On Primary Decomposition

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All rings considered are commutative with 1. Let k be a an algebraically closed field, most of the time we will be dealing with the polynomial ring $k[x_1, \ldots, x_n]$ and when $n = 2$, we take $k[x, y]$ and $k[x, y, z]$ when $n = 3$. We deonte by $\mathbb{A}^n$ the affine n-space over k with the usual Zariski topology. From Hilbert's basis theorem we know that $k[x_1, \ldots, x_n]$ is Noetherian.

Although most of the exposition is done using the ring $k[x_1, \ldots, x_n]$ because it has a strong correspondence with the geometric picture, all the theorems and lemmas are proved in the setting of an arbitrary commutative ring with unity.

1 The problem of decomposition

Suppose X is an algebraic set described by an ideal I in $k[x_1, \ldots, x_n]$. Recall that a dimension of X is defined as the maximal length of chains of irreducible subvarieties. This definition forces the dimension of X to be the same as the maximum of dimensions of irreducible components of X. The irreducible components of X correspond to minimal primes over I. We know that if $\mathfrak{p}$ is a prime and if $\mathfrak{p} = J_1 \cap J_2$ is the intersection of two ideals, then $\mathfrak{p} = J_1$ or $\mathfrak{p} = J_2$, in other words irreducible subvarieties of $\mathbb{A}^n$ are in a bijective correspondence with prime ideals in $k[x_1, \ldots, x_n]$.

Theorem 1. *Let A be a Noetherian ring, then any ideal I has finitely many minimal primes over it.*

Proof. Let I be an ideal, then an application of Zorn's lemma shows that the collection of primes containing I has minimal elements. Therefore, it makes sense to talk about the set of minimal primes over any ideal I. Suppose there is an ideal for which this collection is infinite. Then the collection of all such ideals (which is a well defined set) is non empty, and by the Noetherian condition on A, has maximal elements. Let I be one such maximal ideal. Then I cannot be prime, as then I would be the only minimal prime over itself. Therefore, $\exists\, x, y \notin I$ such that $xy \in I$.

The ideals $I + (x), I + (y)$ are strictly larger than I and, by the maximality condition on I, have finitely many minimal primes over them, say $\mathfrak{p}_1, \ldots, \mathfrak{p}_k$. If $\mathfrak{p}$ is any minimal prime over I, then $I + (x) \subset \mathfrak{p}$ or $I + (y) \subset \mathfrak{p}$ and by minimality, it follows that $\mathfrak{p} = \mathfrak{p}_i$ for some i. Therefore, I has finitely many primes over itself which is a contradiction. Therefore, every ideal has finitely many minimal primes over it. $\square$

By this theorem, we know that X is going to be a finite union of irreducible components. So, it is of interest to find independently, the irreducible components of a given algebraic set X, which amounts to finding the minimal primes over I.

Suppose we consider $n = 1$, in this case, I is generated by some polynomial f. To find minimal primes over I, we need to find the prime factors of f. Let us say $f = f_1^{n_1} \ldots f_m^{n_m}$ where $f_1, \ldots, f_m$ are irreducible. Then (f_i) is a prime containing $I = (f)$ and it is minimal for if any prime contains I, it must contain some f_i.

To find the prime divisors of f, one takes arbitrary polynomials g and sees if g divides f. This way, one can reduce the task to polynomials of smaller degrees until we arrive at irreducible polynomials. If $g \mid f$, then there is an h, such that $gh = f$, which means that the colon ideal $(f : g) = \{h \in k[x] | hg \in (f)\}$ is strictly larger than f. Therefore, in order to find the divisors f_i of f, we look at all the ideals $(I : g)$ and search for the largest ones.

Ultimately we obtain ideals $(f_1^{n_1}), \ldots, (f_m^{n_m})$ and

$$(f) = \prod (f_i^{n_i}) = \cap (f_i^{n_i}) = \prod (f_i)^{n_i}.$$

However, the case $n = 1$ is misleading because the ring $k[x]$ is much nicer than other polynomial rings in that it is a Dedekind domain. Simply, it is a PID and every ideal is a product of primes. The general case is not so simple.

In the general case, suppose we write $X = X_1 \cup \cdots \cup X_m$ where the X_i are irreducible. In the algebraic picture, this amounts to writing I as an intersection of ideals. However, on the algebraic side, we must take care of multiplicity, for ideals with the same radical describe the same algebraic set. For example, $Z(x^2)$ is a single point in $\mathbb{A}^1$, hence irreducible, but I is not an intersection of primes, nor is itself a prime.

Therefore, what we seek is to write $I = I_1 \cap \cdots \cap I_k$ where the ideal I_i corresponds to the variety X_i. If I itself is radical, then this is trivial for radical ideals are the intersection of primes that contain it. Note that if we are to find such a decomposition of I, then the I_i have prime radicals for they correspond to the irreducible X_i.

2 Irreducible ideals

We transfer the notion of irreducibility directly from the geometric picture. On the geometric side, we defined a closed set to be irreducible if it cannot be written as the union of two proper closed subsets. For example, if $X = Z(xy, xz)$, i.e., the union of the x-axis and the yz-plane, then the irreducible components are the x-axis and the yz-plane. If we translate this to the algebraic side, then

Definition 1. *An ideal I of a ring A is irreducible if whenever $I = J_1 \cap J_2$ is the intersection of two ideals, then $I = J_1$ or $I = J_2$.*

Note that prime ideals are always irreducible. If we now take an irreducible ideal, then it appears that it should describe an irreducible set and this is in fact true.

Lemma 1. *Let A be a Noetherian ring, if I is irreducible, then its radical $r(I)$ is prime. Moreover, whenever $ab \in I$ for $a, b \in A$, then $a \in A$ or $b \in r(I)$.*

Proof. Suppose $ab \in r(I)$, say $(ab)^k \in I, k \geq 1$. The increasing chain $(I : b) \subseteq (I : b^2) \subseteq \ldots$ must stabilize by the noetherian condition, therefore, there is some N such that $(I : b^N) = (I : b^{N+t}) \, \forall \, t \geq 1$. If $z = x + ra^k = y + sb^N \in (I + (a^k)) \cap (I + (b^N))$, where $x, y \in I, r, s \in A$, then by multiplying by b^k, we observe that $xb + ra^k b^k \in I$, therefore

$$sb^{N+k} \in I \Rightarrow s \in (I : b^{N+k}) \Rightarrow sb^N \in I.$$

But this means that $z = y + sb^N \in I$, therefore $I = (I + (a)) \cap (I + (b^N))$.

By irreducibility, we must have $I = I + (a^k)$ or $I = I + (b^N)$ which means that $a^k \in I$ or $b^N \in I$. In both cases, $a \in r(I)$ or $b \in r(I)$, and hence $r(I)$ is prime. The second conclusion follows similarly with $k = 1$. ☐

Now given an algebraic set X, we can write it as a union of its irreducible components, does this translate to every ideal being written as an intersection of irreducible ideals?

Theorem 2. *Let A be a Noetherian ring and I any ideal of A, then I can be written as a finite intersection of irreducible ideals.*

Proof. Suppose there is an ideal I that cannot be written as a finite intersection of irreducible ideals. For the collection $\mathcal{S}$ of such ideals, this collection is well defined. By the Noetherian condition, it has maximal elements, let I be one such maximal element. Then I is not irreducible for then $I = I$ is a finite intersection of irreducible ideals, therefore, we can write $I = J_1 \cap J_2$ where J_1, J_2 are strictly larger than I. Because I is maximal in $\mathcal{S}$, both J_1, J_2 can be written as a finite intersection of irreducible ideals, which means that I can also be written as such which is a contradiction. Therefore, every ideal can be written as a finite intersection of irreducible ideals. ☐

Okay, so every ideal can be written as a finite intersection of irreducible ideals. Now we ask the following question. Suppose an ideal I in $k[x_1, \ldots, x_n]$ determines an irreducible algebraic set, then is I irreducible as defined above? Our definition of irreducible ideal was a direct translate of the geoemtric scenario, however, the answer to the question is negative. In fact, we have

$$I = (x^2, xy, y^2) = (x^2, y) \cap (x, y^2).$$

Here, the ideal on the left describes the origin of $\mathbb{A}^2$ which is clearly irreducible, however, the ideal is not irreducible. Observe that (x^2, xy, y^2) is not equal to (x^2, y) or (x, y^2). Moreover, these two ideals are irreducible , for suppose $J \supsetneq (x^2, y) = J_1$, pick an $f \in J \setminus J_1$ and write $f = f_d + f_{d-1} + \cdots + f_1 + f_0$ where f_i is the homogenous part of f of degree i. Because $I \subset J_1$ all homogenous parts of degree ≥ 2 are in I, therefore we conclude that there is a linear part $ax + by \in J \setminus J_1$. We must have $a \neq 0$ and it follows that $J = (x, y)$. Thus, J_1, and similarly J_2, are irreducible for there is only one ideal strictly larger than them, namely (x, y).

We conclude from this example that even though an ideal may describe an irreducible algebraic set, the ideal may not be irreducible.

Now, we go one step further. In the geoemtric side, we also have uniqueness, for suppose $X = X_1 \cup \cdots \cup X_m = Y_1 \cup \cdots \cup Y_n$ is written as the union of two finite collections of irreducible closed subsets and moreover, the unions are minimal in the sense that no smaller union gives X.

Then, given $1 \leq i \leq m$, we have

$$X_i = \cup_{1 \leq j \leq n} Y_j \cap X_i$$

so by irreducibility, we must have $X_i \subseteq Y_j$ for some $1 \leq j \leq n$. Similarly each Y_j is contained in some X_i. Now, suppose $X_1 \subseteq Y_1 \subseteq X_i$ for some i. By minimality, we must have $X_1 = X_i$ for otherwise $X_2 \cup \cdots \cup X_m = X$. Therefore, $X_1 = Y_1$. In other words, if $X_i \subseteq Y_j$, then $X_i = Y_j$. In this way, we see that $m = n$, and by reordering if necessary, $X_i = Y_i \, \forall \, 1 \leq i \leq m$.

However, we do not have such a uniqueness in the algebraic setting. Consider the same ideal I as above, take $J_1' = (x^2, x + y), J_2' = (y^2, y - x)$, then we claim that

$$I = (x^2, xy, y^2) = J_1' \cap J_2'.$$

- Intersection is indeed I : It is quite clear that $I \subseteq J_1' \cap J_2'$. Now, given $f \in J_1'$, say $f = f_1 x^2 + f_2(x + y)$. Now, we know that homogenous parts of degree ≥ 2 are in I, therefore we may write $f = f' + c(x + y)$ where $c \in k$ and $f' \in I$. Similarly, if $g \in J_2'$, then we may write $g = g' + d(y - x)$ where $d \in k, g' \in I$. Now if $f \in J_1' \cap J_2'$, then we need to look only at the linear part and it is easy to see that the linear part must be zero. Therefore, $I = J_1' \cap J_2'$.

- J_1', J_2' are irreducible : Suppose J is an ideal with $J_1' \subsetneq J$, then pick an $f \in J \setminus J_1'$. Again, we may remove the quadratic and higher terms leaving us with some $f' = ax + by + c \in J \setminus J_1'$. If $c \neq 0$, then $f'x = ax^2 + bxy + cx \in J \Rightarrow cx \in J$. It follows that $x, y \in J$, hence $J = (x, y)$. If $c = 0$, then $a \neq b$ for $f' \notin J_1'$, hence $(a - b)x \in J$ and again $J = (x, y)$. Therefore, (x, y) is the only ideal strictly larger than J_1', hence J_1' is irreducible. Similarly J_2' is irreducible.

- J_1, J_2, J_1', J_2' are all distinct : It is easy to see that each is properly contained in (x, y). If any two are equal, then one can show that they will be equal to (x, y) which is a contradiction.

We conclude the following:

1. Even if an ideal describes an irreducible algebraic set, it need not be irreducible.

2. Every ideal can be written as an intersection of irreducible ideals, however such a decomposition need not be unique.

If we were to think of (x^2, xy, y^2) as the intersection of lines in the plane, then we could say that the origin as multiplity 6 - 2 from two copies of y-axis, 2 from the union of x, y-axes and 2 from two copies of x-axis. Interestingly enough, in both decompositions above, the multiplicty has divided evenly. Although our notion of multiplicty is vague (how to extend it when the generators are non homogenous and describe more than points), we ask whether the number of irreducible ideals is unique. This is indeed the case, and the reader is referred to Exercise 7.19 in *Introduction to Commutative Algebra* by Atiyah and MacDonald.

Next observe that in the decompositions $I = J_1 \cap J_2 = J_1' \cap J_2'$, all ideals involved have the same radical (x, y). This also gives us another notion of multiplicity. If each irreducible ideal contributes a multiplicity of 1, then we could say that I contributes a multiplicity of 2 and the irreducible ideals are there to separate out the multiplicities.

One way to correct this is to take the intersection itself as an ideal rather than considering the irreducible ideals separately, i.e., when decomposing I above, we do get J_1, J_2, but we ignore them and simply consider I on its own for it describes an irreducible sets. In other words, we treat the intersection of irreducible ideals with the same radical as a single ideal without separating out the irreducible ideals.

3 Primary ideals

Definition 2. *Let A be a ring, and I an ideal. I is said to be primary if whenever $xy \in I$ for $x, y \in A$ either $x \in I$ or $y \in r(I)$.*

We have previously shown that irreducible ideals are primary. It is quite clear that if I is primary, then $r(I)$ is a prime ideal, say $\mathfrak{p}$, we then say I is $\mathfrak{p}$-primary.

Lemma 2. *If $\mathfrak{q}_1, \ldots, \mathfrak{q}_n$ are $\mathfrak{p}$-primary, then so is $\mathfrak{q} = \cap_{i=1}^n \mathfrak{q}_i$.*

Proof. Suppose $xy \in \mathfrak{q}$, then if $x \in \mathfrak{q}_i$ for every i, we have $x \in \mathfrak{q}$, else we know that $y \in \mathfrak{p}$. In this case, there are positive integers n_i such that $y^{n_i} \in \mathfrak{q}_i$. Taking $N = \max\{n_i\}$, we have $y^N \in \mathfrak{q}_i$ for all i, hence $y^N \in \mathfrak{q}$. Therefore $\mathfrak{q}$ is a primary ideal. Because $r(\mathfrak{q}) = \cap r(\mathfrak{q}_i)$, $\mathfrak{q}$ is $\mathfrak{p}$-primary. □

The converse is also true:

Lemma 3. *Suppose $I_1, \ldots, I_n$ are ideals with prime radicals and suppose $I = I_1 \cap \cdots \cap I_n$ is $\mathfrak{p}$-primary. Furthermore, suppose the collection $\{I_1, \ldots, I_n\}$ is minimal in the sense that no subcollection intersects to give I. Then each $r(I_i) = \mathfrak{p}$.*

Proof. Let $\mathfrak{p}_i = r(I)$, then $\mathfrak{p} = \cap \mathfrak{p}_i$, hence at least one $\mathfrak{p}_i = \mathfrak{p}$. Suppose $\mathfrak{p}_1 \neq \mathfrak{p}$. By minimality, choose $x \in I_2 \cap \cdots \cap I_n \setminus I_1$ and $y \in \mathfrak{p}_1 \setminus \mathfrak{p}$. Then some $y^N \in I_1$, hence $xy^N \in I$. Now, however, $x \notin I$ for $x \notin I_1$ and $y \notin r(I) = \mathfrak{p}$. This is a contradiction because I was assumed to be primary. □

Now we continue with our previous discussion. The introduction of primary ideals helps to fix some of the problems we had with decomposition into irreducible ideals. First of all, by grouping together irreducible ideals with the same radical, we prevent the separation of multiplicities (keep in mind that this is not a rigorous defintion, the notion of multiplicity is, at the moment, only to aid intuition). Therefore, in our earlier examples, we can group $(x^2, y) \cap (x, y^2)$ as one primary ideal.

However, this is not enough, for example we have $(x^2, xy) = (x) \cap (x^2, y)$. Here $(x), (x^2, y)$ are irreducible with different prime radicals (so we cannot take their intersection as one ideal) and (x^2, xy) describes an irreducible set, namely the y-axis. However, all is not lost, for in the zero set of (x^2, xy), the origin has a different multiplicity from the rest of the points and this is the reason why even though it describes an irreducible set, it is neither irreducible nor primary. This example also shows that in Lemma 3, it is important that I be primary.

Now we proceed to describe another decomposition of ideals. Suppose I is an ideal in a Noetherian ring A, then we may write $I = I_1 \cap \cdots \cap I_n$ where the I_i are irreducible. Next, we group together the irreducibles with the same radical to write $I = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_m$ where the $\mathfrak{q}_i$ are primary and $r(\mathfrak{q}_i) \neq r(\mathfrak{q}_j)$. Moreover, we may suppose that the collection $\{\mathfrak{q}_1, \ldots, \mathfrak{q}_k\}$ is minimal. Such a decompostion is called a minimal primary decomposition.

We ask the same questions that we had for decomposition into irreducible ideals.

- Is every ideal in a Noetherian ring decomposible into primary ideals? Yes, every ideal is decomposable into irreducible ideals and they are primary.
- Are ideals that describe irreducible sets primary? No, as we have seen (x^2, xy) is not primary but describes an irreducible set. Although, the origin is different from other points.
- Is the primary decomposition unique?

Before we answer the last question we study primary ideals.

Lemma 4. *Let A be a ring and I an ideal, if $r(I)$ is maximal, then I is primary.*

Proof. Suppose $xy \in I$, then if $y \in r(I) = \mathfrak{m}$, we are done, so suppose $y \notin \mathfrak{m}$. Then $x \in \mathfrak{m}$ and there is some $r \in A, z \in \mathfrak{m}$ such that $ry + z = 1$ because $\mathfrak{m}$ is maximal. Suppose $z^t \in I$, then by taking the tth power, we get an $s \in A$ such that $sy + z^t = 1$. Multiplying by x, we have $s(xy) + z^t x = x \in I$. Thus I is $\mathfrak{m}$-primary. □

Lemma 5. *Let $\mathfrak{q}$ be $\mathfrak{p}$-primary, then for $x \in A$, we have*

- *If $x \in \mathfrak{q}$, then $(\mathfrak{q} \colon x) = 1$.*
- *If $x \notin \mathfrak{q}$, then $(\mathfrak{q} \colon x)$ is $\mathfrak{p}$-primary.*
- *If $x \notin \mathfrak{p}$, then $(\mathfrak{q} \colon x) = \mathfrak{q}$.*

Proof. Given $x \in A$, if $x \in \mathfrak{q}$, then it is obvious that $(\mathfrak{q} \colon x) = A$. Similarly, if $x \notin \mathfrak{p}$, then $yx \in \mathfrak{q}$ if and only if $y \in \mathfrak{q}$.

Lastly, suppose $x \notin \mathfrak{q}$. If $xy \in \mathfrak{q}$, then we must have $y \in \mathfrak{p}$, therefore $\mathfrak{q} \subseteq (\mathfrak{q} \colon x) \subseteq \mathfrak{p}$ and taking radicals we get $r(\mathfrak{q} \colon x) = \mathfrak{p}$. Next, if $yz \in (\mathfrak{q} \colon x)$ and $z \notin \mathfrak{p}$, then $(xy)z \in \mathfrak{q}$ hence $xy \in \mathfrak{q}$ which means $y \in (\mathfrak{q} \colon x)$, therefore $(\mathfrak{q} \colon x)$ is $\mathfrak{p}$-primary. □

The previous lemma gives us, in some sense, indicator functions of primary ideals and their radicals. Recall how in the $k[x]$ case, the irreducible and primary ideals were found using quotient ideals.

Definition 3. *Let A be a ring. An ideal I is said to be decomposable if it can be written as a finite intersection of primary ideals, $I = \cap \mathfrak{q}_i$. Such a decomposition is said to be minimal if the radicals $r(\mathfrak{q}_i)$ are distinct prime ideals and if no smaller intersection gives I.*

Note that using the lemma above, we can always arrive at a minimal decomposition. In a Noetherian ring, as we have seen above, all ideals are decomposable, however in a general ring this is not guaranteed.

Theorem 3. (*First uniqueness theorem*) *Let A be a ring and I a decomposable ideal with minimal decomposition $I = \cap_{i=1}^n \mathfrak{q}_i$. Let $\mathfrak{p}_i = r(\mathfrak{q}_i), 1 \leq i \leq n$, then $\mathfrak{p}_i$ are precisely the prime ideals appearing in the collection $r((I \colon x))$, hence are independent of the minimal primary decomposition.*

Proof. First observe that given $x \in A$, we have $r((I \colon x)) = \cap_{i=1}^n r((\mathfrak{q}_i \colon x))$. Therefore, if $r((I \colon x))$, then it must be equal to some $r(\mathfrak{q}_i \colon x)$ (which is always either $\mathfrak{p}_i$ or A). Conversely, given $1 \leq i \leq n$, we choose, by minimality, $x \in (\cap_{j \neq i} \mathfrak{q}_j) \setminus \mathfrak{q}_i$, then $r((I \colon x)) = r((\mathfrak{q}_i \colon x)) = \mathfrak{p}_i$. □

This theorem supports our intuition. We wanted the primary ideals to describe the irreducible components of the algebraic set defined by I. We know from the geometric picture that the irreducible components should not depend on which decomposition we use, and this show precisely the same. The radicals of the primary ideals, which are prime, describe the same sets that the primary ideal describes and we now have proof that these primes (hence, the irredcible components) do not depend on the decomposition chosen.

Suppose we have a minimal decomposition $I = \cap \mathfrak{q}_i$, then the primes $\mathfrak{p}_i = r(\mathfrak{q}_i)$ are said to *belong to I* or be *associated with I*. Among those primes, the minimal ones are said to *minimal* or *isolated* primes, while the others are said to be *embedded*. In the geometric picture, the minimal primes describe the irreducible components of $Z(I)$ while the embedded primes are going to be subsets of the irreducible components, hence embedded in them.

In some sense, the embedded primes account for varying multiplicities of points in the irreducible components. If we go back to a previous example, we had $(x^2, xy) = (x) \cap (x^2, y)$. Note that both ideals on the right side are primary and irreducible. The associated primes are (x) and (x, y) which describe the y-axis and the origin respectively. The prime (x) is minimal and describes the irreducible component, while (x, y) is embedded and describes the origin which has a different multiplicity (again, we haven't formalized this intuition) from the rest of the y-axis.

We also observe that even though the primes are uniquely determined, the primary ideals need not be, for example $(x^2, xy) = (x) \cap (x^2, y) = (x) \cap (x, y)^2$. However, it turns out that in the minimal primary decomposition $I = \cap_{i=1}^{n} \mathfrak{q}_i$, the primary ideals whose radicals are minimal primes are uniquely determined. For a proof of this we refer the reader to *Introduction to Commutative Algebra* by Atiah and MacDonald.

This shows that even though the embedded primes may arrive from different primary ideals, the minimal primes arise from fixed primary ideals, in the geometric picture, this means that the irreducible components are determined by unique primary ideals while the other subvarieties with different multiplicities need not be. In some sense this says that the subvariety of higher multiplicity, the origin in the example above, may obtain its multiplicity in different ways.

4 Conclusion

Given an ideal I in $k[x_1, \dots, x_n]$ we sought to find the irreducible components of the set described by I. In order to translate the geometric picture, we tried to write $I = I_1 \cap \dots \cap I_m$ where I_i describe the different irreeducible components of $Z(I)$. However, things are not as simple in the algebraic picture, because we need to take care of multiplicities. On way to think of this is that while in the geometric picture all points and surfaces have a uniform thickness, when we view them from the point of view of ideals, the multiplcities transform into varying thickness of the varieties.

We first defined irreducible ideals by directly transfering the notion of irreducible sets from geometry. In a Noetherian ring every ideal can be decomposed into irreducible ideals, but this decomposition is not fully satisfactory. In some sense, it separates out the multiplcities. To correct this, we took the intersection of irreducible ideals with same radical as one ideal, this introduces the notion of primary ideals.

Minimal primary decomposition is a little better behaved than decomposition into irreducible ideals because it bunches up different multiplicities together and points exactly which subvarieties have higher multiplcities (these are given by the embedded primes). Of course, even though our intial motivation comes from algebraic geometry, the notion of irreducible ideals and primary ideals extend to arbitrary commutative rings with unity.

Primary ideals are similar to powers of primes in the integers, however they are not always powers of primes. We have the following notions - ideals with prime radicals, prime powers, irreducible ideals and primary ideals.

- (x^2, xy) has prime radical but is neither a prime power, nor irreducible nor primary.
- (x^2, y) is irreducible and primary, but not a prime power for $(x, y)^2 \subsetneq (x^2, y) \subsetneq (x, y)$.
- (x^2, xy, y^2) is primary and a power of a prime but not irreducible.
- In $k[x, y, z]/(xy - z^2)$, the ideal $\mathfrak{p} = (\bar{x}, \bar{z})$ is prime (where the $^-$ denotes the image in the quotient), $\mathfrak{p}^2$ is a prime power but not primary for $\bar{x}\bar{y} = \bar{z}^2 \in \mathfrak{p}^2$ but $\bar{x} \notin \mathfrak{p}^2$ and $y \notin \mathfrak{p} = r(\mathfrak{p}^2)$.
- In a general ring, prime powers and primary ideals have prime radicals. In a Noetherian ring, irreducible ideals have prime radicals and are primary.

5 Further thoughts

Reader beware, this section is not meant to be rigorous. We look at the primary decomposition from a geometric point of view. Since primary ideals have prime radicals and since an ideal and its radical define the same algebraic set, the primary components of an ideal correspond to the irreducible components of its algebraic set.

We observe that when decomposing an ideal that is radical, the primary ideals we obtain are in fact prime. Indeed, say $\mathfrak{a} = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_n$ is a minimal primary decomposition where $\mathfrak{a}$ is radical and $\mathfrak{q}_i$ are $\mathfrak{p}_i$-primary. Upon taking the radicals, we get $\mathfrak{a} = \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_n$. We can restrict this intersection to the intersection of isolated primes, say $\mathfrak{p}_1, \ldots, \mathfrak{p}_m$ are isolated and the others are embedded.

We have $\mathfrak{a} = \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_m$ and it is easy to see that $\mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_m = \mathfrak{a}$. From minimality, it follows that $\mathfrak{a}$ doesn't have any embedded primes and because the isolated primary components are uniquely determined we must have $\mathfrak{q}_i = \mathfrak{p}_i$. Intuitively, from a geometric point of view, this makes sense because radical ideals are given by intersection of primes (and we have finitely many primes under the assumption of decomposability).

When studying variety as a topological space, we would like to know continuous functions. My intuition is to try to carry forward notions of differential geometry to varieties. The initial focus is on getting continuous maps from a variety X over a field k to k. We focus on these functions because when considering affine varieties X, Y any map $X \to Y$ would have to be specified by the coordinates.

If general point $(x_1, \ldots, x_n) \in X$ maps to a point $(f_1, \ldots, f_m)$ where f_i are functions $X \to k$, then we would obviously require f_i to satisfy whatever polynomial equations determine Y. And since these f_i are functions of $x_1, \ldots, x_n$, it seems natural to require f_i to themselves be polynomials, in fact we can have f_i to be rational polynomials. By the definition of Zariski topology, it is easy to see that polynomial functions and rational functions (restricted to X from $\mathbb{A}^n$) are continuous.

However, note that not all continuous functions $X \to k$ need to be polynomial. For example, consider the function $\mathbb{R} \to \mathbb{R}$ that is identity except sending 1 to 0. This map is continuous under Zariski topology but not given by any polynomial map (not even locally).

So, we look at rational functions $X \to k$. Now which functions can be inverted depend on X, anything not in the vanishing ideal of X can be inverted. What is happening is that we have a family of functions $\mathbb{A}^n \to k$ and we are first passing from X to $\mathbb{A}^n$ and then to $\mathbb{A}^m$ and then restricting the image to Y. A priori we do not have a way to go from X to Y without invoking the fact that these spaces are really subspaces of the much nicer family of affine planes.

While functions not in the vanishing ideal of X can be inverted on all of X, a general function f can be inverted outside $X \cap Z(f)$ which is an open subset of X. Not all points in a variety are identical, for example the origin in $Z(xy)$ is markedly different from other points. This difference comes from the multiplicities. The ideal (xy) has a decomposition $(x) \cap (y)$ and the origin is in the zero sets of both these prime ideals.

The functions that can be inverted at a point p of X is captured by what points don't vanish at p. If $\mathfrak{a}$ is the vanishing ideal of X (it is radical) and has a decomposition $\mathfrak{a} = \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_n$ and p is in the irreducible sets determined by $\mathfrak{p}_1, \ldots, \mathfrak{p}_m$ but not in the others, then we can invert functions outside $\mathfrak{p}_1 \cup \cdots \cup \mathfrak{p}_m$ at p.

So, it seems natural to look at the collection of rational functions defined in a neighbourhood of p. Also, the number of primes in which p appears (m in the notation above) might be a meaningful notion of the multiplicity of p in X for p appears in m irreducible components. This m is unique by the uniqueness theorems unlike embedded primes (which don't exist for radical ideals).