Week 7 - graded assignment Solution

(The numbers in your assignment may differ)

1) Let $\{a_n\}$ and $\{b_n\}$ be two sequences of real numbers.

Let $a_n = 1$ for all n and $b_n = -1$ for all n.

$\{a_n\}$ converges to 1, $\{b_n\}$ converges to -1

$a_n + b_n = 0$ for all n.

$\{a_n + b_n\}$ converges to 0.

Hence option 1 is not correct.

Let $\{a_n\}$ be an increasing sequence.

$a_n = n$ for all n.

$(-1)^n a_n = (-1)^n n$.

Hence $\{(-1)^n a_n\}$ is not a decreasing sequence in this case.

Hence option 2 is not correct.

If $\{a_n\} \to a$, $\{b_n\} \to b$

then, $\{a_n b_n\} \to ab$.

If both a and b are non-zero, then ab must be non-zero.

Hence option 3 is correct.

If $\{a_n\} \to a$ and $\{b_n\} \to a$

then $\{a_n - b_n\} \longrightarrow a - a = 0$.

Hence option 4 is correct.

Let $a_n = \begin{cases} n & \text{when } n = \text{odd.} \\ 1 & \text{when } n = \text{even.} \end{cases}$

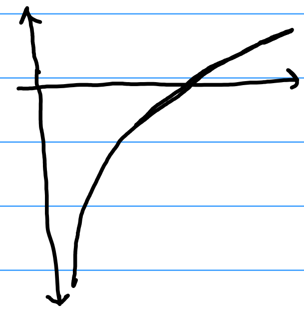
Hence, a_n is a divergent sequence.

But $\{a_{2n}\}$ is a constant sequence $\{1\}$,

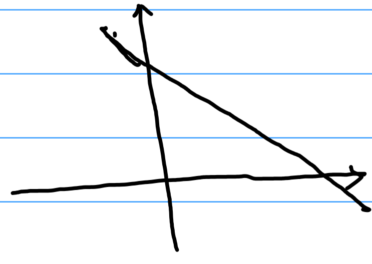
which is obviously a convergent subsequence of $\{a_n\}$.

Hence option 5 is not correct.

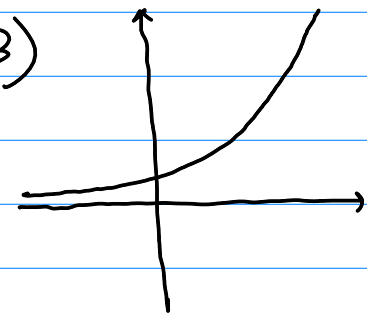
2) i) $f(x) = 3\ln x - 2$ d) logarithmic function 3)



ii) $f(x) = 10 - 4x$ c) linear function 4)

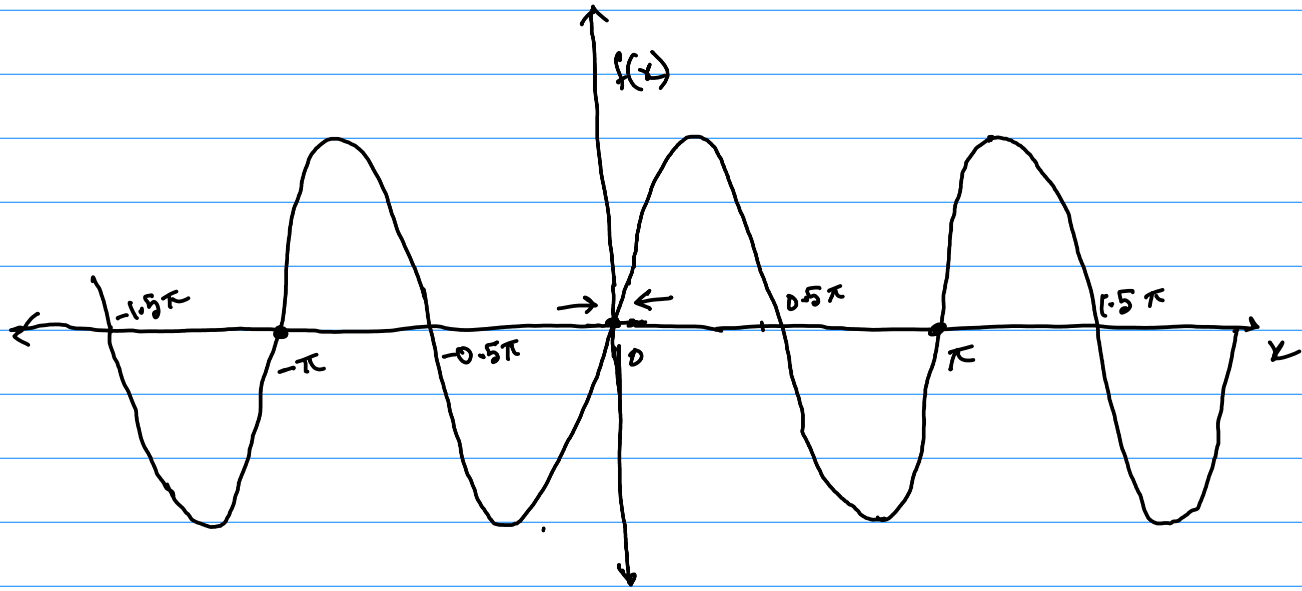


iii) $f(x) = 2^x + 7$ a) exponential function 3)



iv) $f(x) = x^2 - 4x + 4$ b) quadratic function 1)

3)



$\lim_{x \to 0^+} f(x) = 0 = \lim_{x \to 0^-} f(x)$. Hence option 2 is correct.

At $x = \pi$ and $x = -\pi$ there are no sharp corner at the given curve. So, option 5 is correct.

In the interval $[-0.5\pi, 0.5\pi]$ the function is oscillatory (neither monotonically increasing nor monotonically decreasing).

4) $$\lim_{x\to 0} \frac{\log(1+x)}{\sin x} = \lim_{x\to 0} \frac{\frac{\log(1+x)}{x}}{\frac{\sin x}{x}} = \frac{\lim_{x\to 0}\frac{\log(1+x)}{x}}{\lim_{x\to 0}\frac{\sin x}{x}}$$

$$= \frac{1}{1} = 1.$$

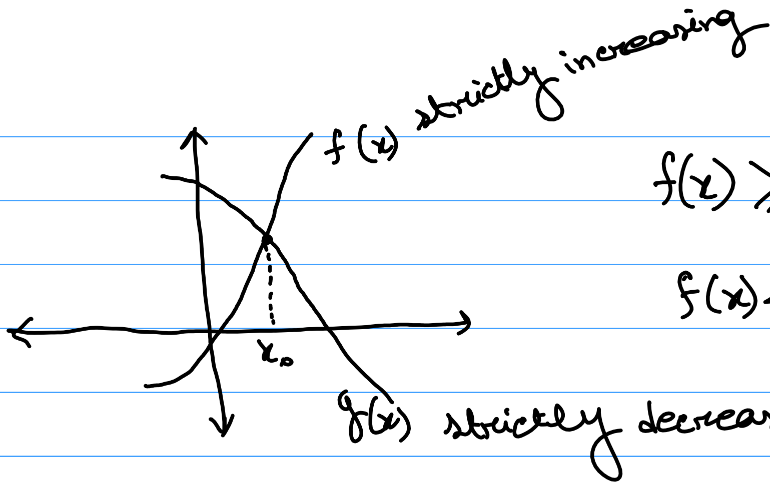
$$\lim_{x\to 0} \frac{\sin 5x}{x} = \lim_{x\to 0} \frac{\sin 5x}{5x} \times 5 = 5 \lim_{x\to 0} \frac{\sin 5x}{5x}$$

$$= 5 \times 1 = 5$$

$$\lim_{x\to 0} \frac{e^{x/2}-1}{\sin 2x} = \lim_{x\to 0} \frac{\frac{e^{x/2}-1}{x/2} \times \frac{x}{2}}{\frac{\sin 2x}{2x} \times 2x}$$

$$= \lim_{x\to 0} \frac{\frac{e^{x/2}-1}{x/2}}{\frac{\sin 2x}{2x}} \times \frac{1}{4}$$

$$= 1 \times 1/4 = 1/4$$

5) 

$f(x) \geq g(x)$ for all $x \geq x_0$.

$f(x) \leq g(x)$ for $x \leq x_0$.

We have, $f(x_0) = g(x_0)$. But for any $x > x_0$, $f(x)$ and $g(x)$ will never intersect. So, option 2 is incorrect.

6) $$a_n = \frac{12n^2}{3n+5} - \frac{4n^2+7}{n+3}$$

$$= \frac{12n^2(n+3) - (4n^2+7)(3n+5)}{(3n+5)(n+3)}$$

$$= \frac{\cancel{12n^3} + 36n^2 - \cancel{12n^3} - 21n - 20n^2 - 35}{3n^2+5n+9n+15}$$

$$= \frac{16n^2 - 21n - 35}{3n^2+14n+15}$$

$$= \frac{16 - \frac{21}{n} - \frac{35}{n^2}}{3 + \frac{14}{n} + \frac{15}{n^2}}$$

as, $n \to \infty$, $a_n \to \frac{16}{3}$.

7) $$R(\omega) = \frac{50e^{\omega}}{10+e^{\omega}} = \frac{50}{\frac{10}{e^{\omega}}+1}$$

If $\omega_1 \geq \omega_2$ then, $e^{\omega_1} \geq e^{\omega_2}$

$$\Rightarrow \frac{10}{e^{\omega_1}} \leq \frac{10}{e^{\omega_2}}$$

$$\Rightarrow \frac{10}{e^{\omega_1}}+1 \leq \frac{10}{e^{\omega_2}}+1$$

$$\Rightarrow \frac{50}{\frac{10}{e^{\omega_1}}+1} \geq \frac{50}{\frac{10}{e^{\omega_2}}+1}$$

$$\Rightarrow R(\omega_1) \geq R(\omega_2)$$

$R(\omega)$ is increasing function.

$$\lim_{\omega \to \infty} R(\omega) = 50$$

Hence, the minimum possible value of r such that $R(\omega) < r$, for all ω, where $r \in \mathbb{Z}$, is 50.

8) $$\lim_{n\to\infty} e\sqrt[n]{n!}\left[\log(1+6/n) - \frac{e^{1/n}-1}{(\sqrt{2\pi n})^{1/n}}\right]$$

$$= \lim_{n\to\infty} e\sqrt[n]{n!}\left[\frac{\log(1+6/n)}{6/n}\times\frac{6}{n} - \frac{\frac{e^{1/n}-1}{1/n}\times 1/n}{\frac{n(\sqrt{2\pi n})^{1/n}}{(n!)^{1/n}}\cdot\frac{(n!)^{1/n}}{n}}\right]$$

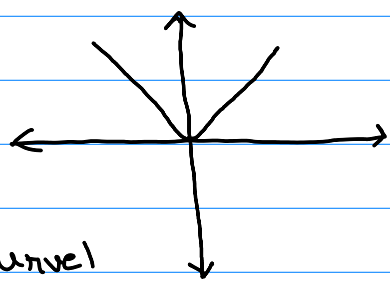
$$= \lim_{n\to\infty} e\left[\frac{\log(1+6/n)}{6/n}\times\frac{6}{\frac{n}{\sqrt[n]{n!}}} - \frac{\frac{e^{1/n}-1}{1/n}}{n\left(\frac{\sqrt{2\pi n}}{n!}\right)^{1/n}}\times\frac{1/n\cdot\sqrt[n]{n!}}{\frac{\sqrt[n]{n!}}{n}}\right]$$

Let $1/n = x$. As $n\to\infty$, $x\to 0$

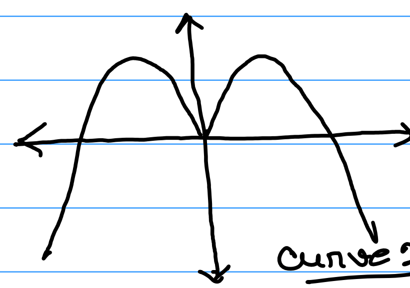
$$= e\left[\lim_{x\to 0}\frac{\log(1+6x)}{6x}\times\frac{6}{\lim_{n\to\infty}\frac{n}{\sqrt[n]{n!}}} - \frac{\lim_{x\to 0}\frac{e^x-1}{x}}{\lim_{n\to\infty} n\left(\frac{\sqrt{2\pi n}}{n!}\right)^{1/n}}\times 1\right]$$

$$= e\left[1\times\frac{6}{e} - \frac{1}{e}\times 1\right] = 6-1 = 5$$

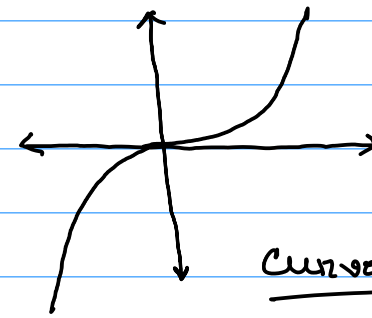
9) Both curve 1 and curve 2 have sharp corners at the origin $(0,0)$. Hence, at the origin these two curves do not have tangents at the origin.



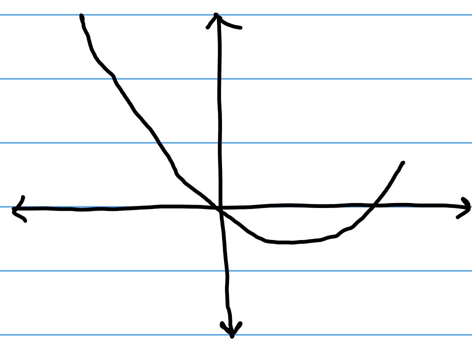
curve1



curve 2



curve3



curve 4

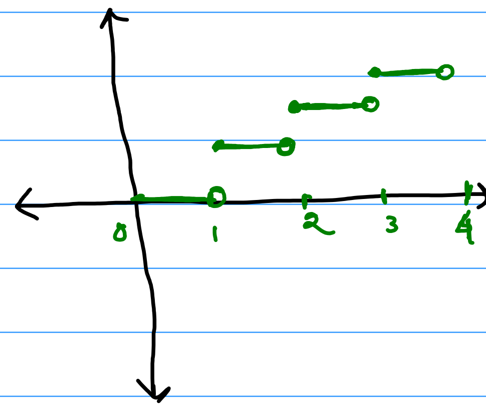
10) $$a_n = \frac{9+15+21+\cdots+3(2n-1)}{n^2}$$

$$= \frac{3\left(3+5+7+\cdots+(2n-1)\right)}{n^2}$$

$$= \frac{3\left(n^2-1\right)}{n^2} = 3\left(1-\frac{1}{n^2}\right)$$

$$\lim a_n = \lim_{n\to\infty} 3\left(1-\frac{1}{n^2}\right) = 3$$

11) $5 \lim_{x \to 3^+} \lfloor x \rfloor - 3 \lim_{x \to 1^-} \lfloor x \rfloor$



$= 5 \times 3 - 3 \times 0 = 15$

Comprehension Type Question:

12) Errorinestimation by Algorithm 1:

$$a_n = \frac{n^2+5n}{6n^2+1}$$

$$\lim a_n = \lim_{n \to \infty} \frac{n^2+5n}{6n^2+1}$$

$$= \lim_{n \to \infty} \frac{1 + 5/n}{6 + 1/n} = \frac{1}{6} \approx 0.166$$

Errorinestimation by Algorithm 2:

$$b_n = \frac{1}{8} + (-1)^n \frac{1}{n}$$

$$\lim b_n = \frac{1}{8} + \lim_{n \to \infty} (-1)^n \cdot \frac{1}{n}$$

$$= \frac{1}{8} + 0 = \frac{1}{8} = 0.125 \quad \left(\text{As } -\frac{1}{n} \leq \frac{(-1)^n}{n} \leq \frac{1}{n} \right)$$

Error in estimation by Algorithm 3:

$$c_n = \frac{e^n + 4}{7e^n}$$

$$\lim c_n = \lim_{n\to\infty} \frac{e^n+4}{7e^n} = \lim_{n\to\infty} \frac{1+\frac{4}{e^n}}{7}$$

$$= \frac{1}{7} \approx 0.143$$

Maximum error in estimation will be given by Algorithm 1.

Minimum error in estimation will be given by Algorithm 2.

13) Error in estimation by the new algorithm:

$$\lim (a_n - b_n) = \lim a_n - \lim b_n$$

$$= \frac{1}{6} - \frac{1}{8}$$

$$= \frac{4-3}{24} = \frac{1}{24}$$

The error in estimation using the new algorithm is less than the error in estimation using any of the Algorithm 1, Algorithm 2 and Algorithm 3.

14)

$$c_n' = ne^{1/8n} - n$$

$$= \frac{e^{1/8n} - 1}{1/8n} \times \frac{1}{8}$$

$$\lim c_n' = \lim_{n \to \infty} \frac{e^{1/8n} - 1}{1/8n} \times \frac{1}{8} = 1/8$$

$$= 0.125$$