no incremental but direct solve

Governing equation:

$$\begin{cases} -\nabla\cdot(K\nabla\phi) = 0 \\ -K\nabla\phi\cdot n = S \quad \text{on } \Gamma_t \\ \phi = g \quad \text{on } \Gamma_u \end{cases}$$

Assume we know ϕ^m from last iteration, solve for ϕ^{m+1}

$$\begin{cases} -\nabla\cdot(K\nabla\phi^{m+1}) = 0 \\ -K\nabla\phi^{m+1}\cdot n = S(\phi^m) \quad \text{on } \Gamma_t \\ \phi^{m+1} = g \quad \text{on } \Gamma_u \end{cases}$$

variational form:

$$-\int \nabla\cdot(K\nabla\phi^{m+1})\, v\, d\Omega - \int_{\Gamma_u}(\phi^{m+1}-g)\nabla v\cdot n\, d\Gamma_u + \beta\int_{\Gamma_u}(\phi^{m+1}-g)\, v\, d\Gamma_u = 0$$

$$\Rightarrow \int K\nabla\phi^{m+1}\cdot\nabla v\, d\Omega - \int K\nabla\phi^{m+1}\cdot n\, v\, d\Gamma_t$$

$$= \int K\nabla\phi^{m+1}\cdot\nabla v\, d\Omega + \int S(\phi^m)\, v\, d\Gamma_t$$

$$\Rightarrow \int K \nabla \phi^{m+1} \cdot \nabla v \, d\Omega + \int s(\phi^{m})\, v \, d\Gamma_t - \int_{\Gamma_u} (\phi^{m+1} - g)\, \nabla v \cdot n \, d\Gamma_u$$

$$+ \beta \int_{\Gamma_u} (\phi^{m+1} - g)\, v \, d\Gamma_u = 0$$

$$\begin{cases} \phi^{m+1} = \sum_{I}^{nodes} \psi_I(x)\, \phi_I^{m+1}, \quad v = \sum_{I}^{nodes} \psi_I(x)\, v_I \\ s = \sum_{I}^{nodes} \psi_I(x)\, s_I^{m} \\ g = \sum_{I}^{nodes} \psi_I(x)\, g_I \end{cases}$$

$$\Rightarrow \int K \sum_{I}^{nodes} \nabla \psi_I(x)\, \phi_I^{m+1} \sum_{J}^{nodes} \nabla \psi_J(x)\, v_J \, d\Omega$$

$$+ \int \sum_{I}^{node} s_I^{m}\, \psi_{I_{\Gamma_t}}(x) \sum_{J}^{nodes} \psi_{J_{\Gamma_t}}(x)\, v_J \, d\Gamma_t$$

$$- \int_{\Gamma_u} \sum_{I}^{nodes} \psi_{I_{\Gamma_u}} (\phi_I^{m+1} - g_I) \sum_{J}^{nodes} \nabla \psi_{J_{\Gamma_u}} v_J \cdot n \, d\Gamma_u$$

$$+ \beta \int_{\Gamma_u} \sum_{I}^{nodes} \psi_{I_{\Gamma_u}} (\phi_I^{m+1} - g_I) \sum_{J}^{nodes} \psi_{J_{\Gamma_u}} v_J \, d\Gamma_u = 0$$

Satisfy for all v_J $\Rightarrow$

$$\int K \sum_I \nabla \psi_I \sum_J \nabla \psi_J \, \phi_I^{m+1} \, d\Omega$$

$$+ \int_{\Gamma_t} \sum_I \psi_{I\Gamma_t} \sum_J \psi_{J\Gamma_t} \, S_I^m \, d\Gamma_t$$

$$- \int_{\Gamma_u} \sum_I \psi_{I\Gamma_u} \sum_J \nabla \psi_{J\Gamma_u} \cdot n \, (\phi_I^{m+1} - g_I) \, d\Gamma_u$$

$$+ \beta \int_{\Gamma_u} \sum_I \psi_{I\Gamma_u} \sum_J \psi_{J\Gamma_u} (\phi_I^{m+1} - g_I) \, d\Gamma_u = 0.$$

discretization: $\int_{cell} f(x) \, d\Omega = \int_{cell} f[x(\xi,\eta)] \, |J| \, d\xi \, d\eta$

$$= \sum_{ij} f(\xi_i \eta_j) \, |J| \, w_i w_j$$

=)

$$\sum_{n_{G}k} K \sum_I \nabla \psi_I(X_G^k) \sum_J \nabla \psi_J(X_G^k) \, \phi_I^{m+1} \, |J|_k w_k$$

$$+ \sum_k^{n_{G\Gamma_t}} \sum_I \psi_{I\Gamma_t}(X_G^k) \sum_J \psi_{J\Gamma_t}(X_G^k) \, S_I^m \, \cancel{d} \, |J|_k \, w_k$$

$$- \cancel{\int_{\Gamma_u}} \sum_k^{n_{G\Gamma_u}} \sum_I \psi_{I\Gamma_u}(X_G^{k\Gamma_u}) \sum_J \nabla \psi_{J\Gamma_u}(X_G^{k\Gamma_u}) \cdot n \, (\phi_I^{m+1} - g_I) \, |J|_k \, w_k$$

$$+ \beta \sum_k^{n_{G\Gamma_u}} \sum_I \psi_{I\Gamma_u}(X_G^{k\Gamma_u}) \sum_J \psi_{J\Gamma_u}(X_G^{k\Gamma_u}) \, (\phi_I^{m+1} - g_I) \, |J|_k \, w_k = 0$$

$$K = \nabla \psi^T \cdot \nabla \psi \; k \, |J| \, w$$

$$- \psi_{\Gamma_u}^T \nabla \psi_{\Gamma_u} \cdot n \, |J| \, w$$

$$+ \beta \psi_{\Gamma_u}^T \psi_{\Gamma_u} \, |J| \, w$$

$$f = \beta \psi_{\Gamma_u}^T \psi_{\Gamma_u} g_I \, |J| \, w$$

$$- \psi_{\Gamma_u}^T \nabla \psi_{\Gamma_u} \cdot n \, g \, |J| \, w$$

$$- \psi_{\Gamma_t}^T \psi_{\Gamma_t} \, s \, |J| \, w$$

$$\boxed{K \cdot \phi^{m+1} = f}$$

how g_I and s_I are determined ?