Q6 - Assignment 4

Rishabh Shah, Shriram SB, Anmol Mishra
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Exercise. Consider a matrix $\boldsymbol{A}$ of size $m \times n, m \leq n$. Define $\boldsymbol{P} = \boldsymbol{A}^T\boldsymbol{A}$ and $\boldsymbol{Q} = \boldsymbol{A}\boldsymbol{A}^T$. (Note: all matrices, vectors and scalars involved in this question are real-valued).

1. Prove that for any vector $\boldsymbol{y}$ with appropriate number of elements, we have $\boldsymbol{y}^t\boldsymbol{P}\boldsymbol{y} \geq 0$. Similarly show that $\boldsymbol{z}^t\boldsymbol{Q}\boldsymbol{z} \geq 0$ for a vector $\boldsymbol{z}$ with appropriate number of elements. Why are the eigenvalues of $\boldsymbol{P}$ and $\boldsymbol{Q}$ non-negative?

   Answer.

$$
\begin{aligned}
y^tPy &= y^tA^tAy \\
&= (Ay)^t(Ay) \\
&= x^tx && (Using,\ x = Ay) \\
&= \|x\|^2 && (Thus,\ y^TPy \geq 0)
\end{aligned}
$$

$$
\begin{aligned}
z^tQz &= z^tAA^tz \\
&= (A^tz)^t(A^ty) \\
&= x^tx && (Using,\ x = A^tz) \\
&= \|x\|^2 && (Thus,\ z^TQz \geq 0)
\end{aligned}
$$

   Assume there exists negative eigenvalue ($\lambda < 0$) for given matrix P. Let $v \neq 0$ be the corresponding eigenvector. Since we had proved $y^tPy \geq 0 \forall y$, if we substitute $y = v$
   $v^tPv = v^t\lambda v = \lambda\|v\|^2$
   Since $\lambda < 0$ and $v \neq 0$, $R.H.S < 0$ and $L.H.S \geq 0$ which is impossible. Thus, every eigenvalue (for non zero eigenvector) of P is non negative.

   Similarly for Q, if we assume there exists negative eigenvalue ($\lambda < 0$). Let $v \neq 0$ be the corresponding eigenvector. Since we had proved $z^tQz \geq 0 \forall z$, if we substitute $z = v$
   $v^tQv = v^t\lambda v = \lambda\|v\|^2$
   Since $\lambda < 0$ and $v \neq 0$, $R.H.S < 0$ and $L.H.S \geq 0$ which is impossible. Thus, every eigenvalue (for non zero eigenvector) of Q is non negative. □

2. If $\boldsymbol{u}$ is an eigenvector of $\boldsymbol{P}$ with eigenvalue λ, show that $\boldsymbol{Au}$ is an eigenvector of $\boldsymbol{Q}$ with eigenvalue λ. If $\boldsymbol{v}$ is an eigenvector of $\boldsymbol{Q}$ with eigenvalue μ, show that $\boldsymbol{A}^T\boldsymbol{v}$ is an eigenvector of $\boldsymbol{P}$ with eigenvalue μ. What will be the number of elements in $\boldsymbol{u}$ and $\boldsymbol{v}$?

Answer.

$$\begin{aligned} Q \times (Au) &= A \times (A^t A) \times u \\ &= A \times (P) \times u \\ &= A \times (\lambda u) \\ &= \lambda(Au) \end{aligned}$$

(Thus, Au is the eigenvector of Q with same eigenvalue λ)
For the above we also need to ensure $u \neq 0 \implies Au \neq 0$

$$\begin{aligned} Pu &= \lambda u \\ A^t \times (Au) &= \lambda u \end{aligned}$$

So if $Au = 0$ this implies $L.H.S = 0$ and consequently $R.H.S = 0$ but since we assumed $u \neq 0$ this case is not possible. Thus by contradiction $u \neq 0 \implies Au \neq 0$

$$\begin{aligned} P \times (A^t v) &= A^t \times (AA^t) \times v \\ &= A^t \times (Q) \times v \\ &= A^t \times (\lambda v) \\ &= \lambda(A^t v) \end{aligned}$$

(Thus, $A^t v$ is the eigenvector of P with same eigenvalue λ)
For the above we also need to ensure $v \neq 0 \implies A^t v \neq 0$

$$\begin{aligned} Qv &= \lambda v \\ A \times (A^t v) &= \lambda v \end{aligned}$$

So if $A^t v = 0$ this implies $L.H.S = 0$ and consequently $R.H.S = 0$ but since we assumed $v \neq 0$ this case is not possible. Thus by contradiction $v \neq 0 \implies A^t v \neq 0$

A is $m \times n$ matrix, thus for $A \times u$ to exist and to result in a column vector, u must be $n \times 1$ vector. A^t is $n \times m$ matrix, thus for $A^t \times v$ to exist and to result in a column vector, v must be $m \times 1$ vector. Thus, the number of elements in $u = n$ and $v = m$

$\square$

3. If $\boldsymbol{v}_i$ is an eigenvector of $\boldsymbol{Q}$ and we define $\boldsymbol{u}_i \triangleq \dfrac{\boldsymbol{A}^T\boldsymbol{v}_i}{\|\boldsymbol{A}^T\boldsymbol{v}_i\|_2}$. Then prove that there will exist some real, non-negative γ_i such that $\boldsymbol{A}\boldsymbol{u}_i = \gamma_i\boldsymbol{v}_i$.

   Answer.

$$
\begin{aligned}
Au_i &= A \times \frac{A^T v_i}{\|A^T v_i\|_2} \\
&= \frac{(AA^T) \times v_i}{\|A^T v_i\|_2} \\
&= \frac{Q v_i}{\|A^T v_i\|_2} \\
&= \frac{\lambda_i}{\|A^T v_i\|_2} v_i && (where\ Qv_i = \lambda_i v_i) \\
&= \gamma_i v_i && (Using,\ \gamma_i = \frac{\lambda_i}{\|A^T v_i\|_2})
\end{aligned}
$$

   Since $\lambda_i \geq 0$ and $\|A^T v_i\|_2 > 0 => \gamma_i \geq 0$. Hence Proved.

   □

4. It can be shown that $\boldsymbol{u}_i^T\boldsymbol{u}_j = 0$ for $i \neq j$ and likewise $\boldsymbol{v}_i^T\boldsymbol{v}_j = 0$ for $i \neq j$ for correspondingly distinct eigenvalues. Now, define $\boldsymbol{U} = [\boldsymbol{v}_1|\boldsymbol{v}_2|\boldsymbol{v}_3|...|\boldsymbol{v}_m]$ and $\boldsymbol{V} = [\boldsymbol{u}_1|\boldsymbol{u}_2|\boldsymbol{u}_3|...|\boldsymbol{u}_m]$. Now show that $\boldsymbol{A} = \boldsymbol{U}\boldsymbol{\Gamma}\boldsymbol{V}^T$ where $\boldsymbol{\Gamma}$ is a diagonal matrix containing the non-negative values $\gamma_1, \gamma_2, ..., \gamma_m$. With this, you have just established the existence of the singular value decomposition of any matrix $\boldsymbol{A}$. This is a key result in linear algebra and it is widely used in image processing, computer vision, computer graphics, statistics, machine learning, numerical analysis, natural language processing and data mining.

   Answer.

$$
\begin{aligned}
AV &= A[u_1 \mid u_2 \mid ... \mid u_m] \\
&= [Au_1 \mid Au_2 \mid ... \mid Au_m] \\
&= [\gamma_1 v_1 \mid \gamma_2 v_2 \mid ... \mid \gamma_m v_m] && (From\ \ previous\ \ proof) \\
&= [v_1 \mid v_2 \mid ... \mid v_m] \times D && (where\ \ D\ = \begin{bmatrix} \gamma_1 & 0 & \dots & 0 \\ 0 & \gamma_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \gamma_m \end{bmatrix}) \\
&= U \times D && ...\ Equation\ \ (1)
\end{aligned}
$$

$$
\begin{aligned}
V^t \times V &= [u_1 \mid u_2 \mid \dots \mid u_m] \times \begin{bmatrix} u_1^t \\ u_2^t \\ \vdots \\ u_m^t \end{bmatrix} \\
&= \begin{bmatrix} u_1^t u_1 & u_1^t u_2 & \dots & u_1^t u_m \\ u_2^t u_1 & u_2^t u_2 & \dots & u_2^t u_m \\ \vdots & \vdots & \ddots & \vdots \\ u_m^t u_1 & u_m^t u_2 & \dots & u_m^t u_m \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix} && (Since \|u_i\| = 1 \ (from\ previous\ part)\ and \\
& && u_i^t \times u_j = 0 \ \forall \ i \neq j) \\
&= I
\end{aligned}
$$

Also since V is a orthonormal matrix

$$
\begin{aligned}
V^t \times V &= V \times V^t \\
V \times V^t &= I && \dots \ Equation \ (2)
\end{aligned}
$$

$$
\begin{aligned}
AV &= UD && From\ Equation\ (1) \\
A \times (V \times V^t) &= U \times D \times V^t \\
AI &= UDV^t && From\ Equation\ (2)
\end{aligned}
$$

$A = UDV^t$ Singular Value Decomposition for every matrix A exists □