

1 Problem statement

Is there a way to find a complement of a function defined on integers (Both domain and range are integers)?

1.1 Example:

Suppose you have a set of positive integers $= 1, 2, 3, 4, 5, \dots$. Let us say we create a subset of even integers out of it $= 2, 4, 6, 8, \dots$. This set of even numbers can be generated by using the function $f(x) = 2x$, $x = 1, 2, 3, \dots$. The complement of this set is set of odd numbers and they can be generated using $g(x) = 2x - 1$, $x = 1, 2, 3, 4, \dots$. So I define $g(x) = 2x - 1$ to be the complement function of $f(x) = 2x$ over the set of integers.

Similarly for the function $f(x, y) = xy$, the complement function (ignoring non-trivial factors) is over the set of all primes. Is there a generic way to arrive at the complement function given an arbitrary integer function

2 Computational difficulty of the complement

If we reduce halting problem to the question of finding complement then it is undecidable (since it is equivalent to asking if Turing machine computing the complement terminates)

3 Circuit version of above - Construction of a circuit for the complement (for integer valued functions)

Circuit for complement is constructed as below:

1. find the values of f for all $n = 1, 2, 3, 4, 5, \dots$ and make each of them a clause
2. construct DNF of above clauses
3. complement the DNF to get CNF (k -CNF where $k = \log(n)$)

This gives an infinite k -CNF formula which has a circuit of unbounded fanin but exponential size. If the circuit construction algorithm for above terminates is again undecidable.

Thus we have a complement class of TC0 i.e. kind of co-TC0 which synonymizes the complement of $f(x, y) = xy$ and this co-TC0 is the complement language of integer multiplication.