SRM INSTITUTE OF SCIENCE AND TECHNOLOGY

RAMAPURAM PART- VADAPALANI CAMPUS, CHENNAI – 600 026

Department of Mathematics

Sub Title: DISCRETE MATHEMATICS FOR ENGINEERS

Sub Code: 18 MAB 302 T

Unit -I - SET THEORY

1. A collection of all well defined objects is called

   (a) set (b) group (c) coset (d) lattice **Ans: a**

2. Power set of empty set has exactly _________ subset.

   (a) one (b) two (c) zero(d) three **Ans: a**

3. What is the Cartesian product of A = {1, 2} and B = {a, b}?

   a) {(1, a), (1, b), (2, a), (b, b)}
   b) {(1, 1), (2, 2), (a, a), (b, b)}
   c) {(1, a), (2, a), (1, b), (2, b)}
   d) {(1, 1), (a, a), (2, a), (1, b)} **Ans: c**

4. What is the cardinality of the set of odd positive integers less than 10?

   (a) 10 (b) 5 (c) 3 (d) 20 **Ans: b**

5. Which of the following two sets are equal?
   a) A = {1, 2} and B = {1} b) A = {1, 2} and B = {1, 2, 3}
   c) A = {1, 2, 3} and B = {2, 1, 3} d) A = {1, 2, 4} and B = {1, 2, 3} **Ans: c**

6. What is the Cardinality of the Power set of the set {0, 1, 2}?

   (a) 8 (b) 6 (c) 7 (d) 9 **Ans: a**

7. In a class of 120 students numbered 1 to 120, all even numbered students opt for Physics, those whose numbers are divisible by 5 opt for Chemistry and those whose numbers are divisible by 7 opt for Math. How many opt for none of the three subjects?

   a) 19 b) 41 c) 21 d) 57 **Ans : b**

8. Let R be a non-empty relation on a collection of sets defined by ARB if and only if

   $A \cap B = \emptyset$ Then (pick the TRUE statement)

   a). R is reflexive and transitive b). R is an equivalence relation

   c). R is symmetric and not transitive d). R is not relexive and not symmetric **Ans: c**

9. The binary relation S = Φ (empty set) on set A = {1, 2,3} is

a). transitive and relexive b). symmetric and relexive

c). transitive and symmetric d). neither reflexive nor symmetric **Ans: c**

10. Number of subsets of a set of order three is

a) 2 b) 4 c) 6 d) 8 **Ans: d**

11. "n/m" means that n is a factor of m, then the relation T is

a). relexive, transitive and not symmetric b). relexive, transitive and symmetric

c). transitive and symmetric d). relexive and symmetric **Ans: a**

12. Two sets are called disjoint if there _____________ is the empty set.

a) Union b) Difference c) Intersection d) Complement **Ans: c**

13. The set difference of the set A with null set is __________

a) A b) null c) U d) B **Ans: a**

14. An equivalence relation R on a set A is said to posses

(a) reflexive, antisymmetric and transitive (b) reflexive, symmetric and transitive

(c) reflexive, nonsymmetric and antisymmetric (d) irreflexive, symmetric and transitive **Ans: b**

15.Relative complement of S with respect to R is defined as

$(a)\ \{x / x \in R \text{ and } x \notin S\}$ $(b)\ \{x / x \in R \text{ and } x \in S\}$

$(c)\ \{x / x \notin R \text{ and } x \in S\}$ $(d)\ \{x / x \notin R \text{ and } x \notin S\}$ **Ans: a**

16. If the relation R is reflexive, antisymmetric and transitive, then the relation R is called

(a) equivalence relation (b) equivalence class (c) partial order relation

(d) partially ordered set **Ans: c**

17. A digraph representing the partial order relation

(a) Helmut Hasse (b) POSET (c) graph relation (d) Hasse diagram **Ans: d**

18. In a poset, the maximum number of greatest and least members if they exist are

(a) more than one (b) unique (c) zero (d) exactly two **Ans: b**

19. Equivalence class of 'a' is defined by

$(a)\ \{x / (a,x) \in R\}$ $(b)\ \{x / (x,a) \in R\}$ $(c)\ \{a / (a,x) \in R\}$ $(d)\ \{a / (x,a) \in R\}$ **Ans: a**

20. If A is a non-empty set with n elements, then number of possible relations on the set A is

(a) 2^n (b) 2^{n-1} (c) 2^{n^2} (d) 2^{n+1} **Ans: c**

21. Which one of the following relations on the set {1, 2, 3, 4} is an equivalent relation

(a) {(2,4), (4,2)} (b) {(2,2), (2,3), (2,4), (3,2), (3,3), (3,4)}

(c) {(1,3), (1,4), (2,3), (2,4), (3,1), (3,4)}(d) {(1,1), (1,2), (2,1), (2,2), (3,3), (4,4)} **Ans: d**

22. From each of the following relations, determine which is one of the relation is a partial order relation

(a) $R \subseteq Z \times Z$ where aRb if a divides b (b) R is the relation on Z, where aRb if a + b is odd

(c) $R \subseteq Z^+ \times Z^+$, where aRb if a divides b (d) none of these. **Ans: c**

23. Determine which one of the following relations on the set {1, 2, 3, 4} is a function.

(a) $R_1 = \{(1,1),(2,1),(3,1),(4,1),(3,3)\}$ (b) $R_2 = \{(1,2),(2,3),(4,2)\}$

(c) $R_3 = \{(4,4),(3,1),(1,2),(4,2)\}$ (d) $R_4 = \{(1,1),(2,1),(1,2),(3,4)\}$ **Ans: a**

24. How many possible functions we get $f : A \to B$, if $|A| = m$ and $|B| = n$

(a) 2^n (b) 2^m (c) n^m (d) m^n **Ans: c**

25. If $A = \{1,2,3\}$ and f, g are functions from A to A given by f = {(1,2), (2,3), (3,1)}, g = {(1,2), (2,1), (3,3)} then {(1,3), (2,2), (3,1)} is the composition relation of one of the following:

$(a)\ f \circ g$ $(b)\ g \circ f$ $(c)\ f \circ (f \circ g)$ $(d)\ f \circ (g \circ f)$ **Ans: a**

26. If $f(x) = ax + b, g(x) = 1 - x + x^2$ for $x \in R$, and $(g \circ f)(x) = 9x^2 - 9x + 3$. Find the values of a and b.

(a) a = 3, b = -1 (or) a = -3, b = 2 (b) a = 1, b = 3 (or) a = 1, b = 2
(c) a = -3, b = -1 (or) a = -3, b = 2 (d) a = 3, b = 2 (or) a = -3, b = -1 **Ans: a**

27. If $A = \{1,2,3,4\}, B = \{x, y, z\}$ and $f = \{(1,x),(2,y),(3,z),(4,x)\}$, then the function f is

(a) both 1 – 1 and onto (b) 1 – 1 but not onto
(c) onto but not 1 – 1 (d) neither 1 – 1 nor onto **Ans: c**

28. A Relation R is defined on the set of integers as xRy iff (x+y) is even . Which of the following statements is TRUE?

(a) R is not an equivalence relation
(b) R is an equivalence relation having one equivalence class
(c) R is an equivalence relation having two equivalence class
(d) R is an equivalence relation having three equivalence class **Ans: c**

29. The number of equivalence relations of the set {1,2,3,4} is
(a) 4 (c) 16
(b) 15 (d) 24 **Ans: b**

30. If R be a symmetric and transitive relation on a set A, then
(a) R is reflexive and hence an equivalence relation
(b) R is reflexive and hence a parital order
(c) R is not reflexive and hence not an equivalence relation
(d) R is Reflexive **Ans: d**

31. Relation R defined on a set N by R={(a,b) : |a - b| is divisible by 5}, is

(a) reflexive (c) transitive
(b) symmetric (d) Equivalence **Ans: d**

32. The domain and range are same for

(a) constant function (c) absolute value function
(b) identity function (d) greatest integer function **Ans: b**

33. The function f : Z →Z given by $f(x)=x^2$ is

(a) one-one (c) one-one and onto
(b) onto (d) in-to **Ans: a**

34. A relation over the set S=[x,y,z]is defined by : {(x,x),(x,y),(y,x),(x,z),(y,z),(y,y),(z,z)}.

(a) Symmetric (c) Irreflexive
(b) Reflexive (d) Anti-symmetric **Ans: b**

35. If sets A and B have 3 and 6 elements each, then minimum number of elements in A ∪ B is

(a) 3 (c) 18
(b)6 (d) 9 **Ans: b**

36. f : R → R is a function defined by f (x) = 10x -7. If g = f^{-1},then g(x)

(a) $\frac{1}{10x-7}$ (c) $\frac{x+7}{10}$
(b) $\frac{1}{10x+7}$ (d) $\frac{x-7}{10}$ **Ans: c**

37.The set of all Equivalence classes of a set A of cardinality C
(a) has the same cardinality as A
(b) forms a partition of A
(c) is of cardinality 2C
(d) is of cardinality C^2 **Ans: b**

38.Which of the following sets is a null set
i. X={x| x=9, 2x =4} ii Y={x|x=2x.x≠0} iii Z={x|x-8=4}

(a) I and II only (c) I and III only
(b) I,II and III (d) I and III only **Ans: a**

39.Let A={1,2,3........} Define ~ by x ~ y ⇔ x divide y. Then ~ is
(a) reflexive, but not a parital-ordering
(b) symmetric
(c) an equivalence relation
(d) a parital- ordering relation **Ans: d**

40.If A={1,2,3}, then relation S={(1,1),(2,2) is
(a) symmetric only
(b) anti-symmetric only
(c) both symmetric and anti-symmetric only **Ans: c**
(d) an equivalence relation

41.If A={1,2,3,4}. Let ~ = {(1,2),(1,3),(4,2). Then ~ is
(a) not anti-symmetric (c) reflexive
(b)transitive (d) symmetric **Ans: b**

42.Let D_{30}={1,2,3,5,6,10,15,30} and relation I be a parital ordering on D_{30}.The all upper bounds of 10 and 15 respectively is
(a)30 (b) 15 (c) 10 (d) 6 **Ans : a**

43.Let D_{30}={1,2,3,5,6,10,15,30} and relation I be a parital ordering on D_{30}.The lub of 10 and 15 respectively is
(a) 30 (c) 10
(b) 15 (d) 6 **Ans : a**

44.Total number of different partitions of a set having four elements
a). 16 b) 8 (c) 15 d) 4 **Ans : c**

45.Hasse diagrams are drawn for

(a) Partially ordered sets (c) boolean algebra

(b) Lattics (d) Modern Algebra **Ans: a**

46.Let X={2,3,6,12,24},and ≤ be the partial order defined on the set S={x,a_1,a_2,a_3,............,a_n,y} as ≤ a_i for i and a_i ≤ y for all i,where n ≥ 1. Number of total orders on the set S which contain partial order ≤ 1

(a) 1 (c) n+1

(b) n (d) n! **Ans: d**

47..Let X={2,3,6,12,24}, and ≤ be the parital order defined by X ≤ Y if X divides Y. Number of edges in the Hasee diagram of (X, ≤) is

(a) 3 (c) 5

(b) 4 (d) 6 **Ans: b**

SET :

Definition: A set is an unordered collection of different elements. A set can be written explicitly by listing its elements using set bracket. If the order of the elements is changed or any element of a set is repeated, it does not make any changes in the set.

Some Example of Sets

- A set of all positive integers
- A set of all the planets in the solar system
- A set of all the states in India
- A set of all the lowercase letters of the alphabet

Representation of a Set

Sets can be represented in two ways −

- Roster or Tabular Form
- Set Builder Notation

Roster or Tabular Form

The set is represented by listing all the elements comprising it. The elements are enclosed within braces and separated by commas.

Example 1 − Set of vowels in English alphabet, A={a,e,i,o,u}

Example 2 − Set of odd numbers less than 10, B={1,3,5,7,9}

Set Builder Notation

The set is defined by specifying a property that elements of the set have in common. The set is described as A={x:p(x)}

Example 1 − The set {a,e,i,o,u} is written as −

$A=\{x:x \text{ is a vowel in English alphabet}\}$

Example 2 − The set $\{1,3,5,7,9\}$ is written as −

$B=\{x:1\leq x<10 \text{ and } x\neq 0\}$

If an element x is a member of any set S, it is denoted by $x\in S$ and if an element y is not a member of set S, it is denoted by $y\notin S$

Example − If $S=\{1,1.2,1.7,2\}, 1\in S$ but $1.5\notin S$

Some Important Sets

N − the set of all natural numbers = $\{1,2,3,4,.....\}$

Z − the set of all integers = $\{.....,-3,-2,-1,0,1,2,3,.....\}$

Z+ − the set of all positive integers

Q − the set of all rational numbers

R − the set of all real numbers

W − the set of all whole numbers

Cardinality of a Set

Cardinality of a set S, denoted by $|S|$, is the number of elements of the set. The number is also referred as the cardinal number. If a set has an infinite number of elements, its cardinality is ∞.

Example − $|\{1,4,3,5\}|=4, |\{1,2,3,4,5,\ldots\}|=\infty|$

If there are two sets X and Y,

- $|X|=|Y|$ denotes two sets X and Y having same cardinality. It occurs when the number of elements in X is exactly equal to the number of elements in Y. In this case, there exists a bijective function 'f' from X to Y.

- $|X|\leq|Y|$ denotes that set X's cardinality is less than or equal to set Y's cardinality. It occurs when number of elements in X is less than or equal to that of Y. Here, there exists an injective function 'f' from X to Y.

- $|X|<|Y|$ denotes that set X's cardinality is less than set Y's cardinality. It occurs when number of elements in X is less than that of Y. Here, the function 'f' from X to Y is injective function but not bijective.

- If $|X|\leq|Y|$ and $|X|\geq|Y|$ then $|X|=|Y|$. The sets X and Y are commonly referred as equivalent sets.

Types of Sets

Sets can be classified into many types. Some of which are finite, infinite, subset, universal, proper, singleton set, etc.

Finite Set

A set which contains a definite number of elements is called a finite set.

Example – $S=\{x|x\in N$ and $70>x>50\}$

Infinite Set

A set which contains infinite number of elements is called an infinite set.

Example – $S=\{x|x\in N$ and $x>10\}$

Subset

A set X is a subset of set Y (Written as $X\subseteq Y$ if every element of X is an element of set Y.

Example 1 –

Let, $X=\{1,2,3,4,5,6\}$ and $Y=\{1,2\}$. Here set Y is a subset of set X as all the elements of set Y is in set X. Hence, we can write $Y\subseteq X$.

Example 2 –

Let, $X=\{1,2,3\}$ and $Y=\{1,2,3\}$. Here set Y is a subset (Not a proper subset) of set X as all the elements of set Y is in set X. Hence, we can write $Y\subseteq X$.

Proper Subset

The term "proper subset" can be defined as "subset of but not equal to". A Set X is a proper subset of set Y (Written as $X\subset Y$) if every element of X is an element of set Y and $|X|<|Y|$.

Example –

Let, $X=\{1,2,3,4,5,6\}$ and $Y=\{1,2\}$. Here set $Y\subset X$ since all elements in Y are contained in X too and X has at least one element is more than set Y.

Universal Set

It is a collection of all elements in a particular context or application. All the sets in that context or application are essentially subsets of this universal set. Universal sets are represented as U.

Example –

We may define U as the set of all animals on earth. In this case, set of all mammals is a subset of U, set of all fishes is a subset of U, set of all insects is a subset of U, and so on.

Empty Set or Null Set

An empty set contains no elements. It is denoted by $\emptyset$. As the number of elements in an empty set is finite, empty set is a finite set. The cardinality of empty set or null set is zero.

Example – $S=\{x|x\in N$ and $7<x<8\}=\emptyset$

Singleton Set or Unit Set

Singleton set or unit set contains only one element. A singleton set is denoted by $\{s\}$.

Example – $S=\{x|x\in N,\ 7<x<9\}=\{8\}$

Equal Set

If two sets contain the same elements they are said to be equal.

Example –

If $A=\{1,2,6\}$ and $B=\{6,1,2\}$, they are equal as every element of set A is an element of set B and every element of set B is an element of set A.

Equivalent Set

If the cardinalities of two sets are same, they are called equivalent sets.

Example –
If $A=\{1,2,6\}$ and $B=\{16,17,22\}$, they are equivalent as cardinality of A is equal to the cardinality of B. i.e. $|A|=|B|=3$

Overlapping Set

Two sets that have at least one common element are called overlapping sets.

In case of overlapping sets −

- $n(A \cup B) = n(A) + n(B) - n(A \cap B)$
- $n(A \cup B) = n(A-B) + n(B-A) + n(A \cap B)$
- $n(A) = n(A-B) + n(A \cap B)$
- $n(B) = n(B-A) + n(A \cap B)$

Example –
Let, $A=\{1,2,6\}$ and $B=\{6,12,42\}$. There is a common element '6', hence these sets are overlapping sets.

Disjoint Set

Two sets A and B are called disjoint sets if they do not have even one element in common. Therefore, disjoint sets have the following properties −

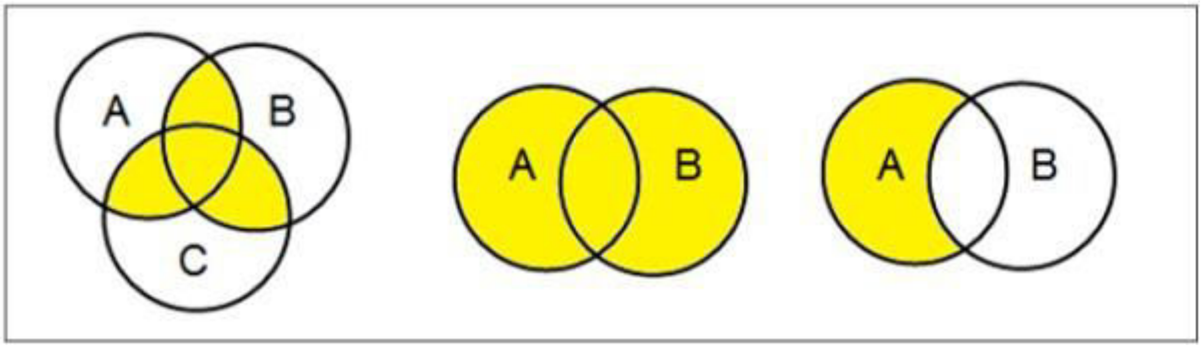
- $n(A \cap B) = \emptyset$
- $n(A \cup B) = n(A) + n(B)$

Example –
Let, $A=\{1,2,6\}$ and $B=\{7,9,14\}$ there is not a single common element, hence these sets are overlapping sets.

Venn Diagrams

Venn diagram, invented in 1880 by John Venn, is a schematic diagram that shows all possible logical relations between different mathematical sets.

Examples



Set Operations

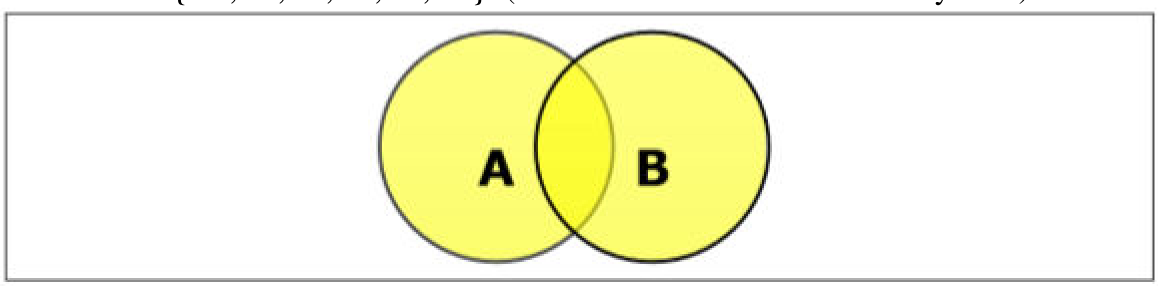
Set Operations include Set Union, Set Intersection, Set Difference, Complement of Set, and Cartesian Product.

Set Union

The union of sets A and B (denoted by $A \cup B$) is the set of elements which are in A, in B, or in both A and B. Hence, $A \cup B = \{x | x \in A \text{ OR } x \in B\}$.

Example –

If $A=\{10,11,12,13\}$A={10,11,12,13} and B = $\{13,14,15\}${13,14,15},
then $A \cup B = \{10,11,12,13,14,15\}$. (The common element occurs only once)

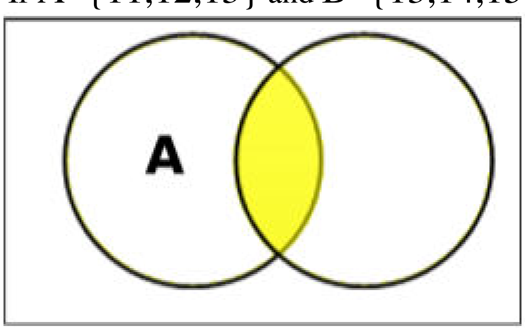


Set Intersection

The intersection of sets A and B (denoted by $A \cap B$) is the set of elements which are in both A and B. Hence, $A \cap B = \{x | x \in A \text{ and } x \in B\}$.

Example –

If $A=\{11,12,13\}$ and $B=\{13,14,15\}$ then $A \cap B = \{13\}$.

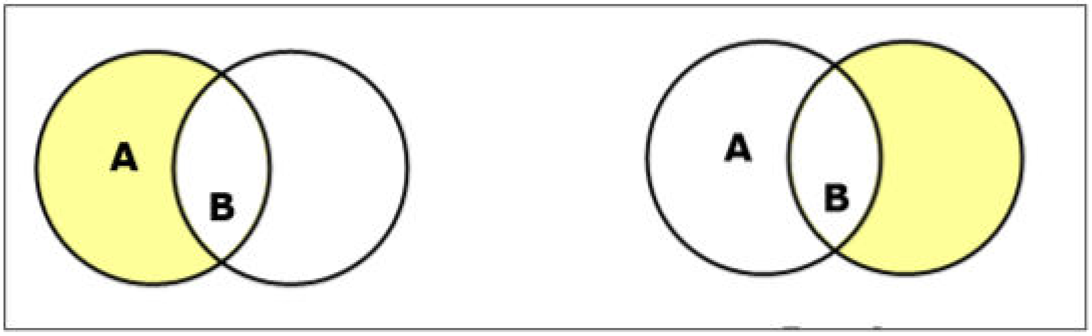


Set Difference/ Relative Complement

The set difference of sets A and B (denoted by $A – B$) is the set of elements which are only in A but not in B. Hence, $A - B = \{x | x \in A \text{ and } x \notin B\}$

Example –

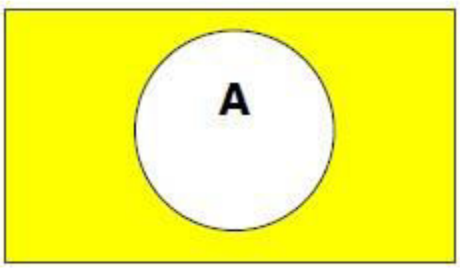
If $A=\{10,11,12,13\}$ and $B=\{13,14,15\}$ then $(A-B) = \{10,11,12\}$ and $(B-A)=\{14,15\}$ Here, we can see $(A-B)\neq(B-A)$



Complement of a Set

The complement of a set A (denoted by A') is the set of elements which are not in set A. Hence, $A'=\{x|x\notin A\}$. More specifically, $A'=(U-A)$ where U is a universal set which contains all objects.

Example – If $A=\{x|x \text{ belongs to set of odd integers}\}$ then $A'=\{y \mid y \text{ does not belong to set of odd integers}\}$



Cartesian Product / Cross Product

The Cartesian product of n number of sets $A_1,A_2,\ldots A_n$ denoted as $A_1\times A_2\cdots\times A_n$ can be defined as all possible ordered pairs $(x_1,x_2,\ldots x_n)$ where $x_1\in A_1$, $x_2\in A_2,\ldots x_n \in A_n$

Example –

If we take two sets $A=\{a,b\}$ and $B=\{1,2\}$

The Cartesian product of A and B is written as – $A\times B =\{(a,1),(a,2),(b,1),(b,2)\}$ The Cartesian product of B and A is written as – $B\times A = \{(1,a),(1,b),(2,a),(2,b)\}$

Power Set

Power set of a set S is the set of all subsets of S including the empty set. The cardinality of a power set of a set S of cardinality n is 2^n. Power set is denoted as $P(S)$.

Example –

For a set $S=\{a,b,c,d\}$ let us calculate the subsets –

- Subsets with 0 elements – $\{\emptyset\}$ (the empty set)
- Subsets with 1 element – $\{a\},\{b\},\{c\},\{d\}$ $\{a\},\{b\},\{c\},\{d\}$

- Subsets with 2 elements − {a,b},{a,c},{a,d},{b,c},{b,d},{c,d}
- Subsets with 3 elements − {a,b,c},{a,b,d},{a,c,d},{b,c,d}
- Subsets with 4 elements − {a,b,c,d}

Hence, P(S) = {{∅},{a},{b},{c},{d},{a,b},{a,c},{a,d},{b,c},{b,d},{c,d},{a,b,c},{a,b,d},{a,c,d},{b,c,d},{a,b,c,d

$|P(S)|=2^4=16$

Note − The power set of an empty set is also an empty set.

$|P(\{\emptyset\})|=2^0=1$

TABLE 1 Set Identities.

Identity	*Name*
$A \cup \emptyset = A$ $A \cap U = A$	Identity laws
$A \cup U = U$ $A \cap \emptyset = \emptyset$	Domination laws
$A \cup A = A$ $A \cap A = A$	Idempotent laws
$\overline{(\overline{A})} = A$	Complementation law
$A \cup B = B \cup A$ $A \cap B = B \cap A$	Commutative laws
$A \cup (B \cup C) = (A \cup B) \cup C$ $A \cap (B \cap C) = (A \cap B) \cap C$	Associative laws
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	Distributive laws
$\overline{A \cup B} = \overline{A} \cap \overline{B}$ $\overline{A \cap B} = \overline{A} \cup \overline{B}$	De Morgan's laws
$A \cup (A \cap B) = A$ $A \cap (A \cup B) = A$	Absorption laws
$A \cup \overline{A} = U$ $A \cap \overline{A} = \emptyset$	Complement laws

Partitioning of a Set

Partition of a set, say *S*, is a collection of *n* disjoint subsets, say $P_1, P_2, \ldots P_n$ that satisfies the following three conditions −

- P_i does not contain the empty set.

  $[P_i \neq \{\emptyset\}$ for all $0 < i \leq n]$
- The union of the subsets must equal the entire original set.

  $[P_1 \cup P_2 \cup \cdots \cup P_n = S]$$[P1 \cup P2 \cup \cdots \cup Pn = S]$
- The intersection of any two distinct sets is empty.

  $[P_a \cap P_b = \{\emptyset\}$, for $a \neq b$ where $n \geq a,b \geq 0]$

Example

Let $S = \{a,b,c,d,e,f,g,h\}$ One probable partitioning is $\{a\},\{b,c,d\},\{e,f,g,h\}$
Another probable partitioning is $\{a,b\},\{c,d\},\{e,f,g,h\}$

On. 1 Show that (A-B)-(B-C)=A-B

$$(A-B)-(B-C) = (A \cap \overline{B}) \cap \overline{(B \cap \overline{C})}$$ (By alternate representation for set difference)

$$= (A \cap \overline{B}) \cap (\overline{B} \cup C)$$ (By De Morgan’s laws)

$$= [(A \cap \overline{B}) \cap \overline{B}] \cup [(A \cap \overline{B}) \cap C]$$ (By Distributive laws)

$$= [A \cap (\overline{B} \cap \overline{B})] \cup [A \cap (\overline{B} \cap C)]$$ (By Associative laws)

$$= (A \cap \overline{B}) \cup [A \cap (\overline{B} \cap C)]$$ (By Idempotent laws)

$$= A \cap [\overline{B} \cup (\overline{B} \cap C)]$$ (By Distributive laws)

$$= A \cap \overline{B}$$ (By Absorption laws)

$$= A - B$$ (By the alternate representation for set difference)

PART-B

1. Let $A = \{1,2,3,4,5,6\}$ with subsets $B_1 = \{1,3,5\}$, $B_2 = \{1,2,3\}$. Write the minsets and partitions of A.

Given $A = \{1,2,3,4,5,6\}$

$B_1 = \{1,3,5\}$, $B_1' = \{2,4,6\}$

$B_2 = \{1,2,3\}$ $B_2' = \{4,5,6\}$

The minsets generated by B_1 & B_2 are

$M_1 = B_1 \cap B_2 = \{1,3\}$

$M_2 = B_1 \cap B_2' = \{5\}$

$M_3 = B_1' \cap B_2 = \{2\}$

$M_4 = B_1' \cap B_2' = \{4,6\}$

Each min set is non-empty and mutually pairwise disjoint. Also $A = M_1 \cup M_2 \cup M_3 \cup M_4$.

$\therefore$ All the minsets are partitions of A.

$\therefore$ The partitions are $\{\{1,3\}, \{5\}, \{2\}, \{4,6\}\}$

2. Show that the relation R defined on the set of real numbers such that aRb if and only if a−b is an integer is an equivalence relation.

Given,

aRb $\Rightarrow$ a-b is an integer $\forall a,b \in R$.

(i) Reflexive:

$a-a=0$ = is a real number.

$\Rightarrow$ aRa $\forall a \in R$.

$\therefore$ R is reflexive.

(ii) Symmetry:

Let $a,b \in R$.

aRb $\Rightarrow$ a-b is an integer.

$\Rightarrow$ b-a is also an integer.

aRb $\Rightarrow$ bRa

$\therefore$ R is symmetric

(iii) Transitive:

Let $a,b,c \in R$.

aRb $\Rightarrow$ a-b is an integer.

bRc $\Rightarrow$ b-c is an integer.

aRb, bRc $\Rightarrow$ $(a-b)+(b-c)$ is also an integer.

($\because$ Sum of two integers is also an integer)

$\Rightarrow$ a-c is an integer.

aRb, bRc $\Rightarrow$ aRc

$\therefore$ R is transitive.

$\Rightarrow$ R is an equivalence relation.

③ If $R=\{(1,2),(2,4),(3,3)\}$ and $S=\{(1,3),(2,4),(4,2)\}$

find (i) $R\cup S$ (ii) $R\cap S$ (iii) $R-S$ iv) $R\oplus S$.

Given $R = \{ (1,2), (2,4), (3,3) \}$

$S = \{ (1,3), (2,4), (4,2) \}$

(i) $R \cup S = \{ (1,2), (2,4), (3,3), (1,3), (4,2) \}$

(ii) $R \cap S = \{ (2,4) \}$

(iii) $R - S = \{ (1,2), (3,3) \}$

iv) $R \oplus S = (R-S) \cup (S-R)$

$S - R = \{ (1,3), (4,2) \}$

$R \oplus S = \{ (1,2), (3,3), (1,3), (4,2) \}$

④ If R is the relation on the set of positive integers such that $(a,b) \in R$ if and only if ab is a perfect square, Show that R is an equivalence relation.

Reflexive:

$(a, a) \in R$, since a^2 is a perfect square.

$\therefore R$ is reflexive.

Symmetric:

$aRb \Rightarrow ab$ is a perfect square.

$\Rightarrow ba$ is also a perfect square.

$\Rightarrow bRa$.

$\therefore R$ is symmetric.

Transitive:

$aRb \Rightarrow ab$ is a perfect square

$ab = x^2$ (say) — (1)

$bRc \Rightarrow bc$ is a perfect square.

$bc = y^2$ (say). —(2)

Multiply ① & ②,

$(ab)(bc) = x^2y^2$

$a(b^2)c = x^2y^2$

$ac = \frac{x^2y^2}{b^2} = \left(\frac{xy}{b}\right)^2$ = a perfect square.

$\therefore aRc$ ie) R is transitive.

Hence R is an equivalence relation.

⑤. Examine if the relation R represented by $M_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ is an equivalence relation, using the properties of M_R.

Given $M_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

Since all the elements in the main diagonal of M_R are equal to 1 each. R is a reflexive relation.

Since M_R is a symmetric matrix ; R is a symmetric relation.

$$M_{R^2} = M_R \cdot M_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = M_R$$

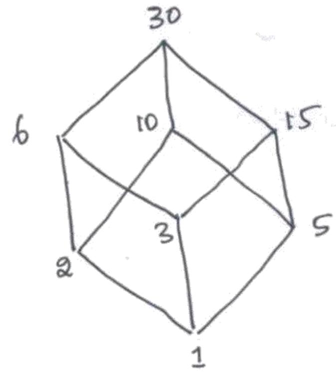
ie) $R^2 \subseteq R$

$\therefore R$ is a transitive relation.

Hence R is an equivalence relation.

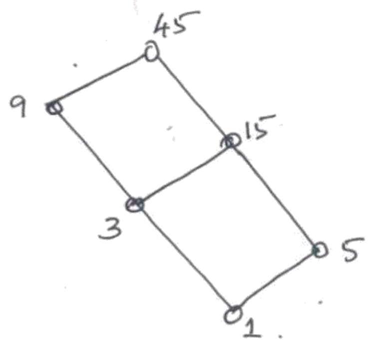
(12) Draw the Hasse diagram for the poset $\{D_{30}, \leq\}$ where $\leq$ is the relation is a divisor of.

$$D_{30} = \{1, 2, 3, 5, 6, 10, 15, 30\}$$



(13) Draw the Hasse diagram for the poset $(D_{45}, \leq)$ where $\leq$ is the relation is a divisor of.

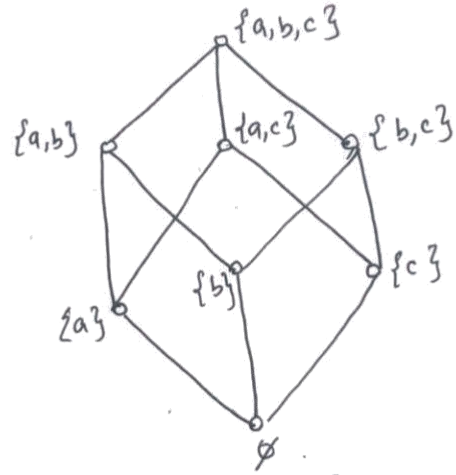
$$D_{45} = \{1, 3, 5, 9, 15, 45\}$$



(14) Draw the Hasse diagram for $\{P(S), \leq\}$ where $S = \{a, b, c\}$

Given $S = \{a, b, c\}$

$$P(S) = \{\{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\}, \varnothing\}.$$



(15) If $R = \{(1,2), (2,4), (3,3)\}$ & $S = \{(1,3), (2,4), (4,2)\}$ find (i) $R \cup S$ (ii) $R \cap S$ (iii) $R - S$ iv) $R \oplus S$.

(i) $R \cup S = \{(1,2), (1,3), (2,4), (3,3), (4,2)\}$

(ii) $R \cap S = \{(2,4)\}$

(iii) $R - S = \{(1,2), (3,3)\}$

iv) $R \oplus S = (R-S) \cup (S-R)$ $\quad S-R = \{(1,3), (4,2)\}$

$= \{(1,2), (1,3), (3,3), (4,2)\}$.

(16) Show that the function $f: Z^+ \to Z^+$ defined by $f(x) = x^2 + 2$ is one to one but not on-to.

Given $f(x) = x^2 + 2$

$$f(x_1) = f(x_2) \Rightarrow x_1^2 + 2 = x_2^2 + 2$$

$$x_1^2 = x_2^2$$

$$x_1^2 - x_2^2 = 0$$

$$(x_1 - x_2)(x_1 + x_2) = 0$$

$x_1 + x_2 \neq 0, \quad x_1 - x_2 = 0 \qquad (\because x_1 + x_2 \neq 0$ as $x_1, x_2 \in Z^+)$

$$\boxed{x_1 = x_2}$$

$\therefore$ f is one to one.

when $y = f(x)$, ie) $y = x^2 + 2$ we have $x^2 = y - 2$

when $y = 1$, x doesnot exist.

Also when $y = 4$, $x = \pm\sqrt{2} \notin Z^+$

$\therefore f(x)$ is not on-to.

(17) Define closure of a relation. Find reflexive and Symmetric closure of $R = \{(1,2), (2,2), (2,3), (3,2), (4,1), (4,4)\}$ define on $A = \{1,2,3,4\}$

The closure of a relation R w.r.to property P is the relation obtained by adding the minimum number of ordered pairs to R to obtain the property P.

$A = \{1,2,3,4\}$

Given $R = \{(1,2), (2,2), (2,3), (3,2), (4,1), (4,4)\}$

Reflexive closure:

$\Delta = \{(1,1), (2,2), (3,3), (4,4)\}$

Reflexive closure $= R \cup \Delta$

$= \{(1,2), (2,2), (2,3), (3,2), (4,1), (4,4), (1,1), (3,3)\}$.

Symmetric closure :

$$R^{-1} = \{ (2,1), (2,2), (3,2), (2,3), (1,4), (4,4) \}$$

Symmetric closure = $R \cup R^{-1}$

$$= \{ (1,2), (2,2), (2,3), (3,2), (4,1), (4,4), (2,1), (1,4) \}.$$

PART- C

2. Let $A = \{1,2,3,4\}$ find the reflexive closure and symmetric closure of the relation $R = \{(1,3), (1,4), (2,2), (3,4), (4,2)\}$

Given $A = \{1,2,3,4\}$

$R = \{(1,3), (1,4), (2,2), (3,4), (4,2)\}$

Reflexive closure: $\Delta = \{(1,1), (2,2), (3,3), (4,4)\}$

Reflexive closure $= R \cup \Delta$

$= \{(1,3), (1,4), (2,2), (3,4), (4,2), (1,1) (3,3), (4,4)\}$

Symmetric closure: $R^{-1} = \{(3,1), (4,1), (2,2), (4,3), (2,4)\}$

Symmetric closure $= R \cup R^{-1}$

$= \{(1,3), (1,4), (2,2), (3,4), (4,2), (3,1), (3,3) (2,4)\}$

3. If R is the relation on $A = \{1,2,3\}$ such that aRb if and only if a+b is even then find M_R, $M_{R^{-1}}$ and M_{R^2}.

$R = \{(1,1), (1,3), (2,2), (3,1), (3,3)\}$

$$M_R = \begin{array}{c c} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \end{array}$$

$$M_{R^{-1}} = (M_R)^T = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$M_{R^2} = M_R \bullet M_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \bullet \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1\vee 0\vee 1 & 0\vee 0\vee 0 & 1\vee 0\vee 1 \\ 0\vee 0\vee 0 & 0\vee 1\vee 0 & 0\vee 0\vee 0 \\ 1\vee 0\vee 1 & 0\vee 0\vee 0 & 1\vee 0\vee 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow M_{R^2} = M_R$$

④ If R is the relation on the set of positive integers such that $(a,b) \in R$ if and only if a^2+b is even. Prove that R is an equivalence relation.

Reflexive:

$a^2+a = a(a+1) =$ even

Since a & $a+1$ are consecutive positive integers

$\therefore (a,a) \in R$

$\Rightarrow R$ is reflexive.

Symmetric:

When a^2+b is even, a & b must be both even or both odd

In either case, b^2+a is even.

$\therefore (a,b) \in R \Rightarrow (b,a) \in R$

$\Rightarrow R$ is symmetric.

Transitive:

when a, b, c are even, a^2+b and b^2+c are even.

Also a^2+c is even.

when a, b, c are odd, a^2+b & b^2+c are even

Also a^2+c is even.

Then $(a,b) \in R$ and $(b,c) \in R \Rightarrow (a,c) \in R$

ie) R is transitive.

$\therefore R$ is an equivalence relation.

(5) Prove that the relation congruence modulo m over the set of positive integers is an equivalence relation.

Reflexive:

$(a-a)$ is a multiple of m.

$\therefore a \equiv a \pmod{m}$.

$\therefore R$ is reflexive.

Symmetric:

when $a-b$ is a multiple of m, $b-a$ is also a multiple of m. ie) $a \equiv b \pmod{m} \Rightarrow b \equiv a \pmod{m}$

$\therefore R$ is Symmetric

Transitive:

when $a-b = k_1 m$ & $b-c = k_2 m$

we get $a-c = (k_1+k_2) m$ (by addition)

$\therefore$ when $a \equiv b \pmod{m}$ and $b \equiv c \pmod{m}$, $a \equiv c \pmod{m}$

$\therefore R$ is transitive.

$\Rightarrow R$ is an equivalence relation.

6) Find the transitive closure of the relation R = { (1,1),(1,2),(2,1),(2,2),(3,3),(3,5),(4,3),(4,4),(4,5),(5,4),(5,5)} defined on the set A = {1,2,3,4,5} using warshall's algorithm.

$$M_R = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

K	In W_{k-1}		W_k has 1's in	W_k
	Position's of 1's in column k	Position's of 1's in row k		
1	1, 2	1, 2	(1,1),(1,2),(2,1),(2,2)	$W_1 = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$
2	1, 2	1, 2	(1,1),(1,2),(2,1),(2,2)	$W_1 = W_2$
3	3 , 4	3 ,4	(3,3),(3,4),(4,3),(4,4)	$W_3 = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$
4	3 , 4 , 5	3 , 4 , 5	(3,3),(3,4),(3,5) (4,3),(4,4),(4,5) (5,3),(5,4),(5,5)	$W_4 = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{pmatrix}$
5	4 , 5	4 , 5	(4,4),(4,5),(5,4),(5,5)	$W_5 = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{pmatrix}$

$\therefore$ The transitive closure of R is { (1.1),(1.2),(2,1),(2,2),(3,3),(3,4),(3,5),(4,3) , (4,4),(4,5),(5,3),(5,4),(5,5)}

7) Find the transitive closure of the relation R = { (1,1),(1,3),(1,5),(2,3),(2,4),(3,3),(3,5),(4,2),(4,4),(5,4)} defined on the set A = {1,2,3,4,5} using warshall's algorithm.

$$M_R = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

K	In W_{k-1}		W_k has 1's in	W_k
	Position's of 1's in column k	Position's of 1's in row k		
1	1	1, 3 , 5	(1,1),(1,3),(1,5)	$W_1 = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$
2	4	3, 4	(4,3),(4,4)	$W_2 = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$
3	1 , 2, 3	3 ,5	(1,3),(1,5),(2,3),(2,5) (3,3), (3,5)	$W_3 = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$
4	2 , 4 , 5	2 , 4	(2,2),(2,4) (4,2),(4,4) (5,2),(5,4)	$W_4 = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{pmatrix}$
5	1 , 3	4	(1,4),(3,4)	$W_5 = \begin{pmatrix} 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{pmatrix}$

$\therefore$ The transitive closure of R is { (1.1),(1.3),(1,4),(1,5),(2,2),(2,3),(2,4), (2,5),(3,3),(3,4),(3,5),(4,2)(4,3) , (4,4),(5,1),(5,4)}

(8) If $f: Z \to N$ defined by $f(x) = \begin{cases} 2x-1, & x>0 \\ -2x, & x \le 0 \end{cases}$ prove that f is one to one and onto and hence find f^{-1}.

Let $x_1, x_2 \in Z$ and $f(x_1) = f(x_2)$

Then either $f(x_1)$ and $f(x_2)$ are both odd or both even [$\because$ an odd number cannot be equal to an even number]

If they are both odd, then

$$2x_1 - 1 = 2x_2 - 1$$
$$2x_1 = 2x_2$$
$$\text{ie)} \quad x_1 = x_2$$

If they are both even, then

$$-2x_1 = -2x_2$$
$$x_1 = x_2$$

Thus whenever $f(x_1) = f(x_2)$, we get $x_1 = x_2$

Hence $f(x)$ is one to one.

Let $y \in N$. If y is odd, its preimage is $\frac{y+1}{2}$.

Since $f\left(\frac{y+1}{2}\right) = 2\left(\frac{y+1}{2}\right) - 1 = y$

If y is even its preimage is $-\frac{y}{2}$. ($\because$ $y = -2x$, $-\frac{y}{2} = x$

Since $f(-\frac{y}{2}) = -2(-\frac{y}{2}) = y$

Thus for any $y \in W$, the preimage is $\frac{y+1}{2} \in Z$ or $-\frac{y}{2} \in Z$

Hence $f(x)$ is on-to.

$\therefore f$ is invertible.

Let $y = f(x) = \begin{cases} 2x-1, & x>0 \\ -2x, & \text{if } x \le 0 \end{cases}$

$$\therefore f^{-1}(y) = x = \begin{cases} \frac{y+1}{2}, & \text{if } y = 1,3,5,\ldots \\ -\frac{y}{2}, & \text{if } y = 0,2,4,6,\ldots \end{cases}$$

$$\text{(or)} \quad f^{-1}(x) = \begin{cases} \frac{x+1}{2}, & \text{if } x = 1,3,5,\ldots \\ -\frac{x}{2}, & \text{if } x = 0,2,4,6\ldots \end{cases}$$

(9) State and prove necessary and sufficient condition for a function is invertible.

The necessary and sufficient condition for the function $f: A \to B$ to be invertible is that f is one to one and on-to.

Proof:

Let $f: A \to B$ be invertible.

Then there exist a unique function $g: B \to A$ such that

$g \circ f = I_A$ and $f \circ g = I_B$ — (1)

Let $a_1, a_2 \in A$ s.t $f(a_1) = f(a_2)$

where $f(a_1), f(a_2) \in B$ [$\because f: A \to B$ is a function.

Since $g: B \to A$ is a function,

$$g(f(a_1)) = g(f(a_2))$$

ie) $(g\circ f)(a_1) = (g\circ f)(a_2)$

$I_A(a_1) = I_A(a_2)$ by ①,

$\boxed{a_1 = a_2}$

Thus whenever $f(a_1) = f(a_2)$, we have $a_1 = a_2$

Hence f is one to one.

Let $b \in B$. Then $g(b) \in A$, since $g: B \to A$ is a function

Now $b = I_B(b) = (f\circ g)(b)$

$= f(g(b))$

Thus, corresponding to every $b \in B$ there is an element $g(b) \in A$ s.t $f(g(b)) = b$.

Hence f is on-to.

Thus necessary part of the condition is proved.

<u>Sufficient part:</u>

Let $f: A \to B$ is bijective.

Since f is on-to, for each $b \in B$ there exist an $a \in A$ such that $f(a) = b$.

Hence we define a function $g: B \to A$ by $g(b) = a$ where $f(a) = b$. — (2)

If possible let $g(b) = a_1$ and $g(b) = a_2$ where $a_1 \neq a_2$

This means that $f(a_1) = f(a_2) = b$

which is not possible since f is one to one.

Thus $g: B \to A$ is a unique function.

Also from ② we get $g\circ f = I_A$ and $f\circ g = I_B$

ie) f is invertible.

Thus the sufficient condition is proved.

(10) Prove that if f and g are invertible then $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

Let $f: A \to B$ and $g: B \to C$ be invertible functions.

$\Rightarrow$ f & g are bijective.

Hence $(g \circ f): A \to C$ is also bijective.

$\therefore$ $g \circ f$ is invertible. $(g \circ f)^{-1}: C \to A$ can be formed.

Thus both $(g \circ f)^{-1}$ and $f^{-1} \circ g^{-1}$ are functions from C to A.

Now for any $a \in A$, let $b = f(a)$ and $c = g(b)$ — (1)

$$(g \circ f)(a) = g(f(a)) = g(b) = c$$

$$(g \circ f)^{-1}(c) = a \quad \text{— (2)}$$

By assumption (1), $a = f^{-1}(b)$, $b = g^{-1}(c)$

$$\Rightarrow (f^{-1} \circ g^{-1})(c) = f^{-1}(g^{-1}(c))$$

$$= f^{-1}(b) = a \quad \text{— (3)}$$

From (2) & (3) it follows that

$(g \circ f)^{-1} = f^{-1} \circ g^{-1}$ since f^{-1}, g^{-1} and $(g \circ f)^{-1}$ are bijective.

(11) If $X = \{1,2,3,4,5\}$ and $f, g: X \to X$ given by $f = \{(1,2), (2,1), (3,4), (4,5), (5,3)\}$, $g = \{(1,3), (2,5), (3,1), (4,2), (5,4)\}$ then show that $(f \circ g)^{-1} = g^{-1} \circ f^{-1}$ also check $f \circ g = g \circ f$.

(i) $(f\circ g)(1) = f(g(1)) = f(3) = 4$

$(f\circ g)(2) = f(g(2)) = f(5) = 3$ and so on.

$\therefore f\circ g = \{(1,4), (2,3), (3,2), (4,1), (5,5)\}$ — (1)

$(g\circ f)(1) = g(f(1)) = g(2) = 5$

$(g\circ f)(2) = g(f(2)) = g(1) = 3$ and so on.

$\therefore g\circ f = \{(1,5), (2,3), (3,2), (4,4), (5,1)\}$ — (2)

From (1) & (2), $f\circ g = g\circ f$.

(ii) $f^{-1} = \{(2,1), (1,2), (4,3), (5,4), (3,5)\}$ — (3)

and $g^{-1} = \{(3,1), (5,2), (1,3), (2,4), (4,5)\}$ — (4)

From (1) $(f\circ g)^{-1} = \{(4,1), (3,2), (2,3), (4,1) (5,1)\}$ — (5)

From (3) & (4),

$g^{-1}\circ f^{-1} = \{(2,3) (1,4) (4,1), (5,5) (3,2)\}$. — (6)

From (1) & (2),

$f^{-1}\circ g^{-1} = \{g^{-1}\circ f^{-1}\}$

Again from (3) & (4),

$f^{-1}\circ g^{-1} = \{(3,2), (5,1), (1,5), (2,3) (4,4)\}$

$\Rightarrow (f\circ g)^{-1} = f^{-1}\circ g^{-1}$