,  $a = 3t^2$ )

### Applied Mechanics for Engineers

4. The motion of particle is defined by the relation  $x = 2t^3 - 12t^2 - 72t - 80$ ;  $x$  in m and  $t$  in sec. Determine the velocity, the acceleration and total distance travelled when  $x = 0$ .

(Ans.  $288 \text{ m/s}$ ,  $96 \text{ m/s}^2$ ,  $944 \text{ m}$ )

5. The acceleration of a particle in rectilinear motion is directly proportional to  $t^2$ . At  $t = 0$ , its velocity is  $-32 \text{ m/s}$ . At  $t = 2 \text{ sec}$ , the particle bounds to origin and its velocity becomes zero. Express the position  $x$  of the particle as a function of  $t$ .

(Ans.  $x = t^4 - 32t + 48$ )

6. A particle moving in a straight line has an acceleration,  $a = \sqrt{v}$ , its displacement and velocity at time  $t = 2 \text{ sec}$ , are  $\frac{128}{3} \text{ m}$  and  $16 \text{ m/s}$ . Find the displacement velocity and acceleration at time  $t = 3 \text{ sec}$ .

7. A projectile is fired from the edge of a  $150 \text{ m}$  cliff with an initial velocity of  $180 \text{ m/s}$  at an angle of  $30^\circ$  with the horizontal. Neglecting air resistance, find

a) the greatest elevation above the ground.

b) the horizontal distance from the gun to the point where the projectile strikes the ground.

c) the velocity with which it strikes ground.

(Ans. a)  $563 \text{ m}$ , b)  $3103 \text{ m}$ , c)  $188 \text{ m/s}$ ,  $\alpha = -34^\circ$ )

8. A projectile is aimed at a marked on the horizontal plane through a point of projection and falls  $10 \text{ m}$  short when the angle of projection is  $15^\circ$  while overshoots the mark by  $25 \text{ m}$  when the inclination is  $40^\circ$ . Calculate the distance of target and required angle of projection. Neglect air resistance.

(Ans.  $x = 46.09 \text{ m}$ ,  $\theta = 19.85^\circ$ )

9. The velocity of a particle, at its greatest height is  $\sqrt{\frac{2}{5}}$  times of its velocity at half of its greatest height. Show that the angle of projection is  $60^\circ$ .

10. A motorist is travelling on a curved section of highway of radius  $750 \text{ m}$  at the speed of  $90 \text{ km/hr}$ . The motorist suddenly applies the brake, causing the automobile to slow down at a constant rate. Knowing that after  $8 \text{ sec}$ , the speed has been reduced to  $72 \text{ km/hr}$ , determine the

acceleration of the automobile immediately after brake has been applied.

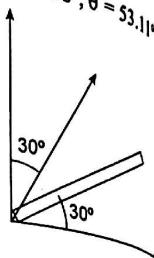
(Ans.  $1.041 \text{ m/s}^2, \theta = 53.11^\circ$ )

11. A projectile is launched from point A with an initial velocity  $v_0$  of  $40 \text{ m/s}$  at an angle of  $30^\circ$  with the vertical. Determine the radius of curvature of projectile described by the particle

a) At point A

b) At the point in the projectile where the velocity is parallel to inclined.

(Ans. a)  $326.5 \text{ m}$ , b)  $62.8 \text{ m}$ )



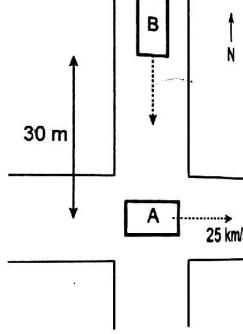
12. The motion of particle is given by the relation  $v_x = 2 \cos t$  and  $v_y = \sin t$ . It is known that initially both x and y coordinate are zero. Determine

a) Total acceleration at the instant of 2 sec.

b) The equation of path.

(Ans. a)  $1.8678 \text{ m/s}^2, \theta = 192.99^\circ$ , b)  $x^2 + 4y^2 - 8y = 0$ )

13. Automobile A is travelling east at the constant speed of  $25 \text{ km/hr}$ . As A crosses the intersection, auto B starts from rest  $30 \text{ m}$  north of intersection and moves south with a constant acceleration of  $1.2 \text{ m/s}^2$ . Determine the position, velocity and acceleration of B relative to A, five seconds after A crosses the intersection.



(Ans.  $x_{B/A} = 37.821$ ,  $v_{B/A} = 9.177$ ,  $a_{B/A} = -1.2(\downarrow)$ )

## Chapter 9

### Kinetics of Particles and Rigid Body

Kinetics is the study of relation existing between the forces acting on a body, the mass of body and the motion of body. It is used to predict the motion caused by given force or to determine the forces required to produce a given motion.

#### 9.1 Newton's second law and momentum

##### 9.1.1 Newton's second law

It states that "If the resultant force acting on a particle is not zero, the particle will have an acceleration proportional to the magnitude of the resultant and in the direction of this resultant force."

$$F \propto a$$

$$\frac{F_1}{a_1} = \frac{F_2}{a_2} = \text{constant}$$

Therefore when particle is subjected to several forces

$$\Sigma F = ma$$

$$\Sigma F_x = ma_x$$

$$\Sigma F_y = ma_y$$

##### 9.1.2 Linear momentum

We know from Newton's second law

$$\vec{\Sigma F} = m \vec{a}$$

$$\vec{\Sigma F} = m \frac{d \vec{v}}{dt}$$

$$\vec{\Sigma F} = \frac{d}{dt} (m \vec{v})$$

The vector  $\vec{m v}$  is called linear momentum or simply momentum. It has the same direction as the velocity and magnitude equal to product of mass and velocity. Denoting linear momentum by  $L$

$$\vec{L} = m \vec{v}$$

Newton's law can be restated as "The resultant of the forces acting on a particle is equal to the rate of change of linear momentum of the particle."

$$\therefore \vec{\Sigma F} = \vec{L}$$

## 9.2 Equation of motion and dynamic equilibrium

### 9.2.1 Equation of motion

#### a) Rectangular component

$$\Sigma F_x = ma_x, \quad \Sigma F_y = ma_y, \quad \Sigma F_z = ma_z$$

#### b) Tangential and normal components

$$\Sigma F_t = ma_t, \quad \Sigma F_n = ma_n, \quad \Sigma F_t = m \frac{dv}{dt}, \quad \Sigma F_n = m \frac{v^2}{\delta}$$

#### c) Radial and transverse component

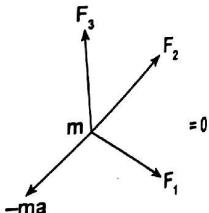
$$\Sigma F_r = a_r, \quad \Sigma F_\theta = a_\theta$$

### 9.2.2 Dynamic equilibrium

From Newton's second law

$$\vec{\Sigma F} = m \vec{a}$$

$$\text{or, } \vec{\Sigma F} - m \vec{a} = 0$$



The vector  $-ma$ , of magnitude  $ma$  and direction opposite to that of acceleration is called inertia vector.

So, when the particle is considered to be in equilibrium under the given forces and inertia vector  $-ma$ ; it is said to be in dynamic equilibrium. The inertia vector measures the resistance that particle offer

when we try to set them in motion or when we try to change the condition of their motion.

$$\Sigma F_x - ma_x = 0 \quad \Sigma F_t - ma_t = 0$$

$$\Sigma F_y - ma_y = 0 \quad \Sigma F_n - ma_n = 0$$

$$\Sigma F_z - mz_z = 0 \quad \Sigma F_r - ar_r = 0$$

$$\Sigma F_\theta - a_\theta = 0$$

## 9.3 Angular momentum and rate of change

### 9.3.1 Angular momentum

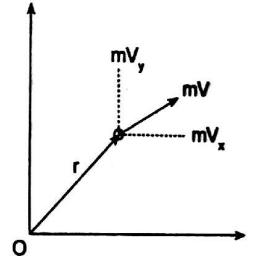
The moment of the vector  $m v$  about a reference point say (O) is called angular momentum of the particle.

$$H_0 = \vec{r} \times \vec{m v}$$

$$\vec{r} = x \vec{i} + y \vec{j} + z \vec{k}$$

$$\vec{v} = v_x \vec{i} + v_y \vec{j} + v_z \vec{k}$$

$$H_0 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x & y & z \\ mv_x & mv_y & mz_z \end{vmatrix}$$



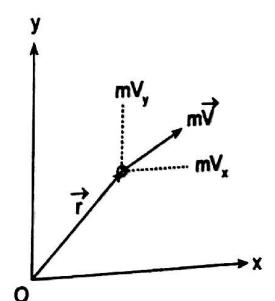
### 9.3.2 Rate of change of angular momentum

Consider a particle of mass  $m$  moving in the  $xy$  plane as shown in figure.

$$H_0 = \vec{r} \times \vec{m v}$$

$$= m v_y x - m v_x y$$

$$\frac{d}{dt}(H_0) = \frac{d}{dt}(m v_y x - m v_x y)$$



$$\dot{H}_O = m[\dot{x}v_y + xv\dot{y} - \dot{v}_x y - \dot{y}v_x]$$

$$\dot{H}_O = [v_x v_y + x a_y - y a_x - v_y v_x]m$$

$$= m(xa_y - ya_x)$$

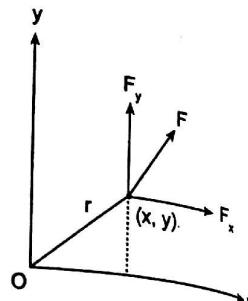
$$= xma_y - yma_x$$

$$= xF_y - yF_x$$

= moment of force about O

So, rate of change of angular momentum of a particle about any point at any

instant is equal to the moment of force ( $\vec{F}$ ) about that point.



#### 9.4 Principle of impulse and momentum

Consider a particle of mass  $m$  acted upon by a force  $F$ , then from Newton's second law

$$\vec{F} = \frac{d}{dt} m \vec{v}$$

where  $m \vec{v}$  is the linear momentum.

$$\vec{F} dt = d(m \vec{v})$$

Integrating both sides;

$$\int_{t_1}^{t_2} \vec{F} dt = \int_{v_1}^{v_2} d(m \vec{v})$$

$$\int_{t_1}^{t_2} \vec{F} dt = \vec{mv}_2 - \vec{mv}_1$$

$$\int_{t_1}^{t_2} \vec{F} dt = \vec{mv}_2 - \vec{mv}_1$$

$$\therefore \int_{t_1}^{t_2} \vec{F} dt + \vec{mv}_1 = \vec{mv}_2$$

Here the  $\int_{t_1}^{t_2} \vec{F} dt$  is called linear impulse or simply impulse.

$$\therefore \text{Imp}_{1-2} = \int_{t_1}^{t_2} \vec{F} dt$$

$$= \vec{i} \int_{t_1}^{t_2} F_x dt + \vec{j} \int_{t_1}^{t_2} F_y dt + \vec{k} \int_{t_1}^{t_2} F_z dt$$

$$\therefore \vec{mv}_1 + \text{Imp}_{1-2} = \vec{mv}_2$$

So, when a particle is acted upon by a force  $F$  during a given time interval, the final momentum  $m \vec{v}_2$  of the particle can be obtained by adding vectorially its initial momentum and the impulse of the force  $F$  during the time interval considered.

When several forces acts on particle

$$\vec{mv}_1 + \sum \text{Imp}_{1-2} = \vec{mv}_2$$

If problem involves two or more particles

$$\sum m \vec{v}_1 + \sum \text{Imp}_{1-2} = \sum m \vec{v}_2$$

If no external forces act

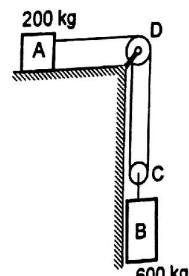
$$\sum m \vec{v}_1 = \sum m \vec{v}_2$$

#### Worked Out Examples

- The two blocks shown start from rest. The horizontal plane and the pulley are frictionless and the pulley is assumed to be of negligible mass. Determine the acceleration of each block and the tension in each cord.

*Solution:*

In such type, we first find the relation between velocity and acceleration of each block and use Newton's second law.



$x_A \rightarrow$  position of block A $x_B \rightarrow$  position of block B $x_C \rightarrow$  position of block CHere  $x_C + k = x_B$ 

$$\therefore x_B = x_C$$

Here ADCO is a continuous cord.

$$AD + DC + CO = k$$

$$x_A + x_C + x_C = k$$

$$x_A + 2x_C = k$$

$$x_A + 2x_B = k$$

$$v_A = -2v_B \text{ and } a_A = -2a_B$$

-ve sign shows that if block A moves towards reference pulley D, then block B will move away from reference pulley D and vice-versa.

**Block A:**Let  $T_1$  be tension on cord 1.

$$\Sigma F_x = ma$$

$$T_1 = m_A \cdot a_A = 200a_A \quad \dots \text{(i)}$$

**Block B**Let  $T_2$  be tension on cord 2.

$$\Sigma F_y = ma$$

$$m_B \cdot g - T_2 = m_B \cdot a_B$$

$$5886 - T_2 = 600a_B \quad \dots \text{(ii)}$$

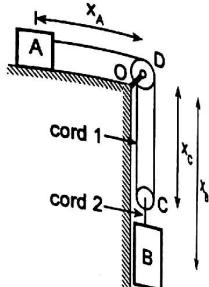
**Pulley C**

$$T_1 + T_1 = T_2$$

$$2T_1 = -T_2 \quad \dots \text{(iii)}$$

Using  $a_A = 2a_B$  and  $2T_1 = T_2$  in eq. (i)

$$\text{or, } \frac{T_2}{2} = 200 \times 2a_B$$



$$\Rightarrow T_2 = 800a_B \quad \dots \text{(iv)}$$

From eq. (ii) and (iv)

$$5886 - 800a_B = 600a_B$$

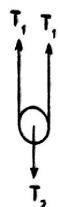
$$1400a_B = 5886$$

$$\therefore a_B = 4.204 \text{ m/s}^2 (\downarrow)$$

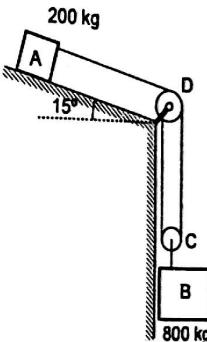
$$\therefore a_A = 2a_B = 8.408 \text{ m/s}^2 (\rightarrow)$$

$$\therefore T_2 = 800a_B = 3363.42 \text{ N}$$

$$\therefore T_1 = \frac{T_2}{2} = 1681.7 \text{ N}$$



2. Two blocks start from rest. The pulley are frictionless and having no mass. If  $u_k$  between block A and inclined plane is 0.4. Determine the acceleration of each block and tension in each cord.

**Solution:** $x_A \rightarrow$  position of block A $x_B \rightarrow$  position of block B $x_C \rightarrow$  position of block C

$$x_C + k = x_B$$

$$\therefore x_B = x_C$$

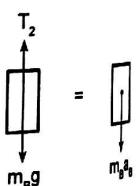
ADCO is a continuous cord.

$$AD + DC + CO = k$$

$$x_A + 2x_C = k$$

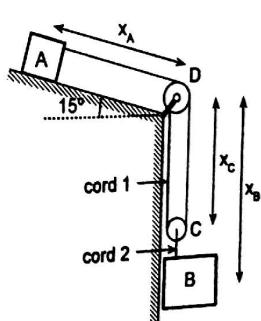
$$x_A + 2x_B = k$$

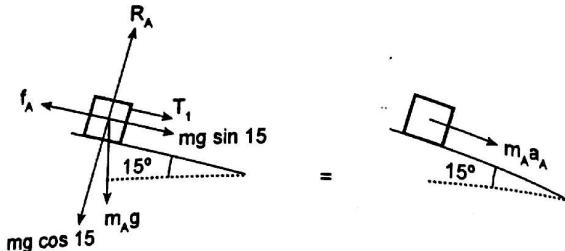
$$v_A = -2v_B \text{ and } a_A = -2a_B$$

**Block A:**

$$\Sigma F_x = ma$$

$$T_1 + m_A \cdot g \sin 15 - f_A = m_A a_A$$





$$T_1 + 507.8 - \mu_k R_A = 200a_A$$

$$T_1 + 507.8 - 0.4 m_A g \cos 15 = 200a_A$$

$$T_1 + 507.8 - 758.05 = 200a_A$$

$$T_1 - 200a_A - 250.25 = 0 \dots\dots (i)$$

**Block B:**

$$\Sigma F_y = ma$$

$$\text{or, } m_B g - T_2 = m_B a_B$$

$$\text{or, } 7848 - T_2 = 800a_B \dots\dots (ii)$$

**Pulley C**

$$T_2 = 2T_1 \dots\dots (iii)$$

Use  $a_A = 2a_B$  and  $T_2 = 2T_1$  in eq. (i)

$$\frac{T_2}{2} - 200(2a_B) - 250.25 = 0$$

$$T_2 - 800a_B - 500.5 = 0 \dots\dots (iv)$$

From eq. (ii) and eq. (iv)

$$T_2 - 800a_B - 500.5 = 0$$

$$-T_2 - 800a_B + 7848 = 0$$

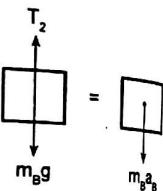
$$-1600a_B = -7347.5$$

$$\therefore a_B = 4.592 \text{ m/s}^2 (\downarrow)$$

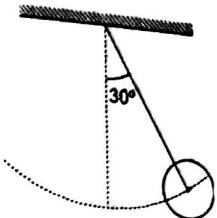
$$a_A = 9.184 \text{ m/s}^2$$

$$\therefore T_1 = 2087.125 \text{ N}$$

$$\therefore T_2 = 4174.25 \text{ N}$$



3. The bob of 6 m pendulum describe an arc of a circle in a vertical plane. If the tension in the cord is 2 times the weight of bob for the position when the bob is displaced through an angle  $30^\circ$  from its mean position. Find the velocity and acceleration of bob for this situation.



**Solution:** Consider a tangential direction,

$$mg \sin 30 = ma_t$$

$$g \sin 30 = a_t$$

$$\therefore a_t = 4.905 \text{ m/s}^2$$

Normal component is directed towards centre

$$T - mg \cos 30 = ma_n$$

$$2mg - mg \cos 30 = ma_n$$

$$2g - g \cos 30 = a_n$$

$$\therefore a_n = 11.12 \text{ m/s}^2$$

$$\therefore \vec{a} = \vec{a}_t + \vec{a}_n$$

$$a = \sqrt{a_t^2 + a_n^2} = \sqrt{(4.905)^2 + (11.12)^2} = 12.15 \text{ m/s}^2$$

$$\tan \beta = \frac{a_t}{a_n} = \frac{4.905}{11.12}$$

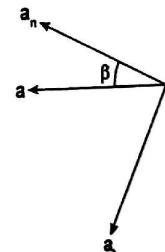
$$\therefore \beta = 23.78^\circ$$

$$\text{Again, } a_n = \frac{v^2}{\delta}$$

$$11.12 = \frac{v^2}{6}$$

$$v^2 = 66.745$$

$$\therefore v = 8.16 \text{ (along } a_t)$$



4. A 2 kg ball revolves in a horizontal circle as shown at a constant speed of 1.5 m/s knowing that  $L = 600 \text{ mm}$ . Determine a)  $\theta$ , b) tension.

**Solution:**

$$\text{Radius } r = L \cos (90 - \theta)$$

$$= L \sin \theta$$

$$= 600 \sin \theta$$

$$\Sigma F_y = ma_y$$

$$T \cos \theta - W = 0$$

$$T \cos \theta = 2g \dots \text{(i)}$$

$$\Sigma F_x = ma_x$$

$$T \sin \theta = ma$$

$$\frac{2g}{\cos \theta} \sin \theta = 2a$$

$$g \tan \theta = a \dots \text{(ii)}$$

$$\text{Also } a_n = \frac{v^2}{r}$$

For uniform circular motion

$$a_n = a$$

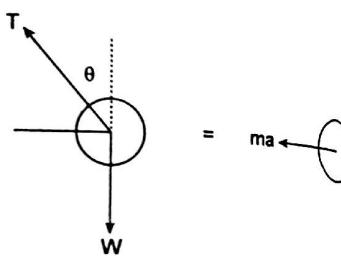
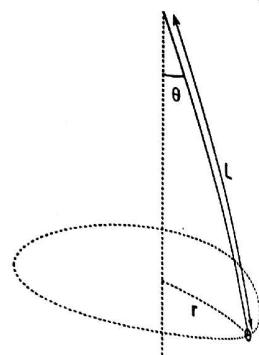
$$\therefore g \tan \theta = \frac{v^2}{\left(\frac{600}{1000}\right) \sin \theta}$$

$$\tan \theta \sin \theta = \frac{(1.5)^2}{9.81 \times 0.6}$$

$$\tan \theta \sin \theta = 0.38226$$

$$\text{solving, } \theta = 34.21^\circ$$

$$\text{Tension } T = \frac{2g}{\cos \theta} = \frac{2 \times 9.81}{\cos 34.21} = 23.72 \text{ N}$$



5. A single wire ACB of length 2 m passes through a ring at C that is attached to a sphere which revolves at a constant speed  $v$  in the horizontal circle shown. Knowing that  $\theta_1 = 60^\circ$  and  $\theta_2 = 30^\circ$  and that the tension is the same in both portions of the wire, determine the speed  $v$ .

**Solution:** Let  $r$  be the radius of horizontal circle. The length of wire is

$$L = \frac{r}{\sin \theta_1} + \frac{r}{\sin \theta_2}$$

$$\therefore r = \frac{L \sin \theta_1 \sin \theta_2}{\sin \theta_1 + \sin \theta_2}.$$

$$\Sigma F_y = 0$$

$$T \cos \theta_1 + T \cos \theta_2 - mg = 0$$

$$\therefore T = \frac{mg}{\cos \theta_1 + \cos \theta_2}$$

$$\Sigma F_x = ma$$

$$T \sin \theta_1 + T \sin \theta_2 = m \frac{v^2}{r}$$

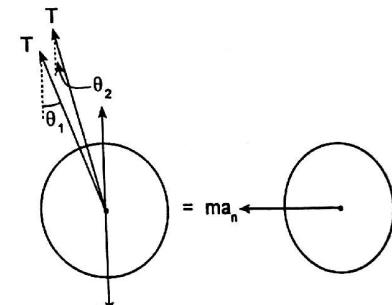
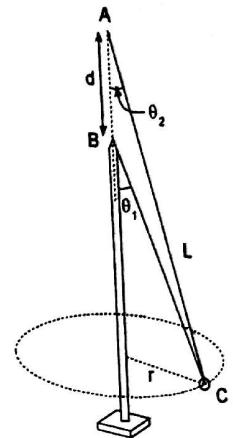
$$mg \frac{\sin \theta_1 + \sin \theta_2}{\cos \theta_1 + \cos \theta_2} = mv^2 \frac{\sin \theta_1 + \sin \theta_2}{2 \sin \theta_1 \sin \theta_2}$$

$$\frac{g L \sin \theta_1 \sin \theta_2}{\cos \theta_1 + \cos \theta_2} = v^2$$

$$v^2 = 6.2193$$

$$\therefore v = 2.49 \text{ m/s}$$

6. The resultant external force acting on a 2 kg particle in space is  $\vec{F} = (12t \vec{i} - 24t^2 \vec{j} - 40t^3 \vec{k}) \text{ N}$ . The particle is at origin and at rest



initially. Determine y-component of acceleration, velocity and position at the instant of 4 sec.

**Solution:**

$$\vec{F} = 12t \hat{i} - 24t^2 \hat{j} - 40t^3 \hat{k} \text{ N}$$

$$\therefore F_x = 12t; F_y = -24t^2; F_z = -40t^3$$

Considering y-component

$$F_y = -24t^2$$

$$m a_y = -24t^2$$

$$a_y = -\frac{24}{2} t^2 = -12t^2$$

$$\text{At } t = 4 \text{ sec; } a_y = -12 \times 16 = -192 \text{ } \hat{j} \text{ m/s}^2$$

Again,

$$a_y = -12t^2$$

$$v_y = -12 \frac{t^3}{3} + c_1$$

$$\text{At } t = 0; v = 0$$

$$\therefore c_1 = 0$$

$$\therefore v_y = -4t^3$$

$$\text{At } t = 4 \text{ sec; } v_y = -256 \text{ } \hat{j} \text{ m/s}$$

$$v_y = -4t^3$$

$$s_y = -4 \frac{t^4}{4} + c_2 = -t^4 + c_2$$

$$\text{At } t = 0, s = 0$$

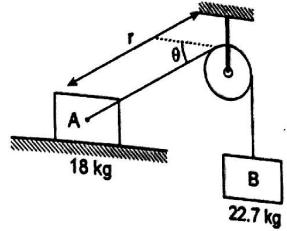
$$c_2 = 0$$

$$s_y = -t^4$$

$$\text{At } t = 4 \text{ sec; } s_y = -256 \text{ } \hat{j} \text{ m}$$

7. The two blocks are released from rest when  $r = 0.73 \text{ m}$  and  $\theta = 30^\circ$ . Neglecting the mass of pulley and the effect of friction in the pulley and between block A and the horizontal surface. Determine

- the initial tension in the cable.
- the initial acceleration of block A
- the initial acceleration of block B



**Solution:**

Let  $y_B$  be the position of block B from reference. Then for cable;

$$r + y_B = k$$

$$r + v_B = 0$$

$$r + a_B = 0$$

$$\therefore r = -a_B$$

For block A

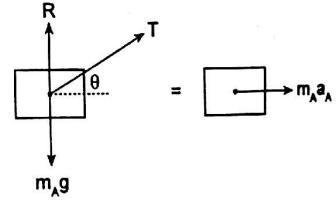
$$\Sigma F = ma$$

$$T \cos \theta = m_A \cdot a_A$$

$$T = m_A \cdot a_A \sec \theta$$

$$= 18 a_A \sec 30$$

$$= 20.78 a_A \dots\dots (i)$$



For block B

$$\Sigma F = ma$$

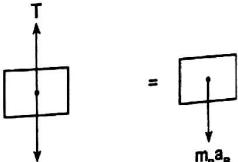
$$m_B g - T = m_B \cdot a_B$$

$$22.7 \times 9.81 - T = 22.7 \times a_B$$

$$222.68 - 20.78 a_A = 22.7 a_B$$

$$\therefore 22.7 a_B + 20.78 a_A - 222.68 = 0 \quad (\text{use eq. (i)})$$

$$\dots\dots (ii)$$



Now it is important to establish relation between  $a_A$  and  $a_B$ .  
For initial case  $a_B = a_A \cos \theta$

But for different case like if velocity of block A is given, the relation has to be calculated using radial and transverse component as below.

#### Proof for initial case

Positive direction for increase in radius and angle.

We know,

$$\cos \theta = \frac{-a_r}{a_A} [-\text{sign w.r.t. A}]$$

$$\therefore a_r = -a_A \cos \theta$$

Also,

$$a_r = \ddot{r} - r \dot{\theta}^2$$

initially  $\dot{\theta} = 0$  [if velocity is given you have to calculate  $\dot{\theta}$  using  $v_r$ ,  $a_A$  and  $v_\theta$ ]

$$\therefore a_r = \ddot{r}$$

$$\text{So, } \ddot{r} = -a_A \cos \theta$$

$$-a_B = -a_A \cos \theta$$

$\therefore a_B = a_A \cos \theta$  hence proved.

From eq. (ii)

$$22.7 a_B + 20.78 a_A - 222.68 = 0$$

$$\text{or, } 22.7 a_A \cos 30 + 20.78 a_A - 222.68 = 0$$

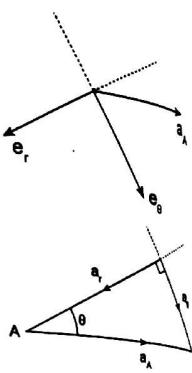
$$\text{or, } 19.658 a_A + 20.78 a_A - 222.68 = 0$$

$$\text{or, } 40.438 a_A = 222.68$$

$$\therefore a_A = 5.5065 \text{ m/s}^2$$

$$\therefore a_B = a_A \cos 30^\circ = 4.768 \text{ m/s}^2$$

$$\therefore T = 20.78 a_A = 114.42 \text{ N}$$



3. The system of particle at a time t is shown in the figure below

$$\begin{aligned}m_1 &= 0.5 \text{ kg} \\v_1 &= 7 \text{ m/s} \\m_2 &= 1.5 \text{ kg} \\v_2 &= 6 \text{ m/s} \\m_3 &= 1 \text{ kg} \\v_3 &= 5 \text{ m/s} \\m_4 &= 0.5 \text{ kg} \\v_4 &= 1.5 \text{ m/s}\end{aligned}$$

Determine:

- total linear momentum of the system.
- angular momentum of system about O.
- angular momentum of system about O.

Solution:

Write the coordinate of all point and find the direction of  $v_1$ ,  $v_2$ ,  $v_3$  and  $v_4$ .

$$m_1(0, 0, 3) \quad m_2(0, 4, 0) \quad m_3(2.5, 4, 0) \quad m_4(0, 0, 0)$$

$$O(0, 0, 0) \quad a(2.5, 4, 3) \quad b(2.5, 0, 3)$$

$$\text{Unit vector along } v_1 = \hat{n}_1 = \frac{\vec{m}_1 a}{|\vec{m}_1 a|} = \frac{\vec{a} - \vec{m}_1}{|\vec{m}_1 a|}$$

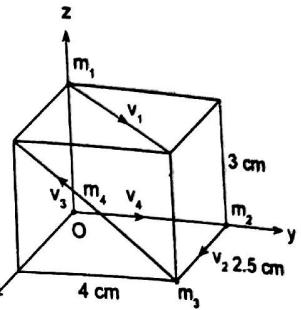
$$= \frac{(2.5, 4, 3) - (0, 0, 3)}{|\vec{m}_1 a|}$$

$$= \frac{2.5 \vec{i} + 4 \vec{j}}{\sqrt{(2.5)^2 + 4^2}}$$

$$= \frac{2.5 \vec{i} + 4 \vec{j}}{4.71}$$

$$\hat{n}_1 = 0.529 \vec{i} + 0.847 \vec{j}$$

$$\therefore \vec{v}_1 = \hat{n}_1 \cdot \vec{v}_1$$



$$= 7(0.529 \vec{i} + 0.847 \vec{j}) \\ = 3.703 \vec{i} + 5.929 \vec{j}$$

Unit vector along  $\vec{v}_2 = \hat{\vec{i}}$

$$\vec{v}_2 = v_2 \hat{\vec{i}} = 6 \vec{i}$$

$$\text{Unit vector along } \vec{v}_3 = \hat{n}_3 = \frac{\vec{m}_3 \vec{b}}{|\vec{m}_3 \vec{b}|} = \frac{\vec{b} - \vec{m}_3}{|\vec{m}_3 \vec{b}|} \\ = \frac{(2.5, 0, 3) - (2.5, 4, 0)}{|\vec{m}_3 \vec{b}|} \\ = \frac{-4 \vec{j} + 3 \vec{k}}{\sqrt{4^2 + 3^2}}$$

$$\hat{n}_3 = -0.8 \vec{j} + 0.6 \vec{k}$$

$$\therefore \vec{v}_3 = v_3 \hat{n}_3 \\ = 5(-0.8 \vec{j} + 0.6 \vec{k}) \\ = -4 \vec{j} + 3 \vec{k}$$

Unit vector along  $\vec{v}_4 = \hat{\vec{j}}$

$$\vec{v}_4 = v_4 \hat{\vec{j}} = 1.5 \vec{j}$$

Total linear momentum

$$\vec{L} = \sum m \vec{v} \\ = m_1 \vec{v}_1 + m_2 \vec{v}_2 + m_3 \vec{v}_3 + m_4 \vec{v}_4 \\ = 0.5(3.703 \vec{i} + 5.929 \vec{j}) + (1.5) 6 \vec{i} + 1.(-4 \vec{j} + 3 \vec{k}) \\ + (0.5).1.5 \vec{j}$$

$$= 10.85 \vec{i} - 0.29 \vec{j} + 3 \vec{k} \text{ kg m/s} \\ |\vec{L}| = \sqrt{(10.85)^2 + (0.29)^2 + 3^2} = 11.2608 \text{ kg m/s}$$

Angular momentum about O

$$\vec{H}_0 = \sum (\vec{r} \times \vec{m} \vec{v})$$

$$\vec{r}_1 = \vec{r}_{m1/O} = \vec{m}_1 - \vec{O} = (0, 0, 3) - (0, 0, 0) = 3 \vec{k}$$

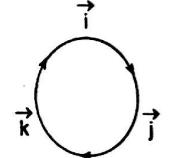
$$\vec{r}_2 = \vec{r}_{m2/O} = \vec{m}_2 - \vec{O} = (0, 4, 0) - (0, 0, 0) = 4 \vec{j}$$

$$\vec{r}_3 = \vec{r}_{m3/O} = \vec{m}_3 - \vec{O} = (2.5, 4, 0) - (0, 0, 0) = 2.5 \vec{i} + 4 \vec{j}$$

$$\vec{r}_4 = \vec{r}_{m4/O} = \vec{m}_4 - \vec{O} = (0, 0, 0) - (0, 0, 0) = 0$$

$$\vec{H}_0 = \sum \vec{r} \times \vec{m} \vec{v}$$

$$= 3 \vec{k} \times 0.5(3.703 \vec{i} + 5.929 \vec{j}) \\ + 4 \vec{j} \times (1.5) 6 \vec{i} + (2.5 \vec{i} + 4 \vec{j}) \\ \times 1.(-4 \vec{j} + 3 \vec{k}) + 0$$



$$= 5.55 \vec{j} - 8.89 \vec{i} - 36 \vec{k} - 10 \vec{k} - 7.5 \vec{j} + 12 \vec{i} \\ = 3.106 \vec{i} - 1.95 \vec{j} - 46 \vec{k}$$

$$|\vec{H}_0| = \sqrt{(3.106)^2 + (1.95)^2 + (46)^2} = 46.146 \text{ kg m}^2/\text{s}^2$$

Angular momentum about a

$$\vec{H}_a = \sum \vec{r} \times \vec{m} \vec{v}$$

$$\vec{r}_1 = \vec{r}_{m1/a} = \vec{m}_1 - \vec{a} = (0, 0, 3) - (2.5, 4, 3) = -2.5 \vec{i} - 4 \vec{j}$$

$$\vec{r}_2 = \vec{r}_{m2/a} = \vec{m}_2 - \vec{a} = (0, 4, 0) - (2.5, 4, 3) = -2.5 \vec{i} - 3 \vec{k}$$

$$\vec{r}_3 = \vec{r}_{m3/a} = \vec{m}_3 - \vec{a} = (2.5, 4, 0) - (2.5, 4, 3) = -3 \vec{k}$$

$$\begin{aligned} \vec{r}_4 &= \vec{r}_{m4/a} = m_4 - \vec{a} = (0, 0, 0) - (2.5, 4, 3) = -2.5 \vec{i} - 4 \vec{j} - 3 \vec{k} \\ H_a &= \sum \vec{r} \times m \vec{v} \\ &= (-2.5 \vec{i} - 4 \vec{j}) \times (0.5)(3.703 \vec{i} + 5.929 \vec{j}) \\ &\quad + (-2.5 \vec{j} - 3 \vec{k}) \times (1.5) \cdot \vec{i} + -3 \vec{k} \times 1.(-4 \vec{j} + 3 \vec{k}) \\ &\quad + (-2.5 \vec{i} - 4 \vec{j} - 3 \vec{k}) \times (0.5)(1.5 \vec{j}) \\ \vec{H}_a &= -9.75 \vec{i} - 27 \vec{j} - 1.875 \vec{k} \end{aligned}$$

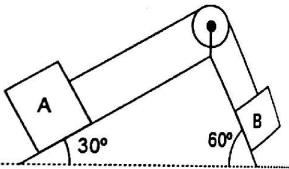
$$|H_a| = \sqrt{(9.75)^2 + 27^2 + (1.875)^2} = 28.767 \text{ kg m}^2/\text{s}^2$$

### Practice Question

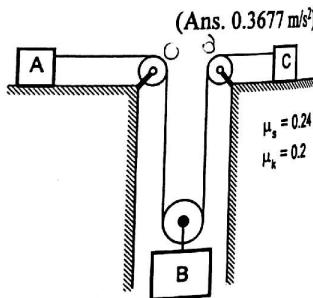
1. A force given by  $\vec{F} = 3t^2 \vec{i} + 5t \vec{j} - (8t^3 + 400) \vec{k}$  acts from  $t=0$  to  $t=10$  sec. Determine the impulse of force.

$$(Ans. 1000 \vec{i} + 250 \vec{j} - 24000 \vec{k} \text{ Ns})$$

2. Two rough plane inclined at  $30^\circ$  and  $60^\circ$  to the horizontal and same height are placed back to back. Mass of A is 12 kg and B is 24 kg and are connected by string. If  $\mu = 0.6$ , find resulting acceleration.



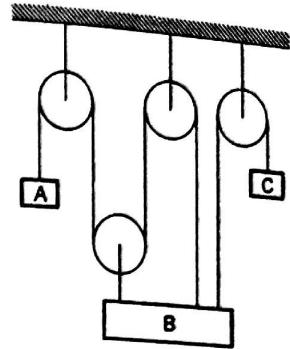
3. Three blocks A ( $m_A = 5 \text{ kg}$ ), B ( $m_B = 10 \text{ kg}$ ), C ( $m_C = 10 \text{ kg}$ ) are connected by rope and pulley arrangement as shown in figure. Neglecting mass of pulley, determine



- a) acceleration of each block  
b) tension in each cable.

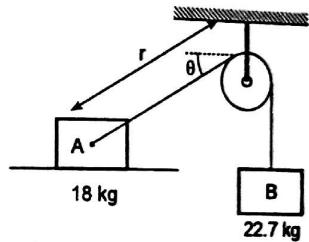
$$(Ans. a_A = 4.76, a_B = 3.08, a_C = 1.401 \text{ m/s}^2; T_1 = 33.6 \text{ N}, T_2 = 67.2 \text{ N})$$

4. The system shown is initially at rest. Neglecting the masses of the pulleys and effect of friction in the pulley.  $m_A = 9 \text{ kg}$ ,  $m_B = 27 \text{ kg}$ ,  $m_C = 9 \text{ kg}$ . Determine  
a) acceleration of each block  
b) tension in each cable.



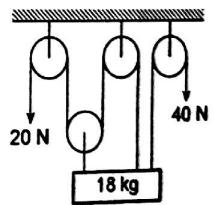
$$(Ans. a_A = 2.3 \text{ m/s}^2, a_B = 0.75 \text{ m/s}^2, a_C = 0.75 \text{ m/s}^2, T_1 = 67 \text{ N}, T_2 = 81 \text{ N})$$

5. The velocity of the block A is 1.8 m/s to the right at the instant when  $r = 0.73$  and  $\theta = 30^\circ$ . Neglecting the mass of pulley and effect of friction, determine at this instant  
a) tension in the cable  
b) acceleration of block A  
c) acceleration of block B



$$(Ans. T = 127.37 \text{ N}, a_A = 6.128 \text{ m/s}^2, a_B = 4.199 \text{ m/s}^2)$$

6. The 18 kg block starts from rest and moves upward when constant forces of 20 N and 40 N are applied to supporting ropes. Neglecting the masses of the pulley and the effect of friction, determine the speed of the block after it has moved 0.457 m.



$$(Ans. 1.5 \text{ m/s})$$

## References

1. "Vectors Mechanics for Engineering- Statics and Dynamics", R. P. Beer and E. R. Johnston. Jr. 4<sup>th</sup> Edition, McGraw Hill, 1987.
2. "Engineering Mechanics- Statics and Dynamics", R.C. Hibbeler, Ashok Gupta. 11<sup>th</sup> Edition, New Delhi, Pearson , 2009.
3. "A Text Book Of Engineering Mechanics", R. S. Khurmi.
4. "Applied Mechanics and strength of materials", R.S.Khurmi.
5. "Engineering Mechanics- Statics and Dynamic", Shame, I.H 3<sup>rd</sup>, New Delhi Prentice Hall of India, 1990.
6. Engineering Mechanics -statics and Dynamics, Dr D.S Kumar, 3<sup>rd</sup>, Katson Books-2012
7. Strength of Material R.K Rajput, S Chand,2007
8. A Text Book of Strength Of Material, Gokarna Bahadur Motra,2010
9. Manual of Statics and Dynamics, Mahendra Raj Dhital, IOE Publication 2000
10. "Engineering Mechanics- Statics and Dynamics", I.C. Jong and B.G. Rogers.
11. Engineering Mechanics- Statics and Dynamics", D. K. Anand and P. F. Cunnif.

## T.U. Syllabus Applied Mechanics CE401

Year: I  
Part: I

Lecture: 3  
Tutorial: 2  
Practical :

### Course objective

This Course is designed to provide basic knowledge of engineering mechanics to the students of all branches of engineering so that it would be helpful for them to understand structural engineering stress analysis principles in later course or to use basics of mechanics in their branch of engineering. This course shall be considered as an introduction: common for all engineering faculties of Tribhuvan University in the first year of undergraduate. Emphasis has been given to Statics.

### Course outline

1. **Introduction:** (2 Hours)
  - 1.1 Definition and Scope of Applied Mechanics
  - 1.2 Concept of Rigid and Deformed bodies.
  - 1.3 Fundamental concept and principle of mechanics: Newtonian Mechanics.
2. **Basic concept in Statics and Static Equilibrium:** (4 Hours)
  - 2.1 Concept of Particles and Free Body Diagram.
  - 2.2 Physical meaning of Equilibrium and its essence in structural application.
  - 2.3 Equation of Equilibrium in two dimensions.
3. **Force acting on particles and rigid body:** (6 Hours)
  - 3.1 Different type of force: points surface traction and Body force – Translational force and rotational force: Relevant Examples.
  - 3.2 Resolution and Composition of force: Relevant examples.
  - 3.3 Principles of Transmissibility and Equivalent force: equivalent force: Relevant Examples.

- 3.4 Moments and couples: Relevant examples.
- 3.5 Resolution of force into forces and a couple: Relevant Examples.
- 3.6 Resultant of force and moment for a system of force: Examples.
- 4. Centre of Gravity, Centroid and Moment of Inertia:** (6 Hours)
- 4.1 Concepts and calculation of Centre of Gravity and Centroid: Examples.
- 4.2 Calculation of Second Moment of Area/Moment of Inertia and Radius of Gyration: and Relevant usages.
- 4.3 Use of parallel axis Theorem: relevant examples.
- 5. Friction:**
- 5.1 Laws of friction, Static and dynamic coefficient of friction, Angle of friction: Engineering example of usage of friction. (2 Hours)
- 5.2 Calculation involving friction in structure: Examples as high Tension Friction Grip bolts and its free body diagram.
- 6. Analysis of beam and Frames:** (9 Hours)
- 6.1 Introduction to structures: Discrete and continuum.
- 6.2 Concept of load estimating and support Idealization: Examples and Standard Symbols.
- 6.3 Use of beams/frames in engineering: Concept of rigid joints/distribute loads in beams/frames.
- 6.4 Concept of statically/kinematic ally Determinate and indeterminate Beams and Frames: Relevant Examples.
- 6.5 Calculation of axial force, Shear force and bending moment for Determinate beams and Frames.
- 6.6 Axial force, Shear force and Bending moment diagram and Examples for drawing it.
- 7. Analysis of Plane trusses:** (4 Hours)
- 7.1 Use of trusses in engineering: Discrete and Continuum.
- 7.2 Calculation of member of force of Truss by method of joints: Simple examples.
- 7.3 Calculation of member of force of truss by method of sections: Simple examples.
- 8. Kinematics of Particles and Rigid Body:** (7 Hours)
- 8.1 Rectilinear kinematics: Continuous Motion.
- 8.2 Position, Velocity and Acceleration of a particle and Rigid body.

## Appendix

8.3  
8.4  
8.5  
8.6

Determination of motion of particle and rigid body.  
 Uniform rectilinear motion of particles.  
 Uniformly Accelerated Rectilinear motion of particles.  
 Curvilinear motion: Rectilinear components with Examples of Particles.

- 9. Kinetics of Particles and Rigid Body: Forces and Acceleration:** (5 Hours)
- 9.1 Newton's second law of motion and momentum.
- 9.2 Equation of motion and Dynamic equilibrium: Relevant Examples.
- 9.3 Angular momentum and Rate of Change.
- 9.4 Equation of motion-Rectilinear and Curvilinear.
- 9.5 Rectilinear: Tangential and Normal Components and polar Coordinates: Radial and Transverse Components.

## Tutorials:

*There shall be related tutorials exercised in class and given as regular homework exercises. Tutorial can be as following for each specified chapters.*

1. Introduction
  - A.Theory; Definition and Concept type questions.
2. Basic concept in Statics and Static Equilibrium.
  - A.Theory; Definition and Concept type questions.
3. Concept of Force acting Structure & force acting on particles and rigid body.
  - A.Practical examples: numerical examples, practical example type questions.
  - B.There can be tutorials for each sub-section.
4. Centre of Gravity, Centroid and Moment of Inertia.
  - A.Concept type, numerical examples and practical examples type questions.
5. Friction.
  - A.Definition type, practical example type and numerical example type questions.
6. Analysis of beam and Frames.
  - A.Concept type, definition type, numerical example type questions.
  - B.There can be tutorial for each sub-sections.
7. Analysis of Plane trusses.
  - A.Definition type, numerical example type questions.
  - B.There can be tutorial for each sub-sections

8. Kinematics of Particles and Rigid Body.  
*A. Concept type, definition type, numerical example type questions.  
 B. There can be tutorial for each sub-sections.*
9. Kinetics of Particles and Rigid Body: Forces and Acceleration.  
*A. Concept type, definition type, numerical example type questions.  
 B. There can be tutorial for each sub-sections.*

**References:**

1. "Mechanics of Engineering- Statics and Dynamics", F. P. Beer and E. R. Johnston. Jr. 4<sup>th</sup> Edition, Mc Graw Hill, 1987.
2. "Engineering Mechanics- Statics and Dynamics", R.C. Hibbeler, Ashok Gupta. 11<sup>th</sup> Edition, New Delhi, Pearson , 2009.
3. "Engineering Mechanics- Statics and Dynamics", I.C. Jong and B.G. Rogers.
4. "Engineering Mechanics- Statics and Dynamics", D. K. Anand and P. F. Cunnif.
5. "A Text Book Of Engineering Mechanics", R. S. khurmi.
6. "Applied Mechanics and strength of materials", R.S.Khurmi. .
7. "A Text Book of Applied Mechanics", I.B Prasad.
8. "Engineering Mechanics- Statics and Dynamic", Shame, I.H 3<sup>rd</sup>, New Delhi Prentice Hall of india, 1990.

# Purbanchal University Syllabus

## APPLIED MECHANICS I (STATICS)

### BEG156CI

Year :I

Semester :I

1. Introduction	3 Hrs
	1.1 Definition & scope of mechanics, Engineering mechanics, and static.
	1.2 Concept of particle, rigid body, deformed & fluid bodies.
	1.3 Equations of static equilibrium in 2D & 3D.
	1.4 Free body Diagram ( Definition, importance & examples)
1.5 System of units	
2. Vector	5 Hrs
	2.1 Introduction ( Vector & scalar quantities, simple operation of vectors & their laws, position vectors)
	2.2 Unit vectors in Cartesian co-ordinates.
	2.3 Dot product (Definition, laws &applications)
	2.4 Cross product (Definition, laws &applications)
2.5 Scalar and vector triple product	
3. Forces	7 Hrs
	3.1 Definition & principles of forces.
	3.2 Types of forces (coplanar, Collinear, concurrent, parallel, external & internal forces)
	3.3 Principle of transmissibility & its limitations.
	3.4 Resolution & composition of forces.
	3.5 Lami's theorem, Varignon's theorem, triangle, parallelogram & polygon law of forces
	3.6 Moment of forces about a point & axis (In scalar & vector form)
	3.7 Definition of couple & prove it as free vector.
	3.8 Resolution of force into force & a couple & vice versa.
	3.9 Resultant of a system of forces. (wrench, parallel, coplanar, concurrent & general)

4. **Distributed force**
  - 4.1 Definition & Derivation of center of gravity & centroid. ( composite figure & Direct Integration method) 5 Hrs
  - 4.2 Centroids of lines, areas and volumes
  - 4.3 Definition of second moment of area (moment of Inertia) and Radius of gyration
  - 4.4 Parallel and perpendicular axis theorem, MOI of common figures (e.g. rectangle, triangle, circle and ellipse) and uniform thin rod.
  - 4.5 MOI of Built up section.
  - 4.6 MOI by Direct integration method.
  
5. **Friction:**
  - 5.1 Introduction ( definition , Types, cause & effect ) 4 Hrs
  - 5.2 Laws of Dry friction
  - 5.3 Static friction, co- efficient of friction & angle of friction
  - 5.4 Condition of sliding or tipping.
  - 5.5 Application to static problems.
  
6. **Introduction to structures :**
  - 6.1 Structural components (Beam, frame, truss, 2-D plate, cable, Arch, grid) 5 Hrs
  - 6.2 Difference between plane & space structures.
  - 6.3 Difference between Mechanism & structures.
  - 6.4 Types of loading & supports.
  - 6.5 Determinacy ( internal & external ) and stability (statical & geometrical ) [General concept]
  
7. **Introduction to Analysis of Beam** 6 Hrs
  - 7.1 Definition & types of beam
  - 7.2 External & internal forces in beam.
  - 7.3 Definition & sign convention of Axial forces, shear forces & bending moment.
  - 7.4 Relationship between load, shear force & Bending moment
  - 7.5 Axial force, shear force & bending moment diagram.
  
8. **Introduction to Analysis of Frame** 4 Hrs
  - 8.1 Definition & type of frame ( rigid, deficient, redundant )
  - 8.2 Determinacy & stability
  - 8.3 Axial force, shear force & Bending moment diagram

Appendix

9. **Introduction to Analysis of plane truss** 4 Hrs
  - 9.1 Definition & types ( according to support condition purpose of utilization, degree of complexity )
  - 9.2 Determinancy & stability
  - 9.3 Analysis of truss ( method of joints & Method of section )
  
10. **Introduction to space truss.** 2 Hrs
  - 10.1 Definition.
  - 10.2 Tension coefficients, shear leg, tripods.
  - 10.3 Analysis of simple space truss.

**Recommended Books**

1. "Engineering Mechanics\_Statics and Dynamics" Shames, I.H, 3<sup>rd</sup> ed., New Delhi, Prentice Hall of India, 1990.
2. "Mechanics for Engineers – Statics and Dynamics" F.P. Beer and E.R. Johnston, JR 4th Edition, Mcgrw-Hill, 1987.

## Pokhara University Syllabus

### MEC 119.3 Applied Mechanics I (3-2-0)

#### Evaluation:

	Theory	Practical	Total
Sessional	50	-	50
Final	50	-	50
Total	100	-	100

#### Course Objectives:

This course has been developed to provide the basic knowledge of engineering mechanics to the students of engineering so that it would be beneficial to understand structural engineering. The knowledge of mechanics can be utilized in wide range of engineering applications using Newton's laws of motion and mechanical equilibrium of different force system. This course shall be considered as a basic for all branches of Engineering of Pokhara University in first year of undergraduate program.

#### Course Contents:

##### 1. Introduction (2 hrs)

- 1.1 Definition and scope of Applied Mechanics
- 1.2 Concept of Statics and Dynamics
- 1.3 Concept of Particle
- 1.4 Concept of Rigid, Deformed and Fluid Bodies
- 1.5 Fundamental Concepts and Principles of Mechanics: Newtonian Mechanics
- 1.6 System of Units

##### 2. Review of Coordinate System (2 hrs)

- 2.1 Cartesian Coordinate System
- 2.2 Polar Coordinate System
- 2.3 Cylindrical Coordinate System
- 2.4 Spherical Coordinate System
- 2.5 Review of Vector Algebra

##### 3. Forces acting on particles and rigid body (7 hrs)

- 3.1 Types of Forces: Point Force, Transitional and Rotational Force- Relevant Examples
- 3.2 Resolution and Composition of Forces- Relevant Examples

#### Appendix

- 3.3 Principle of Transmissibility and Equivalent Forces- Relevant Examples
- 3.4 Moments: Moment of a Force about a point and an axis- Relevant Examples
- 3.5 Theory of Couples:: Relevant Examples
- 3.6 Resolution of a Force into Forces and a Couple- Relevant Examples
- 3.7 Resultant of Force and Moment for a System of Force: Examples

##### 4. Basic Concept of Static Equilibrium (2 hrs)

- 4.1 Concept of Load types, Load Estimation and Support Idealizations- Examples and Standard Symbols
- 4.2 Free Body Diagram- Relevant Examples
- 4.3 Physical Meaning of Equilibrium and its essence in structural application
- 4.4 Equation of Equilibrium in Two/Three Dimensions

##### 5. Friction Forces (3 hrs)

- 5.1 Introduction
- 5.2 Types of Friction and its Coefficients: Static and Dynamic
- 5.3 Laws of Friction
- 5.4 Angle of Friction
- 5.5 Engineering Examples of usage of Friction

##### 6. Center of Gravity, Centroid and Moment of Inertia (6 hrs)

- 6.1 Concept and Calculation of Centre of Gravity and Centroid of Line/Area /Volume – Examples
- 6.2 Concept and Calculation of Second Moment of Area/ Moment of Inertia and Radius of Gyration- Relevant Examples
- 6.3 Use of Parallel Axis Theorem: Relevant Examples

##### 7. Analysis of Beam (5hrs)

- 7.1 Introduction Beam
- 7.2 Use of statically determinant beam
- 7.3 Relationship between Lead, Shearing Force and Bending Moment
- 7.4 Calculation and drawing of Axial Force, Shear Force and Bending Moment

##### 8. Analysis of Truss (4 hrs.)

- 8.1 Introduction to determinant truss
- 8.2 Use of truss in engineering application

- 8.3 Analysis of force by method of joint  
8.4 Analysis of force by method of section

#### 9. Kinematics of Particles and Rigid body (6 hrs)

- 9.1 Rectilinear Kinematics: Continuous Motion  
9.2 Position, Velocity and Acceleration of a Particle and Rigid body  
9.3 Determination of Motion of Particle and Rigid body  
9.4 Uniform Rectilinear Motion of a Particles  
9.5 Uniformly Accelerated Rectilinear Motions of Particles  
9.6 Curvilinear Motion of a Particle  
9.7 Rectangular Components of velocity and Acceleration  
9.8 Introduction of Tangential and Normal Components  
9.9 Radial and Transverse Components

#### 10. Kinetics of Particles and Rigid body: Force and Acceleration (6 hrs)

- 10.1 Newton's Second Law of Motion and Momentum  
10.2 Equation of Motion and Dynamic Equilibrium: Relevant Examples  
10.3 Angular Momentum: Rate of Change and Conservation  
10.4 Motion of Various Particles and Relative Velocity  
10.5 Equation of Motion- Rectilinear and Curvilinear  
10.6 Rectangular Components: Tangential and Normal  
10.7 Polar Coordinates: Radial and Transverse Components

#### 11. Moment and Energy in Rigid body (2 hrs)

- 11.1 Introduction to Moment and Energy  
11.2 Conservation of Linear and Angular Momentum

#### Text Books:

1. "Engineering Mechanics\_Statics and Dynamics", Shames, I. H. 3rd ed., New Delhi, Prentice Hall of India, 1990.
2. "Mechanics of Engineers\_Statics and Dynamics", F. P.Beer and E. R. Johnston, Jr. 4th Edition, McGraw-Hill Book Co., New York, USA (Asia Editions), 1987.

#### References:

1. "Engineering Mechanics\_Statics and Dynamics", R.C. Hibbeler, Ashok Gupta, 11th edition. New Delhi, Pearson, 2009.
2. "Engineering Mechanics- Statics and Dynamics", I.C. Jong and B.G. Rogers.

3. "Engineering Mechanics- Statics and Dynamics", D.K. Anand and P.F. Connif.
4. "Engineering Mechanics of Solids", Egor .P. Popov, 2nd Edition, New Delhi, Prentice Hall of India, 1996.
5. "Engineering Mechanics- Statics & Dynamics", Dr. D.S. Kumar,S.K. Kataria & Sons, New Delhi, Reprint 2011.
6. Practice guide in Applied Mechanics, D. B. Pandit, Ramesh Khanal

## Exam Questions

### Pokhara University Semester – Spring

Level: Bachelor  
Programme: BE

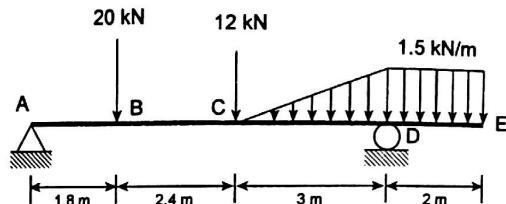
Course: Applied Mechanics I

Candidates are required to give their answers in their own words as far as practicable.

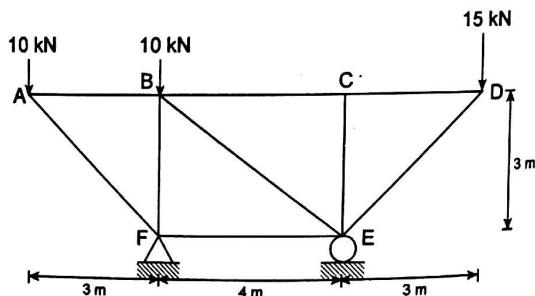
The figures in the margin indicate full marks.

Attempt all the questions.

1. Draw shear force and bending moment diagram of given simply supported beam.



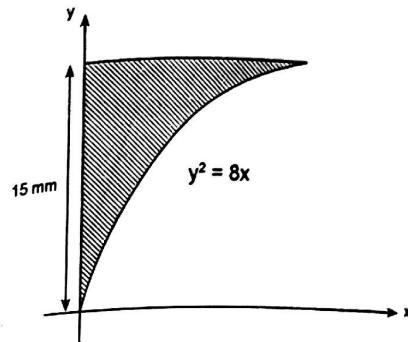
2. a) Analyze the force in each member of following truss.



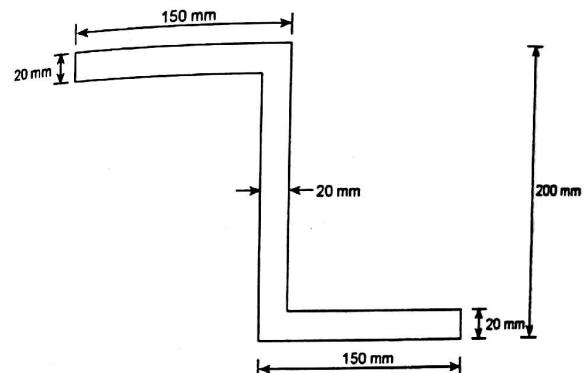
- b) What is the relationship between load, shear force and bending moment?

Appendix  
3:

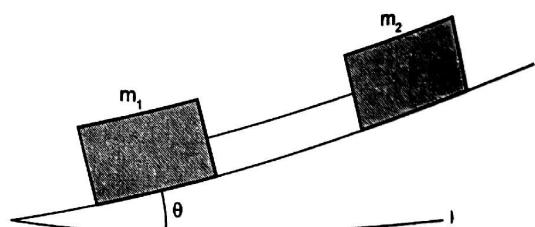
- a) By direct integration method, find centroid of the shown shaded area with respect to given axes.



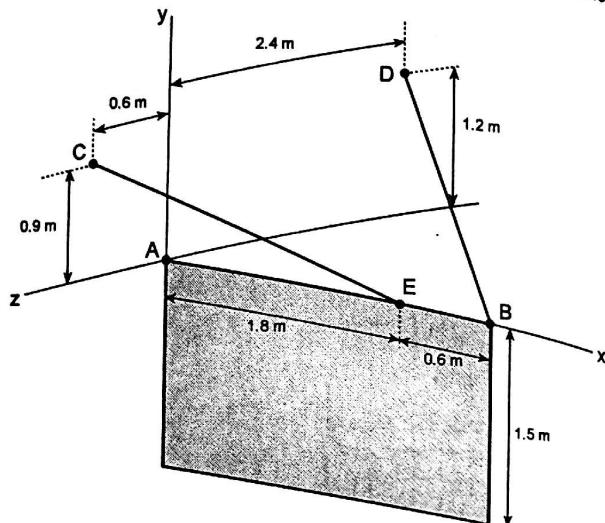
- b) Determine the moment of inertia of the given section about centoidal axes.



4. a) Two masses  $m_1 = 240 \text{ N}$  and  $m_2 = 150 \text{ N}$  are tied together by a rope parallel to the inclined plane surface as shown in figure. If  $\mu_s$  for  $m_1$  is 0.3 and that for  $m_2$  is 0.55; find (a) value of  $\theta$  for which masses will just start slipping (b) tension in the rope.

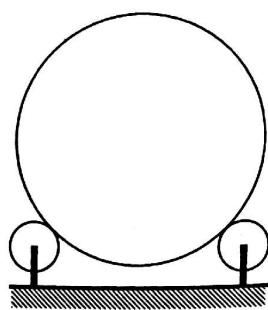


- b) A  $1.5 \text{ m} \times 2.4 \text{ m}$  sign board of uniform density weighs  $1500 \text{ N}$  and is supported by the ball and socket joint at A and by two cables. Determine the tension in each cable and the reaction at A.



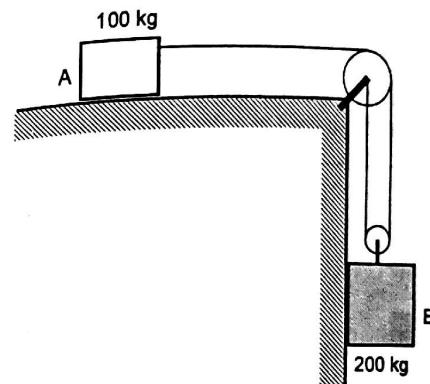
5. a) The motion of a particle is defined by the relation  $x = 3t^3 - 8t^2 + 15$  where  $x$  is expressed in meters and  $t$  in seconds. Determine the time, position and acceleration of a particle when  $v = 0$  ( $v$  is velocity of a particle). 7

- b) A mixing drum of  $150 \text{ mm}$  outside radius rests on two casters each of  $30 \text{ mm}$  radius. The drum executed  $20$  revolutions during the time interval  $t$ , while its angular velocity is being increased uniformly from  $25$  to  $55 \text{ rpm}$ . Knowing that no slipping occurs between the drum and casters determine; (a) angular acceleration of the casters (b) time interval  $t$ .



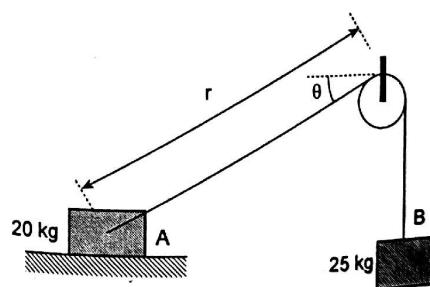
OR

- b) Derive the expression for the determination of tangential and normal component of velocity and acceleration of a particle.  
Two blocks A and B as shown in figure starts from rest. The coefficient of friction between horizontal plane and block A is  $\mu_k = 0.25$  and pulley is assumed to be of negligible mass. Determine acceleration of each block and tension in each cord.



OR

- a) The two blocks are released from rest when  $r = 0.9 \text{ m}$  and  $\theta = 35^\circ$ . Neglecting the mass of the pulley and the effect of friction in the pulley and between block A and the horizontal surface, determine the initial tension in the cable, the initial acceleration of block A and the initial acceleration of block B.



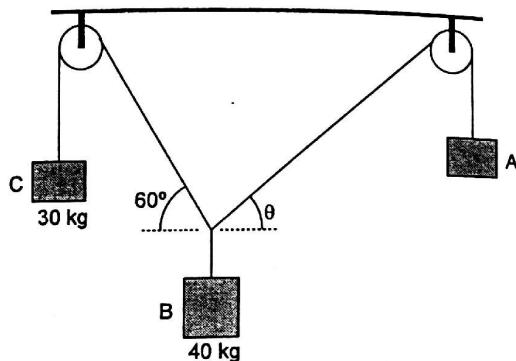
- b) Describe about the conservation linear and angular momentum.

7. Write short notes on (Any TWO):  
 a) Rectilinear motion  
 b) Coordinate system  
 c) Principle of transmissibility

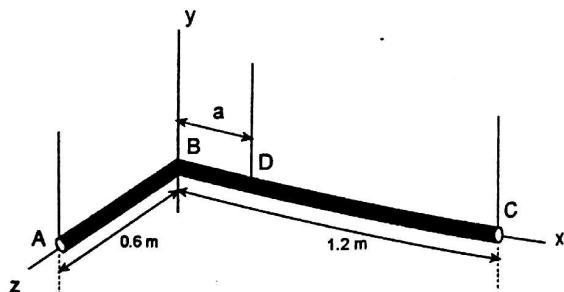
**Pokhara University**  
**2014 – Spring**

Attempt all the questions.

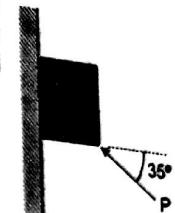
1. a) Determine the mass that must be supported at A and the angle  $\theta$  of the connection cord in order to hold the system in equilibrium.



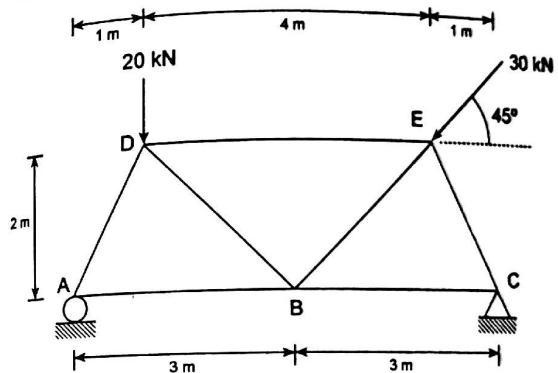
- b) Two steel pipes AB and BC each having a weight per unit length of 35 N/m are welded together at B and are supported by three wires. Knowing at  $a = 0.45$  m, determine the tension in each wire.



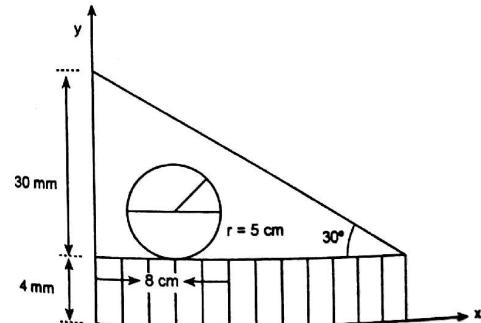
2. a) Determine the range of force P applied on the block A ( $m = 7.5$  kg) for which the block will be in equilibrium against the vertical wall. Take  $\mu_s = 0.35$  and  $\mu_k = 0.3$ .



- b) Analyse the truss given below by using method of joints.

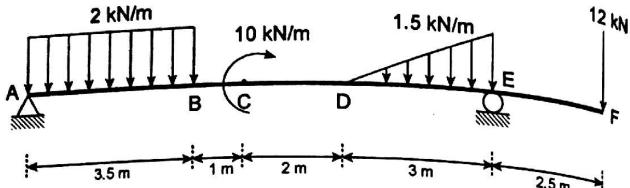


3. a) Find the coordinates of about x and y axes.

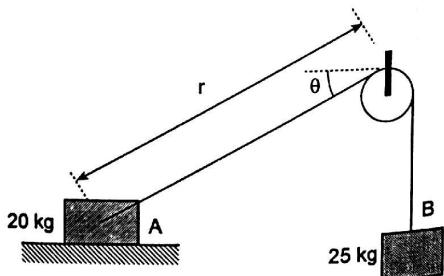


- b) Calculate the moment of inertia of the shaded area about x-axis.

4. Draw the shear force and bending moment diagram of the beam shown below. Show the point of contraflexure, point of zero shear (if any).



5. a) The position of a particle which moves along a straight line is defined by the relation  $x = t^3 - 6t^2 - 15t + 40$ , where  $x$  is expressed in meter and  $t$  in seconds. Determine:
- The time at which the velocity will be zero.
  - The position and distance travelled by the particle at that time.
  - The acceleration of the particle at that time.
  - The distance travelled by the particle from  $t = 4$  sec to  $t = 6$  sec.
- b) A motorist is travelling on a curve section of highway of radius 700 m at the speed of 90 km/hr. The motorist suddenly applied the brakes, causing the automobile to slow down at a constant rate. Knowing that after 8 sec. the speed has been reduced to 70 km/hr, determine the acceleration of the automobile immediately after the brake have been applied.
6. a) The two blocks are released from rest when  $r = 0.8$  m and  $\theta = 30^\circ$ . Neglecting the mass of the pulley and the effect of friction in the pulley and between block A and the horizontal surface, determine the initial tension in the cable, the initial acceleration of block A and the initial acceleration of block B.

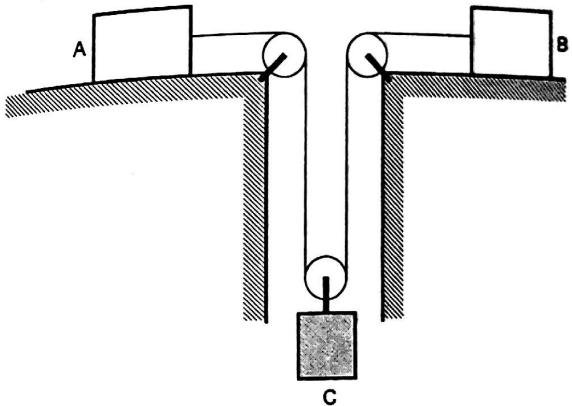


b) Describe in short about conservation of linear and angular momentum with expression.

OR

- b) Three blocks A ( $m_A = 10$  kg), B ( $m_B = 15$  kg) and C ( $m_C = 25$  kg) are connected by rope and pulley arrangement as shown in the figure. The coefficient of kinematic friction between the contact plane and the block A and B is 0.3. Neglecting the masses of pulleys and effect of friction on pulleys. Determine:

- The acceleration of each block.
- Tension in each cable.



7. Write short notes on: (any TWO)
- Branches of applied mechanics
  - Polar Coordinate system
  - Radial and transverse component of velocity and acceleration.

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## Applied Mechanics (CE 401)

Level: BE

Programme: BCE, BME

Year/Part: I/II

Full Marks: 80  
Pass Marks: 32  
Time 3 hrs

Candidates are required to give their answers in their own words as far as practicable.

Attempt ALL questions.

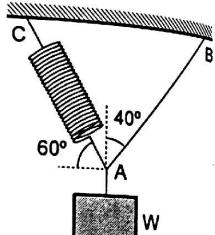
The figures in the margin indicate full marks.

Assume suitable data if necessary.

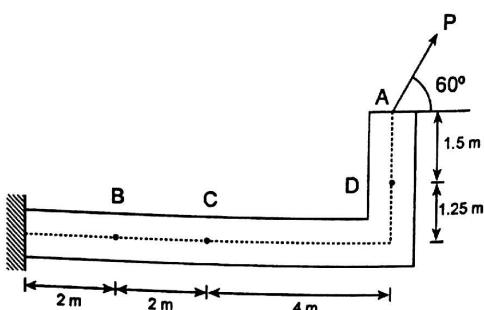
1. Differentiate between rigid body and deformable body.

2. A container of weight  $W$  is subjected from ring A to which cable AB and spring AC are attached. The constant of spring is 100 N/m and its unstretched length is 3 m. Determine the tension in the cable; when

- a)  $W = 120 \text{ N}$ ,  
b)  $W = 160 \text{ N}$ .

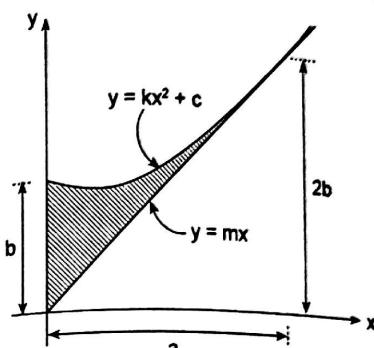


3. A 160 N force  $P$  is applied at point A of a structural member. Replace  $P$  with a) an equivalent force couple system at C, b) an equivalent system consisting of a vertical force at B and a second force at D.



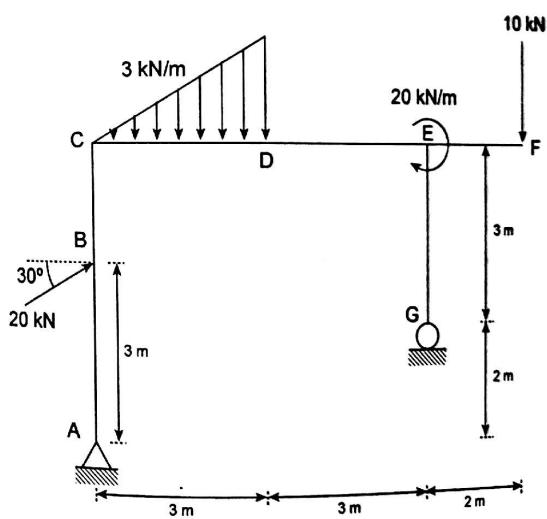
## Appendix

4. Determine the polar moment of inertia and the polar radius of gyration of the shaded area as shown in figure below with respect to centroid.

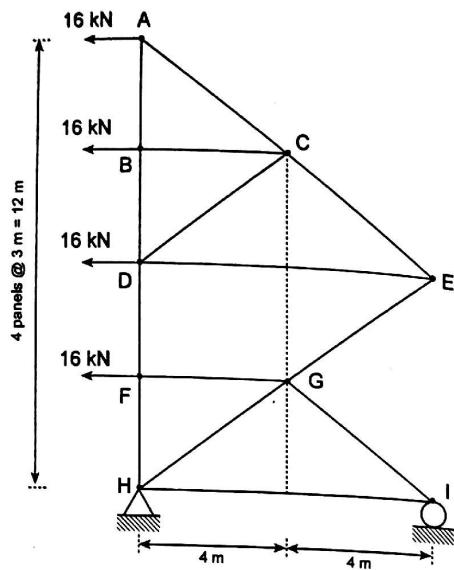


5. Define static friction. Explain why coefficient of static friction is always more than that of kinetic friction coefficient. Support your answer with relevant equations.

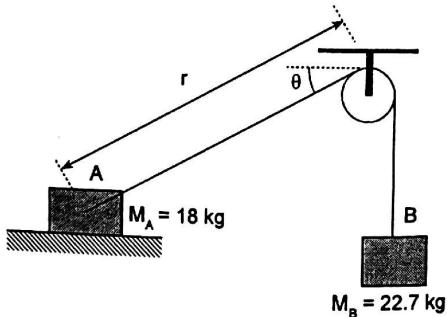
6. Calculate and draw the axial force, shear force and bending moment diagram with its salient features; for the given frame as shown in figure below.



7. Use method of section to determine member forces DE, DF and GI for the given pin jointed truss and also indicate the nature of forces.



8. A particle moving in a straight line has an acceleration  $a = \sqrt{V}$ , its displacement and velocity at time  $t = 2$  sec, are  $\frac{128}{3}$  m and 160 m/s. Find the displacement, velocity and acceleration at time  $t = 3$  sec.
9. The two blocks as shown in figure below are released from rest when  $r = 0.73$  m and  $\theta = 30^\circ$ . Neglecting the mass of the pulley and the effect of the friction in the pulley and between block A and the horizontal surface. Determine: a) The initial tension in the cable; b) Acceleration of the block A and B.

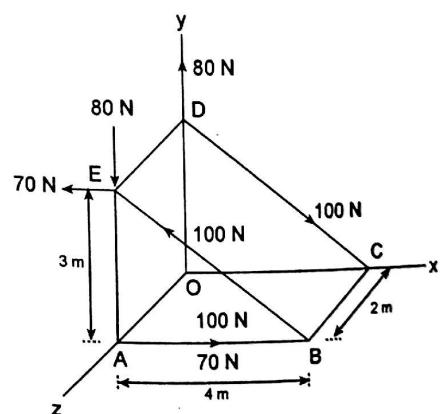
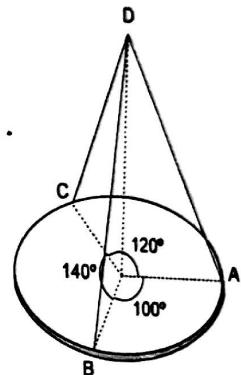


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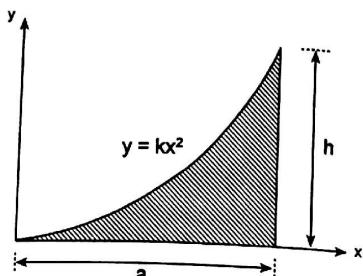
**Applied Mechanics (CE 401)**

Attempt ALL questions.

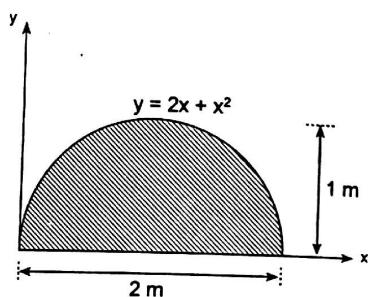
- Differentiate between rigid body and deformable body.
- Explain about the physical meaning of equilibrium. Define free body diagram and concept of particle.
- A homogeneous circular plate of mass 50 kg is supported by three wires. The angular distance between the points of attachment on the circumference of the plane w.r.t. center of the plate makes an angle of  $100^\circ$  while other two angular distances are  $120^\circ$  and  $140^\circ$  as shown in figure below. The three wires are attached to a single point on the ceiling which is 5 m vertically above the centroid of the plate. The plate has diameter of 1 m. Calculate the force developed in each wires.
- Three pairs of couples are acted on the triangular block as shown in figure below. Determine the resultant of them.



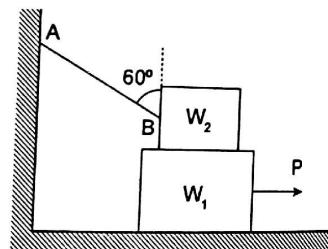
5. Determine the centroidal X and Y coordinates of the shaded area.



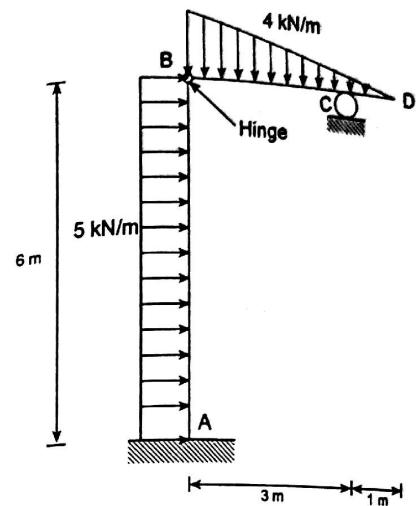
6. Determine the moment of inertia area about X-axis.



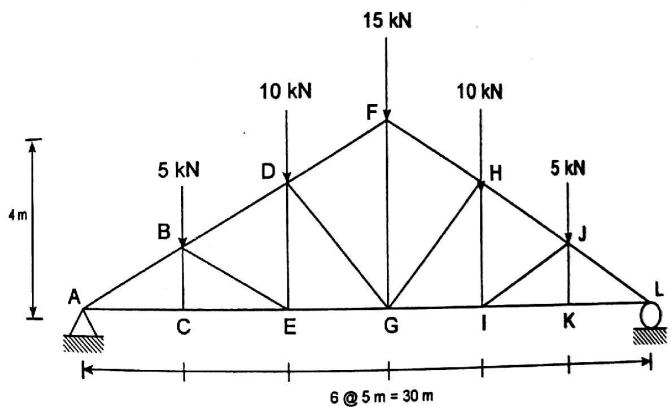
7. A block of weight  $W_1 = 1800$  N rests on a horizontal surface and supports on the top of it another block of weight  $W_2 = 1000$  N as shown in figure below. The block  $W_2$  is attached to a vertical wall by the inclined string AB. Find the magnitude of the horizontal force  $P$ , applied to the lower block as shown, that will be necessary to cause sliding to impend. The coefficient of static friction for all contact surfaces is 0.4.



8. Deduce the relationship between load, shear force and bending moment for a beam section loaded uniformly with intensity of load  $W$ . Draw AFD, SFD and BMD or the given frame loaded as shown in figure below. Indicate also the salient features if any.



9. Determine the member forces in member CE, FH, GH, GI of given truss.



10. Two ships A and B are at a distance of  $4800$  m apart; B being south east of A. Speed of A is  $2.6$  m/s due east and B is travelling at speed of  $4.47$  m/s due north. Determine:
- the relative velocity of B w.r.t. A
  - The shortest distance between them
  - Time taken to reach the shortest distance.

11. What do you mean by principle of impulse and momentum? The resultant external force acting on a  $30 \text{ N}$  particle in space is  $\vec{F} = (12\hat{i} - 24t^2\hat{j} + 30t^3\hat{k}) \text{ N}$ , where  $t$  is the time measured in seconds. Initially, particle is at origin and at rest. Determine Y-component of acceleration, velocity and position at the instant of  $5 \text{ sec}$ .

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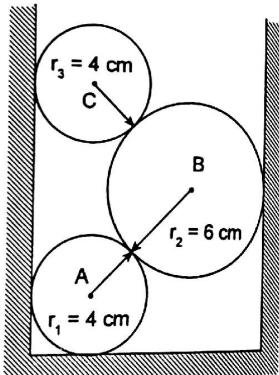
**2070 Chaitra**

### Applied Mechanics (CE 401)

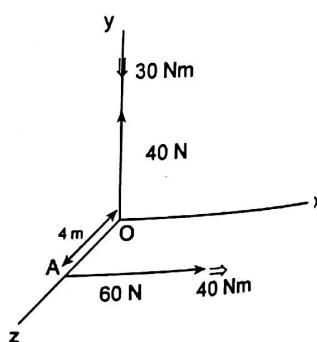
Attempt ALL questions.

1. Describe the scope and importance of applied mechanics in engineering study. Define free body diagram with examples.

2. Determine the reactions at the contact points, if three cylinders are piled in a rectangular ditch as shown in figure. Given that the weight of the cylinders are:  $W_A = 2 \text{ kN}$ ,  $W_B = 5 \text{ kN}$ ,  $W_C = 3 \text{ kN}$ .

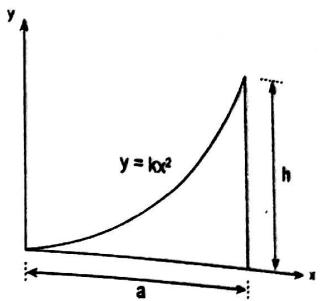


3. How can you reduce a force into a force and couple? Obtain the resultant of the two pairs of wrench shown in the figure. Indicate its line of action.

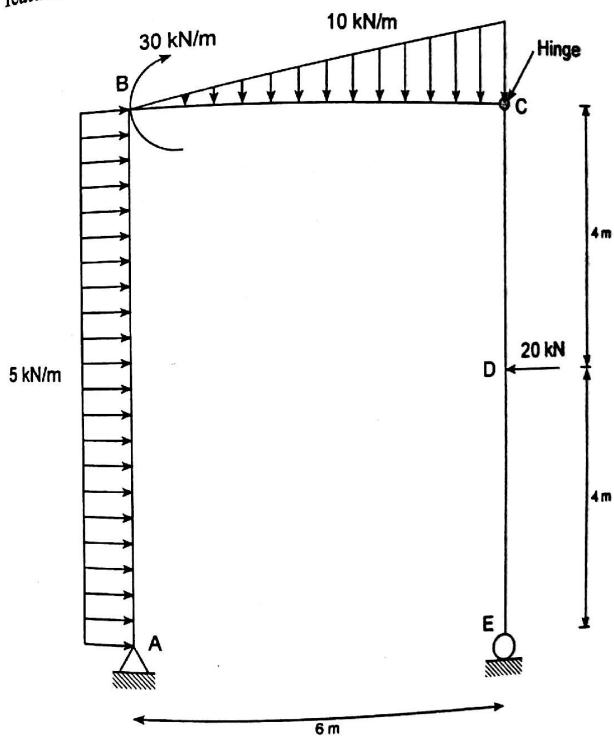


Appendix

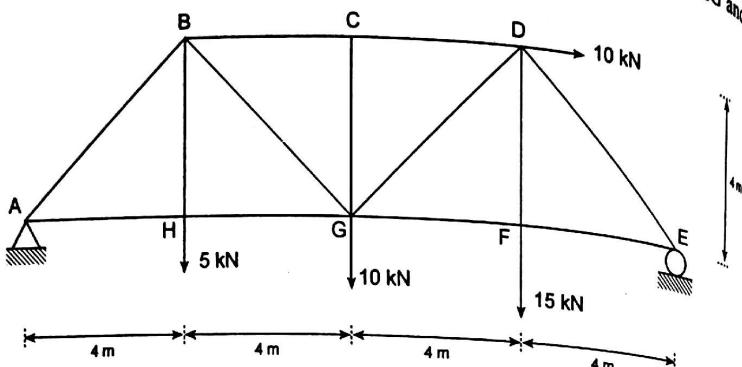
4. Determine centroid of the given plane figure. State and prove parallel axes theorem for moment of inertia. Define centroid and center of gravity.



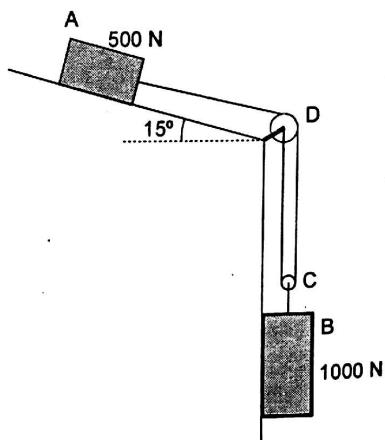
5. Define the angle of friction and also write the laws of static friction.  
6. Draw axial force, shear force and bending moment diagram for the loaded frame as shown in figure below. Also indicate the salient features if any.



7. Determine the total degree of internal, external indeterminacy of given truss. Also determine the member forces in members BC, BG, HG and GD.



8. The acceleration of a particle is given by a relation  $a = v^3$ . It is known that at time  $t = 0$ , position is  $-2 \text{ m}$  and velocity is  $2 \text{ m/sec}$ . Find the displacement, position, velocity and acceleration at instant of  $\frac{1}{2} \text{ sec}$ . What do you mean by projectile and obtain the equations for projectile motion.
9. What do you mean by impulse momentum principle? Two blocks A and B having respective weights  $500 \text{ N}$  and  $1000 \text{ N}$  start from rest. The pulley is frictionless and also practically mass less. The kinetic coefficient of friction between the block A and the inclined surface is  $0.35$ . Determine the acceleration of each block and tension in the cord.



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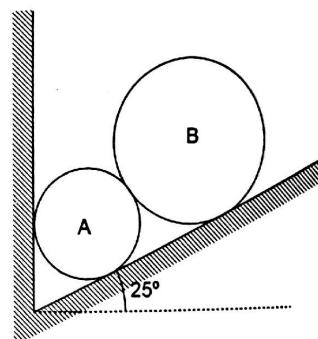
2071 Shawan

## Applied Mechanics (CE 401)

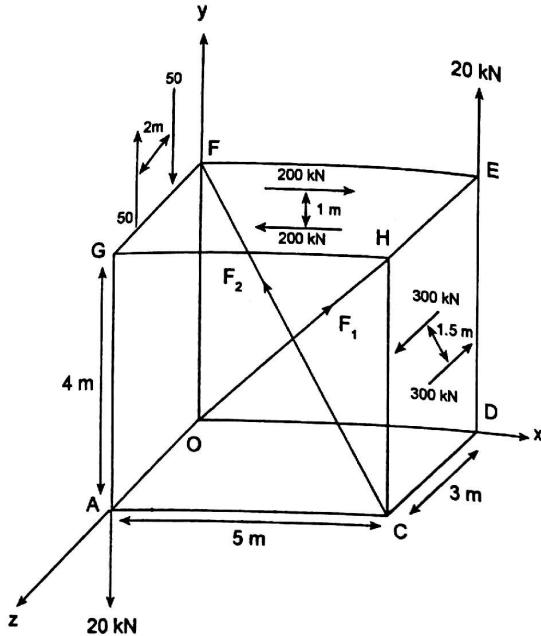
Attempt ALL questions.

1. Why it is necessary to assume a solid body as a perfectly rigid in the engineering study.
2. Why is free body diagram? The cylinder A and B rest in an inclined surface which makes an angle of  $25^\circ$  with horizontal as shown in figure below. Determine reaction at contact points.

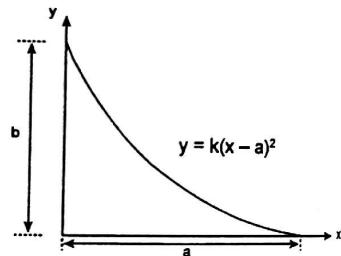
Take:

Weight of cylinder A ( $W_A$ ) =  $100 \text{ N}$ Weight of cylinder B ( $W_B$ ) =  $200 \text{ N}$ Diameter of cylinder A ( $r_A$ ) =  $60 \text{ mm}$ Diameter of cylinder B ( $r_B$ ) =  $90 \text{ mm}$ 

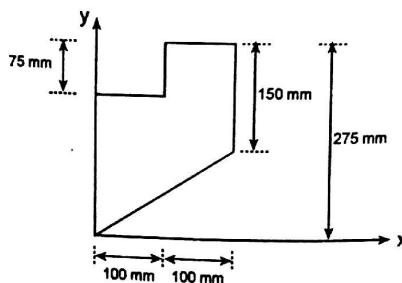
3. Find the resultant of force couple system at point A as shown in figure below. Take  $F_1 = 100 \text{ kN}$ ,  $F_2 = 300 \text{ kN}$ . Define a couple and show that couple is a free vector.



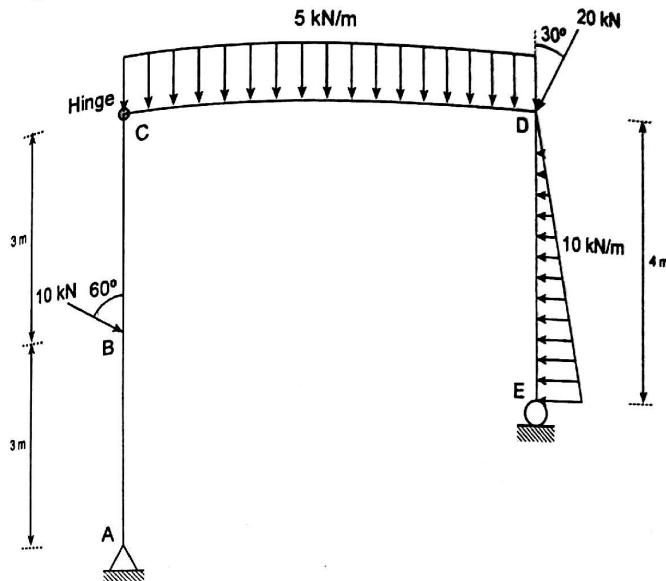
4. Determine by direct integration method, the centroid of the area shown in figure aside:



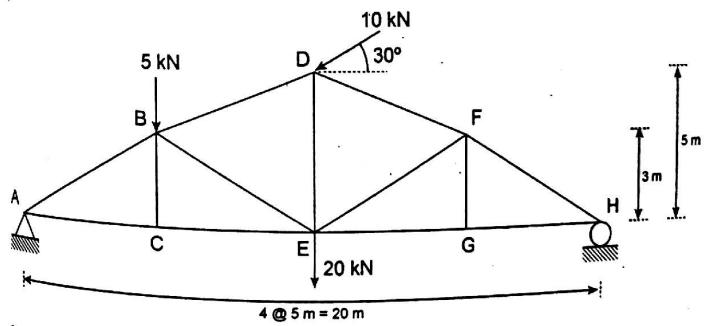
5. Calculate the moment of inertia of the composite area about Y-axis.



- Appendix
6. State laws of dry friction. How can we assume the condition of overturning and sliding of a block? Explain with suitable example.
7. Draw axial force shear force and bending moment diagram for the given frame. Also indicate salient features if any.

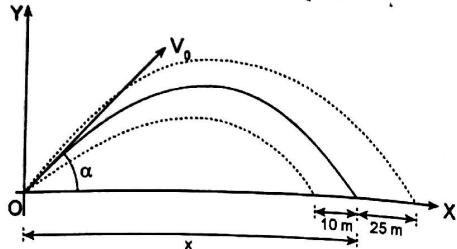


8. Find the member forces in CE, BE, BD and DE for the given truss. Define stability and determinacy of structures with examples.



9. A projectile is aimed at a mark on the horizontal plan through the point of projection and falls 10 shorts when the angle of projection is 15° while overshoots the mark by 25 m when the inclination is 40°.

Calculate the distance of the target and required angle of projection, if the velocity remains constant. Neglecting air resistance, define dependent motion of particle with example.

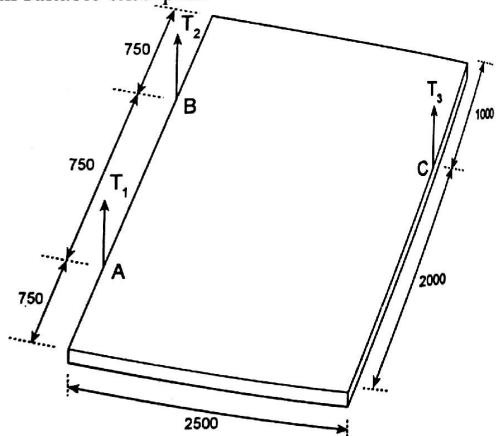


10. Define the dynamic equilibrium. Determine the velocity and acceleration of the particle, if it moves along a curved path defined by  $r = 50$  and  $\theta = t^2/3$ , where  $r$  is in meters and  $t$  is in seconds. Given that the instant angle is  $\theta = \pi/2$ .

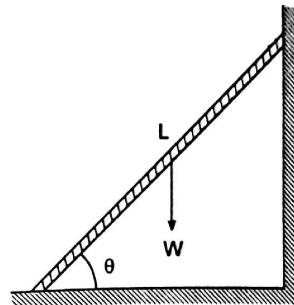
**Tribhuvan University  
2071 Bhadra  
Applied Mechanics (CE 401)**

Attempt ALL questions.

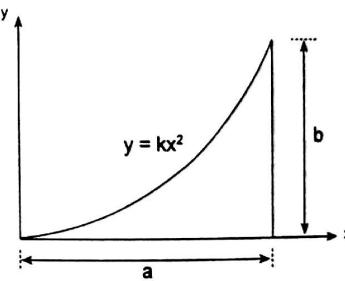
- What is mechanics? Mention scope of Applied Mechanics in engineering.
- Illustrate equilibrium condition of a rigid body and concept of free body diagram with suitable examples.
- Three vertical wires as shown in figure support a plate of 50 kg. Determine the tension in each wire. All dimensions are in mm.



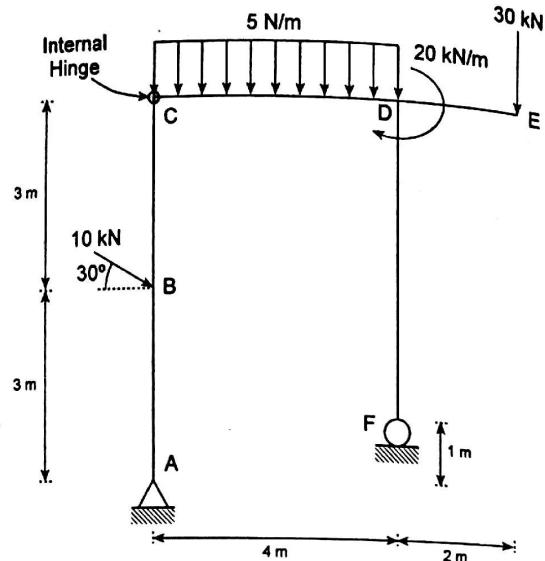
- Force  $\vec{F} = (3\hat{i} - 6\hat{j} + 4\hat{k})$  N passes through point (6, 3, 2) m. Replace this force with an equivalent system, where the force  $\vec{F}$  passes through point (2, 5, 10) m.
- Determine the minimum angle  $\theta$  (made by the ladder AB of length L with the floor) at which a uniform ladder can be placed against a wall without slipping under its own weight (W). The coefficient of friction for all surfaces is 0.2.



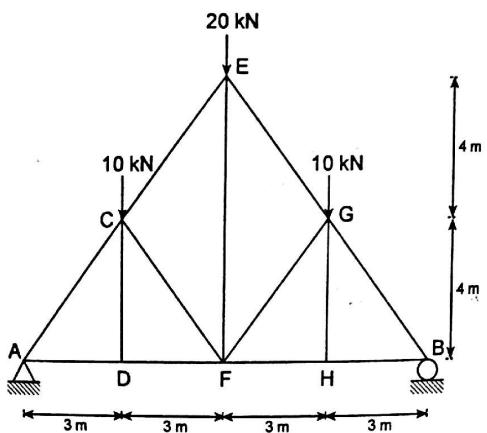
- Determine the moment of inertia about centroidal axis of the shaded plane area by using direct integration method.



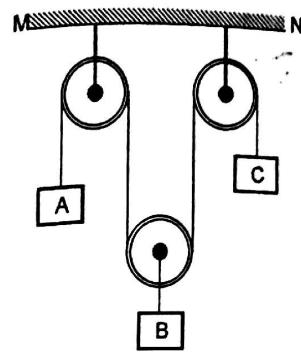
- Draw the axial force, shear force and bending moment diagram of given frame. Indicate also the salient features if any.



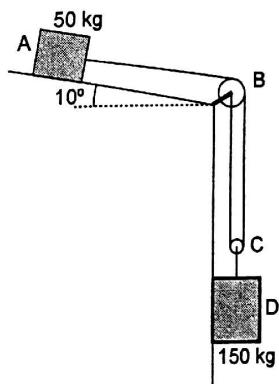
8. Determine the force developed in members CE, DF, EF, GH of given truss loaded as shown in figure.



9. Define uniformly rectilinear motion and uniformly accelerated rectilinear motion. For the pulley systems as shown in figure, calculate the velocity and acceleration of block C. If the velocities and acceleration of block A and B are  $3 \text{ m/s}$  ( $\downarrow$ ),  $2 \text{ m/s}^2$  ( $\uparrow$ ),  $4 \text{ m/s}$  ( $\uparrow$ ) and  $5 \text{ m/s}^2$  ( $\downarrow$ ) respectively.



10. Two blocks shown in figure starts from rest. The pulleys are frictionless and having no mass. The kinetic coefficient of friction between block A and inclined plane is 0.37. Determine the acceleration of each block and tension in each cord. What do you mean by dynamic equilibrium?



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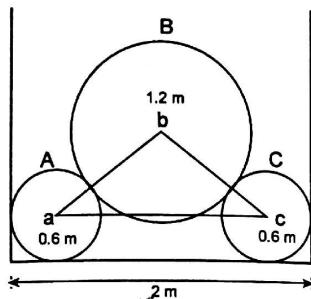
2071 Magh

**Applied Mechanics (CE 401)**

Attempt ALL questions.

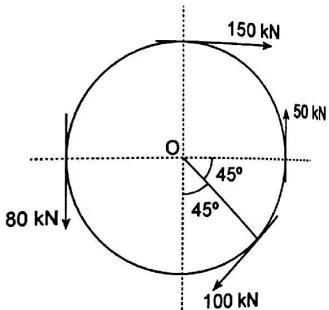
- I. a) Describe the scope of applied mechanics.

- b) The cylinders A and C weigh 1000 N each and the weight of cylinder B is 2000 N. Determine the forces exerted at the contact points.



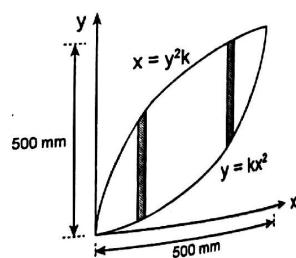
2. a) What is Free Body Diagram and why it is used during analysis of structure

- b) Determine the resultant of the forces acting tangentially to a circle of radius 3 m as shown in figure. What will be the location of the resultant with respect to centre of the circle?

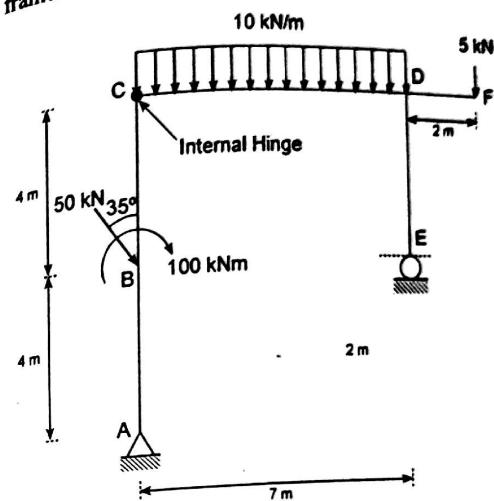


3. Explain the laws of static friction. Also define the limiting friction and angle of friction with suitable example.

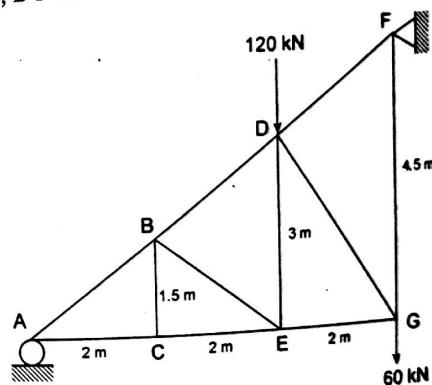
4. Determine the centroid of the hatched area by Direct Integration Method. State and prove the parallel axis theorem for moment of inertia.



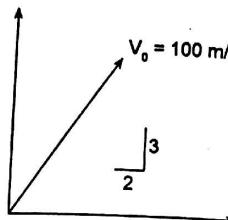
5. Draw the axial force, shear force and bending moment diagram of the given frame. Also show the salient features.



6. Write down the ideal assumption of truss. Determine the member forces BC, DG and EG for given truss.



7. A particle starting from origin is subjected to acceleration such that  $a_x = -4 \text{ m/s}^2$  and  $a_y = -8 \text{ m/s}^2$ . If the initial velocity is 100 m/s directed at a slope of 2:3. Compute the radius of curvature of the path after 5 sec. Also calculate the position at the end of 5 sec. What are the possible equations of motion for a particle in terms of kinetics?

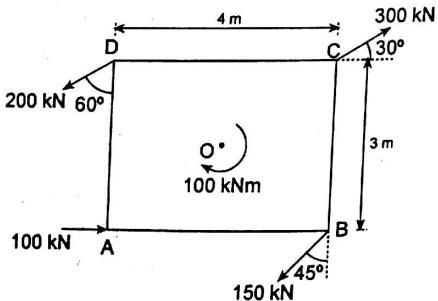


8. a) Define the dynamic equilibrium and impulse momentum principle for particle.
- b) A particle moves along a curved path defined by  $r = 50 + t^{2/3}$  where  $r$  is in meters and  $t$  is in seconds. Determine the velocity and acceleration of the particle when  $\theta = 90^\circ$ .

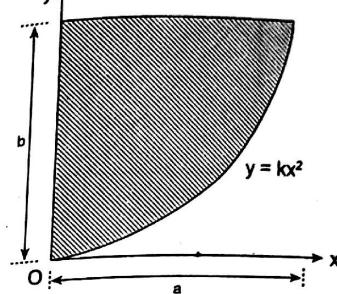
**Tribhuvan University  
2071 Chaitra  
Applied Mechanics (CE 401)**

Attempt ALL questions.

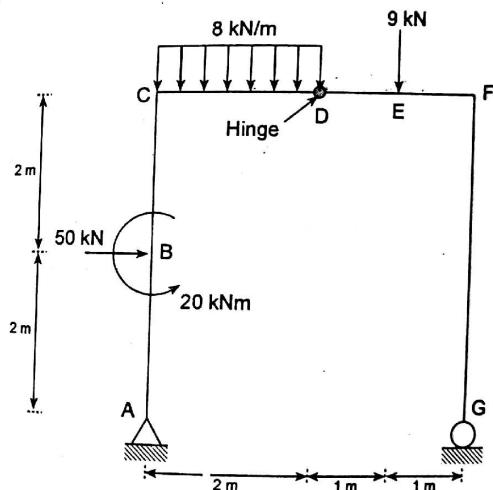
1. Explain about the principle of Applied Mechanics. Why it is necessary to assume a solid body as 'perfectly rigid' for the study of statics?
2. Define Free Body Diagram with example and explain about equations of static equilibrium for 2-D and 3-D analysis of a particle and a rigid body.
3. Describe the principle of transmissibility. Determine the magnitude, direction and position with respect to center O of the resultant of the forces acting on the rectangular plate ABCD as shown in figure below.



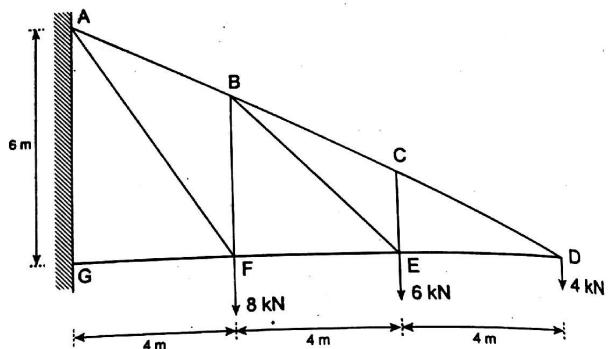
4. Explain radius of gyration. Determine the centroid of the shaded area shown in figure below. Using direct integration method.



5. What are the advantages and disadvantages of friction? Also explain the working principles of high tension friction grip bots.
6. A frame is loaded as shown in figure below. Draw the AFD, SFD and BMD and also show the salient features of each diagram.



7. Describe the use of trusses in engineering. Determine the force developed in BC, BE, EF, AB, AF and BF members of cantilever truss loaded as shown in figure below.



8. a) What is uniformly accelerated rectilinear motion? Also define the angular momentum and its rate of change.

- b) Motion of a particle is defined by a relation  $x = \frac{t^3}{3} - 3t^2 + 8t + 15$ .

Determine the position of particle when velocity is 2.5 m/sec. Also determine the position of particle when acceleration is 3.6 m/sec<sup>2</sup>.

9. a) What are tangential and normal components of velocity? Explain with examples.

- b) A particle moves along a curved path defined by  $r = 4\theta^2$  and  $\theta = \frac{t^2}{2}$  where r is in meters and t in seconds. Determine the velocity and acceleration of the particle when  $\theta = 80^\circ$ .