These things just work!

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In algebraic topology there are many proofs that require the interchanging of certain operations. Often one uses these results without actually proving them, because usually the result will be clear when one draws a picture. What's happening is that most of the time one is working in a very convenient category. In this note, I would like to first see what happens to pushouts and pullbacks under the action of a bunch of functors when one is working concretely with topological spaces. Then we will look at the loop space-suspension adjunction, fibrations and cofibrations and how to convert maps into (co)fibrations. We finish with a short section on k-spaces and some foundational questions.

1 Preliminaries

Definition 1.1. *In any category of topological spaces, given morphisms $B \xrightarrow{f} A, B \xrightarrow{g} C$ the pushout is an object D with morphisms $B, C \to D$ that makes this diagram commute. Moreover D should be universal, i.e., if E is another such object then there should be a unique morphism $D \to E$ making all diagrams commute:*

$$\begin{tikzcd}
B \arrow[r, "f"] \arrow[d, "g"'] & A \arrow[d] \arrow[ddr, bend left] & \\
C \arrow[r] \arrow[drr, bend right] & D \arrow[dr, dashed, "\exists!"] & \\
 & & E
\end{tikzcd}$$

The pullback of $X \xrightarrow{f} Y, Z \xrightarrow{g} Y$ is the dual notion:

$$\begin{tikzcd}
U \arrow[dr, dashed, "\exists!"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
 & W \arrow[r] \arrow[d] & X \arrow[d, "f"] \\
 & Z \arrow[r, "g"'] & Y
\end{tikzcd}$$

Remark. Pushouts and pullbacks (or rather limits and colimits) can be defined in any category, but for the purposes of this article we will mainly focus on topological spaces and their subcategories.

Definition 1.2. *A space X is locally compact if every point has a neighbourhood contained in a compact set.*

Definition 1.3. *A space X is core compact if for every $x \in X$ and neighbourhood $U \ni x$, there exists a neighbourhood $x \in V \subseteq U$ such that every open cover of U has a finite subcollection covering V. Such a V is said to be well below U.*

Locally compact Hausdorff spaces are core compact and Hausdorff spaces are core compact iff locally compact. These two notions (and their pointed versions) form particularly nice subcategories of $\mathsf{Top}, \mathsf{Top}_*$. A finite disjoint union of locally compact (core compact) spaces is locally compact (core compact). However, the quotient of locally compact spaces need not be locally compact. For example, $\mathbb{R}/\mathbb{Z}$ (collapsing $\mathbb{Z}$ to a point; not quotienting the group action) is not locally compact. This is also an example where the quotient of a core compact space is not core compact, because the quotient is Hausdorff.

Definition 1.4. *A space X is called exponentiable if the functor $C(-, X)$ on Top has a right adjoint.*

What this means is that for every space Y we can place a topology (called the exponential topology) on $C(X, Y)$ such that the continuity of maps $f: Z \to C(X, Y)$ is equivalent to the curried map $f^\flat: Z \times X \to Y$ (we will write the other adjoint as $f^\#$) It turns out that the exponential topology, if it exists, is unique.

Even though the definition (not as stated above) works for any category, throughout this article, the word "exponentiable" shall be in the context of the category Top.

Theorem 1.1. *X is exponentiable iff it is core compact.*

When X is core compact, the exponential topology is given by what's called the Isbell topology and when X is locally compact Hausdorff, this is precisely the compact-open topology.

As a corollary, if X is core compact and $\sim$ is an equivalence relation on another space Y, then $(Y/\sim) \times X \cong (Y \times X)/\sim'$ where $\sim'$ is defined by $(a, b) \sim' (c, d)$ if $a \sim c, b = d$. The slogan is that products and quotients commute. For a proof of these results and more see [5], [7].

When possible, we can get the pushout as $A \sqcup_B C$ which is the disjoint union $A \sqcup C$ quotiented by the equivalence relation $f(b) \sim g(b), b \in B$ and the pullback as $X \times_Y Z$ which is the subspace $\{(x, z) : f(x) = g(z)\} \subseteq X \times Y$. It is easy to verify that these satisfy the universal property, provided these objects exist in our category.

Special examples include:

- Disjoint union is the pushout of $\emptyset \to A, B$.
- Product is the pullback of $A, B \to *$
- Quotient X/A is the push out of

$$\begin{tikzcd} A \arrow[r, hook] \arrow[d] & X \arrow[d] \\ * \arrow[r] & X/A \end{tikzcd}$$

- The wedge sum $A \vee B$ of pointed spaces A, B is the pushout of $* \hookrightarrow A, B$.
- The smash product $X \wedge Y$ of pointed spaces X, Y is defined as $X \times Y/X \vee Y$ where $X \vee Y$ is seen as the subspace $X \times \{*_Y\} \cup \{*_X\} \times Y$ (more pedantically, the based points give canonical sections $X, Y \to X \times Y$ and then we use the universal property of the pushout $X \vee Y$).

2 Functors

Fix a space X. We have functors $C(X, -), C(-, X), (-)\times X, (-)\wedge X, (-)\vee X$. Note here that $C(X, -)$ and $C(-, X)$ are functors to Set. If we want them to be functors to Top, then the only obvious choice is the exponential topology (which is unique), which would require additional constraints. It is clear that the first three are functors. Given a map $f: Y \to Z$, we can form

$$\begin{aligned} f \wedge 1: Y \wedge X &\to Z \wedge X \\ (y, x) &\mapsto (f(y), x) \end{aligned} \qquad \begin{aligned} f \vee 1: Y \vee X &\to Z \vee X \\ y \mapsto f(y); &\, x \mapsto x \end{aligned}$$

where we have used the universal properties for quotients, products and disjoint unions. The question we wish to answer is how pushouts and pullbacks behave under these functors.

2.1 Disjoint union

Given a pushout

$$\begin{tikzcd} B \arrow[r] \arrow[d] & A \arrow[d] \\ C \arrow[r] & A \sqcup_B C \end{tikzcd}$$

we easily have the diagram

$$\begin{tikzcd} B \sqcup X \arrow[r] \arrow[d] & A \sqcup X \arrow[d] \arrow[ddr, bend left] & \\ C \sqcup X \arrow[r] \arrow[drr, bend right] & D \arrow[dr, dashed] & \\ & & (A \sqcup_B C) \sqcup X \end{tikzcd}$$

where $D = (A \sqcup X) \sqcup_{B \sqcup X} (C \sqcup X)$ is the pushout after taking disjoint union. Observe that to obtain a continuous map $P \sqcup Q \to R$ it is enough to have continuous maps $P, Q \to R$. Above we use the identity $X \to X$ to obtain the diagram. We claim that $(A \sqcup_B C) \sqcup X$ is the pushout.

Consider $A \to A \sqcup X \to D, C \to C \sqcup X \to D$. These are continuous maps, hence induce $A \sqcup C \to D$. By the properties of quotients, we obtain $A \sqcup_B C \to D$. Similarly, we obtain the continuous map $X \to A \sqcup X \to D$. Together, we get a continuous map $(A \sqcup_B C) \sqcup X \to D$ which is the inverse of the dotted arrow above. The two copies of X are identified via the identity map in D.

Next consider a pullback

$$\begin{tikzcd} A \times_B C \arrow[r] \arrow[d] & A \arrow[d, "f"] \\ C \arrow[r, "g"'] & B \end{tikzcd}$$

then it is easy to see that the fibred product after taking the disjoint union is precisely the subspace $A \times_B C \sqcup X$ of the space $(A \times C) \sqcup (A \times X) \sqcup (X \times C) \sqcup (X \times X)$.

2.2 Products

Given a pullback as above the pullback of $A \times X, C \times X \to B \times X$ is easily seen to be $A \times_B C \times X = \{(a, c, x) \in A \times C \times X : f(a) = g(c)\}$.This holds for any X.

Given a pushout as above, when X is exponentiable, by the relation between quotients mentioned earlier, $(A \sqcup_B C) \times X$ is the pushout of $B \times X \to A \times X, C \times X$.

2.3 Morphisms

Fix a space X, we first consider $C(-, X)$ on pushouts. In order to see this as a functor to Top, we must start with exponentiable spaces. Let A, B, C form a pushout as above, then we get the following diagram

$$\begin{tikzcd} C(B, X) & C(A, X) \arrow[l] & \\ C(C, X) \arrow[u] & C(A \sqcup_B C, X) \arrow[l, "res"'] \arrow[u, "res"'] & \\ & & Y \arrow[ul, dashed, "\exists ! h"] \arrow[uul, bend right, "h_1"'] \arrow[ull, bend left, "h_2"] \end{tikzcd}$$

where the top map is precomposing with f and the left is precomposing with g and the dotted arrow exists by the pasting lemma and is unique for any space Y. The question is whether the dotted arrow is continuous. At this point however, as functors to Set, we have constructed the pullback, i.e., the pushout became a pullback.

Now, suppose A, C are exponentiable and we give $C(A,X), C(C,X)$ the (unique) exponential topology, then by currying, $h_1^\flat \colon Y \times A \to X, h_2^\flat \colon Y \times C \to X$ are continuous. If Y was also exponentiable, then we get the continuous pushout map $h^\flat \colon Y \times (A \sqcup_B C) \to X$ (check that the pushout map is indeed $h^\flat$). If now $A \sqcup_B C$ was exponentiable, then $h \colon Y \to C(A \sqcup_B C, X)$ is continuous.

So, the conditions are that $A, C, A \sqcup_B C, Y$ have to be exponentiable and in that case, $C(A \sqcup_B C, X)$ is a pullback.

Consider the pullback square:

$$\begin{tikzcd} A \times_B C \arrow[r] \arrow[d] & A \arrow[d, "f"] \\ C \arrow[r, "g"'] & B \end{tikzcd}$$

Apply the functor $C(X,-)$. By the universal property of pullbacks, we get a map $G \colon C(X, A \times_B C) \to C(X,A) \times_{C(X,B)} C(X,C)$. In the other direction we proceed as follows. Let $(\phi, \psi) \in C(X,A) \times_{C(X,B)} C(X,C)$, then using the universal property, we have a continuous map $\phi \times \psi \colon X \to A \times_B C$. This gives us

$$\begin{aligned} F \colon C(X,A) \times_{C(X,B)} C(X,C) &\to C(X, A \times_B C) \\ (\phi, \psi) &\mapsto \phi \times \psi \end{aligned}$$

It is clear that the two maps above are inverses of each other, what remains is to show that they are continuous. This requires us to start with an exponentiable space X and give all the mapping spaces the exponential topology. To see continuity of G, it is only needed to know continuity of each factor of G. But the factors are obtained by currying the map $C(X, A \times_B C) \times X \to A \times_B C \to A, C$ where the first map is evaluation and the second is projection. So, G is continuous.

The adjoint of F is

$$(\phi, \psi, x) \mapsto (\phi(x), \psi(x))$$

whose factors are projections followed by evaluation, hence continuous. Therefore, F is also continuous.

It is not clear what $C(X,-)$ does to pushout squares even when we see $C(X,-)$ as a functor to Set; similarly, it's not clear what $C(-,X)$ does to pullback squares.

2.4 Wedge

Let X be a fixed pointed space, and first consider the pushout of A, B, C as above, but this time take them to be from the category of pointed spaces. The pushout after taking wedges is $(A \sqcup_B C) \vee X$. The argument is almost the same as in the case of disjoint union. Note that in this category, to obtain a map $P \vee Q \to R$, it is enough to have continuous (pointed) maps $P, Q \to R$ and use the property of quotients.

Using the properties of fibred products we have the diagram

$$\begin{tikzcd} (A \times_B C) \vee X \arrow[drr, bend left, "h_1"] \arrow[ddr, bend right, "h_2"'] \arrow[dr, dashed, "\exists ! h"] & & \\ & D \arrow[r] \arrow[d] & A \vee X \arrow[d] \\ & C \vee X \arrow[r] & B \vee X \end{tikzcd}$$

where $D = (A \vee X) \times_{B \vee X} (C \vee X)$. To obtain the opposite map, the elements of D are of the form (α, β) with $\alpha \in A$ or in X and $\beta \in C$ or X. Now, because it's an element of the fibred product, we have $\alpha \in X \iff \beta \in X$ and in this case, they are equal, else $(\alpha, \beta) \in A \times_B C$. So, we send (α, β) to (α, β) if it's in $A \times_B C$ and to α if it's in X. This way, we obtain a well defined map

$$k\colon (A \vee X) \times_{B \vee X} (C \vee X) \to (A \times_B C) \vee X$$

It's clear that this is the inverse of the dotted arrow and all that's left is to prove continuity. Open subsets $U \subseteq (A \times_B C) \vee C$ are of the form $U = U_1 \vee U_2$ where U_1, U_2 could be empty. Suppose $(\alpha, \beta) \in k^{-1}(U)$. <u>Now, assume A, C, X are Hausdorff.</u>

- $\alpha \in X, \alpha \neq *$, then there is an open $W \subseteq U_2$ containing α but not $*$. Then $(\alpha, \beta) = (\alpha, \alpha) \in (W \times W) \cap D$ is open in $k^{-1}(U)$.
- $\alpha \in A, \alpha \neq *, \beta = *$ (this happens when α is mapped to the basepoint of b). Let W be a neighbourhood of $*$ contained in U_2 and let $(U_3 \times U_4) \cap D$ a neighbourhood of $(\alpha, *)$ in U_1 not containing $(*, *)$ (with U_3, U_4 open). Then U_3 is open in $A \vee X$ and $U_4 \cup W$ is open in $C \vee X$ and their product (in the fibred product) is contained in $k^{-1}(U)$. Note that W is here only to get an open set. This case also covers $\alpha = *, \beta \in C, \beta \neq *$.
- $\alpha \in A, \alpha \neq *, \beta \neq *$, then proceed as above with U_4 not containing $*$ and we don't need W.
- $(\alpha, \beta) = (*, *)$, then proceed as above, this time taking $U_3 \cup W$ and $U_4 \cup W$.

Therefore, the inverse is continuous and we have a homeomorphism.

2.5 Quotient

Say we have the same pushout diagram and we have subspaces A', B', C' with pushout $A' \sqcup_{B'} C' = D'$. We claim that $(A \sqcup_B C)/(A' \sqcup_{B'} C') \cong (A/A') \sqcup_{B/B'} (C/C')$, where the quotient by $A' \sqcup_{B'} C'$ is actually the quotient by the image $A \sqcup_{B'} C' \to A \sqcup_B C$, obtained by the universal property for pushouts using $A', C' \to A \sqcup C \to D$.

We have a map $A \to (A \sqcup_B C) \to (A \sqcup_B C)/(A' \sqcup B'C')$ which induces a map $A/A' \to D/D'$. Similarly we get a map $C/C' \to D/D'$ and observe that both agree on B/B'. Thus, we get a unique map $E = (A/A') \sqcup_{B/B'} (C/C') \to D/D'$.

In the other direction, consider first maps from $A \sqcup C \to A/A' \sqcup C/C' \to E$. This is continuous (in general, given $P \to P', Q \to Q'$ we get $P \sqcup P' \to Q \sqcup Q'$) and induces $D \to E$. This further induces a map $D/D' \to E$ and it's clear that this is the inverse of the previous map.

Next we look at fibred products. Consider first the scenario when A', C' are singletons, so that we have to consider $A \times_B C, A \times_{B/B'} C$. By the universal property, there is an inclusion $A \times_B C \to A \times_{B/B'} C$. We cannot have a quotient of $A \times_B C$ being homeomorphic to $A \times_{B/B'} C$ via this inclusion because the inclusion need not be surjective.

Instead, look at the case when B, B' are singletons and what we have to do is to obtain $A/A' \times C/C'$ as a quotient of $A \times C$. It isn't the case that we quotient out the subspace $A' \times C'$: consider $A = C = [0, 1], A' = C' = \{0, 1\}$. Instead we need to focus on the equivalence relation

$$\{(a, c) \sim (a', c') \text{ if } a \sim_A a' \text{ or } c \sim_C c'\}$$

and we see that if one wants a nice result, then we would need A, C both to be from a convenient category, say that of locally compact Hausdorff spaces.

In general, if on a space P we have two relations $\sim' \subseteq \sim$ (subset meaning that if $a \sim' b$ then $a \sim b$), then it is easy to see, by using properties of quotient spaces, that $P/\sim \cong (P/\sim')/(\sim/\sim')$ where $\sim/\sim'$ is the relation induced by $\sim$ in $P/\sim'$.

Using this, when A, C are convenient, we may reduce the quotient above in steps to get $(A \times C)/\sim \cong A/\sim_A \times C/\sim_C$ by first quotienting out A' say, and then C', provided $A/\sim_A$ or $C/\sim_C$ is also nice.

It is tempting to generalise a little by taking B' to be a singleton, provided it makes sense to talk about $A/\sim_A \times_B C/\sim_C$ (i.e., the maps $A, C \to B$ should factor through the corresponding quotients). The proof with B a singleton doesn't work because the fibred product is not an actual product whereas the transference of quotients requires actual products (and the proof goes through exponential topologies, which are closely related to the product functor). A possible proof would go as follows: the fibred product is a subspace of $A/\sim_A \times C/\sim_C \cong (A \times C)/\sim$ and $(A \times_B C)/\sim$ (image in the quotient) is a subspace and the isomorphism above maps these subspaces to each other establishing an isomorphism of fibred products. Except this doesn't work because $(A \times_B C)/\sim$ need not be a subspace of $(A \times C)/\sim$!

In general, if we have a map $X \xrightarrow{f} Y$, then using f we can pullback equivalence relations on Y to X and using the universal properties of quotients, we have an induced map from the induced quotient of X to the quotient of Y. Now, if X was a subspace (i.e., when f is injective and the open sets of X are precisely the pullback of open sets of Y), then the induced map between quotients is injective, but need not be a subspace. Consider $[0, 1)$ as a subspace of $[0, 1]$ (we are in as nice a situation as possible for the "big" space) and quotient out $\{0, 1\}$. When pulled back to $[0, 1)$ the quotient doesn't do anything, but when we pass to quotients we don't get a subspace.

2.6 Smash

If A, B, C are as in the default pushout diagram, then we get the pushout diagrams for the product and wedge and by using the result for quotients, we get the pushout $(A \wedge X) \sqcup_{B \wedge X} (C \wedge X) \cong (A \sqcup_B C) \wedge X$. There are no additional adjectives for this statement to hold.

Fibre products are a mess because quotients are a mess. The only thing that we know from the previous subsection is quotient-ing when B is a point, but in this case, because we have a product, we would need both B, X to be points (so their product is), but then there's nothing new to say.

Theorem 2.1. *When X, Z are exponentiable, $X \wedge (Y \wedge Z) \cong (X \wedge Y) \wedge Z$.*

Proof. The idea is as follows: when X is nice enough, we can transfer the quotient on $Y \times Z$ to something on $X \times Y \times Z$. In this way, the left side would be a double quotient. We make it a single quotient using the result on double quotients mentioned above. When Z is nice, we can do the same thing on the right hand side making both sides of the equation a quotient of $X \times Y \times Z$, keeping in mind that product is associative.

By transfering equivalence relations, we have $X \times (Y \wedge Z) \cong (X \times Y \times Z)/\sim$ where $(x_1, y_1, z_1) \sim (x_2, y_2, z_2)$ iff $x_1 = x_2$ and $(y_1, z_1) = (y_2, z_2)$ in $Y \wedge Z$ (which means either the coordinates are equal or both are elements of the wedge). We have to quotient this quotient by $X \vee (Y \wedge Z)$. If we lift this to the product, then we are quotienting $X \times Y \times Z$ by the subset where at least one of the coordinates is the appropriate basepoint (one can guess this by drawing a suitable picture of a cube or something similar and then verify). Note that quotienting by a subset is the same as quotienting by an appropriate equivalence class.

Therefore,

$$X \wedge (Y \wedge Z) \cong (X \times Y \times Z)/(\{*_X\} \times Y \times Z \cup X \times \{*_Y\} \times Z \cup X \times Y \times \{*_Z\}).$$

When Z is exponentiable, the same argument works giving us the desired isomorphism. □

Theorem 2.2. *For spaces X, Y, Z, with X exponentiable, $X \wedge (Y \vee Z) \cong (X \wedge Y) \vee (X \wedge Z)$.*

Proof. We have, by transfering equivalence relations

$$X \times (Y \vee Z) = X \times (Y \sqcup Z)/(*_Y \sim *_Z) \cong (X \times (Y \sqcup Z))/\sim$$

where $(x_1, p_1) \sim (x_2, p_2)$ iff $x_1 = x_2, p_1 = p_2$ in $Y \vee Z$. We always have $X \times (Y \sqcup Z) = X \times Y \sqcup X \times Z$.

We need to further quotient this by $X \vee (Y \vee Z)$. Using universal properties of pushouts, it is easy to see that wedging is associative and $X \vee (Y \vee Z) \cong (X \sqcup Y \sqcup Z)/(*_X \sim *_Y \sim *_Z)$ (i.e., identifying all three basepoints). As in the previous theorem, we need to identify the "big" equivalence relation that is being quotiented.

It is easy to see that the big equivalence relation is one where we collapse all points of $(X \times Y) \sqcup (X \times Z)$ with at least one coordinate being the corresponding basepoint into a single point and keep all others as they are. In other words, $X \vee Y \subseteq X \times Y, X \vee Z \subseteq X \times Z$ are both collapsed into a single point and these points are collapsed into one.

We quotient this out in two steps. Since we have a disjoint union, we can quotient things out separately (there's a homeomorphism happening whether we see the subset as part of one of the spaces or as part of the disjoint space, but this homeomorphism is easy to establish: use properties of disjoint union and the canonical continuous bijection happens to be open as well). Performing the quotient in two steps to first get $(X \wedge Y) \sqcup (X \wedge Z)$ and then $(X \wedge Y) \vee (X \wedge Z)$ as required. $\square$

The thing that we would like to happen is that upon acting these functors we still stay in the same category. So, the question is if X, Y are exponentiable, are $C(X, Y), X \wedge Y, X \times Y, X \vee Y$ exponentiable? But now, if we are going to stick with a subcategory, then exponentiability should be in the context of this subcategory rather than the full $\mathsf{Top}, \mathsf{Top}_*$. We will return to this line of thought towards the end.

3 Suspension

The suspension SX of a space X is obtained by collapsing the subspaces $X \times \{0\}, X \times \{1\}$ to points in $X \times [0,1]$. For a pointed space (X, x_0), the reduced suspension ΣX is obtained by collapsing the subspace $X \times \{0,1\} \cup \{x_0\} \times [0,1]$ to a point.

The suspension SX can be seen as the following pushout

$$\begin{tikzcd} X \times \{0,1\} \arrow[r, hook] \arrow[d, two heads] & X \times [0,1] \arrow[d] \\ \{0,1\} \arrow[r] & SX \end{tikzcd}$$

where the left arrow is the projection onto the second coordinate.

It will be made clear a bit later, but the reason to define it this way lies behind the currying relation of functions. If we were to curry a map $X \times I \to Y$ (i.e., currying a homotopy), then we get a map $X \to C(I, Y)$. If we would like the range to instead be all loops, then we need to start with a map from SX and if we would like the curried map to be a map between pointed spaces as well, then we need to start with a map from ΣX. We shall come to the functoriality of reduced suspension later.

Theorem 3.1. *For an exponentiable space $X, \Sigma X \cong X \wedge S^1$.*

Proof. In $X \times [0,1]$ we are quotienting out the subspace $X \times \{0,1\} \cup \{x_0\} \times [0,1]$. We do this in steps (and this is allowed for any X) by first quotienting the relation $(x,t) \sim (x',t')$ if $x = x', t, t' \in \{0,1\}$ and then quotient out the remaining which would be $X \vee S^1$: The first quotient, when X is exponentiable, is $X \times S^1$. In this quotient $\{x_0\} \times [0,1]$ becomes S^1 and $X \times \{0,1\}$ becoms X and these two subspaces have one point in common, x_0, therefore, this is, as a set, $X \vee S^1$. Because we have the product topology on $X \times S^1$, this subset is the wedge as a topological space as well, hence we get $\Sigma X = X \times S^1 / X \vee S^1 = X \wedge S^1$ as required. $\square$

Lemma 3.1. $SS^{n-1} \cong S^n$.

Proof. Recall that a continuous bijection $X \to Y$ with X compact and Y Hausdorff is a homeomorphism. Start with S^n centred at the origin with the north (N) and south (S) poles removed. Then we have a projection onto the infinite cylinder $S^{n-1} \times \mathbb{R}$ using radial lines and this is easily seen to be a homeomorphism. Since $\mathbb{R} \cong (0,1)$ we have a homeomorphism $f \colon S^{n-1} \times (0,1) \to S^n \setminus \{N, S\}$.

Extend this to $f \colon S^{n-1} \times [0,1] \to S^n$ by sending $(x,1) \mapsto N, (x,0) \mapsto S$. This is a continuous map because the inverse of neighbourhoods of N is a neighbourhood of $S^{n-1} \times \{1\}$ etc. and factors through the quotient to get a continuous bijection $SS^{n-1} \to S^n$ from a compact space to a Hausdorff space. $\square$

Lemma 3.2. $\Sigma S^n = SS^n$ *for* $n \geq 1$.

Proof. The stereographic projection $S^n \setminus \{N\} \to \mathbb{R}^n$ is a homeomorphism. Again, this gives us a homeomorphism $(0,1)^n \to S^n \setminus \{N\}$ which we then extend to I^n by sending the boundary to N. The resulting map is continuous and results in a continuous bijection $I^n/\partial I^n \to S^n$ which is also a homeomorphism.

Instead of collapsing the boundary at once, we do it in steps. We are collapsing $\partial I^{n-1} \times I \cup I^{n-1} \times \partial I$ to a point, but first we quotient out the relation $(x,s) \sim (y,t)$ if $x, y \in \partial I^{n-1}, s = t \in I$. Then, because I is well behaved (it is in fact one of the most well behaved spaces and other spaces would be better off learning a thing or two from I), we can write this quotient as $I^{n-1}/\partial I^{n-1} \times I \cong S^{n-1} \times I$. Let x_0 denote the point that ∂I^{n-1} collapses to in S^{n-1}.

The second quotient involves collapsing $S^{n-1} \times \{0,1\} \cup \{x_0\} \times I$ which gives us the reduced suspension of S^{n-1}. Therefore, $\Sigma S^{n-1} = S^n = SS^{n-1}$ as required. $\square$

4 Homotopy extensions and liftings

Consider a map $X \xrightarrow{f} Y$. Given a space Z, by functoriality, we have maps $C(Y,Z) \to C(X,Z)$ and $C(Z,X) \to C(Z,Y)$. Injectivity of these two maps depend on f being surjective/injective respectively. Surjectivity depends on being able to extend/lift the maps and we can't say much about this without knowing the maps. However, suppose we can extend/lift one such map, then would we be able to extend/lift any other map homotopic to the one that we started with?

The idea is that we are interested in invariants up to homotopy and we would like to get invariants for X from Y and vice versa and we are interested in homotopy classes of maps. Although we shall not really need it, we denote by hTop the category of homotopy classes of topological spaces with morphisms being the homotopy classes of maps between any chosen representatives of the spaces. The morphisms will be denoted by $[X,Y]$.

Definition 4.1. *A map* $X \xrightarrow{f} Y$ *is said to have the* ***homotopy extension property*** *with respect to* Z *if given maps* $g\colon X \to Z, \tilde{g}\colon Y \to Z, G\colon X \times I \to Z$ *such that* $\tilde{g} \circ f = g$ *and* $G|_{X \times \{0\}} = g$ *there is a homotopy* $\tilde{G}$ *of* $\tilde{g}$ *that extends* G *to* Y.

It is said to have the ***homotopy lifting property*** *with respect to* Z *if given maps* $g\colon Z \to Y, \tilde{g}\colon Z \to X, G\colon Z \times I \to Y$ *such that* $f \circ \tilde{g} = g$ *and* $G|_{Z \times \{0\}} = g$ *there is a homotopy* $\tilde{G}$ *of* $\tilde{g}$ *that lifts* G *to* X.

If f *has the homotopy lifting (extension) property for all spaces, then it is called a* ***(co)fibration****. If* f *has the homotopy lifting property for all CW complexes* Z*, then it is called a* ***Serre fibration****.*

Covering spaces form an important class of fibrations and for more on this refer to any of the books listed in the references. It is clear that the homotopy lifting and homotopy extension are dual notions and are nice properties to have. In the subsequent sections we see how to convert maps into (co)fibrations.

Observe that using universal properties, the pushout of a cofibration and the pullback of a fibration are cofibrations and fibrations respectively, i.e., if in our pullback (pushout) square, one of the starting arrows is a (co)fibration, then the parellel arrow is also a (co)fibration. Moreover, the fibre (inverse image of the basepoint) and the cofibre (quotient by image) are homeomorphic (this is an exercise for the reader; use the universal properties to obtain maps in one of the directions, the other direction follows from the definition of (co)fibre). Keep in mind that the pushout square behaves nicely under product with I.

It is also clear that the composition of fibrations and cofibrations are fibrations and cofibrations.

Lemma 4.1. *Let* $f\colon A \to B$ *be a cofibration and* Z *be an exponentiable space. Then* $f \times id\colon A \times Z \to B \times Z$ *is a cofibration.*

Proof. Let a homotopy $G\colon A \times Z \times I \to W$ be given, this is the same as a map $G_1\colon A \times I \to C(Z,W)$. Moreover, let there be an extension g of $G|_{A \times Z \times \{0\}}$ to $B \times Z \times 0$ which corresponds to $g_1\colon B \to C(Z,W)$. Observe that g_1 is an extension of $G_1|_{A \times \{0\}}$. So we can extend the homotopy G_1 to $B \times I$ and currying gives us an extension of G to $B \times Z \times I$. Note that the cofibre is $B/A \times Z$. $\square$

Lemma 4.2. *With A, B exponentiable, let $f\colon A \to B$ is a cofibration. For any Y, $f^*\colon C(B,Y) \to C(A,Y)$ is a fibration with respect to all Z which are exponentiable.*

Proof. Let $G\colon Z \times I \to C(A,Y)$ and $g\colon Z \to C(B,Y)$ be given such that $f^*g = G|_{Z\times\{0\}}$. We want to lift G to $\tilde{G}\colon Z \times I \to C(B,Y)$, i.e., a map $Z \times I \times B \to Y$ extending the curring $Z \times I \times A \to Y$. Because $Z \times A \to Z \times B$ is a cofibration, such an extension is possible and currying gives us the required lifts. Observe that the fibre is, as a set, all those functions which are constant on $f(A)$. From the quotient map, there's an injective map $C(B/f(A), Y) \to C(B,Y)$ and the image is the fibre we're interested in.

The evaluation map $C(B/f(A), Y) \times B \to Y$ factors through the evaluation map for $B/f(A)$, hence would be continuous if $B/f(A)$ is exponentiable. But it's not clear whether this map is a homeomorphism onto its image.

However, in the event that the quotient $B \to B/f(A)$ is a proper map (i.e., inverse of compact sets is compact) and we were looking at the compact-open topology everywhere, then we can say that we have a homeomorphism onto the image, i.e., $C(B/f(A), Y)$ is a subspace of $C(B,Y)$. $\square$

An almost similar argument shows that if $f\colon A \to B$ is a fibration, then, for an exponentiable Y, $C(Y,A) \to C(Y,B)$ is a fibration with fibre $C(Y, f^{-1}(b_0))$. This time the fibre is indeed exactly $C(Y, f^{-1}(b_0))$ because it can be checked that the inclusion $C(Y, f^{-1}(b_0)) \to C(Y,A)$ is a subspace (it is a bijection to the fibre; continuity is seen by currying; homeomorphic to the image because when Y is exponentiable, the Isbell topology is the exponential topology).

A sequence $F \to E \to X$ will be called a fibre sequence if $E \to X$ is a fibration with fibre F. Dually, a sqeuence $A \to B \to C$ will be called a cofibre sequence if $A \to B$ is a cofibration with cofibre C.

4.1 Equivalence of fibres

If $p\colon E \to X$ is a fibration, then the fibre is $E_0 = p^{-1}(x_0)$ for some $x_0 \in X$. Let $x_1 \in X$ be another point with $E_1 = p^{-1}(x_1)$ and $\gamma\colon I \to X$ a path from x_0 to x_1. Consider

$$\begin{tikzcd}
E_0 \times \{0\} \arrow[r, hook] \arrow[d, hook] & E \arrow[d, "p"] \\
E_0 \times I \arrow[r, "\gamma\circ\pi_2"'] & X
\end{tikzcd}$$

where π_2 is the projection onto I. By homotopy extension, we have a (not necessarily unique) continuous map $\psi_\gamma\colon E_0 \times I \to E$. Restricting this to $E_0 = E_0 \times \{1\}$, we get a map $\phi_\gamma\colon E_0 \to E_1$.

Suppose γ_0, γ_1 are any two paths from x_0 to x_1. Let $\psi_0\colon E_0 \times I \to E, \psi_1\colon E_1 \times I \to E$ denote the lifts of $\gamma_0, \bar{\gamma_1}$ where $\bar{\gamma_1}$ is the reverse of γ_1. By pasting lemma, there is a continuous map

$$\begin{aligned}
\psi\colon E_0 \times I &\to E \\
(x,t) &\mapsto \begin{cases} \psi_0(x, 2t) & 0 \le t \le 1/2 \\ \psi_1(\psi_0(x,1), 2t-1) & 1/2 \le t \le 1 \end{cases}
\end{aligned}$$

Projecting this to B gives us the path product $\gamma_0\bar{\gamma_1}$ (γ_0 first, then $\bar{\gamma_1}$), a loop at x_0. Suppose this loop is null homotopic (eg. when γ_0, γ_1 are homotopic paths) through a homotopy $H\colon I \times I \to B$. We can lift this homotopy to a map

$$\psi_H\colon E_0 \times I \times I \to E$$

which is ψ on $E_0 \times I \times \{0\}$.

Restrict ψ_H to the other three edges (which project to x_0) to get a map $E_0 \times I \to E_0$ which is identity at 0 and $\psi(\cdot, 1)$ at 1. This shows that the fibres E_0, E_1 are homotopic. It doesn't prove, however, that any two lifts of a given path γ are homotopic (because the homotopy equivalence proceeds by lifting the reverse path in B).

If we perform a "path product" of lifts as above, then it descends to the usual path product in B. So, fibres over the path components of B are homotopy equivalent.

In particular, when p has the unique path lifting property, there is an action (now that there's a single choice of a lift of loops) of $\pi_1(X, x_0)$ on E_0. This action is useful in the classification of covering spaces, see [1]. While the fibre depends on the base point, it is determined up to homotopy equivalence, whereas the cofibre of a cofibration doesn't really depend on the base point; this is because of the difference between $C(*, X)$ and $C(X, *)$.

4.2 Mapping cylinders and cones

A homotopy of f is a map $X \times I \xrightarrow{H} Y$ from the cylinder over X to Y, which when restricted to $X \times \{0\}$ is f. If H is a null homotopy, then it factors through the cone $C(X) = X \times I / X \times \{1\}$ to Y. We can consider following pushouts

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[d] \\ X \times [0,1] \arrow[r] & Cyl(f) \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[d] \\ C(X) \arrow[r] & C(f) \end{tikzcd}$$

called the mapping cylinder and mapping cones.

The cylinder $Cyl(X)$ is just $X \times [0, 1]$, so homotopies of maps out of X are maps out of $Cyl(X)$ and the cone as defined above. It is clear that Cyl, C are functors on Top. It is also easy to see that these define functors on hTop as well (exercise for the reader; keep in mind that I is nice[1]).

In the context of the pointed category, we would like to consider instead the reduced versions $Cyl_r(X), C_r(X)$ where we also quotient out the interval $\{x_0\} \times I$. The corresponding pushouts will be denoted $Cyl_r(f), C_r(f)$. This notation is not standard.

Lemma 4.3. *The inclusion of a subspace $i\colon A \hookrightarrow X$ is a cofibration if and only if $X \times \{0\} \cup A \times I = Cyl(i)$ is a retract of $X \times I$. Moreover, in this case, A is closed when X is Hausdorff.*

Proof. Assume it is a cofibration and consider the inclusion maps $X, A, A \times I \hookrightarrow Cyl(i)$, then we have an extension $H\colon X \times I \to Cyl(i)$, a retraction. Conversely, composition with the retraction gives us the homotopy extension property: the data given prior to extending a homotopy is a map from the mapping cylinder $Cyl(i)$ to a space Z.

The image of the retract $r\colon X \times I \to Cyl(i) \subseteq X \times I$ is the subset of all x such that $r(x) = x$. When X is Hausdorff, this subset is closed because r is continuous. So, $Cyl(i)$ is closed in $X \times I$, hence A is closed (by intersection with the "top" slice). □

Let $j\colon Y \to Cyl(f)$ and $i\colon x \mapsto [(x, 1)]$ be inclusions and $r\colon Cyl(f) \to Y$ be the retraction obtained by identity on Y and f (after projection) on $X \times I$. The maps j, r, i make sense for $Cyl_r(f)$ as well. The cylinder part in $Cyl(f)$ allows us to tack on homotopies on X by stretching the cylinder. We have

Theorem 4.1. *$Y, Cyl(f)$ are homotopy equivalent via j, r and i is a cofibration.*

Proof. We have $rj = id_Y$, so to prove the homotopy equivalence, we need to show jr is homotopic to $id_{Cyl(f)}$. Consider

$$\begin{aligned} H\colon & Cyl(f) \times I \to Cyl(f) \\ & [y], s \mapsto [y] \\ & [(x, t)], s \mapsto [(x, (1 - s)t)] \end{aligned}$$

this is continuous (by transfering the quotient to the whole product $Y \times I \sqcup X \times I \times I$) and a homotopy between identity and jr as required (basically it squeezes the cylinder down).

[1]Here I refers to the object $[0, 1]$ and not the pronoun for self reference; so "I is" is grammatically correct.

Next, suppose we have the data $g\colon Cyl(f) \to Z, G\colon X \times I \to Z$ such that $gi = G|_{X\times\{0\}}$. Consider the function

$$\begin{aligned} H\colon & Cyl(f) \times I \to Z \\ & [y], s \mapsto g([y]) \\ & [(x,t)], s \mapsto \begin{cases} g([(x, t(1+s)]) & t(1+s) \leq 1 \\ G(x, t(1+s)-1) & t(1+s) \geq 1 \end{cases} \end{aligned}$$

This map is continuous and extends G as required. This map stretches the cylinder along the homotopy G. □

The cofibre of i is the mapping cone of f because we can change the order in which we quotient subspaces. Note that the theorem shows that Y is a strong deformation retract of the mapping cylinder.

The exact same proof works for the reduced version, with cofibre being the reduced mapping cone. Note that in the pointed category, G should map $\{x_0\} \times I$ to the basepoint of Z.

For later use, we note that when $Y = *, Cyl(f), Cyl_r(f)$ are the unreduced and reduced cones of X respectively.

Theorem 4.2. *The mapping cylinders and cones of homotopic maps are homotopy equivalent.*

Proof. Let $f, g\colon X \to Y$ be homotopic via a homotopy $H\colon X \times I \to Y$. Define

$$\begin{aligned} H_1\colon & Cyl(f) \to Cyl(g) \\ & [y] \mapsto [y] \\ & [(x,t)] \mapsto \begin{cases} [H(x, 2t)] & 0 \leq t \leq 1/2 \\ [(x, 2t-1)] & 1/2 \leq t \leq 1 \end{cases} \end{aligned}$$

This is of course well defined. The continuity is as follows: we have a continuous $Y \to Cyl(g)$; we break $X \times I$ as a quotient of $X \times [0, 1/2] \sqcup X \times [1/2, 1]$. We define a map from the first part to Y and from the second to $X \times I$ and then pass to the pushout $Cyl(g)$ giving us a map $X \times [0, 1/2] \sqcup X \times [1/2, 1]$. We can then pass to the quotient identifying the two $1/2$ sections, to get a continuous map $X \times I \to Cyl(g)$. Finally, using the properties of pushouts, we get $H_1\colon Cyl(f) \to Cyl(g)$. Using the reverse of H, we get a map $H_2\colon Cyl(g) \to Cyl(f)$.

Finally define

$$\begin{aligned} G\colon & Cyl(f) \times I \to Cyl(f) \\ & [y], s \mapsto [y] \\ & [(x,t)], s \mapsto \begin{cases} [H(x, 2t)] & 0 \leq t \leq s/2 \\ [H(x, 3s-4t)] & s/2 \leq t \leq 3s/4 \\ [(x, \frac{4t-3s}{4-3s})] & 3s/4 \leq t \leq 1 \end{cases} \end{aligned}$$

Continuity is again verified by breaking it into pieces and using gluing lemma and properties of pushouts. Keeping in mind that pushout and products are well behaved in this case because I is nice. It can also be verified that this is a homotopy beetween $H_2 \circ H_1$ and $id_{Cyl(f)}$. By symmetry, we have the other homotopy, giving $Cyl(f) \simeq Cyl(g)$.

The maps above work for the cones as well (there is nothing new to do, we can freely quotient things out because I is nice enough). It works because H_1, H_2 don't touch the cone point. Therefore, $C(f) \simeq C(g)$.

Intuitively H_1, H_2 are folding the cylinder along the paths determined by H and G folds shorter bits of the cylinder. □

Again, the proof above holds for the reduced versions as well.

4.3 Path space and loop space

A homotopy to $X \times I \to Y$ can be seen as a map $X \to C(I, Y)$ (we could curry X to the other side, when X is nice, but $C(X, Y)$ is a much bigger space to understand when we are only looking at paths in this space). In the pointed category, denote by $P(Y)$ the mapping space $C((I, 0), (Y, y_0))$ with the compact-open topology (which is the same as the subspace topology from $C(I, Y)$).

A homotopy of a map to itself is a map $X \times S^1 \to Y$ (to make this precise, we would need X to be exponentiable so as to quotient I separately), or a map $X \to C(S^1, Y)$. In the pointed category, we take $\Omega Y = C((S^1, 1), (Y, y_0))$. This can be seen as a subspace of $P(Y)$ consisting of all those paths that end at y_0 or as a space with the compact-open topology and these two notions are equivalent (the equivalence is as follows: from the quotient $I \to S^1$ we get the map $\Omega Y \to PY$. This map is clearly continuous, injective and in the compact-open topology it is homeomorphic to its image because the inverse of compact sets under $I \to S^1$ is again compact). It is easy to verify that P, Ω are functors on Top_* (to see continuity of $Pf, \Omega f$ for some $X \to fY$, look at its adjoint).

Observe that the path (loop) space of a wedge sum is not the wedge sum of path (loop) spaces because we can combine loops and pass through the base point multiple times. If we pass from the loop space to the fundamental group by identifying maps up to homotopy, then the fundamental group of a wedge sum is going to be a free product of the two groups, this is the van Kampen theorem. From the result on morphisms, it is clear that path space and loop space functors distribute over pullbacks

Consider the pullback

$$\begin{tikzcd} E_f \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ C(I, Y) \arrow[r, "ev_0"] & Y \end{tikzcd} \qquad \begin{tikzcd} \Omega_f \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ C(S^1, Y) \arrow[r, "ev_1"] & Y \end{tikzcd}$$

Note that evaluation at a particular point is continuous because the $C(I, -)$ etc. is functorial once we use the exponential topologies.

We also have maps $i \colon X \to E_f, \Omega_f$ which takes $x \mapsto (x, f_x)$ where f_x is the constant loop in Y at $f(x)$. Note that $x \mapsto f_x \in C(I, Y), C(S^1, Y)$ is continuous because it's adjoint is continuous.

Theorem 4.3. *Let $p \colon E_f \to X$ denote the projection onto the first coordinate and $e \colon E_f \to Y$ denote the end point evaluation following projection. Then e is a fibration with respect to all exponentiable spaces, $ei = f$ and $X \simeq E_f$.*

Proof. It is clear that all these maps are continuous and $ei = f, pi = id_X$. To show the homotopy equivalence, consider

$$\begin{aligned} H \colon & E_f \times I \to E_f \\ & (x, l), s \mapsto (x, \{t \mapsto l((1-s)t)\}) \end{aligned}$$

This map exists and is continuous by the universal property of pullbacks: the map to X is just projection. The map $E_f \times I \to C(I, Y)$ has adjoint $E_f \times I \times I \to Y$ and is continuous because $(s, t) \mapsto t(1 - s)$ is continuous and so is evaluation. H shows that $ip \simeq id_{E_f}$ as required.

Finally, we need to show that $e \colon E_f \to Y$ is a fibration. A map $g \colon Z \to E_f$ lifting some $f \colon Z \to Y$ is essentially two maps: a map $Z \to C(I, Y)$ whose end points are given by a map $Z \to X \times Y$. So, we look at the following diagram

$$\begin{tikzcd} E_f \arrow[r] \arrow[d, "p \times e_1"'] & C(I, Y) \arrow[d, "e_0 \times e_1"] \\ X \times Y \arrow[r, "f \times id"'] \arrow[d] & Y \times Y \\ Y & \end{tikzcd}$$

where e_0 is the evaluation at 0 etc.

- In this square, E_f is the pullback because the square is two pullback squares stacked together (what happens when you project from $X \times Y, Y \times Y$ to their respective first coordinates?)
- The projection $X \times Y \to Y$ is a fibration and because composition of fibrations is a fibration, it suffices to show that $E_f \to X \times Y$ is a fibration.
- Because pullbacks of fibrations is a fibration, it suffices to show that $C(I, Y) \to Y \times Y$ is a fibration.

Now, $\{0,1\} \hookrightarrow I$ is a cofibration because because of a projection from $I \times I$ to $I \times \{0\} \cup \{0,1\} \times I$, say the radial projection from $(1/2, 2)$ (seeing $I \times I$ as a subset of the plane) (for more details, see any of the books mentioned in the references, I suggest [1]). Since $Y \times Y = C(\{0,1\}, Y)$. From a lemma above, because $\{0,1\}$ and I are exponentiable, we conclude that $C(I, Y) \to Y \times Y$ is a fibration with respect to all exponentiable spaces. $\square$

For later use, we note that if $X = *$, then $E_f = P(Y)$, with fibre $\Omega(Y)$.

When Y is path connected, there is, up to homotopy equivalence, a fibre of e given by $F_f = \{(x, l) : l(0) = f(x), l(1) = y_0\}$ for some fixed $y_0 \in Y$. F_f is called the homotopy fibre of f.

Theorem 4.4. *If $f, g\colon X \to Y$ are homotopic via a map H, then $E_f \simeq E_g, \Omega_f \simeq \Omega_g$.*

Proof. Define

$$\begin{aligned} H_1 \colon E_f &\to E_g \\ (x, l) &\mapsto (x, \{t \mapsto \begin{cases} H(x, 1-2t) & 0 \le t \le 1/2 \\ l(2t-1) & 1/2 \le t \le 1 \end{cases} \}) \end{aligned}$$

Why is this continuous? The map to X is just projection and the map to $C(I, Y)$ is continuous because it is built up of two maps from $E \times [0, 1/2] \sqcup E_f \times [1/2, 1]$ both of which are continuous, hence passes to the quotient. So, the existence of H_1 follows from the properties of quotients and pullbacks. Using the reverse of H, define $H_2 \colon E_g \to E_f$. That these maps are homotopy inverses of each other is verified using a map of the type

$$\begin{aligned} G\colon E_f \times I &\to E_f \\ x, l, s &\mapsto (x, \{t \mapsto \begin{cases} H(x, 2t) & 0 \le t \le s/2 \\ H(x, 3s - 4t) & s/2 \le t \le 3s/4 \\ l(x, \frac{4t-3s}{4-3s}) & 3s/4 \le t \le 1 \end{cases} \}) \end{aligned}$$

It's continuity is again a consequence of a bunch of quotients and pullbacks. We can define a similar map on E_g to conclude that $E_f \simeq E_g$.

A similar argument works for Ω_f, Ω_g, the only difference being that we need to concatenate H on both sides of l so as to close the loops. But the argument is essentially the same with continuity being a consequence of the properties of quotients and pullbacks. $\square$

5 Loop space and suspension functors

We have seen what happens to (co)fibre sequences when acted on by functors$C(X, -)$ and $C(-, X)$. We now look at two important functors: the loop space functor $\Omega X = C((S^1, 1), (X, x_0))$ and the reduced suspension functor ΣX acting on Top_*, the category of pointed spaces. If $X \simeq Y$, then it is easy to see that $\Sigma X \simeq \Sigma Y$ using suspension of maps (and transfering quotients works because I is nice). If $H \colon X \times I \to X$ is some map (mapping the interval over the basepoint to the basepoint), then we have

$$\begin{aligned} \tilde{H} \colon \Omega X \times I &\to \Omega X \\ (\gamma, t) &\mapsto \{s \mapsto H(\gamma(s), t)\} \end{aligned}$$

which is continuous because it's adjoint $\Omega X \times I \times S^1 \to X$ is continuous (it's a composition of a bunch of evaluations). This gives us $\Omega X \simeq \Omega Y$.

Theorem 5.1. *(Loop space - suspension adjunction) These two functors are adjoints of each other.*

Proof. Given spaces $(X, x_0), (Y, y_0)$, curring a map $f\colon \Sigma X \to Y$ gives a map $f^{\#}\colon X \to \Omega Y$. A priori we have a map $X \to C(I, Y)$, but because $X \times \{0,1\} \cup \{x_0\} \times I$ goes to y_0, we get a based map $X \to \Omega Y$.

Taking the adjoint of $g\colon X \to \Omega Y$ and using the universal property of quotients, we get a continuous map $g^{\flat}\colon \Sigma X \to Y$. It is clear that these operations are inverses of each other.

Importantly, note that the path product in ΩY translates to a wedge sum on ΣX when X is nice. Here wedge sum refers to the following: given $f, g\colon \Sigma X \to Y$, we get

$$X \wedge S^1 \to X \wedge (S^1 \vee S^1) = \Sigma X \vee \Sigma X \xrightarrow{f \vee g} Y$$

where $S^1 \to S^1 \vee S^1$ is collapsing the equator.

Lastly, the isomorphism $C(\Sigma X, Y) \cong C(X, \Omega Y)$ above is natural in X, Y in the sense that if we have any map $X \to Z$ or $Y \to Z$ then all diagrams involved commute because the isomorphism is quite simple. The same isomorphism works in $\mathsf{Top}_*, \mathsf{hTop}$. Homotopic maps remain homotopic after currying as well and things work out nicely because of the universal properties of quotients and exponentiability of I. $\square$

From the section on morphisms and pullbacks, it's clear that $\Omega(X \times_Z Y) = \Omega X \times_{\Omega Z} \Omega Y$. Note that the maps in our proof earlier work in the pointed category as well.

Theorem 5.2. *(Reduced suspension and pushouts)* $\Sigma(A \sqcup_B C) = \Sigma A \sqcup_{\Sigma B} \Sigma C$.

Proof. Σ is a covariant functor, so applying it to the pushout square and using the universal property, we get a map $D \to \Sigma(A \sqcup_B C)$ where $D = \Sigma A \sqcup_{\Sigma B} \Sigma C$. For a map in the other direction, consider the diagram

$$\begin{tikzcd}
B \times I \arrow[r] \arrow[d] & A \times I \arrow[d] \arrow[ddr, bend left] & \\
C \times I \arrow[r] \arrow[drr, bend right] & (A \sqcup_B C) \times I \arrow[dr, dashed] & \\
& & \Sigma A \sqcup_{\Sigma B} \Sigma C
\end{tikzcd}$$

where we use the fact that product commutes with pushouts and the various quotient maps while taking suspension. From the universal property, the dotted arrow exists and it's easy to see that the dotted arrow induces a continuous map $\Sigma(A \sqcup_B C) \to D$. It's also easy to see that these arrows are inverses of each other, thus completing the proof. $\square$

As a corollary, if $B \subset A$ then $\Sigma A / \Sigma B = \Sigma(A/B)$. Similarly, if $C(f)$ is the cone of a map $X \xrightarrow{f} Y$, then $C(\Sigma f) = \Sigma(C(f))$ [2].

By currying, if $F \to E \to X$ is a fibre sequence, then so is $\Omega F \to \Omega E \to \Omega X$.

Dually, let $A \to B \to C$ be a cofibre sequence in the category of pointed spaces (meaing the homotopy extensions send the interval over the basepoint to basepoint) and supppose we have maps $G\colon \Sigma A \times I \to Z, g\colon \Sigma B \to Z$ such that $G|_{\Sigma A \times \{0\}} = g|_{\Sigma A}$. Because I is locally compact Hausdorff, we can write $\Sigma A \times I$ as a quotient of $A \times I_s \times I$ where I_s is $[0,1]$ but as the suspension coordinate, so as to not confuse with the other I. Similarly, ΣB is a quotient of $B \times I_s$. We curry the I_s out to get maps $A \times I \to C(I_s, Z)$ and $B \to C(I_s, Z)$.

$$A \times I_s \times I \to \Sigma A \times I \xrightarrow{G} Z \rightsquigarrow A \times I \to C(I_s, Z)$$
$$B \times I_s \xrightarrow{g} Z \rightsquigarrow B \to C(I_s, Z)$$

[2]Once we have proved that loop space and suspension are adjoints, knowing that Ω commutes with pullbacks is enough to deduce that its adjoint Σ commutes with pushouts. This is the famous **adjoint functor theorem** ([4]).

It's easy to see that these maps satisfy the hypothesis for a homotopy extension, therefore we get a map $B \times I \to C(I_s, Z)$. Taking the adjoint of this, we have a map $B \times I_s \times I \to Z$. However, this does not give use what we want. Instead we notice the following, the image inside $C(I_s, Z)$ in the above maps actually lands in the subspace $C(S^1, Z) = \Omega Z$. And when the image lands in a subspace, we can take the codomain to be the said subspace, therefore we get maps

$$A \times I, B \to \Omega Z$$

satisfying the hypothesis for homotopy extension[3]. Keep in mind that a homotopy in the pointed category passes through maps which send base points to base points.

Finally we get a map $B \times I \to C(S^1, Z) \subset C(I_s, Z)$. The adjoint of this is an extension of the map $A \times I_s \times I \to Z$ and allows us to pass to the quotient and get a map $\Sigma B \times I \to Z$ extending G precisely because the image of $B \times I$ lands in $C(S^1, Z)$. Therefore, Σ preserves cofibre sequences.

6 Barratt-Puppe sequence

Here is what we have so far: given a map $X \xrightarrow{f} Y$, we can construct $Cyl(f), C(f), E_f, \Omega_f$ depending only on the homotopy class of f such that $X \simeq E_f, Y \simeq Cyl(f)$ and there is a fibration $E_f \to Y$, cofibration $X \to Cyl(f)$. Let us first look at fibrations:

$$\begin{tikzcd} & X \arrow[r, "f"] \arrow[d, "i"'] & Y \\ F_f \arrow[ur, "p"] \arrow[r, hook] & E_f \arrow[ur, "e_1"'] & \end{tikzcd}$$

We have the following diagram with $e_1 i = f$ where $i(x) = (x, f_x)$ with f_x being the constant loop at $f(x)$. An element $(x, l) \in F_f$ is such that l starts at $f(x)$ and ends at y_0, the basepoint of Y.

Temporarily denote $\bar{P}(Y) = C((I, \{1\}), (Y, \{y_0\}))$, the space of all paths ending at y_0. The evaluation at $0, \bar{P}(Y) \to Y$ is a fibration for the same reasons $P(Y) \xrightarrow{e_1} Y$ was a fibration, and the fibre is $\Omega(Y)$. It is now easy to see that F_f is the pullback of e_0 along f (the topology works out because F_f is a subspace of $X \times C(I, Y)$ etc. and the map $F_f \to X$ is a fibration because e_0 is.

Using the representation of F_f as above it is possible to directly show that $F_f \to X$ is a fibration (and we leave this as an exercise; but the proof is essentially something that we have done above with concatenating paths), but this method doesn't lend itself to be iterated easily.

So, when we convert $X \xrightarrow{f} Y$ to a fibration, the fibre sits as the following pullback

$$\begin{tikzcd} F_f \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \bar{P}(Y) \arrow[r, "e_0"'] & Y \end{tikzcd}$$

From here, the fibre is homeomorphic to $\Omega(Y)$ via the inclusion $\Omega(Y) \ni l \mapsto (x_0, l) \in F_f$.

Now we figure out the fibre we obtain when we convert $\Omega Y \to F_f$ into a fibration. This fibre sits here

$$\begin{tikzcd} & \Omega X \arrow[r, "\Omega f"] \arrow[d, "c"'] & \Omega Y \arrow[d, "i"] \\ & \bar{P}F_f \arrow[r, "e_0"] \arrow[d] & F_f \arrow[d] \\ \Omega X \arrow[ur, "c"] \arrow[r] & \bar{P}X \arrow[r] & X \end{tikzcd}$$

[3]We have taken the reduced suspension ΣX as a quotient of $X \times I$. We could have defined it as $X \wedge S^1$, in which case, we would direcly get $C(S^1, Z)$. The two definitions agree when X is well behaved and our proof above required us to interchange the operation of product and quotients. It is not clear to me whether they agree for any topological space.

where temporarily, i is the inclusion (which is injective!), and c is the map defined by

$$\omega \mapsto \{s \mapsto (w(s), \{t \mapsto f(\omega(s + (1-s)t))\})\}$$

Continuity of c is verified using properties of pullbacks (and the fact that $C(I, Y)$ has exponential topology).

If we had maps $Z \xrightarrow{h_1} \bar{P}F_f, Z \xrightarrow{h_2} \Omega Y$ such that $ih_2 = e_0 h_1$, then following h_1 with the map $\bar{P}F_f \to \bar{P}X$ actually lands in ΩX giving us a map $Z \to \Omega X$ which makes everything commutative (this requires that i is injective) and because there is a commutative triangle at the bottom, the map $Z \to \Omega X$ is actually uniquely determined, making ΩX the pullback.

So, if we convert $\Omega Y \to F_f$ into a fibration, the fibre is homotopic to ΩX and the map to ΩY is precisely Ωf.

The loop space functor preserves fibre sequences and because the map is Ωf, we can continue this process to get the Puppe sequence of fibrations:

$$\cdots \to \Omega F_f \to \Omega X \xrightarrow{\Omega f} \Omega Y \to F_f \to X \xrightarrow{f} Y$$

such that the maps beyond ΩY are obtained by applying Ω and any two consecutive maps form, up to homotopy equivalence of the spaces, a fibre sequence (said differently, it is an exact sequence of spaces with the additional property of being a fibration).

We repeat the construction for cofibre sequences. Starting with $X \xrightarrow{f} Y$ we have seen that $Cyl_r(f) \simeq Y, X \to Cyl_r(f)$ is a cofibration with cofibre $C_r(f)$. Note that although we have proved it for the unreduced versions, all the proofs factor through the appropriate quotients to get the results for the reduced versions. Since $X \to C_r(X)$ is a cofibration, the pushout $Y \to C_r(f)$ is also a cofibration with cofibre ΣX (we are using the fact that we can quotient in steps). What this tells us is that when we convert $X \xrightarrow{f} Y$ into a cofibration, the cofibre (the space and the quotient map) sits as a pushout

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "i"'] & Y \arrow[d] \\ C_r X \arrow[r] & C_r f \end{tikzcd}$$

where i is the inclusion into the zero slice.

The hope is that the next step is ΣY. There is already a map $\Sigma X \xrightarrow{\Sigma f} \Sigma Y$. To get the map from the reduced cylinder of $C_r(f)$, consider

$$\begin{aligned} F \colon & (X \times [0,1] \sqcup Y) \times [0,1] \to Y \times [0,1] \\ & x, s, t \mapsto f(x), \max(s,t) \\ & y, t \mapsto y, t \end{aligned}$$

It is clear that F is continuous. Following this with the quotient to ΣY gives a map $C_r(C_r f) \to \Sigma Y$ such that

$$\begin{tikzcd} C_r f \arrow[r] \arrow[d, "/Y"'] & C_r C_r f \arrow[d, "F"] \\ \Sigma X \arrow[r, "\Sigma f"'] & \Sigma Y \end{tikzcd}$$

commutes, where the map on the left is quotient by Y.

Next, we have a map $Y \times [0,1] \to C_r C_r f$ which actually factors through the reduced cone of Y. Given maps $\Sigma X, C_r C_r f \to Z$ fitting in the pushout diagram, lends us a map $\Sigma Y \to Z$ because the $Y \times \{0\}$ bit lands in $C_r f$ and is collapsed to the basepoint in ΣX. It is also clear that this resulting map fits properly into the pushout square.

Because reduced suspension preserves cofibre sequences and the map above is Σf we get the Puppe sequence of cofibrations:

$$X \xrightarrow{f} Y \to C_r(f) \to \Sigma X \xrightarrow{\Sigma f} \Sigma Y \to \dots$$

where the later maps are obtained by acting Σ and consecutive maps form, up to homotopy equivalence of spaces, a cofibre sequence.

7 A note on k-spaces

There were several instances above where the argument relied on some space being exponentiable in the Top category. Moreover, some operations such as quotients or products might turn our "nice" space into something that is not nice (one such common situation is when the quotient of a Hausdorff space is not Hausdorff). It is therefore important that we work in a suitable category. Core compact spaces is one such category, another is the category of locally compact Hausdorff spaces (which are all core compact).

However, when we change our category, then the exponentiable objects change as well, i.e., a space Z need not be exponentiable in Top, but it could be exponentiable in the category of locally compact Hausdorff spaces for example. How does this affect the results above? The advantage of using universal properties is that they don't rely on the specific structure (eg. compact-open topology). So most proofs should carry forward as long as the notions make sense and the maps are morphisms in our category of choice. There is also a small chicken-egg problem in that we could define certain constructs (quotients, pullbacks etc) just the right way so that the proofs work; but it's not really an issue. What the exercise above ends up showing is that Top is an *example* of a category where such and such nice things happen: top is a category where a, b, and c hold and therefore we can construct the Puppe sequence.

So what one should do is to define a *Model category* where there are notions of the things we are interested in, prove results in such an abstract category and then show that the category we are actually interested in (topological spaces with adjectives) is a concrete example of this abstract category and therefore the results we are interested hold here because their abstract versions hold in the abstract category.

As a small motivating remark before we proceed, compact spaces are nice, homotopy groups are maps from compact spaces and it would be nice if maps from compact spaces could determine the topology of a space.

Definition 7.1. *A space X is called a k-space if its topology is the final topology generated by all continuous maps $K \to X$ where K is compact, i.e., when U open iff the inverse is open under all maps $K \to X$.*

It is often convenient to replace "all compact" in the definition with "all compact Hausdorff" and the resulting spaces are called Hausdorff compactly generated (these spaces themselves need not be Hausdorff). This is not the same as compactly generated Hausdorff spaces (i.e., compactly generated spaces which are also Hausdorff). Similarly we have "core compactly generated spaces" or "locally compactly generated spaces" etc. A comparision of such topologies is found in [6].

Some authors call a Hausdorff compactly generated space as a k-space.

Definition 7.2. *A space is called weakly Hausdorff if the continuous images of compact spaces are closed.*

Definition 7.3. *A space X is compactly generated if it has the property that a subset C is closed iff $C \cap K$ is closed in K for every compact subset $K \subseteq X$. Starting with a space X, similar to kX, we can create the space cX which is compactly generated.*

Weak Hausdorff condition is strictly between T_1 and T_2 separation axioms: being weakly Hausdorff makes all singletons closed, and Hausdorff spaces are weakly Hausdorff. The cocountable topology on any uncountable set is not Hausdorff, but all compact sets are finite, hence closed. See the following thread on MathOverflow for more examples: [8].

Start with a space X, obtain the space kX, $k^H X$ and cX by taking certain final topologies from the compact spaces, Hausdorff compact spaces and inclusions of compact subspaces respectively (these are the set X with open sets being those that satisfy some conditions). We have continuous "identity" maps $kX, k^H X, cX \to X$. A subset in one of these is compact iff it is a compact subset of X. We also have the continuous inclusion $k^H X \to kX$.

Suppose U is open in kX, then because the inclusion $K \hookrightarrow X$ of a compact subset is continuous, $U \cap K$ is open in K. Therefore, U is an open subset of cX which makes the "identity" map $cX \to kX$ continuous.

Suppose U is open in cX and we have an arbitrary continuous function $f\colon K \to X$. Then $f(K)$ is compact in X and $f\colon K \to f(K)$ is continuous. Now, $f^{-1}(U) = f^{-1}(U \cap f(K))$, hence the inverse is open, which means U is open in kX. Therefore, $cX = kX$.

When X is Hausdorff, compact subsets are Hausdorff which means the identity map $cX \to k^H X$ is continuous giving us $kX = cX = k^H X$.

If X is locally compact (this has various definitions, but we mean that every point x has a neighbourhood V contained in a compact set K), then $cX = X$. Therefore, locally compact Hausdorff spaces are HCG. However, there are locally compact non Hausdorff spaces which are not Hausdorff compactly generated, see [10].

At this stage a space X could be LC (locally compact), H (Hausdorff), WH (Weak Hausdorff), CG (compactly generated), HCG (Hausdorff Compactly Generated). We have the following implications

- HCG implies CG
- WH+CG implies HCG: indeed in this case, all compact subsets are Hausdorff
- LC+H implies CG and HCG

Theorem 7.1. *A space is compactly generated iff it is a quotient of a locally compact space.*

Proof. Suppose X is compactly generated. Let $\mathcal{K}$ denote the set [4] of compact subsets of X and Y be their direct sum. Y is locally compact because every point is in some $K \in \mathcal{K}$ and K is open and compact in Y. There is the map $Y \to X$ induced by the inclusions to X and this map is continuous regardless of whether X is compactly generated or not. We then obtain an equivalence relation $\sim$ on Y and a continuous map $Y/\sim \to X$ which is a bijection. When X is compactly generated, this map is a homeomorphism, giving X as a quotient of a locally compact space.

Conversely, let Y be a locally compact space and $\sim$ an equivalence relation on Y such that there is a homeomorphism $Y/\sim \to X$. We take $X = Y/\sim$ and let $p\colon Y \to X$ be the quotient map. Suppose U is a subset of X intersecting every compact subset of X in an open set. To show U is open, we need to show that $p^{-1}(U)$ is open in Y. Let $y \in p^{-1}(U)$. By local compactness, there is an open set V and a compact set K in Y such that $y \in V \subseteq K$.

Since p is continuous, $p(K)$ is compact, hence $p(K) \cap U$ is open in $p(K)$. The restriction $p|_K\colon K \to p(K)$ is continuous, hence, $p|_K^{-1}(p(K) \cap U)$ is open in K. Intersecting this with V gives an open neighbourhood of y contained in $p^{-1}(U)$. Therefore, $p^{-1}(U)$ is opein Y, hence U is open in X completing the proof. □

Proposition 7.1. *A quotient of a compactly generated space is compactly generated.*

Proof. Let $q\colon Y \to X$ be a quotient map with Y compactly generated. Suppose U is a subset of X intersecting every compact subset of X in an open set. Let K be compact in Y, then $p|_K\colon K \to p(K)$ is continuous and $p^{-1}(U) \cap K$ is then open in K. Since Y is compactly generated, it follows that $p^{-1}(U)$ is open in Y, hence U is open in X. □

Theorem 7.2. *A space is Hasudorff compactly generated iff it is a quotient of a disjoint union of compact Hausdorff spaces.*

Proof. If X is a quotient of Y, a disjoint union of compact Hausdorff spaces with quotient map $q\colon Y \to X$. Suppose $U \subseteq X$ such that for every $f\colon K \to X$ where K is compact Hausdorff, the inverse of U is open. To show U is open in X we need to show that $q^{-1}(U)$ is open in Y. But this is obvious because Y is a disjoint union of compact Hausdorff spaces.

For the other direction, it is suggestive to construct a quotient of the compact subsapces of X which are Hausdorff. However, this doesn't work because the image of a Hausdorff compact space in X need not be Hausdorff. Instead, as done in [6], we need to quotient out a much larger space.

Let I denote the set of all non open subsets of X. For each $i \in I$ there is a compact Hausdorff K_i and a continuous map $f_i\colon K_i \to X$ under which the inverse of $i \in I$ is not open. Let S denote the direct sum of these K_i and many one point spaces with various constant maps so as to cover X.

[4] This is a subset of the power set for example. It is important that it is a set because direct sums are defined for sets but not for anything larger.

Too long for a footnote: There is a whole lot of logical steps I am skipping here because I don't know how they work. What follows may be entirely wrong. The choice function seems to require taking choices from "subcategories". So, either we work in a framework with versions of the ZF axioms applying for categories so as to define subcategories through an axiom schema of separation and then use axiom of choice. Or we look at the collection of objects in our category, hope that they form a set and use the usual ZF axioms to choose and the lift our choice back to the category. But, usually our categories are not small and the objects don't form a set. A way around this is to apparently work in the context of Grothendieck universes and inaccessible cardinals so as to constraint the size of the objects in our category (our category being the comma category of compact spaces over X). Once the size is constrained, we get a small category.

Glossing over the foundational questions, assuming S exists, there's a continuous surjection $S \to X$. If $U \subseteq X$ is not open, then it's inverse is not open by choice of our K_is, hence X is a quotient of S. □

Proposition 7.2. *The product of a compactly generated space with a locally compact Hausdorff space is compactly generated.*

Proof. Suppose X is compactly generated and Z is locally compact Hausdorff. X is a quotient of some locally compact space Y. Since Z is core compact, we can transfer the quotient to get $X \times Z$ as a quotient of $Y \times Z$. Now, $Y \times Z$ is locally compact because product of open (compact) sets is open (compact). □

What happens when we don't have the Hausdorff condition in the above theorem? The MathOverflow thread [9] seems interesting.

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