Mathematics for Data Science - 1
Graded Assignment (week-1)
Sep 2022 term

1 Multiple Choice Questions (MSQ)

1. Which of the following are irrational numbers? (1 mark)

   Set of correct options: **(Answer: (a),(b))(1 mark)**

   ○ $\mathbf{3^{1/3}}$

   ○ $(\sqrt{\mathbf{8}}+\sqrt{\mathbf{2}})(\sqrt{\mathbf{12}}-\sqrt{\mathbf{3}})$

   ○ $\dfrac{\sqrt{18}-3}{\sqrt{2}-1}$

   ○ $\dfrac{\sqrt{8}+\sqrt{2}}{\sqrt{8}-\sqrt{2}}$

   Solution:

   (a) $3^{1/3}$, this cannot be written in the form of p/q, where $p, q \in \mathbb{Z},\ q \neq 0$. So $3^{1/3}$ is an irrational number.

   (b) $(\sqrt{8}+\sqrt{2})(\sqrt{12}-\sqrt{3}) = (2\sqrt{2}+\sqrt{2})(2\sqrt{3}-\sqrt{3}) = 3\sqrt{2}\sqrt{3} = 3\sqrt{6}$
   We know that $\sqrt{6}$ is an irrational number. Hence, $(\sqrt{8}+\sqrt{2})(\sqrt{12}-\sqrt{3})$ is also an irrational number.

   (c) $\dfrac{\sqrt{18}-3}{\sqrt{2}-1} = \dfrac{3\sqrt{2}-3}{\sqrt{2}-1} = \dfrac{3(\sqrt{2}-1)}{\sqrt{2}-1} = 3$
   $\implies \dfrac{\sqrt{18}-3}{\sqrt{2}-1}$ is a rational number.

   (d) $\dfrac{\sqrt{8}+\sqrt{2}}{\sqrt{8}-\sqrt{2}} = \dfrac{2\sqrt{2}+\sqrt{2}}{2\sqrt{2}-\sqrt{2}} = \dfrac{3\sqrt{2}}{\sqrt{2}} = 3$
   $\implies \dfrac{\sqrt{8}+\sqrt{2}}{\sqrt{8}-\sqrt{2}}$ is a rational number.

2. Consider the relation $R = \{(x, y) \mid x - y = 0\} \subset \mathbb{R} \times \mathbb{R}$ on the set $\mathbb{R}$. Which of the following is/are true? [Ans: (a) (b), (d), (e)] (1 Mark)

- ◯ R is a transitive relation.
- ◯ R is a function.
- ◯ R is not an equivalence relation.
- ◯ R is a reflexive relation.
- ◯ R is a symmetric relation.

Solution:

Given $R = \{(x, y) \mid x - y = 0\} \subset \mathbb{R} \times \mathbb{R}$,
Reflexivity: Let $a \in \mathbb{R} \implies a - a = 0 \implies (a, a) \in R$. So R is a reflexive relation.

Symmetricity: let $(a, b) \in R \implies a - b = 0 \implies b - a = 0 \implies (b, a) \in R$ so R is symmetric relation.

Transitivity: let $(a, b), (b, c) \in R \implies a - b = 0$ and $b - c = 0 \implies a - c = 0 \implies (a, c) \in R$. So R is a transitive relation.

Hence R is an equivalence relation.

Since we have $R = \{(x, y) \mid x - y = 0\} \subset R \times R$ i.e, $y, x \in \mathbb{R}$ such that $y = x$. For a given input x, we have exactly one output $y = x$. Therefore, R is a function.

3. Which of the following relations is/are one-one function? [Ans: (c)] (1 Mark)

- ◯ $R_1 = \{(x, y) \mid x, y \in \mathbb{R}, x + y > 2\}$
- ◯ $R_2 = \{(x, y) \mid x, y \in \mathbb{R}, x > y\}$
- ◯ $R_3 = \{(x, y) \mid x, y \in \mathbb{R}, x + y = 12\}$
- ◯ $R_4 = \{(x, y) \mid x, y \in \mathbb{R}, y = x^2\}$

Solution:

Option 1: Given $R_1 = \{(x, y) \mid x, y \in \mathbb{R}, x + y > 2\}$

Observe that (1, 2), (1, 3) $\in R_1$, that is, for one input $x = 1$, we are getting two different outputs $y = 2$ and $y = 3$. So R_1 is not a function

Option 2: Given $R_2 = \{(x, y) \mid x, y \in \mathbb{R}, x > y\}$

Observe that (5, 2), (5, 3) $\in R_1$, that is, for one input $x = 5$, we are getting two different outputs $y = 2$ and $y = 3$. So R_2 is not a function

Option 3: Given $R_3 = \{(x, y) \mid x, y \in \mathbb{R}, x + y = 12\}$

We can write $x + y = 12$ as $y(x) = 12 - x$.
For a given function input x, we have only one output $y(x) = 12 - x$. So R_3 is a function.

Another method to check a relation is function:
To check given relation is function, let x_1 and $x_2 \in \mathbb{R}$ such that

$$x_1 = x_2$$

$$\implies 12 - x_1 = 12 - x_2$$

$$\implies y(x_1) = y(x_2).$$

Therefore $y(x) = 12 - x$ is well defined. So R_3 is a function.

To check that the given function is one-one, let x_1 and $x_2 \in \mathbb{R}$ such that
and

$$y(x_1) = y(x_2)$$

$$\implies 12 - x_1 = 12 - x_2$$

$$\implies x_1 = x_1.$$

Therefore the function $y(x) = 12 - x$ is one-one.

Option 4: Given $R_4 = \{(x, y) \mid x, y \in \mathbb{R}, y = x^2\}$

We can write $y = x^2$ as $y(x) = x^2$.
For a given function input x, we have only one output $y(x) = x^2$. So R_4 is a function.

Another method to check a relation is function:

To check given relation is function, let x_1 and $x_2 \in \mathbb{R}$ such that

$$x_1 = x_2$$
$$\implies x_1^2 = x_2^2$$
$$\implies y(x_1) = y(x_2).$$

Therefore $y = x^2$ is well defined. So R_4 is a function.

Observe that for two different inputs $x = -2$ and $x = 2$, we are getting the same out $y = 4$.

So function $y = x^2$ is not one-one.

4. Which of the following is/are true? [Ans:(a), (c)] (1 Mark)

   ◯ Function $f : \mathbb{N} \to \mathbb{R}$ such that $f(x) = x^2$ is not onto.

   ◯ Relation $R = \{(1,1),(1,2),(3,1)\}$ on a set $A = \{1,2,3\}$ is a function.

   ◯ Function $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) = |x|$ is not one-one.

   ◯ Function $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) = (x-1)^2$ is one- one and onto.

Solution:

Option 1: Given $f(x) = x^2$, let $6 \in \mathbb{R}$ (codomain) but there is no natural number such that the square of that number is 6. So given function is not onto.

Option 2: Observe that (1, 1), (1, 2) element of R, that is, for the input 1, we have two different outputs 1 and 2. So this is not a function.

Option 3: Given $f(x) = |x|$, since $f(-1) = 1 = f(1)$ but $-1 \neq 1$, the function is not one- one.

Option 4: Given $f(x) = (x-1)^2$, since $f(0) = 1 = f(2)$ but $1 \neq 2$, the function is not one-one

2 Numerical Answer Type (NAT)

5. Suppose $f : D \longrightarrow \mathbb{R}$ is a function defined by $f(x) = \dfrac{\sqrt{x^2-9}}{x+3}$, where $D \subset \mathbb{Z}$. Let A be the set of integers which are not in the domain of f, then find the cardinality of the set A. **(Answer:6)(2 marks)**

   Solution:

   The function $f(x) = \dfrac{\sqrt{x^2-9}}{x+3}$ is well defined if $x^2 - 9 \geq 0$ and $(x+3) \neq 0$.

   Therefore, the domain of $f(x)$ is $\mathbb{Z} \setminus \{-3, -2, -1, 0, 1, 2\}$.

   By definition, A is the set of integers which are not in the domain of f. Therefore $A = \{-3, -2, -1, 0, 1, 2\}$ and the cardinality of A is 6.

6. Consider a set $S = \{a \mid a \in \mathbb{N},\ a \leq 18\}$. Let R_1 and R_2 be relations from S to S defined as $R_1 = \{(x, y) \mid x, y \in S,\ y = 3x\}$ and $R_2 = \{(x, y) \mid x, y \in S,\ y = x^2\}$. Find the cardinality of the set $R_1 \setminus (R_1 \cap R_2)$. **(Answer: 5)(3 marks)**

Solution:

$S = \{0, 1, 2, 3, ..., 18\}$

$R_1 = \{(0, 0), (1, 3), (2, 6), (3, 9), (4, 12), (5, 15), (6, 18)\}$

$R_2 = \{(0, 0), (1, 1), (2, 4), (3, 9), (4, 16)\}$

$\therefore R_1 \cap R_2 = \{(0, 0), (3, 9)\}$.

Now, $R_1 \setminus (R_1 \cap R_2) = \{(1, 3), (2, 6), (4, 12), (5, 15), (6, 18)\}$.

Hence the cardinality of $R_1 \setminus (R_1 \cap R_2)$ is 5.

7. In a Zoo, there are 6 Bengal white tigers and 7 Bengal royal tigers. Out of these tigers, 5 are males and 10 are either Bengal royal tigers or males. Find the number of female Bengal white tigers in the Zoo. **(Answer: 3) (3 marks)**

Solution:

Let BW be the set of Bengal White tigers and BR be the set of Bengal Royal tigers, and M be the set of male tigers.

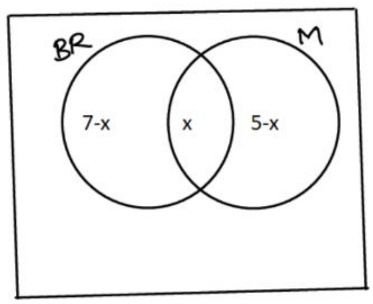


Figure Q7. Tigers in a Zoo

$$n(BR) = 7$$
$$n(M) = 5$$
$$n(BR \cup M) = 10$$

We know that,

$$n(BR \cup M) = n(BR) + n(M) - n(BR \cap M)$$

$$10 = 7 + 5 - n(BR \cap M)$$
$$\implies n(BR \cap M) = 2$$

$\implies$ The number of male Bengal Royal tigers is 2

$\implies$ The number of male Bengal White tigers is $5 - 2 = 3$

$\therefore$ Out of 6 Bengal White tigers, 3 are male.

$\implies$ The number of female Bengal White tigers is $6 - 3 = 3$.

8. Consider the following sets, (1 Mark)
$A = \{x \mid x \in \mathbb{N}\}$
$B = \{x \mid -5 < x < 105, x \in \mathbb{R}\}$
$C = \{x \mid x \text{ is a rational number}, 10 < x \leq 80\}$

Find the cardinality of the set $(A - C) \cap B$. [Ans: 35]

Solution:

Observe that A is set of natural numbers and C is the set of rational numbers which are greater than 10 and less than or equal to 80 and $B = (-5, 105)$

we have,

$A - C = \{0, 1, \ldots 10, 81, 81, \ldots\}$

So, $(A - C) \cap B = \{0, 1, \ldots 10, 81, 81, \ldots\} \cap (-5, 105) = \{0, 1, \ldots 10, 81, 81, \ldots, 104\}$

Hence the cardinality of $(A - C) \cap B$ is 35

3 Comprehension type question

USE THE FOLLOWING INFORMATION FOR QUESTIONS 9 AND 10:

A survey was conducted on pollution of 525 ponds across some cities. It was found that 230 ponds are polluted by fertilisers (F), 245 ponds are polluted by pesticides (P) and 257 ponds are polluted by pharmaceutical products (Ph). 100 ponds are polluted by fertilisers and pesticides, 82 ponds are polluted by fertilisers and pharmaceutical products, 77 ponds are polluted by pesticides and pharmaceutical products.

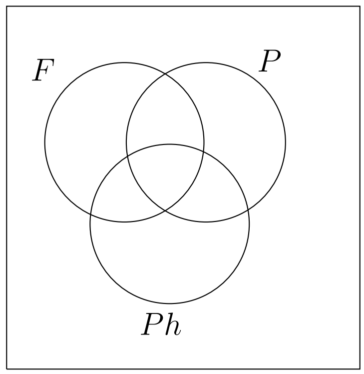


Figure M1T6GA-1

9. Find the number of ponds polluted by all types of water contaminants. [Ans: 52] (1 Mark)

Solution:

Number of ponds polluted by fertilisers is $n(F) = 230$.
Number of ponds polluted by pesticides is $n(P) = 245$
Number of ponds polluted by pharmaceutical products is $n(Ph) = 257$.
Number of ponds polluted by fertilisers and pesticides is $n(F \cap P) = 100$.
Number of ponds polluted by fertilisers and pharmaceutical products is $n(F \cap Ph) = 82$.
Number of ponds polluted by pesticides and pharmaceutical products is $n(P \cap Ph) = 77$.
We have,
$n(F \cup P \cup Ph) = n(F) + n(P) + n(Ph) - n(F \cup P) - n(F \cup Ph) - n(P \cup Ph) + n(F \cap P \cap Ph)$

$$\implies 525 = 230 + 245 + 257 - 100 - 82 - 77 + n(F \cap P \cap Ph)$$

$$\implies n(F \cap P \cap Ph) = 52$$

So number of ponds polluted by all types of water contaminants is $n(F \cap P \cap Ph) = 52$

10. Define a relation on the set of 525 ponds such that two ponds are related if both are polluted by fertilisers and pharmaceutical products. Which of the following is/are true? [Ans: (b), (c)] (1 Mark)

- ◯ Relation is reflexive.
- ◯ Relation is transitive.
- ◯ Relation is symmetric.
- ◯ This is an equivalence relation.

Solution:

Let S be the collection of all 525 ponds. The relation R is defined as

$R = \{(A, B) \mid A, B \in S,\ A \text{ and } B \text{ are polluted by fertilisers and pharmaceutical products}\}$

If $A \in S$ is only polluted by pesticides then (A, A) can not be element of R because A is not polluted by fertiliser and pharmaceutical products. Hence R is not a reflexive relation.

Let $(A, B) \in R$ i.e., A and B are polluted by fertilisers and pharmaceutical products. Then B and A are also polluted by fertilisers and pharmaceutical products. Hence, $(B, A) \in R$ and R is a symmetric relation.

Let $(A, B) \in R$ and $(B, C) \in R$ i.e., A and B are polluted by fertilisers and pharmaceutical products, and B and C are polluted by fertilisers and pharmaceutical products. In particular, A and C are also polluted by fertilisers and pharmaceutical products. Hence $(A, C) \in R$ and R is a transitive relation.

Q2 :

Solution:

Material	Dielectric constant
Air	1
Vecuum	2
Paper	3
Glass	8
Nerve membrance	7
Silicon	13

Observe that elements in domain (Material) has unique output. And each element in codomain (Dieletric constant) has unique preimage.

Hence this function is bijective.

Q3:

Solution: $A = \{x \in \mathbb{N} \mid x \bmod 2 = 0 \text{ and } 1 \le x \le 10\}$

$\Rightarrow A = \{2, 4, 6, 8, 10\}$

$B = \{x \in \mathbb{N} \mid x \bmod 5 = 0 \text{ and } 6 \le x \le 25\}$

$\Rightarrow B = \{10, 15, 20, 25\}$

$C = \{x \in \mathbb{N} \mid x \bmod 7 = 0 \text{ and } 7 \le x \le 29\}$

$\Rightarrow C = \{7, 14, 21, 28\}$

Now $A \setminus (B \cup C) = \{2, 4, 6, 8\}$

$B \setminus (A \cup C) = \{15, 20, 25\}$

$C \setminus (B \cup A) = \{7, 14, 21, 28\}$

$A \setminus (B \cup C) \cup B \setminus (A \cup C) \cup C \setminus (B \cup A)$

$= \{2, 4, 6, 8, 15, 20, 25, 7, 14, 21, 28\}$

so cardinality $= 11$.

Q 4

Soln!

Total number of people $= 180$

Number of people watched Dabang $(N(D)) = 95$

" " " " " Avatar $(N(A)) = 100$

" " " " " RRR $(N(R)) = 100$

" " " " " Dabang and Avatar $(N(D \cup A)) = 50$

" " " " " Avatar and RRR $(N(A \cup R)) = 40$

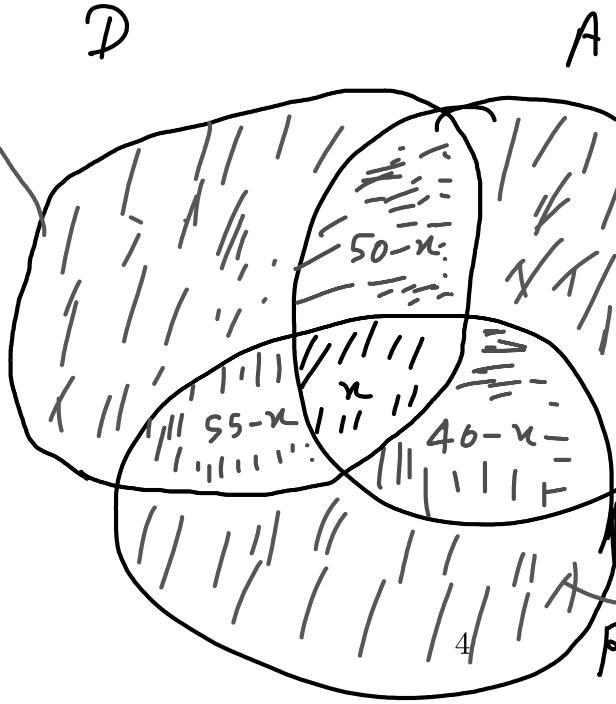
" " " " " Dabang and RRR $(N(D \cup R)) = 55$

Let x number of people watched all 3 movies.

Only D

$95 - (x + 55 - x + 50 - x)$

$= x - 10$



only A.

$100 - (50 - x + x + 40 - x)$

$= 10 + x$

only R

$100 - (55 - x + x + 40 - x)$

$= 5 + x$

So $(x-10)+(10+x)+(5+x)+(55-x)+(50-x)+x$

$+(40-x) = 180$ (Total people)

$\Rightarrow \quad x+150 = 180$

$\Rightarrow \quad x = 30$

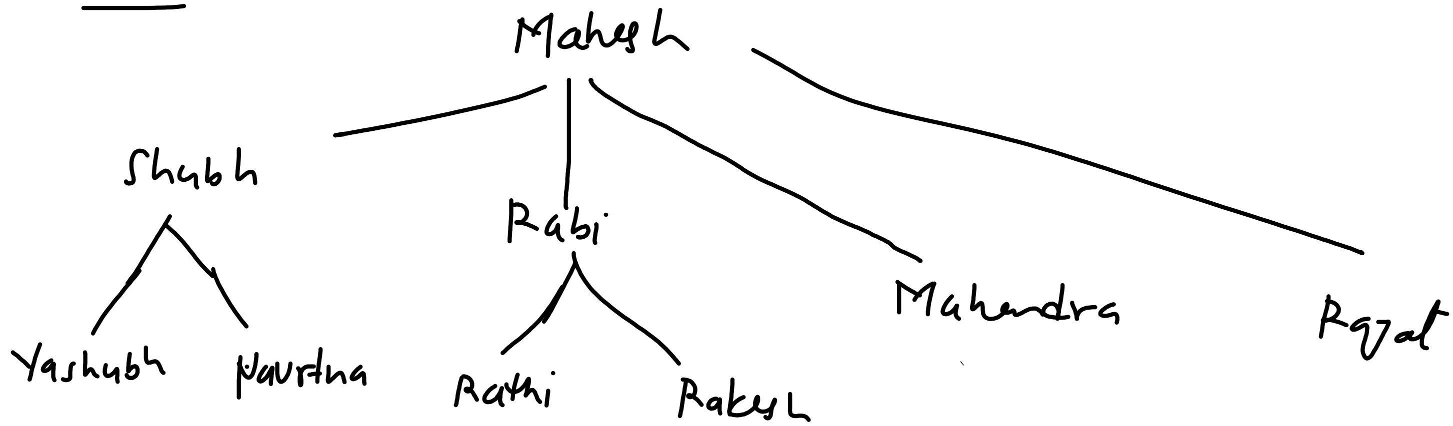
So Number of people only watched RRR and Avatar $= 40-x$

$= 40-30$

$= 10$

Question 8

Solu :



R = { (A, B) | A and B are cousins }

so R = { (Yashubh, Rathi), (Yashubh, Rakesh)
(Navrtha, Rathi), (Navrtha, Rakesh),
(Rathi, Yashubh), (Rakesh, Yashubh),
(Rathi, Navrtha), (Rakesh, Navrtha) }

and S = { (A, B) | A is son of B }

S = { (Shubh, Mahesh), (Rabi Mahesh), (Mahendra, Mahesh)
(Rajat Mahesh), (Yashubh, Shubh),
(Navrtha, Shubh), (Rathi, Rabi), (Rakesh, Rabi) }

So from the elements of R and S, Every option can clearly decided that option is correct or not..

Q9: From solution of Q8, cardinality of R(m) is 8 and of S(n) is also

i.e $m = 8$, $n = 8$

So $m + n = 16$.

Q10: $f = \{ (A, B) \mid A \text{ is son of } B \}$

$\subset P \times Q$ where $P, Q \subset M$.

Option 1: Observe, $f : P \to Q$, in set P, Mahesh $\in P$ does not have image as in Q. So f is not a function.

Option 2: Observe, $f : P \to Q$

$f = \{ ($Yashubh, Shubh$)$, $($Navrtna, Shubh$)$

(Rathi, Rabi), (Rakesh, Rabi) } is a function but Yashubh and Navrtha has same image so f is not one-one.

Option-3: Observe $f: P \to Q$

$f = \{$ (Yashubh, Shubh) (Navrtha, Shubh)
(Rathi, Rabi), (Rakesh, Rabi) $\}$

So f is a function and every element of codomain has preimage so f is onto.

Option 4: Observe $f: P \to Q$

$f = \{$ (Yashubh, Shubh), (Rathi, Rabi) $\}$

So f is a function and for every element in domain has unique image and every element of codomain has preimage.

So f is bijective.

Q 2 :

Solution:

We have $A = \{a \in \mathbb{N} \mid 3 \leq a \leq 8\} = \{3, 4, 5, 6, 7, 8\}$

$B = \{b \in \mathbb{N} \mid 5 \leq b \leq 10\} = \{5, 6, 7, 8, 9, 10\}$

Now, lets find the $A \cup B = \{3, 4, 5, 6, 7, 8\} \cup \{5, 6, 7, 8, 9, 10\}$

$\Rightarrow A \cup B = \{3, 4, 6, 7, 8, 9, 10\}$

Similarly, $A \cap B = \{5, 6, 7, 8\}$

Similarly $A \setminus B = \{3, 4\}$

and $B \setminus A = \{9, 10\}$

Q3: Given $f_1 : D_1 \to C_1$ & $f_2 : D_2 \to C_2$ and $D_1 \subset \mathbb{R}$,

Solution: $D_2 \subset \mathbb{R}$, $C_1 \subseteq \mathbb{R}$, $C_2 \subseteq \mathbb{R}$ (codomains are subset of $\mathbb{R}$)

Now If $x \in D_1$ but $x \notin D_2$, then $f_1(x_1) \in C_1$ but $f_2(x)$ is not defined because $x \notin D_2$. Therefore $f_1(x) + f_2(x)$ is not defined.

Similarly, if $x \notin D_1$ but $x \in D_2$, then the same case will arise as above.

Now, if $x \in D_1$ & $x \in D_2$, then $f_1(x) + f_2(x)$ is well defined. Hence, domain of $f_1 + f_2$ is $D_1 \cap D_2$.

Q 4

Sol^n!

Given $f: \mathbb{N}\setminus\{0\} \to \mathbb{N}\setminus\{0\} \Rightarrow f(x) = x+1$

& $g: \mathbb{N}\setminus\{0\} \to \mathbb{N}\setminus\{0\} \Rightarrow g(x) = \begin{cases} f(x)-1 & \text{if } x>1 \\ 1 & x=1 \end{cases}$

i.e $g(x) = \begin{cases} x+\cancel{1}-\cancel{1} & \text{if } x>1 \\ 1 & \text{if } x=1 \end{cases}$

i.e $g(x) = \begin{cases} x & \text{if } x>1 \\ 1 & \text{if } x=1 \end{cases}$

i.e $g(x) = x,\ x \in \mathbb{N}\setminus\{0\}$

Now, It is very much clear that $g(x)$ is nothing the identity function. Hence $g(x)$ is a bijective function.

We have $f(x) = x+1$

let $x_1, x_2 \in \mathbb{N}\setminus\{0\}$ such that $f(x_1) = f(x_2)$

$\Rightarrow x_1+1 = x_2+1$

$\Rightarrow x_1 = x_2$

So $f(x)$ is one-one.

Let's check $f(x)$ is onto, as we have $1 \in \mathbb{N}\setminus\{0\}$ (co-domain)

Assume, $\exists\ \alpha \in \mathbb{N}\setminus\{0\}$ (domain) such that $f(\alpha) = 1$

$\Rightarrow \alpha+1 = 1$

$\Rightarrow \alpha = 0$

But $0 \notin \mathbb{N}\setminus\{0\}$

hence our assumption is wrong. that is for $1 \in \mathbb{N}\setminus\{0\}$ (codomain of f) does not have pre-image in $\mathbb{N}\setminus\{0\}$ (domain of f) –

Hence, f is one-one but not onto function.

Question 8 Given, $R_1 = \{(a,b) \mid b = a+1, \text{ and } a, b \in \mathbb{N}\}$

So: $R_2 = \{(a,b) \mid b \geq a, \text{ and } a, b \in \mathbb{N}\}$

are relations on $\mathbb{N}$.

Let's check for R_1

Observe that $(1,1) \notin R_1$ as $1 \neq 1+1$

So R_1 is not reflexive.

Also, $(1,2) \in R_1$ i.e $2 = 1+1$, but $(2,1) \notin R_1$ because $1 \neq 2+1$. Hence R_1 is not symmetric

Again, $(1,2) \in R_1$, $(2,3) \in R_1$ but $(1,3) \notin R_1$ because $3 \neq 1+1$.

Hence R_1 is not transitive also.

Lets check for R_2:

Since $a \geq a, \forall a \in \mathbb{N}$, therefore $(a,a) \in R_2$ $\forall a \in \mathbb{N}$. Hence R_2 is reflexive.

Symmetricity: $(1,3) \in R_2$ as $3 > 1$ but $1 \not> 3 \Rightarrow (3,1) \notin R_2$

So R_2 is not symmetric.

Transitivity: let $(a,b) \in R_2 \Rightarrow b \geq a$ & $(b,c) \in R_2 \Rightarrow c \geq b$ $\Bigg\} \Rightarrow c \geq a \Rightarrow (a,c) \in R_2$

Hence R_2 is transitive.

Q9. Given $A = \{3k \mid k \in \mathbb{N}\}$ and a function $f : \mathbb{N} \to A$ as $f(n) = 6n$

Let's check, f is one-one: Let $n_1, n_2 \in \mathbb{N} \ni f(n_1) = f(n_2)$

$$\Rightarrow 6n_1 = 6n_2$$

$$\Rightarrow n_1 = n_2$$

Hence, f is one-one

Let's check, f is onto: Observe that, $9 \in A$, assume that $\alpha \in \mathbb{N} \ni f(\alpha) = 9$

$$\Rightarrow 6\alpha = 9$$

$$\Rightarrow \alpha = 9/6 = 3/2 \notin \mathbb{N}$$

here our assumption is wrong.
i.e $9 \in A$ we don't have pre-image in $\mathbb{N}$.

Hence f is not onto.

Q 10 : Given $f: \mathbb{Q} \rightarrow \mathbb{Z}$ defined as $f(p/q) = p-q$ where $(p,q)=1$

Let's check, f is one-one :-

Observe that $\frac{1}{2} \in \mathbb{Q}$ and $\frac{2}{3} \in \mathbb{Q}$,

note that, $f(1/2) = 1-2 = -1$,

and $f(2/3) = 2-3 = -1$

i.e two different input we are getting out put

hence f is not one-one.

Let's check f is onto : Let $a \in \mathbb{Z}$

So $a-1 \in \mathbb{Z}$

i.e $\frac{a}{1} = a \in \mathbb{Q}$

Observe that, for every element of codomain(a) we have a pre-image (a itself)

Hence f is onto.