Homotopy groups and HELP

Shrivathsa Pandelu

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1 Preliminaries

Denote by e_n the disc in $\mathbb{R}^n$ and $S^n = \partial e_{n+1}$. Observe that $CS^n = e_{n+1}$ where CS^n is the unreduced cone over S^n.
A CW complex is a topological space X with a filtration of subsets $X^0 \subseteq X^1 \subseteq \dots$ whose union is X such that

- X^0 is a discrete space, i.e., a collection of 0-cells
- X^n is obtained by X^{n-1} by attaching n-cells: i.e., there's an indexing set (with discrete topology) J with *characteristic maps* $\phi_\alpha \colon \partial e_n \to X^{n-1}, \alpha \in J$ and

$$X^n = X^{n-1} \sqcup (J \times e_n)/((\alpha, x) \sim \phi_\alpha(x))$$

  i.e., X^n is a pushout.
- X has topology coherent with the X_ns: a subset U is open iff $U \cap X_n$ is open in X^n (subspace topology) for every n

Definition. *Let X be a CW complex, a subspace A is called a subcomplex if it is a closed subset and given as the union of cells of X. If A is a subcomplex, then the image of the cells are contained in A, hence A is a CW complex. The pair (X, A) shall be called a CW pair.*

Guided by minimalism, I now state a few facts about CW complexes whose proofs can be found in the Appendix of [4].

Theorem 1.1.
- *CW complexes are locally compact Hausdorff and compactly generated.*
- *Compact subsets of a CW complex X are contained in finite subcomplexes.*
- *A CW pair (X, A) has the homotopy extension property.*
- *CW complexes are locally contractible.*

Theorem 1.2. *If $A \hookrightarrow X$ has the homotopy extension and A is contractible, then $X \simeq X/A$.*

Proof. There is a nullhomotopy of the inclusion $A \times I \to A \hookrightarrow X$. By homotopy extension we get a map $h \colon X \times I \to X$ such that the subspace $A \times I$ lands in A. Since I is locally compact Hausdorff, if we pass to the quotient on the right, we actually have an induced map $\bar{h} \colon X/A \times I \to X/A$.
On the one hand, we have the quotient map $X \to X/A$. For the other direction, because the top slice of h sends A to a point, we have an induced map $X/A \to X$. The composites are top slices of $h, \bar{h}$, hence homotopic to the corresponding identity maps. □

Remark. If A had a basepoint and the nullhomotopy preserved this basepoint, then it is easy to see that the homotopy equivalence is an equivalence of based spaces as well.

As a corollary, since the reduced suspension is obtained from the unreduced suspension by collapsing $\{x_0\} \times I$, for a CW complex X, $SX \simeq \Sigma X \cong X \wedge S^1$.

2 Groups and cogroups

Given pointed spaces X, Y denote by $[X, Y]_*$ the set of homotopy classes of maps $X \to Y$ preserving base points such that the homotopies also pass through base point preserving maps (this is suggesting that we should be looking at reduced cylinders when talking about basepoint preserving homotopies). Given X, Y, we have functors:

$$[X, -]_*\colon \mathsf{Top}_* \to \mathsf{Set} \qquad [-, Y]_*\colon \mathsf{Top}_* \to \mathsf{Set}.$$

Our first goal is to turn these into functors to Grp. It turns out that this is possible if and only if X, Y have certain additional structure. When X is exponentiable, $[X, -]$ is actually a functor to Top. More generally, since some arguments need these function sets to be topological spaces, , it is often convenient to work in a "convenient category" which happens to be Cartesian closed.

2.1 H-groups

Fix a space W and suppose $[-, W]_*$ is a group functor. For a space X, denote by μ_X the multiplication on $[X, W]_*$. Let $p_1, p_2\colon W^2 = W \times W \to W$ be the projections, let $\mu = \mu_{W^2}(p_1, p_2)$. Given $f, g \in [W, W]_*$, using functoriality and the map $h = f \times g\colon W \to W^2$ we have,

$$\begin{tikzcd}
(f, g) \arrow[r, "\mu_W"] & \mu_W(f, g) = \mu \circ (f \times g) \\
(p_1, p_2) \arrow[u, "h^* \times h^*"] \arrow[r, "\mu_{W^2}"'] & \mu \arrow[u, "h^*"']
\end{tikzcd}$$

If $*$ denotes the identity of $[W, W]_*$, we get

$$\mu \circ (id_W \times *) \simeq id_W \simeq \mu \circ (* \times id_W)$$

By abuse of notation, let p_1, p_2, p_3 denote the three projections $W^3 \to W$. Using associativity on $[W^3, W]_*$,

$$\mu_X(p_1, \mu_X(p_2, p_3)) = \mu_X(\mu_X(p_1, p_2), p_3)$$

where $X = W^3$.
By functoriality $\mu_X(p_1, p_2) \simeq \mu \circ (p_1 \times p_2), \mu_X(p_2, p_3) \simeq \mu \circ (p_2 \times p_3)$. Using these and applying functoriality on the morphism $\mu \times id\colon W^3 \to W^2$ (and $id \times \mu$ as well)

$$\mu \circ (\mu \times id) \simeq \mu_X(\mu_X(p_1, p_2), p_3) \simeq \mu_X(p_1, \mu_X(p_2, p_3)) \simeq \mu \circ (id \times \mu).$$

Lastly, we have the inverse map on the group $[W, W]_*$, let i be the inverse of id. Again using functoriality, we get

$$\mu \circ (id \times i) \simeq \mu_W(id, i) \simeq * \simeq \mu_W(i, id) \simeq \mu \circ (i \times id)$$

With the assumption that $*$ is actually the constant map to the basepoint, these properties define an H-**group** (group up to homotopy). If conversely, W has maps $\mu, i, *$ ($*$ is usually taken to be the constant map and we shall stick to this) satisfying these relations, then we can define a multiplication on $[X, W]_*$ by $\mu_X(f, g) = \mu \circ (f \times g)$ and inverse as $i_X(f) = i \circ f$. These make $[X, W]_*$ into a group up to homotopy) and this is functorial in nature because given an $F\colon X \to Y$, the morphism $[Y, W]_* \to [X, W]_*$ is precomposing with F while the group operations are compositions on the left and composition is associative (that composition of morphisms is associative is a requirement for categories). If W is an abelian H-group, then our group $[X, W]_*$ is also abelian.
There are variations on whether we want our homotopies to be basepoint preserving or not, or whether the identities have to be strict (as opposed to being an identity up to homotopy). However, when our H-space in question is a CW complex, these notions are equivalent, [4].

2.2 H-cogroups

We follow the same train of thought for $[W,-]_*$ to be a group functor. Given based (i.e., base point preserving) maps $f,g\colon W\to X$, we need to get another map $W\to X$ that works as a product of f,g. Using f,g we get a map from two copies $W\sqcup W$ to X. However, to stay in the category of pointed spaces we glue the two copies at the base point, resulting in the wedge $W\vee W$ and a map $f\vee g\colon W\vee W\to X$ (which exists and is continuous by the universal property of quotients). To get a map from W to X function as the product of f,g, it suffices to obtain a map, called the co-multiplication, $W\to W\vee W$[1].
Let $q_1,q_2\colon W\to W\vee W$ be the two copies of W in the wedge sum and $\pi_1,\pi_2\colon W\vee W\to W$ be the two quotients (so that $\pi_1\circ q_1=id, \pi_1\circ q_2=*$ et c). Let π_X denote the multiplication on $[W,X]_*$ and let ν denote the product $\pi_{W\vee W}(q_1,q_2)$. Using the functoriality of p_1 (and likewise p_2) we get for maps $f,g\colon W\to W$

$$(f\vee g)\circ\nu\simeq\pi_W(f,g).$$

Next, with $X=\vee^3W$, the associativity of multiplication gives $\pi_X(\pi_X(q_1,q_2),q_3)\simeq\pi_X(q_1,\pi_X(q_2,q_3))$ (where, as before, q_i refers to inclusion of W in the ith copy). Using functoriality for $q_1\vee q_2\colon W\vee W\to X$ we get $\pi_X(q_1,q_2)\simeq(q_1\vee q_2)\circ\nu$ and similarly $\pi_X(q_2,q_3)\simeq(q_2\vee q_3)\circ\nu$.
Finally using functoriality for $id\vee\nu,\nu\vee id$ we get co-associativity:

$$(id\vee\nu)\circ\nu\simeq\pi_X(q_1,(q_2\vee q_3)\circ\nu)\simeq\pi_X((q_1\vee q_2)\circ\nu,q_3)\simeq(\nu\vee id)\circ\nu.$$

Finally, we come to the co-inverse. Let i be the inverse of id in $[W,W]_*$. Using functoriality and such on $id\vee i, i\vee id$ we get

$$(id\vee i)\circ\nu\simeq *\simeq(i\vee id)\circ\nu.$$

where $*$ is the identity of the group $[W,W]_*$. Typically, $*$ is the constant map and this is what we will stick to. These properties are what make W an H-**co-group** and we can work backwards as well: if W has an H-co-group structure, then $[W,-]_*$ is a group functor. If W is an abelian H-cogroup, then $[W,X]_*$ will be abelian.
From the universal property of pushouts, using $q_2\circ\pi_1, q_1\circ\pi_2$ we obtain the twist morphism

$$\text{twist}\colon W\vee W\to W\vee W$$

which interchanges the two copies of W in the wedge sum.

Theorem 2.1. *(Hilton-Eckmann argument)*[2] *Let W be a space with two (H-)cogroup structures $\nu_1,i_1,*_1$ and $\nu_2,i_2,*_2$ being the comultiplication, coinverse and counits respectively. If*

$$(\nu_2\vee\nu_2)\circ\nu_1=(1\vee twist\vee 1)\circ(\nu_1\vee\nu_1)\circ\nu_2$$

then the two structures are the same and are abelian.

Proof. We have

$$\begin{aligned}
*_1=(*_1\vee *_1)\circ\nu_1 &= (id\vee id)\circ((*_2\vee *_1)\vee(*_1\vee *_2))\circ(\nu_2\vee\nu_2)\circ\nu_1\\
&=(*_2\vee *_1\vee *_1\vee *_2)\circ(1\vee\text{twist}\vee 1)\circ(\nu_1\vee\nu_1)\circ\nu_2\\
&=(*_2\vee *_1\vee *_1\vee *_2)\circ(\nu_1\vee\nu_1)\circ\nu_2\\
&=(id\vee id)\circ((*_2\vee *_1)\vee(*_1\vee *_2))\circ(\nu_1\vee\nu_1)\circ\nu_2\\
&=(id\vee id)((*_2)\vee(*_2))\circ\nu_2\\
&=(*_2\vee *_2)\circ\nu_2\\
&=*_2
\end{aligned}$$

[1]The situation here is dual to the previous situation. Here we use the co-product of two spaces as opposed to the product. Co-products exist in this category because one can define the wedge of two pointed spaces.

[2]This is the dual version of the Eckmann-Hilton argument; name not standard.

where, hopefully, the parentheses indicated whether our codomain was a wedge $W \vee W$ or just W. For the rest of the argument we shall actually need to use the fact that W is a space, $*$ is a constant map to the basepoint. With this knowledge, we have the following huge commutative diagram

$$\begin{tikzcd}
W \vee W \arrow[r, "\nu_2 \vee \nu_2"] & W \vee W \vee W \vee W \arrow[d, "1\vee * \vee * \vee 1"] & \\
W \arrow[u, "\nu_1"] \arrow[d, "\nu_2"'] & W \vee W \vee W \vee W \arrow[r, "(1\vee 1)\vee(1\vee 1)"] & W \vee W \\
W \vee W \arrow[r, "\nu_1 \vee \nu_1"'] & W \vee W \vee W \vee W \arrow[u, "1\vee * \vee * \vee 1"'] &
\end{tikzcd}$$

We require that $*$ is the constant map to the basepoint and we have wedges (rather than disjoint unions). Without this fact, the diagram doesn't commute because the middle two factors are swapped. When $*$ is the constant map, the middle two factors both are mapped to the basepoint, so it doesn't matter whether we twist them or not.
From above we get ν_1 and from below we get ν_2, therefore, $\nu_1 = \nu_2 = \nu$.
Although associativity was assumed, we can prove associativity from the following monstrosity where the pentagon and triangle commute, therefore the left path and right path are the same (and form the condition for associativity).

$$\begin{tikzcd}
 & W \arrow[dl, "\nu"'] \arrow[dr, "\nu"] & \\
W \vee W \arrow[d, "\nu \vee \nu"'] & & W \vee W \arrow[d, "\nu \vee \nu"] \\
W \vee W \vee W \vee W \arrow[rr, no head, "1\vee \mathrm{twist} \vee 1"] \arrow[dr, "(1\vee *)\vee 1 \vee 1"'] & & W \vee W \vee W \vee W \arrow[dl, "1\vee 1 \vee (* \vee 1)"] \\
 & W \vee W \vee W &
\end{tikzcd}$$

again, it is important that $*$ is indeed the constant map to the basepoint for these diagrams to work. Following $(\nu \vee \nu) \circ \nu$ with $(* \vee 1) \vee (1 \vee *)$ gives ν. But placing a mid-twist in between (which shouldn't change the result) gives twist $\circ\, \nu$ and this again requires the observation that the counit is a constant map to the basepoint (to be further pedantic, we need to see the composition as factoring through a suitable subspace, figure things out with the codomain being said subspace and then go back to seeing it in the whole space).
Using associativity, it is now possible to show that $i_1 = i_2$ and we leave this as a short exercise to the reader. □

Theorem 2.2. *(Eckmann-Hilton argument) Let G be a set with two group structures with multiplications $*, \circ$ such that $(a*b)\circ(c*d) = (a\circ c)*(b\circ d)$. Then the two group structures are the same and are commutative.*

Proof. If e_1, e_2 are the identities, then

$$e_1 = (e_1 \circ e_2) * (e_2 \circ e_1) = e_2 \circ e_2 = e_2$$

so the identities are the same, say 1.

$$a * b = (a \circ 1) * (1 \circ b) = a \circ b \forall a, b$$

so the multiplication is the same and we shall denote it by concatenation. We have associativity

$$(ab)c = (ab)(1c) = (a1)(bc) = a(bc)$$

and from associativity, we can conclude that the inverses have to be the same. Commutativity follows because $ab = (1a)(b1) = (1b)(a1) = ba$. □

What makes groups easier is that we can deal with actual elements as opposed to functions and compositions in the case of cogroups. What makes cogroups harder is that comultiplication results in a bigger space whereas multiplication results in a smaller space. However, both arguments above proceed in the same way. In fact the proof for cogroups was based on the proof for groups. However, I have never seen the dual version mentioned anywhere. Probably because a cogroup has a different definition in the algebraic context. The reason I wanted to provide the proof above is to directly prove that there is an abelian cogroup structure on $S^n, n \geq 2$ (as we shall see below) and as a result the higher homotopy groups are abelian. The standard proof is to provide two group structures on the homotopy groups and then use an Eckmann-Hilton argument.

3 Homotopy groups

Given a space Y, define $\Omega Y = C((S^1, 1), (Y, *_Y))$, which is the space of all (based) loops with the compact-open topology.

Theorem 3.1. *If W has an H-cogroup structure, then so does $X \wedge W$ for any exponentiable X.*

Proof. Let $\nu\colon W \to W \vee W$ be the comultiplication, $1\colon W \to W$ the co-unit and $i\colon W \to W$ the co-inverse. We obtain $id \wedge 1, id \wedge i\colon X \wedge W \to X \wedge W$ and $id \wedge \nu\colon X \wedge W \to X \wedge (W \vee W)$. When X is exponentiable, $X \wedge (W \vee W) \cong (X \wedge W) \vee (X \wedge W)$.
Now it is easy to verify that we have an H-cogroup structure. Note that if W is abelian, then so is $X \wedge W$. $\square$

Theorem 3.2. *S^n has an H-cogroup structure which is abelian for $n \geq 2$.*

Proof. S^1 has an H-cogroup structure with comultiplication sending the upper semicircle to the first copy in $S^1 \vee S^1$ and the lower to the second and co-unit being the constant map and co-inverse the complex conjugation. That these maps give S^1 an H-cogroup structure is an exercise for the reader, but can be found in any of the references listed.
Since $S^n = SS^{n-1} = \Sigma S^{n-1} = S^{n-1} \wedge S^1$, all spheres have an H-cogroup structure. Finally, we note that if S^2 has an abelian cogroup structure, then the same is true for any $n \geq 2$ because $S^n = S^{n-2} \wedge S^1 \wedge S^1$ as the smash product is associative (when we have nice spaces).
There are two cogroup structures ν_1, ν_2 on S^2 coming from the two copies of S^1. We have

$$(\nu_2 \vee \nu_2) \circ \nu_1(x,t) = \begin{cases} (2x, 2t) & 0 \leq x, t \leq 1/2 \\ (2x, 2t-1) & 0 \leq x \leq 1/2, 1/2 \leq t \leq 1 \\ (2x-1, 2t) & 1/2 \leq x \leq 1, 0 \leq t \leq 1/2 \\ (2x-1, 2t-1) & 1/2 \leq x, t \leq 1 \end{cases}$$

and

$$(1 \vee \text{twist} \vee 1) \circ (\nu_1 \vee \nu_1) \circ \nu_2 \simeq (\nu_2 \vee \nu_2) \circ \nu_1$$

(in fact we have an equality) where we use the coordinates on S^2 coming as a quotient of I^2. Using the (dual) Eckmann-Hilton argument we are done. $\square$

Now, the suspension-loop space adjunction says that Σ, Ω are adjoints of each other and there is a homeomorphism $C(\Sigma, X, Y) \to C(X, \Omega Y)$ which is given by currying the suspension coordinate. This descends to a homeomorphism of based maps and since I is locally compact Hausdorff, this currying sends homotopic maps to homotopic maps, giving us a bijection

$$[\Sigma X, Y]_* \cong [X, \Omega Y]_*$$

Since S^1 is an H-cogroup, there is an induced group structure on both of these. It is easy to verify that the group structures are the same, therefore this bijection is in fact a group isomorphism.

Definition. *For a space X, the nth homotopy group is $\pi_n(X) = [S^n, X]_*$. Because S^n has an H-co-group structure via quotienting out the equator, $\pi_n(X)$ is a group.*

In the case of homotopy groups and relative homotopy groups defined below, one needs to fix a base point. However, this choice doesn't matter for path connected spaces and the proof of this fact can be found in any of the books mentioned in the references. But the key thing is that the inclusion of a point in S^n is a cofibration (Exercise for the reader: either see S^n as the quotient of a cube and project the cube $S^n \times I$ to the faces; or see S^n as being made of two discs glued at the equator and progressively collapse one of the discs to an interval), so paths of the point can be lifted to homotopies of the sphere whose ends describe elements of the nth homotopy group based at the ends of the path. As a corollary, if two (path connected) spaces are homotopic (without regard to basepoints) then their homotopy groups (which depend on the path components) are isomorphic.

3.1 Long exact sequences associated to (co)fibre sequences

If $A \to B \to C$ is a cofibre sequence, then $[C,Z]_* \to [B,C]_* \to [A,C]_*$ will be an exact (where the kernel is the inverse of the basepoint) sequence of pointed sets/groups/abelian groups (depending on the structure on Z). In the same way, if $F \to E \to B$ is a fibre sequence, then $[Z,F] \to [Z,E] \to [Z,B]$ will be similarly exact.

It is easy to verify that if $A \simeq B$ then the mapping spaces to and from Z are isomorphic as sets/groups/abelian groups. Therefore, given the fibre/cofibre sequences ($C_r f$ being the reduced mapping cone)

$$X \xrightarrow{f} Y \to C_r f \to \Sigma X \xrightarrow{\Sigma f} \Sigma Y \to \Sigma C_r f \to \dots$$

and

$$\cdots \to \Omega F \to \Omega E \to \Omega B \to F \to E \to B$$

we have the induced long exact sequences (of sets/groups/abelian groups as appropriate)

$$\cdots \to [\Sigma Y, Z] \to [\Sigma X, Z] \to [C_r f, Z] \to [Y,Z] \to [X,Z]$$

and

$$\cdots \to [Z,\Omega E] \to [Z,\Omega B] \to [Z,F] \to [Z,E] \to [Z,B].$$

Note here that in the sequences above, any two consecutive maps form a fibre/cofibre sequence up to a homotopy equivalence on the middle term. This lets us prove the exactness at each term.

In particular, when $Z = S^0$, we have a long exact sequence of the homotopy groups by using the suspension-loop space adjunction.

What happens if we start with a fibration $F \hookrightarrow E \xrightarrow{p} B$ where F is the fibre over the basepoint of B? Assume that B is path connected so we don't have to worry about basepoints (needed to define the fibre; under fibration all fibres of a path component are homotopy equivalent). If we follow the program above, we get a long exact sequence of the homotopy groups of Fp, E, B where Fp is the homotopy fibre for p. It is clear that $Fp \simeq F$ via the inclusion $\phi\colon F \to Fp$ (recall that Fp consists of pairs (x,l) such that l is a path in B starting from $p(x)$ and ending at the (fixed) basepoint $* \in B$), and in the long exact sequence from Fp, E, B, if we replace the homotopy groups of Fp by those of F, then the maps from $\pi_n(F) \to \pi_n(E)$ are the ones induced by the inclusion (because $F \to Fp \to E$ is the inclusion map).

When B is path connected, we can lift a path from B to E to show that F intersects every path component of E making $\pi_0(F) \to \pi_0(E)$ surjective.

Next we shall define relative homotopy groups and then we will see that the relative homotopy group $\pi_n(E,F) \cong \pi_n(B)$.

3.2 Relative homotopy groups

Throughout maths, there are techniques of finding out properties of a big space by finding it out for a small space and what's left when removing the small space. Given a subspace A, there are two ways to "remove" it from X, one is to take the complement, the other is to quotient. In the category of based spaces, it is appropriate to quotient.

Ideally we would like to write $\pi_n(X)$ as a sum of $\pi_n(A)$ and some other group which is obtained by "quotienting" out $\pi_n(A)$. This is achieved by making looking at the nullhomotopies of elements of $\pi_n(A)$ as seen in X, and such nullhomotopies would be elements of the next homotopy group.
Following Hatcher's notation, define the relative homotopy groups as follows. Let the coordinates on I^n be $(s_1, \ldots, s_n)$ and see I^{n-1} as the face with $s_n = 0$ and let J^{n-1} be the closure of $\partial I^n - I^{n-1}$. The nth relative homotopy group $\pi_n(X, A)$ consists of homotopy classes of maps $(I^n, I^{n-1}, J^{n-1}) \to (X, A, x_0)$ where the homotopies pass through maps of the same kind.
Remark. $\pi_n(X, x_0)$ is the relative homotopy group of the pair (X, x_0).
Remark. $\pi_1(X, A, x_0)$ consists of homotopy classes of paths in X with one end being x_0 and the other being a point in A and doesn't have a group structure.
The relative homotopy groups for $n \geq 2$ have a group structure obtained by the same formulas but now the coordinate s_n plays a special role. The Eckmann-Hilton argument applies in the same way to give an abelian structure for $n \geq 3$. This construction is functorial on the category of pointed pairs of spaces and the argument is the same as for the usual homotopy groups.

Lemma 3.1. *(Compression criterion) A map $g\colon (I^n, I^{n-1}, J^{n-1}) \to (X, A, x_0)$ is zero in $\pi_n(X, A, x_0)$ iff it is homotopic relative to I^{n-1} to a map whose image lies in A.*

Proof. A null homotopy is a map $I^{n+1} \to X$ such that one face F_1 is g, the opposite face F_2 is the constant map and the bottom face F_3 lands in A. There is a projection of F_1 to the union $F_2 \cup F_3$. This gives a homotopy relative to the base of F_1 of g to a map that lands in A.
Conversely, if g is homotopic, relative to I^{n-1}, to a map landing in A, then we may assume that g itself lands in A. Now by shrinking along the "last" coordinate (i.e., the base of I^n lands in A and the top is the constant map, so bring the base to the top), we obtain a null homotopy (in A itself). □

Restricting an element $(I^n, I^{n-1}, J^{n-1}) \to (X, A, x_0)$ to I^{n-1} gives us an element of $\pi_{n-1}(A)$. Secondly, any element of $\pi_n(X)$ is an element of $\pi_n(X, A, x_0)$ via the inclusion $(X, x_0, x_0) \hookrightarrow (X, A, x_0)$. Thus, we have the sequence

$$\cdots \to \pi_n(A, x_0) \xrightarrow{i_*} \pi_n(X, x_0) \xrightarrow{j_*} \pi_n(X, A, x_0) \xrightarrow{\partial} \pi_{n-1}(A, x_0) \to \cdots \to \pi_0(X, x_0)$$

We can be a bit more general by considering triples of spaces $x_0 \in B \subseteq A \subseteq X$ and having the same maps to get the sequence:

$$\cdots \to \pi_n(A, B, x_0) \xrightarrow{i_*} \pi_n(X, B, x_0) \xrightarrow{j_*} \pi_n(X, A, x_0) \xrightarrow{\partial} \pi_{n-1}(A, B, x_0) \to \cdots \to \pi_1(X, A, x_0)$$

with additional terms for π_0 when $B = x_0$.

Theorem 3.3. *This sequence is exact.*

Proof. The proof is quite simple and involves the Compression Lemma 3.1. Refer [4] Theorem 4.3 for a proof. □

This long exact sequence is also natural in the sense that a map between pointed triples $(X, A, B, x_0) \to (Y, C, D, y_0)$, induces a map between the associated long exact sequences, with commuting squares.
There is a "coordinate free" description of the relative homotopy group as follows. Given a subspace $i\colon A \hookrightarrow X$, the homotopy fibre Fi of the inclusion consists of paths in X starting in A and ending in the basepoint of X. Denote $\pi_n(X, A) = \pi_{n-1}(Fi)$ (note again that π_0 is not defined for the pair) then we have the associated long exact sequence

$$\ldots \pi_{n+1}(X, A) \to \pi_n(A) \to \pi_n(X) \to \pi_n(X, A) \to \ldots \pi_0(A) \to \pi_0(X)$$

Currying things (stacking the maps for the second coordinate of Fi sends us one degree higher) essentially gives the same definition of the relative group as above and the boundary map is the same in both cases (from $Fi \to A$ is is the projection onto the first coordinate, i.e., an element of a homotopy group of smaller index).

Finally, suppose we start with a fibre sequence $F \to E \to B$, then we have two long exact sequences: one for the pair (E,F) and another for the fibre sequence. Since the fibre goes to the basepoint, we have a map of pairs $(E,F) \to (B,b_0)$ inducing a map between their homotopy groups. Together with the identity maps, we have a chain map between the two long exact sequences (verify that it is indeed a chain map) and by the five lemma we get an isomorphism $\pi_n(E,F) \cong \pi_n(B)$. For $n = 1$, path lifting property gives the isomorphism. When B is connected, path lifting also gives a surjection $\pi_0(F) \to \pi_0(E)$.

Remark. The notion of relative homotopy groups here, from [4], is different from that in [1]. The difference being that in [1] $J^{n-1} = \partial I^{n-1} \times I \cup I^{n-1}$, i.e., everything except the top face. We will change to this notation later on, because we will be using [1] rather than [4].
This change only changes j, j_*, ∂ a little bit, but everything else carries through as is.

4 HELP

This section is entirely based[3] on [1] and I only intend to expand on what was concise.

Definition. *A map $e\colon Y \to Z$ is an n-equivalence if for all $y \in Y, e_*\colon \pi_q(Y,y_0) \to \pi_q(Z,e(y))$ is an injection for $q < n$ and a surjection for $q \leq n$; e is said to be a weak equivalence if it is an n-equivalence for all n, i.e., if it is an isomorphism on the homotopy groups.*

A weak equivalence means that the homotopy group functors see Y,Z to be the same object. A homotopy equivalence is a weak equivalence.
We will denote by $C(X)$ the unreduced cone of a space X and $C_r(X)$ shall denote the reduced cone.

Lemma 4.1. *[The technical lemma] The following conditions on a map $e\colon Y \to Z$ are equivalent.*

i) For any $y \in Y, e_*\colon \pi_q(Y,y) \to \pi_q(Z,e(y))$ is an injection for $q = n$ and a surjection for $q = n+1$.
ii) Given maps $f: CS^n \to Z, g\colon S^n \to Y$ and $h\colon S^n \times I \to Z$ such that $f|_{S^n} = h \circ i_0$ and $e \circ g = h \circ i_1$ in the following diagram, there are maps $\tilde{g}, \tilde{h}$ that make the diagram commute.

$$\begin{tikzcd}
S^n \arrow[rr, "i_0"] \arrow[dd] & & S^n \times I \arrow[dl, "h"'] \arrow[dd] & & S^n \arrow[ll, "i_1"'] \arrow[dl, "g"'] \arrow[dd] \\
& Z & & Y \arrow[ll, "e"'] & \\
CS^n \arrow[ur, "f"] \arrow[rr, "i_0"'] & & CS^n \times I \arrow[ul, dashed, "\tilde{h}"] & & CS^n \arrow[ll, "i_1"] \arrow[ul, dashed, "\tilde{g}"]
\end{tikzcd}$$

iii) The conclusion of (ii) holds when $f|_{S^n} = e \circ g$ and h is the constant homotopy at this map.

Here the vertical arrows are inclusion of S^n as the base of the unreduced cone, i_0, i_1 denote the inclusions into the $0,1$ slices of the cylinder respectively.

Proof. Trivially ii) implies iii). Assume iii). When $n = 0$, under based homotopy groups we always have an injection. Without fixing basepoints we need to show that different path components of Y map to different path components in Z. To this end, $S^0 = \{\pm 1\}$ has two points and the unreduced cone is the interval. Given y, y' such that $e(y), e(y')$ are in the same path component, taking f to be a path between $e(y), e(y')$ gives us a path $\tilde{g}$ in Y from y to y'.
When $e \circ g$ is nullhomotopic, the nullhomotopy is of the form $f: CS^n \to Z$ making the hypothesis of iii) hold. Which means there are maps $\tilde{g}, \tilde{h}$ and $\tilde{g}$ is the required null homotopy making e_* injective on π_n.

[3] In these notes the definition of the homotopy fibre is slightly different, so the proofs are the same up to a change of orientation.

An element of $\pi_{n+1}(Z, e(y))$ can be seen as a map $f: CS^n \to Z$ which is constant on the boundary sphere. So taking g to be the constant map and f to be the given element, we obtain a map $\tilde{g}$ which is an element of $\pi_{n+1}(Y, y)$ and $\tilde{h}$ is a homotopy between $f, e \circ \tilde{g}$ making e_* surjective on π_{n+1}.
Now assuming i) we prove ii). The question is about the existence of certain homotopies. We could argue this using the assumptions on e_*, if basepoints were not an issue. The problem is that our maps in the diagram above don't have to fix basepoints, however the homotopy h and the map f essentially give us paths in a fibre.
Let Fe denote the homotopy fibre of e consisting of pairs (y, p) with $y \in Y, p$ a path in Z from $e(y)$ to a basepoint. To figure out which point to use as the basepoint, we pick a basepoint $* \in S^n$ and $y_1 = g(*), z_1 = e(y_1)$.
Let $\bullet \in CS^n$ be the cone point, set $z_0 = f(*, 0), z_{-1} = f(\bullet)$. Let $w = (y_1, *_{z_1})$ denote the basepoint of Fe. In the exact sequece associated to the fibre sequence, we have the segment

$$\cdots \to \pi_{n+1}(Y, y_1) \to \pi_{n+1}(Z, z_1) \to \pi_n(Fe, w) \to \pi_n(Y, y_1) \to \pi_n(Z, z_1) \to \ldots$$

from which we conclude that $\pi_n(Fe, w) = 0$. Nullhomotopy of an element of $\pi_n(Fe)$ will consist of a nullhomotopy of some element of $\pi_n(Y)$ (giving us $\tilde{g}$) and the second coordinate can be curried to give us a map $\tilde{h}$. Elements of $\pi_n(Fe)$ look like n-spheres in Y together with an n-disc in Z whose boundary sphere is the image (under e) of the sphere in Y.
So, our task is to find an appropriate element of $\pi_n(Fe, w)$. Define $k_0: S^n \to Fe$ by

$$k_0(x) = (g(x), \bar{h}_x f_x \bar{f}_* h_*)$$

where the path composition is to be read left to right, h_x denotes the path $h(x, \cdot)$ and f_x the path $f(x, \cdot)$ for $x \in S^n$. Recall that CS^n is $S^n \times I$ with the 1 slice collapsed to the cone point.
The first coordinate is a map to Y and is continuous, the second coordinate is a map to $\overline{P}Z$ (the space of paths ending at the basepoint) and is seen to be continuous by looking at the adjoint, so k_0 is continuous by the properties of pullbacks. However, k_0 is not a based map.
However, observe that there is a path from $k_0(*)$ to the basepoint of Fe (obtained by "shrinking" the second coordinate) and since $* \hookrightarrow S^n$ is a cofibration, there is a homotopy of k_0 into a based map. Concatenating homotopies, k_0 is nullhomotopic in the unbased sense.
Let $k: S^n \times I \to Fe$ be the homotopy from k_0 to the constant map. Projecting to the first coordinate gives us a map $\tilde{g}: S^n \times I \to Y$ which is a homotopes g to the constant map, therefore factors through $\tilde{g}: CS^n \to Y$.
Project to the second coordinate, then take the adjoint to get a map $j: S^n \times I \times I \to Z$ such that

$$\begin{aligned}
j(x, s, 0) &= e(\tilde{g}(x, s)) \\
j(x, s, 1) &= z_1 \\
j(x, 1, t) &= z_1 \\
j(x, 0, t) &= \bar{h}_x f_x \bar{f}_* h_*(t)
\end{aligned}$$

$$\begin{tikzcd}
\cdot \arrow[r, "z_1"] & \cdot \\
\cdot \arrow[u, "\bar{h}_x f_x \bar{f}_* h_*"] \arrow[r, "e\circ\tilde{g}(x,\cdot)"'] & \cdot \arrow[u, "z_1"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\cdot \arrow[r, "e\circ\tilde{g}(x,\cdot)"] & \cdot \\
\cdot \arrow[u, "h_x"] \arrow[r, "f_x"'] & \cdot \arrow[u, "f_* \bar{h}_*"']
\end{tikzcd}$$

For each x we have the square on the left (the arrows indicate the direction of the paths going from 0 to 1) where the horizontal direction is the cone coordinate, vertical is the t coordinate. Sliding things around, we can reparametrize it (this happens on $S^n \times I \times I$ and is a homeomorphism) to get the square on the right. Keep in mind the convention that path products are read left to right (unlike function compositions; and the convention in [1] is different). Since the right edge is the same no matter what the x is, we can pass to a quotient (since the t coordinate I is nice enough we can quotient to $CS^n \times I$) and get $\tilde{h}$ as required. □

Theorem 4.1. *(Homotopy Extension and Lifting Property) Let (X,A) be a relative CW complex of dimension $\leq n$ and let $e\colon Y \to Z$ be an n-equivalence. Then given maps $f: X \to Z, g\colon A \to Y, h\colon A \times I \to Z$ such that $f|_A = h \circ i_0$ and $e \circ g = h \circ i_1$ in the following diagram, there are maps $\tilde{g}, \tilde{h}$ that make the entire diagram commute.*

$$\begin{tikzcd}
A \arrow[rr, "i_0"] \arrow[dd] & & A \times I \arrow[dl, "h"'] \arrow[dd] & & A \arrow[ll, "i_1"'] \arrow[dl, "g"'] \arrow[dd] \\
& Z & & Y \arrow[ll, "e"] & \\
X \arrow[rr, "i_0"'] \arrow[ur, "f"] & & X \times I \arrow[ul, dashed, "\tilde{h}"] & & X \arrow[ll, "i_1"] \arrow[ul, dashed, "\tilde{g}"]
\end{tikzcd}$$

Here a relative CW complex is defined almost the same way as a CW complex via attaching cells, except we start with A and a bunch of 0-cells as opposed to starting with only 0-cells.

Proof. Observe that given $A \subseteq B \subseteq X$, if the result is true when going from A to B and from B to X, then the result is true when going from A to X (restrict f to B and then proceed). So we induct on the skeleta.
The first step is to extend from A to the zero skeleton consisting of A and a bunch of points. The images of a point $p \in X_0 \setminus A$ is determined by the path components that they land in under f and $\tilde{h}$ will be the corresponding paths.
Now it suffices to assume that in going from A to X we attach a family of n cells. For each we have an n sphere and corresponding maps from the sphere and cone (i.e., disc) by passing through A, X using the attaching maps.
For each disc we obtain the corresponding $\tilde{g}, \tilde{h}$ (use the Axiom of Choice here; for each disc we have a set of functions to choose from) and these maps, using g, h, pass through the pushout which is $X, X \times I$ (pushouts and products work nicely because I is nice). By induction we are done. □

Theorem 4.2. *(Whitehead) If X is a CW complex and $e\colon Y \to Z$ is an n-equivalence, then $e_*\colon [X,Y] \to [X,Z]$ is a bijection if $dim X < n$ and a surjection if $dim X = n$.*

Proof. For surjectivity, given $f\colon X \to Z$ we want a $g\colon X \to Y$ such that $f \simeq e \circ g$. We get this by applying HELP to the pair $(X, \emptyset)$.
For injectivity, given $f_1, f_2\colon X \to Y$ if $e \circ f_1 \simeq e \circ f_2$ we want them to be homotopic in Y. Two maps from X to Y is a single map from $X \times \partial I \to Y$. Applying HELP to $(X \times I, X \times \partial I)$ (it hasn't been said in this document, but a product of CW complexes is a CW complex, see [1], [4]) proves injectivity where we take h to be the constant homotopy and f the homotopy between $e \circ f_1, e \circ f_2$ ($\tilde{g}$ gives the homotopy between f_1, f_2). □

Corollary 4.1. *An n-equivalence between CW complexes of dimension less than n is a homotopy equivalence. A weak equivalence between CW complexes is a homotopy equivalence.*

Proof. Let $e\colon Y \to Z$ be a map satisfying either hypothesis. Since $e_*\colon [Z,Y] \to [Z,Z]$ is a bijection we get $f: Z \to Y$ such that $e \circ f \simeq id_Z$. Therefore, $e \circ f \circ e \simeq e$ and since $e_*\colon [Y,Y] \to [Y,Z]$ is a bijection, $f \circ e \simeq id_Y$. □

5 Approximations

Definition. *A space X is said to be n-connected if $\pi_q(X,x) = 0$ for $0 \leq q \leq n$ and all x. A pair (X,A) is said to be n-connected if $\pi_0(A) \to \pi_0(X)$ is surjective and $\pi_q(X,A,a) = 0$ for $1 \leq q \leq n$ and all a, i.e., $A \hookrightarrow X$ is an n-equivalence (by the long exact sequence).*

Lemma 5.1. *A relative CW complex (X,A) with no m-cells for $m \leq n$ is n-connected. In particular (X, X^n) is n-connected for any CW complex X.*

Proof. Let $f\colon (I^q, \partial I^q, J^q) \to (X, A, a)$ be a map, where $q \le n$. The image lies in a compact subset and our goal is to homotope it into a map with image in A. Since compact subsets are contained in a subcomplex with finitely many cells attached to A, if we homotope f inside this subcomplex, it is as good as homotoping it in X. So we may assume that (X, A) has finitely many cells. We can further reduce to the case where $X = A \sqcup_j D^r, r > n$ where j is an attaching map, i.e., X is obtained by attaching a single cell. By induction, this proves the general case.
There is an approximation $f'\colon I^q \to X$ such that $f' = f$ on $\partial I^q, f' \simeq f$ rel ∂I^q and f' misses a point p in the interior of D^r. One can use simplicial approximation giving D^r an appropriate simplicial structure and observing that a simplex of dimension $\le q$ cannot cover an r-simplex, or by using smooth approximations by choosing appropriate neighbourhoods (something like Sard's theorem, Whitney approximations).
Once we have such an f', then by deforming $X \setminus \{p\}$ to A, we deform f' to a map into A. $\square$

Theorem 5.1. *(Cellular approximation) Any map $f\colon (X, A) \to (Y, B)$ between relative CW complexes is homotopic relative to A to a cellular map.*

Proof. Any point in Y is connected by a path to a point in Y^0 (because discs are connected and there are paths to the boundary of discs). Therefore, taking paths of images of points in $X^0 \setminus A$ we obtain a homotopy of $f|_{X^0}$ into Y^0. To proceed with the induction, we will need to use HELP in some form. Suppose we are given $g_n\colon X^n \to Y^n, h_n\colon X^n \times I \to Y$ such that $h_n\colon f|_{X^n} \simeq \iota_n \circ g_n$ where $\iota_n\colon Y^n \to Y$ is the inclusion. Let $j\colon S^n \to X^n$ be an attaching map for the cell $\tilde{j}\colon D^{n+1} \to X$. We have the following diagram (with $g_n \circ j$ being composed with the inclusion $Y^n \to Y^{n+1}$)

$$\begin{tikzcd}
S^n \arrow[rr] \arrow[dd] & & S^n \times I \arrow[dl, "h_n\circ(j\times id)"'] \arrow[dd] & & S^n \arrow[ll] \arrow[dl, "g_n\circ j"'] \arrow[dd] \\
& Y & & Y^{n+1} \arrow[ll] & \\
D^{n+1} \arrow[rr] \arrow[ur, "f\circ\tilde{j}"] & & D^{n+1}\times I \arrow[ul, dashed, "h_{n+1}"'] & & D^{n+1} \arrow[ll] \arrow[ul, dashed, "g_{n+1}"']
\end{tikzcd}$$

HELP applies because, by the previous lemma, ι_{n+1} is an $(n+1)$-equivalence. The map h_{n+1} is the new attaching map and h_{n+1} is the required homotopy. $\square$

Remark. When A, B are themselves CW complexes, something slightly more general can be said. Suppose given f we already had a homotopy $H\colon A \times I \to B$ that homotoped f to a cellular map. Then, the cellular approximation above can be constructed to be an extension of H. The proof works exactly as above with level-by-level homotopies being H on A rather than the constant homotopy of f.

Corollary 5.1. *For CW complexes X, Y any map $X \to Y$ is homotopic to a cellular map, and any two homotopic cellular maps are cellularly homotopic.*

Note that product with I results in a CW complex and applying the cellular approximation to a homotopy gives us a cellular homotopy. These approximation results let us to move to the category of CW complexes with cellular maps as opposed to continuous maps.

Theorem 5.2. *(Approximation by CW complexes) For any space X, there is a CW complex ΓX and a weak equivalence $\gamma\colon \Gamma X \to X$. For a map $f\colon X \to Y$ and another such CW approximation $\gamma\colon \Gamma Y \to Y$, there is a map $\Gamma f\colon \Gamma X \to \Gamma Y$, unique up to homotopy, such that $\gamma \circ \Gamma f = f \circ \gamma$.*
If X is n-connected, $n \ge 1$, then ΓX can be chosen to have a unique vertex and no q-cells for $1 \le q \le n$.

The reader is referred to [1] for a proof of this theorem. Once the approximations have been constructed, the existence and uniqueness (up to homotopy) of Γf follows from Whitehead's theorem, via the bijection $\gamma_*\colon [\Gamma X, \Gamma Y] \to [\Gamma X, Y]$. The construction proceeds as follows: start with one sphere S^q for each generator of $\pi_q(X)$ to obtain a CW complex X_1 and a map $X_1 \to X$ which is surjective on all homotopy groups.

Next we cancel out the relations in π_1 and obtain X_2. Cancelling out relations is the same as attaching discs (the null homotopies) to X_1 resulting in a CW complex X_2 and a map $X_2 \to X$ which is again surjective on all homotopy groups but is also bijective on π_1.
We repeat this construction and take ΓX to be the limit of these X_is and γ to be the map obtained from the universal properties of limits. Note that in going from X_1 to X_2, the fundamental group doesn't change because we are attaching cells of higher dimension (a previous lemma). The homotopy groups are the limits of the homotopy groups of X_is and these are "stable".
If X is n-connected, then the construction outlined above doesn't use any q-cells for $q \leq n$.
Using the same techniques as in the construction above (see [1] for more details), given a pair (X, A) and a CW approximation $\Gamma A \to A$, it is possible to construct an approximation $\Gamma X \to X$ which contains ΓA as a subcomplex. By HELP, this construction is "functorial" up to homotopy.

Definition. *An excisive triad $(X; A, B)$ is a space X with subspaces A, B such that X is the union of the interiors of A, B. A CW triad $(X; A, B)$ is a CW complex X with subcomplexes A, B such that $X = A \cup B$.*

Triads are useful in establishing Mayer-Vietoris type results and an approximation of triads by CW complexes would make certain arguments cleaner (cellular maps have more combinatorial rigidity compared to arbitrary maps).

Theorem 5.3. *[CW approximation of triads] Let $(X; A, B)$ be an excisive triad and let $C = A \cap B$. Then there is a CW triad $(\Gamma X; \Gamma A, \Gamma B)$ and a map of triads*

$$\gamma\colon (\Gamma X; \Gamma A, \Gamma B) \to (X; A, B)$$

such that, with $\Gamma C = \Gamma A \cap \Gamma B$, the maps $\gamma\colon \Gamma C \to C, \Gamma A \to A, \Gamma B \to B, \Gamma X \to X$ are all weak equivalences. If (A, C) is n-connected, then $(\Gamma A, \Gamma C)$ can be chosen to have no q-cells for $q \leq n$. Similarly for (B, C). Up to homotopy, CW approximation of excisive triads is functorial.

Proof. Start with a CW approximation $\gamma\colon \Gamma C \to C$ and extend it to approximate the pairs (A, C) and (B, C). Define ΓX to be the pushout $\Gamma A \sqcup_{\Gamma C} \Gamma B$ (keep in mind that the pushout of CW complexes is a CW complex: the disjoint union is, and the quotient of a CW complex by a *subcomplex* is also CW). It's clear that $(\Gamma X; \Gamma A, \Gamma B)$ is a CW triad and that $\Gamma C = \Gamma A \cap \Gamma B$.
From the universal property of pushouts, there is a map $\gamma\colon \Gamma X \to X$. What remains is showing that this is a weak equivalence. This is established below. □

Definition. *Suppose $i\colon C \to A, j\colon C \to B$ are two maps. The double mapping cylinder $M(i, j)$ is defined as $A \cup (C \times I) \cup B/((c, 0) \sim i(c); (c, 1) \sim j(c))$, i.e., it is the cylinder with A, B glued on the ends.*

Quotienting out the cylinder gives a map $q\colon M(i, j) \to A \sqcup_C B$ (it's continuity follows from the properties of pushouts seeing the mapping cylinder as a certain pushout).
Note that taking $B = C$ and j to be the identity map gives us the usual mapping cylinder. Furthermore, if $i \simeq i'\colon C \to A, j \simeq j'\colon C \to B$, then using an almost similar argument as in the case of the cylinder (this time partitioning $C \times I$ into at least 5 parts), we see that $M(i, j) \simeq M(i', j')$.

Lemma 5.2. *With notation as in the definition above, if $i\colon C \to A$ is a cofibration, then q is a homotopy equivalence.*

Proof. The following proof is considerably less concise than the one in [1], but it's ok. We first record some observations. Denote by $Cyl(i)$ the mapping cylinder of i.

- When i is a cofibration, there is a map $r\colon A \times I \to Cyl(i)$. Let $r_1\colon A \to Cyl(i)$ denote the restriction to $A \times \{1\}$.
- $M(i, j)$ is the pushout $Cyl(i) \sqcup_C B$ where $C \to Cyl(i)$ is the inclusion into the 1-slice.
- $(A \sqcup_C B) \times I$ is the pushout of $i \times id_I, j \times id_I$ because I is exponentiable.

We have the following diagram

$$\begin{tikzcd}
C \arrow[r, "j"] \arrow[d, "i"'] & B \arrow[d] \arrow[ddr, bend left] & \\
A \arrow[r] \arrow[drr, bend right, "r_1"'] & A \sqcup_C B \arrow[dr, dashed, "f"] & \\
& & M(i,j)
\end{tikzcd}$$

where f exists by the universal property of pushouts and r is composed with the inclusion $Cyl(i) \to M(i,j)$. We claim that this is the homotopy inverse of q.
The homotopy $(A \sqcup_C B) \times I \to A \sqcup_C B$ is the dotted arrow in

$$\begin{tikzcd}
C \times I \arrow[r, "j \times id_I"] \arrow[d, "i \times id_I"'] & B \times I \arrow[d] \arrow[ddr, bend left, "\{(b,t) \mapsto b\}"] & \\
A \times I \arrow[r] \arrow[d, "r"'] & (A \sqcup_C B) \times I \arrow[dr, dashed] & \\
Cyl(i) \arrow[r] & M(i,j) \arrow[r, "q"] & A \sqcup_C B
\end{tikzcd}$$

The other homotopy is the dotted arrow in

$$\begin{tikzcd}
C \times I \arrow[r, "j \times id_I"] \arrow[d, "\{(c,t) \mapsto (c,1,t)\}"'] & B \times I \arrow[d] \arrow[ddr, bend left, "\{(b,t) \mapsto [b,1]\}"] & \\
Cyl(i) \times I \arrow[r] \arrow[drr, bend right, "g"'] & M(i,j) \times I \arrow[dr, dashed, "h"] & \\
& & M(i,j)
\end{tikzcd}$$

where g is the following pushout map

$$\begin{tikzcd}
C \times \{0\} \times I \arrow[r, hook] \arrow[d, "\{(c,0,t) \mapsto (i(c),t)\}"'] & C \times I \times I \arrow[d] \arrow[ddr, bend left, "\{(c,s,t) \mapsto (i(c), \max(s,t))\}"] & \\
A \times I \arrow[r] \arrow[drr, bend right, "r"'] & Cyl(i) \times I \arrow[dr, dashed, "g"] & \\
& & M(i,j)
\end{tikzcd}$$

when the t-coordinate is $0, g(\alpha, 0)$ is the inclusion into $M(i,j)$ and so is the map on $B \times \{0\}$, and therefore, $h(\cdot, 0)$ is the identity map. When the t-coordinate is $1, g(\alpha, 1)$ collapses the cylinder of C and acts as r_1 on A, making $h(\cdot, 1)$ the composition $f \circ q$. $\square$

Theorem 5.4. *If $e\colon (X; A, B) \to (X'; A', B')$ is a map of excisive triads such that e is a weak equivalence on $C \to C', A \to A', B \to B'$ where $C = A \cap B, C' = A' \cap B'$, then $e\colon X \to X'$ is a weak equivalence.*

It doesn't have to be mentioned again, but the following proof is from [1], but I have added some notes.

Proof. We shall show that the dotted arrow exists in the following diagram so as to have commutativity up to homotopy for the upper triangle and commutative on the lower triangle.

$$\begin{tikzcd} X \arrow[r, "e"] & X' \\ S^n \arrow[u, "g"] \arrow[r] & D^{n+1} \arrow[u, "f"'] \arrow[ul, dashed, "\tilde{g}"'] \end{tikzcd}$$

If some element of $\pi_n(X)$ given by g is nullhomotopic via f, then it "descends" to X via $\tilde{g}$. Every element of $\pi_{n+1}(X')$ can be seen as a map f which is constant on the boundary and we can take g to be the constant map and the resulting $\tilde{g}$ shows surjectivity on π_{n+1}. This was one of the equivalent conditions in the technical lemma 4.1. The idea is to extend on the intersection first and then to the other parts.

So, let us say we are given f, g. We first extend g to a neighbourhood of S^n, i.e., we will find a neighbourhood U and a function $\hat{g}\colon U \to X$ such that $f|_U = e \circ \hat{g}$. This can be done by a deformation $d\colon D^{n+1} \times I \to D^{n+1}$

$$d(x,t) = \begin{cases} 2x/(2-t) & |x| \le (2-t)/2 \\ x/|x| & |x| \ge (2-t)/2. \end{cases}$$

which enlarges the half disc to the whole disc and shrinks $U = \{x : |x| > 1/2|$ to the boundary. Now define $\hat{g} = g \circ d_1, f' = f \circ d_1$. If we can find the extension for f' then we're done because we're only seeking commutativity up to homotopy (on the upper triangle).

Now define

$$C_A = g^{-1}(X \setminus \mathrm{int}A) \cup \overline{f^{-1}(X' \setminus A')}$$
$$C_B = g^{-1}(X \setminus \mathrm{int}B) \cup \overline{f^{-1}(X' \setminus B')}$$

The purpose of these sets is to break D^{n+1} into pieces where we can obtain extensions by using weak equivalence of e on A, B, C etc. Define $\hat{C}_A, \hat{C}_B$ in a similar way with g being replaced by $\hat{g}$.

We will show $\hat{C}_A, \hat{C}_B$ are disjoint, hence C_A, C_B will be too. The only possible intersection, in view of excisive-ness and the fact that $\overline{f^{-1}(X' \setminus A')} \subseteq f^{-1}(X'\mathrm{int}A')$ is when there is a $v \in \hat{C}_A \cap \hat{C}_B$ such that

$$v \in \hat{g}^{-1}(X \setminus \mathrm{int}A) \cap \overline{f^{-1}(X' \setminus B')} \subseteq \hat{g}^{-1}(\mathrm{int}B) \cap \overline{f^{-1}(X' \setminus B')}.$$

Now, $\hat{g}^{-1}(\mathrm{int}B)$ is an open subset of D^{n+1} and the intersection on the right being nonempty implies there is a point

$$u \in \hat{g}^{-1}(\mathrm{int}B) \cap f^{-1}(X' \setminus B')$$

but this is a contradiction because $e \circ \hat{g} = f|_U$.

So, C_A, C_B are closed subsets of D^{n+1} with empty intersection, which means that their complements form a cover. We can subdivide D^{n+1} as a simplicial complex (see [2] Chapter 3), hence as a CW complex (because finite simplicial complexes are finite CW complexes), such that each cell is contained in one of the complements, i.e., no cell intersects both C_A, C_B.

Having obtained this subdivision of D^{n+1}, all that's left to do is to extend g. Let K_A be the union of all those cells σ such that

$$g(\sigma \cap S^n) \subseteq \mathrm{int}A \text{ and } f(\sigma) \subseteq \mathrm{int}A'$$

and define K_B similarly. Any cell not intersecting C_A is in K_A and similarly for B, so $D^{n+1} = K_A \cup K_B$. Now we extend g in three steps: first on $K_A \cap K_B$, then K_A and finally $K_A \cup K_B$. By HELP, we obtain $\bar{g}$ making the lower triangle commute:

$$\begin{tikzcd} A \cap B \arrow[r, "e"] & A' \cap B' \\ S^n \cap (K_A \cap K_B) \arrow[u, "g"] \arrow[r] & K_A \cap K_B \arrow[u, "f"'] \arrow[ul, dashed, "\bar{g}"'] \end{tikzcd}$$

together with a homotopy $\bar{h}\colon f \simeq e \circ \bar{g} \mathrm{rel} S^n \cap (K_A \cap K_B)$ on $K_A \cap K_B$.

For the next step, define $\bar{g}_A\colon K_A \cap (S^n \cup K_B) \to A$ to be g on $K_A \cap S^n$ and $\bar{g}$ on $K_A \cap K_B$. Since the lower triangle commutes above, by the pasting lemma, this map is continuous.

Since $\bar{h}$ is a homotopy relative to the common intersection and because $f = e \circ g$ on $K_A \cap S^n$ we can glue the identity homotopy with $\bar{h}$ to get a homotopy

$$\bar{h}\colon f|_{K_A \cap (S^n \cup K_B)} \simeq e \circ g_A \text{ rel } S^n \cap K_A.$$

Apply HELP to the pair $(K_A, K_A \cap (S^n \cup K_B))$ to get maps $\tilde{g}_A\colon K_A \to A$ and a homotopy $\tilde{h}_A\colon f|_{K_A} \simeq e \circ \tilde{g}_A$ on $K_A \times I$. Using K_B instead, obtain $\tilde{g}_B, \tilde{h}_B$. On the intersections, these agree with the extensions and homotopy on $K_A \cap K_B$ and therefore can be glued together to get a map $\tilde{g}\colon D^{n+1} \to X$ as required. □

Continuation of proof of 5.3. We were left to show that CW approximation of triads is a weak equivalence. Recall that earlier, given an excisive triad $(X; A, B)$ with $C = A \cap B$, we constructed a CW approximation ΓC of C and extended it to pairs $(A, C), (B, C)$ and took $\Gamma X = \Gamma A \sqcup_{\Gamma C} \Gamma B$. We already have weak equivalences for A, B, C and for various pairs. Weak equivalence for X follows from the above theorem. Above, we had an excisive triad on the left, but the same proof works for a CW triad with int$A$ being replaced by A (which will now be a closed subcomplex). The homotopy functoriality of this approximation follows similar to the homotopy functoriality for spaces and pairs. □

6 Homotopy Excision Theorem

6.1 Some results on Simplicial Complexes and Linearity

The contents of this subsection are mainly from Chapter 12 of [3]. Some preliminary results on simplicial complexes can be found in Spanier Chapter 3 ([2]) or Gray's book ([3]). The geometric realization of a simplicial complex X is denoted by $|X|$.

Definition. *A set $X \subseteq \mathbb{R}^n$ is said to have linear dimension $\leq k$ if it is contained in the union of finitely many affine k-planes. The linear dimension of the emptyset is defined to be -1. Following [3], we denote this by* $\operatorname{lindim} X$.

Lemma 6.1. *If $X \subseteq \mathbb{R}^n$ has linear dimension less than n, then it is nowhere dense.*

Proof. Suppose $X \subseteq A_1 \cup \cdots \cup A_s$, a union of affine k-planes for $k < n$. By taking closure, assume X is closed. Since affine k-planes are nowhere dense, any open $U \subseteq \mathbb{R}^n$ is not contained in A_1. Inductively, suppose $U \setminus (A_1 \cup \cdots \cup A_{i-1})$ is non empty, then, being open in $\mathbb{R}^n$, it is not contained in A_i. □

Lemma 6.2. *Let K be a compact simplicial complex and $f : |K| \to \mathbb{R}^n$ be a linear map. Then* $\dim K \geq \operatorname{lindim} f(|K|)$.

Proof. K has finitely many simplices and by linearity we only need to look at what happens to the vertices. The vertices of each simplex define an affine plane of dimension not more than the dimension of the corresponding simplex and the result follows. □

Lemma 6.3. *Let K be an m-dimensional simplicial complex and $f : |K| \to \mathbb{R}^n$ be linear. Then for every $\epsilon > 0$ there is a point $a \in B_\epsilon(0)$ such that $f^{-1}(a)$ is contained in a subcomplex of dimension $\leq m - n$ after a suitable subdivision.*

Proof. If $m < n$, use the previous lemmas. Assuming $m \geq n$, look at the $n-1$ skeleton K^{n-1}. Its image has linear dimension $< n$, so we may choose an a such that $a \notin f(|K^{n-1}|)$. Inductively we prove that there is a subdivision of the k-skeleta such that $f^{-1}(a) \cap |K^k|$ is in a subcomplex of dimension $\leq k - n$. This is true for $k = n-1$. Suppose it is true for $k-1$ and that we have subdivided K^{k-1}.
We subdivide each simplex of the k-skeleton (after having subdivided all the way up to the $k-1$-skeleton). Then the subdivided simplex will have a subcomplex of dimension $\leq k-n$ containing $f^{-1}(a)$ and the union of these subcomplexes over all the k-simplices finishes the proof.
Suppose σ is a k-simplex (this is a simplex of the original K, but the original boundary has been subdivided). By induction hypothesis, $f^{-1}(a) \cap \partial\sigma$ is contained in a subcomplex τ of $\partial\sigma$ of dimension $\leq k-1-n < k-1$. If there's no point in the interior mapping to a, leave σ unchanged, else pick a point α in the interior mapping to a.
By linearity, any other point of $f^{-1}(a) \cap \sigma$ is a line from α to a point in τ. Joining α to the vertices of σ and τ subdivides the simplex σ (τ has finitely many vertices as it is a compact simplicial complex since σ is compact), and since joining an extra vertex to τ increases the dimension by 1, we have completed the induction argument. □

If K is realized in $\mathbb{R}^N$ for some N, then the proof above works to show that $f^{-1}(a)$ has linear dimension $\leq m - n$ (no subdivisions needed).

The following is from Chapter 13 of [3]. Below we treat the cube I^q as a simplicial complex. D^n_r indicates an n-disc (closed) of radius r.

Lemma 6.4. *Let X be arbitrary and $f : I^q \to X \cup_\alpha D^n$. Then there are subcomplexes N, N' of I^q satisfying*

i) $f^{-1}(D^n_{3/4}) \subset N \subset \mathit{Int}\, N' \subset N' \subset f^{-1}(D^n_{7/8})$

ii) if $\sigma \in N', \operatorname{diam} \chi^{-1}(f(\sigma)) \leq 1/16$

Here χ is the characteristic map $D^n \to X \cup_\alpha D^n$. The numbers are artificial and don't mean much. I am quoting [3], but the point of this lemma is that we can obtain subcomplexes where we could do some local homotopy of maps.

Proof. On the interior of the disc, there is a metric coming from the usual metric on D^n. Cover $X \cup_\alpha D^n$ by open sets such that the open sets intersecting the 7/8-disc have closures in the interior of e^n and all the open sets inside e^n have diameter $< 1/16$. This can be done by taking the union of X with a collar neighbourhood of ∂D^n and covering the rest with neighbourhoods in the interior of the disc. The inverse of this open cover is an open cover of the cube and we can subdivide I^q so that each simplex is contained in some element of this cover.
If σ is a simplex after subdivision and meets $f^{-1}(D^n_{7/8})$, then it is mapped into e^n and its image has diameter $< 1/16$. Let N be the union of all closed simplices meeting $f^{-1}(D^n_{3/4})$ and N' the union of all closed simplices meeting N. $f(N)$ will be contained inside the disc of radius $3/4 + 1/16$ and $f(N')$ inside that of radius $3/4 + 2/16 = 7/8$. Note that N, N' are closed.
The only thing that remains to be seen is that $N \subset \mathrm{Int} N'$. Suppose x_i is a sequence converging to $x \in N$, then we want to show that the sequence is eventually in N'. If a simplex σ contains infinitely many x_is, then it also contains x, hence intersects N and is therefore contained in N'. Since there are only finitely many simplices in our subdivision, eventually the sequence has to be in N'. □

Remark. We may as well have started with a subdivision of I^q rather than the standard simplicial structure.

Lemma 6.5. *(Local Linearization) Given a map $f\colon I^q \to X \cup_\alpha D^n$, there is an open subset $U \subseteq I^q$ and a homotopy $h_t\colon \overline{U} \to D^n$ relative to the boundary ∂U such that*

i) $h_0 = f|_{\overline{U}}$
ii) there is a subcomplex $N \subset U$ such that $h_1|_N$ is linear
iii) Int $N \supset h_1^{-1}(D^n_{1/2})$

Proof. Choose N, N' according to the previous lemma. Under the map f, the vertices of N' are well inside the disc and we can extend it linearly to all of N' to get a map $g\colon N' \to D^n \subset X \cup_\alpha D^n$ (technically, it is in the image of D^n under the characteristic map). Let $U = \mathrm{Int}\, N'$, then $\overline{U} \subseteq N'$ and since $N \subset U, N \cap \partial N' = \emptyset$.
Choose a continuous bump function $\phi\colon N' \to I$ which is 1 on N and 0 on the boundary and define $h_t\colon \overline{U} \to D^n$ by

$$h_t(u) = ((1-t) + t(1-\phi(u)))f(u) + t\phi(u)g(u).$$

This is a convex combination of two points inside a convex set, hence this is well defined and continuous. On the boundary of U this is f and on N, h_1 is linear and given by g. Note that we are freely moving between the disc in the pushout and the disc in a vacuum.
If $\sigma = (v_0, \ldots, v_s)$ is a simplex of N, then $h_1(v_i) = f(v_i)$. Now, the diameter of a simplex is the diameter of its vertex set, therefore diam $h_1(\sigma)$ is bounded by the diameter of $f(\sigma)$ which is bounded by 1/16. For $x \in \sigma$ we have

$$|f(x) - h_1(x)| \leq |f(x) - f(v_0)| + |h_1(v_0) - h_1(x)| \leq 1/8.$$

So, for any x, if $h_1(x)$ is in the 1/2-disc, then $f(x)$ must be in the 5/8-disc. Since Int $f^{-1}(D^n_{3/4}) \subset$ Int N, we get $x \in N$. □

Note that the linearity is not in the sense of linear maps between simplicial complexes because the n-disc D^n doesn't apriori have a simplicial structure. Instead, it is linearity as a map from a simplicial complex to a convex subset of $\mathbb{R}^n$.
In the proof of the homotopy excision theorem, we will need to do something like the above for two disjoint discs simultaneously. This means, we would like to get two disjoint subcomplexes. Let $A = C \cup_\alpha D^m, B = C \cup_\beta D^n$ (where C is arbitrary, but can be taken to be a CW complex) and $X = A \cup_C B$.

Lemma 6.6. *Suppose $h\colon I^q \to X$ and there are complexes M, N of I^q with $h|_M \to D^m, h|_N \to D^n$ linear. Suppose $q \leq m + n - 2$ and the interiors of M, N contain the inverses of the corresponding half discs. Then there exist points $x \in D^m, y \in D^n$ such that if $\pi\colon I^{q-1} \times I \to I^{q-1}$ is the projection, then $K = \pi(h^{-1}(x)), L = \pi(h^{-1}(y))$ are disjoint.*

Once again, linearity here is not as maps between simplicial complexes, but rather that the map is determined by its values on the vertices and convexity of Euclidean discs.

Proof. Firstly, if $q < n$ ($q < m$), then the dimension of N (M) is less than n (m), hence its image in D^n (D^m) is nowhere dense by Lemma 6.1 and we may find a point y (x) whose preimage is empty and the lemma follows. So, assume $q > m, n$.
Applying Lemma 6.3 to $h|_N : N \to D^n$ we can find a point $y \in D^n_{1/2}$ such that $h|_N^{-1}(y)$ is contained in a subcomplex (after a subdivision of N, hence of I^q) of dimension $\leq q - n$. Because $y \in D^n_{1/2}, h^{-1}(y) = h|_N^{-1}(y)$, therefore $L = \pi(h^{-1}(y))$ must have linear dimension $\leq q - n$.
Therefore, as a subset of $I^q, L \times I$ has dimension at most $q - n + 1 \leq m - 1 < m$. In particular, $M \cap (L \times I)$ has a nowhere dense image in D^m. So, we find a $x \in D^m_{1/2}$ such that $x \notin h(M \cap (L \times I))$.
By hypothesis, $h^{-1}(x) \subseteq M$, therefore $h^{-1}(x) \cap (L \times I) = \emptyset$. It follows that L, K are disjoint. $\square$

6.2 Homotopy Excision Theorem

Definition. *A map $f : (A, C) \to (X, B)$ of pairs is an n-equivalence, $n \geq 1$, if*

$$(f_*)^{-1}(im(\pi_0(B) \to \pi_0(X))) = im(\pi_0(C) \to \pi_0(A))$$

and for all choices of basepoints in C, $f_* : \pi_q(A, C) \to \pi_q(X, B)$ is a bijection for $q < n$ and a surjection for $q = n$.

The first condition says that the path components of C as seen in A are mapped to the path components of B as seen in X in a bijective fashion.

Theorem 6.1. *(Homotopy excision). Let $(X; A, B)$ be an excisive triad such that $C = A \cap B$ is non-empty. Assume (A, C) is $(m-1)$-connected and (B, C) is $(n-1)$-connected, where $m \geq 2, n \geq 1$. Then the inclusion $(A, C) \to (X, B)$ is an $(m + n - 2)$-equivalence.*

Recall from the Compression Lemma 3.1 that for a triple $X \supseteq A \supseteq B$ and any basepoint in B, the following sequence is exact:

$$\cdots \to \pi_q(A, B) \xrightarrow{i_*} \pi_q(X, B) \xrightarrow{j_*} \pi_q(X, A) \xrightarrow{k_* \circ \partial} \pi_{q-1}(A, B) \to \ldots.$$

Here $i : (A, B) \to (X, B), j : (X, B) \to (X, A)$, and $k : (A, *) \to (A, B)$ are the inclusions.

Definition. *(Triad homotopy group) For a triad $(X; A, B)$ with basepoint $* \in C = A \cap B$, define*

$$\pi_q(X; A, B) = \pi_{q-1}(P(X; *, B), P(A; *, C)),$$

where $q \geq 2$.

Here $P(X; *, B)$ is all those points that start at the basepoint and end in B. Note that the definition is not symmetric in A, B. An element of this triad homotopy group is a map $I^{q-1} \to P(X; *, B)$ such that one face lands in $P(A; *, C)$ and the other faces land in the constant path. If we curry it out, i.e., shift time as a coordinate of I^q, we get a map $I^q \to X$ such that the "top" (after introducing the time coordinate) lands in B, the "right" lands in A (this originally landed in $P(A; *, C)$) and the other faces go to the base point, i.e., it's a map of tetrads

$$(I^q; I^{q-2} \times \{1\} \times I, I^{q-1} \times \{1\}, J^{q-2} \times I \cup I^{q-1} \times \{0\}) \to (X; A, B, *)$$

where $J^{q-2} = \partial I^{q-2} \times I \cup I^{q-2} \times \{0\} \subseteq I^{q-1}$ **not** as before (see remark before Section 4).
If we carefully unwind what elements of $\pi_{q-1}(P(X; *, B))$ look like, by currying we get an element of $\pi_q(X, B)$. This is a bijection and what's more, it's a group isomorphism (when the notion makes sense) simply because once we fix the "wedging" coordinate (seeing elements of the homotopy groups as maps from cubes), it doesn't matter whether we curry first and then take the sum or sum and then curry.

Remark. Because orientation reversal is a homeomorphism, we can define the same triad homotopy group as

$$\pi_{q-1}(P(X;B,*),P(A;C,*)).$$

Because the group operation is basically placing cubes next to each other (see [4] for an such a description in the case of homotopy groups of pairs) in a compatible manner, this reversal of orientation is a group homomorphism

So, if we use this isomorphism and the long exact sequence for pairs, we get a long exact sequence

$$\cdots \to \pi_{q+1}(X;A,B) \to \pi_q(A,C) \to \pi_q(X,B) \to \pi_q(X;A,B) \to \dots.$$

The homotopy excision theorem is saying that the triad homotopy group is zero for $2 \le q \le m+n-2$. The following proof is from [1], [3], and the latter mentions that this was originally proved by J.M. Boardman.

Move to a CW approximation. In order to prove this, we move to a CW approximation of the given triad (Theorem 5.3). Since we have weak equivalence of pairs and long exact sequence for the CW triads, the triad homotopy groups don't change (this is a consequence of a five lemma between the sequence for the given triad and the long exact sequence from the approximation; the CW approximation gives a map between the triad homotopy groups as well using the "coordinate" description of its elements above).

Relative cells. The connectivity assumptions on our triad lets us choose a CW approximation where X is a union of subcomplexes (post approximation) A,B with intersection C such that the pair (A,C) has no relative q-cells for $q < m$, (B,C) has no relative q-cells for $q < n$.

Reduction to finite complexes. We can assume that X has finitely many cells because the elements of the triad homotopy group land in compact, hence finite, subcomplexes (proving that a particular element is zero in the subcomplex makes it zero in X). Furthermore, we can assume that $(A,C),(B,C)$ have at least one relative cell because otherwise $A = C, B = X$ or $B = C, A = X$ and the map $(A,C) \to (X,B)$ is an isomorphism.

Further reductions. Inductively, we can reduce to the case where A has exactly one relative cell. Suppose $C \subset A' \subset A$ with A being obtained by attaching a cell to A'. Let $X' = A' \sqcup_C B$. We have CW triads $(X';A',B),(X;A,X')$ which are the triads to look at when removing/adding this extra cell. In the first triad, the intersection is C, the pair (A',C) has fewer relative cells than (A,C) and is still $m-1$-connected, the pair (B,C) is still $n-1$-connected. In the second triad, the intersection is A', the pair (A,A') has exactly one cell and is $m-1$-connected (because that's how we chose our CW approximation in the first place) and the pair (X',A') is obtained by attaching the relative cells of (B,C) and hence is $n-1$-connected.

So, if our theorem holds for these two triads, then an application of the five-lemma to the map between long exact sequences associated to the triples $(A,A',C) \hookrightarrow (X,X',B)$ (the second is obtained by attaching B) proves the result for $(X;A,B)$ (if one works through the proof of the five-lemma, surjectivity at the highest level works, but injectivity requires an isomorphism at the next level for the pair $(A',C) \to (X',B)$).

Thus, our goal reduces to proving the result for the case when (A,C) has one relative cell, no matter what B is (observe that X' above can be much larger than B). However, we can reduce again to the case where (B,C) has one relative cell.

Further further reductions. Again, suppose $C \subset B' \subset B$ with B having one more cell than B' and let $X' = A \sqcup_C B'$. Now we have the CW triads $(X';A,B')$ and $(X;X',B)$. Both of these have similar properties as in the hypothesis and the inclusion $(A,C) \to (X,B)$ factors as

$$(A,C) \to (X',B') \to (X,B).$$

If the result holds for these two triads, then both maps are equivalences in the appropriate range, hence so is the composite.

Reduced task. Therefore, we reduce to the case where $A = C \cup D^m, B = C \cup D^n$, where $m \ge 2, n \ge 1$. Let

$$f : (I^q; I^{q-2} \times \{1\} \times I, I^{q-1} \times \{1\}, J^{q-2} \times I \cup I^{q-1} \times \{0\}) \to (X;A,B,*)$$

be a map of tetrads and we wish to show that this is nullhomotopic as a map of tetrads for $2 \leq q \leq m+n-2$.

The idea of the proof is to homotope f (through maps between tetrads) that misses a point in the discs, for then we can radially deform the discs to obtain simpler traids.

Observe that, in general, if $A' \subseteq A$, then the triad homotopy group $\pi_*(A;A,A')$ is zero either by looking at the long exact sequence above, or by noticing that the triad homotopy group is isomorphic to $\pi_*(A,A)$ (because the cube can be deformed so that two adjacent faces become one).

Finishing the proof. We can apply Lemma 6.5 to $f: I^q \to A \cup D^n$ to obtain a subdivision of I^q and a function f_1 homotopic to f such that f_1 is linear on a subcomplex N satisfying

$$\mathrm{Int}N \supseteq f_1^{-1}(D^n_{1/2}).$$

Observe that Lemma 6.5 gives a homotopy on a certain open set relative to the boundary, therefore $f_1 = f$ outside some open set. Because this homotopy only modifies the image in D^n, it is a homotopy through a map of tetrads: if some point initially landed in C, then its value doesn't change, if it initially landed in the interior of D^m, then too its value doesn't change, and if it landed in the interior of D^n, then it stays in the interior throughout the homotopy.

Applying Lemma 6.5 again to $f_1: B \cup D^m$, we can further subdivide I^q and homotope f_1 to a function g which is linear on some subcomplex M such that

$$\mathrm{Int}M \supseteq g^{-1}(D^m_{1/2}).$$

Importantly, g is the same as f_1 outside an open set. Because the interiors of D^n, D^m are disjoint, N, M are disjoint and if one follows the proof of Lemma 6.5, it is clear that $g = f_1$ on N and hence is linear. Similar to the previous homotopy, this too is a homotopy through maps between tetrads, hence g is in the same class as f in the triad homotopy group.

Replace f by g. From Lemma 6.6, we can find points $x \in D^m, y \in D^n$ such that if $\pi\colon I^{q-1} \times I \to I^{q-1}$ is the projection, then $\pi(g^{-1}(x)), \pi(g^{-1}(y))$ are disjoint. Moreover, because g is an element in the triad homotopy group, $\pi(g^{-1}(y))$ is disjoint from $\partial I^{q-1} \times I$.

Recall that an element of the triad homotopy group is one that maps the "top" of I^q to B, the "right" side to A and the rest to the basepoint, chosen to be in C. Since y is a point in D^n, it is part of B, while the faces of $\partial I^{q-1} \times I$ land in A.

In I^{q-1} we have two disjoint closed subsets $\pi(g^{-1}(x)) \cup \partial I^{q-1}$ and $\pi(g^{-1}(y))$. By Urysohn's lemma, there's a continuous function $v\colon I^{q-1} \to I$ which is 0 on the first set and 1 on the second. Define

$$\begin{aligned} h\colon I^{q+1} &\to I \\ (r,s,t) &\mapsto (r, s - stv(r)) \text{ for } r \in I^{q-1}, s,t \in I \end{aligned}$$

- $h(r,s,0) = (r,s)$ is the identity map on I^q
- $h(r,0,t) = (r,0)$, so the bottom face is fixed by the homotopy
- $h(r,s,t) = (r,s)$ for $r \in \partial I^{q-1}$ because $v(r) = 0$ there. Thus, the "sides" are fixed by the homotopy
- $h(r,s,t) = (r,s)$ for $(r,s) \in g^{-1}(x)$. In particular, $h(r,s,t) \in g^{-1}(x)$ iff $(r,s) \in g^{-1}(x)$.
- $h(r,s,t) = (r, s - st)$ for $(r,s) \in g^{-1}(y)$ because $v(r) = 1$ there. So, the preimage of y is pushed down to the bottom face

Denote

$$\begin{aligned} h_1: I^q &\to I^q \\ (r,s) &\mapsto h(r,s,1) \end{aligned}$$

Now, we don't quite know what this does to the "top" face (that given by setting $s = 1$), but because g maps the top face to $B \subset X \setminus \{x\}$, we conclude that $g \circ h$ is a homotopy of maps of tetrads

$$(I^q; I^{q-2} \times \{1\} \times I, I^{q-1} \times \{1\}, J^{q-2} \times I \cup I^{q-1} \times \{0\}) \to (X; A, X \setminus \{x\}, *)$$

And $g \circ h_1$, as a map of tetrads, lands in $(X \setminus \{y\}; A, X \setminus \{x, y\})$.
It is easy to see that there is a deformation retraction $(X \setminus \{y\}; A, X \setminus \{x, y\}, *)$ to the subspace tetrad $(A; A, A \setminus \{x\})$ - and this deformation retraction is a retraction through maps between tetrads. But the triad homotopy groups of $(A; A, A \setminus \{x\})$ are all zero, therefore $g \circ h_1$ is homotopic to a zero element. Note here that we have inclusions of based triads

$$(A; A, A \setminus \{x\}) \subset (X \setminus \{y\}; A, X \setminus \{x, y\}) \subset (X; A, X \setminus \{x\}) \supset (X; A, B)$$

where the first and third induce isomorphisms on the triad homotopy groups via radial deformation retractions.
To summarize,

- Start with a representative f of an element in the triad homotopy group.
- Using Local linearization Lemma 6.5, homotope f to a map g that is linear on some subcomplexes landing in the two discs
- Using Lemma 6.6, find points x, y whose preimages under g have disjoint projections to the base I^{q-1}.
- Using a function v, “push” the part that goes to y down to the base along paths from the basepoint to y
- This results in a representative that misses y, but potentially pushes things that previously missed D^m into D^m (in a way, D^m offers room to move things away from y)
- We can further deformation retract (this time the codomain space rather than the domain I^q) it to land in a triad that has zero triad homotopy group.

Remark. The only reason we need $m \geq 2$ is so that the range of q is at least 2. For $q \leq 1$, there is nothing to prove.
Remark. If one accepts smooth approximations and homotopies to smooth approximations, then one should be able to homotope f into a smooth function and choose y as a regular value (by Sard's theorem, the critical values have measure zero). Then the preimage of y has the dimension we want and the rest carries through.

6.3 Freudenthal suspension isomorphism

We look at some consequences of homotopy excision theorem. We denote by Mf, Cf the mapping cylinder, cones respectively and use $M_r f, C_r f$ for the corresponding reduced constructions.

Theorem 6.2. *Let $f: X \to Y$ be an $(n-1)$-equivalence between $(n-2)$-connected spaces, where $n \geq 2$ (so $\pi_{n-1}(f)$ is surjective). Then the quotient map $\pi: (Mf, X) \to (Cf, *)$ is a $(2n-2)$-equivalence. In particular, Cf is $(n-1)$-connected. If X, Y are $(n-1)$-connected, then π is a $(2n-1)$-equivalence.*

Proof. As typical in applications of the Mayer-Vietoris sequence, we have an excisive triad $(Cf; A, B)$ where A is the bottom 2/3rd of the cone and B the top 2/3 (with the top being the cone point) and the intersection C is homotopy equivalent to X. The projection works as a composite

$$(Mf, X) \xrightarrow{\sim} (A, C) \hookrightarrow (Cf, B) \xrightarrow{\sim} (Cf, *)$$

and the long exact sequence for the pair (Mf, X) looks like

$$\cdots \to \pi_{q+1}(Mf, X) \to \pi_q(X) \to \pi_q(Mf) \to \pi_q(Mf, X) \to \dots$$

Now, Mf is homotopy equivalent to Y and $\pi_{n-1}(f)$ is a surjection, therefore the pair (Mf, X), hence (A, C), is $(n-1)$-connected. Similarly, $(CX, X) \simeq (B, C)$ is also $(n-1)$-connected and n-connected if X is $(n-1)$-connected. Homotopy excision completes the proof. □

Corollary 6.1. *Let $f: X \to Y$ be a based map between $(n-1)$-connected nondegenerately based spaces, $n \geq 2$. Then $C_r f$ is $(n-1)$-connected and $\pi_n(M_r f, X) \to \pi_n(C_r f, *)$ is an isomorphism. Moreover, the canonical map $\eta: Ff \to \Omega C_r f$ induces an isomorphism $\pi_{n-1}(Ff) \to \pi_n(C_r f)$.*

Here "nondegenerately based" (also called "well pointed") means that the inclusion of the basepoint is a cofibration. Intuitively, the basepoint can drag the space along any path. In the based context, we think of the reduced cylinder and cone.

Proof. From the next lemma, the reduced constructions are homotopy equivalent to the unreduced ones and therefore the first statement follows from the theorem.
For the second part, let $j\colon X \to Mf$ be the inclusion. The mapping fibre[4] is $Fj = P(Mf;X,*)$. The retraction $r\colon Mf \to Y$ induces a map

$$Fr\colon Fj \to Ff = P(Y;X,*)$$

We have two long exact sequences coming from j,f with maps between Mf,Y and identity X,X being homotopy equivalences. By five lemma, the induced maps

$$(Fr)_*\colon \pi_q(Mf,X) \to \pi_{q-1}(Ff)$$

is an isomorphism for all q (recall that the homotopy group of a pair was the homotopy group of the mapping fibre of the inclusion shifted by 1).
Recall that the natural map $\eta\colon Ff \to \Omega Cf$ simply joins the path from the cone point to $f(x)$ at the start. Now, the quotient map π factors as $\eta \circ Fr$, therefore, η induces an isomorphism as stated. $\square$

Lemma 6.7. *Let $f\colon X \to Y$ be a based map between nondegenerately based spaces. Then the unreduced mapping cylinder and cone constructions are homotopy equivalent to the reduced constructions. Moreover, the pair (Mf,X) is homotopy equivalent to (M_rf,X).*

Proof. The argument below is parallel to the proof of Theorem 1.2. Let x_0 denote the basepoint of X, then $X\times\{0\}\cup\{x_0\}\times I$ is a retract of $X\times I$. Taking the product with I, there is a retraction

$$X\times I\times J \to X\times\{0\}\times J\cup\{x_0\}\times I\times J$$

where $J=[0,1]$ labelled to be different from I. In particular, $\{x_0\}\times I\subset X\times I$ has the homotopy extension property. Interchange I,J. We can quotient by $X\times\{1\}\times I$ to get a retraction

$$CX\times I \to CX\times\{0\}\cup_{\sim}\{x_0\}\times J\times I$$

where the $\sim$ indicates the identification between the two subspaces. Therefore, $(\{x_0\}\times J/\sim)\subset CX$ has the homotopy extension property.
In both cases, the map is identity on the zero slice of X with respect to I. This means we can attach Y to the bottom with the identity homotopy, i.e., using pushouts and the fact that I is exponentiable (so certain products and quotients can interchange), we get retractions

$$\begin{aligned} Mf\times I &\to (Y\times I)\cup_f(Cyl(X)\times\{0\})\cup\{x_0\}\times J\times I\\ Cf\times I &\to (Y\times I)\cup_f(CX\times\{0\})\cup\{x_0\}\times J\times I\end{aligned}$$

where various identifications have been made on the right.
From the right there is a well defined map to Mf,Cf that shrinks $\{x_0\}\times J\times I$ and is identity on Y (hence on the copy of $Cyl(X),CX$ as well). Composing we get homotopies $Mf\times I\to Mf, Cf\times I\to Cf$ that start at the identity map.
Now, the top slice induces a map $M_rf\to Mf, C_rf\to Cf$ because the segment $\{x_0\}\times I$ collapses to a point. In the other direction we have the corresponding quotient maps and similar to the proof of Theorem 1.2, these provide homotopy equivalences. Note that unlike Theorem 1.2, we don't have a contractible subspace. Secondly, these homotopy equivalences do not affect Y and we did not need Y to be nondegenerately based.
Lastly, the top of Mf (obtained by setting the J-coordinate to be 1) stays in the top slice itself throughout the homotopy. This means that even as maps between pairs $(Mf,X),(M_rf,X)$ we have a homotopy equivalence. This is the same fact that allowed us to pass to the cones. $\square$

[4]Recall that our convention is a little different from [1] in that our paths have the opposite orientation.

Theorem 6.3. *Let $i\colon A \to X$ be a cofibration and an $(n-1)$-equivalence between $(n-2)$-connected spaces, where $n \geq 2$. Then the quotient map $(X,A) \to (X/A,*)$ is a $(2n-2)$-equivalence and a $(2n-1)$-equivalence if A,X are $(n-1)$-connected.*

Proof. The retraction $r\colon Mi \to X$ is a homotopy equivalence of pairs $(Mi,A) \simeq (X,A)$. Our result lets us go from (Mi,A) to $(Ci,*)$. Now, intuitively, Ci is trying to collapse A to a point.
Because i is a cofibration, $X \times \{0\} \cup A \times I$ is a retract of $X \times I$. Imagine $Ci \times I$ as a rectangular base with a prism on top, the base can retract to $X \times \{0\} \cup A \times I$ which means $CA \subset Ci$ is a cofibration. In terms of formulas, we write $Ci = X \cup_A CA$ so

$$Ci \times I = (X \times I) \cup_{A \times I} (CA \times I)$$

and there is a retraction of the first bit that preserves the common subset $A \times I$, thereby giving us our desired retraction.
Because CA is contractible, it follows from Theorem 1.2, that $(Ci,*) \simeq (Ci/CA,*) \cong (X/A,*)$ as based spaces. Let ψ be the quotient map.

$$\begin{tikzcd}
(Mi,A) \arrow[r, "\pi"] \arrow[d, "r"'] & (Ci,*) \arrow[d, "\psi"] \\
(X,A) \arrow[r] & (X/A,*)
\end{tikzcd}$$

The vertical arrows are homotopy equivalences and the square is commutative, therefore the theorem follows from previous results. $\square$

6.3.1 The Hawaiian Earring

When $i\colon A \hookrightarrow X$ is not a cofibration, it is not true that the mapping cone is homotopy equivalent to the quotient. Here is an example from [5] Chapter VII.
Let $A = \{0\} \cup \{1/n\}_{n \geq 1} \subset [0,1]$. This is not a cofibration as there is no retraction $I \times I \to I \times \{0\} \cup A \times I$ for one is locally connected and the other isn't: suppose there was such a retraction r. Take a small neighbourhood V around $(0,1)$ that doesn't intersect $I \times \{0\}$. This neighbourhood consists of many vertical lines and is disconnected. By continuity of r and the fact that r is a retraction, there should be a neighbourhood U around $(0,1) \in I \times I$ that lands in V.
By shrinking, we may assume U is connected. This also contains, as a subset, many vertical lines where r is the identity map. But at the same time, U being connected, $r(U)$ should land in one of the lines of V and we have a contradiction.
Next we show that X/A is not homotopy equivalent to the mapping cone Ci. Here, X/A is the Hawaiian circle: an infinite wedge of circles that go "converge" to a point. Each of these circles is not nullhomotopic: say the circle is the image of the interval $[\frac{1}{n+1}, \frac{1}{n}]$. There is a retraction $X/A \to S^1$ obtained by collapsing all the other circles (this amounts to collapsing $[0, \frac{1}{n+1}] \cup [\frac{1}{n}, 1]$ to a point). Because it is a retraction, the inclusion map $S^1 \to X/A$ is injective on the fundamental groups.
Retraction to each of the circles gives a group homomorphism $\pi_1(X/A) \to \prod_{n=1}^{\infty} \mathbb{Z}$. This is surjective for, given a sequence $(a_1, a_2, \ldots)$, we can construct a loop that goes around the first circle a_1 times during the interval from 0 to 1/2 say, and then a_2 times around the second circle during the interval 1/2 to 3/4 and so on. Therefore, the fundamental group of X/A is uncountable. But note that this projection doesn't take the order of the circles into account, i.e, this group homomorphism isn't injective.
From the arguments above, it follows that X/A is not semilocally simply connected: there's no neighbourhood of the wedge point whose loops are nullhomotopic in X/A (because every such neighbourhood contains a nontrivial circle).
Coming to the cone Ci, this is an infinite broom together with the $[0,1]$ interval as a base. This is locally contractible, i.e., every point has a contractible neighbourhood, hence semilocally simply connected as well. Note that Ci is also an infinite wedge of circles, except these are wedged along sides rather than points.

Suppose there was a homotopy equivalence $X/A \to Ci$, in particular, an isomorphism between the π_1s. Take a contractible neighbourhood V around the image of the wedging point. The preimage U is a neighbourhood of the wedging point, hence has a loop that is nontrivial in X/A. But V being contractible, tells us that this nontrivial loop has trivial image in Ci which is a contradiction.

In fact, we can show that the homotopy group of Ci is countable. Let $f: I \to Ci$ be a loop. By uniform continuity, for $\epsilon < 1/2$ there is a $\delta > 0$ such that two points within δ have images within ϵ of each other. Cover I with finitely many intervals $\{I_j\}_{1\leq j\leq k}$ of length $< \delta$. Under f, these intervals land in a contractible subset of Ci, in particular, cannot contain the circles that make up Ci.
Unlike the earring, one cannot travel through all the circles because near 1 the loop would have to travel up and down too rapidly forcing 1 to not have a well defined image.
In a contractible space, any two paths between two points are homotopic. Homotope the paths I_j to be made up of straight line segments (this is possible because the ϵ-balls are straight line segments). Because the ends are fixed, these piecewise homotopies glue together. Thus f is homotopic to a loop that passes through the cone point a finite number of times and can contain only a finite number of circles that build Ci. It follows that $\pi_1(Ci)$ is countable.

Because reduced suspension is a functor, we have an induced group homomorphism

$$\Sigma\colon \pi_q(X) \to \pi_{q+1}(\Sigma X).$$

It is a group homomorphism simply because Σ commutes with wedge, noting that the reduced suspension is just $\wedge S^1$ when the spaces are compactly generated Hausdorff (or more generally, exponentiable).

Theorem 6.4. *(Freudenthal suspension) Assume that X is nondegenerately based and $(n-1)$-connected, $n \geq 1$. Then Σ is a bijection if $q < 2n-1$ and a surjection if $q = 2n-1$.*

Again, X being nondegenerately based means that the inclusion of a basepoint is a cofibration. Note that all based CW complexes are nondegenerately based by HELP.

Proof. Let $f: (I^q, \partial I^q) \to (X, *)$ be a map. To get Σf, we first look at $f \times id\colon I^q \times I \to X \times I$. On the right we need to collapse $X \times \{0,1\} \cup \{*\} \times I$. We first go to the reduced cone and then collapse one end.
To be consistent with previous notation[5] for elements of relative homotopy groups, we use the reversed cone

$$C_r'X = X \times I/(X \times \{0\} \cup \{*\} \times I).$$

By collapsing the base, $f \times id$ gives a map of triples

$$(I^{q+1}, \partial I^{q+1}, J^q) \to (C_r'X, X, *)$$

whose restriction to $I^q \times \{1\}$ is f and which induces Σf when we quotient the top (i.e., base) of $C'X$. We have the commutative diagram

$$\begin{tikzcd}
& \pi_{q+1}(C_r'X, X, *) \arrow[dl, "\partial"'] \arrow[dr, "\rho_*"] & \\
\pi_q(X) \arrow[rr, "\Sigma"'] & & \pi_{q+1}(\Sigma X)
\end{tikzcd}$$

where ρ_* is induced by the quotient $\rho\colon C_r'X \to \Sigma X$.
Because C_r' is contractible, ∂ is an isomorphism for all q. Above we have constructed an inverse for ∂. This is one of many possible inverses because we aren't operating at the homotopy level, but it suffices to see that the diagram commutes.
Because $X \to C_r'X$ is a cofibration (check!), and an n-equivalence between $(n-1)$-connected spaces (the cone is contractible), ρ is a $2n$-equivalence by Theorem 6.3. From the commutativity of the diagram, it follows that Σ is an isomorphism for $q < 2n-1$ and a surjection for $q = 2n-1$. $\square$

[5] Notation in [1]. If you use the notation in [4], then you can use the usual dunce hat rather than the icecream cone.

Another way of proving the result is to write the suspension ΣX as the union of two (open) reduced cones C_r^+X, C_r^-X (top and bottom) whose intersection is (homotopy equivalent to) X. Then $(C_r^+X, X), (C_r^-X, X)$ are both n-connected by the long exact sequence and the fact that the cones are contractible.
By homotopy excision theorem, $(C_r - X, X) \hookrightarrow (\Sigma X, C_r^+X)$ is a $2n$-equivalence. From the long exact sequences we have isomorphisms

$$\pi_{q+1}(\Sigma X) \cong \pi_{q+1}(\Sigma X, C_r^+X)$$
$$\pi_{q+1}(C_r^-X, X) \cong \pi_q(X)$$

Combining all these together we get the suspension isomorphism (and surjection). Note that the chain of isomorphisms compose to give the suspension morphism because the second isomorphism collapses the bottom of I^{q+1} and the first isomorphism drags the top face along the cone C_r^+X to collapse it (up to an identification of the reduced and unreduced suspensions of the spheres).

Corollary 6.2. *For all $n \geq 1, \pi_n(S^n) = \mathbb{Z}$ and $\Sigma\colon \pi_n(S^n) \to \pi_{n+1}(S^{n+1})$ is an isomorphism.*

Proof. We refer the reader to [1] or [4], but the argument is that using the Hopf bundle one can show $\pi_2(S^2) = \mathbb{Z}$ and then induct upward. For $n = 1$, some care is needed. □

It turns out that the dimension range in the suspension theorem above is sharp: we have $\pi_3(S^2) = \pi_3(S^3) = \mathbb{Z}$ and suspension gives a surjection $\pi_3(S^2) \to \pi_4(S^3)$ but this is not an isomorphism for $\pi_4(S^3) = \mathbb{Z}_2$.
Suspension gives an isomorphism

$$\Sigma\colon \pi_q(S^n) \cong \pi_{q+1}(S^{n+1}) \text{ for } 1 \leq q < 2n - 1.$$

Increasing n by 1 increases the "stable" range by two. When X is highly connected, we have similar stability for the groups $\pi_{q+n}(\Sigma^n X)$.
If one naively takes the suspension isomorphism to hold, then Σ simply shifts the homotopy groups by 1, but going diagonally, i.e., $q + n$th group for the nth suspension, the groups are the same.
In general we get a sequence of groups $\pi_{q+n}(\Sigma^n X)$ with homomorphisms between them given by suspension. Define

$$\pi_q^s(X) = \operatorname{colim} \pi_{q+n}(\Sigma^n X)$$

to be the qth stable homotopy group of X. It is indeed a group with group operation done "eventually" by suspending to a common space and then taking the wedge sum as usual.
In effect, it is all those q dimensional loops that are don't become trivial even after umpteen suspensions. If X is path connected, by Siefert-van Kampen, the suspension is simply connected. While this doesn't always hold as we suspend repeatedly, it offers a picture that suspension closes some "holes" (in a way, homology provides a better picture of holes rather than the homotopy groups) and the stable group measures those holes that are "irremovable".
It turns out that computing the homotopy groups, even for something as simple as a sphere, is quite hard (which makes the sphere, from a certain point of view, exceedingly complicated). We have better hopes for computing the stable homotopy groups of spheres and this has fuelled the growth of algebraic topology during the last many decades.
Moreover, these stable homotopy groups are related to objects called spectra which are in turn related to generalised cohomology theories, i.e., functors that satisfy all the Eilenberg-MacLane axioms for cohomology except the dimension axiom, for example. Furthermore, the computation of these stable homotopy groups are related to deep geometrical problems.

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