Expectation of Poisson distribution

$$
\begin{aligned}
E(X) &= \sum_{x=0}^{\infty} x \frac{e^{-\lambda}\lambda^{x}}{x!} \\
&= \sum_{x=0}^{\infty} x \frac{e^{-\lambda}\lambda}{x} \frac{\lambda^{(x-1)}}{(x-1)!} \\
&= e^{-\lambda}\lambda \sum_{x=1}^{\infty} \frac{\lambda^{(x-1)}}{(x-1)!} \\
&= e^{-\lambda}\lambda \left(\frac{\lambda^{0}}{0!} + \frac{\lambda^{1}}{1!} + \dots \right) \\
&= e^{-\lambda}\lambda e^{\lambda} \\
&= \lambda
\end{aligned}
$$

Variance of Poisson distribution

$$
\begin{aligned}
E(X(X-1)) &= \sum_{x=0}^{\infty} x(x-1)\frac{e^{-\lambda}\lambda^{x}}{x!} \\
&= \sum_{x=0}^{\infty} x(x-1)\frac{e^{-\lambda}\lambda^{2}}{x(x-1)}\frac{\lambda^{(x-2)}}{(x-2)!} \\
&= e^{-\lambda}\lambda^{2}\sum_{x=2}^{\infty}\frac{\lambda^{(x-2)}}{(x-2)!} \\
&= e^{-\lambda}\lambda^{2}\left(\frac{\lambda^{0}}{0!}+\frac{\lambda^{1}}{1!}+\dots\right) \\
&= e^{-\lambda}\lambda^{2}e^{\lambda} \\
&= \lambda^{2}
\end{aligned}
$$

Variance of Poisson distribution

- Now, $E(X^2) = E(X(X-1)) + E(X) = \lambda^2 + \lambda$.
- Hence

$$Var(X) = E(X^2) - (E(X))^2 = \lambda^2 + \lambda - \lambda^2 = \lambda$$

- For a Poisson random variable $X \sim Poisson(\lambda)$, both the expected value and the variance of X are equal to λ.

Examples of Poisson distribution

- Events occurring in fixed interval of time
  1. Number of vehicles passing through a traffic intersection in a fixed time interval of one minute.
  2. Number of people withdrawing money from a bank in a fixed time interval of fifteen minutes.
  3. Number of telephone calls received per minute at a call center
- Events occurring in fixed interval of space
  1. Number of typos (incorrect spelling) in a book.
  2. Number of defects in a wire cable of finite length.
  3. Number of defects per meter in a roll of cloth

Modeling number of accidents

Suppose the number of accidents per week in a factory can be modeled by the Poisson distribution with a mean of 0.5.

1. Find the probability that in a particular week there will be less than two accidents?
   Let X be the number of accidents per week in the factory.
   We have $X \sim Poisson(\lambda = 0.5)$
   Need to find $P(X \leq 2) = \sum_{i=0}^{2} \frac{e^{-0.5} \times 0.5^i}{i!} =$
   $= 0.6065 + 0.3033 + 0.0758 = 0.9856$

Modeling number of killings

The number of dogs that are killed on a particular stretch of road in Chennai in any one day can be modeled by a Poisson(0.42) random variable.

1. Calculate the probability that exactly two dogs are killed on a given day on this stretch of road.
   Let X= number of dogs killed in one day
   $X \sim Poisson(\lambda = 0.42)$
   $$P(X = 2) = \frac{e^{-0.42} \times 0.42^2}{2!} = 0.058$$
2. Find the probability that exactly two dogs are killed over a 5-day period on this stretch of road.
   Let X= number of dogs killed in five day
   $X \sim Poisson(\lambda = 2.1)$
   $$P(X = 2) = \frac{e^{-2.1} \times 2.1^2}{2!} = 0.27$$

Modeling the number of defects

Suppose the number of defects in a wire cable can be modeled by the Poisson distribution with a of 0.5 defects per meter.

1. Find the probability that a single meter of wire will have exactly 3 defects
   Let X be the number of defects per meter
   We have $X \sim Poisson(\lambda = 0, 5)$
   Need to find $P(X = 3) = \dfrac{e^{-3} \times 0.5^3}{3!} = 0.0126$

Modeling typos

A typist makes 1500 mistakes in a book of 500 pages. Let X be number of mistakes per page. Then, $X \sim Poisson(3)$

On how many pages would you expect to find

1. no mistake

   $P(X = 0) = \dfrac{e^{-3} \times 3^0}{0!} = 0.0498$

   Number of pages with no mistakes $= 0.0498 \times 500 \approx 25$ pages
2. one mistake

   $P(X = 1) = \dfrac{e^{-3} \times 3^1}{1!} = 0.1494$

   Number of pages with no mistakes $= 0.1494 \times 500 \approx 75$ pages
3. three or more mistakes

   $P(X >= 31) = 1 - \dfrac{e^{-3} \times 3^0}{0!} - \dfrac{e^{-3} \times 3^1}{1!} - \dfrac{e^{-3} \times 3^2}{2!} = 0.5768$

   Number of pages with no mistakes $= 0.5768 \times 500 \approx 288$ pages

Section summary

Modeling situations using Poisson distribution

- in time
- in space