Topologies on function spaces

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It is natural to look at functions to and from a space X. The functions to a space from loops, spheres and simplices are a starting point for defining the homotopy and homology groups. The functions from a space are useful on the one hand to get at some underlying structure of X (the sheaf of smooth functions, vector fields etc where the geometry of X places some restrictions - hairy ball theorem, for example - on what the functions can look like) and on the other hand to see where we can imbed X. When the domain of a function is a metric space, there is a possible metric topology on the function space. There is the notion of currying which relates maps $Z \times X \to Y$ with maps from Z to the set of functions from X to Y. and we understand on of these (the product topology) reasonably well.

This article started out as some basic notes on the compact open topology which is quite heavily used when talking about the path space of a function (see the last section). However, along the way, I learnt that the compact open topology is not the end of the story, but just the beginning, one started by Ralph H. Fox in his paper [1] "On the topologies of function spaces". Upon looking further, I came across the notes [2] by Tyrone Cutler and learnt about core compact spaces. Here I have tried to provide all the proofs that go into the final result on the sufficient and necessary conditions for a space to be exponentiable (defined later on), although the adjoint functor theorem is a blackbox result which I am not yet familiar with. However, I feel that these notes are in a decent shape now and the main takeaway is that core compact spaces are exponentiable, and most spaces people (i.e., algebraic topologists as I know them at this stage of my education) usually work with (locally compact Hausdorff spaces, locally finite CW complexes etc.) are core compact. We care about exponentiability because it lets us curry functions without bothering about continuity. Our concerns, once abstracted away, are dealt with in a single blow.

1 The compact open topology

This section contains a general discussion to motivate the compact open topology, so assume whatever bare necessities on the spaces involved (for example, first countability).

Let X, Y be topological spaces and let $F = C(X, Y)$ denote the space of all continuous functions from X to Y. Suppose $f_n, n \geq 1$ are functions from X to Y, and we are interested in the notion that this sequence "converges" to $f: X \to Y$. Pointwise convergence is too flimsy a notion because pointwise convergence doesn't always ensure continuity of f. Convergence in the L^p sense is quite restrictive for our purpose as it requires X to be a measured space and Y to have much more structure than is typical in the study of geometry (think of the path space of a manifold).

However, in whatever way we decide to topologise F, we would like to retain - pointwise convergence (or something close to it, such as almost everywhere convergence) and the limit should be continuous (this is a non issue as we are talking about convergence in F). Let's say that the convergence in F is to force pointwise convergence. Let $U \subseteq Y$ be an open set and say $x \in f^{-1}(U)$. Now, pointwise convergence means that eventually all $f_n(x)$ should lie in U. In other words f and a tail of $\{f_n\}$ are functions that send x to U, i.e., if $M(\{x\}, U)$ denotes all those (continuous) functions sending x to U, then $f \in M(\{x\}, U)$ and a tail of $\{f_n\}$ is also inside $M(\{x\}, U)$.

Next consider uniform convergence on subsets, i.e., pointwise convergence "happening at once" on some subset A. This amounts to replacing the point x above with A and everything else stays the same. Arguing similarly, we need f_n to eventually be in $M(A,U)$ where U is now a neighbourhood of $f(A)$. Intuitively, if we have uniform convergence over A, then we have uniform convergence over $\overline{A}$ as well, for we can consider sequences and, in the metric case, consider a standard $\epsilon/2$ argument. Of course this is just a vague statement, but the point is that we shouldn't consider all possible A because it is too restrictive. Over $\mathbb{R}$, we would like to consider all functions and forcing uniform convergence everywhere would severly limit us, consider, for example, the sequence of functions f_n with f_n being 0 on $(-\infty, n]$ and then $x - n$.; this "should" converge to the 0 function, but there is no uniform convergence on $(0, \infty)$ for example.

Anyway, the point is that we restrict what A can be. We only look at compact sets because uniform convergence really makes sense over compact domains. So, we consider

$$M(K,U) = \{f : X \to Y : f(K) \subseteq U\}$$

with K compact and U open. This generates a subbasis for every function is contained in $M(\{x\}, Y)$ for any $x \in X$. Equip F with the topology so generated: the compact open topology.

If Y is T_1, then F is T_1: let $f, g \in F$ with $f(x) \neq g(y)$. Then take a neighbourhood U around $g(y)$ that doesn't contain $f(x)$, then $f \notin M(\{y\}, U)$. Similarly, if Y is T_2, then so is F.

Suppose X, Y, Z are topological spaces, then we have the composition map $C(Y,Z) \times C(X,Y) \to C(X,Z)$. It is fairly easy to see that when each is given the compact-open topology (together with the product topology), then this composition is a continuous map.

There are other ways to topologise F, see [2]. It is not surprising that with additional structure on either of the spaces we may have a topology on F that uniquely suits a particular purpose depending on the field of study, for example, making F a normed space. Here's another example: suppose $f \in C(K,U)$ then one can think of functions in $C(K,U)$ as those functions that send K to "within" U of $f(K)$, or those that nudge $f(K)$ inside U. One way to extend this is to take Y to be a metric space and fix a function $\epsilon \colon Y \to \mathbb{R}_{>0}$ and then look at $C_\epsilon(f,K)$ to be $\{g : d(g(x), f(x)) < \epsilon(f(x)) \forall x \in K\}$ where d is the metric on Y.

2 Some properties of the compact open topology

Given $f, g \colon X \to Y$, we say that f, g are homotopic if there's a continuous map $H \colon X \times I \to Y$ such that $H(\cdot, 0) = f, H(\cdot, 1) = g$. A homotopy can be thought of as a path in $C(X,Y)$, and this thought will be made precise later, although it should be obvious and involves an operation called **currying**. Loops in X are maps $S^1 \to X$ and a homotopy between two loops is a map $S^1 \times S^1 \to X$ or a loop in $C(S^1, X)$. Note, one must prove that the loops and paths so obtained are continuous. In [1] we have the following results.

Lemma 1. *Let X, T, Y be spaces and suppose we have a continuous map $H \colon X \times T \to Y$. Denote by h_t the map $h_t \colon x \mapsto H(x,t)$ for $t \in T$. We then have a map $h \colon T \to C(X,Y)$ which is continuous.*

Proof. Given an open $U \subset Y$, suppose $x \in h_t^{-1}(U)$, then there are open sets V, W such that $(x,t) \in V \times W \subset H^{-1}(U)$ which proves that each h_t is continuous. Next, suppose $K \subset X$ is compact and U is open in Y. Let $t \in h^{-1}(M(K,U))$ which means that h_t maps K into U. With the tube lemma, we obtain an open neighbourhood around t which is mapped into $M(K,U)$ by h, thus proving its continuity. □

Theorem 1. *With the notation as in the lemma, the converse is true if X is locally compact regular.*

Proof. Let U be open in Y, given that h and each h_t is continuous, we need to show that H is continuous. Let $(x,t) \in H^{-1}(U)$, then there is an open set V such that $x \in V \subseteq h_t^{-1}(U)$. Now, X is locally compact regular, so we obtain an open V_1 with compact closure such that $x \in V_1 \subseteq \overline{V_1} = K \subseteq V$. Now $h_t \in M(K,U)$ so there is a neighbourhood W of t such that $W \subseteq h^{-1}(M(K,V))$, which means that any $s \in W$ maps K into U. Therefore, $V_1 \times W \subseteq H^{-1}(U)$ which means H is continuous. □

Theorem 2. *With the notation as above, if X, T are first countable, then continuity of H is equivalent to that of h.*

Proof. Suppose h is continuous, X,T first countable, U open in Y and $(x,t) \in H^{-1}(U)$. Since $X \times T$ is first countable, it suffices to show that H is sequentially continuous. So, let (x_n, t_n) be a sequence converging to (x,t). In particular t_n converges to t, so h_{t_n} converges to h_t in the compact open topology. Now, there is an open V such that $x_0 \in V \subseteq h_{t_0}^{-1}(U)$. Because x_0 is a limit point, $A = V \cap \{x_n\}$ is a compact set, and $h_{t_0} \in M(A,U)$. It follows that there is an N such that for $n \geq N, (x_n, t_n) \in H^{-1}(U)$, hence H is continuous. The converse is true as in the lemma. □

[1] has the following theorem whose proof we shall not provide here

Theorem 3. *If X is separable metrizable and Y the real line. Then in order for $C(X,Y)$ to be topologized such that the continuity of h,H are equivalent, it is necessary and sufficient that X be locally compact.*

3 Exponentiable spaces

In this section (primarily based on [2], but [3] is another source for the interested) we take a step back and abstract everything away. The main question is about having a relation between $C(Z \times X, Y)$ and $C(Z, C(X,Y))$ obtained by the currying process, which, a priori, happens at the level of maps between sets. The question is whether currying can happen at the level of continuous maps.

Definition. *A space X is said to be exponentiable if the functor*

$$\mathsf{Top} \xrightarrow{(-)\times X} \mathsf{Top}$$
$$Z \mapsto Z \times X$$

has a right adjoint R_X.

Here adjoint means that there's a natural bijection between $C(Z \times X, Y)$ and $C(Z, R_X Y)$ where "natural" means that if you fix one of Z, Y and get a map f to the other from some other space W, then the resulting diagrams (obtained by pre/post composing with f) commute. So, our question is to find which spaces are exponentiable. We shall not provide proofs here, but they can be found in [2] or [3].

Now, currying gives a map at the level of sets and prompts the next definition:

Definition. *Let X, Y be spaces and τ a topology on $C(X,Y)$, we write $C_\tau(X,Y)$. We say τ is*

- ***splitting*** *if for every space Z and every continuous $f: Z \times X \to Y$, the adjoint $f^\#: Z \to C_\tau(X,Y)$ is continuous.*
- ***cosplitting*** *if for every space Z and every continuous $f: Z \to C_\tau(X,Y)$, the adjoint $f^\flat: Z \times X \to Y$ is continuous.*
- ***exponential*** *if it is both splitting and cosplitting.*

In the previous section we have shown that the compact open topology is splitting and if X is locally compact Hausdorff (this implies regularity), then it is cosplitting as well. More generally, one sees that the indiscrete topology on $C(X,Y)$ is always splitting and the discrete topology is always cosplitting. Next we show that exponential topologies, if they exist, are unique.

Proposition 1. *For spaces X, Y, a topology τ on $C(X,Y)$ is cosplitting if and only if the evaluation map $ev: C_\tau(X,Y) \times X \to Y$ is continuous.*

Proof. Suppose the evaluation map is continuous, then given $f: Z \to C_\tau(X,Y)$ continuous,

$$Z \times X \xrightarrow{f \times id} C_\tau(X,Y) \xrightarrow{ev} Y$$

is continuous, hence τ is cosplitting. Conversely, if it is cosplitting then the evaluation map is obtained by curring the identity map on $C_\tau(X,Y)$, hence is continuous. □

Observe that the evaluation map being continuous implies that the evaluation at any particular $x \in X$ is also continuous.

Proposition 2. *Let X, Y be spaces and σ, τ topologies on $C(X, Y)$.*

1. *If τ is splitting and $\sigma \subseteq \tau$, then σ is splitting.*
2. *If τ is cosplitting and $\tau \subseteq \sigma$, then σ is cosplitting.*
3. *If σ is splitting and τ is cosplitting, then $\sigma \subseteq \tau$.*

Proof. 1. In this case, $id\colon C_\tau(X, Y) \to C_\sigma(X, Y)$ is continuous, hence given $f: Z \times X \to Y$ observe that $f_\sigma^\# = id \circ f_\tau^\#$ is continuous.

2. Again, $id\colon C_\sigma(X, Y) \to C_\tau(X, Y)$ is continuous, hence given a continuous $f: Z \to C_\sigma(X, Y)$, the composition $f: Z \to C_\tau(X, Y)$ is continuous, hence $f^\flat$ is continuous.

3. We have two evaluation maps ev_σ, ev_τ and τ is cosplitting, hence ev_τ is continuous. Since σ is splitting, the map $id\colon C_\tau(X, Y) \to C_\sigma(X, Y)$ is continuous for it sends $f \mapsto \{x \mapsto ev_\tau(f, x) = f(x)\} = f$. Therefore, τ is finer than σ. □

As a corollary, the exponential topology, if it exists, is unique and is the largest splitting and smallest cosplitting topology. We shall denote it by Y^X. Next we relate X being an exponentiable space and $C(X, Y)$ having an exponential topology. We shall need the following proposition, also from [2].

Proposition 3. *Let X, Y, Z be spaces.*

1. *let $f: Y \to Z$ be continuous. If $C_\sigma(X, Y)$ carries a cosplitting topology and $C_\tau(X, Z)$ carries a splitting topology, then the induced map $f_*\colon C_\sigma(X, Y) \to C_\tau(X, Z)$ is continuous.*
2. *Let $g\colon X \to Y$ be continuous. If $C_\sigma(Y, Z)$ carries a cosplitting topology and $C_\tau(X, Z)$ carries a splitting topology, then the induced map $g^*\colon C_\sigma(Y, Z) \to C_\tau(X, Z)$ is continuous.*

Proof. 1.
$$C_\sigma(X, Y) \times X \xrightarrow{ev} Y \xrightarrow{f} Z$$
is continuous because σ is cosplitting. Because τ is splitting, $C_\sigma(X, Y) \to C_\tau(X, Z)$ is continuous.

2.
$$C_\sigma(Y, Z) \times X \xrightarrow{id \times g} C_\sigma(Y, Z) \times Y \xrightarrow{ev} Z$$
is continuous because σ is cosplitting. Because τ is splitting, $C_\sigma(Y, Z) \to C_\tau(X, Y)$ is continuous. □

Theorem 4. *A space X is exponentiable if and only if the exponential object Y^X exists for each space Y.*

Proof. First suppose X is an exponentiable space, which means there's a right adjoint R_X as in the definition. For every Y, Z there is a bijection
$$T_{Z,Y}\colon C(Z, R_X Y) \to C(Z \times X, Y)$$
and this bijection is natural.

With $Z = *$, the singleton space, we get a bijection $C(*, R_X Y) \xrightarrow{T_{*,Y}} C(* \times X, Y)$. There are canonical bijections LHS $\approx R_X Y$ and RHS $\approx C(X, Y)$. Of these, $R_X Y$ is a topological space. So, topologise the other three spaces so that all the bijections become homeomorphisms. It is in this sense that $R_X Y$ is $C(X, Y)$ with some topology τ, which we show is exponential.

Let $f: Z \times X \to Y$ be given. for any $z \in Z$, we have the map $z: * \to Z$. Now we use naturality to get the following diagram

$$\begin{tikzcd} C(* \times X, Y) \arrow[r] & C(*, R_X Y) \\ C(Z \times X, Y) \arrow[r] \arrow[u] & C(Z, R_X Y) \arrow[u] \end{tikzcd} \qquad \begin{tikzcd} \{(*,x) \mapsto f(z,x)\} \arrow[r, mapsto] & T^{-1}_{*,Y} f(z) \\ f \arrow[r, mapsto] \arrow[u, mapsto] & T^{-1}_{Z,Y} f \arrow[u, mapsto] \end{tikzcd}$$

This diagram says that f goes to the function $T^{-1}_{Z,Y} f \in C(Z, R_X Y)$ which corresponds to the map $\{z \mapsto \{x \mapsto f(z,x)\}\} \in C(Z, C(X,Y))$ and this map is continuous in the topology on $C(X,Y)$ because it is topologized to be homeomorphic (under $T_{*,Y}$) to $R_X Y$. Therefore, τ is splitting.

Using the same diagram, but going backwards along the chain of implications we see that τ is also cosplitting, i.e., an element of $C(Z, C_\tau(X,Y))$ corresponds to a continuous map $Z \to R_X Y$ which corresponds to a continuous $Z \times X \to Y$ and the relation is given precisely by currying. Thus, τ is exponential.

Next, assume an exponential object Y^X exists for every Y. We then have a potential covariant functor Top $\to$ Top sending Y to Y^X. By Proposition 3, this is indeed a functor. Exponentiality makes the currying a bijection between $C(Z \times X, Y)$ and $C(Z, Y^X)$, what's left is to prove that this is natural, but this is an easy thing to verify. Thus $(-)^X$ is a right adjoint to $(-) \times X$ and X is exponentiable. □

We have proved in the previous section that the compact-open topology is splitting (Lemma 1) and that a locally compact Hausdorff space is exponentiable (Theorem 1).

3.1 Some more topologies

Isbell topology

Given a map $H: Z \times X \to Y$, we proved in Lemma 1 that the map $H_z: x \mapsto H(z,x)$ is continuous resulting in a map $h: Z \to C(X,Y)$. In Lemma 1 we showed that this is continuous if we use the compact-open topology. We look closely at the proof of continuity: Take the basic open set $M(K,U)$ as in the lemma, then if $z \in h^{-1}(M(K,U))$ then H maps $\{z\} \times K$ into U. From here we covered $\{z\} \times K$ with some open sets of $Z \times X$ and then used the compactness of K to obtain a finite subcover and then we obtain a neighbourhood of z from this finite cover of K. We use this as a motivation to define the **Isbell topology**:

Let $\mathcal{O}_X$ denote the lattice of open sets of X. A subset $\mathbb{U}$ of $\mathcal{O}_X$ is said to be Scott open if whenever $U \in \mathbb{U}$ and $U \subseteq V$, then $V \in \mathbb{U}$ and if for some indexing set $I, \cup_{i \in I} U_i$, a union of open sets, is in $\mathbb{U}$ then some finite sub-union is in $\mathbb{U}$. This is the property we used in the proof of Lemma 1.

The Isbell topology $C_{Is}(X,Y)$ is the topology generated by the sets $M(\mathbb{U}, V) = \{f \in C(X,Y) : f^{-1}(V) \in \mathbb{U}\}$ as $\mathbb{U}$ runs over all Scott open sets and V is open in Y.

Observe that for K compact, the neighbourhoods of K form a Scott open set $\mathbb{U}_K$ and $M(K,V) = M(\mathbb{U}_K, V)$, therefore the Isbell topology is finer than the compact-open topology. Arguing as in Lemma 1, we see that the Isbell topology is splitting.

Natural topology

There is another way to force a splitting topology: topologize $C(X,Y)$ by all functions $Z \to C(X,Y)$ that come from currying functions $Z \times X \to Y$ over all possible Z, i.e., it is the topology consisting of only those sets whose inverse under all such functions is open in Z (for any Z). This topology, called the **natural topology** $C_{nat}(X,Y)$, is a splitting topology. If σ is any topology on $C(X,Y)$, then the identity $C_{nat}(X,Y) \to C_\sigma(X,Y)$ is continuous iff

$$Z \to C_{nat}(X,Y) \to C_\sigma(X,Y)$$

is continuous for every function $Z \to C_{nat}(X,Y)$ obtained by currying, but this is equivalent to σ begin splitting. Therefore, $C_{nat}(X,Y)$ is the finest splitting topology. We next claim that the natural topology is the intersection of all cosplitting topologies (that it is contained in the intersection is clear). To see this, let $U \subseteq C(X,Y)$ be not open in the natural topology. We wish to construct a cosplitting topology τ on $C(X,Y)$ where U is not open.

Given that U is not open in $C_{nat}(X,Y)$, there is some space Z and a continuous map $f\colon Z \times X \to Y$ such that $f^{\#-1}(U)$ is not open in Z. This means that there is some $z \in f^{\#-1}(U)$ whose no neighbourhood is contained in $f^{\#-1}(U)$.

The neighbourhoods of z form a poset $\mathcal{I}$ and we obtain a net $\{z_i\}_{i\in\mathcal{I}} \subseteq Z$ of elements from the neighbourhoods of z none of which are in $f^{\#-1}(U)$. This net converges to z. Now we have an idea for a topology τ on $C(X,Y)$ where U is not open: call all those subsets V open if either $f_z \notin V$ or $f_{z_i} \in V$ for "sufficiently large" i, i.e., the net, a directed set, should land in V. It is easy to see that such a collection forms a topology and in this topology f_{z_i} converge to f_z and U is not open by definition. We are left to show that τ is cosplitting.

Let $V \subseteq Y$ be open and say $ev(g,x) \in V$ for some $(g,x) \in C_\tau(X,Y) \times X$. If $g \neq f_z$, then $\{g\} \times W \subseteq ev^{-1}(V)$ is open in $C_\tau(X,Y) \times X$ where $W \subseteq g^{-1}(V)$ is a neighbourhood of x.

If $g = f_z$, then there is an open neighbourhood $(z,x) \in V_1 \times V_2 \subseteq f^{-1}(V)$. Consider $R = \{h \in C(X,Y) : h(W \cap V_2) \subseteq V\}$ where W is as before. Then R includes f_z and f_{z_i} for sufficiently large i which are contained in V_1, hence is open in $C_\tau(X,Y)$ and it's clear that $R \times (W \cap V_2) \subseteq ev^{-1}(V)$. Thus, the evaluation map is continuous, hence τ is cosplitting. Since U is not open in $C_\tau(X,Y)$ we conclude that the natural topology is indeed the intersection of all cosplitting topologies (this doesn't necessarily make it cosplitting).

Therefore, if an exponential topology exists, then it must be the natural topology.

Converging evenly

Following [2] we give one example of a cosplitting topology. The idea is to make the evaluation map $C_\tau(X,Y) \times X \to Y$ continuous using the notion of convergence of nets. On the left side we have a product topology, however by continuity, the first factor more or less controls the second factor when looking at inverses of open sets of Y. We say a net (a map from a poset to $C(X,Y)$) *converges evenly* to f if for every $V \overset{open}{\subseteq} Y$, given $y \in V$ there exists W_y, neighbourhood of y, such that $f^{-1}(W_y) \subseteq f_i^{-1}(V)$ for "sufficiently large" i. Call a subset $U \subseteq C(X,Y)$ open if given a net f_i converging evenly to some $f \in U$ the net is eventually in U. This forms a topology $C_{ev}(X,Y)$. Given a net (f_i, x_i) in $C_{ev}(X,Y) \times X$ converging to (f,x) the net (f_i) must converge to f and (x_i) to x. Let $y = f(x)$, then from the definition of even convergence (and the fact that our indexing set is directed) it is clear that $f_i(x_i)$ converges to y. Therefore, the evaluation map is continuous, hence $C_{ev}(X,Y)$ is cosplitting.

Continuous convergence

A net (f_i) in $C(X,Y)$ is said to *converge continuously* to $f \in C(X,Y)$ if for every net (x_j) converging to some $x \in X$, the net $(f_i(x_j))$ in Y converges to $f(x)$. Let σ be a topology on $C(X,Y)$, then

1. σ is splitting iff continuous convergence implies σ-convergence.
2. σ is cosplitting iff σ-convergence implies continuous convergence.
3. σ is exponential iff σ-convergence is equivalent to continuous convergence.

The third statement follows directly from the other two. For the first two statements, the $\Longleftarrow$ direction follows from the sequential (using nets) definition of continuity. The second statement follows from the result that σ is cosplitting iff the evaulation map is continuous (and the sequential form of continuity).

For the first statement, suppose $(f_i)_{i\in I}$ converges continuously to f, take $Z = \{f_i\} \cup \{f\}$ with each $\{f_i\}$ being open and neighbourhoods of f are U such that there is some $i \in I$ such that for $j \geq i, f_j \in U$. This is a topology (verification of this requires I to be a directed poset) and the net (f_i) converges to f by construction. We claim that the evaluation $ev\colon Z \times X \to Y$ is continuous.

Given this, because σ is splitting, $Z \to C_\sigma(X,Y)$ would be continuous. Since convergent nets map to convergent nets, we have σ-convergence of the net (f_i).

It is clear that ev is continuous at (f_i, x) for every $i \in I, x \in X$. Suppose $f(x) = y$ and V is a neighbourhood of Y. If for every open $(f,x) \in U \times W \subseteq Z \times X$ with $W \subseteq f^{-1}(V)$ and U a basic neighbourhoof of f, there was some $g_{U\times W} \in U, x_{U\times W} \in W$ with $g_{U\times W}(x_{U\times W}) \notin V$ ($g_{U\times W}$ must necessarily be different from f), then we obtain[1] a net $(x_{U\times W})_{U\times W}$

- This is a directed poset with $U \times W \leq U_1 \times W_1$ if $U \supseteq U_1, W \supseteq W_1$; to get the directed-ness, observe that each W contains x and I is directed - so we can take intersections of the factors.
- Convergence: given open neighbourhood $W \ni x$, anything larger (in the poset) than $Z \times W$ works.

This means that $(f_i(x_{U\times W}))$ converges to y, hence there is some $i, U \times W$ such that for $j \geq i, U_1 \times W_1 \geq U \times W, f_j(x_{U_1\times W_1}) \in V$.

But now we have a contradiction because $U \times W \ni (f,x)$ where $U = (\{f_j\}_{j\geq i} \cup \{f\})$ is a neighbourhood of (f,x), hence there is a $g_{U\times W} \in U$[2] such that $g_{U\times W}(x_{U\times W}) \notin V$ but such a $g_{U\times W}$ must necessarily be f_j for some $j \geq i$, contradicting continuous convergence. So, ev is continuous, hence σ is splitting.

3.2 Core compactness

Definition. *Let X be a space.*

1. *Given subsets $A, B \subseteq X$, A is said to be well below B, written $A \ll B$ if $A \subseteq B$ and from each covering of B we can extract a finite subcovering of A.*
2. *X is said to be core compact if whenever $x \in V \in \mathcal{O}_X$, there is $x \in U \in \mathcal{O}_X$ with $U \ll V$.*

The motivation being that in the proof of Theorem 1 what we really needed was that from any cover of V we can obtain a cover of V_1, although this doesn't quite give us what we want when we are working with the compact open topology.

It is easy to see that locally compact spaces are core compact. We have the following

Proposition 4. *A regular or Hausdorff space is core compact if and only if it is locally compact.*

Proof. If X is locally compact, then it is core compact. Conversely, suppose X is a core compact regular space. In this case, given $x \in U \ll V$, by regularity there is a closed set $x \in C \subseteq U$ which means that $C \ll V$. This implies that C is compact (an open cover of C together with $V \setminus C$ is an open cover of V).

Next, suppose X Hausdorff and core compact. We show that X is regular. Let $x \in U \ll V$, it suffices to show that $\overline{U} \subseteq V$. Suppose $y \notin V$, then by Hausdorff-ness, for each $z \in V$ there are disjoint open sets $P_z \ni z, Q_z \ni y$. We obtain a finite subcover $U \subseteq P_{z_1} \cup \cdots \cup P_{z_n}$ then $y \in Q_{z_1} \cap \cdots \cap Q_{z_n}$ disjoint from U. Therefore, $y \notin \overline{U}$, thus completing the proof. $\square$

The reader is referred to [2] for an example of a core compact space which isn't locally compact.

Consider X to be the wedge of countably many intervals, say it is all the closed unit radii in the plane at rational angles to the x-axis. The topology on X is one coherent with the topologies on the intervals, i.e., U is open (closed) iff the intersection with each interval is open (closed). It is fairly easy to see that X is Hausdorff, and compact subsets of the intervals are compact in X. It is also compactly generated: for if V is a subset whose intersection with each compact K is open, then the same is true when intersected with the intervals, and since the intervals are compact Hausdorff (hence compactly generated), V is open in each of the intervals, hence in X.

Now, take X to be a neighbourhood of the origin and suppose V is an open neighbourhood of X. Label the intervals I_n, so $V \cap I_n$ contains an interval $[0, a_n)$ for some $a_n > 0$. Denote $b_n = a_n/3$ and

[1] Using the Axiom of Choice

[2] Again, axiom of choice lets us choose all of this before proceeding with the argument.

consider the open cover $\{(b_n,1])\}_n \cup \{\cup[0,2b_n)\}$. This is an open cover of X and no finite subcollection covers V. Therefore, X is compactly generated, but not core-compact. In algebraic topology, CW complexes play an important role and moreover, we would like to have the exponentiable property. Here X is a CW complex, but is not exponentiable and the cause of this failure is that X is not locally finite.

Lemma 2. *For a space X*

1. *If $A_i \ll B_i$ for $i = 1,2$ then $A_1 \cup A_2 \ll B_1 \cup B_2$.*
2. *If X is core compact and $U \ll V$ are open sets then there is an open W with $U \ll W \ll V$.*

Proof. The first part is clear. For 2, for each $x \in V$ iteratively choose $\tilde{W}_x, W_x$ open such that $x \in \tilde{W}_x \ll W_x \ll V$. $\{\tilde{W}_x\}$ covers V, hence we obtain a finite subcover $\cup_{i=1}^n \tilde{W}_{x_i} \supseteq U$. We then have $U \subseteq \cup_{i=1}^n \tilde{W}_{x_1} \ll \cup_{i=1}^n W_i \ll V$, so putting $W = \cup_{i=1}^n W_i$ we have $U \ll W \ll V$. □

Theorem 5. *If X is core compact then the Isbell topology on $C(X,Y)$ for any Y is exponential.*

Proof. We are left to prove that it is cosplitting, or equivalently, the evaluation map $C_{Is}(X,Y) \times X \to Y$ is continuous. Let $V \subseteq Y$ be open and say $(f,x) \in ev^{-1}(V)$. Since f is continuous, $f^{-1}(V)$ is open and there is a neighbourhood $U \ni x$ well below $f^{-1}(V)$. Let $\mathbb{U} = \{W \in \mathcal{O}_X : U \ll W\}$. It is clear that if $V_1 \in \mathbb{U}$ and $V_1 \subseteq V_2$, then $V_2 \in \mathbb{U}$. Suppose some union $\cup_i V_i \in \mathbb{U}$, from the lemma we obtain a $\tilde{W}$ such that $U \ll \tilde{W} \ll \cup_i V_i$. From here we obtain a finite union $\cup_{j=1}^k V_{i_j}$ covering $\tilde{W}$ and U is well below this finite union (but $\tilde{W}$ need not be) for $U \ll \tilde{W}$. Therefore, $\mathbb{U}$ is a Scott open set.

Now $(f,x) \in M(\mathbb{U},V) \times U \subseteq ev^{-1}(V)$, hence ev is continuous, thus $C_{Is}(X,Y)$ is exponential, and hence, X is exponentiable. □

Suppose $C(X,Y)$ has a cosplitting topology, then $ev\colon C(X,Y) \times X \to Y$ is continuous. Let $V \in \mathcal{O}_Y$ and say $f(x) \in V$, then there is an open $\mathcal{N} \subseteq C(X,Y)$ and a neighbourhood U of x such that $ev(\mathcal{N} \times U) \subseteq V$. In other words, $g^{-1}(V) \supseteq U$ for every $g \in \mathcal{N}$. So, it is suggestive to look at the lattice $\mathcal{O}_X$ so that $\mathcal{N}$ is some special subset of this lattice. The points of this lattice are open sets of X. Now, define the Sierpinski space $\mathbb{S}$ to be the space $\{0,1\}$ with topology $\{\emptyset,\{1\},\mathbb{S}\}$. The continuous maps $X \to \mathbb{S}$ are precisely the indicators on open sets, i.e., the points of $\mathcal{O}_X$.

If X is exponentiable, then $C(X,\mathbb{S})$ has an exponential topology (and in that case, it must be the natural topology). From the previous paragraph, it seems that there is some connection between the lattice structure $\mathcal{O}_X$ and cosplitting topologies on $C(X,Y)$. Moreover, if X is core-compact, then it is exponentiable. Being core-compact has something to do with the lattice $\mathcal{O}_X$, so we study $C(X,\mathbb{S})$.

Topologies on $C(X,\mathbb{S})$ are basically topologies on the lattice $\mathcal{O}_X$. What are the cosplitting topologies on $C(X,\mathbb{S})$? One obvious topology is the "lower limit" topology which is generated by subsets of the form $L_U = \{V \supseteq U\}$ as U varies. This is a cosplitting topology, but it seems intuitive that the open sets are larger than necessary for the evaluation $C(X,\mathbb{S}) \times X \to \mathbb{S}$ to be continuous. Clearly, we could leave some χ_U functions isolated and it wouldn't hurt the continuity, i.e., there's no need to involve everything larger than U.

Secondly, what should a meaningful notion of convergence in $\mathcal{O}_X$ look like? We would be looking at a sequence of open sets and it seems natural to look at their *finite* unions and see where that goes. Intuitively, allowing for arbitrary unions leads to arbitrary sets, but, as we have seen before, it's the finiteness that in some sense leads to the correct version of a tube lemma (either by compactness or more generally core compactness). The finiteness allows for small sets (countable collections also seem small, but again, we are interested in getting a version of something like the tube lemma). Another thing to notice is that if we start with an open cover of some open set, then the well below open subsets are hiding in finite sub-unions.

To make notations clear, for a subset $A \subseteq X$, we define $\langle A,\{1\}\rangle = \{f \in C(X,\mathbb{S}) : A \subseteq f^{-1}(1)\}$, i.e. all open sets containing A. Now observe that $\langle A,\{1\}\rangle \cap \langle B,\{1\}\rangle = \langle A \cup B,\{1\}\rangle$.

Given a collection $\omega = \{W_i\}$ of open subsets, set $W = \cup W_i$ and define a topology $C_\omega(X,\mathbb{S})$ as follows. If $U \neq W$, then χ_U is isolated. The basis around χ_W is given by $\langle W_{i_1} \cup \cdots \cup W_{i_n},\{1\}\rangle$ where

$\{i_1, \ldots, i_n\}$ is a finite subset of our indexing set. This is basically "looking for" core compact spaces ([2] defines this a little differently) hiding in this particular cover of W. It is obvious that $C_\omega(X, \mathbb{S})$ is a cosplitting topology.

The study of lattices and topologies is quite a rich subject and it turns out that X is core-compact iff $\mathcal{O}_X$ is a continuous lattice. For more on continuous lattices, the reader is referred to [5].

We come to the following theorem:

Theorem 6. *The following are equivalent:*

1. *X is exponentiable.*
2. *An exponential topology exists on $C(X, Y)$ for each Y.*
3. *X is core compact.*
4. *An exponential topology exists σ on $C(X, \mathbb{S})$, where $\mathbb{S}$ is the Sierpinski space.*
5. *If $q\colon Y \to Z$ is a quotient map, then $q \times id_X\colon Y \times X \to Z \times X$ is a quotient map.*

Proof. We already have $1 \iff 2$ and $3 \implies 2 \implies 4$. Now assume 4, so $ev\colon C_\sigma(X, \mathbb{S}) \times X \to \mathbb{S}$ is continuous. The inverse of $\{1\}$ is the set $\{(f, x) : f = \chi_U, x \in U \in \mathcal{O}_X\}$. Pick an $x \in X, U \ni x$ an open neighbourhood. Then there is some basic open set $\mathcal{N} \times V \ni (\chi_U, x) \in C_\sigma(X, \mathbb{S}) \times X$ and this would mean $U \supseteq V \ni x$ and we claim that $V \ll U$. Let $\omega = \{W_i\}$ be a cover and look at the corresponding topology $C_\omega(X, \mathbb{S})$. Since this is cosplitting, it is finer than $C_\sigma(X, \mathbb{S})$, hence there is some basic open set $\chi_U \in \langle W_{i_1} \cup \cdots \cup W_{i_k}, \{1\}\rangle \subseteq \mathcal{N}$ and this gives us a finite subcover for V. Therfore, we get $4 \implies 3$.

Next, suppose $q\colon Y \to Z$ is a quotient map and X is exponentiable. We use the next lemma as a characterization of quotient maps. $q \times id_X\colon Y \times X \to Z \times X$ is certainly a continuous surjection. Suppose we had maps $h\colon Y \times X \to P, \tilde{h}\colon Z \times X \to P$ such that $h = \tilde{h} \circ (q \times id_X)$. Observe that upon currying, $h^\# = \tilde{h}^\# \circ q$. Continuity of $\tilde{h}$ implies the continuity of h. If h was continuous, then by the splitting part, $h^\#$ would be continuous. Because q is a quotient map, $\tilde{h}^\#$ will be continuous, by the cosplitting part, $\tilde{h}$ should be continuous, which makes $q \times id_X$ a quotient map. That proves $2 \implies 5$.

The fact that 5 implies the other statements seems to require more category theory than the author is currently equipped with, but the main result is the adjoint functor theorem (see [6] chapter 3 for a careful discussion) which says that because Top is somewhat nice, and $- \times X$ preserves quotients (along with other adjectives), it has a right adjoint. $\square$

Lemma 3. *Let $q\colon Y \to Z$ be a continuous sujection. Then q is a quotient map iff for all spaces P, whenever there are maps $h\colon Y \to P, \tilde{h}\colon Z \to P$ such that $\tilde{h} \circ q = h$, the continuity of $h, \tilde{h}$ are equivalent.*

Proof. Continuity of $\tilde{h}$ always implies continuity of h. Suppose q is a quotient map, h is continuous. Take V open in P, then $q^{-1}(\tilde{h}^{-1}(V)) = h^{-1}(V)$. So $\tilde{h}^{-1}(V)$ is open, hence $\tilde{h}$ is continuous.

For the other implication, take $P = Y/\sim$ where $\sim$ is an equivalence relation given by $a \sim b \iff q(a) = q(b)$. Give P the quotient topology. There is a map $Z \to P$ such that the triangle commutes and the quotient map $Y \to P$ is continuous by definition of the topology on P. Therefore, $Z \to P$ is a continuous bijection. It is not hard to see that the induced map $Z \to P$ is always open, hence q is a quotient map. $\square$

Theorem 7. *Suppose X is locally compact Hausdorff and $\sim$ is an equivalence relation on a space Y. On $Y \times X$ we have the relation $(a, b) \sim' (c, d)$ if $b = d$ and $a \sim c$. This is an equivalence relation and we have $Y/\sim \times X \cong (Y \times X)/\sim'$.*

Proof. Follows from the fact that locally compact Hausdorff spaces are core compact. $\square$

4 Applications

This section assumes that the reader is somewhat familiar with the path space construction and fibrations. Given a pointed space (Y, y_0), the path space $P(Y, y_0)$ is all those paths in Y starting from y_0. Intuitively, the constant path is a strong deformation retract of the path space because all paths can be simultaneously shrinked by shrinking the origin. We prove it next.

Theorem 8. *Let x_0 be a strong deformation retract of a normal space X and consider the space P of all continuous maps $(X, x_0) \to (Y, y_0)$. With the compact open topology, the constant map is a strong deformation retract of P.*

Proof. Let $r\colon X \times I \to X$ be the strong deformation retract. Consider

$$\begin{aligned} h\colon P \times I &\to P \\ (f, t) &\mapsto \{x \mapsto f(r(x,t))\} \end{aligned}$$

Note that $f(r(x,t))$ is a continuous function on X and maps x_0 to y_0 for every t_0 because r is a strong deformation retract. The constant path is fixed by h and when $h(f, 0) = f$. We just need to show that h is continuous. To that end, let K be a compact subset of X and U open and suppose $(f, t) \in h^{-1}(M(K, U))$. Let $K_1 = r_t(K)$ which is compact because r_t is continuous (see lemma). By normality and continuity of f, there is an open set V with compact closure such that

$$K_1 \subset V \subset \overline{V} \subset f^{-1}(U).$$

Now, $r^{-1}(V)$ is an open set containing $K \times \{t\}$, so by the tube lemma there is an open neighbourhood W of t such that $K \times W \subseteq r^{-1}(V)$.

Now observe that $M(\overline{V}, U) \times W$ is open, contains (f, t) for $\overline{V} \subseteq f^{-1}(U)$ and is contained in $h^{-1}(M(K, U))$ for given any $g \in M(\overline{V}, U)$ and $s \in W$, we know that r_s maps K into V which is then mapped into U by g for g maps all of $\overline{V}$ into U. This completes the proof. In particular, the constant path is a strong deformation retract of the path space. □

Theorem 9. *A connected path connected space Y is a quotient of its path space $P(Y, y_0)$.*

Proof. There is the obvious map $\phi\colon P(Y, y_0) \to Y$ seding a path ω to $\omega(1)$, and this is continuous as the inverse of an open $U \subseteq Y$ is $M(\{1\}, U)$. Since Y is connected, locally path connected, it is path connected, therefore ϕ is surjective. We just need to show that ϕ is open. Consider a basis element $W = \cap_{i=1}^{n} M(K_i, U_i)$. Index these sets so that $1 \in K_1 \cap \cdots \cap K_k$ and $1 \notin K_{k+1} \cup \cdots \cup K_n$. We know that $\omega(1) \in U_1 \cap \cdots \cap U_k$, so by local path connectedness, there is a path connected neighbourhood V of $\omega(1)$ contained in this intersection. The idea is to perturb ω at the end to end at points in V while keeping it fixed before it enters V (to accomodate for the other compact sets).

Find a t' such that $[t', 1] \cap (K_{k+1} \cup \cdots \cup K_n = \emptyset$ and $\omega([t', 1]) \subset V$. Now, given $y' \in V$, let ω' be a path in V from $\omega(t')$ to y', and consider the path

$$\omega_1(t) = \begin{cases} \omega(1), & 0 \le t \le t'. \\ \omega'(t - t'/1 - t'), & t' \le t \le 1. \end{cases} \tag{1}$$

For $i > k$, it is clear that $\omega_1(K_i) \subseteq U_i$ and for $i \le k$, we have

$$\omega_1(K_i) = \omega(K \cap [0, t']) \cup \omega_1(K_i \cap [t', 1]) \subseteq U_i \cup V = U_i.$$

It follows that $\omega_1 \in W$ and $V \subseteq \phi(W)$, thus proving that ϕ is an open map. □

Remark. The map ϕ identifies paths by their end points and "forgets" the homotopy classes of paths.

Theorem 10. *(Lifting theorem) Let $p\colon (\tilde{X}, \tilde{x_0}) \to (X, x_0)$ be a fibration with unique path lifting. Let Y be a connected locally path connected space. A necessary and sufficient condition that a map $f\colon (Y, y_0) \to (X, x_0)$ have a lifting $(Y, y_0) \to (\tilde{X}, \tilde{x_0})$ is that in $\pi(X, x_0)$, $f_\# \pi(Y, y_0) \subseteq p_\# \pi(\tilde{X}, \tilde{x_0})$.*

Proof. For definition of fibrations etc. see [4]. Here $*$ denotes the path product. If a lift exists, then it is clear that the pushforward of the fundamental group of Y should be contained in that of $\tilde{X}$. We are left to prove the other direction. We have maps

$$(P(Y,y_0),\omega_0) \xrightarrow{\phi} (Y,y_0) \xrightarrow{f} (X,x_0)$$

where ω_0 is the constant path. Since $P(Y,y_0)$ is contractible, by homotopy lifting property, $f\circ\phi$ lifts to a map $\tilde{f}\colon (P(Y,y_0),\omega_0) \to (\tilde{X},\tilde{x_0})$. It suffices to show that $\tilde{f}$ factors through the quotient map ϕ, i.e., if $\phi(\omega)=\phi(\omega')$, then $\tilde{f}(\omega)=\tilde{f}(\omega')$. The idea is to consider the loop formed by ω,ω' and look at it's life in $\tilde{X}$.

Consider the paths $\omega_1(t)(t')=\omega(tt'),\omega_1'(t)(t')=\omega'(tt')$ in $P(Y,y_0)$ from ω_0 to ω,ω' respectively. Then $\tilde{f}\circ\omega_1,\tilde{f}\circ\omega_1'$ are paths in $\tilde{X}$ from $\tilde{x_0}$ to $\tilde{f}(\omega),\tilde{f}(\omega')$, respectively, such that

$$p\circ\tilde{f}\circ\omega_1 = f\circ\phi\circ\omega_1 = f\circ\omega$$

and similarly with ω'. Now, $\omega*\omega'^{-1}$ is a loop in Y at y_0, so there is a loop $\tilde{\omega}$ in $\tilde{X}$ at $\tilde{x_0}$ which pushes down in X to the push forward of this loop from Y, i.e., $p\circ\tilde{\omega}\sim(f\circ\omega)*(f\circ\omega')^{-1}$. This means that in $\tilde{X}$, the paths

$$\tilde{f}\circ\omega_1 \text{ and } \tilde{w}*\tilde{f}\circ\omega_1'$$

are homotopic when pushed down to X via p. By homotopy extension property, these two paths are homotopic in $\tilde{X}$ which means that their end points must be the same (for in X, their end points are equal, namely $\omega(1)=\omega'(1)$, and p has unique path lifting). Therefore, $\tilde{f}(\omega)=\tilde{f}(\omega')$ which means that $\tilde{f}$ factors through ϕ. By the previous theorem, we get a continuous lift of f to $\tilde{X}$ as required. □

Corollary. *If X is connected, locally path connected, then a section of p exists iff $\pi(X,x_0)=p_\#\pi(\tilde{X},\tilde{x_0})$. Moreover, if the equality holds and if $\tilde{X}$ is path connected, then p is a homeomorphism.*

Proof. The first part is obvious (see [4] section 2.4 for more details). For the second part, by unique path lifting, there is a right action of $\pi(X,x_0)$ on the fibres of x_0 giving by the monodromy action (see [4] section 2.3). This is given by lifting the loops to start at a given point in the fibre $p^{-1}(x_0)$ and looking at the other end point. Because $\tilde{X}$ is assumed to be path connected, this action is transitive (requires unique path lifting as well) and the stabilizers of any point in the fibre is the image of the fundamental group of $\tilde{X}$ based at that point in $\pi(X,x_0)$. If we assume equality, then the stabilizer of any point in the fibre is the whole group, therefore, the fibres have to be singletons. This means that p is a bijection (note that by the path lifting property, p is surjective since X is path connected). By the first part of the corollary, we have a section of p which implies that p is an open map, thus a homeomorphism. □

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