Limits for functions of one variable

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However, if one takes a sequence a_n increasing to 2, then $g(a_n) \to 1$.

Note also that for $g(x)$ this happens at each integer value and if a is a non-integer value, then indeed $g(a_n) \to g(a)$ whenever $a_n \to a$.

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Then $f(a_n) = 1 \forall n$ whereas $f(\sqrt{2}) = 0$. Thus even though $a_n \to a$, $f(a_n) \not\longrightarrow f(\sqrt{2})$.

$\sqrt{2} = 1.414 \cdots$

$a_1 = 1.4$, $a_2 = 1.41$, $a_3 = 1.414$

a_n = the number obtained by taking the first n terms after the decimal point in $\sqrt{2}$.

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If there is no such number L then we say that the limit of f at a from the left does not exist.

An equivalent way of thinking of $\lim\limits_{x \to a^-} f(x) = L$ is that as x comes closer and closer to a from the left, $f(x)$ eventually comes closer and closer to L.

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Similarly, if there is a real number R such that $f(a_n) \to R$ for all sequences a_n such that $a_n \to a$ and $a_n > a$, then we say that the limit of f at a from the right exists and equals R.

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We denote it by $\lim_{x\to a} f(x)$.

$\lim_{x\to a^-} x^2 = a^2 = f(a)$; $\lim_{x\to a^+} x^2 = \lim_{x\to a} x^2$: $f(x)=x^2$

$g(x) = \lfloor x \rfloor$: $\lim_{x\to a^-} \lfloor x \rfloor = \lim_{x\to a^+} \lfloor x \rfloor = \lfloor a \rfloor = g(a)$; if $a \notin \mathbb{Z}$

$\lim_{x\to 2^-} \lfloor x \rfloor = 1$, $\lim_{x\to 2^+} \lfloor x \rfloor = 2$: $\lim_{x\to 2} g(x)$ DNE

$f(x) = \begin{cases} 0 & \text{if } x\in\mathbb{Q} \\ 1 & \text{if } x\notin\mathbb{Q} \end{cases}$

$\lim_{x\to a^-} f(x)$, $\lim_{x\to a^+} f(x)$ DNE : $\lim_{x\to a} f(x)$ DNE

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Suppose for all sequences x_n diverging to $-\infty$, there exists L such that $f(x_n)$ converges to L (i.e. as x becomes smaller and smaller, $f(x)$ eventually gets closer and closer to L).

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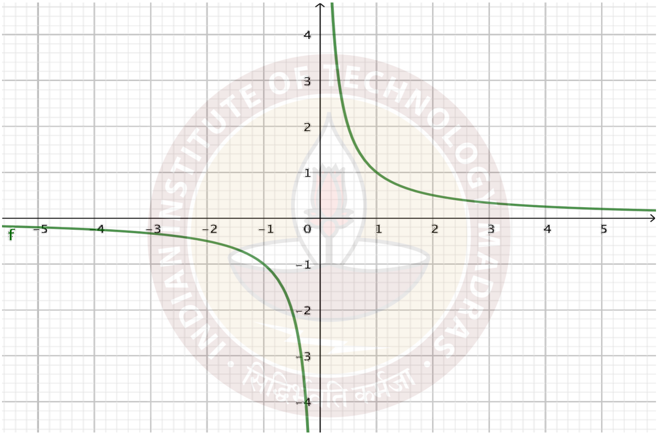
Suppose for all sequences x_n diverging to $-\infty$, there exists L such that $f(x_n)$ converges to L (i.e. as x becomes smaller and smaller, $f(x)$ eventually gets closer and closer to L). Then we say that $\lim_{x\to-\infty} f(x)$ exists and equals L.

The function $\frac{1}{x}$



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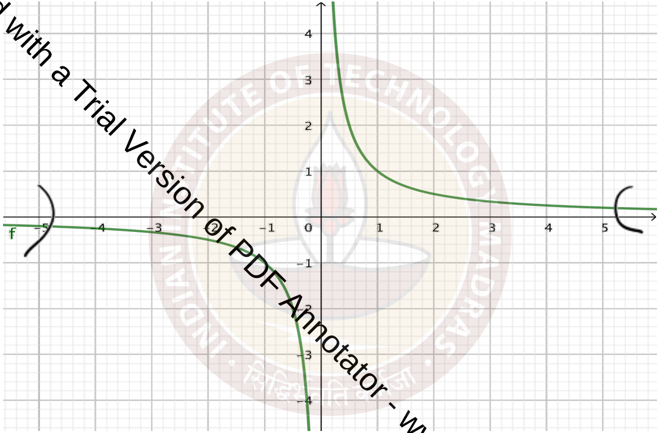

$$\lim_{x\to\infty}\frac{1}{x}=\lim_{x\to-\infty}\frac{1}{x}=0.$$

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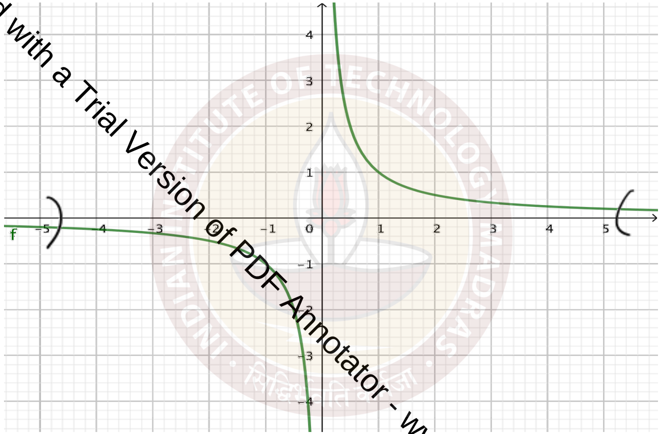

$$\lim_{x \to \infty} \frac{1}{x} = \lim_{x \to -\infty} \frac{1}{x} = 0.$$ Both $\lim_{x \to -0^-} \frac{1}{x}$ and $\lim_{x \to -0^+} \frac{1}{x}$ do not exist.

Some basic examples

1. $\lim_{x \to a} x^k; k \geq 0$

   $= a^k$

2. $\lim_{x \to a} x^x; k < 0, a \neq 0$

   $= a^k$

3. $\lim_{x \to a} e^x$

   $= e^a$

4. $\lim_{x \to a} log_e(x); a > 0$

   $= \log_e(a)$

5. $\lim_{x \to a} sin(x)$

   $= \sin(a)$

6. $\lim_{x \to a} tan(x); a \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

   $= \tan(a)$

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Examples :

$\lim_{x \to 2} \frac{x^2 - 5x + 6}{x - 2}$.

$\frac{2^2 - 5 \times 2 + 6}{2 - 2} = \frac{0}{0}$

$= \frac{(x-2)(x-3)}{(x-2)} = x - 3.$

$-1.$

$\lim_{x \to 0} \frac{1}{x}$ DNE

Cannot substitute

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unit circle

$\ldots \le \frac{\sin(x)}{x} \le 1.$

2. $\lim\limits_{x \to 0} \frac{1 + log_e(x)}{x} = 1.$

3. $\lim\limits_{x \to \infty} \frac{a + be^x}{c + de^x} = \frac{b}{d}.$

$= \lim\limits_{x \to \infty} \frac{ae^{-x} + b}{ce^{-x} + d}$ $\frac{a \times 0 + b}{c \times 0 + d} = b/d.$

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i.e. continuity means "the limit at a can be obtained by evaluating the function at a.

$$f(x) = \begin{cases} \dfrac{\sin(x)}{x} & x \neq 0 \\ 1 & x = 0 \end{cases}$$

Thank you