(2)

In this question, I adapt the "anisotropic" variant of the total variation image denoising model using an ADMM scheme for minimization that was implemented in HW3 Q5. In particular, the python code is adjusted so that instead of working on 2D image data, it handles a 1D function. And the energy functional and regularization parameter lambda are changed to work with the specifications in question 1 above.

Compared to before, our energy functional is now

$$J_f(u) := \frac{1}{2\lambda} \int_0^1 (u-f)^2 dx + |Du|((0,1)), u \in BV(0,1) .$$

So the Augmented Lagrangian now becomes:

$$L_\rho(u, z, y) = \frac{1}{2\lambda} ||u-f||^2 + ||z||_1 + y^T(Du - z) + \frac{\rho}{2} ||Du - z||^2$$

Thus, following the steps to derive the updates as in homework 3,
we now update the u ('x' variable) according to:

$$\left(\frac{1}{\lambda} I + \rho Q^T Q \right) x = \frac{1}{\lambda} f + Q^T(\rho z - y)$$

The 'z' update is similar to before, except now the regularization parameter λ is removed, and the 'y' update is exactly the same.

The full updates and code to run the experiments with user specified parameters are included in the code submission file for this question.

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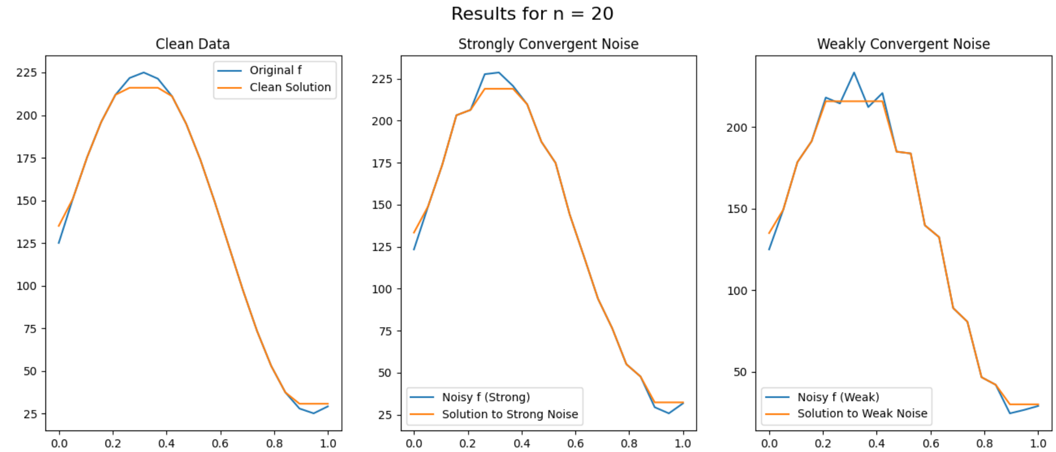
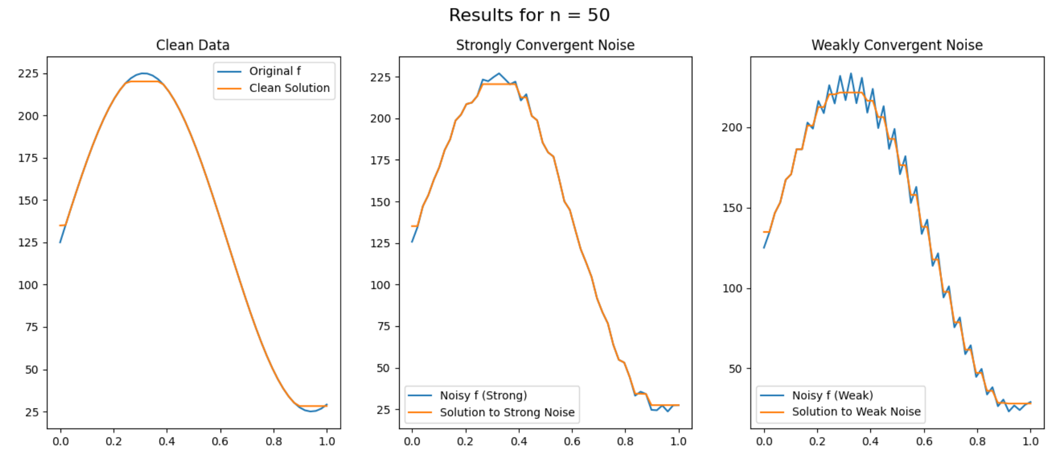
Below, we run the experiments for varying data f, varying types of sequences of noise, and varying n for the discretization. For all the experiments below, we set lambda = 10 and rho = 1.

First we investigate the case where **f is a sin wave**, and we use scalings of

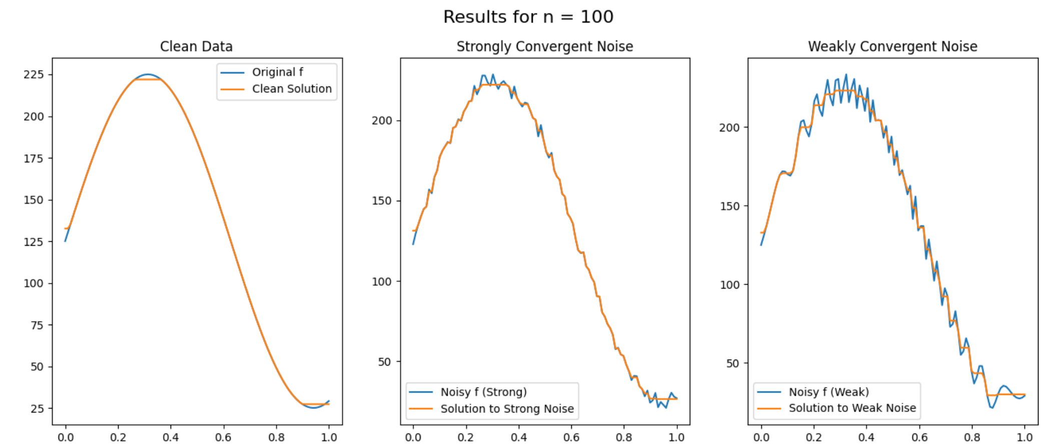
- Gaussian noise for the strongly convergent noise
- $\sin(n\pi x)$ for the weakly convergent noise





For n = 20, we observe that both solutions with noise don't do a great job of reconstructing the original f, however this can be explained by the fact that n is small, so the number of steps in the discretization of the original signal are insufficient to accurately represent f. However, when comparing the solutions to the noisy functions, it is apparent that the solution with the strong noise more closely matches the noisy f than the respective solution with weak noise.

This is even more apparent when we increase n to 50, now the solution to the strong noise does a fairly good job of reconstructing the original f, and an approximate sine wave shape is formed. However, for the weakly convergent noise, we see the "staircasing" effect in the solution where there are many segments that appear to be constant in signal throughout the solution.

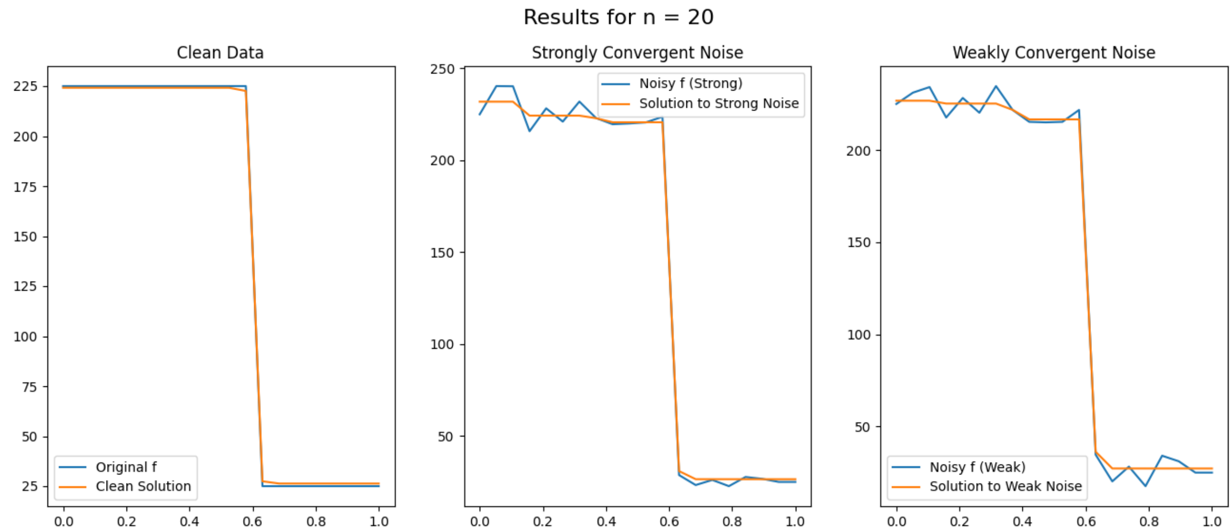


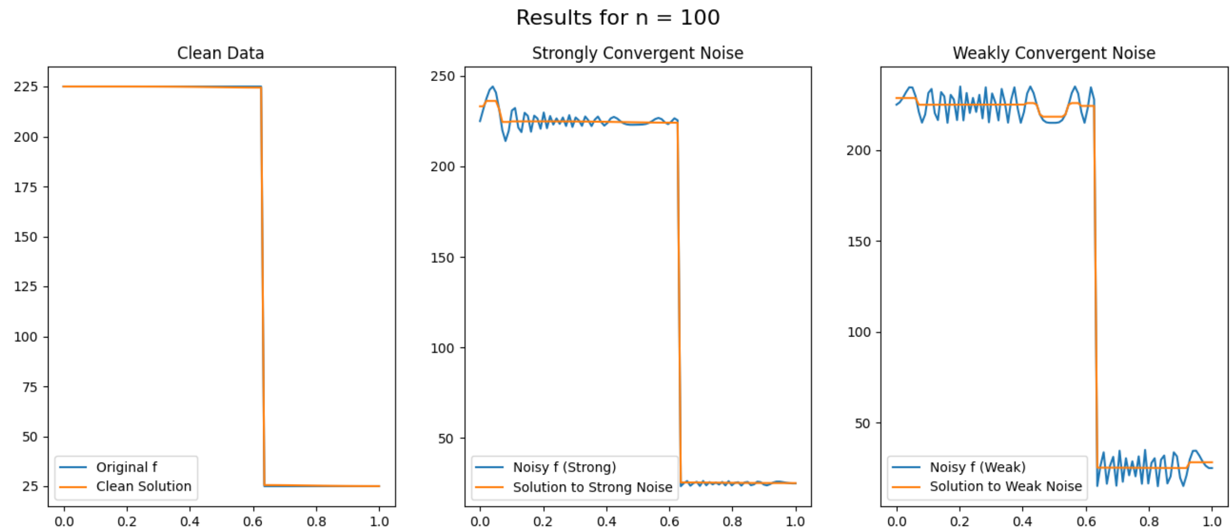
Although as n increases, the weakly convergent solution does approach the overall sine wave shape, it always exhibits these staircasing effects which does not seem to be as apparent with the strongly convergent noise.

Next we investigate the case where **f is a square wave**, and we use scalings of

- $\dfrac{\sin(n\pi x)}{n}$ for the strongly convergent noise
- $\sin(n\pi x)$ for the weakly convergent noise

We observe the following:





Again we observe that for small n, solutions to both the perturbed signals do not manage to reconstruct the original f. However as n increases to 100, it is apparent that the solution to the perturbation with strongly convergent noise does a much better job at reconstructing the original square wave than the one with the weakly convergent noise which still exhibits some oscillations and “staircasing effects” around the noise.

Overall, these experiments seem to align with the theoretical results we’ve studied in class regarding denoising perturbed images with “zero noise limits”. That is, subject to uniform estimates and strong convergence assumptions, the unique minimizers u_n of $J_{f_n}(u)$ converge weakly to the unique minimum of $J_f(u)$. And similarly we observed, if instead η_n converges weakly to 0, the solution also approaches the unique minimum, subject to uniformity constraints, however there are still perturbing effects in the noise that do not diminish uniformly. So we have observed that variational problems can indeed be stable subject to small perturbations.