

# These things just work!

Shrivathsa Pandelu

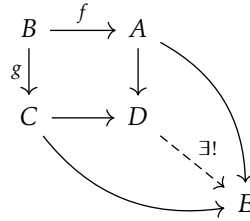
January 2, 2024

In algebraic topology there are many proofs that require the interchanging of certain operations. Often one uses these results without actually proving them, because usually the result will be clear when one draws a picture. What's happening is that most of the time one is working in a very convenient category. In this note, I would like to first see what happens to pushouts and pullbacks under the action of a bunch of functors when one is working concretely with topological spaces. Then we will look at the loop space-suspension adjunction, fibrations and cofibrations and how to convert maps into (co)fibrations. We finish with a short section on  $k$ -spaces and some foundational questions.

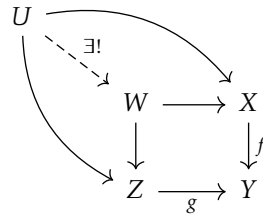
To be clear, we will be working in the context of the category  $\mathbf{Top}$  of topological spaces or  $\mathbf{Top}_*$  of based topological spaces, although our objects in consideration will be certain nice spaces only and the results will not be applicable to all objects in these categories.

## 1 Preliminaries

**Definition 1.1.** In any category of topological spaces, given morphisms  $B \xrightarrow{f} A, B \xrightarrow{g} C$  the pushout is an object  $D$  with morphisms  $B, C \rightarrow D$  that makes this diagram commute. Moreover  $D$  should be universal, i.e., if  $E$  is another such object then there should be a unique morphism  $D \rightarrow E$  making all diagrams commute:



The pullback of  $X \xrightarrow{f} Y, Z \xrightarrow{g} Y$  is the dual notion:



*Remark.* Pushouts and pullbacks (or rather limits and colimits) can be defined in any category, but for the purposes of this article we will mainly focus on topological spaces and their subcategories.

**Definition 1.2.** A space  $X$  is locally compact if every point has a neighbourhood contained in a compact set.

**Definition 1.3.** A space  $X$  is core compact if for every  $x \in X$  and neighbourhood  $U \ni x$ , there exists a neighbourhood  $x \in V \subseteq U$  such that every open cover of  $U$  has a finite subcollection covering  $V$ . Such a  $V$  is said to be well below  $U$ .

Locally compact Hausdorff spaces are core compact and Hausdorff spaces are core compact iff locally compact. These two notions (and their pointed versions) form particularly nice subcategories of  $\text{Top}, \text{Top}_*$ . A finite disjoint union of locally compact (core compact) spaces is locally compact (core compact). However, the quotient of locally compact spaces need not be locally compact. For example,  $\mathbb{R}/\mathbb{Z}$  (collapsing  $\mathbb{Z}$  to a point; not quotienting the group action) is not locally compact. This is also an example where the quotient of a core compact space is not core compact, because the quotient is Hausdorff.

**Definition 1.4.** A space  $X$  is called exponentiable if the functor  $C(-, X)$  on  $\text{Top}$  has a right adjoint.

What this means is that for every space  $Y$  we can place a topology (called the exponential topology) on  $C(X, Y)$  such that the continuity of maps  $f: Z \rightarrow C(X, Y)$  is equivalent to the curried map  $f^b: Z \times X \rightarrow Y$  (we will write the other adjoint as  $f^\#$ ) It turns out that the exponential topology, if it exists, is unique.

Even though the definition (not as stated above) works for any category, throughout this article, the word “exponentiable” shall be in the context of the category  $\text{Top}$ .

**Theorem 1.1.**  $X$  is exponentiable iff it is core compact.

When  $X$  is core compact, the exponential topology is given by what’s called the Isbell topology and when  $X$  is locally compact Hausdorff, this is precisely the compact-open topology.

As a corollary, if  $X$  is core compact and  $\sim$  is an equivalence relation on another space  $Y$ , then  $(Y/\sim) \times X \cong (Y \times X)/\sim'$  where  $\sim'$  is defined by  $(a, b) \sim' (c, d)$  if  $a \sim c, b = d$ . The slogan is that products and quotients commute. For a proof of these results and more see [5], [7].

When possible, we can get the pushout as  $A \sqcup_B C$  which is the disjoint union  $A \sqcup C$  quotiented by the equivalence relation  $f(b) \sim g(b), b \in B$  and the pullback as  $X \times_Y Z$  which is the subspace  $\{(x, z) : f(x) = g(z)\} \subseteq X \times Y$ . It is easy to verify that these satisfy the universal property, provided these objects exist in our category.

Special examples include:

- Disjoint union is the pushout of  $\emptyset \rightarrow A, B$ .
- Product is the pullback of  $A, B \rightarrow *$
- Quotient  $X/A$  is the push out of

$$\begin{array}{ccc} A & \hookrightarrow & X \\ \downarrow & & \downarrow \\ * & \longrightarrow & X/A \end{array}$$

- The wedge sum  $A \vee B$  of pointed spaces  $A, B$  is the pushout of  $* \hookrightarrow A, B$ .
- The smash product  $X \wedge Y$  of pointed spaces  $X, Y$  is defined as  $X \times Y / X \vee Y$  where  $X \vee Y$  is seen as the subspace  $X \times \{*_Y\} \cup \{*_X\} \times Y$  (more pedantically, the based points give canonical sections  $X, Y \rightarrow X \times Y$  and then we use the universal property of the pushout  $X \vee Y$ ).

## 2 Functors

Fix a space  $X$ . We have functors  $C(X, -), C(-, X), (-) \times X, (-) \wedge X, (-) \vee X$ . Note here that  $C(X, -)$  and  $C(-, X)$  are functors to  $\text{Set}$ . If we want them to be functors to  $\text{Top}$  (i.e., we want a subcategory with an *internal Hom*), then the only obvious choice is the exponential topology (which is unique), which

requires additional constraints. It is clear that the first three are functors. Given a map  $f: Y \rightarrow Z$ , we can form

$$\begin{aligned} f \wedge 1: Y \wedge X &\rightarrow Z \wedge X & f \vee 1: Y \vee X &\rightarrow Z \vee X \\ (y, x) &\mapsto (f(y), x) & y &\mapsto f(y); x \mapsto x \end{aligned}$$

where we have used the universal properties for quotients, products and disjoint unions. The question we wish to answer is how pushouts and pullbacks behave under these functors.

## 2.1 Disjoint union

Given a pushout

$$\begin{array}{ccc} B & \longrightarrow & A \\ \downarrow & & \downarrow \\ C & \longrightarrow & A \sqcup_B C \end{array}$$

we easily have the diagram

$$\begin{array}{ccc} B \sqcup X & \longrightarrow & A \sqcup X \\ \downarrow & & \downarrow \\ C \sqcup X & \longrightarrow & D \\ & \searrow & \downarrow \\ & & (A \sqcup_B C) \sqcup X \end{array}$$

(Note: A curved arrow also goes from  $B \sqcup X$  to  $(A \sqcup_B C) \sqcup X$ , and a dotted arrow goes from  $D$  to  $(A \sqcup_B C) \sqcup X$ .)

where  $D = (A \sqcup X) \sqcup_{B \sqcup X} (C \sqcup X)$  is the pushout after taking disjoint union. Observe that to obtain a continuous map  $P \sqcup Q \rightarrow R$  it is enough to have continuous maps  $P, Q \rightarrow R$ . Above we use the identity  $X \rightarrow X$  to obtain the diagram. We claim that  $(A \sqcup_B C) \sqcup X$  is the pushout.

Consider  $A \rightarrow A \sqcup X \rightarrow D, C \rightarrow C \sqcup X \rightarrow D$ . These are continuous maps, hence induce  $A \sqcup C \rightarrow D$ . By the properties of quotients, we obtain  $A \sqcup_B C \rightarrow D$ . Similarly, we obtain the continuous map  $X \rightarrow A \sqcup X \rightarrow D$ . Together, we get a continuous map  $(A \sqcup_B C) \sqcup X \rightarrow D$  which is the inverse of the dotted arrow above. The two copies of  $X$  are identified via the identity map in  $D$ .

Next consider a pullback

$$\begin{array}{ccc} A \times_B C & \longrightarrow & A \\ \downarrow & & \downarrow f \\ C & \xrightarrow{g} & B \end{array}$$

then it is easy to see that the fibred product after taking the disjoint union is precisely the subspace  $A \times_B C \sqcup X$  of the space  $(A \times C) \sqcup (A \times X) \sqcup (X \times C) \sqcup (X \times X)$ .

## 2.2 Products

Given a pullback as above the pullback of  $A \times X, C \times X \rightarrow B \times X$  is easily seen to be  $A \times_B C \times X = \{(a, c, x) \in A \times C \times X : f(a) = g(c)\}$ . This holds for any  $X$ .

Given a pushout as above, when  $X$  is exponentiable, by the relation between quotients mentioned earlier,  $(A \sqcup_B C) \times X$  is the pushout of  $B \times X \rightarrow A \times X, C \times X$ .

## 2.3 Morphisms

Fix a space  $X$ , we first consider  $C(-, X)$  on pushouts. In order to see this as a functor to  $\text{Top}$ , we must start with exponentiable spaces. Let  $A, B, C$  form a pushout as above, then we get the following

diagram

$$\begin{array}{ccc}
C(B, X) & \longleftarrow & C(A, X) \\
\uparrow & & \uparrow \text{res} \\
C(C, X) & \xleftarrow{\text{res}} & C(A \sqcup_B C, X) \\
& \nwarrow h_2 & \nearrow h_1 \\
& & Y
\end{array}$$

(A dotted arrow  $\exists! h$  points from  $Y$  to  $C(A \sqcup_B C, X)$ )

where the top map is precomposing with  $f$  and the left is precomposing with  $g$  and the dotted arrow exists by the pasting lemma and is unique for any space  $Y$ . The question is whether the dotted arrow is continuous. At this point however, as functors to  $\text{Set}$ , we have constructed the pullback, i.e., the pushout became a pullback.

Now, suppose  $A, C$  are exponentiable and we give  $C(A, X), C(C, X)$  the (unique) exponential topology, then by currying,  $h_1^b: Y \times A \rightarrow X, h_2^b: Y \times C \rightarrow X$  are continuous. If  $Y$  was also exponentiable, then we get the continuous pushout map  $h^b: Y \times (A \sqcup_B C) \rightarrow X$  (check that the pushout map is indeed  $h^b$ ). If now  $A \sqcup_B C$  was exponentiable, then  $h: Y \rightarrow C(A \sqcup_B C, X)$  is continuous.

So, the conditions are that  $A, C, A \sqcup_B C, Y$  have to be exponentiable and in that case,  $C(A \sqcup_B C, X)$  is a pullback.

Consider the pullback square:

$$\begin{array}{ccc}
A \times_B C & \longrightarrow & A \\
\downarrow & & \downarrow f \\
C & \xrightarrow{g} & B
\end{array}$$

Apply the functor  $C(X, -)$ . By the universal property of pullbacks, we get a map  $G: C(X, A \times_B C) \rightarrow C(X, A) \times_{C(X, B)} C(X, C)$ . In the other direction we proceed as follows. Let  $(\phi, \psi) \in C(X, A) \times_{C(X, B)} C(X, C)$ , then using the universal property, we have a continuous map  $\phi \times \psi: X \rightarrow A \times_B C$ . This gives us

$$\begin{aligned}
F: C(X, A) \times_{C(X, B)} C(X, C) &\rightarrow C(X, A \times_B C) \\
(\phi, \psi) &\mapsto \phi \times \psi
\end{aligned}$$

It is clear that the two maps above are inverses of each other, what remains is to show that they are continuous. This requires us to start with an exponentiable space  $X$  and give all the mapping spaces the exponential topology. To see continuity of  $G$ , it is only needed to know continuity of each factor of  $G$ . But the factors are obtained by currying the map  $C(X, A \times_B C) \times X \rightarrow A \times_B C \rightarrow A, C$  where the first map is evaluation and the second is projection. So,  $G$  is continuous.

The adjoint of  $F$  is

$$(\phi, \psi, x) \mapsto (\phi(x), \psi(x))$$

whose factors are projections followed by evaluation, hence continuous. Therefore,  $F$  is also continuous.

It is not clear (to me) what  $C(X, -)$  does to pushout squares even when we see  $C(X, -)$  as a functor to  $\text{Set}$ ; similarly, it's not clear (to me) what  $C(-, X)$  does to pullback squares.

## 2.4 Wedge

Let  $X$  be a fixed pointed space, and first consider the pushout of  $A, B, C$  as above, but this time take them to be from the category of pointed spaces. The pushout after taking wedges is  $(A \sqcup_B C) \vee X$ . The argument is almost the same as in the case of disjoint union. Note that in this category, to

obtain a map  $P \vee Q \rightarrow R$ , it is enough to have continuous (pointed) maps  $P, Q \rightarrow R$  and use the property of quotients.

Using the properties of fibred products we have the diagram

$$\begin{array}{ccccc}
 (A \times_B C) \vee X & & \xrightarrow{h_1} & & A \vee X \\
 \downarrow \text{dashed } \exists! h & & \searrow & & \downarrow \\
 D & \longrightarrow & & & A \vee X \\
 \downarrow h_2 & & \downarrow & & \downarrow \\
 C \vee X & \longrightarrow & & & B \vee X
 \end{array}$$

where  $D = (A \vee X) \times_{B \vee X} (C \vee X)$ . To obtain the opposite map, the elements of  $D$  are of the form  $(\alpha, \beta)$  with  $\alpha \in A$  or in  $X$  and  $\beta \in C$  or  $X$ . Now, because it's an element of the fibred product, we have  $\alpha \in X \iff \beta \in X$  and in this case, they are equal, else  $(\alpha, \beta) \in A \times_B C$ . So, we send  $(\alpha, \beta)$  to  $(\alpha, \beta)$  if it's in  $A \times_B C$  and to  $\alpha$  if it's in  $X$ . This way, we obtain a well defined map

$$k: (A \vee X) \times_{B \vee X} (C \vee X) \rightarrow (A \times_B C) \vee X$$

It's clear that this is the inverse of the dotted arrow and all that's left is to prove continuity. Open subsets  $U \subseteq (A \times_B C) \vee C$  are of the form  $U = U_1 \vee U_2$  where  $U_1, U_2$  could be empty. Suppose  $(\alpha, \beta) \in k^{-1}(U)$ . Now, assume  $A, C, X$  are Hausdorff.

- $\alpha \in X, \alpha \neq *$ , then there is an open  $W \subseteq U_2$  containing  $\alpha$  but not  $*$ . Then  $(\alpha, \beta) = (\alpha, \alpha) \in (W \times W) \cap D$  is open in  $k^{-1}(U)$ .
- $\alpha \in A, \alpha \neq *, \beta = *$  (this happens when  $\alpha$  is mapped to the basepoint of  $b$ ). Let  $W$  be a neighbourhood of  $*$  contained in  $U_2$  and let  $(U_3 \times U_4) \cap D$  a neighbourhood of  $(\alpha, *)$  in  $U_1$  not containing  $(*, *)$  (with  $U_3, U_4$  open). Then  $U_3$  is open in  $A \vee X$  and  $U_4 \cup W$  is open in  $C \vee X$  and their product (in the fibred product) is contained in  $k^{-1}(U)$ . Note that  $W$  is here only to get an open set. This case also covers  $\alpha = *, \beta \in C, \beta \neq *$ .
- $\alpha \in A, \alpha \neq *, \beta \neq *$ , then proceed as above with  $U_4$  not containing  $*$  and we don't need  $W$ .
- $(\alpha, \beta) = (*, *)$ , then proceed as above, this time taking  $U_3 \cup W$  and  $U_4 \cup W$ .

Therefore, the inverse is continuous and we have a homeomorphism.

## 2.5 Quotient

Say we have the same pushout diagram and we have subspaces  $A', B', C'$  with pushout  $A' \sqcup_{B'} C' = D'$ . We claim that  $(A \sqcup_B C) / (A' \sqcup_{B'} C') \cong (A/A') \sqcup_{B/B'} (C/C')$ , where the quotient by  $A' \sqcup_{B'} C'$  is actually the quotient by the image  $A \sqcup_{B'} C' \rightarrow A \sqcup_B C$ , obtained by the universal property for pushouts using  $A', C' \rightarrow A \sqcup_B C \rightarrow D$ .

We have a map  $A \rightarrow (A \sqcup_B C) \rightarrow (A \sqcup_B C) / (A' \sqcup_{B'} C')$  which induces a map  $A/A' \rightarrow D/D'$ . Similarly we get a map  $C/C' \rightarrow D/D'$  and observe that both agree on  $B/B'$ . Thus, we get a unique map  $E = (A/A') \sqcup_{B/B'} (C/C') \rightarrow D/D'$ .

In the other direction, consider first maps from  $A \sqcup_B C \rightarrow A/A' \sqcup_{B/B'} C/C' \rightarrow E$ . This is continuous (in general, given  $P \rightarrow P', Q \rightarrow Q'$  we get  $P \sqcup P' \rightarrow Q \sqcup Q'$ ) and induces  $D \rightarrow E$ . This further induces a map  $D/D' \rightarrow E$  and it's clear that this is the inverse of the previous map.

Next we look at fibred products. Consider first the scenario when  $A', C'$  are singletons, so that we have to consider  $A \times_B C, A \times_{B/B'} C$ . By the universal property, there is an inclusion  $A \times_B C \rightarrow A \times_{B/B'} C$ . We cannot have a quotient of  $A \times_B C$  being homeomorphic to  $A \times_{B/B'} C$  via this inclusion because the inclusion need not be surjective.

Instead, look at the case when  $B, B'$  are singletons and what we have to do is to obtain  $A/A' \times C/C'$  as a quotient of  $A \times C$ . It isn't the case that we quotient out the subspace  $A' \times C'$ : consider  $A = C = [0, 1], A' = C' = \{0, 1\}$ . Instead we need to focus on the equivalence relation

$$\{(a, c) \sim (a', c') \text{ if } a \sim_A a' \text{ or } c \sim_C c'\}$$

and we see that if one wants a nice result, then we would need  $A, C$  both to be from a convenient category, say that of locally compact Hausdorff spaces.

In general, if on a space  $P$  we have two relations  $\sim' \subseteq \sim$  (subset meaning that if  $a \sim' b$  then  $a \sim b$ ), then it is easy to see, by using properties of quotient spaces, that  $P/\sim \cong (P/\sim')/(\sim/\sim')$  where  $\sim/\sim'$  is the relation induced by  $\sim$  in  $P/\sim'$ .

Using this, when  $A, C$  are convenient, we may reduce the quotient above in steps to get  $(A \times C)/\sim \cong A/\sim_A \times C/\sim_C$  by first quotienting out  $A'$  say, and then  $C'$ , provided  $A/\sim_A$  or  $C/\sim_C$  is also nice.

It is tempting to generalise a little by taking  $B'$  to be a singleton, provided it makes sense to talk about  $A/\sim_A \times_B C/\sim_C$  (i.e., the maps  $A, C \rightarrow B$  should factor through the corresponding quotients). The proof with  $B$  a singleton doesn't work because the fibred product is not an actual product whereas the transference of quotients requires actual products (and the proof goes through exponential topologies, which are closely related to the product functor). A possible proof would go as follows: the fibred product is a subspace of  $A/\sim_A \times C/\sim_C \cong (A \times C)/\sim$  and  $(A \times_B C)/\sim$  (image in the quotient) is a subspace and the isomorphism above maps these subspaces to each other establishing an isomorphism of fibred products. Except this doesn't work because  $(A \times_B C)/\sim$  need not be a subspace of  $(A \times C)/\sim$ !

In general, if we have a map  $X \xrightarrow{f} Y$ , then using  $f$  we can pullback equivalence relations on  $Y$  to  $X$  and using the universal properties of quotients, we have an induced map from the induced quotient of  $X$  to the quotient of  $Y$ . Now, if  $X$  was a subspace (i.e., when  $f$  is injective and the open sets of  $X$  are precisely the pullback of open sets of  $Y$ ), then the induced map between quotients is injective, but need not be a subspace. Consider  $[0, 1]$  as a subspace of  $[0, 1]$  (we are in as nice a situation as possible for the "big" space) and quotient out  $\{0, 1\}$ . When pulled back to  $[0, 1]$  the quotient doesn't do anything, but when we pass to quotients we don't get a subspace.

## 2.6 Smash

If  $A, B, C$  are as in the default pushout diagram, then we get the pushout diagrams for the product and wedge and by using the result for quotients, we get the pushout  $(A \wedge X) \sqcup_{B \wedge X} (C \wedge X) \cong (A \sqcup_B C) \wedge X$ . There are no additional adjectives for this statement to hold.

Fibre products are a mess because quotients are a mess. The only thing that we know from the previous subsection is quotient-ing when  $B$  is a point, but in this case, because we have a product, we would need both  $B, X$  to be points (so their product is), but then there's nothing new to say.

**Theorem 2.1.** *When  $X, Z$  are exponentiable,  $X \wedge (Y \wedge Z) \cong (X \wedge Y) \wedge Z$ .*

*Proof.* The idea is as follows: when  $X$  is nice enough, we can transfer the quotient on  $Y \times Z$  to something on  $X \times Y \times Z$ . In this way, the left side would be a double quotient. We make it a single quotient using the result on double quotients mentioned above. When  $Z$  is nice, we can do the same thing on the right hand side making both sides of the equation a quotient of  $X \times Y \times Z$ , keeping in mind that product is associative.

By transferring equivalence relations, we have  $X \times (Y \wedge Z) \cong (X \times Y \times Z)/\sim$  where  $(x_1, y_1, z_1) \sim (x_2, y_2, z_2)$  iff  $x_1 = x_2$  and  $(y_1, z_1) = (y_2, z_2)$  in  $Y \wedge Z$  (which means either the coordinates are equal or both are elements of the wedge). We have to quotient this quotient by  $X \vee (Y \wedge Z)$ . If we lift this to the product, then we are quotienting  $X \times Y \times Z$  by the subset where at least one of the coordinates is the appropriate basepoint (one can guess this by drawing a suitable picture of a cube or something similar and then verify). Note that quotienting by a subset is the same as quotienting by an appropriate equivalence class.

Therefore,

$$X \wedge (Y \wedge Z) \cong (X \times Y \times Z) / (\{*_X\} \times Y \times Z \cup X \times \{*_Y\} \times Z \cup X \times Y \times \{*_Z\}).$$

When  $Z$  is exponentiable, the same argument works giving us the desired isomorphism.  $\square$

**Theorem 2.2.** For spaces  $X, Y, Z$ , with  $X$  exponentiable,  $X \wedge (Y \vee Z) \cong (X \wedge Y) \vee (X \wedge Z)$ .

*Proof.* We have, by transferring equivalence relations

$$X \times (Y \vee Z) = X \times (Y \sqcup Z) / (*_Y \sim *_Z) \cong (X \times (Y \sqcup Z)) / \sim$$

where  $(x_1, p_1) \sim (x_2, p_2)$  iff  $x_1 = x_2, p_1 = p_2$  in  $Y \vee Z$ . We always have  $X \times (Y \sqcup Z) = X \times Y \sqcup X \times Z$ .

We need to further quotient this by  $X \vee (Y \vee Z)$ . Using universal properties of pushouts, it is easy to see that wedging is associative and  $X \vee (Y \vee Z) \cong (X \sqcup Y \sqcup Z) / (*_X \sim *_Y \sim *_Z)$  (i.e., identifying all three basepoints). As in the previous theorem, we need to identify the “big” equivalence relation that is being quotiented.

It is easy to see that the big equivalence relation is one where we collapse all points of  $(X \times Y) \sqcup (X \times Z)$  with at least one coordinate being the corresponding basepoint into a single point and keep all others as they are. In other words,  $X \vee Y \subseteq X \times Y, X \vee Z \subseteq X \times Z$  are both collapsed into a single point and these points are collapsed into one.

We quotient this out in two steps. Since we have a disjoint union, we can quotient things out separately (there’s a homeomorphism happening whether we see the subset as part of one of the spaces or as part of the disjoint space, but this homeomorphism is easy to establish: use properties of disjoint union and the canonical continuous bijection happens to be open as well). Performing the quotient in two steps to first get  $(X \wedge Y) \sqcup (X \wedge Z)$  and then  $(X \wedge Y) \vee (X \wedge Z)$  as required.  $\square$

The thing that we would like to happen is that upon acting these functors we still stay in the same category. So, the question is if  $X, Y$  are exponentiable, are  $C(X, Y), X \wedge Y, X \times Y, X \vee Y$  exponentiable? But now, if we are going to stick with a subcategory, then exponentiability should be in the context of this subcategory rather than the full  $\mathbf{Top}, \mathbf{Top}_*$ . We will return to this line of thought towards the end.

### 3 Suspension

The suspension  $SX$  of a space  $X$  is obtained by collapsing the subspaces  $X \times \{0\}, X \times \{1\}$  to points in  $X \times [0, 1]$ . For a pointed space  $(X, x_0)$ , the reduced suspension  $\Sigma X$  is obtained by collapsing the subspace  $X \times \{0, 1\} \cup \{x_0\} \times [0, 1]$  to a point.

The suspension  $SX$  can be seen as the following pushout

$$\begin{array}{ccc} X \times \{0, 1\} & \hookrightarrow & X \times [0, 1] \\ \downarrow & & \downarrow \\ \{0, 1\} & \longrightarrow & SX \end{array}$$

where the left arrow is the projection onto the second coordinate.

It will be made clear a bit later, but the reason to define it this way lies behind the currying relation of functions. If we were to curry a map  $X \times I \rightarrow Y$  (i.e., currying a homotopy), then we get a map  $X \rightarrow C(I, Y)$ . If we would like the range to instead be all loops, then we need to start with a map from  $SX$  and if we would like the curried map to be a map between pointed spaces as well, then we need to start with a map from  $\Sigma X$ . We shall come to the functoriality of reduced suspension later.

**Theorem 3.1.** For an exponentiable space  $X, \Sigma X \cong X \wedge S^1$ .

*Proof.* In  $X \times [0, 1]$  we are quotienting out the subspace  $X \times \{0, 1\} \cup \{x_0\} \times [0, 1]$ . We do this in steps (and this is allowed for any  $X$ ) by first quotienting the relation  $(x, t) \sim (x', t')$  if  $x = x', t, t' \in \{0, 1\}$  and then quotient out the remaining which would be  $X \vee S^1$ : The first quotient, when  $X$  is exponentiable, is  $X \times S^1$ . In this quotient  $\{x_0\} \times [0, 1]$  becomes  $S^1$  and  $X \times \{0, 1\}$  becomes  $X$  and these two subspaces have one point in common,  $x_0$ , therefore, this is, as a set,  $X \vee S^1$ . Because we have the product topology on  $X \times S^1$ , this subset is the wedge as a topological space as well, hence we get  $\Sigma X = X \times S^1 / X \vee S^1 = X \wedge S^1$  as required.  $\square$

**Lemma 3.1.**  $\Sigma S^{n-1} \cong S^n$ .

*Proof.* Recall that a continuous bijection  $X \rightarrow Y$  with  $X$  compact and  $Y$  Hausdorff is a homeomorphism. Start with  $S^n$  centred at the origin with the north ( $N$ ) and south ( $S$ ) poles removed. Then we have a projection onto the infinite cylinder  $S^{n-1} \times \mathbb{R}$  using radial lines and this is easily seen to be a homeomorphism. Since  $\mathbb{R} \cong (0, 1)$  we have a homeomorphism  $f: S^{n-1} \times (0, 1) \rightarrow S^n \setminus \{N, S\}$ .

Extend this to  $f: S^{n-1} \times [0, 1] \rightarrow S^n$  by sending  $(x, 1) \mapsto N, (x, 0) \mapsto S$ . This is a continuous map because the inverse of neighbourhoods of  $N$  is a neighbourhood of  $S^{n-1} \times \{1\}$  etc. and factors through the quotient to get a continuous bijection  $\Sigma S^{n-1} \rightarrow S^n$  from a compact space to a Hausdorff space.  $\square$

**Lemma 3.2.**  $\Sigma S^n = S^{n+1}$  for  $n \geq 1$ .

*Proof.* The stereographic projection  $S^n \setminus \{N\} \rightarrow \mathbb{R}^n$  is a homeomorphism. Again, this gives us a homeomorphism  $(0, 1)^n \rightarrow S^n \setminus \{N\}$  which we then extend to  $I^n$  by sending the boundary to  $N$ . The resulting map is continuous and results in a continuous bijection  $I^n / \partial I^n \rightarrow S^n$  which is also a homeomorphism.

Instead of collapsing the boundary at once, we do it in steps. We are collapsing  $\partial I^{n-1} \times I \cup I^{n-1} \times \partial I$  to a point, but first we quotient out the relation  $(x, s) \sim (y, t)$  if  $x, y \in \partial I^{n-1}, s = t \in I$ . Then, because  $I$  is well behaved (it is in fact one of the most well behaved spaces and other spaces would be better off learning a thing or two from  $I$ ), we can write this quotient as  $I^{n-1} / \partial I^{n-1} \times I \cong S^{n-1} \times I$ . Let  $x_0$  denote the point that  $\partial I^{n-1}$  collapses to in  $S^{n-1}$ .

The second quotient involves collapsing  $S^{n-1} \times \{0, 1\} \cup \{x_0\} \times I$  which gives us the reduced suspension of  $S^{n-1}$ . Therefore,  $\Sigma S^{n-1} = S^n = \Sigma S^{n-1}$  as required.  $\square$

## 4 Homotopy extensions and liftings

Consider a map  $X \xrightarrow{f} Y$ . Given a space  $Z$ , by functoriality, we have maps  $C(Y, Z) \rightarrow C(X, Z)$  and  $C(Z, X) \rightarrow C(Z, Y)$ . Injectivity of these two maps depend on  $f$  being surjective/injective respectively. Surjectivity depends on being able to extend/lift the maps and we can't say much about this without knowing the maps. However, suppose we can extend/lift one such map, then would we be able to extend/lift any other map homotopic to the one that we started with?

The idea is that we are interested in invariants up to homotopy and we would like to get invariants for  $X$  from  $Y$  and vice versa and we are interested in homotopy classes of maps. Although we shall not really need it, we denote by  $\mathbf{hTop}$  the category of homotopy classes of topological spaces with morphisms being the homotopy classes of maps between any chosen representatives of the spaces. The morphisms will be denoted by  $[X, Y]$ .

**Definition 4.1.** A map  $X \xrightarrow{f} Y$  is said to have the **homotopy extension property** with respect to  $Z$  if given maps  $g: X \rightarrow Z, \tilde{g}: Y \rightarrow Z, G: X \times I \rightarrow Z$  such that  $\tilde{g} \circ f = g$  and  $G|_{X \times \{0\}} = g$  there is a homotopy  $\tilde{G}$  of  $\tilde{g}$  that extends  $G$  to  $Y$ .

It is said to have the **homotopy lifting property** with respect to  $Z$  if given maps  $g: Z \rightarrow Y, \tilde{g}: Z \rightarrow X, G: Z \times I \rightarrow Y$  such that  $f \circ \tilde{g} = g$  and  $G|_{Z \times \{0\}} = g$  there is a homotopy  $\tilde{G}$  of  $\tilde{g}$  that lifts  $G$  to  $X$ .

If  $f$  has the homotopy lifting (extension) property for all spaces, then it is called a **(co)fibration**. If  $f$  has the homotopy lifting property for all CW complexes  $Z$ , then it is called a **Serre fibration**.



Covering spaces form an important class of fibrations and for more on this refer to any of the books listed in the references. It is clear that the homotopy lifting and homotopy extension are dual notions and are nice properties to have. In the subsequent sections we see how to convert maps into (co)fibrations.

Observe that using universal properties, the pushout of a cofibration and the pullback of a fibration are cofibrations and fibrations respectively, i.e., if in our pullback (pushout) square, one of the starting arrows is a (co)fibration, then the parallel arrow is also a (co)fibration. Moreover, the fibre (inverse image of the basepoint) and the cofibre (quotient by image) are homeomorphic (this is an exercise for the reader; use the universal properties to obtain maps in one of the directions, the other direction follows from the definition of (co)fibre). Keep in mind that the pushout square behaves nicely under product with  $I$ .

It is also clear that the composition of fibrations and cofibrations are fibrations and cofibrations.

*Remark.* All of the definitions above work nicely in the context of this article. However, if we were to abstract everything to some other category where we would like to do homotopy theory, then one needs to ask how  $X \times I$  relates to  $X$ . There are two ways to define a homotopy, one is as a map  $X \times I \rightarrow Y$  and the other is  $X \rightarrow C(I, Y)$  and as we know these need not be equivalent to each other. In the language of model theory, these are called left and right homotopies with  $X \times I$  called a cylinder object (and there need not be a unique one: there's no reason to stick with  $I = [0, 1]$  and we can change things up to homotopy) and  $C(I, Y)$  a path object. What fails is that firstly, we won't have a distinguished object like  $I$  and secondly, we may not have a product/internal hom (and these are dual to each other in some ways; although a product exists if we assume that our category has small limits).

**Lemma 4.1.** *Let  $f: A \rightarrow B$  be a cofibration and  $Z$  be an exponentiable space. Then  $f \times id: A \times Z \rightarrow B \times Z$  is a cofibration.*

*Proof.* Let a homotopy  $G: A \times Z \times I \rightarrow W$  be given, this is the same as a map  $G_1: A \times I \rightarrow C(Z, W)$ . Moreover, let there be an extension  $g$  of  $G|_{A \times Z \times \{0\}}$  to  $B \times Z \times 0$  which corresponds to  $g_1: B \rightarrow C(Z, W)$ . Observe that  $g_1$  is an extension of  $G_1|_{A \times \{0\}}$ . So we can extend the homotopy  $G_1$  to  $B \times I$  and currying gives us an extension of  $G$  to  $B \times Z \times I$ . Note that the cofibre is  $B/A \times Z$ .  $\square$

**Lemma 4.2.** *With  $A, B$  exponentiable, let  $f: A \rightarrow B$  is a cofibration. For any  $Y$ ,  $f^*: C(B, Y) \rightarrow C(A, Y)$  is a fibration with respect to all  $Z$  which are exponentiable.*

*Proof.* Let  $G: Z \times I \rightarrow C(A, Y)$  and  $g: Z \rightarrow C(B, Y)$  be given such that  $f^*g = G|_{Z \times \{0\}}$ . We want to lift  $G$  to  $\tilde{G}: Z \times I \rightarrow C(B, Y)$ , i.e., a map  $Z \times I \times B \rightarrow Y$  extending the currying  $Z \times I \times A \rightarrow Y$ . Because  $Z \times A \rightarrow Z \times B$  is a cofibration, such an extension is possible and currying gives us the required lifts. Observe that the fibre is, as a set, all those functions which are constant on  $f(A)$ . From the quotient map, there's an injective map  $C(B/f(A), Y) \rightarrow C(B, Y)$  and the image is the fibre we're interested in.

The evaluation map  $C(B/f(A), Y) \times B \rightarrow Y$  factors through the evaluation map for  $B/f(A)$ , hence would be continuous if  $B/f(A)$  is exponentiable. But it's not clear whether this map is a homeomorphism onto its image.

However, in the event that the quotient  $B \rightarrow B/f(A)$  is a proper map (i.e., inverse of compact sets is compact) and we were looking at the compact-open topology everywhere, then we can say that we have a homeomorphism onto the image, i.e.,  $C(B/f(A), Y)$  is a subspace of  $C(B, Y)$ .  $\square$

An almost similar argument shows that if  $f: A \rightarrow B$  is a fibration, then, for an exponentiable  $Y$ ,  $C(Y, A) \rightarrow C(Y, B)$  is a fibration with fibre  $C(Y, f^{-1}(b_0))$ . This time the fibre is indeed exactly  $C(Y, f^{-1}(b_0))$  because it can be checked that the inclusion  $C(Y, f^{-1}(b_0)) \rightarrow C(Y, A)$  is a subspace (it is a bijection to the fibre; continuity is seen by currying; homeomorphic to the image because when  $Y$  is exponentiable, the Isbell topology is the exponential topology).

A sequence  $F \rightarrow E \rightarrow X$  will be called a fibre sequence if  $E \rightarrow X$  is a fibration with fibre  $F$ . Dually, a sequence  $A \rightarrow B \rightarrow C$  will be called a cofibre sequence if  $A \rightarrow B$  is a cofibration with cofibre  $C$ .

## 4.1 Equivalence of fibres

If  $p: E \rightarrow X$  is a fibration, then the fibre is  $E_0 = p^{-1}(x_0)$  for some  $x_0 \in X$ . Let  $x_1 \in X$  be another point with  $E_1 = p^{-1}(x_1)$  and  $\gamma: I \rightarrow X$  a path from  $x_0$  to  $x_1$ . Consider

$$\begin{array}{ccc} E_0 \times \{0\} & \hookrightarrow & E \\ \downarrow & & \downarrow p \\ E_0 \times I & \xrightarrow{\gamma \circ \pi_2} & X \end{array}$$

where  $\pi_2$  is the projection onto  $I$ . By homotopy extension, we have a (not necessarily unique) continuous map  $\psi_\gamma: E_0 \times I \rightarrow E$ . Restricting this to  $E_0 = E_0 \times \{1\}$ , we get a map  $\phi_\gamma: E_0 \rightarrow E_1$ .

Suppose  $\gamma_0, \gamma_1$  are any two paths from  $x_0$  to  $x_1$ . Let  $\psi_0: E_0 \times I \rightarrow E, \psi_1: E_1 \times I \rightarrow E$  denote the lifts of  $\gamma_0, \bar{\gamma}_1$  where  $\bar{\gamma}_1$  is the reverse of  $\gamma_1$ . By pasting lemma, there is a continuous map

$$\begin{aligned} \psi: E_0 \times I &\rightarrow E \\ (x, t) &\mapsto \begin{cases} \psi_0(x, 2t) & 0 \leq t \leq 1/2 \\ \psi_1(\psi_0(x, 1), 2t - 1) & 1/2 \leq t \leq 1 \end{cases} \end{aligned}$$

Projecting this to  $B$  gives us the path product  $\gamma_0 \bar{\gamma}_1$  ( $\gamma_0$  first, then  $\bar{\gamma}_1$ ), a loop at  $x_0$ . Suppose this loop is null homotopic (e.g., when  $\gamma_0, \gamma_1$  are homotopic paths) through a homotopy  $H: I \times I \rightarrow B$ . We can lift this homotopy to a map

$$\psi_H: E_0 \times I \times I \rightarrow E$$

which is  $\psi$  on  $E_0 \times I \times \{0\}$ .

Restrict  $\psi_H$  to the other three edges (which project to  $x_0$ ) to get a map  $E_0 \times I \rightarrow E_0$  which is identity at 0 and  $\psi(\cdot, 1)$  at 1. This shows that the fibres  $E_0, E_1$  are homotopic. It doesn't prove, however, that any two lifts of a given path  $\gamma$  are homotopic (because the homotopy equivalence proceeds by lifting the reverse path in  $B$ ).

If we perform a "path product" of lifts as above, then it descends to the usual path product in  $B$ . So, fibres over the path components of  $B$  are homotopy equivalent.

In particular, when  $p$  has the unique path lifting property, there is an action (now that there's a single choice of a lift of loops) of  $\pi_1(X, x_0)$  on  $E_0$ . This action is useful in the classification of covering spaces, see [1]. While the fibre depends on the base point, it is determined up to homotopy equivalence, whereas the cofibre of a cofibration doesn't really depend on the base point; this is because of the difference between  $C(*, X)$  and  $C(X, *)$ .

## 4.2 Mapping cylinders and cones

A homotopy of  $f$  is a map  $X \times I \xrightarrow{H} Y$  from the cylinder over  $X$  to  $Y$ , which when restricted to  $X \times \{0\}$  is  $f$ . If  $H$  is a null homotopy, then it factors through the cone  $C(X) = X \times I / X \times \{1\}$  to  $Y$ . We can consider following pushouts

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow \\ X \times [0, 1] & \longrightarrow & Cyl(f) \end{array} \quad \begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow \\ C(X) & \longrightarrow & C(f) \end{array}$$

called the mapping cylinder and mapping cones. The mapping cylinder is also denoted by  $Mf$ .

The cylinder  $Cyl(X)$  is just  $X \times [0, 1]$ , so homotopies of maps out of  $X$  are maps out of  $Cyl(X)$  and the cone as defined above. It is clear that  $Cyl, C$  are functors on  $\mathbf{Top}$ . It is also easy to see that these define functors on  $\mathbf{hTop}$  as well (exercise for the reader; keep in mind that  $I$  is nice<sup>1</sup>).

In the context of the pointed category, we would like to consider instead the reduced versions  $Cyl_r(X), C_r(X)$  where we also quotient out the interval  $\{x_0\} \times I$ . The corresponding pushouts will be denoted  $Cyl_r(f) = M_r f, C_r(f)$ . This notation is **not** standard. We will also use the notation  $Cf, C_r f$ .

<sup>1</sup>Here  $I$  refers to the object  $[0, 1]$  and not the pronoun for self reference; so " $I$  is" is grammatically correct.

**Lemma 4.3.** *The inclusion of a subspace  $i: A \hookrightarrow X$  is a cofibration if and only if  $X \times \{0\} \cup A \times I = \text{Cyl}(i)$  is a retract of  $X \times I$ . Moreover, in this case,  $A$  is closed when  $X$  is Hausdorff.*

*Proof.* Assume it is a cofibration and consider the inclusion maps  $X, A, A \times I \hookrightarrow \text{Cyl}(i)$ , then we have an extension  $H: X \times I \rightarrow \text{Cyl}(i)$ , a retraction. Conversely, composition with the retraction gives us the homotopy extension property: the data given prior to extending a homotopy is a map from the mapping cylinder  $\text{Cyl}(i)$  to a space  $Z$ .

The image of the retract  $r: X \times I \rightarrow \text{Cyl}(i) \subseteq X \times I$  is the subset of all  $x$  such that  $r(x) = x$ . When  $X$  is Hausdorff, this subset is closed because  $r$  is continuous. So,  $\text{Cyl}(i)$  is closed in  $X \times I$ , hence  $A$  is closed (by intersection with the “top” slice).  $\square$

Let  $j: Y \rightarrow \text{Cyl}(f)$  and  $i: x \mapsto [(x, 1)]$  be inclusions and  $r: \text{Cyl}(f) \rightarrow Y$  be the retraction obtained by identity on  $Y$  and  $f$  (after projection) on  $X \times I$ . The maps  $j, r, i$  make sense for  $\text{Cyl}_r(f)$  as well. The cylinder part in  $\text{Cyl}(f)$  allows us to tack on homotopies on  $X$  by stretching the cylinder. We have

**Theorem 4.1.**  *$Y, \text{Cyl}(f)$  are homotopy equivalent via  $j, r$  and  $i$  is a cofibration.*

*Proof.* We have  $rj = \text{id}_Y$ , so to prove the homotopy equivalence, we need to show  $jr$  is homotopic to  $\text{id}_{\text{Cyl}(f)}$ . Consider

$$\begin{aligned} H: \text{Cyl}(f) \times I &\rightarrow \text{Cyl}(f) \\ [y], s &\mapsto [y] \\ [(x, t)], s &\mapsto [(x, (1-s)t)] \end{aligned}$$

this is continuous (by transferring the quotient to the whole product  $Y \times I \sqcup X \times I \times I$ ) and a homotopy between identity and  $jr$  as required (basically it squeezes the cylinder down).

Next, suppose we have the data  $g: \text{Cyl}(f) \rightarrow Z, G: X \times I \rightarrow Z$  such that  $gi = G|_{X \times \{0\}}$ . Consider the function

$$\begin{aligned} H: \text{Cyl}(f) \times I &\rightarrow Z \\ [y], s &\mapsto g([y]) \\ [(x, t)], s &\mapsto \begin{cases} g([(x, t(1+s))]) & t(1+s) \leq 1 \\ G(x, t(1+s)-1) & t(1+s) \geq 1 \end{cases} \end{aligned}$$

This map is continuous and extends  $G$  as required. This map stretches the cylinder along the homotopy  $G$ .  $\square$

The cofibre of  $i$  is the mapping cone of  $f$  because we can change the order in which we quotient subspaces. Note that the theorem shows that  $Y$  is a strong deformation retract of the mapping cylinder.

The exact same proof works for the reduced version, with cofibre being the reduced mapping cone. Note that in the pointed category,  $G$  should map  $\{x_0\} \times I$  to the basepoint of  $Z$ .

For later use, we note that when  $Y = *, \text{Cyl}(f), \text{Cyl}_r(f)$  are the unreduced and reduced cones of  $X$  respectively.

**Theorem 4.2.** *The mapping cylinders and cones of homotopic maps are homotopy equivalent.*

*Proof.* Let  $f, g: X \rightarrow Y$  be homotopic via a homotopy  $H: X \times I \rightarrow Y$ . Define

$$\begin{aligned} H_1: \text{Cyl}(f) &\rightarrow \text{Cyl}(g) \\ [y] &\mapsto [y] \\ [(x, t)] &\mapsto \begin{cases} [H(x, 2t)] & 0 \leq t \leq 1/2 \\ [(x, 2t-1)] & 1/2 \leq t \leq 1 \end{cases} \end{aligned}$$

This is of course well defined. The continuity is as follows: we have a continuous  $Y \rightarrow \text{Cyl}(g)$ ; we break  $X \times I$  as a quotient of  $X \times [0, 1/2] \sqcup X \times [1/2, 1]$ . We define a map from the first part to  $Y$  and

from the second to  $X \times I$  and then pass to the pushout  $Cyl(g)$  giving us a map  $X \times [0, 1/2] \sqcup X \times [1/2, 1]$ . We can then pass to the quotient identifying the two  $1/2$  sections, to get a continuous map  $X \times I \rightarrow Cyl(g)$ . Finally, using the properties of pushouts, we get  $H_1: Cyl(f) \rightarrow Cyl(g)$ . Using the reverse of  $H$ , we get a map  $H_2: Cyl(g) \rightarrow Cyl(f)$ .

Finally define

$$G: Cyl(f) \times I \rightarrow Cyl(f)$$

$$[y], s \mapsto [y]$$

$$[(x, t)], s \mapsto \begin{cases} [H(x, 2t)] & 0 \leq t \leq s/2 \\ [H(x, 3s - 4t)] & s/2 \leq t \leq 3s/4 \\ [(x, \frac{4t-3s}{4-3s})] & 3s/4 \leq t \leq 1 \end{cases}$$

Continuity is again verified by breaking it into pieces and using gluing lemma and properties of pushouts. Keeping in mind that pushout and products are well behaved in this case because  $I$  is nice. It can also be verified that this is a homotopy between  $H_2 \circ H_1$  and  $id_{Cyl(f)}$ . By symmetry, we have the other homotopy, giving  $Cyl(f) \simeq Cyl(g)$ .

The maps above work for the cones as well (there is nothing new to do, we can freely quotient things out because  $I$  is nice enough). It works because  $H_1, H_2$  don't touch the cone point. Therefore,  $C(f) \simeq C(g)$ .

Intuitively  $H_1, H_2$  are folding the cylinder along the paths determined by  $H$  and  $G$  folds shorter bits of the cylinder.  $\square$

Again, the proof above holds for the reduced versions as well.

### 4.3 Path space and loop space

A homotopy to  $X \times I \rightarrow Y$  can be seen as a map  $X \rightarrow C(I, Y)$  (we could curry  $X$  to the other side, when  $X$  is nice, but  $C(X, Y)$  is a much bigger space to understand when we are only looking at paths in this space). In the pointed category, denote by  $P(Y)$  the mapping space  $C((I, 0), (Y, y_0))$  with the compact-open topology (which is the same as the subspace topology from  $C(I, Y)$ ).

A homotopy of a map to itself is a map  $X \times S^1 \rightarrow Y$  (to make this precise, we would need  $X$  to be exponentiable so as to quotient  $I$  separately), or a map  $X \rightarrow C(S^1, Y)$ . In the pointed category, we take  $\Omega Y = C((S^1, 1), (Y, y_0))$ . This can be seen as a subspace of  $P(Y)$  consisting of all those paths that end at  $y_0$  or as a space with the compact-open topology and these two notions are equivalent (the equivalence is as follows: from the quotient  $I \rightarrow S^1$  we get the map  $\Omega Y \rightarrow PY$ . This map is clearly continuous, injective and in the compact-open topology it is homeomorphic to its image because the inverse of compact sets under  $I \rightarrow S^1$  is again compact). It is easy to verify that  $P, \Omega$  are functors on  $\text{Top}_*$  (to see continuity of  $Pf, \Omega f$  for some  $X \rightarrow fY$ , look at its adjoint).

Observe that the path (loop) space of a wedge sum is not the wedge sum of path (loop) spaces because we can combine loops and pass through the base point multiple times. If we pass from the loop space to the fundamental group by identifying maps up to homotopy, then the fundamental group of a wedge sum is going to be a free product of the two groups, this is the Seifert-Van Kampen theorem. From the result on morphisms, it is clear that path space and loop space functors distribute over pullbacks

Consider the pullback

$$\begin{array}{ccc} Ef & \longrightarrow & X \\ \downarrow & & \downarrow f \\ C(I, Y) & \xrightarrow{ev_0} & Y \end{array} \quad \begin{array}{ccc} \Omega_f & \longrightarrow & X \\ \downarrow & & \downarrow f \\ C(S^1, Y) & \xrightarrow{ev_1} & Y \end{array}$$

Note that evaluation at a particular point is continuous because the  $C(I, -)$  et c. is functorial once we use the exponential topologies.

*Remark.* The notation  $Ef, Ff$  (to be defined later) is to indicate that these are functors on maps. We would have used  $\Omega f$  too but this notation already has a meaning with respect to the loop space functor  $\Omega$ . The notation  $\Omega_f$  is not quite standard.

We also have maps  $i: X \rightarrow Ef, \Omega_f$  which takes  $x \mapsto (x, f_x)$  where  $f_x$  is the constant loop in  $Y$  at  $f(x)$ . Note that  $x \mapsto f_x \in C(I, Y), C(S^1, Y)$  is continuous because it's adjoint is continuous.

**Theorem 4.3.** *Let  $p: Ef \rightarrow X$  denote the projection onto the first coordinate and  $e: Ef \rightarrow Y$  denote the end point evaluation following projection. Then  $e$  is a fibration with respect to all exponentiable spaces,  $ei = f$  and  $X \simeq Ef$ .*

*Proof.* It is clear that all these maps are continuous and  $ei = f, pi = id_X$ . To show the homotopy equivalence, consider

$$H: Ef \times I \rightarrow Ef$$

$$(x, l), s \mapsto (x, \{t \mapsto l((1-s)t)\})$$

This map exists and is continuous by the universal property of pullbacks: the map to  $X$  is just projection. The map  $Ef \times I \rightarrow C(I, Y)$  has adjoint  $Ef \times I \times I \rightarrow Y$  and is continuous because  $(s, t) \mapsto t(1-s)$  is continuous and so is evaluation.  $H$  shows that  $ip \simeq id_{Ef}$  as required.

Finally, we need to show that  $e: Ef \rightarrow Y$  is a fibration. A map  $g: Z \rightarrow Ef$  lifting some  $f: Z \rightarrow Y$  is essentially two maps: a map  $Z \rightarrow C(I, Y)$  whose end points are given by a map  $Z \rightarrow X \times Y$ . So, we look at the following diagram

$$\begin{array}{ccc} Ef & \longrightarrow & C(I, Y) \\ p \times e_1 \downarrow & & \downarrow e_0 \times e_1 \\ X \times Y & \xrightarrow{f \times id} & Y \times Y \\ \downarrow & & \\ Y & & \end{array}$$

where  $e_0$  is the evaluation at 0 etc.

- In this square,  $Ef$  is the pullback because the square is two pullback squares stacked together (what happens when you project from  $X \times Y, Y \times Y$  to their respective first coordinates?)
- The projection  $X \times Y \rightarrow Y$  is a fibration and because composition of fibrations is a fibration, it suffices to show that  $Ef \rightarrow X \times Y$  is a fibration.
- Because pullbacks of fibrations is a fibration, it suffices to show that  $C(I, Y) \rightarrow Y \times Y$  is a fibration.

Now,  $\{0, 1\} \hookrightarrow I$  is a cofibration because because of a projection from  $I \times I$  to  $I \times \{0\} \cup \{0, 1\} \times I$ , say the radial projection from  $(1/2, 2)$  (seeing  $I \times I$  as a subset of the plane) (for more details, see any of the books mentioned in the references, I suggest [1]). Since  $Y \times Y = C(\{0, 1\}, Y)$ . From a lemma above, because  $\{0, 1\}$  and  $I$  are exponentiable, we conclude that  $C(I, Y) \rightarrow Y \times Y$  is a fibration with respect to all exponentiable spaces.  $\square$

For later use, we note that if  $X = *$ , then  $Ef = P(Y)$ , with fibre  $\Omega(Y)$ .

When  $Y$  is path connected, there is, up to homotopy equivalence, a fibre of  $e$  given by

$$Ff = \{(x, l) : l(0) = f(x), l(1) = y_0\}$$

for some fixed  $y_0 \in Y$ .  $Ff$  is called the homotopy fibre of  $f$ .

*Remark.* What should be considered the basepoint of  $I$ ? In [2],  $Ff$  has the roles of 0, 1 reversed. As far as the arguments go, we can work with either definitions as long as we are consistent. The notation here is the one from [3].

**Theorem 4.4.** *If  $f, g: X \rightarrow Y$  are homotopic via a map  $H$ , then  $Ef \simeq Eg, \Omega_f \simeq \Omega_g$ .*

*Proof.* Define

$$H_1: Ef \rightarrow Eg$$

$$(x, l) \mapsto (x, \{t \mapsto \begin{cases} H(x, 1-2t) & 0 \leq t \leq 1/2 \\ l(2t-1) & 1/2 \leq t \leq 1 \end{cases} \})$$

Why is this continuous? The map to  $X$  is just projection and the map to  $C(I, Y)$  is continuous because it is built up of two maps from  $E \times [0, 1/2] \sqcup Ef \times [1/2, 1]$  both of which are continuous, hence passes to the quotient. So, the existence of  $H_1$  follows from the properties of quotients and pullbacks. Using the reverse of  $H$ , define  $H_2: Eg \rightarrow Ef$ . That these maps are homotopy inverses of each other is verified using a map of the type

$$G: Ef \times I \rightarrow Ef$$

$$x, l, s \mapsto (x, \{t \mapsto \begin{cases} H(x, 2t) & 0 \leq t \leq s/2 \\ H(x, 3s-4t) & s/2 \leq t \leq 3s/4 \\ l(x, \frac{4t-3s}{4-3s}) & 3s/4 \leq t \leq 1 \end{cases} \})$$

It's continuity is again a consequence of a bunch of quotients and pullbacks. We can define a similar map on  $Eg$  to conclude that  $Ef \simeq Eg$ .

A similar argument works for  $\Omega_f, \Omega_g$ , the only difference being that we need to concatenate  $H$  on both sides of  $l$  so as to close the loops. But the argument is essentially the same with continuity being a consequence of the properties of quotients and pullbacks.  $\square$

## 5 Loop space and suspension functors

We have seen what happens to (co)fibre sequences when acted on by functors  $C(X, -)$  and  $C(-, X)$ . We now look at two important functors: the loop space functor  $\Omega X = C((S^1, 1), (X, x_0))$  and the reduced suspension functor  $\Sigma X$  acting on  $\text{Top}_*$ , the category of pointed spaces. If  $X \simeq Y$ , then it is easy to see that  $\Sigma X \simeq \Sigma Y$  using suspension of maps (and transferring quotients works because  $I$  is nice). If  $H: X \times I \rightarrow X$  is some map (mapping the interval over the basepoint to the basepoint), then we have

$$\tilde{H}: \Omega X \times I \rightarrow \Omega X$$

$$(\gamma, t) \mapsto \{s \mapsto H(\gamma(s), t)\}$$

which is continuous because it's adjoint  $\Omega X \times I \times S^1 \rightarrow X$  is continuous (it's a composition of a bunch of evaluations). This gives us  $\Omega X \simeq \Omega Y$ .

**Theorem 5.1.** (Loop space - suspension adjunction) *These two functors are adjoints of each other.*

*Proof.* Given spaces  $(X, x_0), (Y, y_0)$ , currying a map  $f: \Sigma X \rightarrow Y$  gives a map  $f^\#: X \rightarrow \Omega Y$ . A priori we have a map  $X \rightarrow C(I, Y)$ , but because  $X \times \{0, 1\} \cup \{x_0\} \times I$  goes to  $y_0$ , we get a based map  $X \rightarrow \Omega Y$ .

Taking the adjoint of  $g: X \rightarrow \Omega Y$  and using the universal property of quotients, we get a continuous map  $g^b: \Sigma X \rightarrow Y$ . It is clear that these operations are inverses of each other.

Importantly, note that the path product in  $\Omega Y$  translates to a wedge sum on  $\Sigma X$  when  $X$  is nice. Here wedge sum refers to the following: given  $f, g: \Sigma X \rightarrow Y$ , we get

$$X \wedge S^1 \rightarrow X \wedge (S^1 \vee S^1) = \Sigma X \vee \Sigma X \xrightarrow{f \vee g} Y$$

where  $S^1 \rightarrow S^1 \vee S^1$  is collapsing the equator.

Lastly, the isomorphism  $C(\Sigma X, Y) \cong C(X, \Omega Y)$  above is natural in  $X, Y$  in the sense that if we have any map  $X \rightarrow Z$  or  $Y \rightarrow Z$  then all diagrams involved commute because the isomorphism is quite simple. The same isomorphism works in  $\text{Top}_*, \text{hTop}$ . Homotopic maps remain homotopic after currying as well and things work out nicely because of the universal properties of quotients and exponentiability of  $I$ .  $\square$

From the section on morphisms and pullbacks, it's clear that  $\Omega(X \times_Z Y) = \Omega X \times_{\Omega Z} \Omega Y$ . Note that the maps in our proof earlier work in the pointed category as well.

**Theorem 5.2.** (*Reduced suspension and pushouts*)  $\Sigma(A \sqcup_B C) = \Sigma A \sqcup_{\Sigma B} \Sigma C$ .

*Proof.*  $\Sigma$  is a covariant functor, so applying it to the pushout square and using the universal property, we get a map  $D \rightarrow \Sigma(A \sqcup_B C)$  where  $D = \Sigma A \sqcup_{\Sigma B} \Sigma C$ . For a map in the other direction, consider the diagram

$$\begin{array}{ccc}
 B \times I & \longrightarrow & A \times I \\
 \downarrow & & \downarrow \\
 C \times I & \longrightarrow & (A \sqcup_B C) \times I \\
 & \searrow & \nearrow \text{dotted} \\
 & & \Sigma A \sqcup_{\Sigma B} \Sigma C
 \end{array}$$

where we use the fact that product commutes with pushouts and the various quotient maps while taking suspension. From the universal property, the dotted arrow exists and it's easy to see that the dotted arrow induces a continuous map  $\Sigma(A \sqcup_B C) \rightarrow D$ . It's also easy to see that these arrows are inverses of each other, thus completing the proof.  $\square$

As a corollary, if  $B \subset A$  then  $\Sigma A / \Sigma B = \Sigma(A/B)$ . Similarly, if  $C(f)$  is the cone of a map  $X \xrightarrow{f} Y$ , then  $C(\Sigma f) = \Sigma(C(f))$ <sup>2</sup>.

By currying, if  $F \rightarrow E \rightarrow X$  is a fibre sequence, then so is  $\Omega F \rightarrow \Omega E \rightarrow \Omega X$ .

Dually, let  $A \rightarrow B \rightarrow C$  be a cofibre sequence in the category of pointed spaces (meaning the homotopy extensions send the interval over the basepoint to basepoint) and suppose we have maps  $G: \Sigma A \times I \rightarrow Z, g: \Sigma B \rightarrow Z$  such that  $G|_{\Sigma A \times \{0\}} = g|_{\Sigma A}$ . Because  $I$  is locally compact Hausdorff, we can write  $\Sigma A \times I$  as a quotient of  $A \times I_s \times I$  where  $I_s$  is  $[0, 1]$  but as the suspension coordinate, so as to not confuse with the other  $I$ . Similarly,  $\Sigma B$  is a quotient of  $B \times I_s$ . We curry the  $I_s$  out to get maps  $A \times I \rightarrow C(I_s, Z)$  and  $B \rightarrow C(I_s, Z)$ .

$$\begin{aligned}
 A \times I_s \times I &\rightarrow \Sigma A \times I \xrightarrow{G} Z \sim A \times I \rightarrow C(I_s, Z) \\
 B \times I_s &\xrightarrow{g} Z \sim B \rightarrow C(I_s, Z)
 \end{aligned}$$

It's easy to see that these maps satisfy the hypothesis for a homotopy extension, therefore we get a map  $B \times I \rightarrow C(I_s, Z)$ . Taking the adjoint of this, we have a map  $B \times I_s \times I \rightarrow Z$ . However, this does not give us what we want. Instead we notice the following, the image inside  $C(I_s, Z)$  in the above maps actually lands in the subspace  $C(S^1, Z) = \Omega Z$ . And when the image lands in a subspace, we can take the codomain to be the said subspace, therefore we get maps

$$A \times I, B \rightarrow \Omega Z$$

satisfying the hypothesis for homotopy extension<sup>3</sup>. Keep in mind that a homotopy in the pointed category passes through maps which send base points to base points.

Finally we get a map  $B \times I \rightarrow C(S^1, Z) \subset C(I_s, Z)$ . The adjoint of this is an extension of the map  $A \times I_s \times I \rightarrow Z$  and allows us to pass to the quotient and get a map  $\Sigma B \times I \rightarrow Z$  extending  $G$  precisely because the image of  $B \times I$  lands in  $C(S^1, Z)$ . Therefore,  $\Sigma$  preserves cofibre sequences.

<sup>2</sup>Once we have proved that loop space and suspension are adjoints, knowing that  $\Omega$  commutes with pullbacks is enough to deduce that its adjoint  $\Sigma$  commutes with pushouts. This is the famous **adjoint functor theorem** ([4]).

<sup>3</sup>We have taken the reduced suspension  $\Sigma X$  as a quotient of  $X \times I$ . We could have defined it as  $X \wedge S^1$ , in which case, we would directly get  $C(S^1, Z)$ . The two definitions agree when  $X$  is well behaved and our proof above required us to interchange the operation of product and quotients. It is not clear to me whether they agree for any topological space.

## 6 Barratt-Puppe sequence

Here is what we have so far: given a map  $X \xrightarrow{f} Y$ , we can construct  $Cyl(f), C(f), Ef, \Omega_f$  depending only on the homotopy class of  $f$  such that  $X \simeq Ef, Y \simeq Cyl(f)$  and there is a fibration  $Ef \rightarrow Y$ , cofibration  $X \rightarrow Cyl(f)$ . Let us first look at fibrations:

$$\begin{array}{ccc} & X & \xrightarrow{f} Y \\ p \nearrow & \downarrow i & \nearrow e_1 \\ Ff & \hookrightarrow & Ef \end{array}$$

We have the following diagram with  $e_1 i = f$  where  $i(x) = (x, f_x)$  with  $f_x$  being the constant loop at  $f(x)$ . An element  $(x, l) \in Ff$  is such that  $l$  starts at  $f(x)$  and ends at  $y_0$ , the basepoint of  $Y$ .

Temporarily denote  $\bar{P}(Y) = C((I, \{1\}), (Y, \{y_0\}))$ , the space of all paths ending at  $y_0$ . The evaluation at 0,  $\bar{P}(Y) \rightarrow Y$  is a fibration for the same reasons  $P(Y) \xrightarrow{e_1} Y$  was a fibration, and the fibre is  $\Omega(Y)$ . It is now easy to see that  $Ff$  is the pullback of  $e_0$  along  $f$  (the topology works out because  $Ff$  is a subspace of  $X \times C(I, Y)$  et c.) and the map  $Ff \rightarrow X$  is a fibration because  $e_0$  is.

Using the representation of  $Ff$  as above it is possible to directly show that  $Ff \rightarrow X$  is a fibration (and we leave this as an exercise; but the proof is essentially something that we have done above with concatenating paths), but this method doesn't lend itself to be iterated easily.

So, when we convert  $X \xrightarrow{f} Y$  to a fibration, the fibre sits as the following pullback

$$\begin{array}{ccc} Ff & \longrightarrow & X \\ \downarrow & & \downarrow f \\ \bar{P}(Y) & \xrightarrow{e_0} & Y \end{array}$$

From here, the fibre of  $Ff \rightarrow X$  is homeomorphic to  $\Omega(Y)$  via the inclusion  $\Omega(Y) \ni l \mapsto (x_0, l) \in Ff$ .

Now we figure out the fibre we obtain when we convert  $\Omega Y \rightarrow Ff$  into a fibration. It is reasonable to take the basepoint of  $Ff$  as  $(x_0, l_{y_0})$  where  $l_{y_0}$  is the constant loop in  $Y$  at  $y_0$ . We have the following diagram

$$\begin{array}{ccccc} \Omega X & \xrightarrow{-\Omega f} & \Omega Y & & \\ c \downarrow & & \downarrow i & & \\ & \bar{P}Ff & \xrightarrow{e_0} & Ff & \\ c \nearrow & \downarrow & & \downarrow & \\ \Omega X & \hookrightarrow & \bar{P}X & \longrightarrow & X \end{array}$$

where temporarily,  $i$  is the inclusion (which is injective!), and  $c$  is the map defined by

$$\omega \mapsto \{s \mapsto (w(1-s), \{t \mapsto f(\omega(1-s - (1-s)t))\})\}.$$

Because we are using  $\bar{P}X$  instead of the usual  $PX$ , we include  $\Omega X$  with reversed orientation (or rather, it is appropriate to use a  $\bar{\Omega}X$  defined similar to  $\bar{P}X$ , but that is the same subspace of  $C(I, X)$ ). A neat way to see this is to see  $\Omega X$  as the fibre of the end point evaluation  $PX \rightarrow X$ , so here we need to reverse orientations. Because of this reversed orientation, the definition of  $c$  uses the reversed  $\Omega$  and to make things commutative, we need to use  $-\Omega f$ , i.e.,  $\Omega f$  with reversed orientation. You can use the usual inclusion and  $\Omega f$  by switching the signs on  $c$  to end up with homeomorphic pullbacks, but we stick with this to get the same sequence in [2].

Continuity of  $c$  is verified using properties of pullbacks (and the fact that  $C(I, Y)$  has exponential topology).

If we had maps  $Z \xrightarrow{h_1} \bar{P}Ff, Z \xrightarrow{h_2} \Omega Y$  such that  $i h_2 = e_0 h_1$ , then following  $h_1$  with the map  $\bar{P}Ff \rightarrow \bar{P}X$  actually lands in  $\Omega X$  giving us a map  $Z \rightarrow \Omega X$  which makes everything commutative



(this requires that  $i$  is injective) and because there is a commutative triangle at the bottom, the map  $Z \rightarrow \Omega X$  is actually uniquely determined, making  $\Omega X$  the pullback.

So, if we convert  $\Omega Y \rightarrow Ff$  into a fibration, the fibre is homotopic to  $\Omega X$  and the map to  $\Omega Y$  is precisely  $-\Omega f$ .

The loop space functor preserves fibre sequences and because the map is  $-\Omega f$ , we can continue this process to get the Puppe sequence of fibrations:

$$\cdots \rightarrow \Omega Ff \rightarrow \Omega X \xrightarrow{-\Omega f} \Omega Y \rightarrow Ff \rightarrow X \xrightarrow{f} Y$$

such that the maps beyond  $\Omega Y$  are obtained by applying  $\Omega$  with a sign in every alternating groups of tree maps and any two consecutive maps form, up to homotopy equivalence (clarified later) of the spaces, a fibre sequence (said differently, it is an exact sequence of spaces with the additional property of being a fibration).

We repeat the construction for cofibre sequences. Starting with  $X \xrightarrow{f} Y$  we have seen that  $Cyl_r(f) \simeq Y, X \rightarrow Cyl_r(f)$  is a cofibration with cofibre  $C_r(f)$ . Note that although we have proved it for the unreduced versions, all the proofs factor through the appropriate quotients to get the results for the reduced versions. Since  $X \rightarrow C_r(X)$  is a cofibration, the pushout  $Y \rightarrow C_r(f)$  is also a cofibration with cofibre  $\Sigma X$  (we are using the fact that we can quotient in steps). What this tells us is that when we convert  $X \xrightarrow{f} Y$  into a cofibration, the cofibre (the space and the quotient map) sits as a pushout

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ i(f) \downarrow & & \downarrow \\ C_r X & \longrightarrow & C_r f \end{array}$$

where  $i(f)$  is the inclusion into the zero slice constructed for  $f$ . The inclusion of  $Y$  in the (base of the) cone gives us a quotient  $\pi(f)$  to the reduced suspension  $\Sigma f$ . Similar to the fibration sequence, we wish to continue this map forwards and the pattern suggests that the next step should be  $\Sigma Y$  (we already have  $\Sigma f: \Sigma X \rightarrow \Sigma Y$ .)

To continue, we need the cone of  $Y \rightarrow C_r f$ , which naturally requires us to join  $C_r Y, C_r X$  along the zero slice such that  $C_r X$  ends up being “inverted” (we assume all cones are drawn with 0 as the base; these drawings don’t have any real mathematical significance). Quotienting  $C_r Y$  gives us an inverted  $\Sigma X$  and quotienting  $C_r X$  gives  $\Sigma Y$ , suggesting that we use  $-\Sigma f$  to continue the sequence.

To get the map from the reduced cylinder of  $C_r(f)$ , consider

$$\begin{aligned} F: (X \times [0, 1] \sqcup Y) \times [0, 1] &\rightarrow Y \times [0, 1] \\ x, s, t &\mapsto f(x), 1 - \max(s, t) \\ y, t &\mapsto y, 1 - t \end{aligned}$$

It is clear that  $F$  is continuous. Following this with the quotient to  $\Sigma Y$  gives a map  $C_r(C_r f) \rightarrow \Sigma Y$  such that

$$\begin{array}{ccc} C_r f & \longrightarrow & C_r C_r f \\ \downarrow /Y & & \downarrow F \\ \Sigma X & \xrightarrow{-\Sigma f} & \Sigma Y \end{array}$$

commutes, where the map on the left is quotient by  $Y$ .

Next, we have a map  $Y \times [0, 1] \rightarrow C_r C_r f$  which actually factors through the reduced cone of  $Y$ . Given maps  $\Sigma X, C_r C_r f \rightarrow Z$  fitting in the pushout diagram, lends us a map  $\Sigma Y \rightarrow Z$  because the  $Y \times \{0\}$  bit lands in  $C_r f$  and is collapsed to the basepoint in  $\Sigma X$ . It is also clear that this resulting map fits properly into the pushout square.

Because reduced suspension preserves cofibre sequences and the map above is  $-\Sigma f$  we get the Puppe sequence of cofibrations:

$$X \xrightarrow{f} Y \rightarrow C_r(f) \rightarrow \Sigma X \xrightarrow{-\Sigma f} \Sigma Y \rightarrow \dots$$

where the later maps are obtained by acting  $\Sigma$  with signs alternating in groups of three and consecutive maps form, up to homotopy equivalence of spaces, a cofibre sequence.

In both cases above we have a sequence of spaces and maps between them. If you take any two consecutive maps, then it is either a true (co)fibration, or you could change the middle term to something homotopically equivalent to get a true (co)fibration. In both constructions above, the (co)fibre was a certain (co)limit. These objects are then unique up to a homeomorphism and we have picked a specific convenient (co)limit and the corresponding maps. Obviously, we could replace any object in the sequence with a homeomorphic (in fact, homotopically equivalent) one (and change the maps appropriately).

#### On signs

In our presentation above, the signs in the fibre sequence look more natural on account of orientation reversals. While the signs in the cofibre sequence may look forced, it reflects the duality better than not having signs. From what I've written about the cone being inverted, it seems somewhat reasonable to want signs. Hatcher [3] doesn't use signs in either sequence. As far as the sequence itself is concerned, it is a matter of choice whether one wants to include signs or not and these choices may show themselves when considering naturality of these sequences or certain sequences of homology/homotopy group.

Another source of this sign comes from how we define the path space and cones. Our path space consists of paths starting from the base point and cones have 1 as the cone point. This is the same convention in [2]. However, if one is mindful of the loop space suspension adjunction, it might be better to invert one of these. But that in itself doesn't really change the signs in the Puppe sequence because, as we have seen above, those signs are more or less a consequence of the definitions of the maps  $c, F$  above, which are up to the readers. As long as one is consistent, there should be no issues except for certain signs here and there.

## 6.1 Connections between cofibre and fibre sequences

Because  $\Sigma, \Omega$  are adjoint functors, the adjunction gives unit and counit maps

$$\eta: X \rightarrow \Omega \Sigma X \quad \epsilon: \Sigma \Omega X \rightarrow X.$$

Let  $f: X \rightarrow Y$  be a map. We then have the mapping fibre  $Ff$  which is all paths in  $Y$  that start at some  $f(x), x \in X$  and end at the basepoint of  $Y$ , and we have the mapping cone  $Cf$ . The construction indicates that these should be dual to each other (it's simply sending the time interval from one side to another). We can define

$$\eta: Ff \rightarrow \Omega Cf \quad \epsilon: \Sigma Ff \rightarrow Cf$$

by

$$\eta(x, \gamma)(t) = \epsilon(x, \gamma, t) = \begin{cases} (x, 2t) & 0 \leq t \leq 1/2 \\ \gamma(2t - 1) & 1/2 \leq t \leq 1 \end{cases}$$

which basically attaches the path from the cone point to the starting point<sup>4</sup>. These  $\eta, \epsilon$  are adjoints of each other.

**Lemma 6.1.** *Let  $f: X \rightarrow Y$  be a map of based spaces. Then the following diagram<sup>5</sup>, in which the top row is the suspension part of the fibre sequence of  $f$  and the bottom row is the loops on part of the cofibre*

<sup>4</sup>Our conventions for the mapping fibre is different from that in [2], so we have to reverse the orientation.

<sup>5</sup>This is definitely the largest commutative diagram I've ever  $\text{\LaTeX}$ ed to date.

sequence of  $f$ , is homotopy commutative:

$$\begin{array}{ccccccccccc}
& & \Sigma\Omega Ff & \xrightarrow{\Sigma\Omega p} & \Sigma\Omega X & \xrightarrow{\Sigma\Omega f} & \Sigma\Omega Y & \xrightarrow{\Sigma\iota} & \Sigma Ff & \xrightarrow{\Sigma p} & \Sigma X \\
& & \downarrow \epsilon & & \downarrow \epsilon & & \downarrow \epsilon & & \downarrow \epsilon & & \Downarrow \\
\Omega Y & \xrightarrow{\iota} & Ff & \xrightarrow{p} & X & \xrightarrow{f} & Y & \xrightarrow{i} & Cf & \xrightarrow{\pi} & \Sigma X \\
\Downarrow & & \downarrow \eta & & \downarrow \eta & & \downarrow \eta & & \downarrow \eta & & \\
\Omega Y & \xrightarrow{\Omega\iota} & \Omega Cf & \xrightarrow{\Omega\pi} & \Omega\Sigma X & \xrightarrow{\Omega\Sigma f} & \Omega\Sigma Y & \xrightarrow{\Omega\Sigma i} & \Omega\Sigma Cf & & 
\end{array}$$

*Proof.* The commutativity (up to homotopy) is easy to verify and we refer the reader to [2].  $\square$

**Lemma 6.2.** Let  $f: X \rightarrow Y$  be a map of based spaces. Then the following diagram is homotopy commutative, where  $j: X \rightarrow \text{Cyl}_r(f)$  is the inclusion,  $r: \text{Cyl}_r(f) \rightarrow Y$  is the retraction and  $\pi$  is induced by the quotient map  $\text{Cyl}_r(f) \rightarrow C_r f$ .

$$\begin{array}{ccc}
Fj = X \times_j PCyl_r(f) & \xrightarrow{Fr=id \times Pr} & X \times_f PY = Ff \\
& \searrow \pi & \swarrow \eta \\
& \Omega Cf & 
\end{array}$$

*Proof.* The quotient will send both ends of paths in  $Fj$  to the basepoint inducing  $\pi$ . Technically, this is obtained using the pullbacks that define  $\Omega Cf$  and how it relates to  $Mf$  and the fact that  $j(X)$  is collapsed to the basepoint.

Going along  $\eta \circ Fr$  essentially projects the path onto the base of the cone and attaches a path along the cone from the cone point. We can lift this along the cone to get  $\pi$  showing homotopy commutativity of the diagram.  $\square$

## 7 A note on k-spaces

There were several instances above where the argument relied on some space being exponentiable in the  $\mathbf{Top}$  category. Moreover, some operations such as quotients or products might turn our “nice” space into something that is not nice (one such common situation is when the quotient of a Hausdorff space is not Hausdorff). It is therefore important that we work in a suitable category. Core compact spaces is one such category, another is the category of locally compact Hausdorff spaces (which are all core compact).

However, when we change our category, then the exponentiable objects change as well, i.e., a space  $Z$  need not be exponentiable in  $\mathbf{Top}$ , but it could be exponentiable in the category of locally compact Hausdorff spaces for example. How does this affect the results above? The advantage of using universal properties is that they don’t rely on the specific structure (eg. compact-open topology). So most proofs should carry forward as long as the notions make sense and the maps are morphisms in our category of choice. There is also a small chicken-egg problem in that we could define certain constructs (quotients, pullbacks etc) just the right way so that the proofs work; but it’s not really an issue. What the exercise above ends up showing is that  $\mathbf{Top}$  is an *example* of a category where such and such nice things happen:  $\mathbf{Top}$  is a category where  $a$ ,  $b$ , and  $c$  hold and therefore we can construct the Puppe sequence.

So what one should do is to define a *Model category* where there are notions of the things we are interested in, prove results in such an abstract category and then show that the category we are actually interested in (topological spaces with adjectives) is a concrete example of this abstract category and therefore the results we are interested hold here because their abstract versions hold in the abstract category.

As a small motivating remark before we proceed: compact spaces are nice, homotopy groups are maps from compact spaces and it would be nice if maps from compact spaces could determine the topology of a space.

**Definition 7.1.** A space  $X$  is called a *q-compactly generated* if its topology is the final topology generated by all continuous maps  $K \rightarrow X$  where  $K$  is compact, i.e., when  $U$  open iff the inverse is open under all maps  $K \rightarrow X$ .

The  $q$  in  $q$ -compact stands for quasi; this terminology is not standard. It is often convenient to replace “all compact” in the definition with “all compact Hausdorff” and the resulting spaces are called Hausdorff compactly generated (these spaces themselves need not be Hausdorff). This is not the same as compactly generated Hausdorff spaces (i.e., compactly generated spaces which are also Hausdorff). Similarly we have “core compactly generated spaces” or “locally compactly generated spaces” et c. A comparison of such topologies is found in [6].

Some authors call a Hausdorff compactly generated space as  $k$ -space.

**Definition 7.2.** A space is called *weakly Hausdorff* if the continuous images of compact spaces are closed.

**Definition 7.3.** A space  $X$  is *compactly generated* if it has the property that a subset  $C$  is closed iff  $C \cap K$  is closed in  $K$  for every compact subset  $K \subseteq X$ .

*Remark.* Some authors add on the weakly Hausdorff axiom on top of compactly generated. Moreover, if compact spaces are assumed to be Hausdorff, then the notion of weak Hausdorff is different.

Weak Hausdorff condition is strictly between  $T_1$  and  $T_2$  separation axioms: being weakly Hausdorff makes all singletons closed, and Hausdorff spaces are weakly Hausdorff. The cocountable topology on any uncountable set is not Hausdorff, but all compact sets are finite, hence closed. See the following thread on MathOverflow for more examples: [8].

Start with a space  $X$ , obtain the space  $qX$ ,  $kX$  and  $cX$  by taking certain final topologies from the compact spaces, Hausdorff compact spaces and inclusions of compact subspaces respectively (these are the set  $X$  with open sets being those that satisfy some conditions). We have continuous “identity” maps  $qX, kX, cX \rightarrow X$ . A subset in one of these is compact iff it is a compact subset of  $X$ . We also have the continuous inclusion  $kX \rightarrow qX$ .

Suppose  $U$  is open in  $qX$ , then because the inclusion  $K \hookrightarrow X$  of a compact subset is continuous,  $U \cap K$  is open in  $K$ . Therefore,  $U$  is an open subset of  $cX$  which makes the “identity” map  $cX \rightarrow qX$  continuous.

Suppose  $U$  is open in  $cX$  and we have an arbitrary continuous function  $f: K \rightarrow X$ . Then  $f(K)$  is compact in  $X$  and  $f: K \rightarrow f(K)$  is continuous. Now,  $f^{-1}(U) = f^{-1}(U \cap f(K))$ , hence the inverse is open, which means  $U$  is open in  $qX$ . Therefore,  $cX = qX$ . Henceforth, we shall forget the notion of a  $q$ -compactly generated space and stick with compactly generated instead.

When  $X$  is Hausdorff, compact subsets are Hausdorff which means the identity map  $cX \rightarrow kX$  is continuous giving us  $qX = cX = kX$ .

If  $X$  is locally compact (this has various definitions, but we mean that every point  $x$  has a neighbourhood  $V$  contained in a compact set  $K$ ), then  $cX = X$ . Therefore, locally compact Hausdorff spaces are HCG. However, there are locally compact non Hausdorff spaces which are not Hausdorff compactly generated, see [10].

At this stage a space  $X$  could be LC (locally compact), H (Hausdorff), WH (Weak Hausdorff), CG (compactly generated), HCG (Hausdorff Compactly Generated). We have the following implications

- HCG implies CG
- WH+CG implies HCG: indeed in this case, all compact subsets are Hausdorff
- LC+H implies CG and HCG

**Theorem 7.1.** *A space is compactly generated iff it is a quotient of a locally compact space.*

*Proof.* Suppose  $X$  is compactly generated. Let  $\mathcal{K}$  denote the set <sup>6</sup> of compact subsets of  $X$  and  $Y$  be their direct sum.  $Y$  is locally compact because every point is in some  $K \in \mathcal{K}$  and  $K$  is open and compact in  $Y$ . There is the map  $Y \rightarrow X$  induced by the inclusions to  $X$  and this map is continuous regardless of whether  $X$  is compactly generated or not. We then obtain an equivalence relation  $\sim$  on  $Y$  and a continuous map  $Y/\sim \rightarrow X$  which is a bijection. When  $X$  is compactly generated, this map is a homeomorphism, giving  $X$  as a quotient of a locally compact space.

Conversely, let  $Y$  be a locally compact space and  $\sim$  an equivalence relation on  $Y$  such that there is a homeomorphism  $Y/\sim \rightarrow X$ . We take  $X = Y/\sim$  and let  $p: Y \rightarrow X$  be the quotient map. Suppose  $U$  is a subset of  $X$  intersecting every compact subset of  $X$  in an open set. To show  $U$  is open, we need to show that  $p^{-1}(U)$  is open in  $Y$ . Let  $y \in p^{-1}(U)$ . By local compactness, there is an open set  $V$  and a compact set  $K$  in  $Y$  such that  $y \in V \subseteq K$ .

Since  $p$  is continuous,  $p(K)$  is compact, hence  $p(K) \cap U$  is open in  $p(K)$ . The restriction  $p|_K: K \rightarrow p(K)$  is continuous, hence,  $p|_K^{-1}(p(K) \cap U)$  is open in  $K$ . Intersecting this with  $V$  gives an open neighbourhood of  $y$  contained in  $p^{-1}(U)$ . Therefore,  $p^{-1}(U)$  is open in  $Y$ , hence  $U$  is open in  $X$  completing the proof.  $\square$

**Corollary 7.1.** *A quotient of a compactly generated space is compactly generated.*

**Theorem 7.2.** *A space is Hausdorff compactly generated iff it is a quotient of a disjoint union of compact Hausdorff spaces.*

*Proof.* If  $X$  is a quotient of  $Y$ , a disjoint union of compact Hausdorff spaces with quotient map  $q: Y \rightarrow X$ . Suppose  $U \subseteq X$  such that for every  $f: K \rightarrow X$  where  $K$  is compact Hausdorff, the inverse of  $U$  is open. To show  $U$  is open in  $X$  we need to show that  $q^{-1}(U)$  is open in  $Y$ . But this is obvious because  $Y$  is a disjoint union of compact Hausdorff spaces.

For the other direction, it is suggestive to construct a quotient of the compact subspaces of  $X$  which are Hausdorff. However, this doesn't work because the image of a Hausdorff compact space in  $X$  need not be Hausdorff. Instead, as done in [6], we need to quotient out a much larger space.

<sup>6</sup>This is a subset of the power set for example. It is important that it is a set because direct sums are defined for sets but not for anything larger.

Let  $I$  denote the set of all non open subsets of  $X$ . For each  $i \in I$  there is a compact Hausdorff  $K_i$  and a continuous map  $f_i: K_i \rightarrow X$  under which the inverse of  $i \in I$  is not open. Let  $S$  denote the direct sum of these  $K_i$  and many one point spaces with various constant maps so as to cover  $X$ .

Too long for a footnote: There is a whole lot of logical steps I am skipping here because I don't know how they work. What follows may be entirely wrong. The choice function seems to require taking choices from "subcategories". So, either we work in a framework with versions of the ZF axioms applying for categories so as to define subcategories through an axiom schema of separation and then use axiom of (global) choice. Or we look at the collection of objects in our category, hope that they form a set and use the usual ZF axioms to choose and the lift our choice back to the category. But, usually our categories are not small and the objects don't form a set. A way around this is to apparently work in the context of Grothendieck universes and inaccessible cardinals so as to constraint the size of the objects in our category (our category being the comma category of compact spaces over  $X$ ). Once the size is constrained, we get a small category.

Glossing over the foundational questions, assuming  $S$  exists, there's a continuous surjection  $S \rightarrow X$ . If  $U \subseteq X$  is not open, then it's inverse is not open by choice of our  $K_i$ 's, hence  $X$  is a quotient of  $S$ .  $\square$

**Proposition 7.1.** *The product of a compactly generated space with a locally compact Hausdorff space is compactly generated.*

*Proof.* Suppose  $X$  is compactly generated and  $Z$  is locally compact Hausdorff.  $X$  is a quotient of some locally compact space  $Y$ . Since  $Z$  is core compact, we can transfer the quotient to get  $X \times Z$  as a quotient of  $Y \times Z$ . Now,  $Y \times Z$  is locally compact because product of open (compact) sets is open (compact).  $\square$

What happens when we don't have the Hausdorff condition in the above theorem? The Math-Overflow thread [9] seems interesting.

It turns out that compactly generated weakly Hausdorff spaces together with continuous maps form a nice category. It is cartesian closed (i.e., everything is exponentiable), has products and pushouts and subspaces (all of these are not the same as that in Top). Practically everything that has been done in this document carry forward as it is, more or less, because the arguments were purely formal. For more, [2] is a good resource, so is nLab. However, keep in mind that the terminology differs from source to source.

A last thing to mention is that there is the  $k$ -ification functor  $k$  from  $WH^7$  to CGWH which assigns to a topological space the space obtained by the final topology from all compact Hausdorff spaces. The product of two spaces will be the  $k$ -ification of the usual product and when  $Z$  is in CGWH, a map  $Z \rightarrow kX$  is continuous iff the map  $Z \rightarrow X$  is continuous. Subspaces are subsets with the  $k$ -ified subspace topology and in this sense pullbacks exist and any of our constructions above carry forward. Quotients and pullbacks need to have closed relations if we want to stay in CGWH ([2]).

## References

- [1] Spanier, Algebraic Topology
- [2] May, J. Peter. A Concise Course in Algebraic Topology. University of Chicago Press, 1999.
- [3] Hatcher, Algebraic Topology
- [4] nLab article on adjoint functor theorem: [Adjoint functor theorem](#)  
Notes by Kevin Buzzard: [Adjoint functor theorem](#)
- [5] Tyrone Cutler, [Exponentiable spaces](#)  
<https://www.math.uni-bielefeld.de/~tcutler/pdf/Exponentiable%20Spaces.pdf>

---

<sup>7</sup>I start with weak Hausdorff spaces, but I suppose conventions/notions differ.

- [6] Escardó, Martín, et al. “Comparing Cartesian Closed Categories of (Core) Compactly Generated Spaces.” *Topology and Its Applications*, vol. 143, no. 1–3, Aug. 2004, pp. 105–45. DOI.org (Crossref), <https://doi.org/10.1016/j.topol.2004.02.011>.
- [7] [My notes on Exponentiability](#)  
<https://shrivathsap.github.io/resources/pdfs/topgeom/cotop.pdf>
- [8] MathOverflow thread on Weak Hausdorff spaces which are not Hausdorff: [link](#)  
<https://mathoverflow.net/questions/88420/>
- [9] MathOverflow thread on product of compact space with a compactly generated space: [link](#)  
<https://mathoverflow.net/questions/85912/>
- [10] MathStackExchange answer of a locally compact space which is not Hausdorff compactly generated: [link](#)  
<https://math.stackexchange.com/questions/2971379/>
- [11] All commutative diagrams were made using <https://tikzcd.yichuanshen.de/>