Averaging Systems with Applications to Opinion Dynamics

Final Project

ECEN 5008: Coordinated Control of Multi Agent Systems

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## Motivation

The paper provides a lucid explanation for concepts covered in ECEN 5008: Coordinated control of Multi Agent Systems taught at CU Boulder. With the presented material, I hope to achieve the following-

- Demonstrate a clear understanding of topics covered in class.
- A quick and handy refresher without having to go back to the textbook
- Additional course material/resource for future students enrolling into ECEN 5008.

If the textbook is food for engineers, the presented manuscript is the digested version of the textbook- essentially the same but broken down into simpler components.

The document is divided into two parts, part A and part B. Part A covers the theoretical aspects of graph theory and matrix algebra applied to averaging systems. In part B, results from part A are used to validate numerical simulations of some popular opinion dynamics models, namely the French-Harary-DeGroot model, the Friedkin and Johnsen model, the Metropolis-Hastings model, and the Equal neighbor model.

## PART A

### 1. Strongly Connected graphs

Consider the summation matrix $S = \sum_{i=0}^{n-1} A^k$ which is central to algebraic graph theory. Following section studies each part of $S$ in detail to arrive at the Perron-Frobenius theorem and the convergence properties of the adjacency matrix A.

#### 1.1. What is $A^k$?

Let A be the adjacency matrix for a weighted graph G, i.e. $a_{ij} > 0$ if there is a path from node i to j and $a_{ij} = 0$ if there isn't. Here onwards, $m \to n$ represents a path from node m to node n. What do the powers of A represent?

Consider $A$,

$$\begin{bmatrix} a_{11} & a_{12} & . & . & . & a_{1n} \\ a_{21} & & & & & \\ . & & & & & \\ . & & & & & \\ . & & & & & \\ a_{n1} & & & & & a_{nn} \end{bmatrix}$$

$a_{11} > 0$ if there is a self-loop or path $1 \to 1$ from node 1 to itself. Similarly, $a_{12} > 0$ if there is a path $1 \to 2$ from node 1 to node 2 … (1)

Naturally, $A^2$ *will be*

$$\begin{bmatrix} * & * & * & . & . & * \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \end{bmatrix} \begin{bmatrix} * & & \\ * & & \\ * & & \\ * & & \\ * & & \\ * & & \end{bmatrix}$$

Let examine the connection between node 1 and node 2.

The element at index (1,2) of $A^2$, $a'_{12}$ will be a result of multiplying row 1 and column 2 of A (Incidentally, the result is the calculation of cross correlation between row 1 and column 2, which is a fantastic indicator of how much information sent out by node 2 enter node 1). What will be cause $a'_{12}$ to be positive? Consider the expression given below.

$$(1 \to 1)\&(1 \to 2)\ OR\ (1 \to 2)\&(2 \to 2)\ OR\ \dots\ (1 \to N)\&(N \to 2)$$

Using (1), $a'_{12}$ will be positive if there is a path from 1 to 1 & then from 1 to 2 OR a path from 1 to 2 & then a self-loop at 2 OR a path from node 1 to 3 & then from 3 to 2, and so on.

Continuing, $A^3$ *would be given by*

$$\begin{bmatrix} * & * & * & . & . & * \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \end{bmatrix} \begin{bmatrix} * & & \\ * & & \\ * & & \\ * & & \\ * & & \\ * & & \end{bmatrix} \begin{bmatrix} * & & \\ * & & \\ * & & \\ * & & \\ * & & \\ * & & \end{bmatrix}$$

The element at index (1,2) of $A^3$, $a''_{12}$ will be positive if you traverse the self-loop at 1 twice & then have a path from 1 to 2 OR have a path from 1 to 2, then from 2 to 1, then again from 1 to 2, and so on. All possible ways are summarized in the expression below.

$$[(1 \to 1)\&(1 \to 1)\ OR\ (1 \to 2)\&(2 \to 1)\ OR\ \dots\ (1 \to N)\&(N \to 1)](1 \to 2) +$$
$$[(1 \to 1)\&(1 \to 2)\ OR\ (1 \to 2)\&(2 \to 2)\ OR\ \dots\ (1 \to N)\&(N \to 2)](2 \to 2) +$$
$$\dots$$
$$[(1 \to 1)\&(1 \to N)\ OR\ (1 \to 2)\&(2 \to N)\ OR\ \dots\ (1 \to N)\&(N \to N)](N \to 2)$$

The discussion above paints a clear picture of what the multiplication machine does. It sets up all the number of ways

in which you can reach from node 1 to node 2 in $k$ steps and add them up. This leads to our first important result:

*Element at the index $(i, j)$ of $A^k$ is positive if there exists a path of length k between node i and node j* … (2)

### 1.2. The Summation matrix S

Now that we know what powers of the adjacency matrix represent, let's analyze the sum of powers of $A$. If S is a positive matrix, or $s_{ij} > 0 \; \forall \; i, j$, it means that for some arbitrary $k$ there exists a path between node $i$ and node $j$. In other words, one can start from any node $i$ and end at any node $j$. Therefore, the graph is strongly connected. For the graphs which are not strongly connected, globally reachable nodes correspond to the positive columns of S.

*Why does the summation start at zero index?*

Zero index simply means you don't have to travel or need a path to stay at where you are. Unless you're beyond the event horizon of a black hole, in which case no one really knows.

*Why does the summation end at n-1 index?*

Consider node 1 and node n. The longest path you can take to reach n from 1 is of length n-1, shown below

$$1 \rightarrow 2 \rightarrow 3 \ldots n-1 \rightarrow n$$

If there exists no path of length $n-1$, or if the entry at index $(1, n)$ of $A^{n-1}$ $is$ 0, there is no existence of paths longer than n-1 since higher powers simply traverse the loops multiple times. Hence, its sufficient to go up to index $n-1$ in summation.

### 1.3. Irreducible matrices

Matrix A is irreducible if its summation matrix S is positive. Therefore, we have a fast and easy way to check the connectivity of a graph. The digraph G is strongly connected if its corresponding adjacency matrix A is irreducible.

Another definition one might find in the literature regarding irreducible matrices is as follows: -

The matrix A is irreducible if there exist no permutation matrix P such that $PAP^T$ is block upper triangular. Conversely, A is reducible if for some matrices $P, B, C, D$,

$$PAP^T = \begin{bmatrix} B & C \\ \mathbf{0} & D \end{bmatrix}$$

This is the math equivalent of calling gold as bright, shiny yellow. Let's look at what the similarity transformation is doing. Divide the block triangular matrix in 4 sections: - $B, C, D$ and the zero matrix. We've segregated the nodes into colonies $B$ and $D$. $C$ shows the routes we can take to reach D from B. But since there is a big chunk of zero matrix in lower left section, there is no way to reach colony B from D. Therefore, the graph is not strongly connected, and the corresponding adjacency matrix is reducible.

### 1.4. Primitive matrices

A matrix is called primitive if its $k^{th}$power is positive. i.e., $A$ is primitive if $A^k > 0$ for some $k \in N$. The definition makes it obvious that every primitive matrix is irreducible and therefore, every digraph with a primitive adjacency matrix is strongly connected. But can we extract some more information about the structure of the graph? Since $A^k > 0$, every entry in the diagonal of $A^k$ is also positive which confirms the existence of self-loops, and as a result the graph *aperiodic*. Therefore, a primitive adjacency matrix corresponds to a strongly connected and aperiodic digraph.

## 2. Convergence Results

Having understood the properties of a graph and its adjacency matrix, we are ready to apply some useful properties and theorems taken from matrix algebra. Mostly, we will be dealing with *semi-convergence*, or when the system converges to a value other than 0.

### 2.1. Row Stochastic matrices

Row stochastic matrices are non-negative matrices wherein all rows sum up to 1, which implies that each element is at most 1. 1 is always an eigenvalue for any row stochastic matrix.

Such matrices find applications in probability and averaging systems because of their convergence properties. Let's find out why it is so with an example.

Consider the matrix $A = \begin{bmatrix} 0.3 & 0.7 \\ 0.1 & 0.9 \end{bmatrix}$

Cleary, all its rows sum up to 1. What happens when you multiply it with a vector, say $x = [1 \; 2]^T$? Elements of $x$ could represent states or transition probabilities.

$$Ax = \begin{bmatrix} 0.3 & 0.7 \\ 0.1 & 0.9 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0.3(1) + 0.7(2) \\ 0.1(1) + 0.9(2) \end{bmatrix} = \begin{bmatrix} 1.7 \\ 1.9 \end{bmatrix}$$

Evolving our state by multiplying it with $A$ at every step,

Second Iteration

$$A(Ax) = \begin{bmatrix} 0.3 & 0.7 \\ 0.1 & 0.9 \end{bmatrix} \begin{bmatrix} 1.7 \\ 1.9 \end{bmatrix} = \begin{bmatrix} 0.3(1.7) + 0.7(1.9) \\ 0.1(1.7) + 0.9(1.9) \end{bmatrix} = \begin{bmatrix} 1.84 \\ 1.88 \end{bmatrix}$$

Third Iteration

$$A(A(Ax)) = \begin{bmatrix} 0.3 & 0.7 \\ 0.1 & 0.9 \end{bmatrix} \begin{bmatrix} 1.84 \\ 1.88 \end{bmatrix} = \begin{bmatrix} 0.3(1.84) + 0.7(1.84) \\ 0.1(1.88) + 0.9(1.88) \end{bmatrix} = \begin{bmatrix} 1.868 \\ 1.876 \end{bmatrix}$$

After every step, it seems that our states move closer to each other. The initial difference between our states was 1 (2-1) which reduces to 0.008 in mere three iterations thanks to the structure of A.

*Convex Combination*

Consider an arbitrary row stochastic matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, where $a + b = 1, \& \, c + d = 1$.

When $A$ is multiplied by a vector $x = [x_1 \; x_2]^T$, we get a special class of linear combination of $A's$ columns, called as $convex\ combination$.

$$Ax = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} ax_1 + bx_2 \\ cx_1 + dx_2 \end{bmatrix}$$

A $convex\ combination$ of points $x_1, x_2, x_3, \dots, x_k$ is given by $\theta_1 x_1 + \theta_2 x_2 + \theta_3 x_3 + \cdots + \theta_k x_k$, where $\theta_1 + \theta_2 + \theta_3 + \cdots + \theta_k = 1$.

$ax_1 + bx_2 \; \& \; cx_1 + dx_2$ are convex combinations of $x_1 \; \& \; x_2$, since $(a + b) = (c + d) = 1$. Convex combination of $x_1, x_2$ generates a new point which divides the line segment joining $x_1, x_2$ in the ratio $b{:}a$ internally. On taking the combination once, you get a set of points $x'_1, x'_2$ between $x_1$ and $x_2$. On taking the convex combination of $x'_1, x'_2$, we get another set of points $x''_1$ and $x''_2$ which lie between $x'_1, x'_2$. Therefore, doing the process iteratively, starting from $x_1 \; \& \; x_2$, we keep reducing the length of line segment $\overline{x_1 x_2}$ untill it reaches 0 at the point of convergence, say $x^*$, and the system converges ($A$ is then a *semi-convergent* matrix since $x^* \neq 0$). Figure 1 convey the process for $A = \begin{bmatrix} 0.3 & 0.7 \\ 0.1 & 0.9 \end{bmatrix}$ & $x = [1 \; 2]^T$.

Convex combination of $x_1 \; \& \; x_2$ gives a point between $x_1 \; \& \; x_2$. This is the key idea for convergence. If it were a simple linear combination, we could get a result well outside $x_1 \; \& \; x_2$, and when done iteratively, the solution may eventually blow up.

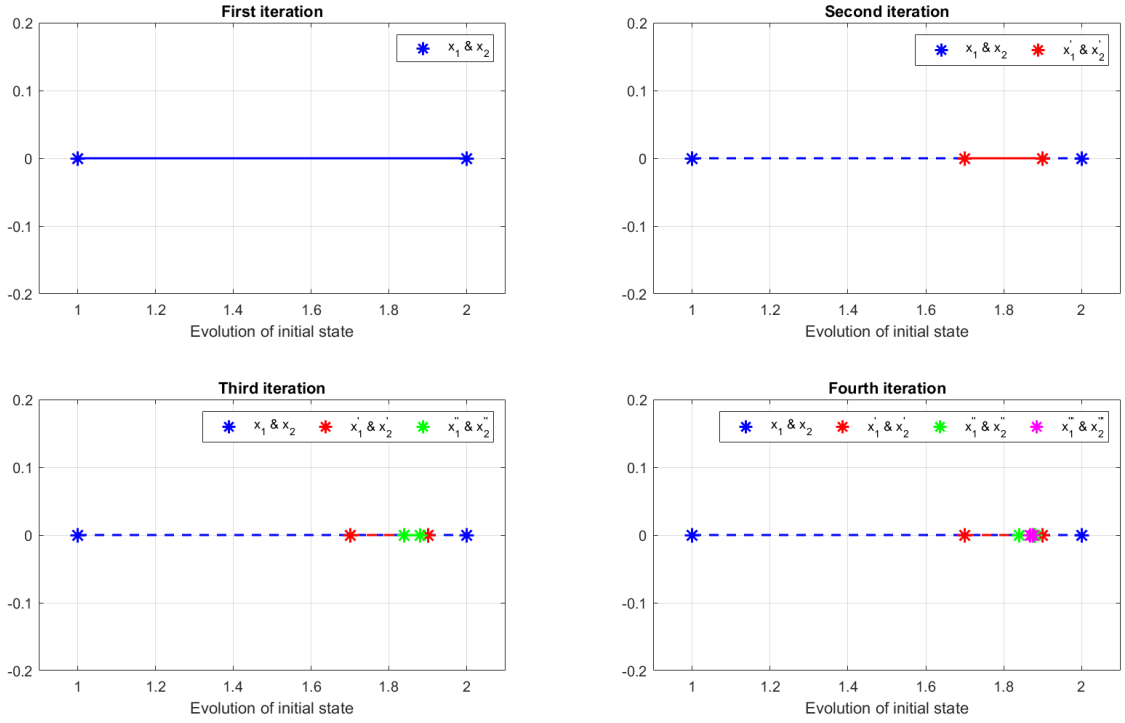



*Figure 1*

If $b > a$, we weigh $x_2$ more than $x_1$ and the new point generated will be closer to $x_2$. If $b > a$ and $d > c$, Both $x'_1, x'_2$ will be closer to $x_2$ than $x_1$ and consequently, $x^*$ will be closer to $x_2$, or $x_2$ has more weightage, or their opinion matters more. A 2x2 doubly stochastic adjacency matrix (one in which rows and columns sum to 1) is symmetric and of the form

$$A = \begin{bmatrix} a & b \\ b & a \end{bmatrix}$$

The points generated after first iteration are $x'_1 \; \& x'_2$. $x'_1$ would be as far away from $x_1$ as $x'_2$ is from $x_2$. This process continues after each iteration till the states meet bang in the middle, at $x^*$. Therefore, the states reach $average\ consensus$. Indeed, it is the case for number of states $n > 2$ that $average\ consensus$ is reached when the Adjacency matrix is doubly stochastic.

*Are all row stochastic matrices semi-convergent?*

No. Take $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ for example.

$$A^k = \begin{cases} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, k\ is\ odd \\ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, k\ is\ even \end{cases}$$

Hence, $\lim_{k\to\infty} A^k$ does not exist. Given below are convergence conditions (Theorem 2.7, *Lectures on Network Systems* by *Francesco Bullo*), proof of which can be found in any standard linear algebra textbook. -

$For\ a\ square\ matrix\ A, the\ following\ holds$:

$(i) A\ is\ convergent\ iff\ \rho(A) < 1.$

$(ii) A\ is\ semi-convergent\ iff$

$(a) 1\ is\ a\ semi\ simple\ eigen\ value$
$(b) All\ other\ eigenvalues\ have\ magnitude\ less\ than\ 1.$

### 2.2. Summarizing

Let's bring together everything we've studied about row stochastic matrices, irreducible matrices, primitive matrices and graph theory using the $Perron-Frobenius\ theory$ (theorem 2.12, *Lectures on Network Systems* by *Francesco Bullo*) which states that-

$For\ A \epsilon R^{nXn}, n \geq 2\ be\ non-negative, then$

$(i) there\ exists\ a\ real\ eigenvalue\ \lambda > |\mu| > 0$

$(ii) the\ right\ and\ left\ eigenvectors\ v\ and\ w\ of\ \lambda\ can\ be\ selected\ to\ be\ non-negative.$

$Additionally, if\ A\ is\ irreducible\ then,$

$(iii)\ \lambda\ is\ strictly\ positive\ and\ simple.$

$(iv)\ v\ and\ w\ are\ unique\ and\ positive, upto\ resacling.$

$Additionally, if\ A\ is\ primitive, then$

$(v)\ \lambda\ satisfies\ \lambda > |\mu|\ for\ all\ other\ eigenvalues\ \mu.$

The adjacency matrix $A$ in an averaging system would most likely be row stochastic. Additionally, if $A$ is irreducible, then the eigenvalue 1 is simple from theorem *2.12.(iii)*. Finally, if $A$ is also primitive, all other eigenvalues of $A$ are less than 1 from theorem *2.12.(v)*. Therefore, Theorem *2.7.(ii)* can be applied to conclude that $A$ is *semi-convergent*, and the states converge to values given by Theorem *2.13*

$$\lim_{k\to\infty} A^k = \mathbf{1}_n w^T \text{ and } \lim_{k\to\infty} x(k) = \big(w^T x(0)\big)\mathbf{1}_n, \ldots (3)$$
$$where\ \ w^T\ is\ the\ dominant\ left\ eigenvector\ of\ A$$

Now, most of the results given in the textbook *Lectures on Network Systems* by *Francesco Bullo* can be stated trivially:

Theorem 5.1

"*Consensus for row-stochastic matrices with a globally-reachable aperiodic strongly-connected component*" or in other words the adjacency matrix $A$ is row stochastic and primitive, in which case $\lim_{k\to\infty} x(k)$ is given by (3).

If $A$ is doubly stochastic, the system will reach *average consensus* as discussed in section 2.1 and we have

$$\lim_{k\to\infty} x(k) = average\big(x(0)\big)\mathbf{1}_n \ldots (4)$$

Theorem 5.2

*"Convergence for row-stochastic matrices with multiple aperiodic sinks"* or in other words the adjacency matrix is row stochastic but not irreducible. From Theorem *2.7.(ii)*, the system can still converge to a solution with the solution given as

$$\lim_{k\to\infty} x(k) = \begin{cases} (w^p)^T x(0), if\ node\ i\ belongs\ to\ sink\ p \\ \sum_{p=1}^{n_s} z_{i,p}\big((w^p)^T x(0)\big), otherwise \end{cases} \ldots (5)$$

$where\ n_s\ is\ the\ number\ of\ sinks\ in\ the\ condensation\ graph$

$and\ z_{i,p}\ is\ the\ convex\ combniation\ coefficient\ if\ there\ is\ a\ directed\ path\ \ from\ node\ i\ to\ sink\ p.$

## PART B

In this section, a conversation between 5 panelists is modelled using some of the popular opinion dynamic models. Numerical simulation is followed by analysis and discussion of the results.

### 1. French-Harary-DeGroot panel

French-Harary-DeGroot panel is probably the most widely used opinion dynamics model because of its simplicity. In this model, each panelist is appraised of other panelist's opinion through weights dictated by matrix entries. Opinion at the iteration $k$ is related to the opinion at previous iteration by the following relation,

$$x(k) = Ax(k-1), where\ A\ is\ the\ adjacency\ matrix$$

Each panelist takes weighted average of the nodes they are connected to. Weights are good indicator of attachment to own opinion and interpersonal influence between two connected nodes.

Example 1: let's take

$$A_{panel} = \begin{bmatrix} 0.15 & 0.15 & 0.1 & 0.2 & 0.4 \\ 0 & 0.55 & 0 & 0 & 0.45 \\ 0.3 & 0.05 & 0.05 & 0 & 0.6 \\ 0 & 0.4 & 0.1 & 0.5 & 0 \\ 0 & 0.3 & 0 & 0 & 0.7 \end{bmatrix}$$

With the digraph and condensation graph as follows,

$$S = \begin{bmatrix} 1.2314 & 1.2253 & 0.1708 & 0.4472 & 1.9253 \\ 0 & 2.7992 & 0 & 0 & 2.2008 \\ 0.3856 & 1.0438 & 1.1029 & 0.1268 & 2.341 \\ 0.0634 & 1.6435 & 0.2035 & 1.9507 & 1.1399 \\ 0 & 1.4672 & 0 & 0 & 3.5328 \end{bmatrix}$$

Cleary $S \not> 0$, and therefore the graph is not strongly connected. Drawing the digraph and condensation graph (Figure 2 & Figure 3),

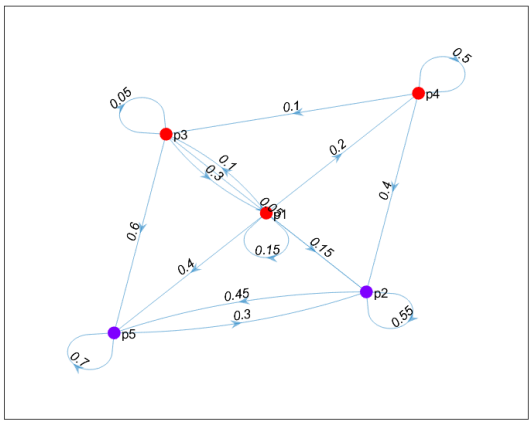



*Figure 2*

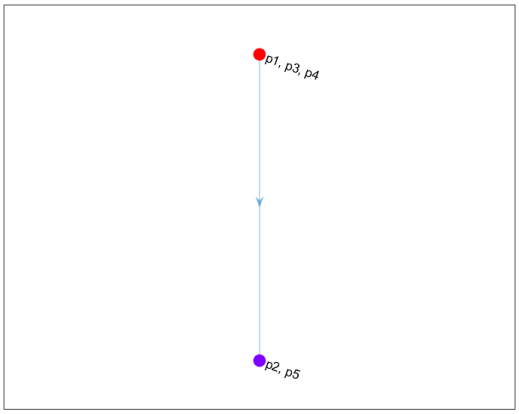



*Figure 3*

For a row stochastic and reducible adjacency matrix, (5) applies and we have

$$\lim_{k\to\infty} x(k) = 0.64\mathbf{1}_n$$

$$w^T = [0, 0.4, 0, 0, 0.6],$$
$$x(0) = [0.15, 0.55, 0.05, 0.5, 0.7]^T$$

Let's go ahead with a numerical simulation and see what the results are in Figure 4.

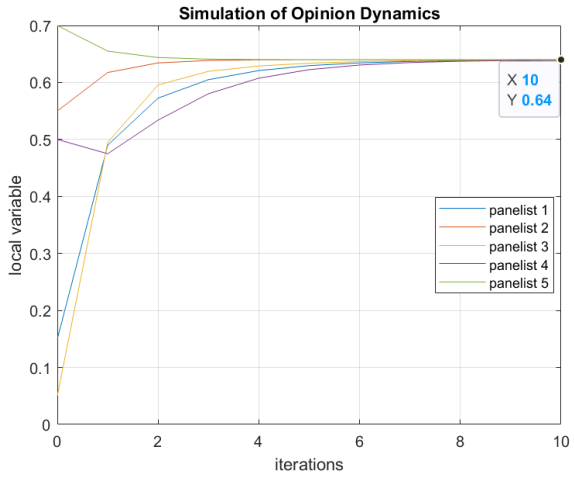



*Figure 4*

The condensation graph has two sinks. Sink 1 contains panelists 1, 3, & 4, and sink 2 contains panelists 2 & 5. There is a directed edge from sink 1 to sink 2 or sink 1 asks for information from sink 2. This behavior is clearly seen from the simulation. Panelist 2 & 5 agree with each other and the members of sink 1 catch up to the agreement reached by sink 2. Numerical simulation verifies that the consensus value reached is 0.64.

## 2. Friedkin and Johnsen panel

The Friedkin and Johnsen model is a persistent disagreement model wherein the states do converge, but not to a common value because of the 'disagreement'. Each node has an openness level $\lambda_i \in [0,1]$, which is a measure of the person's attachment to their own opinion. The model is formulated as

$$x(k+1) = \Lambda A x(k) + (I - \Lambda)x(0)$$

$$where\ \Lambda = diag(\lambda_1, \dots, \lambda_n)\ \&\ A\ is\ the\ adjacency\ matrix$$

If there was no second term $(I-\Lambda)x(0)$ or if $\lambda_i$ =1 (no attachment to initial opinion), we would revert to the French-Harary-DeGroot dynamics. States reach consensus due to the first term $\Lambda A$. The second term adds unequal biases to states in consensus to cause disagreement.

Let's understand the Friedkin and Johnsen model using 2 different examples.

Example 2: $A = \begin{bmatrix} 0.2 & 0.2 & 0.2 & 0.2 & 0.2 \\ 0.2 & 0.2 & 0.2 & 0.2 & 0.2 \\ 0.2 & 0.2 & 0.2 & 0.2 & 0.2 \\ 0.2 & 0.2 & 0.2 & 0.2 & 0.2 \\ 0.2 & 0.2 & 0.2 & 0.2 & 0.2 \end{bmatrix}$

$$\Lambda = diag([0.5, 1, 1, 0.2, 0.8])$$

Numerical simulation and discussion follow in Figure 5 and Figure 6.

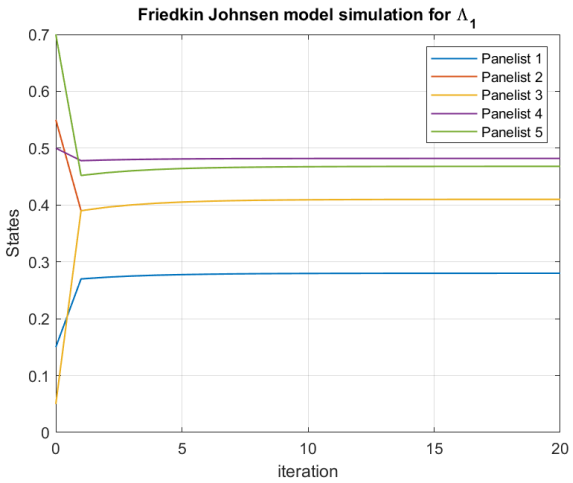



*Figure 5*

Clearly, $\Lambda A$ is semi-convergent and panel will reach consensus. Panelist 2 and 3 have $\lambda = 1$, or no have no attachment to their initial opinion. Subsequently, both agree to a value. All other panels have varied $\lambda$s which causes disagreement. Panelist 4 has the smallest $\lambda$ or highest attachment to their initial opinion and therefore the plot shows that Panelist 4's state has a shallow horizontal trajectory. The final steady state value for panelist 4 is close to the initial opinion.

Example 3: $A = \begin{bmatrix} 0.2 & 0.2 & 0.2 & 0.2 & 0.2 \\ 0.2 & 0.2 & 0.2 & 0.2 & 0.2 \\ 0.2 & 0.2 & 0.2 & 0.2 & 0.2 \\ 0.2 & 0.2 & 0.2 & 0.2 & 0.2 \\ 0.2 & 0.2 & 0.2 & 0.2 & 0.2 \end{bmatrix}$

$$\Lambda = diag([0.25, 0.75, 0.2, 0.8, 1])$$

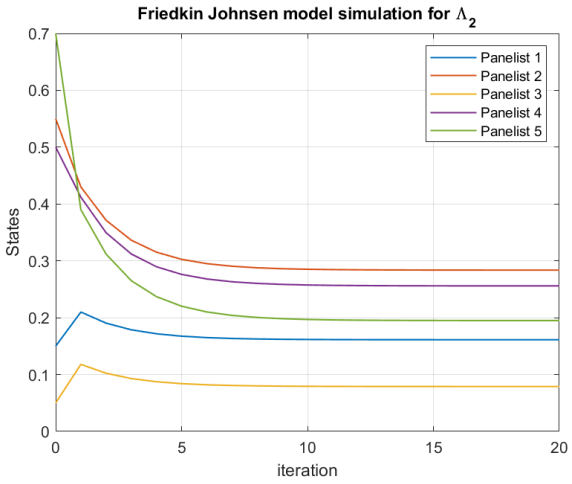



*Figure 6*

Continuing from Example 3's discussion, since all $\lambda$s are unique for Example 4, all the panelists disagree as seen on the plot.

### 3. The Metropolis-Hastings panel

Metropolis-Hastings model is used to design averaging algorithms which achieve average consensus. For a given adjacency matrix A, edge set E and out degree matrix D, the Metropolis-Hastings adjacency matrix is given by

$$(A_{Metr-Hast})_{ij} = \begin{cases} \dfrac{1}{1+\max\{d_i, d_j\}}, \{i,j\} \in E, i \neq j \\ 1 - \displaystyle\sum_{\{i,h\}\in E, i\neq h} (A_{Metr-Hast})_{ih}, if\ i = j \\ 0, otherwise \end{cases}$$

Example 4: $A = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{bmatrix}, D = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}$

Gives $A_{Metr-Hast} = \begin{bmatrix} 0.55 & 0.2 & 0.25 & 0 & 0 \\ 0.2 & 0.2 & 0.2 & 0.2 & 0.2 \\ 0.25 & 0.2 & 0.3 & 0 & 0.25 \\ 0 & 0.2 & 0 & 0.55 & 0.25 \\ 0 & 0.2 & 0.25 & 0.25 & 0.3 \end{bmatrix}$

Also, ${A_{Metr-Hast}}^2 = \begin{bmatrix} 0.4050 & 0.2 & 0.2525 & 0.04 & 0.1025 \\ 0.2 & 0.2 & 0.2 & 0.2 & 0.2 \\ 0.2525 & 0.2 & 0.255 & 0.1025 & 0.19 \\ 0.04 & 0.2 & 0.1025 & 0.55 & 0.25 \\ 0.1025 & 0.2 & 0.19 & 0.2525 & 0.255 \end{bmatrix}$

$\therefore A_{Metr-Hast}$ is doubly stochastic and primitive. The digraph is strongly connected and aperiodic(Figure 7 and Figure 8), and from (4), average consensus would be reached.

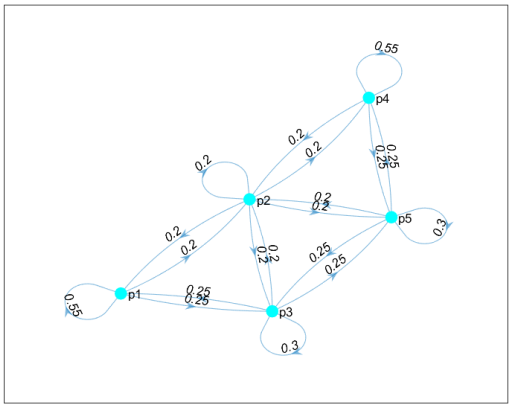



*Figure 7*

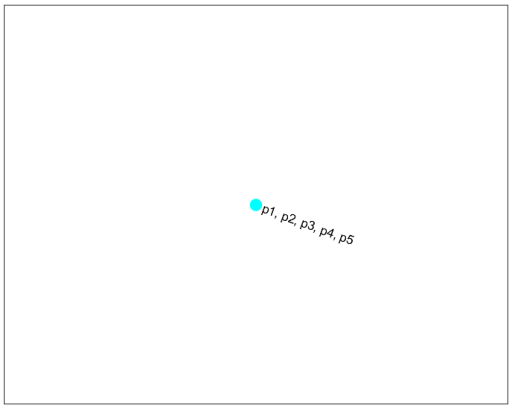



*Figure 8*

Going ahead with a numerical simulation in Figure 9, we can see that all states converge to $\overline{x(0)} = 0.39$.

$$x(0) = [0.15, 0.55, 0.05, 0.5, 0.7]^T$$

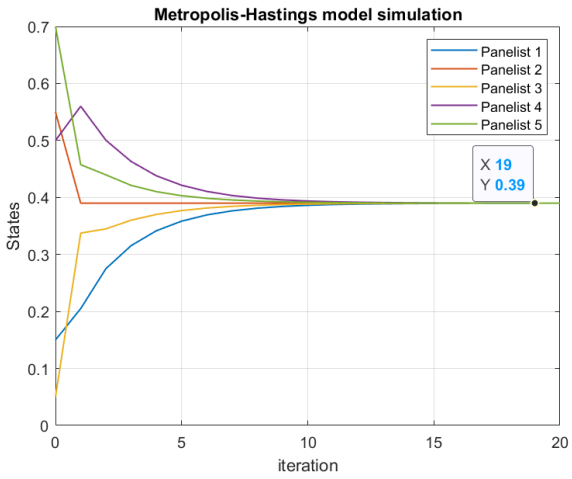



*Figure 9*

### 4. Equal neighbor panel

The equal neighbor model is similar to Metropolis-Hastings model. Difference being each row of binary adjacency matrix

is divided by the out degree of the respective row. For a given binary adjacency matrix A and out degree matrix D. The equal neighbor adjacency matrix is given by

$$A_{equal-Neighbor} = D^{-1}A$$

Example 5: $A = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{bmatrix}, D = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}$

Gives $A_{equal-Neighbor} =$

$$\begin{bmatrix} 0 & 0.5 & 0.5 & 0 & 0 \\ 0.25 & 0 & 0.25 & 0.25 & 0.25 \\ 0.33 & 0 & 0.33 & 0 & 0.33 \\ 0 & 0.5 & 0 & 0 & 0.5 \\ 0 & 0.33 & 0.33 & 0.33 & 0 \end{bmatrix}$$

With $A_{equal-Neighbor}$ as the adjacency matrix, we get the following digraph and condensation graph in Figure 10 and Figure 11.

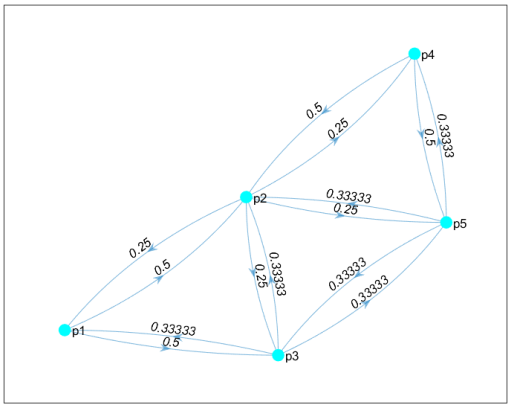



*Figure 10*

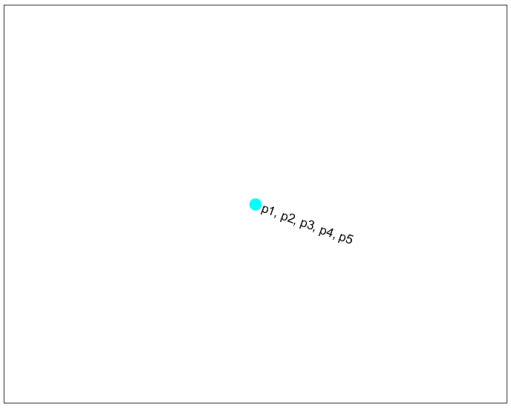



*Figure 11*

The digraph contains no self-loops but is strongly connected. This should hint that $A_{equal-Neighbor}$ is irreducible. Indeed, we can see that $S > 0$.

$$S = \begin{bmatrix} 1.5567 & 1.2685 & 1.0868 & 0.3993 & 0.6887 \\ 0.6343 & 1.956 & 0.8877 & 0.6343 & 0.8877 \\ 0.7245 & 1.1836 & 1.7623 & 0.4591 & 0.8704 \\ 0.3993 & 1.2685 & 0.6887 & 1.5567 & 1.0868 \\ 0.4591 & 1.1836 & 0.8704 & 0.7245 & 1.7623 \end{bmatrix}$$

Numerical simulation produces the following results(Figure 12),

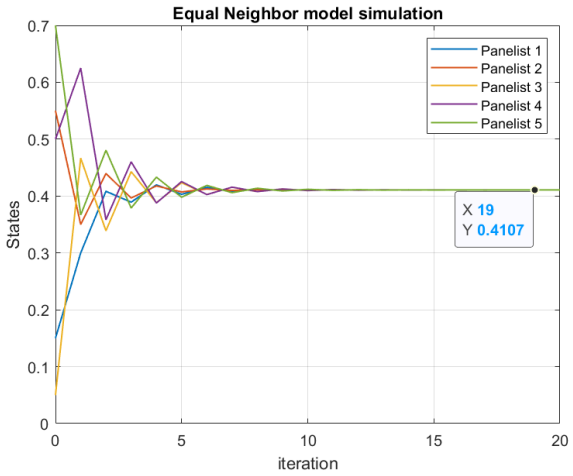



*Figure 12*

For row stochastic irreducible matrices, (3) applies and we get $\lim_{k\to\infty} x(k) = 0.4107\mathbf{1_n}$, where $w^T = [0.1429, 0.2857, 0.2143, 0.1429, 0.2143]$ has been scaled according to $w_{equal-neighbor} = \frac{1}{\sum_{i=1}^{n} d_i}[d_1 \quad . \quad . \quad . \quad d_5]^T$.

## Moving Forward

If you're still into the math

Your matrix is not going to converge all the time in which case algorithms are developed to achieve consensus. Convex optimization is the next topic you should be looking at. It is broadly applied to networked systems and swarms, aimed at minimizing a cost function, getting a required profile and to have fast running software for systems having large number of nodes.

If you're done with math

I feel you. Here's an interesting thought exercise. People usually don't agree to each other's opinions, they agree to schools of thought. It's the overlapping and intersection of these schools which brings about consensus.

Leonardo, Botticelli, and Michelangelo were the greatest artists during the *Italian renaissance.* All three of them probably met at the *platonic academy.* They were not individual identities, but rather represented schools of thought in the field of art. They had different ideas regarding colors, composition, and structure. If the Sistine chapel was to be white-washed and painted again, who's style do you think will prevail? Or will the fans of Raphael cause disagreement for being left out?

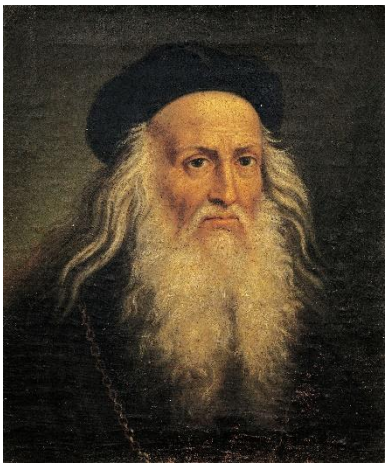
Botticelli

*Botticelli is known for his intimate compositions, depicting beautiful, melancholic figures with naturalness in a closed architectural setting. His drawings have clear contours with slight contrast of light and shadow and are made using soft pastel colors. Female figures dominate with the main group in the focus, with flanks showing different incident.*

Michelangelo

*Michelangelo had an eye for the detail. Being a sculptor, his works are known for their three-dimensional qualities. His paintings bring out a dynamic reality into the subject. They depict the thought process involved in a human state. David is famous for his "fight or flight" look. The paintings maximize contrast range with vivid depth, achieved using shadowing. Michelangelo believed drawings were superior to colors.*

Da Vinci



*Leonardo was an expert of human anatomy, botany, and geology. His works showed how humans registered emotion in expression and gesture. He used subtle gradation of tone, indistinguishable brush strokes and dramatic landscapes with figures. His works are famous for depicting humanity's role in god's incarnation*