An overview of Morse Theory

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January 2, 2022

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1 Morse Lemma

Let M be a smooth manifold without boundary, and let $f: M \to \mathbb{R}$ be a smooth function. The critical points of f are those points $p \in M$ where $df_p = 0$. If $(U, x^1, \ldots, x^n)$ is a chart about p, then p is a critical point if $\partial f / \partial x^i = 0 \forall i$. This notion is of course chart independent. Now, suppose p is a critical point, then we may define the Hessian of f at p locally as the matrix $(\partial^2 f / \partial x^1 \partial x^j)_{i,j}$. We need to check that this definition is chart independent.
Let $v, w \in T_p M$ and let $\tilde{v}, \tilde{w}$ be local extensions of v, w respectively (such extensions exist for example by considering "constant" vector fields in a local chart). Then their Lie Bracket satisfies

$$0 = [\tilde{v}, \tilde{w}]_p f = \tilde{v}_p(\tilde{w} f) - \tilde{w}_p(\tilde{v} f).$$

It follows that the bilinear function $(v, w) \mapsto \tilde{v}_p(\tilde{w} f)$ is symmetric and independent of the local extensions $\tilde{v}, \tilde{w}$: it is independent of $\tilde{v}$ because $\tilde{w}_p(\tilde{v} f) = \tilde{v}_p(\tilde{w} f) = v(\tilde{w} f)$ and the right side is independent of $\tilde{v}$.
By a local calculation it is easy to see that the matrix corresponding to this bilinear form in the chart above is precisely the Hessian of f at p. We say that p is a non-degenerate critical point if p is a critical point and the Hessian H is non-degenerate, i.e., if $H(v, w) = 0 \forall w$, then $v = 0$.
In general, given a bilinear form H on a (real) vector space V, the index of H is the maximal dimension of subspaces on which H is negative definite and the nullity of H is the dimension of the nullspace of H, i.e., all those v for which $H(v, w) = 0 \forall w$. It is a theorem of linear algebra that there is a basis of V such that H has the matrix

$$\begin{bmatrix} -I_r & & \\ & I_s & \\ & & 0 \end{bmatrix}$$

where r is the index, $r + s$ is the rank of H.

Lemma 1.1. *(Morse lemma) Let p be a non-degenerate critical point of f. Then there is a local coordinate system $(U, y^1, \ldots, y^n)$ around p with $y^i(p) = 0 \forall i$ and*

$$f = f(p) - (y^1)^2 - \cdots - (y^k)^2 + (y^{k+1})^2 + \cdots + (y^n)^2$$

on U where k is the index of f at p.

Proof sketch. In the vector space case we use an argument similar to the Gram-Schmidt procedure. Here we do the same, but shrink the neighbourhood so all the steps (particularly those of divisions and square roots) are well defined. To start, one uses Hadamard lemma (this is the version in [1]) which says the following : if f is a smooth function on a convex neighbourhood V of 0 in $\mathbb{R}^n$, with $f(0) = 0$, then

$$f(x_1, \ldots, x_n) = \sum_{i=1}^{n} x_i g_i(x_1, \ldots, x_n)$$

for some smooth functions g_i with $g_i(0) = \frac{\partial f}{\partial x_i}(0)$.
We apply the Hadamard lemma twice(because p is critical, the derivatives vanish) to $f - f(p)$ to obtain a chart where

$$f(x_1, \ldots, x_n) = \sum_{i,j} x_i x_j h_{ij}(x_1, \ldots, x_n).$$

Now, we can assume $h_{ij} = h_{ji}$ by replacing h_{ij} with $\frac{1}{2}(h_{ij} + h_{ji})$. Proceed with diagonalization. k is well defined as it is the index of the Hessian at p. □

As a corollary, non-degenerate critical points of f are isolated. A function f is said to be Morse if all its critical points are non-degenerate. If M is compact, it follows that a Morse function has finitely many critical points (this is not true for non-Morse functions: consider the height function on a torus lying on the plane).

2 Pseudo-gradients

Let M be a manifold, $f: M \to \mathbb{R}$ a Morse function. A pseudo-gradient for f is a vector field X on M such that

- $df(X) \leq 0$ everywhere with equality only at critical points of f.
- In a Morse chart around critical points, X agrees with the negative of the usual Euclidean gradient of f.

Given a critical point p, by Morse lemma, there is a chart $(U, x^1, \ldots, x^n)$ such that on $U x^i(p) = 0$ and

$$f = f(p) - (x^1)^2 - \cdots - (x^k)^2 + (x^{k+1})^2 + \cdots + (x^n)^2$$

where k is the index of f at p. Denote by V_-, V_+ to be the span of $\{x^1, \ldots, x^k\}, \{x^{k+1}, \ldots, x^n\}$ respectively intersected with U. Note that we only require X to agree with $-\mathrm{grad} f$ on some Morse chart around critical points.
If M has a Riemannian metric, then it is known that any function f has gradient, which is given by "dualising" the differential of f, see [3]. However, on a general manifold, we do not have a canonical choice of a Riemannian metric, so it is not a priori guaranteed that a pseudo gradient exists. However, just as every manifold can be given a Riemannian metric, we can construct a pseudo-gradient for any given f. Both the results involve a local construction patched up using a partition of unity.
Intuitively, the gradient is the direction of maximum increase of f, so a trajectory along a pseudo-gradient corresponds to travelling along a line where f decreases. The trajectories at critical points are constants, and under the assumption that M is compact, since every flow is complete, we expect that the trajectories start and end at critical points, because f cannot keep decreasing. We shall prove this soon.

Theorem 2.1. *(Existence of pseudo-gradient) Given a compact manifold M and a morse function f, a pseudo-gradient of f exists.*

Proof. Observe that the morse lemma implies that the critical points of f are isolated. Since the set of critical points is a closed set, by compactness, it is finite. Let $c_1, \ldots, c_r$ be the critical points. Around each, take a Morse chart $(U_i, \phi_1)_{1 \leq i \leq r}$. Next, include some more charts $(U_i, \phi_i)_{r < i \leq N}$ covering M. We may shrink these other charts so that they contain no critical points (there are finitely many critical points). On each U_i, we have the function $f \circ \phi_i^{-1}$, and it's corresponding (pullback of) negative gradient X_i on $\phi_i^{-1}(U_i)$. Let ψ_i be a partition of unity subordinate to the given cover, and extend X_i to $\tilde{X}_i$ by using ψ_i and set $X = \sum \tilde{X}_i$. We claim that X is a pseudo-gradient for f.
By construction, each critical point is in a single U_i, so on a smaller Morse chart X agrees with the negative gradient. Next, at each point $x \in M$,

$$(df)_m(X) = \sum \phi_i(m) df_m(X_i(m)) \leq 0$$

and if it is equal to zero, then each term must be zero, which means that either m is a critical point or $\phi_i(m) = 0 \forall i$ which is impossible. Therefore, equality holds only at the critical points. Thus, X is a pseudo-gradient for f. □

Given a smooth vector field, we have the corresponding flows $\phi_t(x)$ for each $x \in M$. Since M is compact, $\phi_t(x)$ is defined for all $t \in \mathbb{R}$. Given $a \in M$, define

$$W^s(a) = \{x \in M | \lim_{t \to \infty} \phi_t(x) = a\},$$

$$W^u(a) = \{x \in M | \lim_{t \to -\infty} \phi_t(x) = a\}.$$

$W^s(a)$ is called the stable manifold of a and $W^u(a)$ the unstable manifold. The flow for X is a path where f is decreasing, so if a point is in the stable manifold of a, then $\phi^t(x)$ is a path of decreasing f from x to a.

2.1 The standard ball

Suppose p is a critical point of a morse function f on a compact manifold M. Let $(U, x^1, \ldots, x^n)$ be a Morse chart around p where $f(x) = f(p) - (x^1)^2 - \cdots - (x^k)^2 + (x^{k+1})^2 + \cdots + (x^n)^2$ with k, the index of f at p. Denote by Q the quadratic form $-\|x_-\|^2 + \|x_+\|^2$ where x_- is the vector $(x^1, \ldots, x^k)$ and x_+ is $(x^{k+1}, \ldots, x^n)$. The standard balls around p is defined as

$$U(\epsilon, \eta) = \{x : |Q(x)| \leq \epsilon, \|x_-\|^2 \|x_+\|^2 \leq \eta(\epsilon + \eta)\}.$$

The idea is as follows. Given a pseudo-gradient for f, we want to analyse its flow. Since M is compact, the flows are complete, i.e., they exist for all time. However, by definition/construction, the value of f decreases along the flow lines, but because M is compact we cannot expect f to decrease forever. So, intuitively, we expect the flow lines to start from and end at critical points.
To prove our intuition, we would like to know how the flow behaves near critical points. To this end, on a neighbourhood as above, the pseudo-gradient X matches with the actual negative gradient (shrink U if necessary), so in coordinates we have

$$X = \sum_{i=1}^{k} 2x^i \partial_i + \sum_{i=k+1}^{n} -2x^i \partial_i$$

Then by the uniqueness of solutions to ODEs, the flow starting from $(c^1, \ldots, c^n) \in U$ is given by

$$\gamma : t \mapsto (c^1 e^{2t}, \ldots, c^k e^{2t}, c^{k+1} e^{-2t}, \ldots, c^n e^{-2t}) \tag{1}$$

as long as it lies in U. This is a curve of the form $\|x_-\|^2 \|x_+\|^2 = \text{const}$.
On U, consider the continuous map $\theta \colon x \mapsto (\|x_-\|, \|x_+\|) \in \mathbb{R}^2$. In $\mathbb{R}^2$ we consider the region $S = |u^2 - v^2| \leq \epsilon, u^2 v^2 \leq \eta(\epsilon + \eta)$ so thatl $U(\epsilon, \eta)$ is $\theta^{-1}(S)$. Because θ is continuous, the boundary of $U(\epsilon, \eta)$ is the inverse of the boundary of S, and this boundary divides U into 2 regions. It is easy to verify that the boundary is given by

$$\partial_\pm U = \{x : Q(x) = \pm\epsilon, \|x_\mp\|^2 = \eta\}$$
$$\partial_0 U = \{x : \|x_-\|^2 \|x_+\|^2 = \eta(\epsilon + \eta)\}$$

Our vector field is $(2x_-, -2x_+)$ and the flow through $c = (c_-, c_+)$ is given by $(e^{2t} c_-, e^{-2t} c_+)$. Under θ, the flow is mapped to $(e^{2t}\|c_-\|, e^{-2t}\|c_+\|)$. In dimension 2, the flows of $(u, -v)$ are precisely the level sets of $(u, v) \mapsto u^2 v^2$ (this is not true in higher dimensions), so the quadratic form $-u^2 + v^2$ strictly decreases along the level sets of $u^2 v^2$.

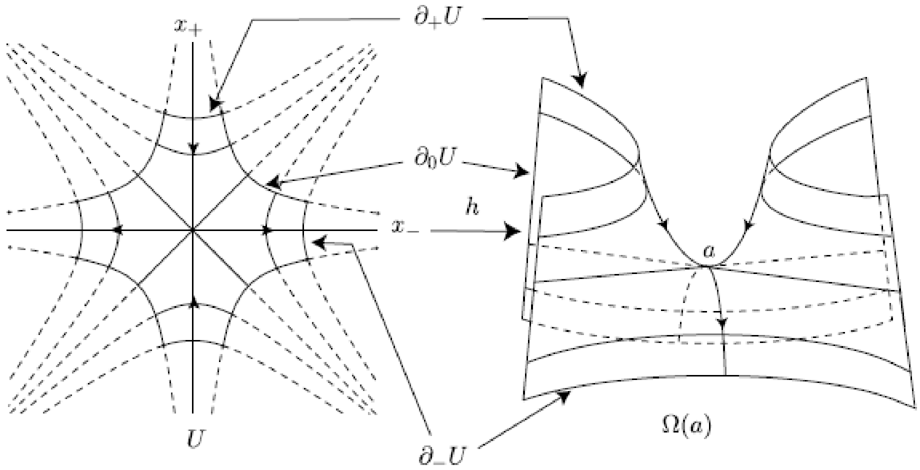

Figure 2.1: The standard ball, h is the diffomorphism from the Morse chart to the neighbourhood $\Omega(a)$ around critical point a. Figure taken from [2].

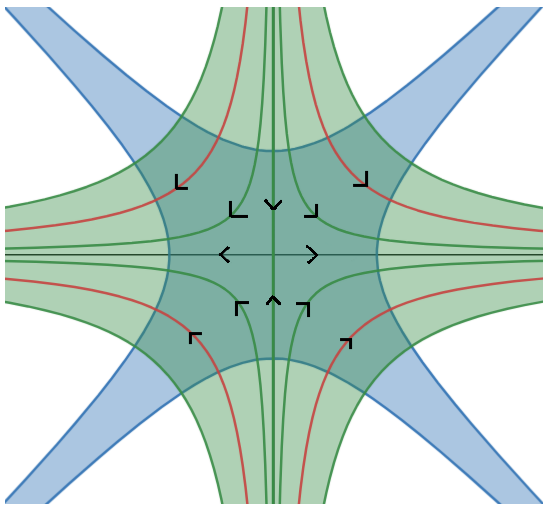

Figure 2.2: Here we see the regions in the standard ball and the flow lines are marked with arrows. Figure made using Desmos: [10].

With all this background, suppose γ is any arbitrary flow to X that enters $U(\eta,\epsilon)$ (for small enough η,ϵ this standard ball is contained inside a usual Euclidean ball). Then we look at the image of this curve under θ. By looking at the flows of $(u,-v)$ in $\mathbb{R}^2$, we conclude that either γ must reach the origin in infinite time, or it reaches the $Q=-\epsilon$ region. And any point on the $Q=-\epsilon$ region cannot enter $U(\eta,\epsilon)$. From this analysis we conclude that if γ enters and leaves any $U(\eta,\epsilon)$ then it cannot re-enter the same region. From here we get the following theorem.

Theorem 2.2. *Let M be compact and γ a trajectory for X, then there exist critical points c,d of f such that* $\lim_{t\to\infty}\gamma(t)=c, \lim_{t\to-\infty}\gamma(t)=d$.

Proof. We consider the limit towards $t=\infty$, the other end is similar. There are finitely many critical points, say $c_1,\dots c_r$, and suppose none of these are a limit point. Then around each c_i we get a $U_i(\eta_i,\epsilon_i)$ which γ doesn't enter or enters and leaves (by the analysis above, it cannot oscillate within the bounds of the standard ball). Let Ω be the union of the interiors of these standard balls. Observe that $df(X)$ is a continuous negative function which vanishes precisely at the c_i, therefore outside Ω, there is a positive number ϵ_0 such that $df(X)\leq -\epsilon_0$.
Since f decreases along γ, and, by assumption, there is a time t_0 such that for $t\geq t_0, \gamma\cap\Omega=\emptyset$, we conclude that

$$f(\gamma(t)) = f(\gamma(t_0)) + \int_{t_0}^{t} df_{\gamma(s)}(X(\gamma(s))ds \leq f(\gamma(t_0)) - \epsilon_0(t-t_0) \to -\infty$$

which is impossible. Therefore, γ must tend towards critical points as $t\to\pm\infty$. □

2.2 Stable and unstable manifolds

Since M is compact, X generates a one-parameter family of diffeomorphisms $\phi_t\colon M\to M$ corresponding to the flows along X. We defined the sets

$$W^s(a) = \{x\in M | \lim_{t\to\infty}\phi_t(x)=a\},$$

$$W^u(a) = \{x\in M | \lim_{t\to-\infty}\phi_t(x)=a\}.$$

In this subsection we prove that these are actually submanifolds of M. We will prove that the stable manifold is a manifold, the unstable part is similar. On a chart such as U as above, the points that terminate at p are the points corresponding $x_-=0$ and these lie on a sphere S^{n-k-1} of the form $\|x_+\|^2=\epsilon, x_-=0$ in U (for a small ϵ). Consider

$$\begin{aligned}\Phi\colon S^{n-k-1}\times\mathbb{R} &\to M\\ (x,t) &\mapsto \phi_t(0,x)\end{aligned}$$

The image of Φ is precisely $W^s(p)\setminus\{p\}$ because any (non constant) flow line terminating at p has to pass through the boundary of $U(\epsilon,\eta)$ for some η and since it terminates at p, it must actually pass through S^{n-k-1}, which means, by the properties of a one-parameter family of diffeomorphisms, it is in the image of Φ.
We claim that Φ is an embedding. If $\Phi(z_1,t_1)=\Phi(z_2,t_2)$, then $\phi_{t_1-t_2}(0,z_1)=(0,z_2)$ but this is impossible unless $(z_1,t_1)=(z_2,t_2)$ as z_1,z_2 are on the sphere and the flow is radially inward. Therefore Φ is injective.
Next, the pseudo-gradient is not tangential to the sphere since the sphere is a level set of f. Thus, the push forward of the tangent space from a point $(x,t)\in S^{n-k-1}\times\mathbb{R}$ is injective because ϕ_t is a diffeomorphism (so the map is injective on tangent space of S^{n-k-1}) and X is not tangential at $t=0$, so it continues to be not in the span of the push forward of the tangent space at any time (because ϕ_t is a diffeomorphism).
More precisely, let $v_1,\dots,v_{n-k-1}$ be a basis for the tangent space of S^{n-k-1} at x. Under Φ, the push forward of these vectors is $(\phi_t)_*v_1,\dots,(\phi_t)_*v_{n-k-1}$ because it's just the restriction of ϕ_t to S^{n-k-1}. Furthermore,

$$\Phi_*(\partial/\partial t) = X(\Phi(x,t)) = (\phi_t)_*(X|_{(0,x)})$$

where $(0,x)$ is the point in U. This equality follows from the fact that ϕ_t is a flow of X. Since $X|_{(0,x)}$ is not tangential to the sphere (because the sphere is a level set of f), and ϕ_t is a diffeomorphism, it follows that Φ_* is injective.

In order to prove that it is an embedding we need to show that it is homeomorphic to its image in M (with subspace topology). We will show that around every $(q,t) \in S^{n-k-1} \times \mathbb{R}$ there is a neighbourhood $V \times (t-\kappa, t+\kappa), \kappa > 0$ and a neighbourhood W of $\phi_t(q)$ in M such that $\Phi \colon V \times (t-\kappa, t+\kappa) \to W \cap W^s(p)$ is a homeomorphism. It will then follow that Φ is an embedding because it is injective. Suppose this is true for $t = 0$, i.e., for a point on $S^{n-k-1} \subset W^s(p)$. Given $T \neq 0$, let l_T denote the translation of $\mathbb{R}$ by T. We have the commutative diagram

$$\begin{tikzcd} S^{n-k-1} \times \mathbb{R} \arrow[r, "\Phi"] \arrow[d, "1 \times l_T"'] & M \arrow[d, "\phi_T"] \\ S^{n-k-1} \times \mathbb{R} \arrow[r, "\Phi"] & M \end{tikzcd}$$

Restricting this to $V \times (-\kappa, \kappa)$ gives the following commutative diagram

$$\begin{tikzcd} V \times (-\kappa, \kappa) \arrow[r, "\Phi"] \arrow[d, "1 \times l_T"'] & W \cap W^s(p) \arrow[d, "\phi_T"] \\ V \times (T-\kappa, T+\kappa) \arrow[r, "\Phi"] & \phi_T(W) \cap W^s(p) \end{tikzcd}$$

Because ϕ_T is a diffeomorphism and $W^s(p)$ is invariant under ϕ_T,

$$\phi_T(W \cap W^s(p)) = \phi_T(W) \cap \phi_T(W^s(p)) = \phi_T(W) \cap W^s(p).$$

In this diagram, all maps except the bottom one are diffeomorphisms, therefore the bottom map is a diffeomorphism.

Now, fix a $q \in S^{n-k-1} \subset W^s(p)$ and let U be the Morse chart as above. On U, recall that the flow is given by 1

$$\gamma : t \mapsto (c^1 e^{2t}, \ldots, c^k e^{2t}, c^{k+1} e^{-2t}, \ldots, c^n e^{-2t})$$

So, $W^s(p) \cap U = \{(c_-, c_+) \in U | c_- = 0\}$. Let $q = (0, \ldots, 0, q^{k+1}, \ldots, q^n)$, $\|q\|^2 = \|q_+\|^2 = \epsilon$. Let $W \not\ni 0$ be an open ball around q in M such that $q \in W \subset U$ and observe that $W \cap W^s(p) = \{(c_-, c_+) \in W | c_- = 0\}$. Because Φ is continuous and $\Phi(q,0) \in W \cap W^s(p)$, we can get a neighbourhood $V \times (-\kappa, \kappa)$ of $(q,0)$ such that the restriction

$$g = \Phi \colon V \times (-\kappa, \kappa) \to W \cap W^s(p)$$

is defined. Next, on W consider the map

$$\begin{aligned} h \colon W &\to S^{n-k-1} \times \mathbb{R} \\ (x_-, x_+) &\mapsto \left(\frac{\sqrt{\epsilon} x_+}{\|x_+\|}, \frac{1}{2} \ln \left(\frac{\|x_+\|}{\sqrt{\epsilon}} \right) \right) \end{aligned}$$

This is well defined and continuous by the conditions on W, so the restriction h to $W \cap W^s(p)$ is also continuous.

$$\begin{tikzcd} V \times (-\kappa, \kappa) \arrow[r, "g"] \arrow[d, hook] & W \cap W^s(p) \arrow[dl, "h"] \\ S^{n-k-1} \times \mathbb{R} & \end{tikzcd}$$

Whenever defined $g \circ h, h \circ g$ are identities on their respective domains. Therefore, if $\tilde{V} \subset V \times (-\kappa, \kappa)$ is open, then $h(g(\tilde{V})) = \tilde{V}$ and $g(\tilde{V}) = h^{-1}(\tilde{V})$. Therefore, g is an open map. It then follows that g is a homeomorphism onto its image, which is of the form $\tilde{W} \cap W^s(p)$ with $q \in \tilde{W}$ open in M.

So, $W^s(p) \setminus \{p\}$ is an embedded submanifold. Using the diffeomorphism $(0,1) \to \mathbb{R} : s \mapsto \ln(s/1-s)$, we get the embedding $\Psi \colon S^{n-k-1} \times (0,1) \to M$. In local coordinates

$$\Psi \colon (x_{k+1}, \dots, x_n, s) \mapsto (0, \dots, 0, x_{k+1}((1-s)/s)^2, \dots, x_n((1-s)/s)^2)$$

This map extends to $S^{n-k-1} \times (0,1]$ and from there we can quotient out $S^{n-k-1} \times \{1\}$ to get a the diffeomorphism $D^{n-k} \cong W^s(p)$. A similar analysis (or using $-f$) holds for $W^u(p)$, thus $W^s(p), W^u(p)$ are embedded submanifolds diffeomorphic to discs with

$$\dim W^u(p) = \operatorname{codim} W^s(p) = \operatorname{Ind}(p).$$

Soon we shall see the stable and unstable manifolds on a torus corresponding to a height function, see Figure 4.1.

3 Morse Theorems

Given a real valued function f on M, let $M^a = f^{-1}(-\infty, a]$.

Theorem 3.1. *(First Morse theorem) Let f be a smooth real valued function on a manifold M (not necessarily compact; without boundary). Let $a < b$ and suppose $f^{-1}[a,b]$ is compact and without critical points. Then M^a is diffeomorphic to and a deformation retract of M^b and furthermore, the inclusion $M^a \hookrightarrow M^b$ is a homotopy equivalence.*

Proof sketch. The idea is to push down along a pseudo-gradient of f. Let ρ be a smooth function satisfying

$$\rho(x) = \begin{cases} -\frac{1}{df_x(X)} & x \in f^{-1}[a,b] \\ 0 & \text{outside a relatively compact neighbourhood of } f^{-1}[a,b] \end{cases}$$

Such a ρ exists by using bump functions. Note that [1] uses a Riemannian metric on M to get hold of a gradient instead of using a pseudo-gradient as in [2]. Let $Y = \rho X$. Since Y has compact support, it's flow is defined for all time (see [1] for an explanation), so let ϕ_t denote the t-time flow of Y and these form a one-parameter family of diffeomorphisms of M.
For a fixed $q \in M$, it can be verified that the t-derivative of $f(\phi_t(q))$ is 1 as long as $\phi_t(q) \in f^{-1}[a,b]$. From here, it follows that $\phi_{b-a} \colon M \to M$ carries M^a diffeomorphically onto M^b because the derivative is 1, so f varies linearly (one also sees that ϕ_{a-b} is the inverse along which f decreases with unit speed). A deformation retract is given by the one parameter family $r_t \colon M^b \to M^a$

$$r_t(q) = \begin{cases} q & x \in f(q) \leq a \\ \phi_{t(a-f(q))}(q) & a \leq f(q) \leq b \end{cases}$$

Note that by the choice of ρ, points outside a neighbourhood of $f^{-1}[a,b]$ are stationary under ϕ_t, so these flows migrate points in M^a to M^b by "stretching" a neighbourhood of the level set of a. □

Remark. The compact-ness hypothesis on $f^{-1}[a,b]$ cannot be removed as seen in this example from [1] where M^a, M^b are not diffeomorphic. Essentially, the puncture obstructs the flow.

Corollary. *(Reeb's theorem) Let M be a compact manifold (without boundary) admitting a Morse function f with exactly 2 critical points, then M is homeomorphic to the sphere.*

Proof. Being compact M admits a maximum and minimum and these are critical points of index $n, 0$ respectively. Since connected components are closed, hence compact, we conclude that M is connected. By composing with a diffeomorphism, we may assume $f(M) = [0,1]$. By Morse lemma, for sufficiently small $\epsilon > 0$ the sets $f^{-1}[0,\epsilon], f^{-1}[1-\epsilon, 1]$ are discs of dimension n.
By Morse theorem, $M^{1-\epsilon}$ is diffeomorphic to M^ϵ, therefore, $M = M^{1-\epsilon} \cup f^{-1}[1-\epsilon, 1]$ is two discs glued along the boundary $f^{-1}(1-\epsilon)$. It is a standard result that this is homeomorphic to S^n. □

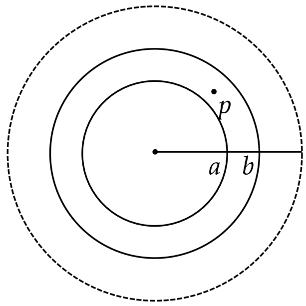

Figure 3.1: $f^{-1}[a,b]$ is not compact and M^a, M^b are not diffeomorphic, here f is the radius and p is a hole. M^a is a closed disc, while M^b is punctured.

Theorem 3.2. *(Second Morse theorem) Let p be a non-degenerate critical point with index k. Let $f(p) = c$ and suppose $f^{-1}[c-\epsilon, c+\epsilon]$ is compact and without any other critical point for some $\epsilon > 0$. Then for all sufficiently small ϵ, the set $M^{c+\epsilon}$ has the same homotopy type as $M^{c-\epsilon}$ with a k-cell attached.*

See [1] or [2] for a proof. Intuitively, imagine that f is a height function (as in the next section), then if p has index k, then it is as if the manifold "grows" in $n-k$ directions (where double derivative is positive) and along the other directions f has peaked. So, upon crossing p we are "closing" of these k directions by attaching a handle.

4 Examples and classification of one manifolds

4.1 Vertical Torus, a classic example

Let T denote a torus embedded in $\mathbb{R}^3$ with a parametrization given by

$$(u,v) \mapsto (r\cos u \cos v + R\cos v, r\cos u \sin v + R\sin v, r\sin u)$$

where $u, v \in [0, 2\pi]$ (closed interval to cover the torus), $R > r > 0$. We take a height function given by the projection onto the x-axis and an easy calculation shows that the critical points of this function are given when (u,v) is one of $(0,0), (\pi,0), (0,\pi), (\pi,\pi)$. One also finds that the indices are 2(max), 1, 1, 0(min) respectively.
Label the critical points A, B, C, D as in the figure. The stable and unstable manifolds are depicted in the figure below. The flow lines move from one critical point to another.

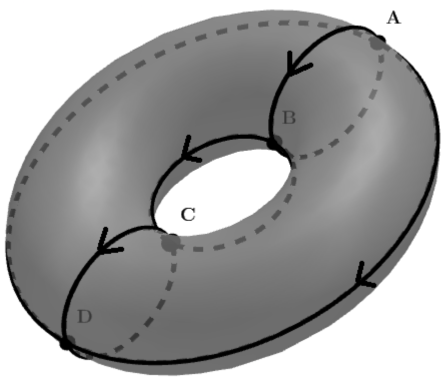

Figure 4.1: Flow lines on the torus viewed obliquely. Figure made using Geogebra [10].

At A, we have a 2-dimensional unstable manifold, consisting of flows going downwards. At B we have a one dimensional stable manifold consisting of flows originating from A and a one dimensional unstable manifold. Other points are similar. The exact relation and number of flows between critical points leads to Morse homology.
By Morse theorems, we get the following decomposition of the torus

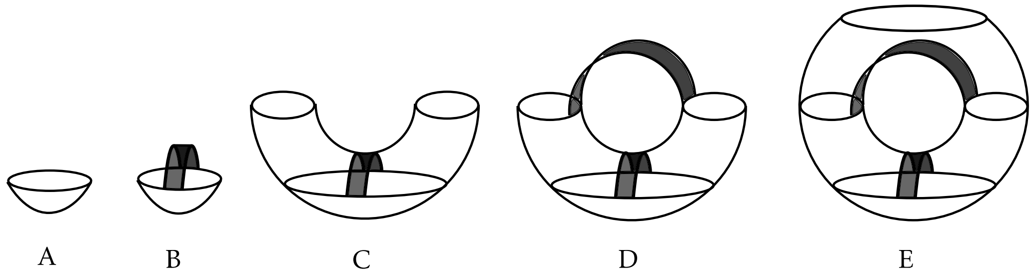

Figure 4.2: Decomposition of the torus

where A is a disc (homotopic to a point), and from A to B we add a 1 cell which results in a half-torus C. To C we add another 1-cell resulting in D which is homotopic to E. Finally to E we add a 2-cell and "close" the torus.

4.2 Sphere

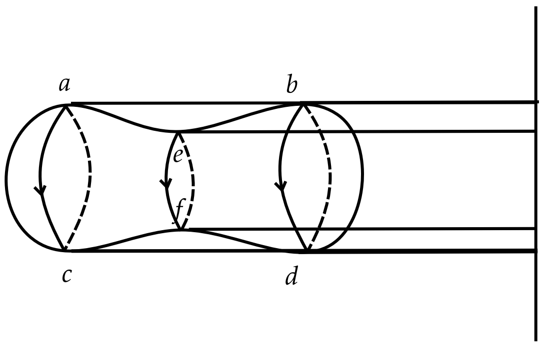

Figure 4.3: Height function on a peanut

The height function on the sphere is easily seen to be Morse with two critical points: one index 0 critical point and one index 2 critical point, with no index 1 critical point. The Morse theorems then give the usual cellular structure on the sphere. We also consider the height function on a peanut shaped object diffeomorphic to the sphere obtained by squishing the sphere along the equator.
Now there are 6 critical points, two each of index $0, 1, 2$. Note that the Morse theorem applies here as well.

4.3 Classification of one manifolds

Let V be a compact manifold with boundary and let X be a vector field defined on a neighbourhood of the boundary. We say X is *incoming* if given a coordinate chart $(U, x^1, \ldots, x^n)$ around some $p \in \partial V$ with the interior being $x^n > 0$ then in local coordinates, $X = \sum a^i \partial/\partial x^i$ with $a^n > 0$. Using partitions of unity, it is easy to see that such vector fields exist.

The idea is that if X is an incoming vector field defined on V, then we can define the integral flow corresponding to X. Indeed, at any boundary point, we have a smooth extension to a neighbourhood where there is a notion of integral flows (which depends smoothly on the initial conditions). Precisely because the vector field is incoming, the points on the boundary flow into the manifold (depending smoothly on intiial conditions). Once the point goes inside, we can use the usual results to continue the flow. Thus, the concept of integral flows, trajectories extend to manifolds with boundary. Moreover, when V is compact, just as in the open case, we can show that the vector fields are complete.
Given an incoming field X (defined on a neighbourhood of ∂V), we can construct a Morse function f on V with $df(X) < 0$ on a neighbourhood of the boundary as follows: There is a flow ϕ corresponding to X defined on $[0,\delta)$ for some $\delta > 0$. Define $f = \phi^s(x), s \in [0,\delta), x \in \partial V$. We then extend this arbitrarily to V and perturb it to obtain a Morse function.
We can then extend X to V to obtain a pseudo-gradient adapted to f. This extension agrees with X on a neighbourhood of ∂V. By observations above, the concept of stable and unstable manifolds extends to the case of manifolds with boundary. Importantly, f has no critical points on the boundary, so all points on the boundary lie on trajectories between critical points on the interior. The theory on manifolds with boundary is a relatively new and exciting area of interest.

Manifolds with boundary
While the original theory was developed in the context of closed manifolds, people have subsequently come up with theories on manifolds with boundary.
In [6], given a manifold with boundary V, a function is Morse if its critical points are in the interior and are non degenerate and its restriction to ∂V is also Morse.
[7] however deals with Morse functions on cobordisms as defined next.

Definition. *Let Σ_0, Σ_1 be compact oriented n-manifolds with nonempty boundaries M_0, M_1 respectively. A cobordism (Ω, Y) between $(\Sigma_0, M_0), (\Sigma_1, M_1)$ is a compact oriented $(n+1)$-manifold Ω with boundary $\partial\Omega = Y \cup \Sigma_0 \cup \Sigma_1$ where Y is nonempty, $\Sigma_0 \cap \Sigma_1 = \emptyset$, and $Y \cap \Sigma_0 = M_0, Y \cap \Sigma_1 = M_1$.*

Remark. Technically, Ω is a manifold with corners. We shall not go much into [7] for this very reason.
Remark. The usual definition : given closed manifolds M, N a cobordism between M, N is a quintuple (W, M, N, i, j) where W is an $n+1, i\colon M \to W, j\colon N \to W$ are embeddings such that $\partial W = i(M) \sqcup j(N)$. In this case, M, N are said to be *cobordant*.
And [7] defines Morse functions as

Definition. *Let $F\colon \Omega \to [0,1]$ be smooth, a critical point is called Morse if its Hessian is nondegenerate. F is called Morse on the cobordism (Ω, Y) if $F(\Sigma_0) = 0, F(\Sigma_1) = 1$ and F has only Morse critical points, the critical points are not in $\Sigma_0 \cup \Sigma_1$ and ∇F is everywhere tangent to Y.*

Note that we require a Riemannian metric on Ω to talk about ∇F.

Theorem 4.1. *Let V be a compact connected 1-manifold, then it is diffeomorphic to S^1 if $\partial V = \emptyset$ and to $[0,1]$ otherwise.*

Proof. The following is from [2]. Let X be an incoming vector field and f a Morse function for which X is an adapted function. The critical points of f are its local minima and maxima. Let $c_1, \ldots, c_k$ be the local minima of f with stable manifolds $W^s(c_i)$. The stable manifold is diffeomorphic to an open interval consisting of two trajectories ending at c_i and the point c_i itself (that there are two trajectories comes from the local picture). The closure A_i of the stable manifolds contains two other points (starting points of the trajectories) and

- either both are maxima (and they may coincide)
- or at least one of them is a boundary point

Being the closure of a connected set, A_i is connected. If the ends coincide, then the end is a maximum and A_i is diffeomorphic to a circle (and can be seen as the one point compactification of $W^s(c_i)$). And if the ends are different, then it is diffeomorphic to a closed interval.
If $x \in V$ is not a maximum, it's trajectory ends in a minimum and when x is a maximum, then it is in the end point of some A_i (more specifically, two A_i). In this way, the union of A_i is V.
If $k = 1$, then we are done. Else, A_1 must intersect some A_i and this intersection can contain only a local maxima. Furthermore, $\partial V \cap (A_1 \cap A_i) = \emptyset$ and the intersection can contain at most two points (because the stable manifolds of different minima cannot intersect).

- If the intersection has two points, then both are maxima and $A_1 \cup A_i$ is diffeomorphic to S^1 (seen as gluing two intervals along the boundary).
- If there is one point, then $A_1 \cup A_i$ is diffeomorphic to $[0,1]$. If the union is V we are done.

If the union is not V, then we continue adding A_is till we cover V. At each stage one of the above must hold, thus completing the proof. □

5 Smale condition

Let f be a Morse function on a manifold and X a pseudo-gradient adapted to f. For a critical point $p, W^s(p), W^u(p)$ are embdded submanifolds diffeomorphic to discs of dimension $n-\mathrm{Ind}p, \mathrm{Ind}p$ respectively. We say that (f,X) satisfies the *Smale condition* if all stable and unstable manifolds intersect transversally, i.e., $W^u(a) \pitchfork W^s(b)$ for all critical points a,b. Observe

- $W^u(p) \pitchfork W^s(p)$
- $W^s(a) \cap W^u(b) = \emptyset$ if $f(a) \leq f(b)$.

Under the Smale condition, $W^u(a) \cap W^s(b)$ is an embedded submanifold with

$$\dim(W^u(a) \cap W^s(b)) = \mathrm{Ind}(a) - \mathrm{Ind}(b).$$

Denote the intersection by $\mathcal{M}(a,b) = \{x | \lim_{t\to\infty} \phi_t(x) = b, \lim_{t\to-\infty} \phi_t(x) = a\}$.

Lemma 5.1. *$\mathbb{R}$ acts smoothly, freely and properly on $\mathcal{M}(a,b)$.*

Proof. The action is $(t,x) \mapsto \phi_t(x)$. This is smooth by the properties of solutions to ODEs. It is free because f strictly decreases along flow lines. Let $K \subset \mathcal{M}(a,b)$ be compact. Suppose $\forall n > 0 \exists x_n \in K$ such that $\phi_n(x_n) \in K$. Because K is compact we get a subsequence x_{n_m} converging to $x \in K$. There is a neighbourhood $U_b \ni b$ disjoint from K and some $N > 0$ such that $\phi_t(x) \in U \forall t > N$. By smooth dependence of solutions, we know that for sufficiently large $m, \phi_{n_m}(x_m) \in U$ which is a contradiction. Therefore $\{t : \phi_t(K) \cap K \neq \emptyset\}$ is bounded. This set is closed by the smoothness of the action, therefore it is compact. It follows (see [4]) that the action is proper. □

From the quotient manifold theorem [4], the quotient $\mathcal{L}(a,b)$ is a smooth manifold of dimension $\mathrm{Ind}(a) - \mathrm{Ind}(b) - 1$.

5.1 Examples

5.1.1 Height function on sphere

The height function on the standard sphere S^n is a Morse function with 2 critical points: one of index n and another of index 0. In this case it is easy to see that the Smale condition is satisfied, because the only nonempty intersection is $S^n \setminus \{N,S\}$ where N,S are the two critical points and we see that the stable and unstable manifolds intersect transversally.
The pinched sphere in Figure 4.3 does not satisfy the Smale condition. Specifically, observe that the stable and unstable manifolds of points e,f in Figure 4.3 are not transversal.

5.1.2 Height function on torus

The height function on the torus and the associated gradient does not satisfy the Smale condition. Indeed, with earlier notation, the $W^u(b) \not\pitchfork W^s(c)$ as the tangent spaces overlap. However, if we instead consider the tilted torus, we get a system satisfying the Smale condition. Specifically, consider the function

$$(r\cos u\cos v + R\cos v, r\cos u\sin v + R\sin v, r\sin u) \mapsto \cos\alpha(r\cos u\cos v + R\cos v) - \sin\alpha(r\sin u)$$

obtained by tilting the torus through some angle $\alpha \in (0,\pi)$ about the y-axis. The critical points and trajectories change slightly and we get the following

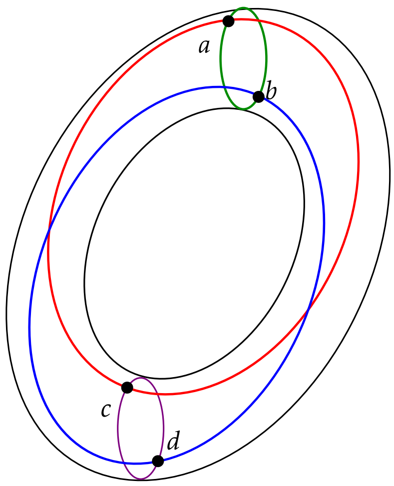

Figure 5.1: Tilted torus

5.2 Kupka-Smale Theorem

Next we consider the existence and genericness of Morse functions and pseudo-gradients satisfying the Smale condition.

Theorem 5.1. *Let V be a manifold with boundary and let f be a Morse function on V with distinct critical values. We fix Morse charts in the neighborhood of each critical point of f. Let Ω be the union of these charts and let X be a pseudo-gradient field on V that is transversal to the boundary. Then there exists a pseudo-gradient field X' that is close to X, equals X on Ω and for which we have $W^u_{X'}(a) \pitchfork W^s_{X'}(b)$ for all critical points a, b of f*

Here, by "close to" X we mean the following: given $\epsilon > 0$, a cover of V by charts U_i and for every compact $K_i \subset U_i$, there is an X' such that

$$\|X - X'\| < \epsilon$$

on K_i (Euclidean norm; i.e., flat metric on the charts).

Idea of proof. Let the critical points be $\{c_1, \ldots, c_k\}$ with $\alpha_i = f(c_i)$ satisfying $\alpha_1 > \cdots > \alpha_k$. Starting with the given vector field X, we perturb it locally in steps in such a way that the stable manifolds become transversal to all the unstable manifolds. Intuitively, if we imagine f to be the height function, then we perturb X starting from the "top" of the manifold to the bottom such that at each stage we "twist" the vector field, thereby twisting the stable manifold, to be transversal to the unstable manifold. To accomplish this, we look at the neighbourhood of a critical point and perturb X in an annulus around it in its Morse chart. The perturbation is done using bump functions. For further details the reader is referred to [2]. □

Remark. If a vector field is sufficiently close as in the sense above and agrees with a pseudo-gradient X on Ω, then X' is also a pseudo-gradient. Also, given a morse function f, since it has finitely many critical points, we can locally perturb f using bump functions to obtain another Morse function whose critical values are all distinct.
Obtain the torus by identifying the edges of a square, the gradient for the height function and it's perturbation are described in the next picture.
Image

6 Morse homology

6.1 Morse homology modulo 2

Let us say M is a compact manifold and (f, X) be a pair of Morse function and pseudo-gradient satisfying the Smale condition. In this case, for critical points a, b, we can define the space of trajectories, $\mathcal{M}(a,b)$ connecting them and take its quotient $\mathcal{L}(a,b)$ which is a manifold of dimensions $\operatorname{Ind}(a) - \operatorname{Ind}(b) - 1$.

Definition. *Let $Crit_k(f)$ denote the set of critical points of index k. For a ring R, define $C_k(f,R) = \{\sum_{c \in Crit_k(f)} a_c c | a_c \in R\}$.*

For $R = \mathbb{Z}/2\mathbb{Z}$, define a boundary map as follows: given $a \in \operatorname{Crit}_k(f)$

$$\partial_X(a) = \sum_{b \in \operatorname{Crit}_{k-1}(f)} n_X(a,b) b$$

where $n_X(a,b)$ is the number of trajectories modulo 2 from a to b along X, i.e., the cardinality of $\mathcal{L}(a,b)$ (modulo 2) which is a 0-dimensional manifold.
The differential extends uniquely to $C_k(f, \mathbb{Z}/2\mathbb{Z}), C_k(f, \mathbb{Z})$. We claim that this defines a complex on M. To check this, we need to verify that $\partial_X^2 = 0$ and that each $\mathcal{L}(a,b)$ where $\operatorname{Ind}(a) = \operatorname{Ind}(b) + 1$ is finite.

6.2 Integer coordinates

When it comes to homology with integer coefficients, we need to keep track of orientations.**More explanation???**

Definition. *Let E be a subspace of a vector space and let $\mathcal{B}_0$ be a basis of E. If $\mathcal{B}$ is a basis for any complement of E, then the relation*

$$\mathcal{B} \sim \mathcal{B}' \iff det_{(\mathcal{B}, \mathcal{B}_0)}(\mathcal{B}', \mathcal{B}_0) > 0$$

is an equivalence relation independent of the basis $\mathcal{B}_0$. A co-orientation on E is a choice of such an equivalence class.

Exercise 14 Chapter 3 [2] Suppose E, F are subspaces of a vector space, with E oriented, F co-oriented and $E \pitchfork F$. Then there is an induced orientation on $E \cap F$: Let $\mathcal{B}_0$ be a basis for $E \cap F$ and extend it to a basis $(\mathcal{B}_0, \mathcal{B}_1)$ of E and a basis $(\mathcal{B}_0, \mathcal{B}_1)$ of F. Now, $\mathcal{B}_1$ is a complement of F, so we can change the sign of one of its vectors to agree with the co-orientation on F. We can then change the sign of one of the vectors in $\mathcal{B}_0$ so that the new basis of E is positively oriented.
Note here that if the intersection is zero dimensional, then the orientation is just a sign associated to the point.

Let c be a critical point, then $W^s(c)$ is diffeomorphic to a disc, hence orientable. A choice of orientation on $W^s(c)$ is a choice of co-orientation on $W^u(c)$. We then have an induced orientation on $\mathcal{M}(a,b)$. Using the pseudo-gradient X, we get an orientation on the quotient $\mathcal{L}(a,b)$ (specifically, a basis is positively oriented, if together with the pseudo-gradient, we have a positively oriented basis fo $\mathcal{M}(a,b)$).
When $\mathrm{Ind}(a) = \mathrm{Ind}(b) + 1$, the space $\mathcal{L}(a,b)$ is zero dimensional and its orientation is a choice of signs to each of its points. Let $N_X(a,b) \in \mathbb{Z}$ denote the sum of these signs (supposing that we have proved it to be finite). Note that $n_X(a,b)$ is this number modulo 2.
We define the boundary map: given $a \in \mathrm{Crit}_k(f)$

$$\partial_X(a) = \sum_{b \in \mathrm{Crit}_{k-1}(f)} N_X(a,b)b.$$

flipping orientation does what?

6.3 Broken trajectories

$\mathcal{L}(a,b)$ denotes the set of trajectories from a to b. We extend this notion to include *broken trajectories* that pass through critical points:

$$\overline{\mathcal{L}}(a,b) = \cup_{c_i \in \mathrm{Crit}(f)} \mathcal{L}(a,c_1) \times \mathcal{L}(c_1,c_2) \times \cdots \times \mathcal{L}(c_{q-1},b).$$

Each $\mathcal{L}(a,b)$ is given the quotient topology and each term in the union is given the product topology. On the union we must define a topology that is induced by these product topologies and moreover, these broken trajectories form a compactification of $\mathcal{L}(a,b)$. **Some more motivation?**
We endow $\overline{\mathcal{L}}(a,b)$ with a topology by describing neighbourhoods around each of its points. Let $\lambda = (\lambda_1,\ldots,\lambda_q)$ be a broken trajectory. Let it connect the critical points $c_0 = a \to c_1 \to \cdots \to c_{q-1} \to b = c_q$. Each critical point admits a standard ball $\Omega(c_i)$. Each λ_i exits $\Omega(c_{i-1})$ and enters $\Omega(c_i)$. Recall that in the Morse charts, the stable and unstable manifolds are described as trajectories of certain level sets (those of the form $\|x_-\|, \|x_+\|$ being constant). Let U^-_{i-1} denote a neighbourhood of the exit point in its level set and likewise U^+_i be a neighbourhood of the entry point in its level set. Let $U^-(U^+)$ denote the collection $U^-_i(U^+_i)$ and we define a neighbourhood $\mathcal{W}(\lambda, U^-, U^+)$ of λ by declaring $\mu = (\mu_1,\ldots,\mu_k) \in \mathcal{W}(\lambda, U^-, U^+)$ if

- $\mu_j \in \mathcal{L}(c_{i_j}, c_{i_{j+1}})$ where $\{c_{i_1},\ldots,c_{i_k}\}$ is some subset of the critical points in λ.
- μ_j exits $\Omega(c_{i_j})$ through U^-_j and enters $\Omega(c_{i_{j+1}})$ through U^+_{j+1}

It is clear that these sets form a basis for a topology on $\overline{\mathcal{L}}(a,b)$. Here $k \le q$, i.e., μ cannot pass through more critical points than λ. If $\lambda \in \mathcal{L}(a,b)$, then its neighbourhood consists of trajectories that leave $\Omega(c_0)$ and enter $\Omega(c_q)$ sufficiently close to λ. Specifically, let's say λ leaves $\Omega(c_0)$ at some x_0 and enters $\Omega(c_q)$ at some y_0 and let U, V be the corresponding neighbourhoods in their level sets.
The trajectories we are interested in are those of points in U that enter $\Omega(c_q)$ through V. Because the stable and unstable manifolds are embeddings of discs, sets of the form $\phi_t(U)$ for U open in the level set of the form $\|x_-\| = \epsilon$ in a Morse chart are open in the subspace topology. Therefore, the collection of paths leaving U and entering V is open in $\mathcal{M}(a,b)$ and hence open in the quotient $\mathcal{L}(a,b)$. Conversely, because of the way $\mathbb{R}$ acts on $\mathcal{M}(a,b)$, the open sets in the quotient $\mathcal{L}(a,b)$ are exactly of this form, i.e., consisting of trajectories leaving a specific open set and entering another. In otherwords, the topology described above agrees with the quotient topology on $\mathcal{L}(a,b)$, i.e., there is a homeomorphic image of the quotient space of trajectories in the space of broken trajectories. It is in this sense that $\overline{\mathcal{L}}(a,b)$ is a compactification of $\mathcal{L}(a,b)$.

6.4 Compactness

In this subsection we show that $\overline{\mathcal{L}}(a,b)$ is compact. Note that the space is second countable because V is second countable, so compactness can be proved using sequential compactness.

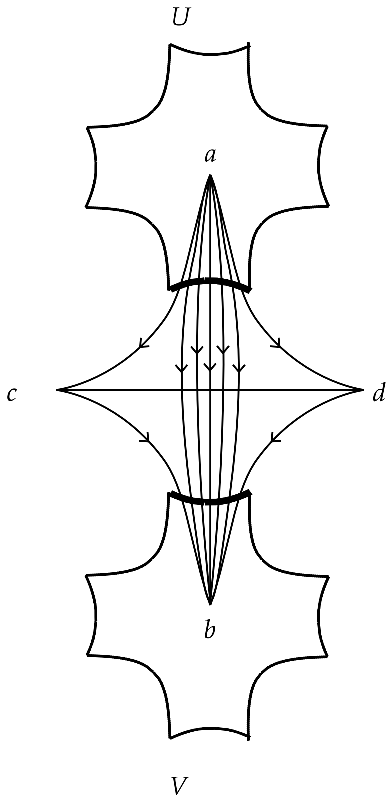

Figure 6.1: In the adjacent figure, U, V are the standard balls around a, b. The set of broken trajectories include paths going through c, d as well and is ultimately the line joining c, d. The set of trajectories between a, b is the open interval between c, d. The sets U^-, V^+ are marked in bold. We see that $\overline{\mathcal{L}}(a,b)$ includes $\mathcal{L}(a,b)$ together with c, d.

Theorem 6.1. *$\overline{\mathcal{L}}(a,b)$ is compact.*

Proof sketch. Let l_n be a sequence in $\overline{\mathcal{L}}(a,b)$. We first assume that $l_n \in \mathcal{L}(a,b)$. Each l_n exits $\Omega(a)$ at some l_n^- and enters $\Omega(b)$ at l_n^+. The l_n^- lie on $W^u(a) \cap \Omega(a)$, i.e., on a sphere which is compact. By going to a subsequence we may assume that $\lim l_n^- = a^-$. Let γ denote the trajectory of a^- and let $c_1 = \lim_{t\to\infty} \gamma(t)$ and suppose γ enters $\Omega(c_1)$ through some d^+.
By smooth dependence on initial conditions, for sufficiently large n, each l_n must enter $\Omega(c_1)$ at some d_n^+. By the consequence of the next lemma, we have $\lim d_n = d^+$. If $c_1 = b$, then we have $\lim l_n = \gamma$ in the space of broken trajectories, else each l_n must exit $\Omega(c_1)$ through some d_n^-. As before, we obtain a further subsequence such that d_n^- converges to some d^-
If d^- is not on the unstable manifold of c_1, then it is on the trajectory of some d^* with $f(d^*) = f(d_n^+)$ which is moreover not in $W^s(c_1)$. Again by the following lemma, we get $\lim d_n^- = d^*$ which gives $d^* = d^+$ but $d^+ \in W^s(c_1)$ giving us a contradiction.
So, from the given sequence, we can, in steps, obtain a subsequence converging to some broken trajectory; the process terminates because there are finitely many critical points. In general, given a sequence $l_n \in \overline{\mathcal{L}}(a,b)$, we can obtain a subsequence l_n such that

$$l_n = (l_n^1, \ldots, l_n^q) \in \mathcal{L}(a, c_1) \times \cdots \times \mathcal{L}(c_{q-1}, b)$$

for some critical points $c_1, \ldots, c_{q-1}$ (because there are finitely many critical points). We then apply the arguments above to l_n^1 and then to l_n^2 and so on, at each stage the sequence of trajectories converges to a broken trajectory. $\square$

Lemma 6.1. *Let $x \in V \setminus Crit(f)$ and let x_n be a sequence converging to x. Let y_n, y be points in the trajectories of x_n, x respectively and moreover suppose $f(y_n) = f(y)$. Then $\lim y_n = y$.*

Idea of proof. The idea involves in normalising the pseudo-gradient outside a neighbourhood of $\mathrm{Crit}(f)$ so that f decreases at unit speed and using the smoothness of the flow, see [2]. $\square$

Corollary. *$\mathcal{L}(a,b)$ is finite when $Ind(a) - Ind(b) = 1$.*

So, the chain maps are well defined. Next we need to prove that $\partial_X^2 = 0$. This requires the following theorem.

Theorem 6.2. *If $Ind(a) = Ind(b) + 2$, then $\overline{\mathcal{L}}(a,b)$ is a compact 1-manifold with boundary.*

The proof requires the following result from [2].

Theorem 6.3. *Let V be a compact manifold and $f : V \to \mathbb{R}$ a Morse function and let X be a pseudo-gradient adapted to f and satisfying the Smale condition. Let a, c, b be 3 critical points of indiced $k-1, k$ and $k+1$ respectively. Let $\lambda_1 \in \mathcal{L}(a,c)$ and $\lambda_2 \in \mathcal{L}(c,b)$. Then there exists a continuous embedding ψ from an interval $[0,\delta)$ to a neighbourhood of $(\lambda_1,\lambda_2) \in \overline{\mathcal{L}}(a,b)$ that is differentiable on $(0,\delta)$ and satisfies*

$$\left\{ \psi(0) = (\lambda_1,\lambda_2) \in \overline{\mathcal{L}}(a,b) \psi(s) \in \mathcal{L}(a,b) fors \neq 0 \right.$$

Moreover, if l_n is a sequence in $\mathcal{L}(a,b)$ converging to (λ_1,λ_2) then l_n is contained in the image of ψ for sufficiently large n.

The proofs are fairly involved and we refer the reader to [2]. Now observe that for a critical point $a \in \mathrm{Crit}_{k+1}(f)$, the coefficient of $b \in \mathrm{Crit}_{k-1}(f)$ in $\partial_X^2(b)$ is

$$\sum_{c \in \mathrm{Crit}_k(f)} N_X(a,c) N_X(c,b).$$

From Theorem 6.3 we see that for a, b as above,

$$\overline{\mathcal{L}}(a,b) = \mathcal{L}(a,b) \sqcup \partial \overline{\mathcal{L}}(a,b)$$

and

$$\partial \overline{\mathcal{L}}(a,b) = \cup_{c \in \mathrm{Crit}_k(f)} \mathcal{L}(a,c) \times \mathcal{L}(c,b).$$

So, the coefficient above is the sum of signed boundary points of $\overline{\mathcal{L}}(a,b)$. Because it is a compact 1-manifold with boundary, each component has 2 boundary points and these two have opposite orientations. Therefore, the sums cancel out (in $\mathbb{Z}$ and $\mathbb{Z}/2\mathbb{Z}$ coefficients) and we have a well defined complex and Morse homology is well defined.

6.5 Independence and Singular homology

So far we have described the Morse homology for a given Morse-Smale system (X, f) on a compact manifold M. It is now natural to ask if this is independent of (X, f) and how does Morse homology relate to Singular homology. And the answer is that the Morse homology is independent of (X, f) and is in fact the same as the singular homology. The first result is based on deforming one Morse function into another.

Theorem 6.4. *Let V be a compact manifold. Let $f_0, f_1 : V \to \mathbb{R}$ be two Morse functions and let X_0, X_1 be pseudo-gradients associated two f_0, f_1 satisfying the Smale condition. Then there exists a morphism of complexes*

$$\Phi_* : (C_*(f_0), \partial_{X_0}) \to (C_*(f_1), \partial_{X_1})$$

inducing an isomorphism in the homology.

Idea of proof. The proof is somewhat categorical and based on interpolations between Morse functions. The interpolations are chosen in a clever manner, specifically we choose them to be as follows: given Morse functions f_0, f_1, we look at smooth functions

$$\begin{aligned} F : V \times [0,1] &\to \mathbb{R} \\ (x,s) &\mapsto F_s(x) = F(x,s) \end{aligned}$$

such that

$$\begin{cases} F_s = f_0 \text{ for } s \in [0,1/3] \\ F_s = f_1 \text{ for } s \in [2/3,1] \end{cases}$$

From such a function, we deduce a morphism of complexes

$$\Phi^F : (C_*(f_0), \partial_{X_0}) \to (C_*(f_1), \partial_{X_1})$$

such that when $(f_1, X_1) = (f_0, X_0)$ and $I_s(x) = f_0(x) \forall (x,s) \in V \times [0,1]$, then $\Phi^I = \mathrm{Id}$.
And finally, given f_2 is another Morse function with adapted gradient X_2 satisfying the Smale condition, if G is an interpolation between f_1, f_2 stationary on $[0,1/3] \cup [2/3,1]$ and H is the composition of F, G in the obvious manner, then the morphisms $\Phi^G \circ \Phi^F$ and Φ^H coincide.
Once alll these properties are satisfied, then if $G_s = F_{1-s}$, we see that Φ^F, Φ^G are inverses of each other and therefore induce isomorphisms at the level of homology.
The full proof can be found in [2]. □

Recall that the Morse theorems give a cellular decomposition of any compact manifold. It turns out that the Morse homology is canonically isomorphic to cellular homology (and therefore the singular homology). This fact in turn proves that the Morse homology is independent of the Morse function and the adapted pseudo-gradient. Moreover, it is independent of the smooth structure on the manifold. Essentially the proof involves in obtaining a cellular decomposition from the Morse-Smale system (with the unstable manifolds being the cells) and proving that the complex so obtained is isomorphic to the Morse complex. Further details can be found in [2].

6.6 Examples of Morse homology

6.6.1 Sphere

For the standard sphere S^n with the height function and the usual gradient, we have 1 critical point each of index $0, n$ and no other critical points. So, the homology is easy to calculate and for $R = \mathbb{Z}, \mathbb{Z}/2\mathbb{Z}$ we have

$$H_k(h, R) = \begin{cases} R & k = 0, n \\ 0 & \text{otherwise} \end{cases}$$

6.6.2 Tilted torus

Here we look at the height function on a tilted torus, essentially we embed the torus in $\mathbb{R}^3$ as before, and before projecting onto the x-axis, we rotate the space about the y-axis. Specfically, we consider the function

$$(r\cos u \cos v + R\cos v, r\cos u \sin v + R\sin v, r\sin u) \mapsto \cos\alpha(r\cos u\cos v + R\cos v) - \sin\alpha(r\sin u)$$

for some $\alpha \in (0, \pi/2)$ say. The critical points and the trajectories change slightly and we get the following picture (where the arrows indicate orientation):
Now we compute the Morse homology. We have 1 index 2 critical point (a), 2 index 1 critical points (b, c) and 1 index 0 critical point (d). We also note that over $\mathbb{Z}$ the boundary operator is

$$\begin{aligned} \partial a &= b - b + c - c \\ \partial b &= d - d \\ \partial c &= d - d \end{aligned}$$

So, the Morse complex becomes

$$\dots 0 \to \mathbb{Z} \to \mathbb{Z}^2 \to \mathbb{Z}$$

over $\mathbb{Z}/2\mathbb{Z}$ and

$$\dots 0 \to \mathbb{Z}/2\mathbb{Z} \to (\mathbb{Z}/2\mathbb{Z})^2 \to \mathbb{Z}/2\mathbb{Z}$$

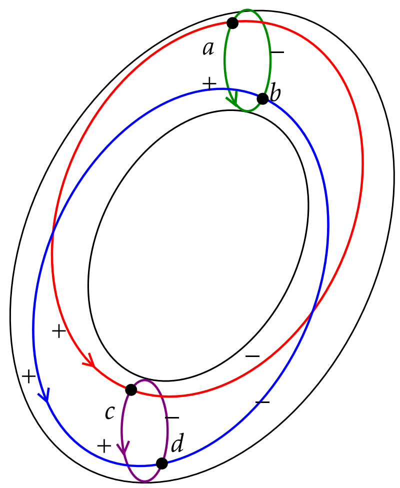

Figure 6.2: Signed trajectories on a tilted torus

with trivial boundary map in both cases and the Morse homology for the height function h (with usual gradient) is

$$H_k(h,\mathbb{Z}) = \begin{cases} \mathbb{Z} & k = 0,2 \\ \mathbb{Z}^2 & k = 1 \\ 0 & k > 2 \end{cases}$$

$$H_k(h,\mathbb{Z}/2\mathbb{Z}) = \begin{cases} \mathbb{Z}/2\mathbb{Z} & k = 0,2 \\ (\mathbb{Z}/2\mathbb{Z})^2 & k = 1 \\ 0 & k > 2 \end{cases}$$

6.6.3 Projective plane

We see $\mathbb{R}P^n$ as a quotient of S^n, with coordinates $[x_0 : \ldots : x_n]$. Consider the function $\sum a_i x_i^2$ on $\mathbb{R}P^n$, with $a_0 < a_1 < \cdots < a_n$. In the hemisphere $x_i > 0$ say, an easy computation shows that $c_i = [0 :: \ldots : 1 : \ldots 0]$ with 1 in the ith coordinate is a critical point. Moreover, by the choice of a_i, c_i has index $n - i$**??**

With coordinates $[x : y : z]$ on $\mathbb{R}P^2$, we look at $f = ax^2 + by^2 + cz^2, a < b < c$. This has 3 critical points: a maximum u (index 2), a saddle point v (index 1) and a minimum w (index 0). We get the following trajectories

There are 2 trajectories from u to v and both are positively oriented. There are 2 trajectories from v to w, but with opposite orientations (similar to the case of the tilted torus). So, the integral Morse complex is

$$\ldots 0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z}$$

and integral Morse homology becomes

$$H_k(f,\mathbb{Z}) = \begin{cases} \mathbb{Z} & k = 0 \\ \mathbb{Z}/2\mathbb{Z} & k = 1 \\ 0 & k \geq 2 \end{cases}$$

The mod 2 complex is

$$\ldots 0 \to \mathbb{Z}/2\mathbb{Z} \xrightarrow{0} \mathbb{Z}/2\mathbb{Z} \xrightarrow{0} \mathbb{Z}/2\mathbb{Z}$$

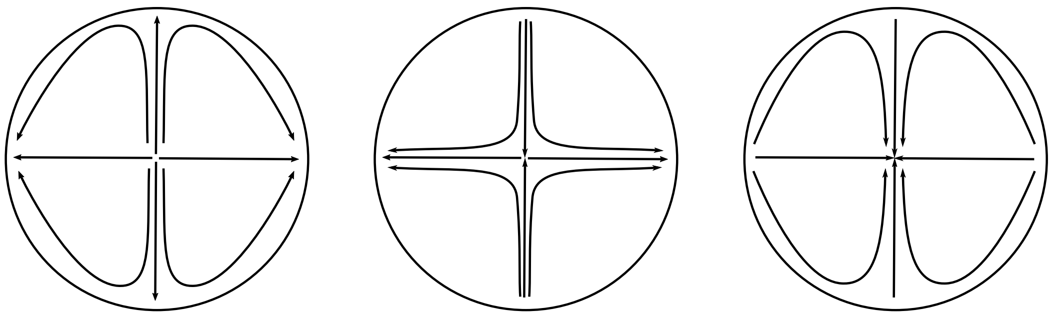

Figure 6.3: Trajectories around maximum, saddle and minimum (left to right)

and mod 2 Morse homology becomes

$$H_k(f, \mathbb{Z}/2\mathbb{Z}) = \begin{cases} \mathbb{Z}/2\mathbb{Z} & k = 0, 1, 2 \\ 0 & k > 2 \end{cases}$$

7 Morse inequalities

Because the Morse homology is the same as singular homology, we directly get results such as the Kunneth formula, Poincare duality, Excision theorem, Mayer-Vietoris sequences, Universal Coefficient theorem etc. There are in fact proofs that rely only on techniques from Morse theory as opposed to translating the results from singular homology to Morse homology. These details can be found in [2]. In this section we focus on Morse inequalities, mainly following [1]. Some background in relative homology and exact sequences can be found in [5].

Definition. *The Betti numbers $b_\lambda(M)$ of a manifold M are the ranks of the λ-th homology groups. When $X \supset Y$, the relative Betti number $b_\lambda(X, Y)$ is the rank of the λ-th relative homology group. The Euler characteristic is defined as $\chi(X, Y) = \sum (-1)^\lambda b_\lambda(X, Y)$.*

While the coefficients can be from any group, we will deal with $\mathbb{Z}$.

Definition. *Let S be a function from certain pairs of spaces to $\mathbb{Z}$. S is subadditive if whenever $X \supset Y \supset Z$ we have $S(X, Z) \le S(X, Y) + S(Y, Z)$. If equality holds, then S is called additive.*

For $X \supset Y \supset Z$, from the exact sequence

$$\cdots \to H_k(Y, Z) \to H_k(X, Z) \to H_k(X, Y) \to \ldots$$

we see that the relative Betti number is subadditive. Similarly, the Euler characteristic is additive.

Lemma 7.1. *Let S be subadditive and let $X_0 \subset \cdots \subset X_n$, then $S(X_n, X_0) \le \sum_{i=1}^n S(X_i, X_{i-1})$. If S is additive, then equality holds.*

Proof. Induction on n. The case $n = 1$ holds trivially and $n = 2$ is true by definition. If the result is true for $n - 1$, then

$$S(X_n, X_0) \le S(X_n, X_{n-1}) + S(X_{n-1}, X_0) \le \sum_{i=1}^n S(X_i, X_{i-1})$$

and the result is true for n. $\square$

Let M be a compact manifold and f a Morse function. Let $a_1 < \cdots < a_k$ be such that M^{a_i} contains exactly i critical points, and $M^{a_k} = M$. Then, by the Morse theorems, we know that going from $M^{a_{i-1}}$ to M^{a_i} is obtained by attaching a λ_i cell where λ_i is the index of the critical point. Therefore,

$$\begin{aligned} H_k(M^{a_i}, M^{a_{i-1}}) &= H_k(M^{a_{i-1}} \cup e^{\lambda_i}, M^{a_{i-1}}) \\ &= H_k(e^{\lambda_i}, \partial e^{\lambda_i}) \text{ by excision} \\ &= \begin{cases} \mathbb{Z} & k = \lambda_i \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

Applying the lemma to $\emptyset = M^{a_0} \subset \cdots \subset M^{a_k} = M$ with $S = b_\lambda$, we get

$$b_\lambda(M) \leq \sum_{i=1}^{n} b_\lambda(M^{a_i}, M^{a_{i-1}}) = C_\lambda$$

where C_λ is the number of critical points of index λ (because each critical point of index λ contributes 1 to the rank). And applying to the Euler characteristic gives

$$\chi(M) = C_0 - C_1 + \cdots \pm C_n.$$

Lemma 7.2. *The function $S_\lambda(X,Y) = b_\lambda(X,Y) - b_{\lambda-1}(X,Y) + \cdots \pm b_0(X,Y)$ is subadditive.*

Proof. The proof is purely linear algebra. Given an exact sequence

$$\xrightarrow{h} A \xrightarrow{i} B \xrightarrow{j} C \xrightarrow{k} \cdots \to D \to 0$$

of vector spaces, we have

$$\begin{aligned} \mathrm{Rank} h &= \mathrm{Rank} A - \mathrm{Rank} i \\ &= \mathrm{Rank} A - \mathrm{Rank} B + \mathrm{Rank} j \\ &\vdots \\ &= \mathrm{Rank} A - \mathrm{Rank} B + \cdots \pm \mathrm{Rank} D. \end{aligned}$$

Hence, the last expression is ≥ 0. In the relative homology exact sequence corresponding to a triple $X \supset Y \supset Z$, there is a boundary map $H_{\lambda+1}(X,Y) \xrightarrow{\partial} H_\lambda(Y,Z)$. Applying the equations above to this map gives

$$\mathrm{Rank}\partial = b_\lambda(Y,Z) - b_\lambda(X,Z) + \ldots b_\lambda(X,Y) - b_{\lambda-1}(Y,Z) \geq 0.$$

Collecting the terms gives us

$$S_\lambda(Y,Z) - S_\lambda(X,Z) + S_\lambda(X,Y) \geq 0$$

completing the proof. □

And from here, using the previous lemma to S_λ on $\emptyset \subset M^{a_1} \subset \cdots \subset M^{a_k}$, we get

$$S_\lambda(M) \leq \sum_{i=1}^{k} S_\lambda(M^{a_i}, M^{a_{i-1}}) = C_\lambda - C_{\lambda-1} + \cdots \pm C_0$$

or

$$b_\lambda(X,Y) - b_{\lambda-1}(X,Y) + \cdots \pm b_0(X,Y) \leq C_\lambda - C_{\lambda-1} + \cdots \pm C_0.$$

Theorem 7.1. *(Morse inequalities) If C_λ denotes the number of critical points of index λ on a compact manifold M, then*

$$b_\lambda \leq C_\lambda \tag{2}$$

$$\sum (-1)^\lambda b_\lambda(M) = \sum (-1)^\lambda C_\lambda \tag{3}$$

$$b_\lambda(X,Y) - b_{\lambda-1}(X,Y) + \cdots \pm b_0(X,Y) \leq C_\lambda - C_{\lambda-1} + \cdots \pm C_0 \tag{4}$$

Equation 4 is called the strong Morse inequality, and it's clear that this implies Equation 2 which is called the weak Morse inequality. While we have followed [1] and proved this using the concept of subadditive functions, a proof using only Morse homology (for C_λ is the ranks of the vector spaces appearing in the Morse complex) is given in [2] and [9]. Simple algebraic manipulation of Equation 4 for different values of λ gives

Corollary. *If $C_{\lambda+1} = C_{\lambda-1} = 0$ then $b_\lambda = C_\lambda$ and $b_{\lambda+1} = b_{\lambda-1} = 0$.*

7.1 Complex Projective Space

We think of $\mathbb{C}P^n$ as the equivalence classes of $(n+1)$-tuples $[z_0 : \ldots : z_n]$ of complex numbers such that $\sum |z_j|^2 = 1$. Just as in the case of the real projective space, consider the function

$$f(z_0 : \ldots : z_n) = \sum c_j |z_j|^2$$

where c_j are distinct real numbers. In the open set U_0 defined by $z_0 \neq 0$, setting $x_j + iy_j = |z_0| z_j / z_0$, the functions $x_1, y_1, \ldots, x_n, y_n : U_0 \to \mathbb{R}$ form a set of coordinates and on this chart,

$$f = c_0 + \sum_{j=1}^{n} (c_j - c_0)(x_j^2 + y_j^2)$$

The only critical point is $p_0 = [1 : 0 : \ldots : 0]$ and here the Hessian is non-degenerate with index equal to twice the number of j with $c_j < c_0$. Similarly, the other critical points are $p_1 = [0 : 1 : \ldots : 0], \ldots, p_n = [0 : \ldots : 1]$ with index of p_k being twice the number of j with $c_j < c_k$. So, every possible even index between $0, 2n$ appears exactly once.
So, directly via Morse inequalities or by using the cellular decomposiiton from Morse theorems we see that $\mathbb{C}P^n$ has a CW structure $e^0 \cup e^2 \cup \cdots \cup e^{2n}$ and

$$H_k(\mathbb{C}P^n, \mathbb{Z}) = \begin{cases} \mathbb{Z} & k = 0, 2, 4, \ldots, 2n \\ 0 & \text{otherwise} \end{cases}$$

8 Further topics

8.1 Manifolds with boundary

We follow [6]. As mentioned breifly above, a function f on a manifold M with boundary ∂M is said to be Morse if all its critical points lie in the interior and are non degenerate and the restriction $f|_{\partial M}$ is also Morse. The critical points of the restriction of f to the boundary are of two types : N and D. A critical point $p \in \partial M$ is of type N (resp. D) if $\langle df(p), n(p) \rangle$ is negative (resp. positive) where $n(p)$ is the outward normal to M at p. It turns out that the homotopy type of the sublevel sets changes only when crossing a critical point in the interior of M or a critical point of type N.
Let

- C_k denote the critical points of $f : \mathrm{Int}M \to \mathbb{R}$ of index k;
- N_k denote the critical points of $f : \partial M \to \mathbb{R}$ of type N and index k;
- D_k denote the critical points of $f : \partial M \to \mathbb{R}$ of type D and index k.

In [6] we have the following theorem regarding the Morse complex on manifolds with boundary:

Theorem 8.1. *Let F_*^N be the free graded $\mathbb{Z}$-module generated by $C_* \cup N_*$. There exists a differential $\partial : F_*^N \to F_{*-1}^N$ making (F_*^N, ∂) a chain complex whose homology is isomoprhic to the singular homology.*

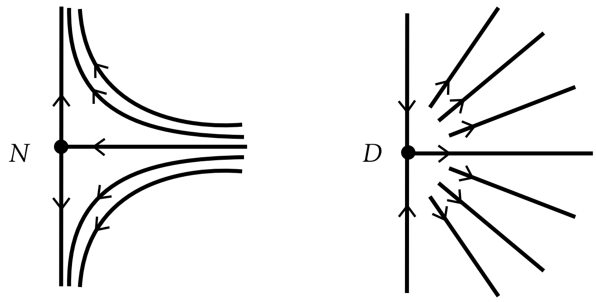

Figure 8.1: Boundary critical points

On manifolds without boundary, the Morse polynomial is defined as

$$\mathcal{M}_f(t) = \sum_{c \in \mathrm{Crit}(f)} t^{\mathrm{Ind} c}$$

and captures most of the data of the Morse complex. In the case of manifolds with boundary, we define $\mathcal{M}_f^N(t)$ similarly by summing over critical points in $C_* \cup N_*$. The Poincare polynomial is defined as

$$\mathcal{P}_M(t) = \sum_k b_k(M) t^k.$$

Theorem 8.2. *We have $\mathcal{M}_f^N(t) - \mathcal{P}_M(t) = (1+t)Q^N(t)$ where Q^N is a polynomial with non-negative coefficients.*

In this way the theory is developed for manifolds with boundary.

8.2 Cancellation theorem

8.3 h-cobordism

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