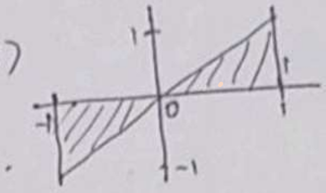
MATHEMATICS FOR DATA SCIENCE - 1

WEEK 9 GRADED ASSIGNMENT

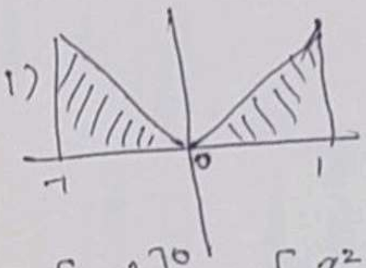
(1) Match functions with graphs and area under the curve.

(i) $f(x) = x$

3) 

$$\int_{-1}^{1} f(x)\,dx = \int_{-1}^{1} x\,dx = \left[\frac{x^2}{2}\right]_{-1}^{1} = 0.$$

(i) $\longrightarrow$ (b) $\longrightarrow$ (3)

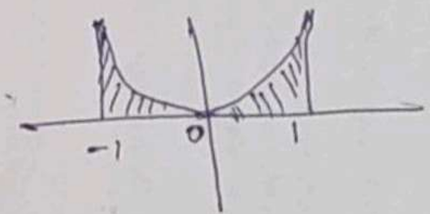
1) 

(ii) $f(x) = |x|$

$$\int_{-1}^{1} f(x)\,dx = \int_{-1}^{1} |x|\,dx$$

$$= \int_{-1}^{0} (-x)\,dx + \int_{0}^{1} x\,dx = \left[-\frac{x^2}{2}\right]_{-1}^{0} + \left[\frac{x^2}{2}\right]_{0}^{1} = 1$$

(ii) $\longrightarrow$ (c) $\longrightarrow$ (1)



(iii) $f(x) = x^2$

$$\int_{-1}^{1} x^2\,dx = \left[\frac{x^3}{3}\right]_{-1}^{1} = \frac{2}{3}$$

(iii) $\longrightarrow$ (a) $\longrightarrow$ (2)

(2) Which of the following curves enclose a negative area on the X-axis in the interval [0, 1]?

Area enclosed above X-axis (+ve direction of Y-axis) is positive and area enclosed below X-axis (-ve direction of Y-axis) is negative. So if the area enclosed below the X-axis is more than the area enclosed above, then the area enclosed by the curve is negative.

Curve 2 and Curve 4 enclosed negative area.

(3) cylinder of radius x and height $2h$ inscribed in a circle of radius R.

Soln: From the right angled Δ^{le} OAB, we have $h^2+x^2=R^2$.

For the volume, $V = 2\pi x^2 h = 2\pi(R^2-h^2)h = 2\pi R^2 h - 2\pi h^3$

$$\frac{dV}{dh} = 2\pi R^2 - 6\pi h^2$$

$$\frac{dV}{dh} = 0 \Rightarrow 2\pi R^2 = 6\pi h^2 \Rightarrow h^2 = \frac{R^2}{3} \Rightarrow h = \pm\frac{R}{\sqrt{3}} \text{ are critical points}$$

But since h is height of the cylinder, $h = \frac{R}{\sqrt{3}}$

$$\frac{d^2V}{dh^2} = -12\pi h. \quad \frac{d^2V}{dh^2}\left(\frac{R}{\sqrt{3}}\right) = -12\pi\cdot\frac{R}{\sqrt{3}} < 0.$$

$\therefore$ Max volume is attained when $h = \frac{R}{\sqrt{3}}$.

$|||^{ly}$ for the surface area, $S = 4\pi x h = 4\pi h\sqrt{R^2-h^2}$

$$\frac{dS}{dh} = 4\pi\left[\sqrt{R^2-h^2} + h\cdot\frac{1}{2\sqrt{R^2-h^2}}(-2h)\right]$$

$$= 4\pi\left[\sqrt{R^2-h^2} - \frac{h^2}{\sqrt{R^2-h^2}}\right]$$

$$\frac{dS}{dh} = 0 \Rightarrow R^2-h^2 = h^2 \Rightarrow h^2 = \frac{R^2}{2} \Rightarrow h = \pm\frac{R}{\sqrt{2}} \text{ are the critical pts}$$

But, again since h is the height, $h = \frac{R}{\sqrt{2}}$.

$$\frac{d^2S}{dh^2} = 4\pi\left[\frac{-2h}{\sqrt{R^2-h^2}} + \frac{h^2(-2h)}{2(R^2-h^2)^{3/2}} - \frac{2h}{\sqrt{R^2-h^2}}\right] = 4\pi\left[\frac{-4h}{\sqrt{R^2-h^2}} - \frac{h^3}{(R^2-h^2)^{3/2}}\right]$$

$$\frac{d^2S}{dh^2}\left(\frac{R}{\sqrt{2}}\right) = 4\pi\left[\left(-4\times\frac{R}{\sqrt{2}}\times\frac{\sqrt{2}}{R}\right) - \left(\frac{R^3}{2\sqrt{2}}\times\frac{2\sqrt{2}}{R^3}\right)\right] = -20\pi < 0.$$

$\therefore$ Max surface area is attained when $h = \frac{R}{\sqrt{2}}$

$f_1(x) = x^3$, $f_2(x) = x$; $g_1(x) = \sqrt{x}$, $g_2(x) = e^x$

(4) Note that $f_2(x)$ and $g_2(x)$ are increasing functions.
Thus the minimum is attained at 0 (in the interval $[0,1]$)
$f_2(0) = 0$ and $g_2(0) = 1$.
$\therefore$ The difference is 1.

(5) Error in prediction for company A will be the difference in areas enclosed ~~between~~ by curves f_1 and g_1

$$= \left| \int_0^1 (f_1 - g_1)(x) \right| = \left| \int_0^1 f_1(x)\,dx - \int_0^1 g_1(x)\,dx \right|$$

$$= \left| \int_0^1 x^3\,dx - \int_0^1 \sqrt{x}\,dx \right| = \left| \left[\frac{x^4}{4}\right]_0^1 - \left[\frac{x^{3/2}}{3/2}\right]_0^1 \right|$$

$$= \left| \frac{1}{4} - \frac{2}{3} \right| = \frac{5}{12} \,\|$$

Error in prediction for company B will be the difference in areas enclosed by f_2 and g_2

$$= \left| \int_0^1 (f_2 - g_2)(x)\,dx \right| = \left| \int_0^1 f_2(x)\,dx - \int_0^1 g_2(x)\,dx \right|$$

$$= \left| \int_0^1 x\,dx - \int_0^1 e^x\,dx \right| = \left| \left[\frac{x^2}{2}\right]_0^1 - \left[e^x\right]_0^1 \right| = e - \frac{3}{2} \,/\!/$$

Clearly $e - \frac{3}{2} > \frac{5}{12}$. So error in prediction for company B is greater than the error in prediction for company A.

(6) $f(x) = x^3 - 3x + 1 \Rightarrow f'(x) = 3x^2 - 3 \Rightarrow f''(x) = 6x$
$f'(x) = 0 \Rightarrow x = \pm 1$ (critical points)
$f''(1) = 6 > 0$ – local minimum $\qquad f(1) = 1 - 3 + 1 = -1 \,/\!/$
$f''(-1) = -6 < 0$ – local maximum

(7) $f(x) = 2x^2 + \frac{5}{6}$, $0 \le x \le 6$

Dividing $[0,6]$ into 3 sub-intervals of equal lengths

$[0,2]$, $[2,4]$, $[4,6]$.

Riemann sum $= \sum_{i=1}^{3} f(x_i^*) \Delta x_i$, x_i^* - left end point of the interval

$= 2f(0) + 2f(2) + 2f(4)$

$= 2\left[\frac{5}{6} + \left(8 + \frac{5}{6}\right) + \left(32 + \frac{5}{6}\right)\right] = 2\left(40 + \frac{5}{2}\right) = 85$ //

(8) $f(x) = \begin{cases} -2x+3 & 0 \le x \le 10 \\ x^2 & 10 < x \le 20 \end{cases}$

$f'(x) = \begin{cases} -2 & 0 < x < 10 \\ 2x & 10 < x < 20 \end{cases}$

$f'(x) \neq 0$ for $x \in (0,10)$. similarly, $f'(x) \neq 0$ for $x \in (10,20)$.

$f(x)$ is not cont. at $x = 10$ (hence not differentiable)

So $x = 10$ is a critical point.

$f(0) = 3$; $f(10) = -17$; $f(20) = 400$ (0 & 20 end pts, 10-critical pt)

Global min. attained at $x = 10$. Min. value $= \underline{\underline{-17}}$.

(9) $x - y = 5 \Rightarrow y = x - 5$

$f(x) = 2xy = 2x(x-5) = 2x^2 - 10x$

$f'(x) = 4x - 10$. $f'(x) = 0 \Rightarrow x = \frac{5}{2}$.

$f''\left(\frac{5}{2}\right) = 4 > 0 \Rightarrow$ ~~4~~ $x = \frac{5}{2}$ is a local minimum

$f\left(\frac{5}{2}\right) = 2 \times \frac{25}{4} - 10 \times \frac{5}{2} = \frac{25}{2} - 25 = -\frac{25}{2}$ //.