Computing integrals and areas

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Upshot : If one knows the integral of a continuous function, then one can use it to compute the area between the graph of the function and an interval on the X-axis.

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- Integration by parts : $\int (fg')(x)dx = (fg)(x) - \int (f'g)(x)dx$

$$\frac{d}{dx}(fg)(x) = f'(x)g(x) + g'(x)f(x) = (f'g)(x) + (g'f)(x)$$

$$\therefore \int (f'g + g'f)(x)\,dx = (fg)(x)$$

$$\therefore \int (f'g)(x)dx + \int (g'f)(x)\,dx = (fg)(x)$$

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- For any $c \in \mathbb{R}$, $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$
- If $f(x) \geq g(x)$ for all but finitely many points on the interval $[a, b]$, then $\int_a^b f(x)dx \geq \int_a^b g(x)dx$

Examples

$$\int_1^3 x^2 - 4x + 2 \quad dx$$

$$= \int_1^3 x^2\,dx - 4\int_1^3 x\,dx + 2\int_1^3 1\,dx$$

$$= \left[\frac{x^3}{3}\right]_1^3 - 4\left[\frac{x^2}{2}\right]_1^3 + 2\left[x\right]_1^3$$

$$= \frac{1}{3}(3^3-1^3) - \frac{4}{2}(3^2-1^2) + 2(3-1)$$

$$= \frac{26}{3} - 16 + 4 = \frac{26}{3} - 12 = \frac{-10}{3}.$$

$$\int_{-2}^2 x^2 sin(x) dx$$

$$= \boxed{\int_{-2}^0 x^2 \sin(x)\,dx} + \int_0^2 x^2 \sin(x)\,dx$$

$$= \int_2^0 x^2 \sin(x)\,dx + \int_0^2 x^2 \sin(x)\,dx$$

$$= -\int_0^2 x^2 \sin(x)\,dx + \int_0^2 x^2 \sin(x)\,dx = 0.$$

$f(x) = x^2 \sin(x)$

$f(-x) = (-x)^2 \sin(-x)$

$= -x^2 \sin(x)$

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If f is defined piecewise on subintervals of $[a, b]$ then its definite integral from a to b can be computed by computing the definite integrals on each subinterval and adding them up.

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$$\text{Example : } f(x) = \begin{cases} x & \text{if } 0 \leq x \leq 1 \\ 3 - x & \text{if } 1 \leq x \leq 2 \end{cases}$$

What is $\int_0^2 f(x)dx$?

$$\int_0^2 f(x)\,dx = \int_0^1 f(x)\,dx + \int_1^2 f(x)\,dx$$

$$= \int_0^1 x\,dx + \int_1^2 (3-x)\,dx$$

$$= \left[\frac{x^2}{2}\right]_0^1 + \left[3x - \frac{x^2}{2}\right]_1^2 = \frac{1}{2} + 3(2-1) - \frac{1}{2}(4-1)$$

$$= 2.$$

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$$\int_a^b (fg')(x)dx = (fg)(b) - (fg)(a) - \int_a^b (f'g)(x)dx$$

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Example : $\int_0^\infty 3xe^{-3x}dx$ $= \lim_{b\to\infty} \underbrace{-be^{-3b}}_{\to 0} - \frac{1}{3}e^{-3b} + \frac{1}{3} = \frac{1}{3}.$

$f(x) = x$

$g'(x) = 3e^{-3x}$

$\Rightarrow g(x) = -e^{-3x}$

$$\int_0^b 3xe^{-3x}dx$$

$$= \underline{\left[-xe^{-3x}\right]_0^b} - \int_0^b 1 \times (-e^{-3x})\,dx$$

$$= -be^{-3b} - \left[\frac{e^{-3x}}{3}\right]_0^b = -be^{-3b} - \left(\frac{e^{-3b}}{3} - \frac{1}{3}\right)$$

$$= -be^{-3b} - \frac{1}{3}e^{-3b} + \frac{1}{3}.$$

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$\cos(2x)$
$= \cos^2(x) - \sin^2(x)$
$= 2\cos^2(x) - 1.$

Example : $\int_0^a \sqrt{a^2 - u^2} du$ $= \pi a^2/4.$

$g(x)$

$u = a \sin(x)$

Limits are : 0 to $\pi/2$.

$g(0) = 0$

$g(\pi/2) = a.$

$f(u) = \sqrt{a^2 - u^2}$

$f(g(x)) = \sqrt{a^2 - g(x)^2}$
$= \sqrt{a^2 - a^2 \sin^2 x}$

$$= \int_0^{\pi/2} \sqrt{a^2 - a^2 \sin^2(x)}\, a\cos(x)\,dx$$

$$= \int_0^{\pi/2} a\cos(x)\, a\cos(x)\,dx$$

$$= \int_0^{\pi/2} a^2\cos^2(x)\,dx = a^2 \int_0^{\pi/2} \cos^2(x)\,dx$$

$$= a^2 \int_0^{\pi/2} \frac{1+\cos(2x)}{2}\,dx$$

$$= \frac{a^2}{2}\left[x + \frac{\sin(2x)}{2}\right]_0^{\pi/2} = \frac{a^2}{2}\left(\frac{\pi}{2} + \frac{\sin(\pi)}{2}\right)$$

Back to computing areas

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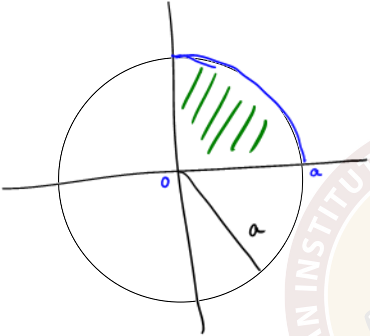

$$x^2 + y^2 = a^2.$$

$$y = \sqrt{a^2 - x^2}.$$

$$f(x) = \sqrt{a^2 - x^2}$$

$$\int_0^a f(x)\,dx = \frac{\pi a^2}{4}.$$

Area of the circle $= 4 \times \frac{\pi a^2}{4}$

$= \pi a^2.$

Thank you