Q 1 (a) for a rigid body of N particles, The total no. of degrees of freedom is given by

$$N\left(\begin{matrix}\text{degree of each} \\ \text{particle}\end{matrix}\right) - (\text{constraints})$$

each particle is move 3 - directions

$3N$

Now, considering constraints. let us choose a first particle. it can move in any direction without constraints, for a second particle one co-ordinate is now constrined due to ~~first~~ first particle.

chusing third particle 2 co-coordinates are constrained due to parevious two particles.

Now, rest $N-3$ particle have all 3 co-ordinates constraied.

$\therefore$ Total number of constraints

$$= 0 + 1 + 2 + 3(N-3) = 3N - 6$$

$$\Rightarrow \text{Degree of freedom} = 3N - (3N-6)$$

$$= 6$$

A sphere ~~have~~ has less degrees of freedom then 6 Becouse it has symmetry.

Now, consider two particles 1, and 2 in a rigid body.

Let force due to 2 on 1 = $\vec{F}_{12}$

force due to 1 on 2 = $\vec{F}_{21}$

The force $\vec{F}_{12}$ is along $\vec{r}$ ($\because$ central forces)

from Newton's third law of motion

$$\vec{F}_{12} = -\vec{F}_{12}$$

Let us assume due to the action of the forces, $\vec{r}_1$ changes by $d\vec{r}_1$ and $\vec{r}_2$ changes by $d\vec{r}_2$

Now, since this is a rigid body $\therefore$ $|r|$ = constant.

$\Rightarrow$ $(\vec{r}_2-\vec{r}_1)(\bar{r}_2-\bar{r}_1) = r^2$ = constant

$\Rightarrow$ r^2 = constant $\Rightarrow$ $\boxed{2r\,dr = 0}$

$\Rightarrow$ $r \cdot dr = 0$ $\Rightarrow$ $(\bar{r}_2-\bar{r}_1)\cdot(d(\bar{r}_2-\bar{r}_1)) = 0$

$\Rightarrow$ $d(\bar{r}_2-\bar{r}_1)$ is perpendicular to $(\bar{r}_2-\bar{r}_1)$

Now, work done by both the forces will be given by

$$W = \vec{F}_{12}\, dr_1 + \vec{F}_{21}\, dr_2$$

$= \vec{F}_{21} (dr_2 - dr_1)$

since the displacement due to the forces is $\perp$ to the direction of force

$$\vec{F}_{21} \cdot d(r_2 - r_1) = 0$$

$\Rightarrow W = 0$

$\Rightarrow$ Total work due to internal forces that are central is 0.

$\Rightarrow$ (O LOG, we can say that due to central internal forces b/w any two particles ends up as zero.

(b) The Langrandian is given by

$$L = \frac{I_1}{2}(\dot\phi^2 \sin^2\theta + \dot\theta^2) + \frac{I_3}{2}(\dot\psi + \dot\phi\cos\theta)^2 - Mgl\cos\theta$$

The generalized co-ordinates are

θ, ψ, ϕ

The conjugate momentum are given by

$$p_i = \frac{\partial L}{\partial \dot q_i}$$

$$\Rightarrow p_\phi = \frac{\partial L}{\partial \dot\phi} = \frac{\partial}{\partial \dot\phi}\Big(\frac{I_1}{2}(\dot\phi^2 \sin^2\theta + \dot\theta^2) + \frac{I_3}{2}(\dot\psi + \dot\phi\cos\theta)^2 - Mgl\cos\theta\Big)$$

$$p_\phi = (I_1 \sin^2\theta + I_2 \cos^2\theta)\,\dot\phi + I_3 \dot\psi \cos\theta$$

Similarly

$$p_\psi = \frac{\partial}{\partial\dot\psi}\left(\frac{I_2}{2}(\dot\phi^2 \sin^2\theta + \dot\theta^2) + \frac{I_2}{2}(\dot\psi + \dot\phi\cos\theta)^2 - Mgl\cos\theta\right)$$

$$= \boxed{I_3(\dot\psi + \dot\phi\cos\theta) = p_\psi}$$

and $\boxed{p_\theta = \frac{\partial L}{\partial\dot\theta} = I_2\dot\theta}$

The Hamiltonians are given by

$$H_i = \frac{d}{dt}\left(\frac{\partial L}{\partial\dot q_i}\right) + \frac{\partial L}{\partial q_i}$$

$$h_q = \frac{d}{dt}\left(\frac{\partial L}{\partial\dot\phi}\right) + \frac{\partial L}{\partial\phi}$$

$$= \frac{\partial}{\partial t}\Big((I_2 \sin^2\theta + I_3 \cos^2\theta)\,\dot\phi + (I_3\dot\psi\cos\theta)\Big)$$

$$= (I_1\, 2\sin\theta\cos\theta\cdot\dot\theta + I_3\cdot 2\cos\theta(-\sin\theta)\cdot\dot\theta)\,\dot\phi$$

$$+ \ddot\phi(I_1 \sin^2\theta + I_3\cos^2\theta)$$

$$+ I_3\ddot\psi\cos\theta + (-I_3\dot\psi\sin\theta)\,\dot\theta$$

$$H_\theta = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) + \frac{\partial L}{\partial \theta}$$

$$= I_1 \ddot{\theta} + \frac{I_1}{2}(\dot{\phi}^2 2\sin\theta\cos\theta) + \frac{I_3}{2} 2(\dot{\psi} + \dot{\phi}\cos\theta)(-\sin\theta)$$

$\therefore$ EOM is given by

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \left(\frac{\partial L}{\partial q_i}\right) = 0 \Rightarrow H_i - 2\left(\frac{\partial L}{\partial q_i}\right) = 0$$

$\Rightarrow$ The equation of motion are

$$H_\phi = 0 \quad \ldots (i)$$

$$H_\psi = 0 \quad \ldots (ii)$$

$$H_\theta = I_1 (\dot{\phi}^2 - 2\sin\theta\cos\theta)$$
$$+ I_3 \cdot 2(\dot{\psi} + \dot{\phi}\cos\theta)(-\sin\theta)$$

Q 2> b) lets start with a gen 2×2 matrix.

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$\det(A) = 1$

$\Rightarrow ad - bc = 1 \Rightarrow ad = 1 + bc \Rightarrow d = \frac{1+bc}{z}$

$$\therefore A = \begin{bmatrix} a & b \\ c & \frac{1+bc}{a} \end{bmatrix}$$

$$AA^T = I$$

$$\begin{bmatrix} a & b \\ c & \frac{1+bc}{a} \end{bmatrix} \begin{bmatrix} a & c \\ b & \frac{1+bc}{a} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} a^2+b^2 & ac + \frac{b+b^2c}{a} \\ ac + \frac{b+b^2c}{a} & c^2 + \frac{(1+bc)^2}{a^2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$a^2 + b^2 = 1$

$a^2 c = \sqrt{1-a^2} + (1+a^2)c = 0$

$c = -\sqrt{1-a^2}$

substshilA into A

$$A = \begin{bmatrix} a & \sqrt{1-a^2} \\ \sqrt{1-a^2} & a \end{bmatrix}$$

Now finding eige values

(2.2)

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} a-\lambda & \sqrt{1-a^2} \\ -\sqrt{1-a^2} & a-\lambda \end{vmatrix} = 0$$

$$(a-\lambda)^2 - \left(\left(-\sqrt{1+a^2}\right)\left(\sqrt{1-a^2}\right)\right) = 0$$

$$a^2 - 2a\lambda + \lambda^2 + 1 - a^2 = 0$$

$$\lambda^2 - 2a\lambda + 1 = 0$$

$$\lambda = \frac{2a \pm \sqrt{4a^2-4}}{2} = a \pm \sqrt{a^2-1}$$

$\therefore$ eigen values are $a + \sqrt{a^2-1}$ and

$a - \sqrt{a^2-1}$

c)

$$\mathcal{L} = \frac{1}{2} \partial_\mu \varphi \, \partial_\nu \varphi \, \eta^{\mu\nu} - \frac{1}{2} m^2 \varphi^2 - \frac{\lambda}{4!} \varphi^4$$

$\therefore$ the EOM is

$$\frac{\partial \mathcal{L}}{\partial \varphi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial (\partial_\mu \varphi)} \right) = 0$$

$$\frac{\partial \mathcal{L}}{\partial \varphi} = -\frac{1}{2}(2m^2\varphi) - \frac{4\lambda\varphi^3}{4!} = -m^2\varphi - \frac{\lambda\varphi^2}{3!}$$