RESULTS ON FINITE COLLECTION OF POLYGONS AND A PROOF OF THE JORDAN CURVE THEOREM

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Abstract. We introduce the notion of polygons and Jordan curves. We first provide a proof of the Jordan Curve Theorem for polygons, and then we answer the following questions: given a finite collection of polygonal regions in the plane, can we write their union as an almost disjoint union of polygonal regions? What do the boundaries of the connected components in the complement of these polygons look like? Having answered these questions, we construct a "regular" polygonal cover for arcs in the plane and use such a covering to prove a separation result about arcs inside discs. In the last section we provide a proof of the Jordan Curve Theorem using the methods developed in the previous sections.

Contents

Key words and phrases. Jordan Curve Theorem, finite collection of polygons, separation theorems in the plane.

1. Introduction

In this paper, we provide a proof of the classic Jordan Curve Theorem. This is one of those theorems that is very simple to state and understand but notoriously difficult to prove. The first proof was given by Jordan (although there were some doubts, [1] claims that the original proof was indeed correct), and since then there have been many other proofs.

In the following two sections we provide a few definitions and a proof of the Jordan Curve Theorem for polygons using a parity function. In section 4 we answer the following question: can we write a finite union of polygonal regions as an almost disjoint union of other polygonal regions. The answer is yes, provided the collection satisfies a regularity condition as defined later. Furthermore, we also prove that under said regularity, all connected components in the complement of a union of polygons (just the boundaries) have polygonal boundaries.

In section 5 we spend some time proving certain Jordan like construction and results for finite collection of polygons, their boundaries and arcs connecting points on these boundaries.

In section 6 we prove a rather interesting theorem about arcs, which doesn't seem to be a direct consequence of the Jordan Curve Theorem. Suppose we have two arcs intersecting only at the end points. A bulk of this section is spent in explicitly constructing a polygonal covering of one of these arcs meeting the aforementioned regularity conditions and more. We then show that under a rather mild condition on these two arcs (see Theorem 6), given a point, say x, not on these arcs, we can cover one of these arcs by polygons such that x is in the unbounded component of the complement of these polygons (we shall prove that there is only one unbounded component in section 3).

In section 7 we prove that an arc lying inside the unit disc with ends on the boundary circle separates the disc into two regions and makes up their common boundary. This theorem has been proved before, but we provide a proof using the methods developed in the previous sections. Finally, in section 8 we provide a proof of the Jordan Curve theorem, mainly relying on the methods developed in the previous sections.

The core of the paper is contained in sections 4 through 6 and these concern finite collections of polygons. The results and methods involved here, specifically Theorems 4, 5, 6 and the construction detailed in section 6.1, are novel to the best of the author's knowledge.

While the end goal of this article is to prove the Jordan Curve Theorem, we mention here that the bulk of it is spent in developing certain results for finite collections of polygons and polygonal covers of arcs/curves. As far as proofs of the theorem go, [1] contains the original proof and [2] is a much shorter proof and both involve approximations of curves by polygons. The proofs detailed in [3, 4] are slightly different, but still concentrated on curves. As mentioned before, the focus of this article is on finite collections of polygons and polygonal covers and came out of an attempt by the author to provide a proof of Jordan Curve Theorem.

Throughout we shall use the same label to refer to a map or its image. It is assumed that the reader is familiar with basic real analysis and topology.

2. Definitions

A Jordan curve is a homeomorphic image of S^1 in $\mathbb{R}^2$. An arc is a homeomorphic image of $[0, 1]$ in any $\mathbb{R}^n$, while a Jordan arc is an arc in $\mathbb{R}^2$. A path is any continuous image of $[0, 1]$ in any $\mathbb{R}^n$, although we will mostly be confined to the plane.

A polygon P is a Jordan curve that is piecewise linear i.e., a map $P : S^1 \to \mathbb{R}^2$ with finitely many points in S^1 between which P is a line. Similarly, we define a polygonal arc.

Given a polygon P, let $V(P) = \{v_1, \ldots, v_n\}$ be a minimal set such that P is a line between $v_i, v_{i+1}, i = 1, \ldots, n$ where $v_{n+1} = v_1$. By minimality, the two lines at v_i must be non parallel. The points in V are called the vertices of P, and the lines are called the edges of P.

Let $\mathcal{P} = \{P_1, \ldots, P_n\}$ be a finite collection of polygons. We define the finite set of "new vertices" $V(\mathcal{P})$ to be the set that contains

- $V(P_i), i = 1, \ldots, n$.
- Points of intersections of non parallel edges from different polygons in $\mathcal{P}$.

An edge of $\mathcal{P}$ refers to any edge of any $P \in \mathcal{P}$. Given $v \in V = V(\mathcal{P})$, take an open ball U around v disjoint from other points of V, and edges that don't contain v. Any such a ball shall be called the *zone* of v, and exists because both V and the set of edges of $\mathcal{P}$ are finite. Edges that pass through v induce a radius (or diameter) in the zone of v because U is an open disc with centre v.

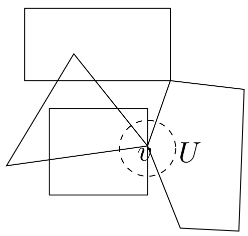

Figure 2.1. Zone U of vertex v with 6 radii

Two radii are *adjacent* if there is no other radii between them in at least one orientation (i.e., clockwise or anticlockwise). By the choice of $U, U \cap \mathcal{P}$ contains only radial lines. Similarly define a zone for $v \notin \cup_{P \in \mathcal{P}} P \setminus V(\mathcal{P})$ by avoiding all points in V and edges not passing through v. This time, there is only one diameter in U as only overlapping edges of P_i pass through v.

Suppose C_1, C_2 are compact subsets of $\mathbb{R}^2$. The distance d between C_1, C_2 refers to the minimum attained by the distance map (which is continuous) on $C_1 \times C_2$. It is zero if and only if $C_1 \cap C_2 \neq \emptyset$, and when $d \neq 0$, an open ball of radius d (or less) around any $x \in C_1$ does not intersect C_2.

3. Jordan curve theorem for polygons

In this section we prove that if P is a polygon, then $\mathbb{R}^2 \setminus P$ has two components, one bounded and the other unbounded, both having P as their boundaries.

3.1. **Parity function.**

Let P be a polygon and f be a direction not parallel to any of the edges in P. Since there are finitely many edges, such an f always exists. Because rotation is a linear homeomorphism, sending polygons to polygons, we may suppose that f is along the positive x-axis. We will now define a parity function $n\colon \mathbb{R}^2 \setminus P \to \{0,1\}$.

The zones at each point of P have two sectors given by non parallel radii at vertices, and a diameter at non vertices. For $x \notin P$, take R_x to be the ray (half-line) originating at x parallel to f. For each $p \in R_x \cap P$, we define a contribution $c(p)$ to $n(x)$ to be 1 if R_x intersects both sectors in the zone of p and 0 otherwise. Define

$$n(x) = \sum_{p \in R_x \cap P} c(p) \mod 2.$$

Here the empty sum is taken to be 0. The sum is finite because R_x is not parallel to any edge of P. Note that if p is not a vertex, then its contribution(see Figure 3.1) is 1, and if p is a vertex, it is 1 if and only if the edges at p lie on both sides of R_x.

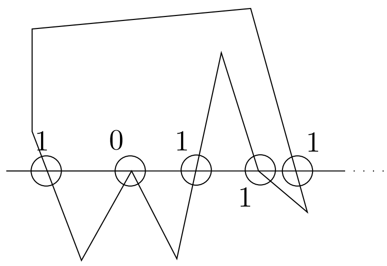

FIGURE 3.1. Zones and their contributions

Lemma 3.1.1. *The parity function constructed above is locally constant.*

Proof. Take $x = (a,b) \notin P$ and π to be the projection onto the y-axis. Suppose $R_x \cap P = \emptyset$. Consider the closed set $P_x = P \cap \{(c,d) \in \mathbb{R}^2 | c \geq a\}$, then $\pi(P_x)$ doesn't contain b, hence there is a neighbourhood $(b \pm \delta)$ disjoint from $\pi(P_x)$. It follows for p sufficiently close to x with $\pi(p) \in (b \pm \delta)$, $R_p \cap P = \emptyset$, so n is identically zero in a neighbourhood of x.

Suppose $R_x \cap P = \{a_1, \ldots, a_m\}$. In the zone of a_i, the edges of P induce two radii, neither of which is parallel to the x-axis. Projecting both to the y-axis we get two positive lengths, of which r_i is the smaller one. Take $r = \min\{r_1, \ldots, r_m\}$.

Next, let η be the smallest y-length of the edges of P. Since no edge is parallel to the x-axis, $\eta > 0$. Lastly, let $\delta > 0$ be such that $B(x,\delta) \cap P = \emptyset$. Now, shift the line R_x vertically within $\epsilon = \min\{r, \eta, \delta\}$ of the original to get a ray R'_x, i.e., R'_x is the half-line parallel to positive x-axis originating from a point (a, b') with $|b' - b| < \epsilon$. When comparing $R'_x \cap P$ with $R_x \cap P$, by the choice of r, we see that (see Figure 3.1)

- If a_i is not a vertex, then it is replaced by another non vertex
- If a_i is a vertex that contributes 1 to $n(x)$, then it is replaced by a non vertex
- If a_i is a vertex that contributes 0 to $n(x)$, then either it is replaced by two non vertices or removed altogether

So this was about the edges that both R_x, R'_x intersect. Now, suppose R'_x was shifted upwards to $y = b'$ and intersects an edge e that R_x did not.

It follows that e lies above the line $y = b$. Let v be the lower vertex of e and e' the other edge of v. Let N be the union of R_x, R'_x and the vertical line segement l from (a,b) to (a,b'). Observe that N divides the plane into two parts.

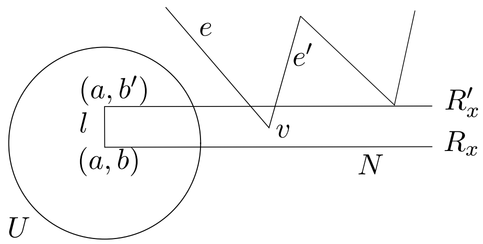

FIGURE 3.2. Parity is invariant under small vertical shifts

Since e touches R'_x but not R_x, v is either inside N or on its boundary. Note that e, e' cannot intersect l because l lies in $U = B(x, \epsilon)$. Since $\epsilon < \eta$, the other vertex of e' must be outside N. If e' doesn't intersect R_x, R'_x, then it intersects l as the other vertex is outside N. Thus, either R_x intersects e' or R'_x intersects e'.

If R_x intersects e', then by the choice of r, N cannot contain v. Thus, e' intersects R'_x but not R_x. So, e, e' together contribute 0 to n, by two non vertices or by v. Note that the lower vertex for e' is also v, so the edges that R'_x intersects but not R_x come in pairs.

So, n doesn't change under this vertical shift. Since every point of U is obtained by a vertical shift followed by a horizontal shift from x, and n remains invariant under these perturbations (by choice of ϵ), it is constant on U. The parity doesn't change under horizontal shifts because, by the choice of δ, the intersection points do not change. $\square$

Lemma 3.1.2. *Parity function is surjective.*

Proof. Let p be a point on some edge e of P that is not a vertex. There are two sectors in the zone of p. The parity for points in those sectors differ by 1 because the horizontal ray passes through an extra edge, namely e, which contributes 1. $\square$

As a consequence of the lemma, any small neighbourhood around non vertices has points of both parity. It follows that $\mathbb{R}^2 \setminus P$ is disconnected.

3.2. **Two components.** The following is taken from [2]. Cover P with a zone U_p at each $p \in P$ and obtain a finite subcover $U_1, \ldots, U_n$. Each U_i has two connected sectors, say U'_i, U''_i. Label the sectors so that

$$U'_i \cap U'_{i+1} \neq \emptyset \text{ and } U''_i \cap U''_{i+1} \neq \emptyset.$$

Then the unions $U'_1 \cup \cdots \cup U'_n, U''_1 \cup \cdots \cup U''_n$ are connected sets and there is a line from any $p \notin P$ to one of these sets that doesn't intersect P (draw the line from p to any edge and look at the first time it meets P). Thus, $\mathbb{R}^2 \setminus P$ has at most two components, hence exactly two components. Since n is continuous, the components are given by $n^{-1}(0), n^{-1}(1)$.

Of these, $n^{-1}(0)$ is unbounded for we can enclose P in a rectangle whose outside remains connected after removing P and is part of $n^{-1}(0)$, whereas $n^{-1}(1)$ is inside, hence bounded.

The proof of the lemma above shows that all points of P that are not vertices lie in the boundary of both components. Since boundary is closed, both components have P as their boundary.

Thus, $\mathbb{R}^2 \setminus P$ has two components, the bounded "inside" $i(P)$ and the unbounded "outside" $o(P)$ with P as their common boundary. Observe that these are path components of the complement, hence independent of the choice of the ray used to compute parity, so we may choose any convenient ray and check whether the parity is 0 (outside) or 1 (inside).

Given a collection of polygons $\mathcal{P}$ outside of $\mathcal{P}$ refers to $\cap_{P\in\mathcal{P}} o(P)$ and inside refers to $\cup_{P\in\mathcal{P}} i(P)$.

3.3. **Approximations.**

Lemma 3.3.1. *Suppose U is an open set in $\mathbb{R}^n$ and $J\colon [0,1] \to U$ a continuous path, with $J(0) \neq J(1)$. Given $\epsilon > 0$, there is a polygonal arc $P\colon [0,1] \to U$ with $P(0) = J(0), P(1) = J(1)$ such that every point of P is within ϵ of some point of J.*

Proof. For $x \in J$ choose $\mu_x > 0$ such that $B(x, 2\mu_x) \subseteq U$. From the cover $\{B(x,\mu_x)\}_{x\in J}$ obtain a finite subcover $\{B(x_1,\mu_1),\dots,B(x_m,\mu_m)\}$ and let $\mu = \min\{\mu_1,\dots,\mu_m\}$. Then for $x \in B(x_i,\mu_i)$, $B(x,\mu) \subseteq B(x_i, 2\mu_i) \subseteq U$.

We may take $\epsilon < \mu$. By uniform continuity of J, choose N such that

$$|t_1 - t_2| < 2/N \Rightarrow |J(t_1) - J(t_2)| < \epsilon.$$

Take $J(0), J(1/N), \dots, J(1)$ and draw line segments between consecutive points to get a piecewise linear path from $J(0)$ to $J(1)$. Observe that some of these lines may be degenerate because it is possible that $J(\frac{i}{N}) = J(\frac{i+1}{N})$, but $J(0) \neq J(1)$, so P is not altogether degenerate.

For $1 \le i \le N$,

$$\left| J\left(\frac{i-1}{N}\right) - J\left(\frac{i}{N}\right)\right| < \epsilon \Rightarrow J\left(\frac{i-1}{N}\right) \in B\left(J\left(\frac{i}{N}\right), \epsilon\right)$$

so the line between them is in $B(J(\frac{i}{N}),\epsilon) \subseteq U$ by the choice of $\epsilon < \mu$.

Thus, every point on the resulting union of lines is in U and within ϵ of some point of J. Next, replace any two overlapping lines by their union, this way the lines intersect at only finitely many points. Remove the loops as we go from $J(0)$ to $J(1)$ along the union of lines. Since there are finitely many loops (finitely many intersection points) this process terminates.

We will be left with a polygonal arc, i.e., a Jordan arc, $P \subset U$ such that every point of P is within ϵ of some point of J and ends of P are the ends of J. It is easy to obtain P as an injective image of $[0,1]$ using piecewise definitions. $\square$

Corollary 3.3.1. *If P is a polygon and $J \subset i(P)$ (or $J \subset o(P)$), then there is a polygonal approximation of J within given $\epsilon > 0$ that lies in $i(P)$ ($o(P)$).*

Corollary 3.3.2. *A connected open $U \subseteq \mathbb{R}^n$, is path connected, hence arc connected. In particular, for a polygon $P, i(P), o(P)$ are arc connected.*

Proof. If S_x denotes the path component of $x \in U$, then one can show that it is both closed and open, hence $S_x = U$. The rest follows from Lemma 3.3.1. $\square$

Corollary 3.3.3. *For a polygon P, given $x, y \in \overline{i(P)}$, there is a polygonal arc between them which lies entirely in $i(P)$, except possibly at the ends.*

Proof. For $x \in P$ there is a straight line from x to a point $x' \in i(P)$ that, except for x, lies in $i(P)$. Similarly obtain a point $y' \in i(P)$ for y. Using Corollary 3.3.2, obtain a polygonal arc from x' to y'. We get the required polygonal arc by joining these paths. $\square$

4. Some results about polygons

Let $\mathcal{P} = \{P_1, \ldots, P_n\}$ be a collection of polygons, $V = V(\mathcal{P})$. Let U be a zone of some fixed $v \in V$. The edges of $\mathcal{P}$ determine radii in U which in turn determine open sectors between adjacent radii. These sectors remain connected in the complement of $\mathcal{P}$ and are either inside or outside $\mathcal{P}$.

We say that v is a *regular* vertex if for every component outside $\mathcal{P}$, there is at most one sector that intersects it, otherwise v is *singular.* This notion doesn't depend on U as long as it has it is disjoint from other points of V and edges that v doesn't lie on, i.e., as long as U is a zone of v.

When v is not a vertex, its zone has just a diameter, and one of the sectors lies in some $i(P_i)$, so non vertices are always regular. We say that $\mathcal{P}$ is regular if all points in $\mathcal{P}$ are regular.

Polygons P, Q are said to have *shallow intersection* if $i(P) \cap i(Q) = \emptyset$. The collection $\mathcal{P}$ is said to have shallow intersection if $i(P_j) \cap i(P_k) = \emptyset$, $\forall j \neq k$. In this case, no edge of P_i goes into $i(P_j)$ for any j. The union $\overline{i(P_1)} \cup \cdots \cup \overline{i(P_n)}$ is said to be a *shallow union* when $\mathcal{P}$ has a shallow intersection.

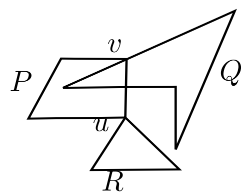

Figure 4.1. Here v is a regular vertex, u is singular, P, Q don't have shallow intersection, P, R and Q, R do have shallow intersection.

Theorem 4.0.1. *Let G be a graph. If every vertex of G has degree 2, then G is a disjoint union of cycles.*

For a proof see [5].

Theorem 4.0.2. *Let P, Q be two polygons, $V = V(\{P, Q\})$. Suppose $P \not\subset \overline{i(Q)}, Q \not\subset \overline{i(P)}$. Then $\overline{i(P)} \cup \overline{i(Q)}$ is a shallow union $\overline{i(R_1)} \cup \cdots \cup \overline{i(R_m)}$ for some polygons $R_1, \ldots, R_m$ whose vertices come from V with edges subsegments of the edges of P, Q.*

Lemma 4.0.1. *With the setting as above, let $v \in P \cup Q$, then its zone has at most 4 sectors and no two of them are in the same component of $i(P) \setminus \overline{i(Q)}$.*

Proof. The zone U of v has at most 4 radii - two each from P, Q and therefore at most 4 sectors. Suppose a sector S bounded by r_1, r_2 is in $i(P) \setminus \overline{i(Q)} = i(P) \cap o(Q)$. Because r_1, r_2 come from P, Q the sectors adjacent to S cannot be in $i(P) \cap o(Q)$. For example, if r_1 is from P, then the other side of r_1 is in $o(P)$. Therefore, if U has 2 or 3 sectors then at most one is in $i(P) \setminus \overline{i(Q)}$.

We are left with the case when U has 4 sectors, two of which, S_1, S_2 are in $i(P) \cap o(Q)$ (by the discussion above, it is at most two). This can happen only when the radii from Q at v and the corresponding sector inside Q are all in $i(P)$. Now, suppose S_1, S_2 are in some component R of $i(P) \setminus o(Q)$, then there is a polygonal path from a point in S_1 to S_2 that lies inside R. Using radii from v, we construct a polygon $\tilde{P}$ that lies in $\overline{R \cup \{v\}}$, hence in $i(P) \cup \{v\}$ with $\tilde{P} \cap P = \{v\}$.

We claim that $Q \subset \overline{i(\tilde{P})} \subseteq \overline{i(P)}$ which is a contradiction. For the first inclusion, observe that the radii from Q at v are in $i(\tilde{P})$, and Q doesn't intersect $\tilde{P}$ (except

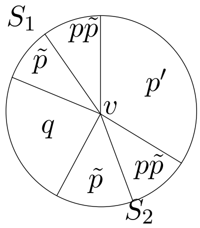

FIGURE 4.2. In the adjacent figure we use lower case letters to indicate sectors inside polygons, product to denote intersection and prime to denote outside.

at v), therefore by connectedness, Q must be in $\overline{i(\tilde{P})}$. For the second inclusion, let $x \in i(\tilde{P})$, then there is a path from x to any $y \in \tilde{P}$ lying in $i(\tilde{P})$. This is a path from x to a point in $i(P)$, so it suffices to prove that it doesn't intersect P.

To this end, if it did intersect P, then because the path lies in $i(\tilde{P})$, we conclude that there is some $x_0 \in o(P) \cap i(\tilde{P})$. However, this is impossible for $o(P) \subset o(\tilde{P})$. $\square$

Proof of Theorem 4.0.2. The proof has the following steps: we show that components of $i(P) \setminus \overline{i(Q)}$ have boundaries that are unions of polygons, we then show that this boundary is actually a single polygon and the last step completes the proof.

Step 1: Boundary of components is a union of polygon

Let R be a component of $i(P) \setminus \overline{i(Q)}$. Given $x \in \partial R$, we know $x \in P \cup Q$ and let U be a zone of x. From the lemma, U has at most 4 sectors and exactly one is in R (because $x \in \partial R$).Let r_1, r_2 be the radii bounding this sector. If $x \notin V$, then $r_1 \parallel r_2$, hence an open line segment is part of $\partial R¿$ When l is maximal, the ends must be in $\partial R \cap V$ as otherwise we can extend it. If $x \in V$, then $r_1 \not\parallel r_2$.

Thus, ∂R is a union of line segments with ends in V and at these points there are exactly two non parallel line segments that are part of ∂R (because of the previous lemma). Therefore, ∂R is a graph with vertices from V each of degree 2. By Theorem 4.0.1 it is a union of disjoint cycles, in this case polygons. Obeserve that if P' is one such boundary polygon, then one of the sectors at every point in P' is in R.

Step 2: Boundary has exactly one polygon

With R as above, suppose Q_1 is a boundary polygon of R. Then R must either lie inside or outside Q_1 because R is path connected. Suppose it lies outside Q_1. Any path from $i(Q_1)$ to $o(Q_1)$ must then pass through R, in particular through $i(P)$. However, observe that $o(P) \cap o(Q_1) \neq \emptyset$, so we conclude that $i(Q_1) \cap o(P) = \emptyset$.

The edges of Q_1 are subsegments of edges in $P \cup Q$ and for any such edge, we must have both sides in $i(P)$ (because the outside part is in R, hence in $i(P)$ and the inside part is disjoint from $o(P)$). Therefore, these edges must be subsegements of edges of Q which forces $Q = Q_1$ and in this case, Q is inside $\overline{i(P)}$ which is a contradiction. Therefore, R must be inside Q_1 and in this case, by connectedness of $R, \partial R = Q_1$ is a polygon.

Step 3:

So the components of $i(P) \setminus \overline{i(Q)}$ lie inside their polygonal boundaries whose vertices come from V. Since V is finite, there can be only finitely many polygons, say $R_1, \ldots, R_{m-1}$. The inside regions of R_i, R_j cannot intersect for $i \neq j$ because they would then describe the same component. By construction of R_i, the inside regions of R_i, Q cannot intersect. Set $R_m = Q$, then the collection $\{R_1, \ldots, R_m\}$

has shallow intersection. It is clear that $\overline{i(P)} \cup \overline{i(Q)} = \overline{i(R_1)} \cup \cdots \cup \overline{i(R_m)}$ and that the edges of R_i are subsegments of the original edges. $\square$

In the proof, we partitioned $i(P) \setminus \overline{i(Q)}$ into polygonal regions we call this the *reduction* of P by Q.

Theorem 4.0.3. *Let $P_1, \ldots, P_n$ be polygons, set $V = V(\{P_1, \ldots, P_n\})$. Suppose $P_i \not\subset \overline{i(P_j)}$ for $i \neq j$. Then $\overline{i(P_1)} \cup \cdots \cup \overline{i(P_n)}$ is a shallow union of $\overline{i(R_1)}, \ldots, \overline{i(R_m)}$ for some polygons $R_1, \ldots, R_m$ with vertices coming from V and edges subsegement of the original edges.*

Proof. By the previous theorem, the statment is true for $n = 2$. Assume $n \geq 3$ and that the statment is true for $n - 1$. First reduce P_1 by P_2 to obtain polygons $R^2_1, \ldots, R^2_k$. The set of new vertices is a subset of the original and the collection $R^2_1, \ldots, R^2_k, P_2$ has shallow intersection.

Ignoring any R^2_i for which $R^2_i \subset \overline{i(P_3)}$, which means $i(R^2_i) \subseteq i(P_3)$ we take $R^2_i \not\subset \overline{i(P_3)}$. By construction, $\overline{i(R^2_i)} \subset \overline{i(P_1)}$, so $P_3 \not\subset \overline{i(R^2_i)}$. Reduce each R^2_i by P_3 to obtain polygons $R^{23}_{i1}, \ldots, R^{23}_{ik_i}$ such that $i(R^{23}_{ij}) \subset i(R^2_i)$. Since $R^2_1, \ldots, R^2_k$ had shallow intersection, the collection $\{R^{23}_{ij} | 1 \leq i \leq k, 1 \leq j \leq k_i\}$ has shallow intersection. In fact, these R^{23}_{ij} also have shallow intersection with P_2, P_3.

Continuing this way we next reduce each R^{23}_{ij} by P_4 and so on to arrive at a collection $R_1, \ldots, R_p$ each contained in $\overline{i(P_1)}$ having shallow intersection among themselves and with $P_2, \ldots, P_n$. At each stage the union $\overline{i(P_1)} \cup \cdots \cup \overline{i(P_n)}$ is preserved and the set of vertices is a subset of the original.

Using induction hypothesis, obtain polygons $R_{p+1}, \ldots, R_m$ with vertices from $V(\{P_2, \ldots, P_n\}) \subset V$ and shallow intersection such that $\overline{R_{p+1}} \cup \ldots \overline{i(R_m)} = \overline{i(P_2)} \cup \cdots \cup \overline{i(P_n)}$.

At each stage above, the set of vertices is a subset of the original V and the edges are subsegments of the original edges. Together $R_1, \ldots, R_m$ satisfy the requirements of the statement, proving the theorem for n and by induction for every n. $\square$

Remark. Since the set of new vertices is a subset of the original V and all edges are subsegments of the original edges, the zones around each new vertex doesn't change from the original. As a consequence, if $\{P_1, \ldots, P_n\}$ is regular, then so is the new collection of polygons (the common outside does not change from the original).

Given a collection of polygons $\mathcal{P}$, the *intersection graph* is a graph with a vertex v_P for every $P \in \mathcal{P}$ and an edge between v_P, v_Q if $\overline{i(P)} \cap \overline{i(Q)} \neq \emptyset$. If there is an edge between v_P, v_Q, then there is a path from any point in $\overline{i(P)}$ to any point in $\overline{i(Q)}$ that lies entirely in their union.

Theorem 4.0.4. *Let $\mathcal{P} = \{P_1, \ldots, P_n\}$ be a regular collection of polygons with shallow intersection and connected intersection graph. Then all components in the complement of $\mathcal{P}$ have polygonal boundaries.*

Proof. Since $\mathcal{P}$ has shallow intersection, for every $j, i(P_j)$ is unaffected by the presence of other polygons, so it stays connected. If R is a component in the complement of $P = P_1 \cup \cdots \cup P_n$ that intersects some $i(P_j)$, then $R = i(P_j)$ for $i(P_j)$ is connected and there are no paths to points outside. Clearly $i(P_j)$ has a polygonal boundary.

Let R be a component on the outside and let $V = V(\mathcal{P})$. For $x \in \partial R \setminus V$, say from some edge e, the zone has two sectors. Since $x \in \partial R$, one of these sectors lies in the common outside. The other sector lies inside some P_i (that which e is a part of). Reasoning as before, there is an open segment around x in e that is part of ∂R.

For $x \in \partial R \cap V$ its zone has exactly one sector lying in R by regularity. As before, two radii at v are part of ∂R. We conclude that ∂R is a union of disjoint polygons.

Suppose Q_1, Q_2 are two boundary polygons, then they are disjoint, so either $Q_1 \subset i(Q_2)$ or $Q_1 \subset o(Q_2)$. In the first case R must lie between Q_1, Q_2 (although there may be other boundary polygons in between as well) and this contradicts the connectedness of the intersection graph because a path from inside of Q_1 (hence inside some P_i) to outside of Q_2 must pass through R (which lies in the common outside of $\mathcal{P}$). In the second case, R must lie outside both Q_1, Q_2 again contradicting the path connectedness of the intersection graph (this time there is no path from inside Q_1 to inside Q_2 which doesn't pass through R). Therefore R must have exactly one polygon in its boundary. □

Remark. Observe that the boundaries of components outside $\mathcal{P}$ are polygonal regardless of whether $\mathcal{P}$ has a shallow intersection.

5. Some more results about polygons

Let a, b be points in the plane, l_1, l_2, l_3 be polygonal arcs from a to b that intersect only at the ends. Using two of the three arcs, we form polygons P_1, P_2, P_3, where P_i doesn't use l_i. Let n_1, n_2, n_3 be the corresponding parity functions, calculated using a direction not parallel to any of the edges in l_1, l_2, l_3.

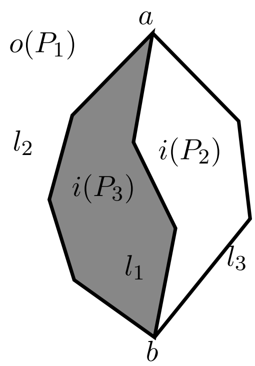

Figure 5.1.

Lemma 5.0.1. $n_1 + n_2 + n_3 = 0$.

Proof. Take $x \notin L = l_1 \cup l_2 \cup l_3$ and let R_x be the ray originating from x used to compute the parity.

Suppose R_x passes through a. At a, take a zone U small enough to avoid x. Denote by r_1, r_2, r_3 the radii in U induced by l_1, l_2, l_3 respectively. The diameter d induced by R_x cuts U in half. If two of the three radii, say r_1, r_2, lie on the same side then the contribution by a is 0 to n_3 and 1 to n_1, n_2. Otherwise all three radii are on the same half and the contribution to each n_i is 0. In both cases the contribution to the sum is 0. The same holds for b.

If p is a point other than a, b, then it appears in exactly one of l_1, l_2, l_3 and its zone has two sectors. Since each l_i is part of two polygons, the contribution from p

is counted twice in the sum $n_1(x) + n_2(x) + n_3(x)$. It follows that $n_1 + n_2 + n_3$ is identically zero. $\square$

As a consequence, every point in the complement of L is either inside exactly two polygons or outside all three. In the zone of a, there are three sectors and it is easy to see that one of the three sectors, say the one bounded by r_2, r_3, must lie outside all P_i. Then the radius r_1 lies inside P_1 and by connectedness, l_1 excluding a, b, lies inside P_1. So, some $l_i \setminus \{a, b\}$ is inside P_i.

Suppose l_1 lies inside P_1. Bound all the polygons in a large square and take a point p outside this square, so $p \in o(P_1) \cap o(P_2) \cap o(P_3)$. For $x \in o(P_1)$, there is a path from x to p lying outside P_1. Since l_1 is inside P_1, this path cannot intersect l_1, hence P_2, P_3. It follows that $x \in o(P_2) \cap o(P_3)$. Therefore, $o(P_1) \subseteq o(P_2) \cap o(P_3)$ and $n_1(x) = 0 \Rightarrow n_2(x) = n_3(x) = 0$.

Thus there are three components $o(P_1), i(P_2), i(P_3)$ in the complement of L with boundaries P_1, P_2, P_3 respectively. The zones at a, b have three sectors, one for each component. For points in l_1 other than a, b both sectors are in $i(P_1)$.

Corollary 5.0.1. *With the setting as above, if $\phi\colon [0,1] \to \mathbb{R}^2$ is a path with $\phi(0) \in o(P_1), \phi(1) \in i(P_2)$ that doesn't intersect $i(P_3)$, then it intersects l_3.*

Proof. Because $o(P_1) \subseteq o(P_2), \phi \cap P_2 \neq \emptyset$. Suppose $\phi \cap l_3 = \emptyset$, then it must intersect $l_1 \setminus \{a, b\}$. Let $t_1 = \inf\{\phi^{-1}(\phi \cap l_1)\}$. Since $\phi(0) \notin i(P_1)$, we have $t_1 \neq 0$. Let U be a zone of $\phi(t_1) \in l_1$, by continuity there is an interval (t_0, t_2) such that $\phi((t_0, t_1)) \subset U$.

One of the sectors of U is in $i(P_3)$ and the other in $i(P_2)$. For $t_0 < t < t_1$, by minimality of $t_1, \phi(t)$ cannot lie on the radii in U. By hypothesis, $\phi(t)$ cannot lie in the sector contained in $i(P_3)$. So, $\phi(t)$ is in the sector that is contained in $i(P_2)$ and we can find a $t' < t$ such that $\phi(t') \in P_2$. Since $\phi \cap l_3 = \emptyset$ by assumption, $\phi(t') \in l_1 \setminus \{a, b\}$ contradiction minimaity of t_1. So, $\phi \cap l_3 \neq \emptyset$. $\square$

Corollary 5.0.2. *If $l_1, \ldots, l_n$ are n polygonal paths between a, b that intersect only at a, b, then the complement of $L = l_1 \cup \cdots \cup l_n$ has $n - 1$ bounded components and one unbounded.*

Proof. Induction. $\square$

Lemma 5.0.2. *Suppose P, Q are polygons with $P \not\subset \overline{i(Q)}$. Suppose $Q \cap i(P)$ is a non empty, connected (open) path whose ends are different. Then we can reduce P by Q in the sense that $i(P) \setminus \overline{i(Q)} = i(R)$ for some polygon R.*

Proof. $L = \overline{Q \cap i(P)}$ is a polygonal arc between two points $a, b \in P, a \neq b$. Let L_1, L_2 be the two paths between a, b along P. So L_1, L, L_2 are three polygonal arcs that intersect only at the ends and we know that L (except for the ends) is contained inside P. Let $P_1 = L_1 \cup L, P_2 = L_2 \cup L$.

We have $i(P_1) \cap Q = \emptyset$ because $i(P_1) \cap Q \subset i(P) \cap Q \subset L$. So, $i(P_1)$ is connected in the complement of Q, hence inside or outside Q. Similarly $i(P_2)$ is inside or outside Q. Take $z \in L \setminus \{a, b\}$, then the zone of z has two sectors, one from each $i(P_1), i(P_2)$ and these two sectors should also come from $i(Q), o(Q)$. So, one of $i(P_1), i(P_2)$ is inside Q and the other outside. Assume $i(P_2) \subset i(Q)$, then

$$i(P) \setminus \overline{i(Q)} = (i(P_1) \cup i(P_2) \cup L) \setminus \overline{i(Q)} = i(P_1).$$

So, the reduction of P by Q is the polygon P_1. $\square$

5.1. **Removing singular vertices.** Let $\mathcal{P} = \{P_1, \ldots, P_n\}$ be a collection of polygons, $V = V(\mathcal{P})$. If $v \in V$ is a singular vertex, with zone U, then there is more than one sector in U that comes from a component C outside $\mathcal{P}$.

Choose radii r_1, r_2 such that all sectors of U lying in C lie in a sector S determined by r_1, r_2. Let S to be minimal in the sense that no smaller sector in S contains all the sectors of U that intersect C. Then the sectors in S that have r_1 or r_2 in their boundary must lie in C. We note $r_1 \neq r_2$ and that they cannot lie inside any P_i.

By assumption $S \not\subseteq C$, so it contains some radii of U. In fact, there must be at least two radii in S as if there is only one radius s, then S has two parts - sectors sr_1, sr_2. By the assumptions on C, v at least two of the sectors in S must be in C outside $\mathcal{P}$, which means that both sides of s are outside $\mathcal{P}$ which is impossible.

As we go from r_1 to r_2 through S, let s_1, s_2 be the first and second radii we meet before reaching r_2. The sector $r_1 s_1$ in S must lie in C, therefore outside $\mathcal{P}$ and the sector S' determined by s_1, s_2 must lie in some $i(P_i)$ because s_1 cannot have both its sides outside $\mathcal{P}$.

Let a, b be the midpoints of s_1, s_2 respectively. Let l be a polygonal path from a to b that lies entirely in S'. Form the polygon $Q = av \cup l \cup bv$.

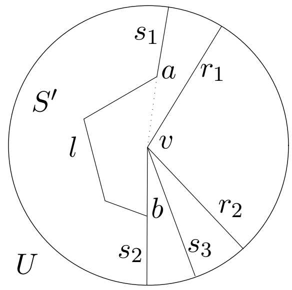

Figure 5.2. Line bv may also be removed, av is removed

For each i, by the choice of s_1, s_2 either $S' \subset \overline{i(P_i)}$ or $S' \subset o(P_i)$. Therefore, the intersection $Q \cap i(P_i)$ is either empty or the open path l or the union $l \cup bv$ in case s_2 is inside P_i. In the last two cases, the end points of the intersection are different, so we may apply Lemma 5.0.2 to reduce P_i by Q and obtain a polygon P'_i ($= P_i$ when $Q \cap i(P_i) = \emptyset$).

One side of av is outside $\mathcal{P}$ and the other in $i(Q)$, so av is outside all P'_i, hence $i(Q)$ is now part of C. The set $V(P'_1, \ldots, P'_n)$ includes the vertices of Q in addition to the original V. At the zones of $a(b)$ we have three radii from $l, s_1(s_2)$. So, a, b are regular. The other points of Q other than v are regular as they have only two radii from l in their zones.

Lastly, note that the loss of interior from P_i is just $i(Q)$. So, anything not in $i(Q)$ covered by $\mathcal{P}$ is still covered by $\overline{i(P'_1)} \cup \cdots \cup \overline{i(P'_n)}$. Therefore, if w is a regular vertex outside U, it cannot become singular upon these reductions.

Now, shrink U to avoid l. The sectors contained in C still lie in S. Since av is removed, the number of radii between r_1, r_2 has reduced by at least 1. Repeating the steps above, we make sure that S has no other radius in between, giving us one sector in U that intersects C. Repeating this process for other sectors of U makes v a regular point.

The process terminates as there are finitely many radii and we are left with one less singular point. Modify $\mathcal{P}$ in a finite number of steps so that the resulting V has no singular points. Note that at each stage the cardinality of $\mathcal{P}$ does not change.

5.2. **Jordan-like results.** Let $\mathcal{P} = \{P_1, \ldots, P_n\}$ be a regular collection of polygons with shallow intersection and a connected intersection graph. In this case, $V = V(\mathcal{P})$ is the collection of vertices in $\mathcal{P}$. Let C be a bounded component outside $\mathcal{P}$ with polygonal boundary R (Theorem 4.0.4).

Suppose $u_1, u_3 \in R$ are distinct points. Let L_1, L_2 be the two paths along R from u_1 to u_3. Fix polygons $R_1, R_3 \in \mathcal{P}$ that contain u_1, u_3 respectively. We may have $R_1 = R_3$. Traversing L_1 from u_1 to u_3 gives each $p \in L_1$ a backward edge and a forward edge. In the zone of p, these edges determine radii b, f respectively and two sectors one of which is inside R.

All other radii are in the other sector S. As we go from b to f through S, form the sequence of sectors that lie in $i(P)$ for some $P \in \mathcal{P}$. By shallow intersection, such P are unique. If this sequence has just one term, then points on b, f also have the same sequence, so an open segment of R around p also has the same associated sequence. The ends of a maximal such segment must lie in V.

So the only $p \in L_1$ that can have more than one element in their associated sequence of polygons are those from $V(R) = V \cap R$, a finite set. Now start at the sector corresponding to R_1 at u_1 and go through the sequence of polygons associated to points in $V(R) \cap L_1$ along L_1 in the manner described, backward edge to forward edge outside R, till we reach the sector corresponding to R_3 at u_3. This sequence we call $S(L_1)$. Similarly define $S(L_2)$ starting at u_1 and going to u_3.

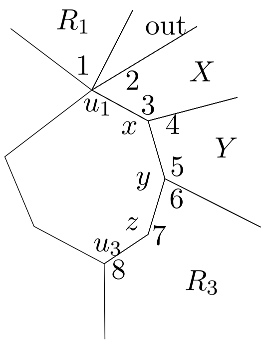

FIGURE 5.3. Obtaining $S(L_1)$

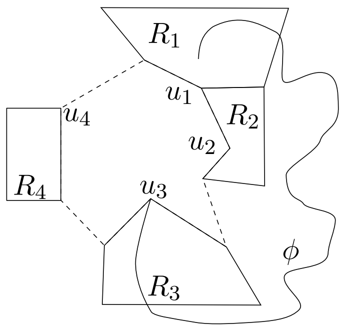

FIGURE 5.4. Set up for Theorem 5

For example, in the figure above, the sequence $S(L_1)$ will contain the sectors numbered in order, i.e., the sectors $1, 2$ at u_1 (skipping the sector lying outside the cover) then sectors $3, 4$ at x and so on till sector 8 at u_3.

Suppose we can choose polygons $R_2 \in S(L_1), R_4 \in S(L_2)$ different from R_1, R_3 (we can have $R_2 = R_4$) and points $u_2 \in L_1 \cap V(R), u_4 \in L_2 \cap V(R)$ that are part of R_2, R_4 respectively. Assume u_1, u_2, u_3, u_4 are distinct.

Let ϕ be a path (continuous image of $[0, 1]$) from a point in $\overline{i(R_1)}$ to one in $\overline{i(R_3)}$ that lies outside R, except possibly for the ends and if an end does lie on R, then it is u_1 or u_3. Let ψ be a path from a point in $\overline{i(R_2)}$ to one in $\overline{i(R_4)}$ with similar conditions. Furthermore, assume

- ϕ doesn't intersect $\overline{i(R_2)}, \overline{i(R_4)}$ with the exception of u_1, u_3

- ψ doesn't intersect $\overline{i(R_1)}, \overline{i(R_3)}$ with the exception of u_2, u_4

Theorem 5.2.1. *The paths ϕ, ψ should intersect.*

Proof. Suppose they do not intersect. Assume $\phi(0) \in \overline{i(R_1)}, \phi(1) \in \overline{i(R_3)}$. If $\phi(0) \neq u_1$, let l_1 be a polygonal arc from u_1 to $\phi(0)$ that, except for the ends, lies in $i(R_1)$. Note that l_1 doesn't intersect $\overline{i(R_2)}, \overline{i(R_4)}$ except possibly at u_1. Extend ϕ by travelling along l_1 from u_1 till the first time we meet ϕ and from there along ϕ. If $\phi(1) \neq u_3$, extend ϕ using a similar polygonal arc l_3 from u_3 to $\phi(1)$.

Since the open segments of l_1, l_3 lie inside R_1, R_3, they cannot intersect ψ. So, ψ cannot intersect the extension of ϕ, which we continue to denote by ϕ. Even after the extension, ϕ lies outside R except for the ends and the ends are u_1, u_3 because l_1, l_3 can intersect R only at u_1, u_3 respectively. Furthermore, ϕ continues to not intersect $\overline{i(R_2)}, \overline{i(R_4)}$ except possibly at u_1, u_3.

Let $d > 0$ be the distance between (the extended) ϕ and ψ and $\epsilon = \min\{d, |u_1 - u_3|\}$. Around u_1, take a zone of radius $< \epsilon/2$ and a radial line r_1 from u_1 to a point $x \in \phi$ in this zone. Note that $x \neq u_3$ is outside R and does not lie in any sector corresponding to $\overline{i(R_2)}, \overline{i(R_4)}$. So, the line r_1 doesn't intersect $\overline{i(R)}, \overline{i(R_2)}, \overline{i(R_4)}$ except at u_1. Similarly, take a line r_3 from u_3 to a point $y \in \phi$ that does't intersect $\overline{i(R)}, \overline{i(R_2)}, \overline{i(R_4)}$ except at u_3.

Let ϕ' be the restriction of ϕ between x, y. Note that $x \neq y$ by the choice of ϵ. We know that ϕ' lies outside R and is at a distance of some $\mu > 0$ from the closed set $\overline{i(R_2)} \cup \overline{i(R_4)}$. Approximate ϕ' within $\min\{\epsilon, \mu\}$ by a polygonal arc P that lies outside R.

Let Q be the polygonal arc $r_1 + P + r_3$ from u_1 to u_3. We make the following observations

- Q lies outside R except for the ends u_1, u_3 which lie on R
- Q doesn't intersect $\overline{i(R_2)}, \overline{i(R_4)}$ except possibly at u_1, u_3. In particular $Q \cap i(R_2) = Q \cap i(R_4) = \emptyset$
- By the choice of ϵ, ψ doesn't intersect r_1, r_3 and P, hence it doesn't intersect Q

Extend ψ similar to ϕ using polygonal arcs lying in $i(R_2), i(R_4)$ to get a path from u_2 to u_4. Since Q doesn't intersect $\overline{i(R_2)}, \overline{i(R_4)}$ and u_i are distinct, Q doesn't intersect this extension of ψ.

The paths L_1, L_2, Q are all polygonal arcs between u_1, u_3 that intersect only at the ends and Q lies outside $L_1 \cup L_2$. We may assume that L_2 is inside $L_1 \cup Q$. When looking at L_1, L_2, Q, ψ, the zone of u_4 has two parts, one lying in $i(R)$ and the other in $i(L_2 \cup Q)$, and ψ enters the second sector. Similarly, u_2 has two sectors in its zone and ψ enters that sector lying outside $L_1 \cup Q$ (the other is inside R). By Corollary 5.0.1, ψ should intersect Q which is a contradiction. $\square$

The main idea of the proof is to use Corollary 5.0.1. We can extend the result using a few assumptions. If ϕ doesn't have u_1 as an end for example, then the extension of ϕ must start at u_1 and enter $i(R_1)$. At the same time, if ψ doesn't have u_2 as an end, then the extension of ψ must start at u_2 and enter $i(R_2)$. It is easy to see that, in this case, we can deal with $u_1 = u_2$, because of the order in $S(L_1)$ which allows the use of Corollary 5.0.1. Similarly, we can have (observe that u_1, u_3 can be interchanged and intuitively this is just turning the diagram upside down)

- $u_1 = u_4$ when ϕ doesn't have u_1 as an end and ψ doesn't have u_4 as an end
- $u_3 = u_2$ when ϕ doesn't have u_3 as an end and ψ doesn't have u_2 as an end etc.

An extreme case is $u_1 = u_2 = u_4$ when ϕ doesn't have u_1 as an end and ψ doesn't have u_2, u_4 as its ends.

However, the proof doesn't apply directly to the case when all u_i are the same, because to construct L_1, L_2 we need $u_1 \neq u_3$. Although, such an extension can be similarly proved by looking at the sequence of polygons at u_1 and assuming that the R_i come in the appropriate order : R_1 between R_4, R_2 and R_2 between R_1, R_3. We will also need ϕ, ψ to be outside R. If they don't intersect, we obtain Q as before, and this time it is directly a polygon. At u_1 we have two sectors determined by Q and by the order forced on R_i, we see that ψ is a path from a point in one of these sectors to the other, i.e., a path from $i(Q)$ to $o(Q)$. So it must intersect Q, a contradiction.

Remark. Observe that we are close to having a version of the Jordan curve theorem because ϕ above corresponds to a curve and ψ is a path which we have forced to go from "inside" ϕ to outside and we have shown that ψ must intersect ϕ.

6. A theorem about arcs

Suppose we have points a, b in the plane and arcs ϕ_1, ϕ_2 from a to b that intersect only at a, b. Set I_1, I_2 to be $[0,1]$ and suppose $\phi_i \colon I_i \to \mathbb{R}^2$ with $\phi_i(0) = a, \phi_i(1) = b, i = 1, 2$. Since ϕ_1 is a homeomorphism, ϕ_1 and its inverse are uniformly continuous. Fix $\epsilon > 0$, and choose $\delta > 0, \epsilon > \epsilon' > 0$ such that

$$|t_1 - t_2| < 3\delta \Rightarrow |\phi_1(t_1) - \phi_1(t_2)| < \epsilon$$

$$|\phi_1(t_1) - \phi_1(t_2)| \leq \epsilon' \Rightarrow |t_1 - t_2| < \delta.$$

6.1. A special covering.

At a, b take open squares with disjoint closures and diameter $< \epsilon'$. For $t \in (0,1)$, pick an open square of diameter $< \epsilon'$ around $\phi_1(t)$ whose closure doesn't intersect ϕ_2. By compactness of ϕ_1 obtain a finite open subcover $\mathcal{P} = \{P_1, \dots, P_n\}$.

We refine this cover in steps. At each step we ensure that there are unique polygons T_1, T_2 such that $a \in \overline{i(T_1)}, b \in \overline{i(T_2)}, \overline{i(T_1)} \cap \overline{i(T_2)} = \emptyset$ and if $\phi_2 \cap \overline{i(P)} \neq \emptyset$, then $P = T_1$ or $P = T_2$. This is true for $\mathcal{P}$.

(1) First we remove singular vertices (5.1) in the cover $\mathcal{P}$. A point $v \in \phi_1$ is inside some polygon $\tilde{P}$ which means that a zone of v has no sectors outside $\mathcal{P}$, hence v is regular. If $v \notin \phi_1$ is singular, then we choose a zone that avoids ϕ_1. The modifications to $\mathcal{P}$ while making v regular involves removing a part of this zone from each $P \in \mathcal{P}$, therefore even after these modifications ϕ_1 is covered by the resulting polygons (and their insides).

   If the original collection was $\{P_1, \dots, P_n\}, T_1 = P_1, T_2 = P_n$, then each P_i is replaced (as detailed in 5.1) with a $P'_i, \overline{i(P'_i)} \subseteq \overline{i(P_i)}$. It is easy to see that $T_1 = P'_1, T_2 = P'_n$ after making v regular. Similarly remove other singular vertices. At each stage we can find suitable T_1, T_2.

(2) Remove redundant polygons to arrive at a minimal cover $\{P_1, \dots, P_n\}$ with $T_1 = P_1, T_2 = P_n$. Note that T_1, T_2 cannot be redundant. Now if $P_i \subset \overline{i(P_j)}$, then $i(P_i) \subset i(P_j)$ and P_i is redundant, so no $P_i \subset \overline{i(P_j)}$. Take the collection $\overline{P_2, \dots, P_{n-1}}$ and apply Theorem 4.0.3 to obtain a collection $R_1, \dots, R_m$. Now $\overline{i(R_j)} \subseteq \overline{i(P_k)}$ for some $2 \leq k \leq n-1$, so $\overline{i(R_j)} \cap \phi_2 = \emptyset$.

(3) So, we have the collection $\{P_1, R_1, \ldots, R_m, P_n\}$. We cannot have $P_1 \subset \overline{i(R_j)}$ and remove any R_j for which $R_j \subset \overline{i(P_1)}$. So we may take $R_j \not\subset \overline{i(P_1)}, 1 \leq j \leq m$. Reduce (Theorem 4.0.2) each R_j by P_1 to get a collection $\{P_1, R'_1, \ldots, R'_s, P_n\}$.
(4) As in Step 3, reduce each R'_j by P_n and obtain $\{P_1, R''_1, \ldots, R''_t, P_n\}$. The sets $i(P_1), i(P_n)$ are unaltered and $\overline{i(P_1)} \cap \overline{i(P_n)} = \emptyset$. The union $\cup \overline{i(P_j)} \supset \phi_1$ is preserved and the collection has shallow intersection.

Furthermore, for any i, there are $j, k, 2 \leq l \leq n-1$ such that

$$\overline{i(R''_i)} \subseteq \overline{i(R'_j)} \subseteq \overline{i(R_k)} \subseteq \overline{i(P_l)} \Rightarrow \phi_2 \cap \overline{i(R''_i)} = \emptyset.$$

So, we can take $T_1 = P_1, T_2 = P_n$. For the sake of convenience let continue to call this collection $\mathcal{P} = \{P_1, \ldots, P_n\}$ and set $V = V(\mathcal{P})$.

Lastly, we make sure that if two polygons intersect, then the intersection contains points of ϕ_1. For polygons P, Q with shallow intersection, the intersection of edges $e \in P, f \in Q$ is either a point (short intersection), or a closed segment (long intersection). Henceforth short and long intersections refer to those that don't contain points of ϕ_1. This is a two step process.

First we make sure that long intersections have points of ϕ_1. Then we look at short intersections in $P \cap Q$ not part of any long intersection in $P \cap Q$. Henceforth, short intersections refer only to this specific type. The notion of short intersection now depends on the polygons P, Q for $e \cap f$ may be short in $P \cap Q$, but not in $P' \cap Q'$ for some P', Q' (see Figure 6.1). The zone of a short intersection $v \in P \cap Q$ has 2 radii each from P, Q.

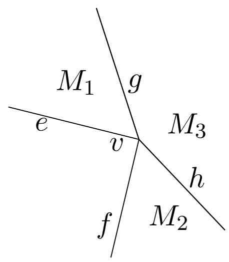

FIGURE 6.1. In the adjacent figure, the intersection $g \cap g$ is a long intersection. The vertex $v = e \cap f$ is a short intersection when considered as a point in $M_1 \cap M_2$, but the same vertex is not a short intersection in $M_1 \cap M_3$ and $M_2 \cap M_3$ because it appears in the long intersections $g \cap g, h \cap h$ respectively.

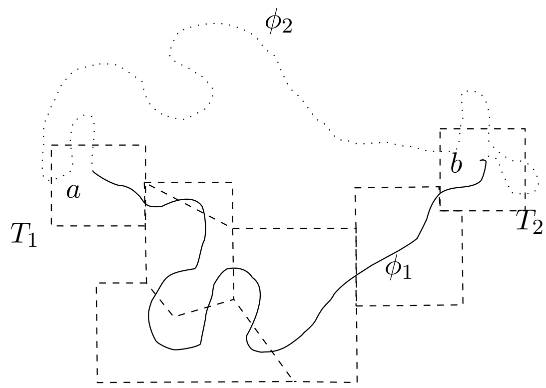

FIGURE 6.2. Example of a special covering

Removing long intersections:

Consider edges $e \in P, f \in P$, where $P, Q \in \mathcal{P}$ are arbitrary, and suppose $e \cap f$ is a long intersection. The ends of $e \cap f$ are in V and since one side of this segment lies in $i(P)$ and the other in $i(Q)$, by shallow intersection, there is no point of V in $e \cap f$ other than the ends.

Around $e \cap f$ take a rectangle R with one side parallel to e such that $\overline{i(R)}$ is disjoint from

- ϕ_1 and
- edges $e' \in \mathcal{P}$ with $e' \cap \overline{e \cap f} = \emptyset$ and
- $V \setminus e \cap f$, so that $V \cap \overline{i(R)} = V \cap e \cap f \subset i(R)$.

This is possible because $e \cap f$ is compact, so we take a finite cover by squares with one side parallel to e and then take a "minimum height" rectangle. The initial cover of $e \cap f$ is one that avoids $\phi_1 \cup (V \setminus (e \cap f))$ and edges of $\mathcal{P}$ that don't intersect $e \cap f$. Such a cover exists by the assumptions on $\mathcal{P}$ and $e \cap f$.

Now $V \cap R = V \cap e \cap f$ has 2 elements, so R does not contain any $\tilde{P} \in \mathcal{P}$. We have

$$e \cap f \subset i(R) \Rightarrow i(R) \cap i(P) \neq \emptyset, i(R) \cap i(Q) \neq \emptyset$$

so by shallow intersection $R \not\subseteq \overline{i(\tilde{P})}$ for any $\tilde{P} \in \mathcal{P}$. Reduce (Theorem 4.0.2) each $\tilde{P} \in \mathcal{P}$ by R giving us in place of $\tilde{P}$, a shallow union of some polygons. We show that each reduction gives one polygon.

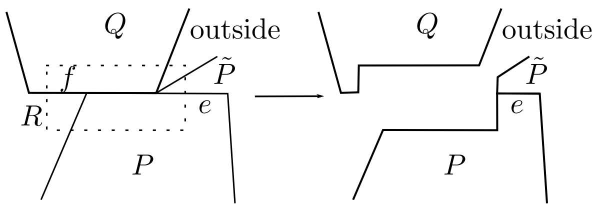

FIGURE 6.3. Removing long intersection $e \cap f$

Number of polygons doesn't change:

$P \cap i(R)$ is a connected path (involving 3 or fewer segments) and divides R into polygons R_1, R' with $i(R_1) \subset i(P)$. By shallow intersection $Q \cap i(R) \subset \overline{i(R')}$. By Lemma 5.0.2, $Q \cap i(R)$ divides R' into 3 or fewer polygons R_2, R_3, R_4 with $i(R_2) \subset i(Q)$ where R_3, R_4 may be degenerate, $\overline{i(R_3)} \cap \overline{i(R_4)} = \emptyset$. At each end of $e \cap f$, there is one sector each for R_1, R_2 and one for R_3 or R_4.

For $\tilde{P} \in \mathcal{P}$, if $\tilde{P} \cap \overline{i(R)} = \emptyset$ the reduction by R doesn't change $\tilde{P}$, so assume $\tilde{P} \cap \overline{i(R)} \neq \emptyset$, then we must have $\tilde{P} \cap e \cap f \neq \emptyset$. If $e \cap f \subset \tilde{P}$, by shallow intersection $\tilde{P} = P$ or $\tilde{P} = Q$ and it is easy to see that $R \cap i(\tilde{P})$ is a connected path.

For $\tilde{P} \neq P, Q$, $\tilde{P} \cap e \cap f$ contains only the ends of $e \cap f$. At each point of $\tilde{P} \cap e \cap f$, there are two segments induced by $\tilde{P}$ and both must be in $\overline{i(R_3)}$ or $\overline{i(R_4)}$ by shallow intersection. The reduction of $\tilde{P}$ by R happens in two steps - by R_3, R_4 separately. Observe that $R_3 \cap i(P), R_4 \cap i(\tilde{P})$ are connected in the step-wise reduction. After reducing by R_3 for example, $R_4 \cap i(\tilde{P})$ doesn't change.

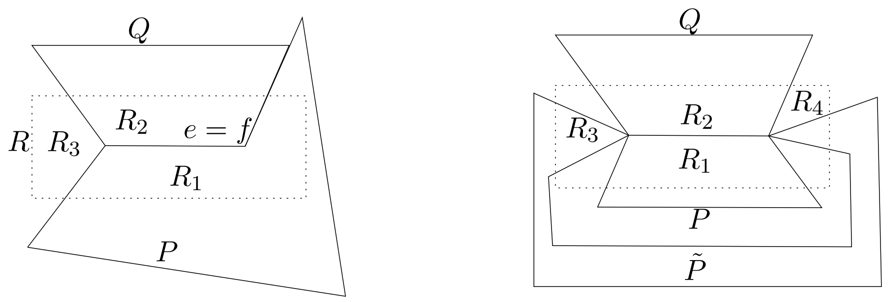

FIGURE 6.4. Examples of what R looks like. R_4 is degenerate in the first one

In both cases, by Lemma 5.0.2, the number of polygons doesn't change. Each $\tilde{P} \in \mathcal{P}$ is replaced by a P' forming a collection $\mathcal{P}'$. Similar to step 1, we can find T_1, T_2.

Regularity:

$\mathcal{P}'$ covers ϕ_1 because $\overline{R} \cap \phi_1 = \emptyset$ and has a shallow intersection because the interiors have shrunk. So $V(\mathcal{P}') = V'$ is the collection of vertices in $\mathcal{P}'$ and consists of points of R that lie in $\overline{i(P_j)}, P_j \in \mathcal{P}$ and points of V outside R. As a consequence $i(R)$ is now in the common outside.

- Because R can intersect only those edges that intersect $e \cap f$, any edge intersecting R must go inside R. If $x \in R$ lies on an edge of P_i, then $x \notin V$ by construction of R. Its original zone had one diameter and now contains a radius on either side introduced by R. After reducing P_i by R, the sector that was in $i(R)$ is now in the common outside and the new zone has three sectors, with two radii from R and one from half of the original diameter.
- If $x \in R$ lies inside P_i, then its original "zone" (a ball around x that lies inside the polygon will do) had no radii and the new zone has two.

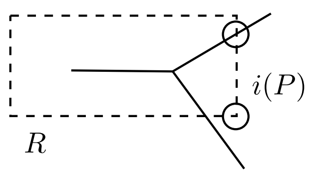

FIGURE 6.5. After reducing by R, the vertices introduced by R are regular.

The other points of V' are points of V outside R. If x is such a point, then the zone of x is unaltered because the line segments passing through x are unaltered (for they are outside R). Note that any loss of interiors comes from $i(R)$, so the sectors at x do not change. So, $\mathcal{P}'$ is regular.

Number of long intersections:

Every long intersection in $\mathcal{P}'$ must come from a long intersection in $\mathcal{P}$ because the new edges of $\mathcal{P}'$ lie inside polygons of $\mathcal{P}$ and cannot have long intersections by shallow intersection of $\mathcal{P}$.

Suppose e_1, f_1 are two edges in $\mathcal{P}$ such that $e_1 \cap f_1$ is long. If $e_1 \cap f_1 \subset i(R)$, then by the choice of R, we have $e_1 \cap f_1 = e \cap f$, which is removed. If $e_1 \cap f_1 \subset o(R)$, it is unaffected and is a long intersection in $\mathcal{P}'$. Lastly, if $e_1 \cap f_1$ enters R but not contained in it, then one end of $e_1 \cap f_1$ is an end of $e \cap f$ and the other is outside R. Because $i(R)$ is a convex shape, $\overline{e_1 \cap f_1 \setminus i(R)}$ is a connected closed segment, so $e_1 \cap f_1$ gives rise to exactly one long intersection in $\mathcal{P}'$.

Thus, the number of long intersections has decreased by 1. Repeating this process, we arrive at a collection $\mathcal{P}$ such that $\phi_1 \subset \cup_{P \in \mathcal{P}} \overline{i(P)}$ with shallow intersection and every long intersection of edges having points from ϕ_1. At each step, we can find T_1, T_2.

Removing short intersections:

Suppose $v = e \cap f$ is a short intersection for edges $e \in P, f \in Q$. Let U be a zone of $v \in V$ that avoids ϕ_1. In U, the sectors that lie inside P, Q are disjoint and since $v \in V$ one of them must be convex. Without loss of generality, assume the sector in $i(P)$ bounded by radii r_1, r_2 is convex. Join the midpoints of r_1, r_2 to get a triangle W contained inside $\overline{i(P)}$.

Reducing $\tilde{P} \in \mathcal{P} \setminus \{P\}$ doesn't change it as $\overline{i(W)} \cap i(\tilde{P}) = \emptyset$. Reducing P by W gives it two vertices in place of v. As before, these vertices and those outside U are regular. The component $i(W)$ is either a new component outside, or it merges with some other component outside all $\tilde{P}$ (depending on whether the other side of r_i is inside or outside $\mathcal{P}$). In either case, v stays regular. So, after reduction, points in $V(\mathcal{P}) \cup V(W)$ are regular.

Thus, upon reducing $\mathcal{P}$ by W, we have a regular polygonal cover (by the choice of U) of ϕ_1 with shallow intersection. Moreover, any long intersection is a subsegment of one in $\mathcal{P}$ and, by the choice of U, contains a point of ϕ_1.

The zones of a, b have two or three radii, so they cannot be short intersections. Moreover, v is no longer a vertex of the reduced P as there is a ball around v that doesn't intersect $i(P) \setminus \overline{i(W)}$. Thus, the number of short intersections has reduced by 1 as $v \in P \cap Q$ is no longer an intersection in the new collection. Again, since the number of polygons hasn't changed, we can find T_1, T_2.

This process terminates and we obtain a regular $\mathcal{P}$ containing ϕ_1 in its shallow union and if $P \cap Q \neq \emptyset, P, Q \in \mathcal{P}$, then $P \cap Q$ (in fact, every component of $P \cap Q$) contains a point of ϕ_1. Furthermore, there are polygons T_1, T_2 as described in the beginning.

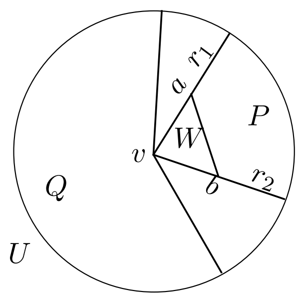

FIGURE 6.6. Removing short intersection $v \in P \cap Q$

6.2. **Consequence of the special covering.** Suppose we have the covering $\mathcal{P} = \{P_1, \ldots, P_n\}$ as above and C is a bounded component outside $\mathcal{P}$. Let $Q_1, \ldots, Q_k \in \mathcal{P}$ be the polygons that intersect the polygon $R = \partial C$. To each Q_i, by minimality of the cover, we have the interval $[a_i, b_i] \subseteq I_1$ where a_i is the first time ϕ_1 intersects $\overline{i(Q_i)}$ and b_i last. By the refinement in the last subsection, if $Q_i \cap Q_j \neq \emptyset$, then $[a_i, b_i] \cap [a_j, b_j] \neq \emptyset$. We started with a cover where the diameter was less than ϵ' and refined this cover. At each step of the refinement the diameter never increased, so $b_i - a_i < \delta$ (assumptions made at the start of section 6).

Suppose in $\mathcal{P}$, the polygon P_1 contains a and P_n contains b, i.e., P_1 plays the role of T_1 and P_n that of T_2. We know that $\overline{i(P_1)} \cap \overline{i(P_n)} = \emptyset$ and $\phi_2 \cap \overline{i(P_j)} = \emptyset, 2 \leq j \leq n-1$. Now, $\overline{i(P_1)} \cap \phi_2, \overline{i(P_n)} \cap \phi_2$ are two non empty (because they contain a, b) disjoint closed sets in ϕ_2. Because ϕ_2 is connected, there is an $s \in (0,1) \subset I_2$ such that $\phi_2(s) \notin \overline{i(P_1)} \cup \overline{i(P_n)}$, hence it is outside $\mathcal{P}$, in particular outside $Q_1, \ldots, Q_k$.

Set

$$\alpha = \min_{1 \leq i \leq k} a_i, \beta = \max_{1 \leq i \leq k} b_i.$$

Look at the path $\phi_1(\alpha) \xrightarrow{\phi_1} a \xrightarrow{\phi_2} \phi_2(s)$. The first part, i.e. $\phi_1(\alpha) \xrightarrow{\phi_1} a$ intersects $\cup_i \overline{i(Q_i)}$ only at $\phi_1(\alpha)$ by definition, hence intersects R at most once. If the second part, i.e., $a \xrightarrow{\phi_2} \phi_2(s)$, intersects R, hence any of the $\overline{i(Q_j)}$, then, by construction of $\mathcal{P}$, that $Q_j = T_1$ or T_2 and we must have $\alpha = 0$ or $\beta = 1$.

Since $\phi_2(s) \notin R$, we can obtain a subpath, or more properly sub-arc, not intersecting R, except possibly at one end. This arc, ψ_1, goes from some Q_i to $\phi_2(s)$. By the arguments in the preceding paragraph, ψ_1 can intersect only those $\overline{i(Q_j)}$ which contain $\phi_1(\alpha)$ or $\phi(\beta)$.

Next, using the path $\phi_1(\beta) \xrightarrow{\phi_1} b \xrightarrow{\phi_2} \phi_2(s)$, obtain an arc ψ_2 not intersecting R, except possible at one end and intersecting only those $\overline{i(Q_j)}$ that contain $\phi(\alpha)$ or $\phi_1(\beta)$. Then the arc $\psi = \psi_1 \cup \psi_2$ goes from a point in some $\overline{i(Q_i)}$ to one in $\overline{i(Q_j)}$ and intersects only those $\overline{i(Q_l)}$ that contain $\phi_1(\alpha)$ or $\phi_1(\beta)$.

Now assume $\phi_2(s) \in o(R)$. Then the path ψ (except for the ends) must also lie outside R by definition of ψ_1, ψ_2 for if they meet $i(R)$ then ψ_1 or ψ_2 must meet R.

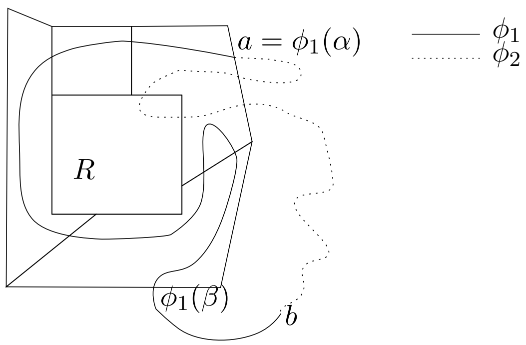

FIGURE 6.7. Set up for Lemma 7, only the necessary polygons are drawn

Lemma 6.2.1. *$\beta - \alpha < 3\delta$.*

Proof. Choose polygons R_1, R_3 from $Q_1, \ldots, Q_k$ with associated intervals $[\alpha, \alpha'], [\beta', \beta]$ respectively where α', β' are chosen so that $\alpha' - \alpha, \beta - \beta'$ are maximal. The ends of ψ are in these polygons, say $\psi(0) \in \overline{i(R_1)}, \psi(1) \in \overline{i(R_3)}$. If $\psi(0) \in R$, take $u = \psi(0)$, otherwise take u to be a vertex of R_1 in $V(R)$. Similarly, if $\psi(1) \in R$, take $v = \psi(1)$, otherwise take it to be a vertex of R_3 in $V(R)$.

If $u = v$, then these intervals should intersect (they may even be equal) and it follows that $\beta - \alpha < 2\delta < 3\delta$. So, assume that $u \neq v$ and that $[\alpha, \alpha'] \cap [\beta', \beta] = \emptyset$.

Let the two paths from u to v along R be L_1, L_2. Starting at u traverse the sectors in the sequence $S(L_1)$ (with ends R_1, R_3), till the first point in $V(R) \cap L_1$ that has a polygon R_2 with $[a_i, b_i], b_i > \alpha'$. Going along L_2, arrive at a point in $V(R) \cap L_2$ that has a polygon R_4 with interval $[a_j, b_j], b_j > \alpha'$.

By the maximality assumption, we must have $a_i, a_j > \alpha$. As we move sequentially from R_1, each polygon shares an edge or vertex with the previous one, so their intervals intersect. Therefore, we must have $a_i, a_j < \alpha'$. If $b_i \geq \beta'$ (or similarly $b_j \geq \beta'$), then

$$\beta - \alpha = \beta - b_i + b_i - a_i + a_i - \alpha < 3\delta.$$

So, assume $b_i, b_j < \beta'$ and without loss of generality, assume $b_i \leq b_j$.

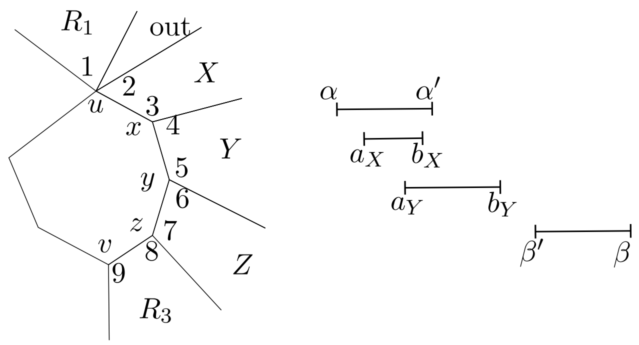

FIGURE 6.8. Example

For example, in Figure 6.8, we traverse the sectors from u to v according to the numbering, i.e., sectors $1, 2$ at u, then $3, 4$ at x. Sector numbered 4 is from the polygon Y whose associated interval is the first to go beyond $[\alpha, \alpha']$. Notice that in the sequence each polygon shares an edge or vertex with the previous one.

Let ϕ' be the restriction of ϕ_1 to $[b_i, b_j]$. By assumption (on b_i, b_j), ϕ' doesn't intersect $\overline{i(R_1)}, \overline{i(R_3)}$. If $\phi' \cap L_1 \neq \emptyset$, then let x be the last time (while going from b_i to b_j) that it hits L_1. Being closed, $\phi_1(x) \in L_1$ and by assumption $\phi_1(x) \neq u, v$. Otherwise, set $x = b_i$.

Next, if $\phi' \cap L_2 \neq \emptyset$, then set y to be the first time it hits L_2 while going from x to b_j. Again, $\phi_1(y) \in L_2$ and $\phi_1(y) \neq u, v$. Otherwise set $y = b_j$.

This way we obtain a path ϕ, given by the restriction of ϕ' to go from x to y, from a polygon R_2' with a vertex in L_1 to a polygon R_4' with a vertex in L_2, say $\phi(0) \in \overline{i(R_2')}, \phi(1) \in \overline{i(R_4')}$. Take $u_2 = \phi(0)$ if $\phi(0) \in R$, otherwise take it to be a vertex of R_2' in $V(R)$. Similarly, take $u_4 = \phi(1)$ if it is in R, otherwise take it to be a vertex of R_4' in $V(R)$. Note that R_2', R_4' are different from R_1, R_3 (because ϕ', hence ϕ, doesn't intersect $\overline{i(R_1)}, \overline{i(R_3)}$). Observe

(1) Both ψ, ϕ lie outside R and the ends may lie on R or outside. We assumed ψ is outside, but ϕ is outside by definition of R. If the ends do lie on R, then it is u, v or u_2, u_4 respectively.
(2) By assumptions on b_i, b_j, ϕ doesn't intersect $\overline{i(R_1)}, \overline{i(R_3)}$. In particular, ϕ cannot have u or v as its ends.

(3) ψ intersects only those $\overline{i(Q_i)}$ that contain $\phi_1(\alpha)$ or $\phi_1(\beta)$. Neither $\overline{i(R'_2)}, \overline{i(R'_4)}$ have these points by the maximality conditions on α', β' so ψ doesn't intersect $\overline{i(R'_2)}, \overline{i(R'_4)}$.
(4) By 3, if $\psi(0) = u$ ($\psi(1) = v$), then R'_2, R'_4 cannot have u (v) as a vertex.
(5) Lastly, note that ϕ, ψ do not intersect

We are in a possition to apply Theorem 5.2.1 and its extensions and we have a contradiction. Therefore, one of b_i, b_j is larger than β' and we conclude that $\beta - \alpha < 3\delta$. □

Theorem 6.2.1. *Suppose ϕ_1, ϕ_2 are two arcs meeting only at the ends a, b. Let B be a circle or a polygon such that $\phi_1 \cup \phi_2 \subset i(B)$. Suppose there is a point $c \in \phi_2 \setminus \{a, b\}$ with a path ϕ going from c to a point outside B such that $\phi \cap \phi_1 = \emptyset$. Then given any $x \notin \phi_1 \cup \phi_2$, there is a polygonal cover $\mathcal{P}$ (with each polygon inside B) of ϕ_1 such that x is in the unbounded component of $\mathbb{R}^2 \setminus \mathcal{P}$.*

Proof. Let $4\epsilon = \inf_{y \in \phi_1} |x - y| > 0$. With ϵ', δ defined as in the start of Section 6, around each point of ϕ_1 take open squares such that

- Each square has diameter $< \epsilon'$.
- The closure of inside of each square is inside B (possible because $\phi_1 \subset i(B)$).
- The closure of inside of each square is disjoint from ϕ (possible because $\phi \cap \phi_1 = \emptyset$), in particular c.
- a, b are in different squares whose interiors have disjoint closures.

From this cover obtain a finite subcover $\mathcal{S}$. Refine $\mathcal{S}$ and obtain a special covering $\mathcal{P}$. Now c is in the unbounded component of $\mathcal{P}$ because $o(B)$ is in the unbounded component of $\mathcal{P}$ and we have a path, namely ϕ, going from c to $o(B)$ avoiding $\mathcal{P}$. Notice that ϕ avoids the squares in $\mathcal{S}$ and $\mathcal{P}$ is obtained by shrinking these squares, so ϕ avoids $\mathcal{P}$ as well. It is clear that each polygon in $\mathcal{P}$ is inside B.

By the choice of ϵ, ϵ', the point x is outside $\mathcal{P}$. Suppose x is in a bounded component C (outside $\mathcal{P}$) with boundary polygon R. Continuing with the notation above, let $Q_1, \ldots, Q_k$ be the polygons surrounding R and $[a_i, b_i]$ be the interval associated to Q_i. By the lemma $\beta - \alpha < 3\delta$ where α, β are as defined above. Because c is in the unbounded component in the complement of $\mathcal{P}, c \in o(R)$ and the lemma is applicable.

Now, pass a line through x and let p_1, p_2 be the first time the two rays hit R (so x is between p_1, p_2). Since $x \in i(R)$, both rays must hit R. Suppose $p_1 \in Q_1$. Since Q_1 contains a point of ϕ_1 and has diameter less than ϵ', we know that p_1 is within ϵ' of some $q_1 \in \phi_1$. Similarly, p_2 is within ϵ' of some $q_2 \in \phi_1$. Since $\beta - \alpha < 3\delta$, we have $|q_1 - q_2| < \epsilon$ and

$$|p_1 - p_2| < \epsilon' + \epsilon + \epsilon' < 3\epsilon.$$

Since x is on the line segment between p_1, p_2, it is a convex linear combination of p_1, p_2. So,

$$|x - q_1| \le |x - p_1| + |p_1 - q_1| \le |p_1 - p_2| + |p_1 - q_1| < 4\epsilon \Rightarrow 0 < \inf_{y \in \phi_1} |x - y| < 4\epsilon.$$

This contradicts the definition of 4ϵ, therefore x must be in the unbounded component in the complement of $\mathcal{P}$. □

7. Arcs in discs

Let $\overline{\mathbb{D}}$ be the open unit disc and $J\colon [0,1] \to \mathbb{R}^2$ be an arc such that $J(t) \in \mathbb{D}\, \forall\, t \in (0,1)$ and $J(0), J(1) \in S^1$. We will show that $\overline{\mathbb{D}} \setminus J$ has two components, determined by the arcs in $S^1 \setminus \{J(0), J(1)\}$, with common boundary J.

7.1. **Separation theorem.** Let A_1, A_2 be the two arcs of $S^1 \setminus \{J(0), J(1)\}$ and take $c \in A_1, d \in A_2$. Suppose there is a path ϕ from c to d in $\overline{\mathbb{D}}$ that avoids J. Let ϵ be the minimum distance between ϕ and J. Approximate ϕ by a polygonal path P within ϵ so that $P \subset \overline{\mathbb{D}}$ ($\overline{\mathbb{D}}$ is convex, so this is possible). Draw tangents at c, d to $S^1 = C$. If they do not intersect, then use a perpendicular line outside C to join them. This way, we get a polygonal path between c, d lying outside the circle. Together with P we have a polygon Q.

Using rays that go outside the circle, we conclude that $J(0), J(1)$ are on different sides of Q. More generally what we notice is that the two arcs determined by c, d are on different sides of Q, hence in particular $J(0), J(1)$ are on different sides. Since J is a path going from inside Q to the outside, it must intersect Q. Because J is inside the disc it must intersect P which is impossible by the choice of ϵ. Therefore, c, d, hence A_1, A_2, are in different components of $\overline{\mathbb{D}} \setminus J$.

7.2. **Exactly two components.** Let $x \in \mathbb{D} \setminus J$. Fix a $c \in A_1$ and take a normal to C going outwards at c. The arcs J, A_1 meet only at the ends (taking the closure of A_1). Applying Theorem 6.2.1 (taking B to be a circle of radius 2 centered at the origin for example), we obtain a polygonal cover $\mathcal{P}$ such that x is in the unbounded component in the complement of $\mathcal{P}$.

Then, there are paths from x to points far outside the circle that avoid $\mathcal{P}$ and hence J. Since $x \in \mathbb{D}$, any such path must pass through S^1 and therefore A_1 or A_2. Thus, x is in the same component of $\overline{\mathbb{D}} \setminus J$ as A_1 or A_2 and $\overline{\mathbb{D}} \setminus J$ has exactly two components.

7.3. **Boundary.** We have two components, C_1, C_2 corresponding to A_1, A_2 respectively. We will show that J is part of the boundary of C_1. First, $J(0), J(1) \in \partial C_1, \partial C_2$ because any neighbourhood of both these points intersects A_1, A_2 and $\partial C_1, \partial C_2$ are closed.

Lemma 7.3.1. *Let $R \subset \mathbb{R}^2$ with $Int(R), Ext(R) \neq \emptyset$, then any path from $Int(R)$ to $Ext(R)$ intersects ∂R.*

Proof. Suppose $\phi\colon [0,1] \to \mathbb{R}^2$ is a path from $x = \phi(0) \in Int(R)$ to $y = \phi(1) \in Ext(R)$. There is a neighbourhood around x that lies in $Int(R)$, hence a $t > 0$ such that $\phi([0,t)) \subseteq Int(R)$. Take

$$t_0 = \sup_{[0,1]}\{t : \phi([0,t)) \subseteq Int(R)\}.$$

If $\phi(t_0) \in Int(R)$, then $t_0 \neq 1$ and we can increase t_0. If $\phi(t_0) \in Ext(R)$, there is a $t < t_0$ such that $\phi((t, t_0)) \subseteq Ext(R)$. Both contradict the definition of t_0, hence $\phi(t_0) \in \partial R$. □

By this lemma, we conclude that ∂R disconnects $Int(R)$ from $Ext(R)$.

Corollary 7.3.1. *Let $R \subset \mathbb{R}^2$ with $Int(R), Ext(R) \neq \emptyset$. Suppose there is a path from $x \in Int(R)$ to $y \in Ext(R)$ in $(\mathbb{R}^2 \setminus \partial R) \cup S$ for some subset S of the plane. Then $S \cap \partial R \neq \emptyset$.*

Suppose $x \in J$ is not in ∂C_1. Then there is a neighbourhood of x in J that is not in ∂C_1. We will show that adding this neighbourhood to $\overline{\mathbb{D}} \setminus J$ connects C_1, C_2. Essentially, we want to show that if $J' = J([0,1] \setminus (t_1, t_2))$, then $\overline{\mathbb{D}} \setminus J'$ is connected, where $0 < t_1 < t_2 < 1$.

Around each point of $J_1 = J([0, t_1])$ take open squares whose

- closures avoid $J_2 = J([t_2, 1])$ and
- for $0 < t \leq t_1$, the closure is contained in $\mathbb{D}$.

Obtain a finite subcover $\mathcal{S}$ of J_1. In $\mathcal{S}$, there is only one square that intersects S^1. We now remove singular vertices in such a way as to obtain a cover of J_1 by polygons $\mathcal{P}$ such that

- The boundary of polygons in $\mathcal{P}$ do not intersect J_2.
- Every point of J_1 lies inside at least one of the new polygons.
- Only one polygon intersects S^1 at exactly two points.

This is true for $\mathcal{S}$. Now the vertices that can be singular are inside C, as only one square in $\mathcal{S}$ goes outside, so points outside are regular. Since we are going to shrink the polygons, it is clear that the boundaries of the resulting polygons do not intersect J_2.

Given a singular vertex v, if $v \in J_1$, then take a zone that lies inside one of the polygons, else take a zone that avoids J_1. Ensure that the zone lies inside the circle C. Since the new edges and the consequent loss of the interiors happen inside this zone, the new edges do not intersect the circle and J_1 stays inside the polygons. The intersection with S^1 still has the original two points.

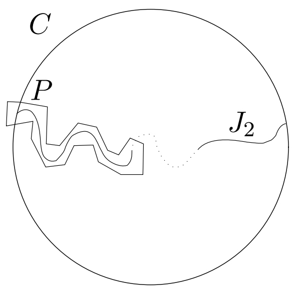

FIGURE 7.1. $J \subset \partial C_1 \cap \partial C_2$

Since J_1 is connected, the intersection graph of (the modified) $\mathcal{S}$ is connected. So the unbounded component of $\mathcal{S}$ has a polygonal boundary P which has edges going into the disc $\mathbb{D}$. Since every point of J_1 lies inside some $Q \in \mathcal{P}, P \cap J_1 = \emptyset$. By construction $P \cap J_2 = \emptyset$.

In $\mathcal{P}$ there is exactly one polygon that contains $J(0)$ and hence intersects A_1, A_2. We know that there are edges of P that go into $\mathbb{D}$, so these edges give a path from a point in A_1 to one in A_2 that lies entirely in $\mathbb{D}$ and avoids $J_1 \cup J_2$.

It lies entirely in $\mathbb{D}$ because $P \cap S^1$ has only two points. So, $\overline{\mathbb{D}} \setminus (J_1 \cup J_2)$ is connected, therefore $J((t_1, t_2)) \cap \partial C_1 \neq \emptyset$. This path along P must intersect $J((t_1, t_2))$ and the first point of intersection is then arcwise accessible from A_1.

We conclude that $J \subseteq \partial C_1, \partial C_2$ and that any open segment of J has a point accessible from A_1, i.e., an x with an arc from a point in A_1 to x that avoids $J \setminus \{x\}$.

Let J be a Jordan curve. Since J is bounded, it is inside a circle C. Take two different points on J and extend the line between them to a chord of the circle. We can talk of the "first" and "last" points on this chord, which corresponds to the points of J that lie farthest apart on this chord (such points exist because $J \cap \text{chord}$ is closed). Let these points be a, b and the ends of the chord c, d with a being between c, b and b between a, d. The line segments ca, bd are disjoint and $ac \cap J = \{a\}, bd \cap J = \{b\}$.

The curve J, being homeomorphic to S^1, gives us two arcs from a to b, call them ϕ_1, ϕ_2. Together with the segments ac, bd, we get two arcs

$$J_1 = ac \cup \phi_1 \cup bd; J_2 = ac \cup \phi_2 \cup bd$$

from c to d: they intersect only on the segements ac, bd. Let the arcs of $C \setminus \{c, d\}$ be A_1, A_2.

For $x \in ca, x \neq a, c$, there is a ball U around x that avoids J and the line segment bd. Because $U \cap (J_1 \cup J_2) = U \cap ac$ is a diameter in $U, U \setminus (J_1 \cup J_2)$ has two components. Around c, there is an open ball that avoids J, bd. This ball has three components of which one lies outside the circle C. We see that all three parts are connected in the complement of $J_1 \cup J_2$. Similar results hold true for points on bd different from b.

8.1. **At least two components.** In $\overline{\mathbb{D}} \setminus J_1$ let the components of A_1, A_2 be C_1, C_2 respectively. Being connected in $\overline{\mathbb{D}} \setminus J_1$, ϕ_2 must lie in one of these components, say C_2. Now, a point of ϕ_1 is accessible from A_1 so this path, say ψ, therefore lies in C_1. So, $\psi \cap \phi_2 = \emptyset$ and $\psi \cap J_2 = \emptyset$ giving us $\psi \subset \overline{\mathbb{D}} \setminus J_2$.

In $\overline{\mathbb{D}} \setminus J_2$, let the components corresponding to A_1, A_2 be D_1, D_2 respectively. Since ψ is a path in $\overline{\mathbb{D}} \setminus J_2$, a point of ϕ_1 is in D_1 via ψ. We conclude that $\phi_1 \subset D_1$ as ϕ_1 is connected in $\overline{\mathbb{D}} \setminus J_2$.

Similarly, we have $C_1 \subseteq D_1$ and $D_2 \subseteq C_2$. For $x \in \phi_1$, choose an open U such that $x \in U \subset D_1$. Since ϕ_1 is contained in the boundaries of C_1, C_2, this neighbourhood has points from both C_1, C_2. In particular, U has a point from $C_2 \cap D_1$, so $C_2 \cap D_1 \neq \emptyset$. Note that $C_1 \cap D_2 = \emptyset$ as $C_1 \subseteq D_1$. Thus,

$$\overline{\mathbb{D}} \setminus (J_1 \cup J_2) = C_1 \sqcup (C_2 \cap D_1) \sqcup D_2.$$

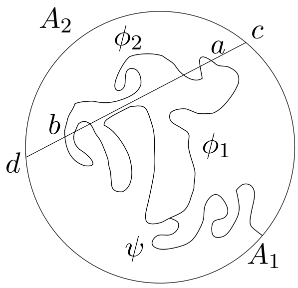

Figure 8.1. J inside C

Let $x \in C_2 \cap D_1$ and suppose ϕ is a path from x to a point $y \in C$ that avoids J. If ϕ goes outside the circle C, look at the first time it hits C (since x is inside, it must hit C at least once). With this as our new y, we may assume that ϕ, except for y, lies inside C.

Case 1: $y \in A_1$

Since $x \in C_2$ and ϕ is inside C, we have $\phi \cap J_1 \neq \emptyset$. Since $\phi \cap J = \emptyset$, it must intersect one of ca, db, say ac. Let $z \in ac$ be the first point of intersection. By the choice of $y, z \neq c$.

Parametrize the restricted path by $[0,1]$ with $\phi(0) = x, \phi(1) = z$. Take a ball U around z, disjoint from J and a $t_1 < 1$ such that $\phi(t_1) \in U$. By the choice of z, for any $t < 1, \phi(t) \notin J_1$ and since $\phi(t) \notin J$, we have $\phi(t) \notin J_1 \cup J_2$. Let ϕ' be the restriction of ϕ between $x, \phi(t_1)$.

$U \setminus (J_1 \cup J_2)$ has two components that lie inside the circle, say H_1, H_2. Because $z \in \partial C_1 \cap \partial C_2$, one of these lies in C_1 and the other in C_2, say $H_1 \subset C_1$. Because $C_1 \subset D_1$, we have $H_1 \subset D_1, H_2 \subset D_2$.

If $\phi(t_1) \in H_1 \subset C_1$ or $\phi(t_1) \in H_2$, then ϕ' must meet J_1, J_2 respectively, which is impossible.

Case 2: $y = c$

Take a ball U around y that avoids J, bd. As mentioned above, $U \setminus (J_1 \cup J_2)$ has three components, of which one lies outside C. Parametrize ϕ by $[0,1]$ with $\phi(0) = x, \phi(1) = y$ and pick a $t < 1$ such that $\phi(t) \in U$. We know that $\phi(t)$ must either be on the radius of U induced by line ac, or in one of the two components of $U \setminus (J_1 \cup J_2)$ inside C.

If it lies on the radius, then ϕ intersects ac and we continue as in case 1. If it lies in one of the other components, then as in case 1, $\phi(t)$ is in C_1 or in D_2 which also impossible.

All other cases, i.e., $y \in A_2, y = d$ can be treated similarly. We conclude that there is no path from x to a point in C avoiding J.

For $x \in C_1$ there is a path from x to points on A_1 avoiding J_1. Since $\phi_2 \in C_2$, such paths also avoid ϕ_2 and hence J. Similarly, for points in D_2, there are paths to A_2 that avoid J. Lastly, for points on ac, bd different from a, b the line itself is a path to the circle that avoids J.

So, we have the "inside" of J which is the set $C_2 \cap D_1$ and the "outside", which is everything else in the complement of J. Fix a point p outside the circle C. If x is on or outside the circle, there is a path from x to p avoiding the inside of C and hence J. If $x \in C_1 \cup D_2$ or $x \in ac \cup bd, x \neq a, b$ first go to C and then to p. So, the outside of J is path connected.

Let $i(J)$ denote the inside and $o(J)$ the outside. We have shown above that there is no path from a point in $i(J)$ to one in $o(J)$ that avoids J, so $\mathbb{R}^2 \setminus J$ has at least two components.

8.2. **Boundary.** Next, we show that all components in the complement of J have J as the boundary. Since the boundary is closed, it suffices to show that all components have $\phi_1 \cup \phi_2$ as the boundary. For points in ϕ_1, take neighbourhoods that avoid ϕ_2. We know that these neighbourhoods contain points from C_1, hence from $o(J)$. It follows that $J = \partial o(J)$.

Let $x \in i(J)$ and $y \in \phi_1$. We will show that in any neighbourhood of y, there is a point that is accessible from the component C_x of x in the complement of J. Then y is in the boundary of this component.

Parametrize ϕ_1 by $(0,1)$ with $\lim\limits_{t\to 0} \phi_1(t) = a, \lim\limits_{t\to 1} \phi_1(t) = b$. Suppose $y = \phi_1(t_1)$ where $t_1 \in (t_0, t_2) \subset (0,1)$ is from any neighbourhood of y in ϕ_1. We have arcs l_1 via ϕ_1 and l_2 through ϕ_2 between points $a_1 = \phi_1(t_0), b_1 = \phi_1(t_2)$.

Some point z in the open l_1 is accessible from A_1, through some arc ψ. This arc doesn't intersect J, hence $\overline{l_2}$. We can apply Theorem 6.2.1 (with $B = C$) to conclude that there is a polygonal covering $\mathcal{P}$ of l_2 such that x is in the unbounded component in the complement of $\mathcal{P}$. Thus, there are polygonal paths from x to any point outside the circle avoiding $\mathcal{P}$ and its inside, hence avoiding l_2.

Let ψ' be one such path. We chose $x \in i(J)$, so ψ' must intersect J. Since it cannot intersect $\overline{l_2}$, it must intersect the open l_1. Suppose ψ' is parametrized by $[0, 1]$ with $\psi'(0) = x$. Let $t_3 > 0$ be the first time ψ' intersects (closed) l_1, then observe that

$$\psi'([0, t_3)) \cap J = \emptyset.$$

Therefore, $\psi'|_{[0,t_3)}$ is a connected path in the complement of J, hence lies in C_x. Since $\psi'(t_3) \in l_1$, it follows that the neighbourhood $\phi_1((t_0, t_2))$ of y has points accessible from C_x and from $o(J)$ which lies exterior to C_x.

We conclude that $y \in \partial C_x$ and that any neighbourhood has a point (arcwise) accessible from x. It follows that ϕ_1, ϕ_2 and hence J are in the boundary of C_x. Since $\partial C_x \subseteq J$, we have $J = \partial C_x$.

8.3. **Two components.** This subsection is based on [4]. So far we have shown that if J is a Jordan curve, then $\mathbb{R}^2 \setminus J$ has one unbounded component, $o(J)$ and bounded components in $i(J)$. We know that all components have J as the boundary. All that is left is to show that there is exactly one bounded component.

Lemma 8.3.1. *Let γ_1, γ_2 be two Jordan curves. If*

$$\gamma_2 \cap i(\gamma_1) \neq \emptyset \text{ and } \gamma_2 \cap o(\gamma_1) \neq \emptyset,$$

then

$$\gamma_1 \cap i(\gamma_2) \neq \emptyset \text{ and } \gamma_1 \cap o(\gamma_2) \neq \emptyset.$$

Proof. First take $x \in \gamma_2 \cap i(\gamma_1)$ and $y \in \gamma_2 \cap o(\gamma_1)$. Choose neighbourhoods U_x, U_y of x, y respectively such that $U_x \subset i(\gamma_1), U_y \subset o(\gamma_1)$. Observe that $U_x \cap U_y = \emptyset$.

Since x, y are in the boundary of every component of the complement of γ_2, pick $z_1 \in U_x, z_2 \in U_y$ that lie in the same component. There is a path from z_1 to z_2 that avoids γ_2. Since $z_1 \in i(\gamma_1), z_2 \in o(\gamma_1)$, such a path intersects γ_1. However, this path is in a component of $\mathbb{R}^2 \setminus \gamma_2$, therefore γ_1 intersects every component in the complement of γ_2. $\square$

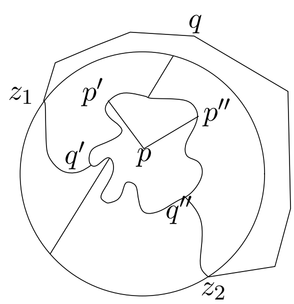

FIGURE 8.2. Curve J'

Take $p \in i(J)$ and draw two rays through it. Both rays intersect J because $p \in i(J)$ and J is bounded. Let $p', p'', p' \neq p''$ be the first points of intersection. Let $q' \in \phi_1, q'' \in \phi_2$ be accessible from A_1, A_2 respectively, say there is an arc from

$z_1 \in A_1$ to q' and $z_2 \in A_2$ to q''. Fix a q outside the circle and take disjoint polygonal paths to z_1, z_2 outside the circle.

Together we get the Jordan curve $J' : q \to z_1 \to q' \to p' \to p \to p'' \to q'' \to z_2 \to q$. Since $p \in i(J), q \in o(J)$, by the lemma above, J' must intersect every component in the complement of J. However, by the choice of p', p'', the $p' \to p \to p''$ arcs lie in the component of p and the other arcs lie on or outside J. So, there can be only two components, that of p and the outside of J. In particular the inside of $J, i(J)$ is a connected open set.

In conclusion, $\mathbb{R}^2 \setminus J$ has two connected components, one bounded and the other unbounded, both having J as the boundary. This completes the proof of the Jordan Curve Theorem.

9. Epilogue

In this article we have given proofs of some intuitive results on finite collection of polygons, using which we proved a few theorems on arcs in the plane and then the Jordan Curve Theorem. However, unlike most other proofs, we did not need the complete Jordan Arc Theorem. What we used was a special case when the arcs are parts of Jordan Curves.

Using stronger approximation theorems, we can envelope this polygonal arc to prove the connectedness of the complement. Proofs of the arc theorem can be found in the references listed.

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