Statistics for Data Science -1

Lecture: Poisson Distribution

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Learning objectives

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1. Derive the formula for the probability mass function for Poisson distribution.

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2. Expectation and variance of the Poisson distribution.

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1. Derive the formula for the probability mass function for Poisson distribution.
2. Expectation and variance of the Poisson distribution.
3. To understand situations that can be modeled as a Poisson distribution.

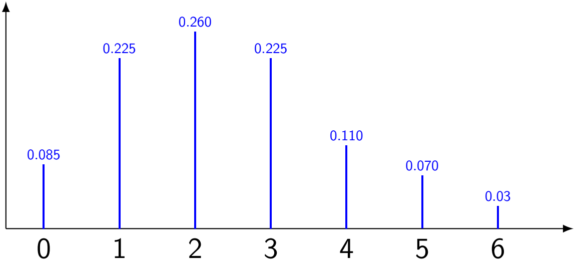
Introduction

- The Poisson probability distribution gives the probability of a number of events occurring in a fixed interval of **time** or **space**.
- We assume that these events happen with a known average rate, λ, and independently of the time since the last event.
- Let X denote the number of times an event occurs in an interval of time (or space).
- We say $X \sim Poisson(\lambda)$, in other words, X is a random variable that follows Poisson distribution with parameter λ.
- The Poisson distribution may be used to approximate the Binomial distribution if the probability of success is "small" and the number of trials is "large" .

Motivation example

Consider a researcher who is observing the number of vehicles that pass a busy traffic intersection in a day. She collects data comprising of 1000 one minute intervals and tabulates the same in form of a frequency table given below.

Number of vehicles	0	1	2	3	4	5	> 6
Count	80	225	260	225	110	70	30



Tabular summary

x	Freq f	Rel Freq f_r	$f_r x$	$f_r x^2$
0	80	0.08	0	0
1	225	0.225	0.225	0.225
2	260	0.26	0.52	1.04
3	225	0.225	0.675	2.025
4	110	0.11	0.44	1.76
5	70	0.07	0.35	1.75
6	30	0.03	0.18	1.08
	1000	1	2.39	7.88

- Mean $= 2.39$
- Variance $= 7.88 - 2.39^2 = 2.16$

Observations

- Number of vehicles passing a traffic intersection are at random and independently of each other
- The average number of vehicles per minute is about 2.39 which is equivalent to 143 per hour.
- Question: What is the appropriate probability distribution to model the number of vehicles passing a traffic intersection?
    - Poisson

Derivation

Let X denote the number of events in a given interval (time or space). Then X follows a Poisson distribution with parameter λ

1. The number of events occurring in non-overlapping intervals are independent.
2. The probability of exactly one event in a short interval of length, δt, is equal to $\lambda \delta t$.
3. The probability of exactly two or more events in a short interval is essentially zero.

What is the Probability of n events happening in interval of length t?

Poisson as Binomial approximation

- Define "success" as exactly one event happening in a short interval of length δt
- The n events happening in interval of length t can be viewed as n successes happening in n intervals of length δt, with each one of them being an independent and identical trial.
- Hence the problem can be viewed as a $Bin\left(n, p = \dfrac{\lambda}{n}\right)$ experiment.

Derivation- contd

$$
\begin{aligned}
&= \binom{n}{x}\left(\frac{\lambda}{n}\right)^x\left(1-\frac{\lambda}{n}\right)^{n-x} \\
&= \frac{n(n-1)\dots(n-x+1)}{x!}\left(\frac{\lambda}{n}\right)^x\left(1-\frac{\lambda}{n}\right)^n\left(1-\frac{\lambda}{n}\right)^{-x} \\
&= \frac{\lambda^x}{x!}\left(\frac{n(n-1)\dots(n-x+1)}{n}\right)^x\left(1-\frac{\lambda}{n}\right)^n\left(1-\frac{\lambda}{n}\right)^{-x} \\
&= \frac{\lambda^x}{x!}\left(\frac{n^x(1-\frac{1}{n})\dots(1-\frac{(x-1)}{n})}{n^x}\right)\left(1-\frac{\lambda}{n}\right)^n\left(1-\frac{\lambda}{n}\right)^{-x}
\end{aligned}
$$

Now let's make the intervals very small, i.e, $\delta t \longrightarrow 0$ or $n \longrightarrow \infty$

Derivation- contd

$$
\begin{aligned}
&= \lim_{n\to\infty} \binom{n}{x} \left(\frac{\lambda}{n}\right)^x \left(1-\frac{\lambda}{n}\right)^{n-x} \\
&= \lim_{n\longrightarrow\infty} \frac{\lambda^x}{x!} \underbrace{\left(\frac{n^x(1-\frac{1}{n})\ldots(1-\frac{(x-1)}{n})}{n^x}\right)}_{\to 1 \; as \; n \to \infty} \underbrace{\left(1-\frac{\lambda}{n}\right)^n}_{\to e^{-\lambda} \text{ as } n\to\infty} \underbrace{\left(1-\frac{\lambda}{n}\right)^{-x}}_{\to 1 \; as \; n \to \infty} \\
&= \frac{\lambda^x}{x!} e^{-\lambda}
\end{aligned}
$$

Probability mass function of Poisson

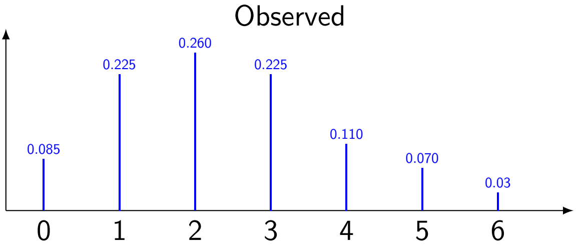
The distribution, with an average number of λ events per interval, is defined as Poisson discrete random variable, $X \sim Poisson(\lambda)$, with the p.m.f given by

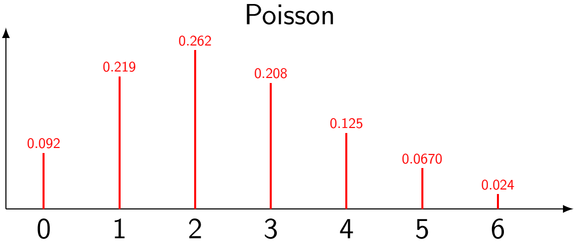
$$P(X = x) = \frac{e^{-\lambda}\lambda^x}{x!}, x = 0, 1, 2, \ldots$$

- X represents the random variable number of events per time interval (In the example: number of vehicles passing per minute)
- e is the mathematical constant 2.718

Going back to the example

x	freq	Prob
0	80	0.092
1	225	0.219
2	260	0.262
3	225	0.208
4	110	0.125
5	70	0.060
6	30	0.024

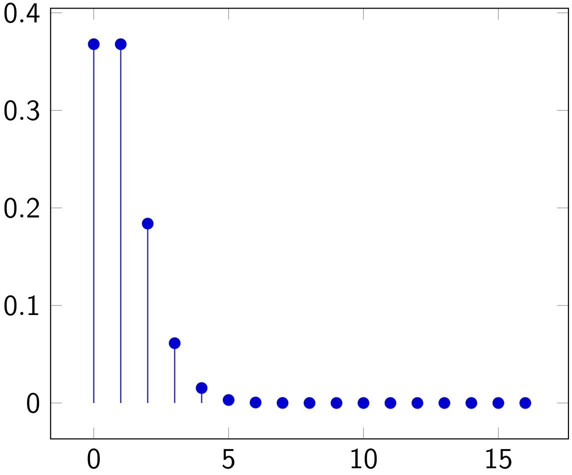

Observed
0.085
0.225
0.260
0.225
0.110
0.070
0.03
0
1
2
3
4
5
6


Poisson
0.092
0.219
0.262
0.208
0.125
0.0670
0.024
0
1
2
3
4
5
6

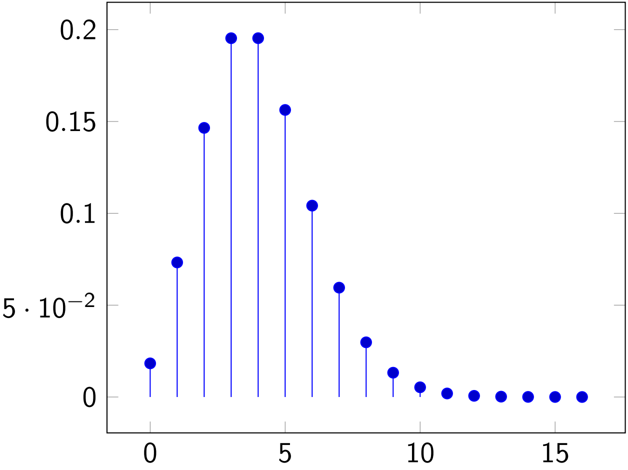
Shape of pmf versus λ

- The shape of the Poisson distribution depends on the value of the parameter λ.
- If λ is small the distribution has positive skew, but as λ increases the distribution becomes progressively more symmetrical.

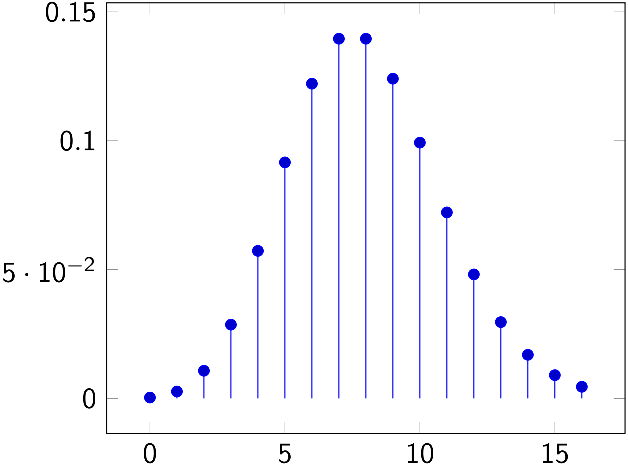
Graph of pmf for $\lambda = 1$



Graph of pmf for $\lambda = 4$



Graph of pmf for $\lambda = 8$



Section summary

- pmf of Poisson distribution
- shape of pmf versus λ