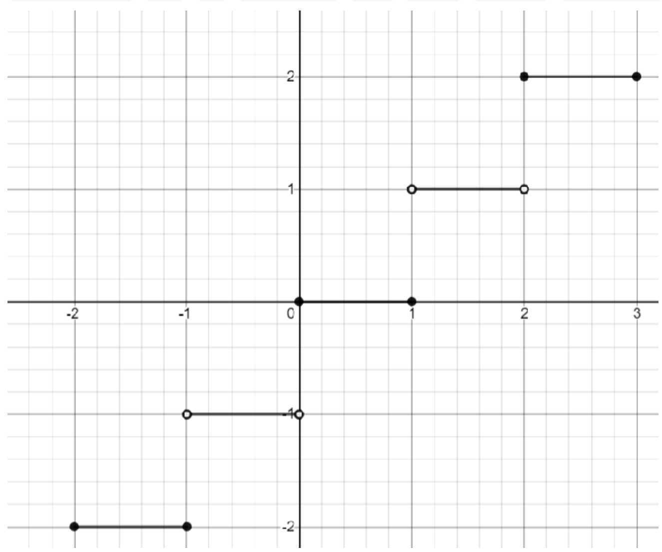
Week 5 Graded Assignment Question
Mathematics for Data Science - 1

1. A graph is shown in Figure M1W8A-8.1, ∘ symbol signifies that the straight line does not touch the point and the • symbol signifies that the line touches the point.
Choose the correct option.



M1W8A-8.1

- ◯ The graph cannot be a function, because it fails the vertical line test.
- ◯ The graph cannot be a function, because it passes the horizontal line test but fails the vertical line test.
- ◯ **The graph can be a function, because it passes the vertical line test.**
- ◯ The graph cannot be a function, because it passes the vertical line test but fails the horizontal line test

Solution
First, to check the given graph represents a function, we have to use vertical line test. Now, $x = c$ where c is a constant, shown already(in grid vertical lines) in the figure M1W8A-8.1, crosses the graph once (including • and ∘ per definition). Therefore, the given graph represents a function.
Set of correct options:

- The graph can be of a function, because it passes the vertical line test.
- The graph fails the horizontal line test.

- The graph represents the graph of neither even function nor odd function.
- The given graph is not invertible in the given domain.

Set of incorrect options:

- The graph cannot be of a function, because it passes the vertical line test but fails the horizontal line test.
- The graph cannot be of a function, because it fails the vertical line test.
- The graph cannot be of a function, because it passes the horizontal line test but fails the vertical line test.
- The graph fails the horizontal line test thus it can be an injective function.
- The graph represents the graph of either even function or odd function.

2. For $y = x^n$,where n is a positive integer and $x \in \mathbb{R}$, which of the following statement is true?

   - ◯ For all values of n, y is not a one-to-one function.
   - ◯ For all values of n, y is an injective function.
   - ◯ y is not a function.
   - ◯ **If n is an even number, then y is not an injective function. If n is an odd number, then y is an injective function.**

Solution:
To check y is a function for all positive integer, let there exist two elements in the domain $x_1, x_2 \in \mathbb{R}$ such that

$$x_1 = x_2$$

taking the power n on both sides, we get,

$$\implies x_1^n = x_2^n$$

$$\implies y(x_1) = y(x_2).$$

Observe that for single input we get a unique output. So, $y = x^n$ is a function for all $n \in \mathbb{Z}^+$.(For vertical line test, see Figure M1W8AS-8.1 and M1W8AS-8.2.
If $n > 1$ is odd positive integer then graph of the function is similar to Figure M1W8AS-8.1, where vertical and horizontal lines are for vertical and horizontal line test respectively.

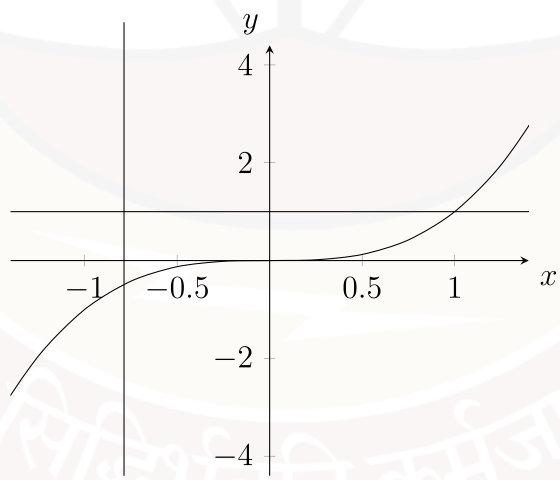


Figure M1W8AS-8.1

If n is even positive integer then graph of the function is similar to Figure M1W8AS-8.2, where vertical and horizontal lines are for vertical and horizontal line test respectively).

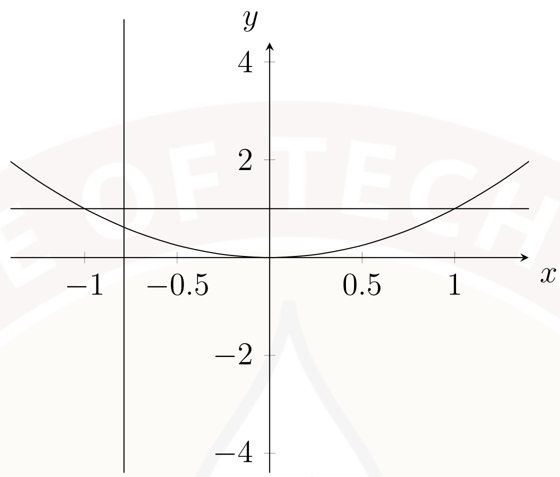


Figure M1W8AS-8.2

So, third option is not correct.
If $n = 1$ which is a positive integer then given function becomes $y = x$.
Using horizontal line test, this function is one-to-one.
So, first option is not correct.
If $n = 2$ which is an even positive integer then given function becomes $y = x^2$.
Using the horizontal line test, this function is not one-to-one(For horizontal line test, see the Figure M1W8AS-8.2).
So, second option is not true.
Now, to check for odd positive integer n, $y = x^n$ is injective. Let $n = 2m + 1$ be an odd integer and $m \in \mathbb{Z}$ (For horizontal line test, see the Figure M1W8AS-8.1). Let there exist two distinct elements in the domain $x_1, x_2 \in \mathbb{R}$ such that

$$f(x_1) = f(x_2)$$
$$\implies x_1^{2m+1} = x_2^{2m+1}$$

Taking the power $\frac{1}{2m+1}$ on both sides we get,

$$x_1 = x_2.$$

From above we see that as no two distinct elements in the domain give the same image. Hence $x_1, x_2 \in \mathbb{R}$ can't be distinct. So, this shows that this function is injective.
Now, to check for even positive integer $n, y = x^n$ is injective or not. Let $n = 2m$ be an even integer and $m \in \mathbb{Z}$, $y = x^{2m} = (x^2)^m$ (See Figure M1W8AS-8.2).
Now, for $x = a, -a$, we get same output $y = a^m$.
Therefore, for even positive integer $n, y = x^n$ not one to one function.
Hence, fourth option is correct.

3. If $4m - n = 0$, then the value of

$$\left(\frac{16^m}{2^n} + \frac{27^n}{9^{6m}}\right)$$

is

Answer: **2**

Solution:

Given $4m - n = 0$.

Now

$$\begin{aligned} &\frac{16^m}{2^n} + \frac{27^n}{9^{6m}} \\ &= \frac{(2^4)^m}{2^n} + \frac{(3^3)^n}{(3^2)^{6m}} \\ &= \frac{2^{4m}}{2^n} + \frac{3^{3n}}{3^{2\times 6m}} \\ &= 2^{4m-n} + 3^{3n-12m} \\ &= 2^{4m-n} + 3^{-3(4m-n)} \\ &= 2^0 + 3^0 \\ &= 1 + 1 = 2 \end{aligned}$$

4. Half-life of an element is the time required for half of a given sample of radioactive element to change to another element.The rate of change of concentration is calculated by the formula $A(t) = A_o(\frac{1}{2})^{(\frac{t}{\gamma})}$ where γ is the half-life of the material, A_o is the initial concentration of the radioactive element in the given sample, $A(t)$ is the concentration of the radioactive element in the sample after time t.
If Radium has a half-life of 1600 years and the initial concentration of Radium in a sample was 100%, then calculate the percentage of Radium in that sample after 2000 years.
   - ◯ 35%
   - ◯ **42%**
   - ◯ 19%
   - ◯ 21%

Solution
Given $A(t) = A_o(\frac{1}{2})^{(\frac{t}{\gamma})}$ and half life of radium is 1600 i.e $\gamma = 1600$

$$\implies A(2000) = A_o(\frac{1}{2})^{(\frac{2000}{1600})}$$

$$= A_o(\frac{1}{2})^{(\frac{5}{4})}$$

At initial time($t = 0$) the concentration of Radium is 100%
$\implies$ At initial time the concentration of Radium is A_o
So, the percentage of Radium in that sample after 2000 years is

$$(\frac{A_o(\frac{1}{2})^{(\frac{5}{4})}}{A_0} \times 100)\% = ((\frac{1}{2})^{(\frac{5}{4})} \times 100)\% \approx 42\%$$

5. If $f(x) = (1-x)^{\frac{1}{2}}$ and $g(x) = (1-x^2)$, then find the domain of the composite function $g \circ f$.

- ○ $\mathbb{R}$
- ○ $\boldsymbol{\big((-\infty, 1] \cap [-2, \infty)\big) \cup (-\infty, -2)}$
- ○ $[1, \infty)$
- ○ $\boldsymbol{\mathbb{R} \setminus (1, \infty)}$

Answer: Option 2, Option 4

Solution:

Given $f(x) = (1-x)^{\frac{1}{2}}$.

To define $f(x)$, $1 - x \geq 0 \implies x \leq 1$.

So, the domain of $f(x)$ is $(-\infty, 1] = \mathbb{R} \setminus (1, \infty)$.

Now, the domain of $g(x) = (1-x^2)$ is $\mathbb{R}$ and the range of $f(x)$ is $[0, \infty)$.

Hence, when we use the two rules as in the video lecture to determine the domain of $g \circ f$.

Here, the domain of $g \circ f$ = The domain of $f = (-\infty, 1] = \mathbb{R} \setminus (1, \infty)$.

Now, $\big((-\infty, 1] \cap [-2, \infty)\big) \cup (-\infty, -2) = [2, 1] \cup (-\infty, -2) = (-\infty, 1]$.

Hence, second and fourth option is true.

6. Find the domain of the inverse function of $y = x^3 + 1$.

   - ◯ $\mathbb{R}$
   - ◯ $\mathbb{R} \setminus \{1\}$
   - ◯ $[1, \infty)$
   - ◯ $\mathbb{R} \setminus [1, \infty)$

   Answer: Option 1

   Solution:

   We know that the domain of the inverse of a given function is the range of the given function.

   Since the range of the function $y = x^3 + 1$ is $\mathbb{R}$, domain of the inverse function of $y = x^3 + 1$ is $\mathbb{R}$.

   Hence, first option is correct.

7. If $f(x) = x^3$, then which of the following options is the set of points where the graphs of the functions $f(x)$ and $f^{-1}(x)$ intersect each other?

- ◯ {(-1,1),(0,0),(1,-1)}
- ◯ {(-2,-8),(1,1),(2,8)}
- ◯ **{(-1,-1),(0,0),(1,1)}**
- ◯ {(-2,-8),(0,0),(2,8)}

Solution:
Let $g(x) = x^{\frac{1}{3}}$ be a function such that $f \circ g = (x^{\frac{1}{3}})^3 = x$ and $g \circ f = (x^3)^{\frac{1}{3}} = x$
Hence, g is the inverse function of f.
To get intersection point

$$
\begin{aligned}
& f = g \\
\implies & x^3 = x^{\frac{1}{3}} \\
\implies & x^9 = x \\
\implies & x^9 - x = 0 \\
\implies & x(x^8 - 1) = 0 \\
\implies & x((x^4)^2 - 1) = 0 \\
\implies & x(x^4 + 1)(x^4 - 1) = 0 \\
\implies & x(x^4 - 1)(x + 1)(x - 1)(x^2 + 1) = 0
\end{aligned}
$$

Observe that for real value 0, -1, 1, $f = g$ and $g(0) = 0, g(-1) = -1, g(1) = 1$.
It follows that the set of points where the graphs of the functions $f(x)$ and $f^{-1}(x)$ intersect each other is {(-1,-1),(0,0),(1,1)}

8. In a survey, the population growth in an area can be predicted according to the equation $\alpha(T) = \alpha_o(1 + \frac{d}{100})^T$ where d is the percentage growth rate of population per year and T is the time since the initial population count α_o was taken. If in 2015, the population of Adyar was 30,000 and the population growth rate is 4% per year, then what will be the approximate population of Adyar in 2020?($T = 0$ corresponds to the year 2015, $T = 1$ corresponds to the year 2016 and so on..)

   - ◯ 60251
   - ◯ 71255
   - ◯ 91000
   - ◯ **36500**

   Solution:
   Given $\alpha(T) = \alpha_o(1 + \frac{d}{100})^T$ and initial population count $\alpha_o = 30000$.
   The population growth rate $d = 4\%$ per year and $T = 5$.
   Hence, the approximate population of Adyar in 2020 is $\alpha(5) = 30000 \times (1 + \frac{4}{100})^5 \approx 36500$

9. An ant moves along the curve whose equation is $f(x) = x^2 + 1$ in the restricted domain $[0, \infty)$. Let a mirror be placed along the line $y = x$. If the reflection of the ant with respect to the mirror moves along the curve $g(x)$, then which of the following options is(are) correct?

   - ◯ $\boldsymbol{g(x) = f^{-1}(x)}$
   - ◯ $g(x) = f(x)$
   - ◯ $\boldsymbol{g(x) = \sqrt[2]{(x-1)}}$
   - ◯ $g(x) = \sqrt[2]{(x+1)}$

   Solution:
   Given $f(x) = x^2 + 1$ in the restricted domain $[0, \infty)$.
   Using horizontal test, given function f is one to one function(See Figure M1W8AS-8.3).

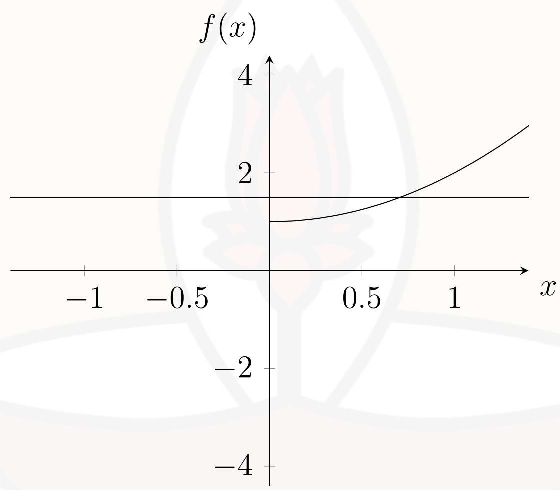


Figure M1W8AS-8.3

Hence, f is reversible function.
We know that the graph of the inverse of a function is symmetric along the line $y = x$ and the mirror placed along the the line $y = x$.
It follows that $g(x)$ is inverse of the function f i.e $g(x) = f^{-1}(x)$.
Now, consider a function $f^{-1}(x) = \sqrt{x-1}$.
Since, $f \circ f^{-1} = (\sqrt{x-1})^2 + 1 = x = I$, similarly, $f^{-1} \circ f = x = I$, where I is the identity function.
Therefore, inverse function of $f(x) = x^2 + 1$ is $f^{-1}(x) = \sqrt{x-1}$,
where f^{-1} is just a notation of inverse function.
Hence, first and third options are correct.

10. Suppose a textile shop provides two different types of offers during a festival season. The first offer(D_1) is "shop for more than ₹14,999 and pay only ₹9,999". The second offer(D_2) is "avail 30% discount on the total payable amount". If Shalini wants to buy two dresses each of which costs more than ₹8,000 and she is given the choice to avail both offers simultaneously, then which of the following options is(are) correct?

- ◯ The minimum amount she should pay after applying two offers cannot be determined because the exact values of the dresses she wanted to buy are unknown.
- ◯ **The minimum amount she should pay after applying the two offers simultaneously is approximately ₹6,999.**
- ◯ The amount she is supposed to pay after applying D_2 only is approximately ₹11,199.
- ◯ **The amount she is supposed to pay after applying D_1 only is approximately ₹9,999.**
- ◯ **Suppose the total payable amount is ₹17,999 for the two dresses. In order to pay minimum amount Shalini should avail offer D_1 first and offer D_2 next.**
- ◯ **Suppose the total payable amount is ₹17,999 for the two dresses. If Shalini avails offer D_2 first, then she cannot avail offer D_1.**
- ◯ Suppose the total payable amount is ₹17,999 for the two dresses. In order to pay minimum amount Shalini should avail offer D_2 first and offer D_1 next.

Solution:
Given each dress cost more than ₹8000.
If Shalini wants to buy two dresses then total payable amount is greater than ₹16000 which is greater than ₹14999.
So she can avail offer D_1.
If she avails offer D_1 first and offer D_2 next, then the total payable amount = ₹9999$(1 - \frac{30}{100}) \approx$ ₹6999
If she avails offer D_2 first, then the total payable amount can be less than ₹14999 or greater than or equal to ₹14999. After that she may or not avail the offer D_1. If she avails offer D_1 after D_2, then also the total payable amount can not be less than ₹6999. In any case the minimum amount she should pay after applying the two offers(without any order) simultaneously is at least ₹6,999.
Hence, first option is not correct and second option is correct.
Since the total payable amount is unknown, therefore we can not say how much she needs to pay after applying the offer D_2.
Hence, third option is not correct.
As we see above if Shalini avails offer D_1 only, then payable amount = ₹9999.
Hence, fourth option is correct.
Suppose the total payable amount is ₹17,999 for the two dresses (which is greater than

₹14999).
If Shalini avails offer D_2 first then payable amount $=$ ₹$17999(1-\frac{30}{100}) \approx$ ₹12599 which is less than ₹14999.
So, she can not avail offer D_1 next and she has to pay $\approx$ ₹12599.
Hence, sixth option is correct and seventh option is not correct.
And if Shalini avails offer D_1 first then payable amount $=$ ₹9999
and then offer D_2 then total payable amount $=$ ₹$9999(1-\frac{30}{100}) \approx$ ₹6999
Hence, from above in order to pay minimum amount Shalini should avail offer D_1 first and offer D_2 next.
So, fifth option is correct.

11. If $f(x) = x^2$ and $h(x) = x - 1$, then which of the following options is(are) incorrect?

- ◯ $f \circ h$ is not an injective function.
- ◯ **$f \circ h$ is an injective function**
- ◯ **$f(f(h(x))) \times h(x) = (x-1)^4$**
- ◯ $f(f(h(x)) \times h(x) = (x-1)^5$

Solution:
Given $f(x) = x^2$ and $h(x) = x - 1$.
Using horizontal line test, $f \circ h = (x-1)^2$ is not a injective function.
Again, $f(f(h(x))) \times h(x) = ((x-1)^2)^2 \times (x-1) = (x-1)^{2\times 2+1} = (x-1)^5$.
Hence, second and third options are incorrect.

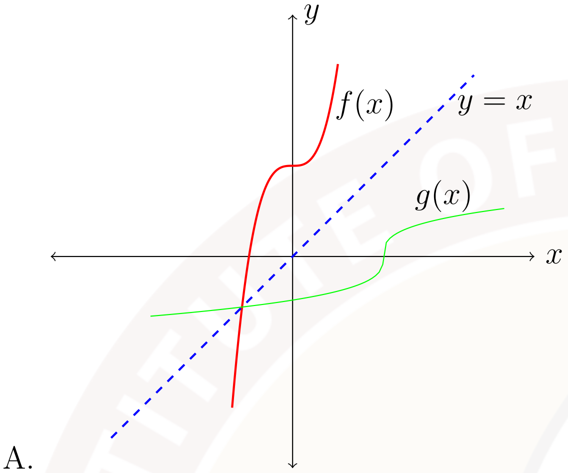
Likewise, the set of other options given below can be solved accordingly:
Set of correct options:

- $f \circ h$ is not an injective function.
- $f(f(h(x)) \times h(x) = (x-1)^5$
- $h \circ f$ is not an injective function.
- There are two distinct solution for $h(h(f(x)) = 0$.

Set of incorrect options:

- $f \circ h$ is an injective function.
- $f(f(h(x))) \times h(x) = (x-1)^4$.
- $h \circ f$ is an injective function.
- There is only one solution for $h(h(f(x)) = 0$.
- Let $q(x) = f(x) - h(x)$, then $q(x)$ is bijective function if $q(x) : \mathbb{R} \to [-1.25, \infty)$
- Let $q(x) = f(x) + h(x)$, then $q(x)$ is bijective function if $q(x) : \mathbb{R} \to [0.75, \infty)$
- Let $q(x) = f(x) + h(x)$, then $q(x)$ is bijective function if $q(x) : \mathbb{R} \to [-1.25, \infty)$
- Let $q(x) = f(x) - h(x)$, then $q(x)$ is bijective function if $q(x) : \mathbb{R} \to [0.75, \infty)$

12. Let $f(x)$, $g(x)$, $p(x)$ and $q(x)$ be the functions defined on $\mathbb{R}$. Refer Figure 3 (A and B) and choose the correct option(s) from the following.
(MSQ), (Answer: Option (a)(b)(c)(d))



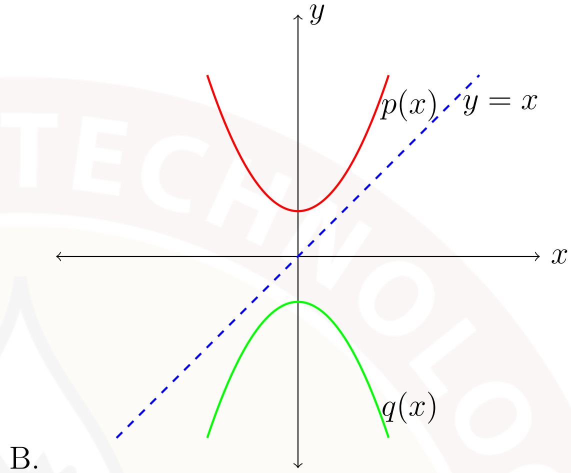


Figure 3

- ◯ $g(x)$ may be the inverse of $f(x)$.
- ◯ $p(x)$ and $q(x)$ are even functions but $f(x)$ and $g(x)$ are neither even functions nor odd functions.
- ◯ $q(x)$ could not be the inverse function of $p(x)$.
- ◯ $p(x)$, $q(x)$ can be an even degree polynomial functions and $f(x)$ can be an odd degree polynomial functions.

Solution: {See Figure 3 for reference}

• $f(x)$ and $g(x)$ are approximately symmetric across $y = x$. Therefore, $g(x)$ may be inverse of $f(x)$.

• $p(x)$ and $q(x)$ are approximately symmetric across $y-$axis. Therefore, $p(x)$ and $q(x)$ are even functions.

• $p(x)$ and $q(x)$ are not symmetric across $y = x$. Therefore, $q(x)$ could not be the inverse of $p(x)$.

• From the end behaviors of the graphs, we can claim that $p(x)$ and $q(x)$ are even-degree polynomials and $f(x)$ can be an odd-degree polynomial.