JPMC Quant Mentorship Case Study

Shriya Harlapur, IIT Kharagpur
Third-Year Undergraduate Student
Mathematics and Computing (Int. M.Sc.)

29th January 2023

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1 Finance

1.1 Part A

1.1.1 Question 1

a) Total bushels of wheat $= 500,000$
One futures contract $= 5000$ bushels of wheat
Number of bushels of wheat $= \frac{500000}{5000} = 100$
The farmer is planning to sell in the future, hence he needs to go **short** on all the **100 futures contracts**

b) The decision of whether to exercise a futures contract depends largely on how the price of wheat is projected to grow by the end of the year. If the price of wheat is projected to rise, then the farmer is much better off selling his wheat at market price when he harvests it. Futures contracts will mostly be used when the price of wheat is projected to fall by the end of the year and the farmer wants to hedge against it. While formulating the contract, the farmer will try to maximise the price he is being paid by the futures contract and will try to negotiate a deal that is closest to how he sees the price growing or falling in the future.

That means, if the current price if \$100 but the farmer believes that the price of wheat will rise to \$150 by the end of the year, he would try to negotiate for a price closer to \$150 instead of accepting the current rate of \$100. Similarly, the bank will try to negotiate a price closer to the current rate if they feel that prices will rise by the end of the year.

If the prices are projected to fall by the end of the year, say to roughly \$90, the farmer will try to negotiate a futures price which is closer to the current market rate of \$100 while the bank (CME group) will try to negotiate a rate closer to the projected price of 0.

In either case, neither party will get exactly the rate they want, and the rate will lie between $\$100 - \$150$ or $\$90 - \$100$ respectively. Assume the farmer agrees to 100 futures contracts (as calculated above), each to sell 5000 bushels of wheat at \$120 per bushel, with the current market rate being \$100, and the prices projected to rise. If the price at the end of the year is below \$120, he would have made a profit of the difference anyway but if the price rises above \$120, his losses will not be too high. This is assuming that wheat is a relatively stable commodity whose prices don't fluctuate drastically. Similarly, if the prices are projected to fall and the farmer agrees to sell each bushel at \$95, if the price falls below \$95, he makes a profit on the difference, and even if the market price at the end of the year is over \$95, his losses are not very drastic.

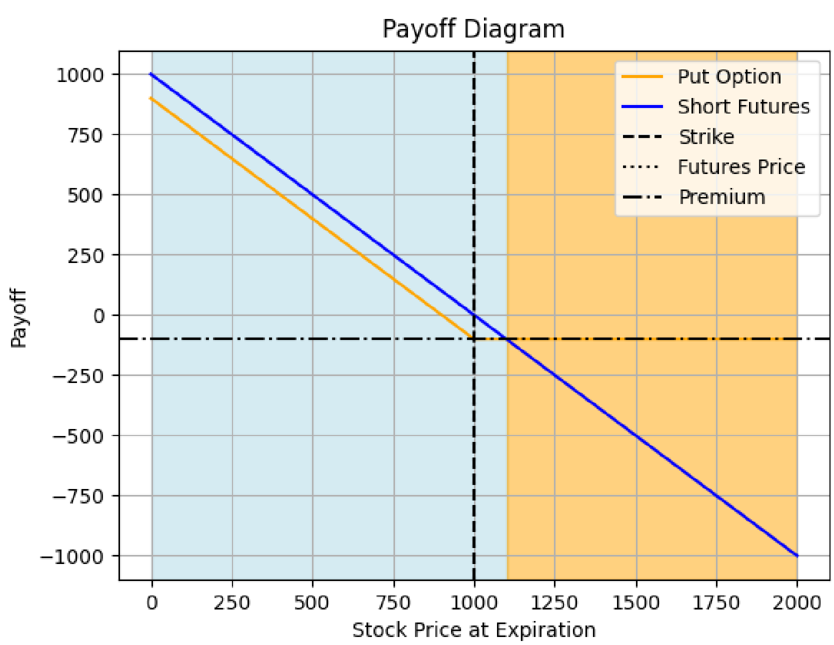
To summarize, futures contracts are a good way to try to minimise losses in the market because they guarantee at least a certain fixed rate.

1.1.2 Question 2

Date	Apr 2021	Sept 2021	Feb 2022	July 2022
Spot Price	120.50			117.10
Oct 2021	119.80	119.20		
March 2022		118.80	118.00	
July 2022			117.60	

Payoff per barrel of oil = Difference in option price and spot price Payoff
$= (119.80 - 119.20) + (118.80 - 118.00) + (117.60 - 117.10)$
$= 0.60 + 0.80 + 0.50$
Payoff per barrel $= 1.90$
Total payoff for $100,000$ barrels $= 100000 \times 1.9 = 190,000$

1.1.3 Question 3



i. Strike of the option = \$1000

ii. Premium of the option = \$100

iii. Futures price = \$1000

The futures payoff (shown in blue) is a straight line which decreases with a constant slope of -1. It meets the x-axis at the point $(1000, 0)$, and so \$1000 is the breakeven point for the futures contract. If the stock price falls below this, Trader X makes a profit on the difference. However, if the stock price rises above the breakeven point, Trader X is at risk of making an unlimited loss as the prices rise.

The put-option payoff (shown in orange) is slightly less than the futures payoff in the profitable region, but it has limited liability and the losses are limited to the premium paid at the beginning. The breakeven point for the put option contract is \$900, which is lower than the breakeven point for the futures contract. Trader Y is lossmaking even in the \$900 − \$1000 whereas Trader X still makes a profit in this region. In the \$900 − \$1000 region, Trader Y makes a loss which is the difference of that price and \$900, and the loss then becomes constant (and equal to the premium of \$100) when the stock price is \$1000 and above.

When the stock price reaches \$1100, both Trader X and Trader Y suffer an equal loss of \$100. However, as the stock price rises above \$1100, Trader Y's loss still remains \$100 whereas Trader X's loss increases (or his profit decreases) linearly with the growth in the stock price as shown.

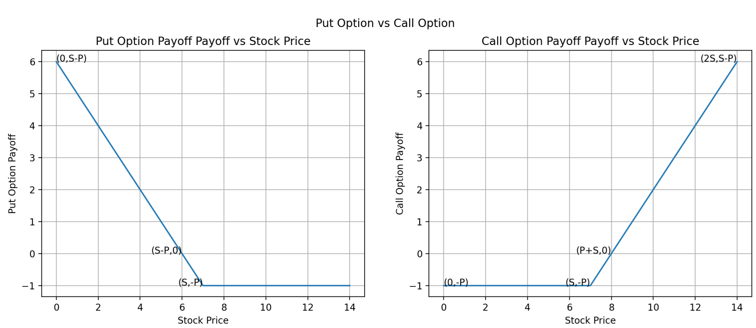
The futures strategy (Trader X) is more profitable when the stock price is in the range \$0 − \$1100 (shaded in blue), and Trader Y is more profitable (or rather less lossmaking) when the stock price rises above \$1100 (shaded in orange).

Codefile to generate plot:

`SHRIYA_HARLAPUR_FINANCE_PARTA_Q3.py`

1.2 Part B

1.2.1 Question a



The above graphs show the payoff vs stock price of a put and a call option, both with strike $S = 7$ and premium $P = 1$. We denote the stock price by x (since it is the independent variable lying on the x axis) and the payoff by y (since it is the dependent variable lying on the y axis).

In order to calculate the combined effect of both these options, we first calculate the equations of each strategy separately:

Put Option:
The points $(0, S-P)$ and $(S, -P)$ form the endpoints of the line segment in the range $0 \le x \le S$. For $x > S$ we have $y = -P$ as a constant ray. Thus, the final equation of the curve for the put option becomes:

$$y = \begin{cases} -x + S - P, & \text{if } \ 0 \le x \le S \\ -P, & \text{if } x > S \end{cases} \tag{1}$$

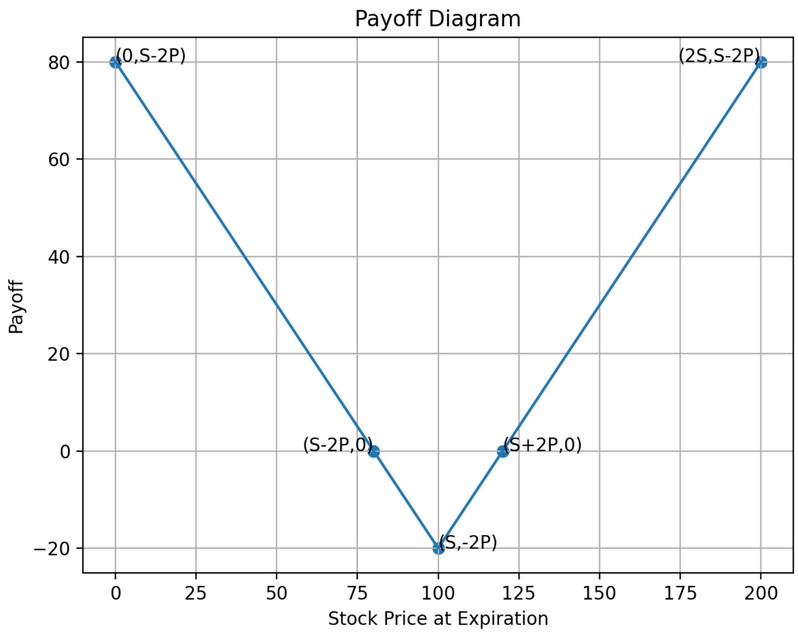
Call Option:
The points $(P + S, 0)$ and $(S, -P)$ lie on the line for $x > S$. For $0 \le x \le S$ we have $y = -P$ as a constant line segment. Thus, the final equation of the curve for the call option becomes:

$$y = \begin{cases} -P, & \text{if } \ 0 \le x \le S \\ x - S - P, & \text{if } x > S \end{cases} \tag{2}$$

Combining Put Option and Call Option:
Adding the corresponding values for each of the cases $0 \le x \le S$ and $x > S$ above, we obtain:

$$y = \begin{cases} -x + S - 2P, & \text{if } \ 0 \le x \le S \\ x - S - 2P, & \text{if } x > S \end{cases} \tag{3}$$

This is graphed below, taking strike $S = 100$ and premium $P = 10$.



From the above graph, we can see that this strategy is profitable when the expiry stock price lies in the in the regions $[0, S-2P)$ and $(S+2P, \infty)$. When the stock price lies in the range $(S-2P, S+2P)$, the strategy is lossmaking. However, this loss is capped at double the value of the premium paid for either option ($-2P$, also the sum of the premiums paid for both options) and this minimum value is reached when the stock price becomes exactly equal to the strike price S. At the points $S-2P$ and $S+2P$, there is neither profit nor loss and these are the breakeven points.

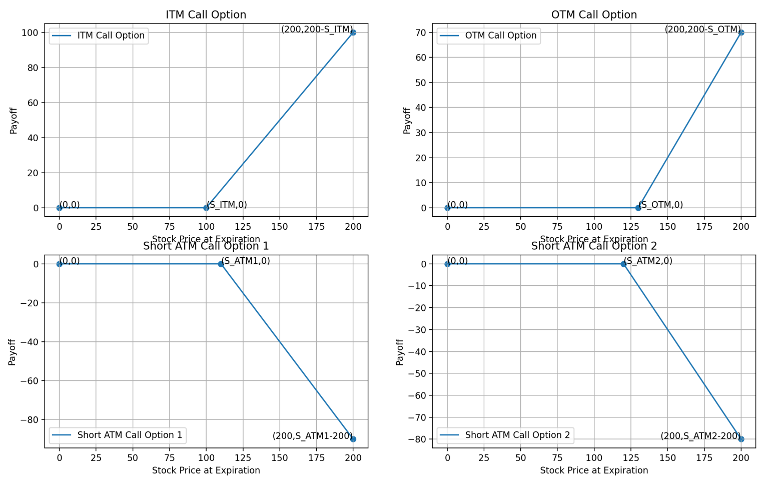
Thus, this strategy is a great way to realise profits while minimising loss.

Codefiles to generate plots:

`SHRIYA_HARLAPUR_FINANCE_PARTB_Q1_PLOT1.py`

`SHRIYA_HARLAPUR_FINANCE_PARTB_Q1_PLOT2.py`

1.2.2 Question b



Similar to the above procedure, we formulate payoff equations for each of the separate options and then add them together.
Definitions:
S_{ITM} = Strike price of ITM Call Option (here, we take S_{ITM}=100)
S_{OTM} = Strike price of OTM Call Option (here, we take S_{OTM}=130)
S_{ATM1} = Strike price of ATM Call Option 1 (here, we take S_{ATM1}=110)
S_{ATM2} = Strike price of ATM Call Option 2 (here, we take S_{ATM2}=120)

In all the equations, we take x =stock price at expiration date and y = payoff.

ITM Call Option:

$$y = \begin{cases} 0, & \text{if } \ 0 \leq x \leq S_{ITM} \\ x - S_{ITM}, & \text{if } x > S_{ITM} \end{cases} \tag{4}$$

OTM Call Option:

$$y = \begin{cases} 0, & \text{if } \ 0 \leq x \leq S_{OTM} \\ x - S_{OTM}, & \text{if } x > S_{OTM} \end{cases} \tag{5}$$

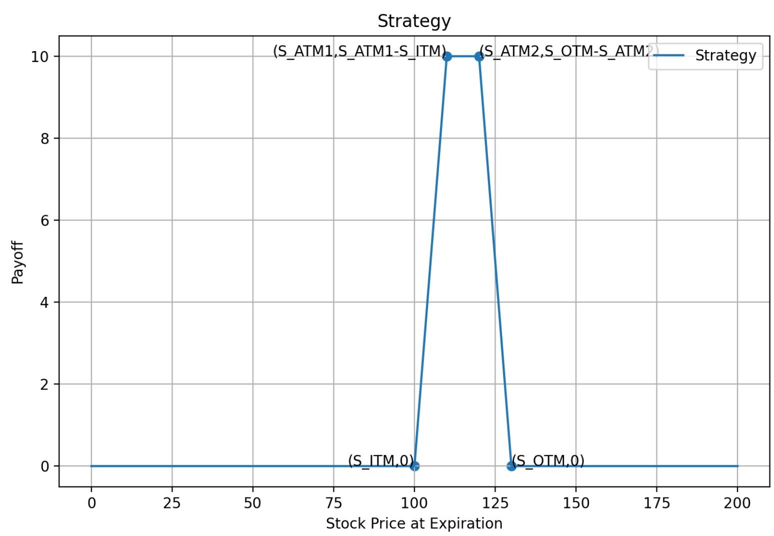
ATM Call Option 1:

$$y = \begin{cases} 0, & \text{if } \ 0 \leq x \leq S_{ATM1} \\ S_{ATM1} - x, & \text{if } x > S_{ATM1} \end{cases} \tag{6}$$

ATM Call Option 2:

$$y = \begin{cases} 0, & \text{if } \ 0 \leq x \leq S_{ATM2} \\ S_{ATM2} - x, & \text{if } x > S_{ATM2} \end{cases} \tag{7}$$

Total Payoff:



We have $S_{ITM} < S_{ATM1} < S_{ATM2} < S_{OTM}$ and $S_{ATM1} - S_{ITM} = S_{ATM2} - S_{ATM1} = S_{OTM} - S_{ATM2}$
Thus, adding the y values in each of the ranges $0 \leq x \leq S_{ITM}$, $S_{ITM} \leq x \leq S_{ATM1}$, $S_{ATM1} \leq x \leq S_{ATM2}$, $S_{ATM2} \leq x \leq S_{OTM}$ and $x \geq S_{OTM}$ separately we have:

$$y = \begin{cases} 0, & \text{if } \ 0 \leq x \leq S_{ITM} \\ x - S_{ATM1}, & \text{if } S_{ITM} \leq x \leq S_{ATM1} \\ S_{ATM1} - S_{ATM2}, & \text{if } S_{ATM1} \leq x \leq S_{ATM2} \\ S_{OTM} - x, & \text{if } S_{ATM2} \leq x \leq S_{OTM} \\ 0, & \text{if } \ x \geq S_{OTM} \end{cases} \tag{8}$$

Codefile to generate plots:

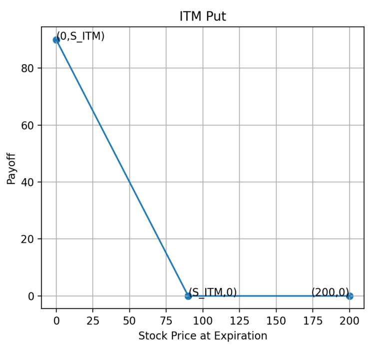
`SHRIYA_HARLAPUR_FINANCE_PARTB_Q2.py`

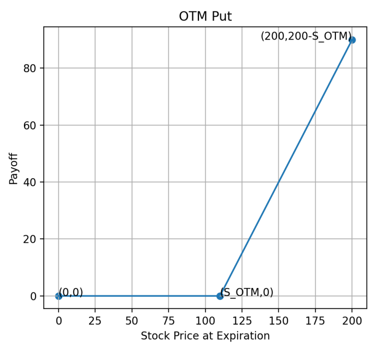
1.2.3 Question c

Similar to the above procedure, we formulate payoff equations for each of the separate options and then add them together.
Definitions:
S_{ITM} = Strike price of ITM Put Option (here, we take S_{ITM}=110)
S_{OTM} = Strike price of OTM Put Option (here, we take S_{OTM}=90)

In all the equations, we take x =stock price at expiration date and y = payoff.

Also, in order to be profitable, we must take $S_{ITM} > S_{OTM}$. This guarantees that the payoff is always ≥ 0. If we reverse the values, payoff will become < 0





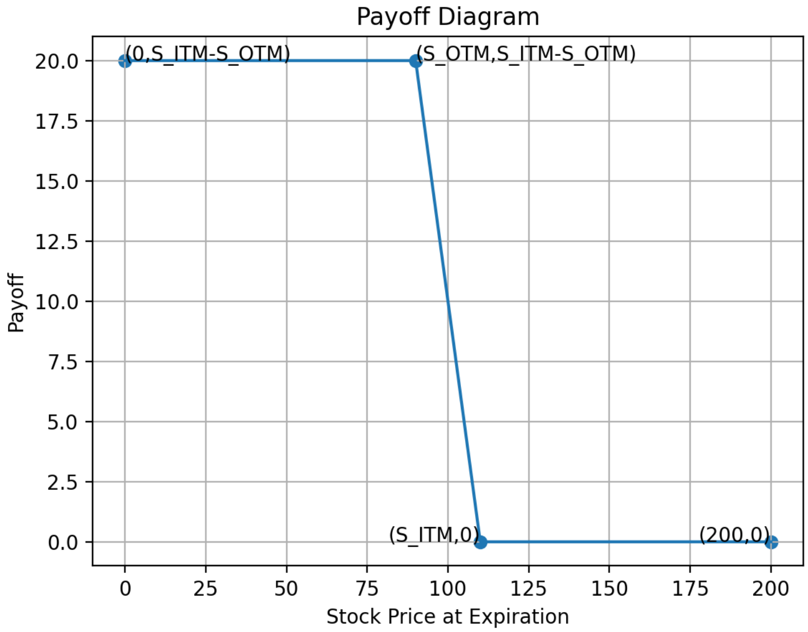
ITM Put Option:

$$y = \begin{cases} S_{ITM} - x, & \text{if } \ 0 \leq x \leq S_{ITM} \\ 0, & \text{if } x > S_{ITM} \end{cases} \tag{9}$$

OTM Put Option:

$$y = \begin{cases} 0, & \text{if } \ 0 \leq x \leq S_{OTM} \\ x - S_{OTM}, & \text{if } x > S_{OTM} \end{cases} \tag{10}$$

Total Payoff:



Adding the ITM Put and OTM Put equations together, we get:

$$y = \begin{cases} S_{ITM} - S_{OTM}, & \text{if } \ 0 \leq x \leq S_{OTM} \\ S_{ITM} - x, & \text{if } \ S_{OTM} \leq x \leq S_{ITM} \\ 0, & \text{if } x > S_{ITM} \end{cases} \tag{11}$$

Codefiles to generate plots:

SHRIYA_HARLAPUR_FINANCE_PARTB_Q3_PLOT1.py

SHRIYA_HARLAPUR_FINANCE_PARTB_Q3_PLOT2.py

2 Mathematics

2.1 Problem 1

2.1.1 Final Answer: 1st Quadrant

Since the probability of going either upwards or eastwards is equal, it means that after an infinite number of steps, "half" of them would have been upward and "half" of them would have been eastward (although half of infinity is still infinity).

This means that at least 1 step is taken in the eastward direction and at least 1 step is taken in the upward direction. Since the ant never moves in the $-x$ and $-y$ directions, it remains in the region bounded by the $+x$ and $+y$ directions. Thus the ant is in the first quadrant where $x > 0$ and $y > 0$.

2.1.2 Final Answer: 2

Please see the answer to Question 2.1.4 to see my explanation for why the endpoints will lie on the line $x + y = 2$ with a uniform probability distribution on the x co-ordinate.

$$x_1 = 0$$

$$y_1 = 0$$

$$x_2 = x, 0 \leq x \leq 2$$

$$y_2 = 2 - x, 0 \leq x \leq 2, 0 \leq y \leq 2$$

$$\mid x_1 - x_2 \mid = \mid 0 - x \mid = \mid -x \mid = x (since x \geq 0)$$

$$\mid y_1 - y_2 \mid = \mid 0 - (2 - x) \mid = \mid x - 2 \mid = 2 - x (since 0 \leq x \leq 2)$$

$$d_m = \mid x_1 - x_2 \mid + \mid y_1 - y_2 \mid = x + 2 - x = 2$$

Probability of each point (x, y) being selected $= \frac{1}{2}$

We obtain this from section 2.1.4

Now,

$$p(x) = \frac{1}{2}$$

$$d_m = 2$$

$$E(d_m) = \int_0^2 p(x) d_m dx$$

$$E(d_m) = \int_0^2 2 \frac{1}{2} dx$$

$$E(d_m) = \int_0^2 dx$$

$$E(d_m) = [x]_0^2 = 2 - 0$$

$$E(d_m) = 2$$

2.1.3 Final Answer: $8/3$

Similar to the above section, we have

$$p(x) = \frac{1}{2}$$
$$x_1 = 0$$
$$y_1 = 0$$
$$x_2 = x, 0 \leq x \leq 2$$
$$y_2 = 2 - x, 0 \leq x \leq 2, 0 \leq y \leq 2$$
$$x_1 - x_2 = 0 - x = -x$$
$$y_1 - y_2 = 0 - (2 - x) = x - 2$$
$$d_e^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2 = (-x)^2 + (x - 2)^2 = 2x^2 - 4x + 4$$
$$E(d_e^2) = \int_0^2 p(x) d_e^2 dx$$
$$E(d_e^2) = \int_0^2 2 \times \frac{2x^2 - 4x + 4}{2} dx$$
$$E(d_e^2) = \int_0^2 x^2 - 2x + 2dx$$
$$E(d_e^2) = [\frac{x^3}{3} - x^2 + 2x]_0^2$$
$$E(d_e^2) = \frac{2^3}{3} - 4 + 4$$
$$E(d_e^2) = \frac{8}{3}$$

2.1.4 Final Answer: $x + y = 2, x > 0, y > 0$

The series of the lengths of the steps is given by $1, \frac{1}{2}, \frac{1}{4}, \ldots$
The length of the i^{th} step is given by $l_i = 2^{1-i}$
The cumulative length of all the steps taken (irrespective of the direction) is given by the summation of the series:

$$\sum_{i=1}^{\infty} 2^{1-i} = 2$$

Thus, the sum of the x co-ordinate and the y co-ordinate is always 2, irrespective of which specific steps are taken upwards and which specific steps are taken eastwards.
More formally, we can formulate a sequence given below:

Let the indices of all the steps taken in the +x direction be denoted by the set λ_x. Then, the indices of all the steps taken in the +y direction are denoted by $\lambda_y = N - \lambda_x$ or $\lambda_x \cup \lambda_y = N$ and $\lambda_x \cap \lambda_y = \phi$, where N is the set of natural numbers.
Thus, the final x co-ordinate is the sum of the terms with indices belonging to λ_x and the final y co-ordinate is the sum of the terms with indices belonging to λ_y. More formally,

$$x = \sum_{k \in \lambda_x} 2^{1-k}$$

and

$$y = \sum_{k \in \lambda_y} 2^{1-k}$$

Adding the two together, we obtain

$$x + y = \sum_{k \in \lambda_x \cup \lambda_y} 2^{1-k}$$

$$x + y = \sum_{k \in N} 2^{1-k}$$

which is precisely the sum calculated earlier. Thus, we obtain

$$x + y = 2$$

Additionally, as discussed in section 2.1.1, the domain is restricted to the first quadrant, so we also impose the conditions $x > 0, y > 0$.

Probability of ending at each point on the line:
Each point on the line $x + y = 2$ can be expressed as an ordered pair

$$(x, y) = (\sum_{k \in \lambda_x} 2^{1-k}, \sum_{k \in \lambda_y} 2^{1-k})$$

where all the symbols have the same meaning as above. The probability of any particular index k going into either λ_x or λ_y is equal. Thus,

$$P(i \in \lambda_x) = P(i \in \lambda_y) = \frac{1}{2}$$

Also, each point in the range $0 < x < 2$ can be expressed as a unique sum of terms from a subsequence of the above series. That is, each x in the given range has a unique indexing set λ_x and its corresponding y co-ordinate has a unique indexing set λ_y. Since the probability of each index set going into a particular indexing set is $\frac{1}{2}$, the probability of the ant ending up on any point on the line (i.e. the probability of obtaining a particular x co-ordinate and its corresponding y co-ordinate) follows a uniform distribution. Thus,

$$P((x, y)|0 < x < 2, y = 2 - x) = \frac{1}{2 - 0} = \frac{1}{2}$$

We use this probability in sections 2.1.2 ans 2.1.3 to calculate the expected value of the given distance functions.

2.2 Problem 2

2.2.1 Final Answer: 70

The total number of cards $= 4 + 4 = 8$
If all the cards had been of different colours, then there would have been 8! distinct permutations of these cards.
However, since 4 are identical red cards and the other 4 are identical black cards, each group of identical cards decreases the total number of distinct permutations by a factor of 4!.
Another way to think about this is that we have to choose 4 positions out of the given 8 for the red cards, and the other 4 will automatically go to the black cards, bringing the number to $^8C_4$
Thus, the total number of distinct permutations is now:

$$^8C_4 = \frac{8!}{4!4!} = 70$$

2.2.2 Final Answer: Expected Payoff is 1.32857

The problem of generating all the possible permutations of 4 identical black cards and 4 identical red cards is the same as the problem of generating all binary sequences with exactly 4 1s and 4 0s. To do this, we initialise an array $\{0, 0, 0, 0, 1, 1, 1, 1\}$, process each array and then generate the next lexicographically greater permutation till we reach an array that is reverse sorted and has no lexicographic successor.

Generating the next permutation: We traverse the array from the back until we reach an index i where the element at the $i - 1^{th}$ index is less than the element at the i^{th} index. Then we find an index $j < i$ such that the element at index j is less than the element at index i. We now swap these 2 elements and reverse the entire subarray between these 2 elements to get the next permutation. The idea behind this is that the element at index j is the element that needs to be pushed to the position where it generates the next permutation.

Calculating maximum payoff for each permutation: For each permutation, traverse the array and add 1 to the current score if it is a red card and subtract 1 if it is a black card. Keep track of the maximum score and update it when the current score becomes higher.

Source code for the above program:

`SHRIYA_HARLAPUR_MATHS_Q2_2_MAIN.cpp`

2.2.3 Final Answer: $\{0, 1, 2, ..., 24, 25, 26\}$

The minimum value is 0 for the permutation BB...BBRR...RR (26 consecutive black cards and then 26 consecutive red cards). The maximum value will be 26 for the opposite permutation, i.e. RR...RRBB...BB (26 consecutive red cards followed by 26 consecutive black cards). All the other values in the range 0-26 are obtained by changing the permutations accordingly. One cannot obtain a negative value for the maximum payoff since one can always stop at position 0, i.e. before picking any cards, and that will give us a payoff of 0. Another way to think about this is that one can always pick all 52 cards and this will guarantee a payoff of 0, which means that any strategy that has a negative payoff is never the optimum. Similarly, one cannot obtain a value > 26 for the payoff, since the maximum positive points (red cards) available in the game are 26. Values between 0 and 26 are obtained by pushing black cards into the front of the permutation one-by-one. Thus, the set of all values obtained is

$$\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26\}$$

2.2.4 Proof of $n_0 = \binom{2n}{n} - \binom{2n}{n+1}$

Maximum payoff is never negative:

The maximum payoff can never be less than 0 since we can always guarantee a payoff of at least 0 for any configuration of the cards using any one of the following two strategies:

- Stop playing at move 0, i.e. don't pick up any cards
- Stop playing after all the cards are picked, i.e. at move $2n$. This guarantees that you have picked up an equal number of red and blue cards and thus your payoff is 0.

Thus, the number of deck configurations for which the payoff is less than or equal to zero is simply the number of deck configurations for which the payoff is equal to zero.

Another way to prove this is as follows:

Let R be the number of red cards remaining and B be the number of black cards remaining. The pay-off for each red card is $R - B$ and the pay-off for each black card is $B - R$. Hence, the expected pay-off is equal to $(R - B) * R - (B - R) * B = R^2 - B^2$.

To maximize the expected pay-off, we want to maximize $R^2 - B^2$. This is equivalent to maximizing R^2. As $R + B = 52, R^2 + 2RB + B^2 = 52^2 = 2704, R^2 + 2RB = 2704 - B^2$. Hence, $R^2 - B^2 = (R + B)(R - B) = 52 * 0 = 0$. This means that the expected pay-off is 0.

Proving the equation:

Therefore, n_0 is equal to the number of deck configurations such that $R^2 - B^2 = 0$. This is equivalent to the number of ways to partition 52 objects into two sets such that the sum of the squares of the sizes of the sets is 2704. By stars and bars, this is equal to $\binom{2n}{n} - \binom{2n}{n+1} = \binom{2*26}{26} - \binom{2*26}{26+1} = \binom{52}{26} - \binom{52}{27} = n_0$.

The reflection principle states that if we have a function f(x) that is symmetric about the origin (i.e., $f(-x) = f(x)$), then the number of solutions to the equation $f(x) = k$ is equal to the number of solutions to the equation $f(x) = -k$, for any value of k.

In this case, we can use the reflection principle to count the number of ways to partition the 26 red cards and 26 black cards into two sets such that the sum of the squares of the sizes of the sets is 676. Let R be the number of red cards in one set and B be the number of black cards in the other set. Then $R + B = 52$ and $R^2 + B^2 = 2704$.

By the reflection principle, the number of solutions to $R^2 + B^2 = 2704$ is equal to the number of solutions to $(-R)^2 + (-B)^2 = 2704$. To see this, note that if (R, B) is a solution to $R^2 + B^2 = 2704$, then $(-R, -B)$ is a solution to $R^2 + B^2 = 2704$.

Since the number of solutions to $R^2 + B^2 = 2704$ is $\binom{2n}{n}$, the number of solutions to $(-R)^2 + (-B)^2 = 2704$ is also $\binom{2n}{n}$. However, the number of solutions to $R^2 + B^2 = 2704$ where $R > B$ is $\binom{2n}{n}$ and the number of solutions to $(-R)^2 + (-B)^2 = 2704$ where $R < B$ is $\binom{2n}{n}$. Hence, the number of solutions to $R^2 + B^2 = 2704$ where $R = B$ is $\binom{2n}{n} - \binom{2n}{n}$.

Since the number of solutions to $R^2 + B^2 = 2704$ where $R = B$ is $\binom{2n}{n} - \binom{2n}{n}$, the number of solutions to $R^2 + B^2 = 2704$ is $\binom{2n}{n} - \binom{2n}{n+1}$. Therefore, $n_0 = \binom{2n}{n} - \binom{2n}{n+1}$.

2.2.5 Proof of $n_k = \binom{2n}{n} - \binom{2n}{n+k+1}$

Let R be the number of red cards in one set and B be the number of black cards in the other set. Then $R + B = 26$ and $R - B = k$.

By the reflection principle, the number of solutions to $R - B = k$ is equal to the number of solutions to $R - B = -k$. To see this, note that if (R, B) is a solution to $R - B = k$, then $(26 - R, 26 - B)$ is a solution to $R - B = -k$.

Since the number of solutions to $R - B = k$ is $\binom{2n}{n}$, the number of solutions to $R - B = -k$ is also $\binom{2n}{n}$. However, the number of solutions to $R - B = k$ where $R > B$ is $\binom{2n}{n}$ and the number of solutions to $R - B = -k$ where $R < B$ is $\binom{2n}{n+k+1}$. Hence, the number of solutions to $R - B = k$ where $R = B + k/2$

is $\binom{2n}{n} - \binom{2n}{n+k+1}$.

Since the number of solutions to $R - B = k$ where $R = B + k/2$ is $\binom{2n}{n} - \binom{2n}{n+k+1}$, the number of solutions to $R - B = k$ is $\binom{2n}{n} - \binom{2n}{n+k+1}$. Therefore, $n_k = \binom{2n}{n} - \binom{2n}{n+k+1}$.

2.2.6 Final Answer: $P(X \leq k) = 1 - \frac{n!n!}{(n+k+1)!(n-k-1)!}$

Given:

$$n_k = \binom{2n}{n} - \binom{2n}{n+k+1}$$

Thus we can write:

$$n_0 = \binom{2n}{n} - \binom{2n}{n+1}$$

$$n_1 = \binom{2n}{n} - \binom{2n}{n+2}$$

$$\vdots$$

$$n_{n-1} = \binom{2n}{n} - \binom{2n}{2n}$$

$$n_n = \binom{2n}{n}$$

The total number of possibilities is therefore n_n
The cumulative distribution function (or cdf) is given by:

$$P(X \leq k) = \frac{n_k}{n_n}$$

$$= \frac{\binom{2n}{n} - \binom{2n}{n+k+1}}{\binom{2n}{n}}$$

$$= 1 - \frac{\binom{2n}{n+k+1}}{\binom{2n}{n}}$$

$$= 1 - \frac{\frac{2n!}{(n+k+1)!(n-k-1)!}}{\frac{2n!}{n!n!}}$$

$$P(X \leq k) = 1 - \frac{n!n!}{(n+k+1)!(n-k-1)!}$$

2.2.7 Final Answer: $\frac{f_k}{f_{k-1}} = \frac{n-k+1}{n+k}$

From section 2.2.6, we have

$$P(X \leq k) = 1 - \frac{n!n!}{(n+k+1)!(n-k-1)!}$$

Therefore,

$$f_k = P(X \geq k) = 1 - P(X \leq k-1) = 1 - (1 - \frac{n!n!}{(n+k)!(n-k)!})$$

$$f_k = \frac{n!n!}{(n+k)!(n-k)!}$$

and similarly

$$f_{k-1} = \frac{n!n!}{(n+k-1)!(n-k+1)!}$$

Thus,

$$\frac{f_k}{f_{k-1}} = \frac{\frac{n!n!}{(n+k)!(n-k)!}}{\frac{n!n!}{(n+k-1)!(n-k+1)!}}$$

$$= \frac{(n-k+1)!(n+k-1)!}{(n-k)!(n+k)!}$$

$$\frac{f_k}{f_{k-1}} = \frac{n-k+1}{n+k}$$

Putting $n = 26$, we obtain

$$\frac{f_k}{f_{k-1}} = \frac{27-k}{26+k}$$

2.2.8 Final Answer: $P(X = k) = \frac{2k+1}{n+k+1} f_k$

From section 2.2.7 we have

$$f_k = P(X \geq k) = \frac{n-k+1}{n+k} f_{k-1}$$

Similarly,

$$f_{k+1} = P(X \geq k+1) = P(X > k) = \frac{n-k}{n+k+1} f_k$$

Thus,

$$P(X = k) = f_k - f_{k+1} = f_k - \frac{n-k}{n+k+1} f_k$$

$$= f_k(1 - \frac{n-k}{n+k+1})$$

$$= f_k(\frac{n+k+1-n+k}{n+k+1})$$

$$P(X=k) = \frac{2k+1}{n+k+1} f_k$$

Putting $n = 26$, we obtain

$$P(X=k) = \frac{2k+1}{27+k} f_k$$

Using this expression in the code, we obtain the following values for P(payoff= k), $0 \le k \le 26$:

k	P(payoff=k)	k	P(payoff=k)	k	P(payoff=k)
0	0.037037	9	0.0233554	18	1.24769 e-06
1	0.103175	10	0.0118604	19	2.28719 e-07
2	0.148239	11	0.00546948	20	3.58114 e-08
3	0.166028	12	0.00228657	21	4.69479 e-09
4	0.158377	13	0.000864324	22	5.01342 e-10
5	0.133081	14	0.000294354	23	4.18899 e-11
6	0.100085	15	8.99013 e-05	24	2.56897 e-12
7	0.0679313	16	2.44817 e-05	25	1.02839 e-13
8	0.0417939	17	5.90125 e-06	26	2.01646 e-15

Source code for the above program:

`SHRIYA_HARLAPUR_MATHS_Q2_8_MAIN.cpp`

2.2.9 Final Answer: Most likely payoff = 3

From the above table, it is clear to see that the payoff with the maximum probability value is $k = 3$, with a probability of 0.166028 or roughly 16.6%.

2.2.10 Final Answer: Expected Payoff = 4.04

The expected value of the game is given by:

$$\sum_{k=0}^{26} k \times P(X=k)$$

$P(X = k)$ is calculated using the same procedure as described in section 2.2.8

As calculated, the expected payoff for a 52 card deck with 26 red cards and 26 black cards is 4.04066, or rounded to 2 decimal places it is 4.04.

Source code for the above program:

`SHRIYA_HARLAPUR_MATHS_Q2_10_MAIN.cpp`