Fuel cell toy problem

- Governing equation:

$$\nabla \cdot (-K \nabla \phi) = 0$$

- Boundary conditions:

$$\begin{cases} -K \nabla \phi = t & \text{on } \Gamma_t \\ \phi = g & \text{on } \Gamma_u \end{cases}$$

- Variational form:

$$0 = \int \nabla \cdot (-K \nabla \phi) v \, dx - \int (\phi - g) \nabla v \cdot n \, d\Gamma_u + \beta \int (\phi - g) \frac{1}{h_E} v \, d\Gamma_u$$

$$0 = \int -K \nabla \phi \cdot n \, v \, d\Gamma_t + \int K \nabla \phi \cdot \nabla v \, dx - \int (\phi - g) \nabla v \cdot n \, d\Gamma_u + \beta \int (\phi - g) \frac{1}{h_E} v \, d\Gamma_u$$

$$-K \nabla \phi \cdot n = t \quad \text{on } \Gamma_t$$

$$0 = \int t \, v \, d\Gamma_t + \int K \nabla \phi \cdot \nabla v \, dx - \int (\phi - g) \nabla v \cdot n \, d\Gamma_u + \beta \int (\phi - g) \frac{1}{h_E} v \, d\Gamma_u$$

- discretization: $t = \sum_I \psi_I(x)\, t_I$

$$\phi = \sum_I \psi_I(x)\, \phi_I$$

$$v = \sum_I \psi_I(x)\, v_I$$

$$g = \sum_I \psi_I(x)\, g_I$$

$$\int \Big(\sum_I \psi_I t_I\Big) \sum_J \psi_J v_J \, d\Gamma_t + \int k \Big(\sum_I \nabla \psi_I \, \phi_I\Big) \sum_J \nabla \psi_J v_J \, dx$$

$$- \int \Big[\sum_I \psi_I (\phi_I - g_I)\Big] \sum_J \nabla \psi_J v_J \cdot n \, d\Gamma_u$$

$$+ \beta \int \Big[\sum_I \psi_I (\phi_I - g_I)\Big] \frac{1}{h_E} \sum_J \psi_J v_J \, d\Gamma_u \quad = 0$$

$\forall\ v_J$ satisfy

$$\Rightarrow \int \Big(\sum_I \psi_I t_I\Big) \psi_J \, d\Gamma_t + \int k\Big(\sum_I \nabla\psi_I \phi_I\Big)\cdot \nabla\psi_J \, dx$$

$$- \int \Big[\sum_I \psi_I (\phi_I - g_I)\Big] \nabla\psi_J \cdot n \, d\Gamma_u$$

$$+ \beta \int \Big[\sum_I \psi_I (\phi_I - g_I)\Big] \frac{1}{h_E} \psi_J \, d\Gamma_u = 0$$

number of nodes equations, n nodes, n equations.
~~th~~ the above is the J^{th} equation.

~~Gauss integral~~

In matrix form

$$\int \sum_J \psi_I \psi_J t_J \, d\Gamma_t + \int \sum_J k \nabla\psi_I \nabla\psi_J \phi_J \, dx$$

$$- \int \sum_J (\nabla\psi_I \cdot n) \psi_J (\phi_J - g_j) \, d\Gamma_u$$

$$+ \int \sum_J \frac{\beta}{h_E} \psi_I \, \psi_J (\phi_J - g_J) \, d\Gamma_u = 0$$

$$\int \psi_I \psi_J \, d\Gamma_t \, t_J + \int k \nabla\psi_I \nabla\psi_J \, dx \, \phi_J$$

$$- \int \nabla\psi_I \cdot n \, \psi_J \, d\Gamma_u \, (\phi_J - g_J)$$

$$+ \int \frac{\beta}{h_E} \psi_I \psi_J \, d\Gamma_u \, (\phi_J - g_J) = 0$$

$$\Rightarrow \quad K = k(\nabla\psi)^T \cdot \nabla\psi - (\nabla\psi_{Bu} \cdot n_{Bu})^T \cdot \psi_{Bu}$$

$$+ \frac{\beta}{h_E} \psi_{Bu}^T \cdot \psi_{Bu}$$

$$f = \frac{\beta}{h_E} \psi_{Bu}^T \cdot \psi_{Bu} \cdot g - (\nabla\psi_{Bu} \cdot n)^T \cdot \psi_{Bu} \cdot g$$

$$- \psi_{Bt}^T \cdot \psi_{Bt} \cdot t$$

$$K \cdot \phi = f$$