Quick Obstruction Theory

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1 Postnikov Towers

Let X be path connected. A Postnikov tower is a series of approximations "from below" of (the homotopy type of) X. Let π_n denote the nth homotopy group of $X, n \geq 0$ (ignoring base points).

For $N \geq 1$, suppose X_n captures all the homotopy groups $\pi_i, 0 \leq i \leq n$. We should then expect X_{n+1} to be obtained from X_n using a $K(\pi_{n+1}, n+1)$. An obvious space to look at is $X_n \times K(\pi_{n+1}, n+1)$ and a more general space to look at would be a fibration $X_{n+1} \to X_n$ with fibre $K(\pi_{n+1}, n+1)$.

If we were to obtain such a sequence of spaces, then a nice thing to have would be that $X = \varprojlim X_n$, i.e., X should be the inverse limit of the Postnikov tower. Recall

Lemma 1.1. *[1, Lemma 4.7] Given a CW pair (X, A) and a map $f : A \to Y$ with Y path connected, then f can be extended to a map $X \to Y$ if $\pi_{n+1}(Y) = 0$ for all n such that $X \setminus A$ has cells of dimension n.*

This is simply an extension by various null homotopies.

Given a CW complex X and an $n \geq 1$, we can obtain X_n by attaching cells of dimension $n+2$ and higher that kill the generators of appropriate homotopy groups. More precisely, kill the generators of $\pi_{n+1}(X)$ to obtain a space Y_1. Then kill the generators of $\pi_{n+2}(X)$ to obtain a space Y_2. Repeat this to obtain a sequence of spaces

$$X \subset Y_1 \subset Y_2 \subset \ldots$$

and let the colimit be X_n. By the very definition of CW complexes, the colimit is a CW complex. The image of any sphere must factor through one of the Y_is and if the sphere has dimension $> n$, then its image in some Y_n is zero. Thus, X_n has trivial homotopy groups in degrees $n+1$ and higher, and the inclusion $X \hookrightarrow X_n$ is an isomorphism on $\pi_i, i \leq n$.

Moreover, by the extension lemma above, the inclusion $X \hookrightarrow X_n$ extends to a map $X_{n+1} \to X_n$ giving us a Postnikov tower

$$\begin{tikzcd}
 & \vdots \arrow[d] \\
 & X_3 \arrow[d] \\
 & X_2 \arrow[d] \\
X \arrow[r] \arrow[ru] \arrow[ruu] & X_1
\end{tikzcd}$$

Converting $X_n \to X_{n-1}$ into a fibration and using the long exact sequence in homotopy shows us that the fibre F_n is a $K(\pi_n, n)$. Recall that using a path space construction, any map $Y \to Z$ can be factored as $Y \simeq Y' \twoheadrightarrow Z$ - a weak equivalence followed by a fibration.

First replace $X_2 \to X_1$ by a fibration $X_2 \to X_2' \to X_1'$. Then modify the composition $X_3 \to X_2'$. Iterate this to replace everything by a fibration. Thus, we have for every path connected CW complex X, a Postnikov tower satisfying

i) $X \to X_n$ induces isomorphisms on $\pi_i, i \leq n$.
ii) $\pi_i(X_n) = 0$ for $i > n$.
iii) The map $X_n \to X_{n-1}$ is a fibration with fibre $K(\pi_n(X), n)$.

Before the next proposition, we need to set up some conventions. We write maps from spheres to a space X as maps $\alpha\colon I^k \to X$ that send the boundary to the basepoint[1]. Null homotopies of α will be considered as maps $I^{k+1} \to X$ that restricts to α at the $(-,1)$ end, and maps to the basepoint on all other boundaries.

The reverse or inverse of a map $I^k \to X$ will be done by reversal of the last coordinate, and composition/concatenation of two such maps is also along the direction of the last coordinate (provided composition makes sense). For $k \geq 2$, there are other equivalent (up to homotopy) conventions.

Proposition 1.1. *[1, Proposition 4.67] For an arbitrary sequence of fibrations $\cdots \to X_2 \to X_1$, (with abelian fundamental groups) there is an exact sequence*

$$0 \to \varprojlim{}^1 \pi_{i+1} X_n \to \pi_i \varprojlim X_n \to \varprojlim \pi_i X_n \to 0.$$

Proof. **Step 1: Exactness at the last term**

Consider the natural map $\lambda\colon \pi_i \varprojlim X_n \to \varprojlim \pi_i X_n$ obtained by projecting onto the coordinates of $\varprojlim X_n$. An element on the right side is represented by a family of maps $f_n\colon S^i \to X_n$. Because $X_n \to X_{n-1}$ is a fibration, we can inductively modify the f_n so that the compositions are equal as maps and not just homotopy classes of maps. This gives surjectivity of λ.

Step 2: Elements of the kernel of λ

Let f be an element in the kernel of λ. Denote by f_n the nth coordinate of f. There is a nullhomotopy $F_n\colon I^{i+1} \to X_n$ of f_n. At the nth level, we have two maps

$$p_{n+1}F_{n+1}, F_n\colon D^{i+1} \to X_n$$

that agree on S^i. Here, $p_{n+1}\colon X_{n+1} \to X_n$ is the fibration. Together, we get a map

$$g_n = (p_{n+1}F_{n+1} * F_n^{-1})\colon S^{i+1} \to X_n$$

where $*$ denotes composition/concatenation and $(-)^{-1}$ denotes reversal.

The collection $(g_n) \in \prod \pi_{i+1}(X_n)$ is well defined up to a choice of all the null homotopies F_n. Let F'_n denote another collection of null homotopies and define g'_n similarly. Set

$$G_n = F_n * (F'_n)^{-1} \in \pi_{i+1}(X_n).$$

It's not too hard to see that

$$p_{n+1}G_{n+1} * g_n * G_n^{-1} = g'_n$$

in $\pi_{i+1}(X_n)$.

Assuming abelian fundamental groups, this suffices to conclude that the collection (g_n) is a well defined element of the $\varprojlim^1$ term. In the nonabelian case, we will have to modify the definition of $\varprojlim^1$.

Step 3: Homomorphism for $i \geq 1$

Thus, we get a map $\ker\lambda \to \varprojlim^1 \pi_{i+1} X_n$. For $i \geq 1$, this map is a homomorphism: the proof of this uses the Eckmann-Hilton argument - composition can be defined using any coordinate (and this is where we need $i \geq 1$ so that $i + 1 \geq 2$.

Step 4: Surjectivity for $i \geq 0$

For surjectivity, let (g_n) be a sequence in $\prod \pi_{i+1}(X_n)$. We need to construct a sequence $\alpha_n\colon S^i \to X_n$ that form an element of $\pi_i(\varprojlim X_n)$, and we need null homotopies β_n of α_n.

[1] We will assume things are nice enough that we won't have to keep track of the basepoints.

To start, let α_1 be the constant map. This is already null homotopic, so we may take β_1 to also be the constant map. Assuming we have constructed all elements up to X_n, the expectation is that g_n is obtained by concatenating $p_{n+1}\beta_{n+1}$ and β_n. As per our construction, we should then expect

$$g_n * \beta_n = p_{n+1}\beta_{n+1}.$$

The left side is another null homotopy of α_n - with one face being α_n and the rest being mapped to the base point of X_n. The rest of the boundary sits cofibrantly in the entire disk, so by $X_{n+1} \to X_n$ being a fibration, we can lift all of $g_n * \beta_n$ to a map $\beta_{n+1} : I^{i+1} \to X_{n+1}$ which restricts to an element α_{n+1} at the face that used to be α.

By construction, $p_{n+1} \circ \alpha_{n+1} = \alpha_n$ and β_{n+1} is a null homotopy of α_{n+1}. Therefore, $(\alpha_n) \in \pi_i(\varprojlim X_n)$ and the corresponding element in the $\varprojlim^1$ term is indeed (g_n).

Step 5: Injectivity for $i \geq 0$ Suppose f, f' are two elements in the kernel of λ mapping to the same element of $\varprojlim^1 \pi_{i+1} X_n$. Let the coordinates be f_n, f_n' respectively, and the null homotopies be F_n, F_n'. Them mapping to the same element in $\varprojlim^1 \pi_{i+1} X_n$ means there is a sequence $(G_n) \in \prod \pi_{i+1} X_n$ such that

$$p_{n+1} G_{n+1} * g_n * G_n^{-1} = g_n' \text{ in } \pi_{i+1} X_n$$

where g_n, g_n' are as before.

Expand g_n and set $H_n = G_n * F_n, n \geq 0$. Rearranging the terms gives

$$p_{n+1} H_{n+1} * (H_n)^{-1} = p_{n+1} F_{n+1}' * (F_n')^{-1} \implies (H_n)^{-1} * F_n' \simeq p_{n+1}(H_{n+1}^{-1} * F_{n+1}')$$

as maps $D^{i+1} \to X_n$. Note

- The left side is a homotopy between f_n and f_n'.
- The collection $(H_n)^{-1} * F_n'$ is coherent up to homotopy.

We will now replace each of the homotopies so as to obtain an honest coherence (thereby making f, f' homotopic). This can be done again by a homotopy lifting property as above. Once this is done, we get a new system of homotopies that lifts to a homotopy $f \simeq f'$ as desired. □

I learnt much of this proof from [2] where Hirschhorn defines $\varprojlim^1$ for a tower of nonabelian groups[2]. It is also shown that when all the X_n are H-spaces and the fibrations are H-maps, then the exact sequence is an exact sequence of groups.

Corollary 1.1. *[1, Corollary 4.68] For a Postnikov tower (X_n) of a connected CW complex X, the natural map $X \to \varprojlim X_n$ is a weak homotopy equivalence. In particular, X is a CW approximation of $\varprojlim X_n$.*

Proof. The $\varprojlim^1$ term is zero because of the Mittag-Leffler condition[3], and the composition $\pi_i(X) \to \pi_i(\varprojlim X_n) \to \varprojlim \pi_i(X_n)$ is an isomorphism for large n. □

1.1 Principal fibrations

Definition 1.1. *A fibration $F \to E \to B$ is called principal if there is a commutative diagram*

$$\begin{tikzcd} & F \arrow[r] \arrow[d] & E \arrow[r] \arrow[d] & B \arrow[d] \\ \Omega B' \arrow[r] & F' \arrow[r] & E' \arrow[r] & B' \end{tikzcd}$$

where the second row is a fibration sequence and the vertical maps are weak homotopy equivalences.

[2] [2] uses the opposite sign in the definition of the map to the $\varprojlim^1$ term.

[3] The Mittag-Leffler condition on an inverse system says that at each stage, the eventual image of the maps stabilizes. In the present case, the eventual image is the homotopy groups of X.

Given X, a Postnikov tower gives a sequence of spaces X_n with fibrations

$$K(\pi_n, n) \to X_n \to X_{n-1}.$$

If this fibration is principal, then X_n must be, up to weak homotopy equivalence, the fibre of a map $k_{n-1} \colon X_{n-1} \to K(\pi_n, n+1)$ which determines a class in $H^{n+1}(X_{n-1}; \pi_n X)$ called the $(n-1)$st k-invariant of X.

In turn, these k-invariants tell us how to build X out of various Eilenberg-MacLane spaces as a sort of twisted product. I will state the following without proof

Theorem 1.1. *[1, Theorem 4.69] A connected CW complex X has a Postnikov tower of principal fibrations iff $\pi_1(X)$ acts trivially on $\pi_n(X)$ for all $n > 1$.*

1.2 Moore-Postnikov tower

Let $f \colon X \to Y$ be a map between connected CW complexes. A Moore-Postnikov tower is a tower $\cdots \to Z_2 \to Z_1$ together with maps $X \to Z_n \to Y$ such that everything commutes and

- Each composition $X \to Z_n \to Y$ is homotopic to f.
- Each $X \to Z_n$ induces an isomorphism on $\pi_i, i < n$ and a surjection for $i = n$.
- Each $Z_n \to Y$ induces an isomorphism on $\pi_i, i > n$ and an injection for $i = n$.
- Each $Z_{n+1} \to Z_n$ is a fibration with fibre a $K(\pi_n F, n)$ where F is the homotopy fibre of f.

Replacing Y with the mapping cylinder M_f of f, we may assume f is an inclusion. Given a pair (X, A), by attaching cells to A (or a CW approximation of A), we obtain a relative n-connected complex (Z, A) with a map $Z \to X$ inducing isomorphism on $\pi_i, i > n$ and an injection for $i = n$. Such a pair is called an n-connected CW model for (X, A). For more details, refer to [1, Proposition 4.15]. Furthermore ([1, Proposition 4.18]) such a construction is functorial up to homotopy relative to A.

Using this result, we successively approximate the pair (M_f, X) to obtain the Moore-Postnikov tower. We can then replace each $Z_n \to Z_{n-1}$ by a fibration inductively. Care must be taken with replacing the maps $Z_n \to Y$ so that things commute strictly.

As for the fibres of the fibration, it is obtained by an inspection of a couple of long exact sequences and the defining properties of the Moore-Postnikov tower. For further details we refer to [1, Proposition 4.71].

1.3 Obstruction classes, a first look

The main question is the following relative lifting problem: given a CW pair (W, A), a fibration $X \to Y$ and maps $A \to X, W \to Y$, when does the following lift exist?

$$\begin{tikzcd}
A \arrow[r] \arrow[d, hook] & X \arrow[d] \\
W \arrow[r] \arrow[ur, dashed] & Y
\end{tikzcd}$$

A Postnikov tower for X is an approximation for X from below. A first approach is to inductively construct lifts $W \to X_n$ extending the map $A \to X_n$. Assume X has a Postnikov tower of principal fibrations, then the inductive stage would look like

$$\begin{tikzcd}
A \arrow[r] \arrow[d, hook] & X_n \arrow[r] \arrow[d] & PK \arrow[d] \\
W \arrow[r] & X_{n-1} \arrow[r] & K = K(\pi_n X, n+1)
\end{tikzcd}$$

with X_n being the pullback of the second square - as it is the (homotopy) fibre of the fibration $X_{n-1} \to K$. A lift amounts to the composition $W \to X_{n-1} \to K$ being null homotopic. Since such a lift exists for A, we can assume we have a nullhomotopy of the map $A \to K$ and we want an extension of this to W. Given the nullhomotopy, we obtain a map $W \cup CA \to K$, i.e., an **obstruction class**

$$\omega_n \in H^{n+1}(W \cup CA; \pi_n X) \cong H^{n+1}(W, A; \pi_n X).$$

This is simply keeping track of the maps we have and we get a cohomology class because the Eilenberg-MacLane spaces represent ordinary cohomology[4].

Proposition 1.2. *[1, Proposition 4.72] A lift $W \to X_n$ extending the given $A \to X_n$ exists iff $\omega_n = 0$.*

Proof. We want to show that $W \cup CA \to K$ extends to a map $CW \to K$ iff $\omega_n = 0$, i.e., iff the map $W \cup CA \to K$ is null homotopic.

If $g_t \colon W \cup CA \to K$ is a null homotopy, then g_1 is a constant map extending to a constant map on CW. Applying the homotopy extension property for the pair $(CW, W \cup CA)$ and using the reverse g_{1-t}, the 0-time map gives the desired extension of $W \cup CA \to K$ to CW.

Conversely, an extension $CW \to K$ is automatically null homotopic and the restriction of the null homotopy to $W \cup CA$ makes $\omega_n = 0$. □

If all the obstruction classes are zero, then we obtain a map $W \to \varprojlim X_n$ extending the composition $A \to X \to \varprojlim X_n$. Let M be the mapping cylinder of the map $X \to \varprojlim X_n$.

The restriction of $W \to \varprojlim X_n \subset M$ to A factors through X (or rather the image of X in $\varprojlim X_n$), we can homotope the restriction to a map $A \to X \subset M$. Then extending this homotopy to all of W, we get a map $(W, A) \to (M, X)$.

Because $X \to \varprojlim X_n$ is a weak homotopy equivalence, the relative homotopy groups of (M, X) are all trivial. By compression lemma ([1, Lemma 4.6]), the map $(W, A) \to (M, X)$ can be homotoped to a map $W \to X$ extending the map $A \to X$ thus solving the extension problem.

As an application, together with Hurewicz theorem, we get

Proposition 1.3. *[1, Proposition 4.74] If X and Y are connected abelian CW complexes, then a map $f \colon X \to Y$ inducing isomorphisms on all homology groups is a homotopy equivalence.*

Proof sketch. Being abelian means that π_1 acts trivially on all π_n. By replacing Y with the mapping cylinder we may assume f is an inclusion. We know $\pi_1(Y, X)$ is zero as $\pi_1(X) \to \pi_1(Y)$ is an isomorphism.

By hypothesis, the relative homology groups $H_*(Y, X)$ are all zero, and by the universal coefficient theorem, this makes all cohomology groups $H^{n+1}(Y, X; \pi_n(X)) = 0$. By obstruction theory, there is an extension of the identity map $X \to X$ to a retraction $Y \to X$ making all the maps $\pi_n(Y) \to \pi_n(Y, X)$ surjective. By naturality of the action of $\pi_1, \pi_1(X)$ acts trivially on $\pi_n(Y, X)$. By the relative Hurewicz theorem, all the relative homotopy groups vanish. □

2 Obstruction cochains

This section is based on the lecture notes by A. Freire [4]. Those notes follow Steenrod's original papers, albeit only consider much simpler cases.

Previously, to extend a map $f \colon A \to X$ to a map $W \to X$, we considered a Postnikov tower over X and constructed a lift to each term of the tower. Another approach would be to extend to W one dimension at a time, keeping the codomain fixed.

Say $\phi \colon S^q \to A$ is an attaching map for some cell of W. An extension to the corresponding $q+1$-cell exists if the composite $f \circ \phi$ is null homotopic. This composite is an element of $\pi_q(X)$; to not keep track of base points, we'll assume is that $\pi_1(X)$ acts trivially on all homotopy groups. Thus f determines a cochain $C_{q+1}(W, A) \to \pi_q(X)$ on the relative $q+1$-cellular chains.

Switching to the notation in [4], let K, Y be abelian CW complexes. For a map $f \colon K^q \to Y$, let $c(f)$ be the **obstruction cochain** as constructed above.

[4] This is related to the Brown representability theorem.

Since $c(f)$ is a certain composition, it is natural under morphisms in the following sense: given $h\colon K \to K', f'\colon (K')^q \to Y$, we have

$$c(f' \circ h) = h^* c(f'),$$

where h^* is the pullback at the cochain level. Moreover, if $f_0, f_1\colon K^q \to Y$ are homotopic, then $c(f_0) = c(f_1)$.

Theorem 2.1. *The obstruction cochain is a cocycle.*

Proof. First, consider the case when K is $(q-1)$-connected. Hurewicz theorem gives an isomorphism $\pi_q(K^q) \to H_q(K^q)$, but as there are no $(q+1)$-cells, $H_q(K^q) = Z_q(K^q)$ (all q-cycles).

Let $\chi\colon S^q \to K^q$ be the attaching map of some $(q+1)$-cell e of K. Then

$$\partial e = \sum_\alpha a_\alpha e_\alpha$$

where the coefficient a_α of the q-cell e_α is given by the degree of the composite

$$S^q \xrightarrow{\chi} K^q \to K^q/(K^q \setminus e_\alpha) \cong S^q.$$

Note that because the quotient map is cellular, it becomes a projection operator at the chain level.

Next, the Hurewicz map sends χ to a sum $\sum_\alpha b_\alpha e_\alpha$. Given a map between spheres of the same dimension, the corresponding morphism at the top integral homology is given by the degree of the map. Therefore, following the Hurewicz homomorphism with the projection operators lets us compute the numbers b_α and these happen to be the degrees of the same maps above, i.e., $b_\alpha = a_\alpha$. We conclude that the Hurewicz map sends χ to ∂e.

Thus, $c(f)$ is the cochain given by the composition

$$C_{q+1}(K) \xrightarrow{\partial} Z_q(K) = Z_q(K^q) \xrightarrow{\cong} \pi_q(K^q) \xrightarrow{f_*} \pi_q(Y).$$

Now, $Z_q(K)$ is a summand of $C_q(K)$, as the latter is free abelian, which means that the composition of the last two arrows can be extended to a homomorphism $h\colon C_q(K) \to \pi_q(Y)$. It follows that $c(f) = \delta h$ is a coboundary.

For the general case, let e be a $(q+2)$-cell. We want to show that $(\delta c)(e) = c(\partial e) = 0$. Let K' be the subcomplex formed from e and all the cells appearing with a nonzero coefficient in ∂e. Because the cochain is obtained by a simple composition, it is clear that the restricted obstruction cochain is the obstruction cochain associated to the restriction of f. Since K' is $(q-1)$-connected, we have our result. □

The obstruction cocycle tell us if a given function can be extended over skeletons. Suppose $f\colon K^q \to Y$ has a vanishing obstruction cocycle, i.e., it can be extended to the $(q+1)$-skeleton, then we ask about the number of such extensions. This can have a nice answer only when we identify extensions up to homotopy. In this case, we may as well allow for f to vary up to homotopy.

Here's the setup. Suppose $f_0, f_1\colon K^q \to Y$ are two maps such that the restrictions

$$f_0|_{K^{q-1}}, f_1|_{K^{q-1}}\colon K^{q-1} \to Y$$

are homotopic via a homotopy $k\colon K^{q-1} \times I \to Y$. Are f_0, f_1 homotopic to each other?

The given data gives a map $F\colon (K \times I)^q \to Y$. The $(q+1)$-skeleton of $K \times I$ contains $K^q \times I$ and we are interested in the extension of F to this subcomplex (the rest of $(K \times I)^{q+1}$ is extra information).

The obstructions to extension of homotopies lie in the relative pair $(K, K \times \{0,1\})$. Following [3], let $\bar{0}, \bar{1}$ be the generators of $C^0(I)$ (integer coefficients) and $\bar{I}$ the generator of $C^1(I)$, chosen so that

$$\delta\bar{0} = -\bar{I}, \delta\bar{1} = \bar{I}.$$

Now, for any $\varphi \in C^p(K; \pi_q(Y))$, the cochain $\varphi \times \bar{I}$ vanishes on $K \times \{0,1\} \subset K \times I$, hence is an element of the relative cochain group. Furthermore, since $\sigma \mapsto \sigma \times I$ is a one-to-one correspondence between the p-cells of K and the $(p+1)$-cells of $(K \times I) \setminus (K \times \{0,1\})$, we conclude that there is an isomorphism

$$C^p(K; \pi_q(Y)) \cong C^{p+1}(K \times I, K \times \{0,1\}; \pi_q(Y)).$$

Furthermore, the isomorphism commutes with the coboundary operator, so there is an induced isomorphism

$$H^p(K;\pi_q(Y)) \cong H^{p+1}(K\times I, K\times\{0,1\};\pi_q(Y)).$$

Returning to obstructions to homotopy, let $c(F)$ be the obstruction coycle associated to F. When restricted, this cochain coincides with $c(f_0)$ on $K\times 0$ and $c(f_1)$ on $K\times 1$, therefore

$$c(F)-c(f_0)\times\bar{0}-c(f_1)\times\bar{1}\in C^{q+1}(K\times I, K\times\{0,1\};\pi_q(Y)).$$

Using the above isomorphism, we define the **deformation cochain** $d(f_0,k,f_1)$ by

$$d(f_0,k,f_1)\times\bar{I}=(-1)^{q+1}[c(F)-c(f_0)\times\bar{0}-c(f_1)\times\bar{1}].$$

If $f_0|_{K^{q-1}}=f_1|_{K^{q-1}}$ and k is the identity homotopy, then we use the notation $d(f_0,f_1)$ and call it the **difference cochain**. If the deformation cochain vanishes, then $f_0\simeq f_1$.

- As with the obstruction cochain, there is a naturality with respect to compositions. Given $h\colon K'\to K$ and f_0,k,f_1, let f_0',k',f_1' be obtained by precomposition with h, then

$$h^*d(f_0,k,f_1)=d(f_0',k',f_1').$$

- Say we have three maps $f_0,f_1,f_2\colon K^q\to Y$ with homotopies k,k' connecting f_0 to f_1 and f_1 to f_2 respectively. We can stack homotopies to get a homotopy $k''\colon f_0\simeq f_2$. If σ is a q-cell of K, then $\partial(\sigma\times I)=\partial\sigma\times I\cup\sigma\times\{0,1\}$. By construction of the obstruction cochain, we see that

$$d(f_0,k'',f_2)=d(f_0,k,f_1)+d(f_1,k',f_2).$$

  Note that we can afford to be sloppy with adding spheres because Y is abelian; so choose the basepoint to be on the "stacking layer" of the homotopies while gluing spheres.

Theorem 2.2 (Coboundary formula)**.** *With the setup as above,*

$$\delta d(f_0,k,f_1)=c(f_0)-c(f_1).$$

Proof. Applying δ to the defining property of d, we get

$$(\delta d(f_0,k,f_1)\times\bar{I}=(-1)^{q+1}[c(f_0)\times\bar{I}-c(f_1)\times\bar{I}]$$

where we use the fact that $c(F),c(f_0),c(f_1)$ are cocycles, and $\delta(u\times v)=\delta u\times v+(-1)^p u\times\delta v$ for cochains u,v with u being a p-cochain. As $\varphi\mapsto\varphi\times\bar{I}$ is an isomorphism, the theorem follows. ☐

Let us take a step back and look at what we have so far. Let K be a CW complex and Y an abelian space, and $f\colon K^{q-1}\to Y$ a map. This gives us an obstruction cocycle $c(f)\in C^q(K,\pi_{q-1}(Y))$ that depends only on the homotopy class of f. There is an extension of f to K^q if and only if $c(f)$ vanishes. If there are two extensions f_0,f_1, then we can construct a difference cochain $d(f_0,f_1)$.

The coboundary formula tells us something strong: the homotopy class of f determines a cohomology class in the next dimension, provided at least one extension of f exists - this is because all the obstruction cocycles associated to extensions of f are cohomologous.

We ask the following questions

- Which cohomology classes are obstruction classes?
- Which cocycles are difference cochains?
- If two functions have the same (or cohomologous) obstruction class, are they homotopic?

Given $f,g\colon K^{q-1}\to Y$ with the same obstruction cocycle, if they had the same restriction to the $(q-2)$-skeleton, then we know that $d(f,g)=0$. We now answer the second question.

Lemma 2.1. *Let $f_0\colon K^q\to Y$ and $d\in C^q(K;\pi_q(Y))$ be given. Then there is an $f_1\colon K^q\to Y$ such that $f_1|_{K^{q-1}}=f_0|_{K^{q-1}}$ and $d(f_0,f_1)=d$.*

Proof. With the given data, we construct a map $F\colon (K^q \times 0) \cup K^{q-1} \times I \to Y$ with the constant homotopy. We want to extend this to the q-skeleton $(K \times I)^q$ and take the "top part" as f_1. We can perform the extension cell-by-cell.

Let e be a q-cell of K. Decompose $e \times I$

$$e \times I = (S^{q-1} \times I) \cup (D^q \times 0) \cup (D^q \times 1) \cup (D^q \times I).$$

The first three terms make a copy of S^q. Let E_0 be the disc represented by the first two terms, and $E_1 = D^q \times 1$, so $S^q = E_0 \cup E_1$.

The value of the extension on this copy of S^q should be represent the element $\phi = (-1)^{q+1} d(e) \in \pi_q(Y)$. At the same time, F is already defined on E_0 as the boundary of e lives on K^{q-1}. Therefore, we need to find a representative of ϕ with the given values on E_0.

Pick any representative $g\colon S^q \to Y$ of ϕ. We have$g|_{E_0} \simeq F|_{E_0}$, and because $E_0 \subset S^q$ is a subcomplex, this homotopy extends to a homotopy on all of S^q giving us another representative g' of ϕ which is F on E_0. Extend cell-by-cell to complete the proof. □

Now that we know every cochain can be a difference cochain (provided extensions exist), we make the following observations. Suppose $f\colon K^{q-1} \to Y$ is given and can be extended to K^q. Then

- If f_0, f_1 are two extensions to K^q, the cochain $d(f_0, f_1)$ is defined and the coboundary formula tells us that $c(f_0), c(f_1)$ are cohomologous.
- If f_0 is an extension and c any cochain cohomologous to $c(f_0)$, say $c(f_0) - c = \delta d$, then there is an f_1 such that $d(f_0, f_1) = d$. By the coboundary formula, we conclude that $c = c(f_1)$ is the obstruction class of an extension of f.

Thus, there is a cohomology class $\bar{c}(f)$ that is represented by a cochain if and only if that cochain is the obstruction class of an extension of f. Moreover, any extension of f further extends if and only if $\bar{c}(f)$ vanishes.

By the homotopy extension property, any homotopy defined on a subcomplex of K extends to all of K, thus we obtain

Theorem 2.3. *Let $f_0, f_1\colon K \to Y$, and assume $f_0 = f_1$ on K^{q-1} (so $d(f_0, f_1)$ is defined). Then there is an $f_1'\colon K \to Y$ homotopic to f_0 on K (rel. K^{q-2}) and coinciding with f_1 on K^q iff the difference cocycle $d(f_0, f_1) \in Z^q(K, \pi_q(Y))$ is a coboundary.*

3 More obstruction theory

Let us assume that Y is path connected, $(q-1)$-connected with abelian π_1 if $q = 1$, and $\pi_q(Y) \neq 0$. With these assumptions, any $f\colon K^0 \to Y$ can be extended all the way up to K^q and for any two $f, g\colon K^q \to Y$, the difference $c(f) - c(g)$ is a coboundary as the restrictions to K^{q-1} are homotopic (by induction/boot strapping).

Definition 3.1. *Given $f\colon K^q \to Y$, the obstruction class $\bar{c}(f) \in H^{q+1}(K; \pi_q(Y))$ is independent of f and is called the* ***primary obstruction class*** *of the pair (K, Y).*

The primary obstruction class is natural under maps $h\colon K \to K'$ and therefore defines a topological invariant independent of the cell decomposition of K.

With the same hypotheses on Y, let $f_0, f_1\colon K \to Y$. We know that the restrictions to K^{q-1} are homotopic. We ask whether $f_0|_{K^q} \simeq f_1|_{K^q}$. By the assumption that f_0, f_1 are defined on all of $K, \bar{c}(f_0), \bar{c}(f_1)$ both vanish.

For any homotopy $k\colon f_0|_{K^{q-1}} \simeq f_1|_{K^{q-1}}$, we get a difference cocycle $d(f_0, k, f_1)$. If k, k' are two such homotopies, then the difference $d(f_0, k, f_1) - d(f_0, k', f_1)$ is a coboundary - this is because the obstructions for f_0, f_1 vanish, there is an isomorphism of cochains/cohomology groups, and the difference of two obstruction cocycles on $(K \times I)^q$ is a coboundary.

Definition 3.2. *Given $f_0, f_1\colon K \to Y$, the* ***primary difference class*** *$\bar{d}(f_0, f_1) \in H^q(K; \pi_q(Y))$ of f_0, f_1 is the cohomology class represented by any difference cochain $d(f_0, k, f_1)$ as above.*

The vanishing of the primary difference class is a necessary and sufficient condition for $f_0|_{K^q} \simeq f_1|_{K^q}$. The additivity formula for the difference cochain carries over to an additivity formula for the primary difference classes. More precisely, if $f_0, f_1, f_2 \colon K \to L$, then

$$\bar{d}(f_0, f_2) = \bar{d}(f_0, f_1) + \bar{d}(f_1, f_2).$$

Using the same technique as in Lemma 2.1, one can show that given $f_0 \colon K \to Y$, each cohomology class in $H^q(K; \pi_q(Y))$ is a primary difference class provided K is of dimension $\leq q+1$. We need this dimensional constraint so that the function we get in the proof of Lemma 2.1 can be extended to all of K.

Theorem 3.1 (Classification theorem). *Let K be q-dimensional. Fix $f_0 \colon K \to Y$. The map*

$$[K, Y] \to H^q(K; \pi_q(Y))$$
$$[f] \mapsto \bar{d}(f, f_0)$$

defines a bijection.

Proof. The map is well defined as if $f \simeq f'$ then $\bar{d}(f, f_0) = \bar{d}(f', f_0)$.

If $\bar{d}(f, f_0) = \bar{d}(f', f_0)$, then by the additivity formula, $\bar{d}(f, f') = 0$ and by the hypothesis on the dimension of $K, f \simeq f'$.

Given any cocycle $d \in Z^q(K; \pi_q(Y))$, by Lemma 2.1, we can find an $f \colon K \to Y$ extending $f_0|_{K^{q-1}}$ such that $\bar{d}(f_0, f) = -[d]$. By the additivity formula, $\bar{d}(f, f_0) = [d]$ proving surjectivity. □

Let $y_0 \in Y$ be fixed (and assume Y is a finite complex), and set $\bar{d}_Y = \bar{d}(c_{y_0}, id_Y) \in H^q(Y; \pi_q(Y))$ where c_{y_0} is the constant map taking the value y_0. Since Y is path connected, this class is independent of the choice of y_0. By naturality, for any $f \colon K \to Y$, we get $\bar{d}(c_{y_0}, f) = f^* \bar{d}_Y$ where, by abuse of notation, c_{y_0} is again the constant map $K \to Y$.

Corollary 3.1 (Hopf-Whitney theorem). *With the setup as aobve, the map $f \mapsto f^* \bar{d}_Y$ defines a bijection*

$$[K, Y] \to H^q(K; \pi_q(Y)).$$

4 What next?

That is about all I want to write on the subject at this time. There are lots more one can do. Firstly, one can talk about the applications of the Postnikov tower and its variations.

With regards to obstruction theory, I have only considered CW complexes. Instead, one could start with relative CW complexes and consider relative cochains etc throughout. In [3], the question of existence of sections of fibre bundles is considered, and one should use local coefficients in this case. Much of the material here is from sections 32-37 of [3] (although I first came across the notes [4]).

Then there's the topic of spectra and generalized cohomologies where there is a version of obstruction theory (refer Goerss-Hopkins obstruction theory etc). Another tool closely related to the ideas presented here is spectral sequences.

Anyway, that's all for now. Extend forth!

References

[1] Hatcher, A. (2002). Algebraic topology. United Kingdom: Cambridge University Press.

[2] Hirschhorn, The homotopy groups of the inverse limit of a tower of fibrations, `https://math.mit.edu/~psh/notes/limfibrations.pdf`

[3] Steenrod, N. (1999). The Topology of Fibre Bundles. United States: Princeton University Press.

[4] Lecture notes by Alex Freire `https://web.math.utk.edu/~afreire/teaching/m664s24/Obstruction_theory_notes.pdf`