# Critical points : local maxima and minima

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A turning point as in 1 above is called a local maximum and a turning point as in 2 above is called a local minimum.

# Example

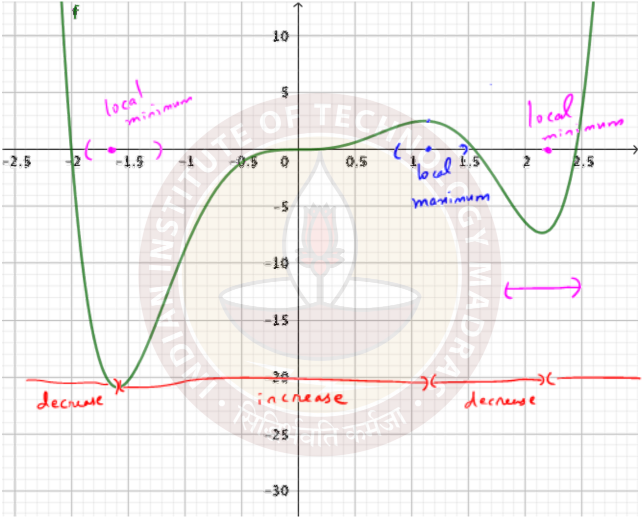


Figure: $\Gamma\left((x^2-4x+3.8)(x+2)x^3\right)$

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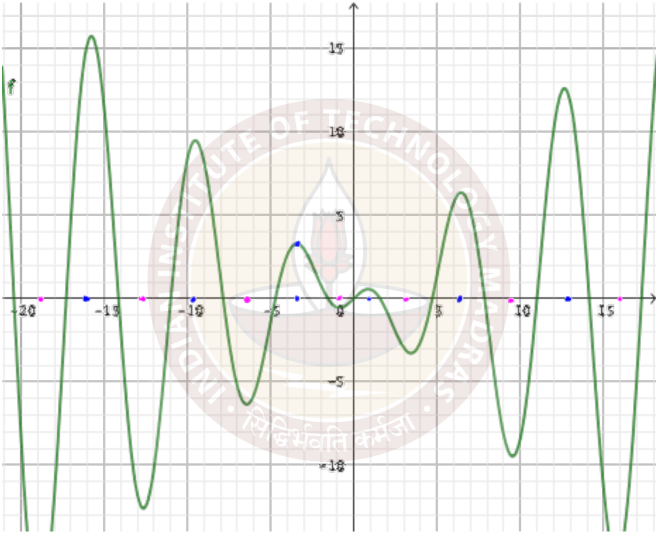


Figure: $\Gamma\left(x\cos(x)\right)$

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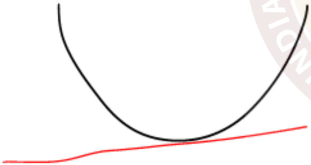
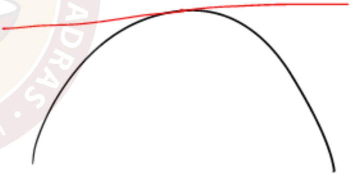
This is a general phenomenon that is easy to visualize.





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Thus, proving that the tangent (if it exists) is horizontal at a turning point is equivalent to showing that $f'(a) = 0$ .

$a$ is a turning point & $f'(a)$ exists.

Suppose $a$ is a local maximum. For some $(a-\epsilon, a+\epsilon)$

if $0 < h < \epsilon$ then $f(a-h) < f(a)$ & $f(a+h) < f(a)$.

$$f'(a) = \lim_{h \to 0^+} \frac{f(a+h) - f(a)}{h} \leq 0 \quad \Rightarrow f'(a) = 0.$$

$$= \lim_{h \to 0^-} \frac{f(a+h) - f(a)}{h} \geqslant 0$$

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A saddle point is a critical point which is not a local maximum or local minimum.

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3. If $a$ is a critical point and $f'(a) = 0$, then the test is inconclusive.

# Examples

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$f(x) = x^3 - 12x$

$f'(x) = 3x^2 - 12$. Setting it to 0, $3x^2 - 12 = 0 \Rightarrow x^2 = 4 \Rightarrow x = \pm 2$.

we obtain Critical points: $\pm 2$.

$f''(x) = 6x$.

$f''(2) = 12 > 0$ $\therefore$ 2 is a local minimum

$f''(-2) = -12 < 0$ & $-2$ is a local maximum

$f(x) = cos(x)$

$f'(x) = -\sin(x)$. Setting it to 0, critical points are $\{k\pi \mid k \in \mathbb{Z}\}$.

$f''(x) = -\cos(x)$.

$f''(k\pi) = -\cos(k\pi) = \begin{cases} -1 & \text{if } k \text{ is an even integer} \\ 1 & \text{if } k \text{ is an odd integer} \end{cases}$

$k\pi$; $k$ is even are local maxima

$k\pi$; $k$ is odd are local minima.

$f(x) = x^3 + x^2 - x + 5$

$f'(x) = 3x^2 + 2x - 1$. Setting it to 0, $3x^2 + 2x - 1 = 0 \Rightarrow$ Roots are $\dfrac{-2 \pm \sqrt{4+12}}{2 \times 3} = \dfrac{-2 \pm \sqrt{16}}{6} = \dfrac{-2 \pm 4}{6}$

we obtain Roots: $-1, \frac{1}{3}$.

$f''(x) = 6x + 2$.

$f''(-1) = -6 + 2 = -4$ : local maximum

$f''(1/3) = 2 + 2 = 4$ : local minimum

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Example : $f(x) = x^2$ on the interval $[-1, 1]$.

$f'(x) = 2x$. Set it to 0.

$f''(x) = 2$. $f''(0) = 2$.

$\therefore$ It is a local minimum.

$-1$ & $1$ are also local maxima.

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We can in fact do this on any function which is defined piecewise continuously with finitely many pieces on a closed and bounded interval.

# Example

$$f(x) = \begin{cases} x^3 + x^2 - x + 5 & \text{if } 0 \le x \le 100 \\ x^3 + 2x^2 + x - 5 & \text{if } -100 \le x < 0 \end{cases}$$

$$f'(x) = \begin{cases} 3x^2 + 2x - 1 & \text{if } 0 < x < 100 \\ 3x^2 + 4x + 1 & \text{if } -100 < x < 0 \end{cases}$$

Critical pts: $\frac{-2 \pm \sqrt{4+12}}{6}$ , $\frac{-2 \pm \sqrt{16-12}}{6}$

Bdry pts: $-100, 0, 100$.

$\frac{-2 \pm 4}{6} = \begin{cases} 1/3 \\ -1 \end{cases}$ , $\frac{-2 \pm 2}{6} = \begin{cases} 0 \\ -2/3 \end{cases}$

local min. $f''(1/3) = 4$ local max. $f''(-1) = -4$

$f''(x) = \begin{cases} 6x + 2 \\ 6x + 4 \end{cases}$ $\boxed{f''(0) = 4}$ sep. $\boxed{f''(-2/3) = 0}$

$f(-1) = -1 + 2 + (-1) - 5 = -5$

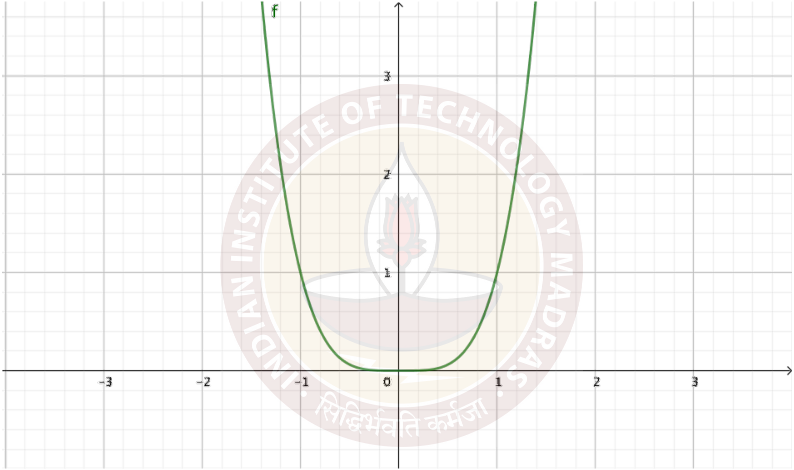
$f(0) = 5$ $f(-2/3) = \frac{-8}{27} + \frac{8}{9} + \frac{(-2)}{3} - 5$

$f(1/3) =$

Global max. $f(100) = 100^3 + 100^2 - 100 + 5$ Inconcl.

Global min. $f(-100) = (-100)^3 + 2 \times 100^2 - 100 - 5$

# Warning example : $f(x) = x^4$




$f'(x) = 4x^3 \Rightarrow$ Critical point is 0.

$f''(x) = 12x^2 \Rightarrow f''(0) = 0$. Inconclusive.

Thank you