Fuzzy Sets

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Introduction -

A classical set is a set with a crisp boundary for example -
a classical set A of real numbers greater than 10 can be expressed as,

$$A = \{x / x > 10\}$$

Here there is a clear, unambiguous boundary 10 such that if x is greater than this number then x belongs to the set.

Although classical sets are suitable for various applications and have proven to be an important tool for mathematics and computer science, they do not reflect the nature of human concepts and thoughts, which tend to be abstract and imprecise.

As an illustration - mathematically we can express the set of tall persons as a ~~grp~~/collection of persons having big height more than 6 ft. as,

$$A = \{x / x > 6\}$$

where,
A → is 'tall person'
x → is 'height.'

Yet this is an unnatural and inadequate way of representing our usual concept of a 'tall person'.

The dichotomous nature of the classical set would classify a person 6.00 ft tall as a 'tall person', but not a person 5.99 ft tall. This distinction is intuitively unreasonable. The flaws comes from the sharp transition between inclusion & exclusion in a set.

In contrast to classical set, a fuzzy set, as a name implies is a set without a crisp boundary. i.e. the transition from "belong to a set" to "not belong to a set" is gradual. It is a smooth transition characterised by membership functions that give fuzzy sets flexibility in modeling commonly used linguistic expressions such as "the water is hot", or "the temp. is high".

→ Fuzziness does not come from the randomness of the constituent members of the sets, but from the uncertain and imprecise nature of abstract thoughts and concepts.

Basic Definitions and Terminology

Let X be the space of objects and x be the generic element of X.
A classical set A, $A \subseteq X$, is defined as a collection of elements or objects $x \in X$, such that each x can either belong or not belong to the set A.
By defining the characteristics function for each element x in X, we can represent a classical set A by a set of ordered pair $(x, 0)$ or $(x, 1)$ which indicates $x \notin A$ or $x \in A$ resp.

A fuzzy set expresses the degree to which an element belongs to a set.
Thus, the characteristic function of a fuzzy set is allowed to have values between 0 & 1. that denotes the degree of membership of an element in a given set.

Def.ⁿ - Fuzzy sets and membership functions.

If X is a collection of objects denoted by x, then fuzzy set A in X is defined as a set of ordered pairs -

$$A = \{ (x, \mu_A(x)) / x \in X \}$$

$\mu_A(x) \rightarrow$ membership of object x for fuzzy set A.

The MF (membership function) maps each element of X to a membership grade (or membership value) between 0 & 1.

→ Thus, the defⁿ of fuzzy set is a simple extension of the defⁿ of classical set in which the characteristic function is permitted to have any value between 0 and 1.

→ If values of $\mu_A(x)$ are restricted to 0 or 1 then A reduces to a classical set. and $\mu_A(x)$ is the characteristic function of A.

→ X : referred to as universe of discourse or simply universe.
It consists of discrete objects (ordered / non-ordered) or continuous space.

ex. 1 fuzzy set with a discrete non ordered universe.

Let X = { mumbai, Pune, Nagpur } be the set of cities one may choose to live in.

The fuzzy set C = "desirable city to live in" may be described as,

C = {(mumbai, 0.9), (Pune, 0.8), (Nagpur, 0.6)}

Here, the universe of discourse X is discrete and it contains non-ordered objects - in this case - three cities in the state of maharashtra.

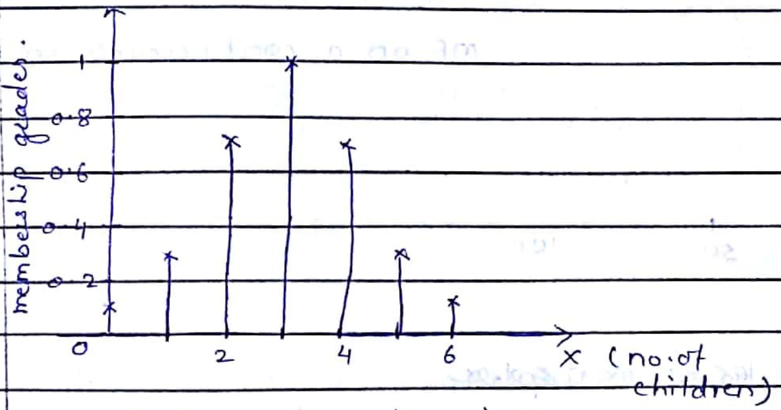
The membership grades listed above are quite subjective, you can come up with three different but legitimate values to reflect his/her preference.

ex 2 : fuzzy set with discrete ordered universe.

Let $X = \{0, 1, 2, 3, 4, 5, 6\}$ be the set of numbers of children, a family may choose to have.
Then,
the fuzzy set A = "sensible number of children in a family".
may be described as

$$A = \{(0, 0.1), (1, 0.3), (2, 0.7), (3, 1), (4, 0.7), (5, 0.3), (6, 0.1)\}.$$

Here universe X is discrete ordered.



MF on discrete universe.

Again here the membership grades are subjective measures.

Ex 3 : Fuzzy sets with a continuous universe.

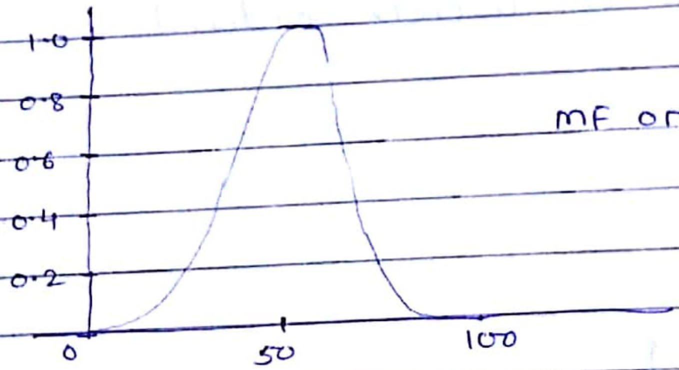
Let $X = R^+$ be the set of possible ages of human beings.
Then the fuzzy set

B = 'about 50 yrs old'

may be expressed as

$$B = \{ (x, \mu_B(x)) / x \in X \}$$

where $\mu_B(x) = \dfrac{1}{1 + \left(\frac{x-50}{10}\right)^4}$



MF on a continuous universe.

From the preceding examples, it is clear that the construction of the set depends on two things - identification of a suitable universe of discourse
- specification of an appropriate MF. membership function.

→ The specification of MF is subjective, that means, the MF specified for the same concept by different persons may vary considerably. (ex. - "sensible no. of children in a family").

→ This subjectivity comes from the individual differences in perceiving or expressing abstract concepts & has little to do with randomness.

∴ the subjectivity and non randomness of fuzzy sets is the primary difference between the study of fuzzy sets and probability theory, which deals with objective treatment of random phenomena.

→ For simplicity of notation, we now introduce an alternative way of denoting a fuzzy set.

A fuzzy set A ~~is~~ can be denoted as,

$$A = \begin{cases} \sum_{x_i \in X} \mu_A(x_i)/x_i & \text{if } X \text{ is a collection of discrete objects.} \\ \int_X \mu_A(x)/x & \text{if } X \text{ is a continuous space (usually the real line } R\text{).} \end{cases}$$

The Σ & $\int$ sign in above equation stand for union of $(x, \mu_A(x))$ pairs.
they do not indicate summation or integration.
Also, "/" sign is only a marker and does not imply division.

∴ alternative expressions →

~~exa~~ $C = 0.9/\text{mumbai} + 0.8/\text{pune} + 0.6/\text{Nagpur}.$

$A = 0.1/0 + 0.3/1 + 0.7/2 + 1.0/3 + 0.7/4 + 0.3/5 + 0.1/6.$

$$B = \int_{R^+} \frac{1}{1+\left(\frac{x-50}{10}\right)^4} \Big/ x$$

In practice, when the universe of discourse X is a continuous space (the real line R or its subset), we usually partition X into several fuzzy sets whose MF's cover X in a more or less uniform manner.

These fuzzy sets, which usually carry names that conform to adjectives appearing in our daily linguistic usage such as "large", "medium" or "small" are called linguistic values or linguistic labels.
Thus the universe of discourse X is often called the linguistic variable.

ex: linguistic variables and linguistic values.

Let X = "age".
then we define fuzzy sets "young", "middle aged" & "old" that are characterised by MF, $\mu_{old}(x)$, $\mu_{middle aged}(x)$ and $\mu_{old}(x)$. respectively.

A fuzzy set is uniquely specified by its membership function. To describe membership functions more specifically, we shall define the nomenclature used in the literature.

Defn: Support –

The support of a fuzzy set A is the set of all points x in X such that $\mu_A(x) > 0$:

$$\text{Support}(A) = \{ x \mid \mu_A(x) > 0 \}$$

Defn – Core –

The core of a fuzzy set A is the set of all points x in X such that $\mu_A(x) = 1$

$$\text{Core}(A) = \{ x \mid \mu_A(x) = 1 \}$$

Defn – Normality –

A fuzzy set A is normal if its core is non-empty.

In other words, we can always find a point $x \in X$ such that $\mu_A(x) = 1$.

Defn – cross over points –

A cross over point of a fuzzy set A is a point $x \in X$ at which $\mu_A(x) = 0.5$

$$\text{crossover}(A) = \{ x \mid \mu_A(x) = 0.5 \}.$$

Defⁿ: fuzzy singleton -

A fuzzy set whose support is a single point in X which with $\mu_A(x) = 1$ is called a fuzzy singleton.

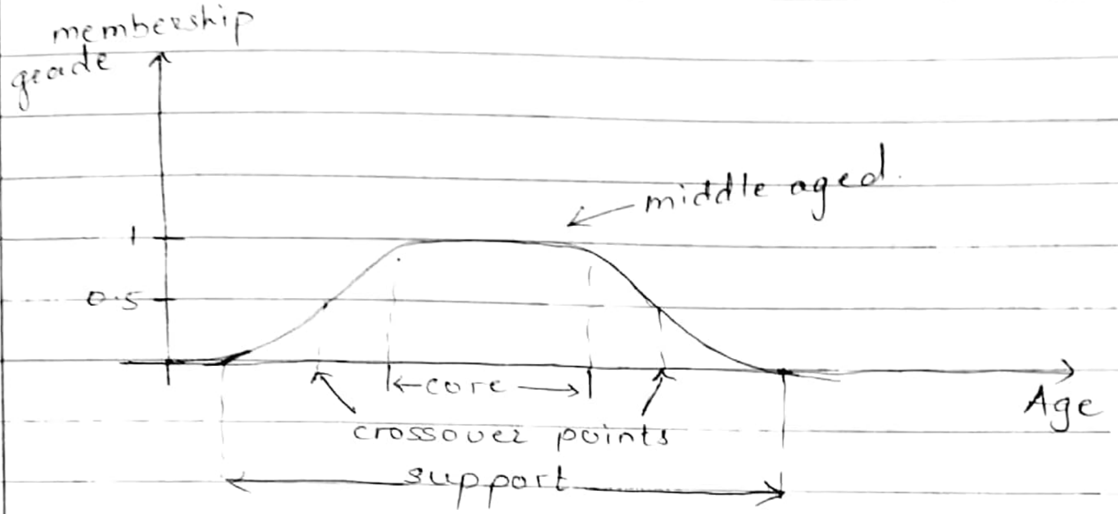


fig. :- fuzzy set "middle aged".

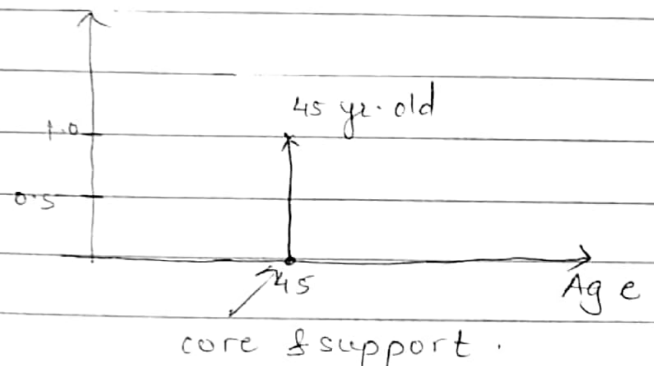


fig. :- Fuzzy singleton "45 yr old".

using the notation for a level set, we can express the support and core of a fuzzy set A as,

$$\text{support}(A) = A_{0'}$$

and

$$\text{core}(A) = A_1$$ respectively.

Def^n - Convexity -

A fuzzy set A is convex iff for any $x_1, x_2 \in X$ and any $\lambda \in [0, 1]$

$$\mu_A(\lambda x_1 + (1-\lambda)x_2) \geq \min\{\mu_A(x_1), \mu_A(x_2)\}$$

Alternatively, A is convex if all its α-level sets are convex.