# Separability and and all that

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## 1 Introduction

Given a topological space, it is of interest to know if the space is separable, or second countable and so on, this, in some sense, tells that the space is quite small. The Stone-Weierstrauss theorem is one important theorem in that direction. We shall also discuss the separability of $L^p$ spaces. Unless mentioned otherwise, our functions are always complex valued continuous functions.

## 2 The space of continuous functions

Let $X$ be a locally compact Hausdorff space, the space of all $F$-valued continuous functions is denoted by$C(X, F)$ where $F = \mathbb{C}$ or $\mathbb{R}$ (this notation applies to any topological space). A function is said to vanish at infinite if given any $\epsilon > 0$ there is a compact set $K$ such that on $|f(x)| < \epsilon \forall x \notin K$. We denote the set of all $F$-valued continuous functions vanishing at infinty by $C_0(X, F)$. Note that when $X$ itself is compact, then all functions vanish at infinity. Both $C(X, F), C_0(X, F)$ are $F$-vector spaces.

If $f \in C_0(X, F)$, then $|f|$ is bounded because continuous functions on compact sets are bounded (and we know that $f$ is bounded by 1, say, outside some compact set $K$). So, on $C_0(X, F)$, we have the sup norm defined as follows

$$\|f\|_{\text{sup}} = \sup_{x \in X} |f(x)|.$$

It is easy to see that $\|\cdot\|_{\text{sup}}$ is indeed a norm. With this norm, the space $C_0(X, F)$ is now a normed linear space. What is also true is that this norm makes $C_0(X, F)$ a Banach space as if $\{f_n\}$ is a Cauchy sequence in $C_0(X, F)$, then for any $x \in X$ the sequence $f_n(x)$ is also Cauchy and since $F$ is complete, it converges to some $f(x)$. Now given $\epsilon > 0$ there is an $N$ such that for any $n, m \geq N$, $\|f_n - f_m\|_{\text{sup}} < \epsilon$. Now, given an $x \in X$, we have

$$|f_n(x) - f_m(x)| \leq \|f_n - f_m\|_{\text{sup}} < \epsilon \implies |f_n(x) - f(x)| \leq \epsilon$$

which is obtained by letting $m \to \infty$. Since $x$ was arbitrary, $\|f_n - f\|_{\text{sup}} \leq \epsilon \forall n \geq N$. Now, the continuity of $f$ follows from a standard $\epsilon/3$ argument and the fact that $f \in C_0(X, F)$ also follows similarly : given $\epsilon > 0, N$ as above, obtain a compact $K$ such that $|f_N| < \epsilon$ outside $K$, which would mean that for any $n \geq N, |f_n(x)| < 2\epsilon$ outside $K$, hence $|f| \leq 2\epsilon$ outside $K$.

Therefore, $C_0(X, F)$ is a Banach space. If $X$ also has a measure $\mu$ on the Borel $\sigma$-algebra, then we can also define the $p$-norms:

$$\|f\|_p = (\int_X |f|^p d\mu)^{1/p}.$$

Using the Holder and Minkowski inequalities, $\|\cdot\|_p$ is a pseudo-norm. Let $N$ be the subspace (of the space of all functions with finite $p$-norm) of functions for which the integral above evalueates to 0. Then, after quotienting by $N$, we obtain the space $L^p(X, \mu)$. This is a normed linear space, and is in fact a Banach space (see [2]).

## 3 Weierstrass Approximation Theorem

**Theorem 1.** *Suppose $f$ is a continuous real valued function on a closed interval $[a, b]$ and $\epsilon > 0$ is given. There is a polynomial $p$ such that $\|f - p\|_{sup} < \epsilon$.*

Without loss of generality we may assume that $[a, b] = [-1, 1]$ (so it is symmetric about $0$) for given any other interval, we may change the variables, without changing the norms (intuitively this amounts to a horizontal stretch of $f$). One idea is to use some kind of interpolating polynomial. Say we divide $[-1, 1]$ into intervals of length $1/n$ for some suitable $n$, and then interpolate a polynomial through the points $(t/n, f(t/n)), -n \leq t \leq n$. To ensure that this polynomial is within $\epsilon > 0$ of $f$ we choose $n$ to be large enough and exploit the uniform continuity of $f$.

However, the interpolation polynomial is rather messy. Instead, it seems easier to consider a linear interpolation. It is rather obvious that the linear interpolation is gong to be within $\epsilon$ of $f$ : if $|f(x) - f(y)| < \epsilon$ when $|x - y| < 2/n$ say, then for any $\lambda \in [0, 1]$, the number $\lambda f(x) + (1 - \lambda)f(y)$ is within $\epsilon$ of $f(x), f(y)$, which are within $\epsilon$ of $f(\lambda x + (1 - \lambda)y)$.

Having obtained the linear interpolation, we would like to smooth it out into a polynomial. This is where the idea of convolution comes into play. Convolutions are a way to take a weighted average of the values of $f$. Given two functions $f, g$ on $\mathbb{R}$, the convolution is the function defined by

$$f * g(x) = \int_{\mathbb{R}} f(x - t)g(t)dt = \int_{\mathbb{R}} f(t)g(x - t)dt$$

where the second equality follows from the translational invariance of the lebesgue measure. Of course one has to make sure that the convolution exists and so on, but as long as $f, g$ are integrable (i.e., $L^1$) the convolution exists. Moreover, it is clear that we can define the convolution of complex valued functions on $\mathbb{R}^n$.

Observe that if $g$ is a polynomial, then so is the convolution $f * g$. Also observe that when $f \equiv 1$, then the convolution $f * g \equiv \int_{\mathbb{R}} g(t)dt$, so one could ask for this integral to be $1$.

The $n$th Landau kernel is defined to be

$$\phi_n(x) = c_n(1 - x^2)^n \chi_{[-1,1]}$$

where $c_n$ is a normalizing coeffiicient so that the integral is $1$.

It turns out that we can directly convolve $f$ with the Landau kernels to obtain a uniform approximation of $f$ by polynomials.

*Proof.* (Using Landau Kernels) Given a continuous function $f$ on $[-1, 1]$ it is clear that the convolutions $f * \phi_n$ exist and are polynomials. We claim that these polynomials converge uniformly to $f$. First observe that

$$\begin{aligned}
\int_{-1}^{1} (1 - x^2)^n dx &= 2 \int_0^1 (1 - x^2)^n dx \\
&\geq 2 \int_0^1 (1 - x)^n dx \\
&= \frac{2}{n + 1}
\end{aligned}$$

So, the normalizing coefficient $c_n$ is at most $(n + 1)/2$.

Now, fix an $x \in [0, 1], \epsilon > 0$, we have

$$\begin{aligned}
|f(x) - f * \phi_n(x)| &= \int_{-1}^{1} |f(x - t) - f(x)|\phi_n(t)dt \\
&= \int_{-1}^{-\delta} |f(x - t) - f(x)|\phi_n(t)dt + \int_{-\delta}^{\delta} |f(x - t) - f(x)|\phi_n(t)dt \\
&+ \int_{\delta}^{1} |f(x - t) - f(x)|\phi_n(t)dt
\end{aligned}$$

where we use the fact that $\int_{-1}^{1} \phi_n = 1$ and $\delta > 0$ is such that for $|t| < \delta, |f(x-t) - f(x)| < \epsilon$. Note that by uniform continuity, this $\delta$ is independent of $x$. Since $f$ is continuous on a compact inteval, it is bounded by some $M > 0$.

It suffices to show that as $n \to \infty$, the integrals $\int_{\delta}^{1} \phi_n$ goes to 0 for a fixed $\delta > 0$. Then the terms at the ends can be made small depending on $n$ and the middle term is small by uniform continuity. To this end, observe that

$$\int_{\delta}^{1} c_n(1-x^2)^n dx \le \frac{n+1}{2} \int_{\delta}^{1} (1-\delta^2)^n dx = \frac{n+1}{2}(1-\delta^2)^n(1-\delta).$$

Since $0 < 1-\delta^2 < 1, (n+1)(1-\delta^2)^n \to 0$ as $n \to \infty$ (for example, look at the series $\sum nx^{n-1}$ which has radius of convergence 1) and this concludes the proof. □

We provide another proof which uses some probabilistic methods.

*Proof.* (Using Bernstein polynomials) Let $f$ be defined on $[0,1]$, and define the $n$th Bernstein polynomial to be

$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} f(\frac{k}{n}) x^k (1-x)^{n-k}.$$

Let $\Omega = \{0,1\}^n$ with the discrete $\sigma$-algebra and binomial measure, $S_n \colon \Omega \to \mathbb{R}$ be the random variable obtained by adding the coordinates. Let $P$ be the associated probability measure.

Note that $B_n(x)$ is the expectation of $f(S_n/n)$. Given $\epsilon > 0$, let $\delta > 0$ be obtained using the uniform continuity of $f$ such that $|x-y| < \delta \implies |f(x) - f(y)| < \epsilon$. Furthermore, $E(S_n/n) = x, V(S_n/n) = x(1-x)/n$ and

$$|B_n(x) - f(x)| = |E(f(S_n/n) - f(E(S_n/n)))|.$$

Let $A$ denote the event $\{S_n/n - x| < \delta$, then

$$\int_A |f(S_n/n) - f(x)| dP \le \epsilon P(A) \le \epsilon$$

and the same integral over the complement $B$ of $A$ is bounded by $2MP(B)$ where $M$ is the bound on $f$. Finally, by Chebyshev-Markov inequelity

$$P(B) = P(|S_n/n - E(S_n/n)| \ge \delta) \le \frac{V(S_n/n)}{de^2} = \frac{x(1-x)}{n\delta^2} \le \frac{1}{n\delta^2}$$

So, adding things up

$$|B_n(x) - f(x)| \le \epsilon + \frac{1}{n\delta^2}$$

independent of $x$. Therefore, $B_n$ converge uniformly to $f$. □

By using the triangle inequality, it is easy to see that polynomials with complex coefficients are dense in the space of complex valued continuous functions on $[a,b]$ and so are polynomials with coefficients from $\mathbb{Q} + i\mathbb{Q}$. This second collection of polynomials is countable, hence the space of complex valued continuous functions on $[a,b]$ with the sup norm is separable.

# 4 Stone-Weierstrauss Theorem

The goal of this section is to generalise the Weierstrass Approximation theorem to spaces other than closed intervals. Let $X$ be a locally compact Hausdorff space, then $C_0(X,F)$, is a Banach space with the sup metric and it is a commutative unital $\mathbb{C}$-algebra. Separability of $C_0(X,F)$ amounts to finding a subset of $C_0(X,F)$ which is dense. In the previous case, this subset was the set of polynomials. Note that the set of polynomials is a unital $\mathbb{C}$ subalgebra.

$A \subset C_0(X, F)$ is dense in $C_0(X, F)$, if given any $f \in C_0(X, F), \epsilon > 0$, there is a $g \in A$ such that $\|f - g\|_{\text{sup}} < \epsilon$, i.e., at every $x \in X, g(x)$ is within $\epsilon$ of $f(x)$. It is then clear that $A$ should not have a vanishing point, i.e., a point $x \in X$ where every $g \in A$ vanishes. Moreover, since $X$ is normal, given any two closed disjoint compact subsets $C, D$ there is a continuous function which attains $0$ on $C$ and $1$ on $D$, this is the Urysohn lemma. In particular, given $x \neq y$, there is an $f \in C_0(X, F)$ such that $f(x) \neq f(y)$. With a suitable choice of $\epsilon$, it is easy to see that there must be a $g \in A$ such that $g(x) \neq g(y)$.

We say that a subset $A$ separates points if given $x \neq y \in X$, there is a $g \in A$ such that $g(x) \neq g(y)$ and vanishes nowhere if for every $x \in X$ there is a $g \in A$ such that $g(x) \neq 0$. From the discussion above, it is clear that if $A$ is to be dense, then it has to separate points and vanish nowhere. Observe that the two conditions are independent of each other.

Before we state (one version of) the Stone-Weierstrass Theorem, we have the following definition and lemma.

**Definition 1.** *A subset $S$ of $C_0(X, \mathbb{R})$ is called a lattice if for every $f, g \in S, f \wedge g = \min\{f, g\} \in S$ and $f \vee g = \max\{f, g\} \in S$. Note that $f \wedge g, f \vee g$ are continuous functions.*

**Lemma 1.** *Let $A$ be a closed subalgebra of $C_0(X, \mathbb{R})$, then*

1. *if $A$ is unital, given $f \in A, f \geq 0$, $\sqrt{f} \in A$*
2. *if $f \in A$, then $|f| \in A$*
3. *$A$ is a lattice*

*Proof.* 1. Without loss of generality we may assume $0 \leq f \leq 1$ for $A$ is closed under multiplication by constants. Now take $g = 1 - f$ so that $\sqrt{f} = \sqrt{1 - g}$. Since $0 \leq g \leq 1$, we have the series expansion

$$\sqrt{1 - g(t)} = 1 - \sum_{n \geq 1} a_n g^n(t)$$

and this convergence is uniform. So, we approximate $\sqrt{f}$ as a limit of elements in $A$. Since $A$ is closed, it follows that $\sqrt{f} \in A$.

2. Let $f$ be given and we may assume that $|f| \leq 1$ as before. Now, from the Weierstrass Approximation theorem there is a polynomial $P$ within $\epsilon$ of $|x|$ on $[-1, 1]$. Then $|P(0)| \leq \epsilon$, so the polynomial $P - P(0)$ is within $2\epsilon$ of $|x|$ and passes through $0$ at $0$. Rename this polynomial as $P$. Then $P \circ f$ is a polynomial in $f$ and doesn't have a constant term, therefore an element of $A$. And $||f| - P \circ f| \leq 2\epsilon$ on $X$. Since $A$ is closed, it follows that $|f| \in A$.
3. This follows from the observation that

$$f \wedge g = \frac{1}{2}(f + g - |f - g|); f \vee g = \frac{1}{2}(f + g + |f - g|). \qquad \square$$

**Theorem 2.** (*Real version*) *Let $X$ be a compact Haudorff space, then a closed subalgebra $A$ of $C(X, \mathbb{R})$ is dense if and only if it separates points and vanishes nowhere.*

*Proof.* From the discussion above, it is clear that if $A$ is dense, then it must separate points and vanish nowhere. Conversely, assume $A$ vanishes nowhere and separates points. Let $f \in C(X, \mathbb{R}), \epsilon > 0$ be given. Now, for every $s, t \in X$ there is a function $h \in A$ such that $h(s) \neq h(t)$ because $A$ separates points.

Since $A$ vanishes nowhere, there is a $g_s \in A$ such that $g_s(s) \neq 0$. Now, given $\mu, \lambda \in \mathbb{R}$, consider

$$g = \frac{\mu}{g_s(s)} g_s + \left(\lambda - \mu \frac{g_s(t)}{g_s(s)}\right) \frac{h - h(s)}{h(t) - h(s)}.$$

Now, $h \in A$ and $h(s) = \mu, h(t) = \lambda$. Since $\mu, \lambda$ are arbitrary, it follows that given $s, t \in X$ there is a function $f_{s,t} \in A$ such that

$$f_{s,t}(s) = f(s), f_{s,t}(t) = f(t).$$

So, $f_{s,t}$ approximate $f$ near $s, t$. Now fix $s$ and vary $t$, and take

$$U_t = \{v \in K : f_{s,t}(v) < f(v) + \epsilon\}$$

By continuity of $f_{s,t}, f$ this is an open set. Moreover, $t \in U_t$, hence as $t$ varies over $X$, this covers $X$. By compactness, we obtain a finite subcover $U_{t_1}, \dots, U_{t_k}$. Put

$$h_s = \min\{f_{s,t_i}\}$$

then $h_s \in A$ and $h_s(v) \leq f(v) + \epsilon \forall v \in K$. Now set $V_s = \{v \in X : f(v) - \epsilon < h_s(v)\}$. Again by continuity, $V_s$ is open and clearly $s \in V_s$ therefore this collection covers $X$ and we obtain a finite subcover $\{V_{s_1}, \dots, V_{s_m}\}$.

Finally set $g = \max\{h_{s_i}\}$. Then $f - \epsilon < g < f + \epsilon$ and $g \in A$ as required. $\square$

*Remark.* Observe that if $A$ vanished at some fixed $x_0$, then the above proof shows that $A$ is dense in $\{f \in C(X, \mathbb{R}) : f(x_0) = 0\}$.

**Corollary 1.** (*Real version - locally compact*) *Let $X$ be a locally comapct Hausdorff space, then a closed subalgebra $A$ of $C_0(X, \mathbb{R})$ is dense if and only if it separates points and vanishes nowhere.*

*Proof.* As before, if $A$ is dense, then it is clear that $A$ should separate points and shouldn't vanish anywhere. To prove the converse, we move to the one point compactification $\tilde{X}$ of $X$. Every $f \in C_0(X, \mathbb{R})$ has a continuous extension $\tilde{f}$ to $\tilde{X}$ by setting $\tilde{f}(\infty) = 0$. Similarly we look at the set $\tilde{A}$ obtained by extending the functions in $A$ to $\tilde{X}$. It is easy to see that $\tilde{A}$ is a closed subalgebra of $C(\tilde{X}, \mathbb{R})$, and we can apply the theorem above to see that $\tilde{A}$ is dense in $\{f \in C(\tilde{X}, \mathbb{R}) : f(\infty) = 0\}$.

Since the metric on $C_0(X, \mathbb{R})$ is the restriction of that on $C(\tilde{X}, \mathbb{R})$, it then follows that $A$ is dense in $C_0(X, \mathbb{R})$ as required. $\square$

**Theorem 3.** (*Complex version*) *Let $X$ be a compact Hausdorff space and $S$ be a separating subset of $C(X, \mathbb{C})$. Then the complex unital $*$-algebra generated by $S$ is dense in $C(X, \mathbb{C})$.*

Here the unital $*$-algebra generated by $S$ is the smallest $\mathbb{C}$-algebra closed under complex conjugation (of functions) that contains $S$.

*Proof.* Any set containing $S$ is a separating set. Let $S$ itself be a unital $*$-algebra. Because $S$ is closed under complex conjugation $f \in S \implies Re(f), Im(f) \in S$. Now, to approximate any $f \in C(X, \mathbb{C})$, it suffices to approximate the real and imaginary parts of $f$. It is easy to see that $S \cap C(X, \mathbb{R})$ is unital and separating, hence dense in $C(X, \mathbb{R})$. It follows by the Stone-Weierstrass theorem that $S$ is dense in $C(X, \mathbb{C})$. $\square$

# 5 Continuous functions in function spaces

This section is based almost entirely on [2] Chapter 5.

Let $(X\mathcal{M}, \mu)$ be a measured space. We have defined the space $L^p(\mu)$ for $1 \leq p < \infty$. We can also define the $L^\infty$-norm as

$$\|f\|_\infty = \inf\{c \in \mathbb{R} : \mu(\{x : |f(x)| > c\}) = 0\}.$$

It is also called the essential supremum of $f$. Note that outside a set of measure $0$, $f$ is bounded by $\|f\|_\infty$. This allows us to define $L^\infty(\mu)$ similar to $L^p(\mu)$ and we have similar versions of Holder and Minkowski inequalities.

These $L^p(\mu)$ spaces are actually Banach spaces (see [2] or any text on measure theory/functional analysis). What we aim to prove here is that when we have a topological space $X$ with the Borel $\sigma$-algebra, the continuous functions are going to be dense in $L^p(\mu)$.

For a locally compact Hausdorff space $X$, let $C_C(X, F)$ denote all continuous $F$-valued functions on $X$ with compact support. We aim to prove that $C_C(X, \mathbb{C})$ is dense in $L^p(\mu), 1 \leq p < \infty$. The intuition is that the functions in $L^p$ are approximated by step functions, and that we can approximate indicator functions with continuous compactly supported functions.

## 5.1 Few preliminaries

Let $X$ be a locally compact Hausdorff space with the Borel $\sigma$-algebra.

**Definition 2.** *Let $X$ be a Hausdorff space, and $\mu$ a Borel measure. Then $\mu$ is said to be*

- *Outer regular if for every Boreal measurable $E$, $\mu(E) = \inf\{\mu(U) : E \subseteq U, U open\}$.*
- *Inner regular or tight if for every open $U$, $\mu(U) = \sup\{\mu(K) : K \subseteq U, K \text{ compact }\}$.*
- *Inner regular on finite measure sets and open sets if the condition of inner regularity hold on open sets as well as measurable $E$ of finite measure.*
- *Locally finite if every $x \in X$ has a neighbourhood $U$ of finite measure.*

*$\mu$ is said to be a Radon measure if it is outer regular, inner regular on open sets and locally finite.*

*On a locally compact Hausdorff space, $\mu$ is said to be regular if it is outer regular, inner regular on open and finite measure sets and finite on compact sets. Note that if $\mu$ is locally finite, then it is finite on compact sets and under the assumption of local compactness, the converse holds as well.*

Next we discuss, briefly, the existence of certain continuous bump functions over a locally comapct Hausdorff space.

Let $X$ be locally compact Hausdorff and let a closed set $A$ and point $x \notin A$ be given. Because $A$ is closed we obtain an open neighbourhood $U$ of $x$ disjoint from $A$. By the local compactness we have an open neighbourhood $W$ of $x$ such that $\overline{W}$ is compact. Now, $\overline{W} \cap A$ is a closed subset of a compact set, hence itself compact, and it is disjoint from $x$, so we can find an open neighbourhood $V$ of $x$ such that $\overline{V}$ is disjoint from $\overline{W} \cap A$.

Consider the neighbourhood $U_1 = U \cap W \cap V$. This is disjoint from $A$, and it's closure is a closed subset of $W$, hence compact and moreover,

$$\overline{U_1} \cap A \subseteq \overline{W \cap V} \cap A \subseteq \overline{V} \cap \overline{W} \cap A = \emptyset.$$

Thus, we have found an open neighbourhood $U_1$ of $x$ with compact closure such that the closure is disjoint from $A$.

Now, if $K$ is a comapct set disjoint from $A$, then it is clear how one can obtain an open $U$ such that $K \subset U \subset \overline{U} \subset X \setminus A$ and $\overline{U}$ is compact. As in the proof of Urysohn lemma (see [4]) one can now obtain a continuous function $f \colon X \to [0,1]$ such that $f|_K = 1, f|_{X\setminus U} = 0 \implies f|_A = 0$. This $f$ then has compact support.

## 5.2 Approximation results

**Theorem 4.** (*Density of continuous functions in $L^p$*) *Let $X$ be a locally compact space, $\mu$ a Radon measure on $X$.*

1. *For any $p \in [1, \infty)$, the image of $C_C(X, \mathbb{C})$ in $L^p(\mu)$ is dense in $L^p(\mu)$.*
2. *For $p = \infty$, the closure of $C_C(X)$ in $L^\infty(\mu)$ is contained in the image of continuous bounded functions.*

*Remark* As mentioned in [2], the closure of $C_C(X)$ in $L^\infty$ is not usually $L^\infty$. For example, take $X = \mathbb{R}^n$. Here, a function is in the closure of $C_C(X)$ if it is the uniform limit of compactly supported continuous functions, say $f = \lim f_n$. Now, given any $x$, we have

$$|f(x)| \leq |f(x) - f_n(x)| + |f_n(x)|$$

which can be made small outside a compact set depending on $n$ which is chosen to be sufficiently so that the first term is also small.

So, $f$ would lie in $C_0(X)$. Conversely, any function in $C_0(X)$ is a uniform limit of compactly supported functions and this is seen to be true using continuous bump functions.

*Proof.* Part 2 is quite clear as in the remark above, so we only prove 1. Let $V$ denote the complex vector space which is the closure of $C_C(X)$ in $L^p(\mu)$ (the image after quotienting by functions whose norm is 0; exercise to the reader - why is it a vector space?)

First we show that characteristic functions are in $V$. Let $E$ be of finite measure and $\epsilon > 0$ be given. By regularity of the Radon measure, there is a compact set $K$ and an open set $V$ such that $K \subset E \subset U$ and $\mu(U \setminus K) < \epsilon$. By Urysohn lemma, there is a continuous function $f$ with compact support such that $\chi_K \leq f \leq \chi_U$ and $\mathrm{supp}(f) \subset U$. We then have

$$\int_X |f - \chi_E|^p d\mu \leq \int_{U \setminus K} |f - \chi_E|^p d\mu \leq \mu(U \setminus K) < \epsilon$$

which shows that $\chi_E \in V$.

By linearity, it follows that $V$ contains all non-negative step functions which are in $L^p(\mu)$. Next, let $f \in L^p(\mu)$ be non-negative, and let $\{s_n\}$ be a sequence of non-negative step functions increasing to $f$ pointwise. Since $s_n \leq f$ everywhere, $s_n \in L^p(\mu)$, hence $s_n \in V \forall n$. Since $V$ is closed, it suffices to prove that $s_n$ converge to $f$ in $L^p$ as well.

Let $g_n = |f - s_n|^p$. Now, $g_n$ converges pointwise to $0$, and we apply the dominated convergence theorem. By a version of the Holder theorem, we have

$$(a+b)^p \leq 2^{p/q}(|a|^p + |b|^p)$$

where $q$ is the dual exponent of $p$. So,

$$|g_n| \leq 2^{p/q}(|f|^p + |s_n|^p) \leq 2^{1+p/q}|f|^p \in L^1(\mu)$$

so, by DCT,

$$\|f - s_n\|_p^p = \int_X |g_n| d\mu \to 0$$

completing the proof.

Finally, given an arbitrary $f \in L^p(\mu)$, we split it into real, imaginary, positive and negative parts and use linearity. $\square$

## 5.3 The question of separability

It is clear that over an arbitrary Borel space, the step functions and continuous compactly supported functions are dense in $L^p(\mu), 1 \leq p < \infty$. Over $\mathbb{R}^n$ we actually have separability. This is because we can approximate all Borel measurable sets using rectangles, which can themselves be approximated by rectangles with rational vertices, and if $R$ is one such rectangle, then any $c\chi_R$ can be well approximated by a $q\chi_R$ where $q$ is rational.

Over sequence spaces, denoted $l^p$, too we have separability. We define

$$l^p = \{(x_1, x_2, \dots) : x_i \in \mathbb{C}, \sum |x_i|^p < \infty\}$$

$$l^\infty = \{(x_1, x_2, \dots) : x_i \in \mathbb{C}, \sup |x_i| < \infty\}$$

which is essentially $L^p(\mathbb{Z}_{\geq 0}, \mu)$ where $\mu$ is the counting measure. So, $l^p$ are Banach spaces (in fact $l^2$ is a Hilbert space). We also define

$$c_0 = \{x \in l^\infty : \lim x_n = 0\}; c = \{x \in l^\infty : \lim x_n \text{ exists }\}; c_{00} = \{x \in l^\infty : x_n = 0 \text{ a.e.}\}.$$

**Theorem 5.** *$c_0, c$ are Banach subspaces of $l^\infty$.*

*Proof.* It is easy to see that they are indeed subspaces. Let $\{x_n = (x_{n1}, x_{n2}, \dots)\}$ be a Cauchy sequence in $c_0$ or $c$, which means, given $\epsilon > 0, \sup |x_{ni} - x_{mi}| < \epsilon$ for large $n, m$. Let $y_m = \lim_n x_{nm}, y = (y_1, y_2, \dots)$. It is easy to see that $x_n \to y$, what we need to show is that $y \in c_0$ or $y \in c$.

First, suppose $x_n \in c_0$ and let $\epsilon > 0$ be given. Choose $N$ such that for $n, m \geq N, \|x_n - x_m\|_{\text{sup}} < \epsilon$. Since $\lim_i x_N i = 0$, there is a $K$ such that $|x_N k| < \epsilon \forall k \geq K$. So, for $n \geq N, i \geq K, |x_{ni}| < 2\epsilon \implies |y_i| \leq 2\epsilon$. Therefore $y \in c_0$.

Next, suppose $x_n \in c$, let $c_n = \lim_i x_{ni}$, we claim that $\lim c_i, \lim y_i$ exist and are equal. We have

$$|x_{nj} - x_{mj}| \leq |x_{ni} - x_{mj}| + |x_{ni} - x_{mi}| + |x_{mi} - x_{mj}|$$

Given $\epsilon > 0$, fix $n, m$ such that the middle term is less than $\epsilon/3$. Hacing fixed $n, m$ take $i, j$ large enough so that the other two terms are also less than $\epsilon/3$. Now, we take the limit over $j$ on the left side to obtain

$$|c_n - c_m| \leq \epsilon.$$

Therefore, $c = \lim c_n$ exists. Next,

$$|y_j - c| \leq |y_j - x_{nj}| + ||x_{nj} - c_n| + |c_n - c|$$

and we find $n$ large such that the first and third terms are $< \epsilon/3$. Having found the $n$, we can make $j$ large enough, depending on the $n$ so that the middle term is also less than $\epsilon/3$ and we have $|y_j - c| < \epsilon$ for sufficiently large $j$. Therefore, $y \in c$ completing the proof. $\square$

However, $c_{00}$ is not a closed subspace, because the sequence $\{(1, 1/2, \dots, 1/n, 0, \dots)\}_{n \geq 1}$ doesn't have a limit in $c_{00}$. Note that this sequence does have a limit in $l^\infty, l^p, p > 1$. We then ask about the closure of $c_{00}$ in $l^\infty$. If $x \in l^\infty$ is in the closure of $c_{00}$, then eventually each term must become arbitrarily small because elements in $c_{00}$ are eventually $0$, therefore $x \in c_0$. Conversely, if $x \in c_0$, then one may take the term-by-term approximation, therefore $\overline{c_{00}} = c_0$.

Lastly, we ask if $l^p, 1 \leq p < \infty$ is separable. And the answer is yes. Simply consider the elements of the form $(x_1, x_2, \dots)$ where each coordinate is rational and only finitely many are nonzero. This is a countable set $S$, say. Given any $y = (y_1, y_2, \dots) \in l^p$, since $\sum |y_i|^p < \infty$, the $p$-norm of a tail can be made arbitrarily small. We may approximate the rest with elements from $S$. Therefore $l^p, 1 \leq p < \infty$ is separable.

# References and Further Reading


[1] Notes by John O'Conner

[2] Notes by E. Kowalski, Measure and Integral

[3] Notes by Roland Speicher

[4] J. R. Munkres, *Topology*