max-product composition

max-product composition of $R(x,y)$ and $S(y,z)$ is defined as

$$T(x,z) = R \bullet S$$

$$R(x,y) \bullet S(y,z) = \max_{y \in Y} \left[\mu_R(x,y) \cdot \mu_S(y,z) \right]$$

$$= \bigvee_{y \in Y} \left[\mu_{\underset{\sim}{R}}(x,y) \cdot \mu_{\underset{\sim}{S}}(y,z) \right]$$

$\forall x \in X,\ z \in Z.$

$$\therefore T = \begin{array}{c} \\ x_1 \\ x_2 \end{array} \begin{array}{c} \begin{array}{ccc} z_1 & z_2 & z_3 \end{array} \\ \begin{bmatrix} 0.6 & 0.3 & 0.21 \\ 0.72 & 0.36 & 0.63 \end{bmatrix} \end{array}$$

$$\mu_{\underset{\sim}{T}}(x_1, z_1) = \max \{ [\mu_{\underset{\sim}{R}}(x_1, y_1) \cdot \mu_{\underset{\sim}{S}}(y_1, z_1)], [\mu_R(x_1, y_2) \cdot \mu_{\underset{\sim}{S}}(y_2, z_1)] \}$$

$$= \max \{ 0.6, 0.24 \}$$

$$= 0.6$$

$$\mu_{\underset{\sim}{T}}(x_1, z_2) = \max \{ 0.3, 0.12 \} = 0.3$$

$$\mu_{\underset{\sim}{T}}(x_1, z_3) = \max \{ 0.18, 0.21 \} = 0.21$$

$$\mu_{\underset{\sim}{T}}(x_2, z_1) = \max \{ 0.2, 0.72 \} = 0.72$$

$$\mu_{\underset{\sim}{T}}(x_2, z_2) = \max \{ 0.10, 0.36 \} = 0.36$$

$$\mu_{\underset{\sim}{T}}(x_2, z_3) = \max \{ 0.06, 0.63 \} = 0.63$$

$$\therefore T = R \circ S = \begin{array}{c|ccc|} & z_1 & z_2 & z_3 \\ \hline x_1 & 0.6 & 0.5 & 0.3 \\ x_2 & 0.8 & 0.4 & 0.7 \\ \hline \end{array}$$

$$\mu_T(x_1, z_1) = \max\{\min[\mu_{\underset{\sim}{R}}(x_1, y_1), \mu_{\underset{\sim}{S}}(y_1, z_1)], \min[\mu_{\underset{\sim}{R}}(x_1, y_2), \mu_{\underset{\sim}{S}}(y_2, z_1)]$$

$$= \max\{0.6, 0.3\}$$

$$= 0.6$$

$$\mu_T(x_1, z_2) = \max\{\min[\mu_{\underset{\sim}{R}}(x_1, y_1), \mu_{\underset{\sim}{S}}(y_1, z_2)] \min[\mu_{\underset{\sim}{R}}(x_1, y_2), \mu_{\underset{\sim}{S}}(y_2, z_2)]\}$$

$$= \max\{0.5, 0.3\}$$

$$= 0.5$$

$$\mu_T(x_1, z_3) = \max\{\min[\mu_{\underset{\sim}{R}}(x_1, y_1), \mu_{\underset{\sim}{S}}(y_1, z_3)], \min[\mu_{\underset{\sim}{R}}(x_1, y_2), \mu_{\underset{\sim}{S}}(y_2, z_3)]\}$$

$$= \max\{0.3, 0.3\}$$

$$= 0.3$$

$$\mu_T(x_2, z_1) = \max\{\min[\mu_{\underset{\sim}{R}}(x_2, y_1), \mu_{\underset{\sim}{S}}(y_1, z_1)], \min[\mu_{\underset{\sim}{R}}(x_2, y_2), \mu_{\underset{\sim}{S}}(y_2, z_1)]\}$$

$$= \max\{0.2, 0.8\}$$

$$= 0.8$$

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and $\underset{\sim}{B} = \left\{ \frac{0.4}{y_1} + \frac{0.9}{y_2} \right\}$

The cartesian product performed over fuzzy sets $\underset{\sim}{A}$ and $\underset{\sim}{B}$ results in relation $\underset{\sim}{R}$

$$\underset{\sim}{R} = \underset{\sim}{A} \times \underset{\sim}{B} = \begin{array}{c c} & \begin{array}{cc} y_1 & y_2 \end{array} \\ \begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array} & \begin{bmatrix} 0.3 & 0.3 \\ 0.4 & 0.7 \\ 0.4 & 0.9 \end{bmatrix} \end{array}$$

$\mu_R(x_1, y_1) = \min(\mu_A(x_1), \mu_B(y_1))$

2) Two fuzzy relations are given by

$$\underset{\sim}{R} = \begin{array}{c c} & \begin{array}{cc} y_1 & y_2 \end{array} \\ \begin{array}{c} x_1 \\ x_2 \end{array} & \begin{bmatrix} 0.6 & 0.3 \\ 0.2 & 0.9 \end{bmatrix} \end{array} \quad \text{and} \quad \underset{\sim}{S} = \begin{array}{c c} & \begin{array}{ccc} z_1 & z_2 & z_3 \end{array} \\ \begin{array}{c} y_1 \\ y_2 \end{array} & \begin{bmatrix} 1 & 0.5 & 0.3 \\ 0.8 & 0.4 & 0.7 \end{bmatrix} \end{array}$$

Obtain fuzzy relation T as a composition between the fuzzy relations.

The composition between two given fuzzy relations is performed in two ways

a) max-min composition
b) max-product composition.

max-min composition of R(X, Y) and S(Y, Z) is defined as,

$$R(X,Y) \circ S(Y,Z) = \max_{y \in Y} \left\{ \min \left[\mu_R(x,y), \mu_{\underset{\sim}{S}}(y,z) \right] \right\}$$

$$= \bigvee_{y \in Y} \left[\mu_{\underset{\sim}{R}}(x,y) \wedge \mu_{\underset{\sim}{S}}(y,z) \right], \quad \forall x \in X, z \in Z.$$

d) Bounded difference $\tilde{T} \odot \tilde{R}$

$$\mu_{\tilde{T} \odot \tilde{R}}(x) = \max\left[0, \mu_{\tilde{T}}(x) - \mu_{\tilde{R}}(x)\right]$$

$$= \max\left\{0, \left[\frac{0}{0} + \frac{0.1}{1} + \frac{0.4}{2} + \frac{0.6}{3} + \frac{0.5}{4} + \frac{0.5}{5}\right.\right.$$

$$= \left\{\frac{0}{0} + \frac{0.1}{1} + \frac{0.4}{2} + \frac{0.6}{3} + \frac{0.5}{4} + \frac{0.5}{5}\right\}$$

d) Bounded difference $\mu_{\underset{\sim}{A} \ominus \underset{\sim}{B}}(x)$

$$= \max[0, \mu_{\underset{\sim}{A}}(x) - \mu_{\underset{\sim}{B}}(x)]$$

$$= \max\left\{0, \left[\frac{0.1}{1} + \frac{0.1}{2} + \frac{0.2}{3} + \frac{0.5}{4}\right]\right\}$$

$$= \left\{\frac{0.1}{1} + \frac{0.1}{2} + \frac{0.2}{3} + \frac{0.5}{4}\right\}.$$

4) The discretised membership functions for a transistor and a registor are

$$\mu_{\underset{\sim}{T}} = \left\{\frac{0}{0} + \frac{0.2}{1} + \frac{0.7}{2} + \frac{0.8}{3} + \frac{0.9}{4} + \frac{1}{5}\right\}$$

$$\mu_{\underset{\sim}{R}} = \left\{\frac{0}{0} + \frac{0.1}{1} + \frac{0.3}{2} + \frac{0.2}{3} + \frac{0.4}{4} + \frac{0.5}{5}\right\}$$

a) Algebraic sum $\underset{\sim}{A} + \underset{\sim}{B}$

$$\mu_{\underset{\sim}{T}} + \mu_{\underset{\sim}{R}} \Rightarrow \mu_{\underset{\sim}{T}+\underset{\sim}{R}}(x) = \mu_{\underset{\sim}{T}}(x) + \mu_{\underset{\sim}{R}}(x) - \mu_{\underset{\sim}{T}}(x) \cdot \mu_{\underset{\sim}{R}}(x)$$

$$= \left[\frac{0}{0} + \frac{0.3}{1} + \frac{1.0}{2} + \frac{1.0}{3} + \frac{1.3}{4} + \frac{1.5}{5}\right\}$$

$$- \left[\frac{0}{0} + \frac{0.02}{1} + \frac{0.21}{2} + \frac{0.16}{3} + \frac{0.36}{4} + \frac{0.5}{5}\right]$$

$$= \left\{\frac{0}{0} + \frac{0.28}{1} + \frac{0.79}{2} + \frac{0.84}{3} + \frac{0.94}{4} + \frac{1.0}{5}\right\}$$

b) Algebraic product $\underset{\sim}{A} \cdot \underset{\sim}{B}$

$$\mu_{\underset{\sim}{T} \cdot \underset{\sim}{R}}(x) = \mu_{\underset{\sim}{T}}(x) \cdot \mu_{\underset{\sim}{R}}(x)$$

$$= \left\{\frac{0}{0} + \frac{0.02}{1} + \frac{0.21}{2} + \frac{0.16}{3} + \frac{0.36}{4} + \frac{0.5}{5}\right\}$$

c) Bounded sum $\underset{\sim}{T} \oplus \underset{\sim}{R}$

$$\mu_{\underset{\sim}{T} \oplus \underset{\sim}{R}} = \min[1, \mu_{\underset{\sim}{T}}(x) + \mu_{\underset{\sim}{R}}(x)]$$

$$= \min\left\{1, \left[\frac{0}{0} + \frac{0.3}{1} + \frac{1.0}{2} + \frac{1}{3} + \frac{1.3}{4} + \frac{1.5}{5}\right]\right\}$$

$$= \left\{\frac{0}{0} + \frac{0.3}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5}\right\}$$

i) $\underset{\sim}{B_1} \cup \underset{\sim}{\overline{B_1}} = \max\left[\mu_{\underset{\sim}{B_1}}(x), \mu_{\underset{\sim}{\overline{B_1}}}(x)\right]$

$= \left\{\frac{1}{1.0} + \frac{0.75}{1.5} + \frac{0.7}{2.0} + \frac{0.85}{2.5} + \frac{1}{3.0}\right\}$

j) $\underset{\sim}{B_2} \cap \underset{\sim}{\overline{B_2}} = \min\left[\mu_{\underset{\sim}{B_2}}(x), \mu_{\underset{\sim}{\overline{B_2}}}(x)\right]$

$= \left\{\frac{0}{1.0} + \frac{0.4}{1.5} + \frac{0.2}{2.0} + \frac{0.1}{2.5} + \frac{1}{3.0}\right\}$

k) $\underset{\sim}{B_2} \cup \underset{\sim}{\overline{B_2}} = \max\left[\mu_{\underset{\sim}{B_2}}(x), \mu_{\underset{\sim}{\overline{B_2}}}(x)\right]$

$= \left\{\frac{1}{1.0} + \frac{0.6}{1.5} + \frac{0.8}{2.0} + \frac{0.9}{2.5} + \frac{1}{3.0}\right\}$

3) Consider two fuzzy sets

$\underset{\sim}{A} = \left\{\frac{0.2}{1} + \frac{0.3}{2} + \frac{0.4}{3} + \frac{0.5}{4}\right\}$

$\underset{\sim}{B} = \left\{\frac{0.1}{1} + \frac{0.2}{2} + \frac{0.2}{3} + \frac{1}{4}\right\}$

a) Algebraic sum $\underset{\sim}{A} + \underset{\sim}{B}$

$\mu_{\underset{\sim}{A}+\underset{\sim}{B}} = \mu_{\underset{\sim}{A}}(x) + \mu_{\underset{\sim}{B}}(x) - \mu_{\underset{\sim}{A}}(x) \cdot \mu_{\underset{\sim}{B}}(x)$

$= \left\{\frac{0.3 - 0.02}{1} + \frac{0.5 - 0.06}{2} + \frac{0.6 - 0.08}{3} + \frac{1.5 - 0.5}{4}\right\}$

$= \left\{\frac{0.28}{1} + \frac{0.44}{2} + \frac{0.52}{3} + \frac{0.5}{4}\right\}$

b) Algebraic product $\underset{\sim}{A} \cdot \underset{\sim}{B}$

$\mu_{\underset{\sim}{A}\cdot\underset{\sim}{B}} = \mu_{\underset{\sim}{A}}(x) \cdot \mu_{\underset{\sim}{B}}(x)$

$= \left\{\frac{0.02}{1} + \frac{0.06}{2} + \frac{0.08}{3} + \frac{0.5}{4}\right\}$

c) Bounded sum $\underset{\sim}{A} \oplus \underset{\sim}{B}$

$\mu_{\underset{\sim}{A}\oplus\underset{\sim}{B}} = \min\left[1, \mu_{\underset{\sim}{A}}(x) + \mu_{\underset{\sim}{B}}(x)\right]$

$= \min\left\{1, \left[\frac{0.3}{1} + \frac{0.5}{2} + \frac{0.6}{3} + \frac{1.5}{4}\right]\right\}$

$= \left\{\frac{0.3}{1} + \frac{0.5}{2} + \frac{0.6}{3} + \frac{1.0}{4}\right\}$

2) Given two fuzzy sets

$$\underset{\sim}{B_1} = \left\{ \frac{1}{1.0} + \frac{0.75}{1.5} + \frac{0.3}{2.0} + \frac{0.15}{2.5} + \frac{0}{3.0} \right\}$$

$$\underset{\sim}{B_2} = \left\{ \frac{1}{1.0} + \frac{0.6}{1.5} + \frac{0.2}{2.0} + \frac{0.1}{2.5} + \frac{0}{3.0} \right\}$$

find

a) $\underset{\sim}{B_1} \cup \underset{\sim}{B_2} = \max\left[\mu_{\underset{\sim}{B_1}}(x), \mu_{\underset{\sim}{B_2}}(x) \right]$

$$= \left\{ \frac{1}{1.0} + \frac{0.75}{1.5} + \frac{0.3}{2.0} + \frac{0.15}{2.5} + \frac{0}{3.0} \right\}$$

b) $\underset{\sim}{B_1} \cap \underset{\sim}{B_2} = \min\left[\mu_{\underset{\sim}{B_1}}(x), \mu_{\underset{\sim}{B_2}}(x) \right]$

$$= \left\{ \frac{1}{1.0} + \frac{0.6}{1.5} + \frac{0.2}{2.0} + \frac{0.1}{2.5} + \frac{0}{3.0} \right\}$$

c) $\overline{\underset{\sim}{B_1}} = 1 - \mu_{\underset{\sim}{B_1}}(x)$

$$= \left\{ \frac{0}{1.0} + \frac{0.25}{1.5} + \frac{0.7}{2.0} + \frac{0.85}{2.5} + \frac{1}{3.0} \right\}$$

d) $\overline{\underset{\sim}{B_2}} = 1 - \mu_{\underset{\sim}{B_2}}(x)$

$$= \left\{ \frac{0}{1.0} + \frac{0.4}{1.5} + \frac{0.8}{2.0} + \frac{0.9}{2.5} + \frac{1}{3.0} \right\}$$

e) $\underset{\sim}{B_1} | \underset{\sim}{B_2} = B_1 \cap \overline{B_2}$

$$= \min\left[\mu_{\underset{\sim}{B_1}}(x), \mu_{\overline{\underset{\sim}{B_2}}}(x) \right]$$

$$= \left\{ \frac{0}{1.0} + \frac{0.4}{1.5} + \frac{0.3}{2.0} + \frac{0.15}{2.5} + \frac{0}{3.0} \right\}$$

f) $\overline{\underset{\sim}{B_1} \cup \underset{\sim}{B_2}} = 1 - (\underset{\sim}{B_1} \cup \underset{\sim}{B_2})$

$$= \left\{ \frac{0}{1.0} + \frac{0.25}{1.5} + \frac{0.7}{2.0} + \frac{0.85}{2.5} + \frac{1}{3.0} \right\}$$

g) $\overline{\underset{\sim}{B_1} \cap \underset{\sim}{B_2}} = 1 - (\underset{\sim}{B_1} \cap \underset{\sim}{B_2})$

$$= \left\{ \frac{0}{1.0} + \frac{0.4}{1.5} + \frac{0.8}{2.0} + \frac{0.9}{2.5} + \frac{1}{3.0} \right\}$$

h) $\underset{\sim}{B_1} \cap \overline{\underset{\sim}{B_1}} = \min\left[\mu_{B_1}(x), \mu_{\overline{B_1}}(x) \right]$

$$= \left\{ \frac{0}{1} + \frac{0.25}{1.5} + \frac{0.3}{2.0} + \frac{0.15}{2.5} + \frac{0}{3.0} \right\}$$

Prob :- Sivanandam pg. 264.

Consider two fuzzy sets

$$\underset{\sim}{A} = \left\{ \frac{1}{2} + \frac{0.3}{4} + \frac{0.5}{6} + \frac{0.2}{8} \right\}$$

$$\underset{\sim}{B} = \left\{ \frac{0.5}{2} + \frac{0.4}{4} + \frac{0.1}{6} + \frac{1}{8} \right\}$$

perform union, intersection, complement & diff. over fuzzy sets A & B.

1) Union :- $\underset{\sim}{A} \cup \underset{\sim}{B} = \max[\mu_{\underset{\sim}{A}}(x), \mu_{\underset{\sim}{B}}(x)]$

$$= \left\{ \frac{1}{2} + \frac{0.4}{4} + \frac{0.5}{6} + \frac{0.1}{8} \right\}$$

2) Intersection $\underset{\sim}{A} \cap \underset{\sim}{B} = \min[\mu_{\underset{\sim}{A}}(x), \mu_{\underset{\sim}{B}}(x)]$

$$= \left\{ \frac{0.5}{2} + \frac{0.3}{4} + \frac{0.1}{6} + \frac{0.2}{8} \right\}$$

3) Complement $\bar{\underset{\sim}{A}} = 1 - \mu_{\underset{\sim}{A}}(x)$

$$= \left\{ \frac{0}{2} + \frac{0.7}{4}, \frac{0.5}{6} + \frac{0.8}{8} \right\}$$

$$\bar{\underset{\sim}{B}} = 1 - \mu_{\underset{\sim}{B}}(x)$$

$$= \left\{ \frac{0.5}{2} + \frac{0.6}{4} + \frac{0.9}{6} + \frac{0}{8} \right\}$$

4) Difference

$$\underset{\sim}{A} | \underset{\sim}{B} = \underset{\sim}{A} \cap \bar{\underset{\sim}{B}} = \min[\mu_{\underset{\sim}{A}}(x), \mu_{\bar{\underset{\sim}{B}}}(x)]$$

$$= \left\{ \frac{0.5}{2} + \frac{0.3}{4} + \frac{0.5}{6} + \frac{0}{8} \right\}$$

$$\underset{\sim}{B} | \underset{\sim}{A} = B \cap \bar{\underset{\sim}{A}} = \min[\mu_{\underset{\sim}{B}}(x), \mu_{\bar{\underset{\sim}{A}}}(x)]$$

$$= \left\{ \frac{0}{2} + \frac{0.4}{4} + \frac{0.1}{6} + \frac{0.8}{8} \right\}$$

Fuzzy set operations

1) Union $(\underset{\sim}{A} \cup \underset{\sim}{B})$

$$\mu_{\underset{\sim}{A} \cup \underset{\sim}{B}}(x) = \max \boxed{\mu_{\underset{\sim}{A}}(x), \mu_{\underset{\sim}{B}}(x)} = \mu_{\underset{\sim}{A}}(x) \vee \mu_{\underset{\sim}{B}}(x), \forall x \in U$$

max operator

2) Intersection $(\underset{\sim}{A} \cap \underset{\sim}{B})$

$$\mu_{\underset{\sim}{A} \cap \underset{\sim}{B}}(x) = \min \left[\mu_{\underset{\sim}{A}}(x), \mu_{\underset{\sim}{B}}(x)\right] = \mu_{\underset{\sim}{A}}(x) \wedge \mu_{\underset{\sim}{B}}(x), \forall x \in U$$

min operator

3) Complement $(\bar{\underset{\sim}{A}})$

$$\mu_{\bar{\underset{\sim}{A}}}(x) = 1 - \mu_{\underset{\sim}{A}}(x) \quad \forall x \in U.$$

4) Algebraic sum $(\underset{\sim}{A} + \underset{\sim}{B})$

$$\mu_{\underset{\sim}{A} + \underset{\sim}{B}}(x) = \mu_{\underset{\sim}{A}}(x) + \mu_{\underset{\sim}{B}}(x) - \mu_{\underset{\sim}{A}}(x) \cdot \mu_{\underset{\sim}{B}}(x)$$

5) Algebraic product $(\underset{\sim}{A} \cdot \underset{\sim}{B})$

$$\mu_{\underset{\sim}{A} \cdot \underset{\sim}{B}}(x) = \mu_{\underset{\sim}{A}}(x) \cdot \mu_{\underset{\sim}{B}}(x).$$

6) Bounded sum $(\underset{\sim}{A} \oplus \underset{\sim}{B})$

$$\mu_{\underset{\sim}{A} \oplus \underset{\sim}{B}}(x) = \min \left[1, \mu_{\underset{\sim}{A}}(x) + \mu_{\underset{\sim}{B}}(x)\right]$$

7) Bounded difference $(\underset{\sim}{A} \odot \underset{\sim}{B})$

$$\mu_{\underset{\sim}{A} \odot \underset{\sim}{B}} = \max \left[0, \mu_{\underset{\sim}{A}}(x) - \mu_{\underset{\sim}{B}}(x)\right]$$

Dr. Zadeh — Principle of complexity and imprecision
— The closer one looks at a real world problem, the fuzzier becomes its solution.

fuzzy logic — paradigm of computing c̄ words
— a tech. to deal c̄ imprecision & info. granularity
— provides a mechanism for representing linguistic constructs such as 'high', low, tall, many, few.

— based on notion of relative graded membership functions of cognitive processes.

Imprecise & vague data → Fuzzy logic system → decisions.

FLS accepting imprecise & vague data & providing a decision.

The utility of fuzzy sets lies in their ability to model uncertain or ambiguous data and to provide suitable decisions.

SONAL.

Fuzzy Set theory

Lotfi Zadeh in 1965.

fuzziness ⟷ vagueness.

entire real world is complex.
this complexity is arises from uncertainty.
uncertainty based on random processes → probability theory.

uncertainty is characterised by non-random processes
- uncertainty due to partial info.
- due to info. that is not fully reliable
- due to inherent imprecision in the language.
- due to receipt of info. from multiple sources about a problem c is conflicting.

→ fuzzy set theory provides effective problem solving approach.

Fuzzy logic is a form of multivalued logic to deal with reasoning that is approximate rather than precise.

crisp logic — fuzzy logic

Is water colorless?	Is ram honest?	
y/n.	extremly honest	(1)
	extremly dishonest	(0)
	very honest	(0.85)
	honest at times	(0.4)
↓	↓	
crisp set	fuzzy set.	