10.6.8

EE25BTECH11052 - Shriyansh Kalpesh Chawda

Question

Construct a pair of tangents to a circle of radius 4cm from a point P lying outside the circle at a distance of 6cm from the centre. (10, 2023)

Solution

Let the center of the circle be at origin and point P be at distance 6 from center along x-axis.

$$\mathbf{O} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{1}$$

$$\mathbf{h} = \begin{pmatrix} 6 \\ 0 \end{pmatrix} \tag{2}$$

The equation of the circle $x^2+y^2 = 16$ can be written in the general conic form (8.1.2.1):

$$\mathbf{x}^\top \mathbf{V}\mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0 \tag{3}$$

The parameters of the circle are:

$$\mathbf{V} = \mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{4}$$

$$\mathbf{u} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{5}$$

$$f = -16 \tag{6}$$

The center and radius of the circle are:

$$\mathbf{c} = -\mathbf{u} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{7}$$

$$r = \sqrt{\|\mathbf{u}\|^2 - f} = \sqrt{0 - (-16)} = 4 \tag{8}$$

Finding Direction Vectors using Eigenvalue Decomposition:

A point $\mathbf{h}$ lies on a tangent to the conic if it satisfies formula (10.1.10.1):

$$\mathbf{m}^\top \left((\mathbf{V}\mathbf{h} + \mathbf{u})(\mathbf{V}\mathbf{h} + \mathbf{u})^\top - \mathbf{V} g(\mathbf{h}) \right) \mathbf{m} = 0 \tag{9}$$

First, calculate $g(\mathbf{h})$:

$$g(\mathbf{h}) = \mathbf{h}^\top \mathbf{V}\mathbf{h} + 2\mathbf{u}^\top \mathbf{h} + f \tag{10}$$

$$= \begin{pmatrix} 6 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 6 \\ 0 \end{pmatrix} + 0 - 16 \tag{11}$$

$$= 36 - 16 = 20 \tag{12}$$

Calculate $\mathbf{Vh} + \mathbf{u}$:

$$\mathbf{Vh} + \mathbf{u} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 6 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 6 \\ 0 \end{pmatrix} \tag{13}$$

Define matrix $\mathbf{Q}$:

$$\mathbf{Q} = (\mathbf{Vh} + \mathbf{u})(\mathbf{Vh} + \mathbf{u})^\top - \mathbf{V}g(\mathbf{h}) \tag{14}$$

$$= \begin{pmatrix} 6 \\ 0 \end{pmatrix} \begin{pmatrix} 6 & 0 \end{pmatrix} - 20 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{15}$$

$$= \begin{pmatrix} 36 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 20 & 0 \\ 0 & 20 \end{pmatrix} \tag{16}$$

$$= \begin{pmatrix} 16 & 0 \\ 0 & -20 \end{pmatrix} \tag{17}$$

Eigenvalue Decomposition of Q:
The matrix $\mathbf{Q}$ is diagonal, so the eigenvalues are:

$$\lambda_1 = 16 \tag{18}$$

$$\lambda_2 = -20 \tag{19}$$

The eigenvector matrix is:

$$\mathbf{P} = \mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{20}$$

From the condition $\mathbf{m}^\top \mathbf{Q} \mathbf{m} = 0$:

$$16m_1^2 - 20m_2^2 = 0 \tag{21}$$

$$\frac{m_1^2}{m_2^2} = \frac{20}{16} = \frac{5}{4} \tag{22}$$

$$\frac{m_1}{m_2} = \pm \frac{\sqrt{5}}{2} \tag{23}$$

The two direction vectors are:

$$\mathbf{m}_1 = \begin{pmatrix} \sqrt{5} \\ 2 \end{pmatrix} \tag{24}$$

$$\mathbf{m}_2 = \begin{pmatrix} \sqrt{5} \\ -2 \end{pmatrix} \tag{25}$$

The corresponding normal vectors (perpendicular to direction vectors):

$$\mathbf{n}_1 = \begin{pmatrix} -2 \\ \sqrt{5} \end{pmatrix} = - \begin{pmatrix} 2 \\ -\sqrt{5} \end{pmatrix} \tag{26}$$

$$\mathbf{n}_2 = \begin{pmatrix} 2 \\ \sqrt{5} \end{pmatrix} \tag{27}$$

Choosing proper signs:

$$\mathbf{n}_1 = \begin{pmatrix} 2 \\ -\sqrt{5} \end{pmatrix} \tag{28}$$

$$\mathbf{n}_2 = \begin{pmatrix} 2 \\ \sqrt{5} \end{pmatrix} \tag{29}$$

Finding Points of Contact using Formula (10.1.8.1):

For a circle, the points of contact are given by:

$$\mathbf{q}_{ij} = \pm r \frac{\mathbf{n}_j}{\left\|\mathbf{n}_j\right\|} - \mathbf{u}, \quad i, j = 1, 2 \tag{30}$$

For $\mathbf{n}_1 = \begin{pmatrix} 2 \\ -\sqrt{5} \end{pmatrix}$:

$$\|\mathbf{n}_1\| = \sqrt{4+5} = 3 \tag{31}$$

$$\mathbf{q}_1 = r\frac{\mathbf{n}_1}{\|\mathbf{n}_1\|} - \mathbf{u} \tag{32}$$

$$= 4 \cdot \frac{1}{3}\begin{pmatrix} 2 \\ -\sqrt{5} \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{33}$$

$$= \begin{pmatrix} \frac{8}{3} \\ -\frac{4\sqrt{5}}{3} \end{pmatrix} \tag{34}$$

For $\mathbf{n}_2 = \begin{pmatrix} 2 \\ \sqrt{5} \end{pmatrix}$:

$$\|\mathbf{n}_2\| = 3 \tag{35}$$

$$\mathbf{q}_2 = 4 \cdot \frac{1}{3}\begin{pmatrix} 2 \\ \sqrt{5} \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{36}$$

$$= \begin{pmatrix} \frac{8}{3} \\ \frac{4\sqrt{5}}{3} \end{pmatrix} \tag{37}$$

Therefore, the points of contact are:

$$\mathbf{q}_1 = \begin{pmatrix} \frac{8}{3} \\ -\frac{4\sqrt{5}}{3} \end{pmatrix} \tag{38}$$

$$\mathbf{q}_2 = \begin{pmatrix} \frac{8}{3} \\ \frac{4\sqrt{5}}{3} \end{pmatrix} \tag{39}$$

Verification:

Check if points lie on circle:

$$\|\mathbf{q}_1\|^2 = \left(\frac{8}{3}\right)^2 + \left(-\frac{4\sqrt{5}}{3}\right)^2 \tag{40}$$

$$= \frac{64}{9} + \frac{80}{9} = \frac{144}{9} = 16 \quad \checkmark \tag{41}$$

Check if tangent from $\mathbf{h}$:

$$(\mathbf{h} - \mathbf{q}_1)^\top \mathbf{q}_1 = \begin{pmatrix} 6 - \frac{8}{3} & 0 + \frac{4\sqrt{5}}{3} \end{pmatrix} \begin{pmatrix} \frac{8}{3} \\ -\frac{4\sqrt{5}}{3} \end{pmatrix} \tag{42}$$

$$= \frac{10}{3} \cdot \frac{8}{3} + \frac{4\sqrt{5}}{3} \cdot \left(-\frac{4\sqrt{5}}{3}\right) \tag{43}$$

$$= \frac{80}{9} - \frac{80}{9} = 0 \quad \checkmark \tag{44}$$

Equations of Tangents using Formula (10.1.2.1):
The equation of tangent at point $\mathbf{q}$ is:

$$(\mathbf{V}\mathbf{q} + \mathbf{u})^\top \mathbf{x} + \mathbf{u}^\top \mathbf{q} + f = 0 \tag{45}$$

For our circle with $\mathbf{V} = \mathbf{I}$ and $\mathbf{u} = \mathbf{0}$:

$$\mathbf{q}^\top \mathbf{x} + f = 0 \tag{46}$$

$$\mathbf{q}^\top \mathbf{x} = 16 \tag{47}$$

Tangent 1 at $\mathbf{q}_1 = \begin{pmatrix} \frac{8}{3} \\ -\frac{4\sqrt{5}}{3} \end{pmatrix}$:

$$\begin{pmatrix} \frac{8}{3} & -\frac{4\sqrt{5}}{3} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 16 \tag{48}$$

$$\frac{8}{3}x - \frac{4\sqrt{5}}{3}y = 16 \tag{49}$$

$$8x - 4\sqrt{5}y = 48 \tag{50}$$

$$2x - \sqrt{5}y = 12 \tag{51}$$

Tangent 2 at $\mathbf{q}_2 = \begin{pmatrix} \frac{8}{3} \\ \frac{4\sqrt{5}}{3} \end{pmatrix}$:

$$\begin{pmatrix} \frac{8}{3} & \frac{4\sqrt{5}}{3} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 16 \tag{52}$$

$$2x + \sqrt{5}y = 12 \tag{53}$$

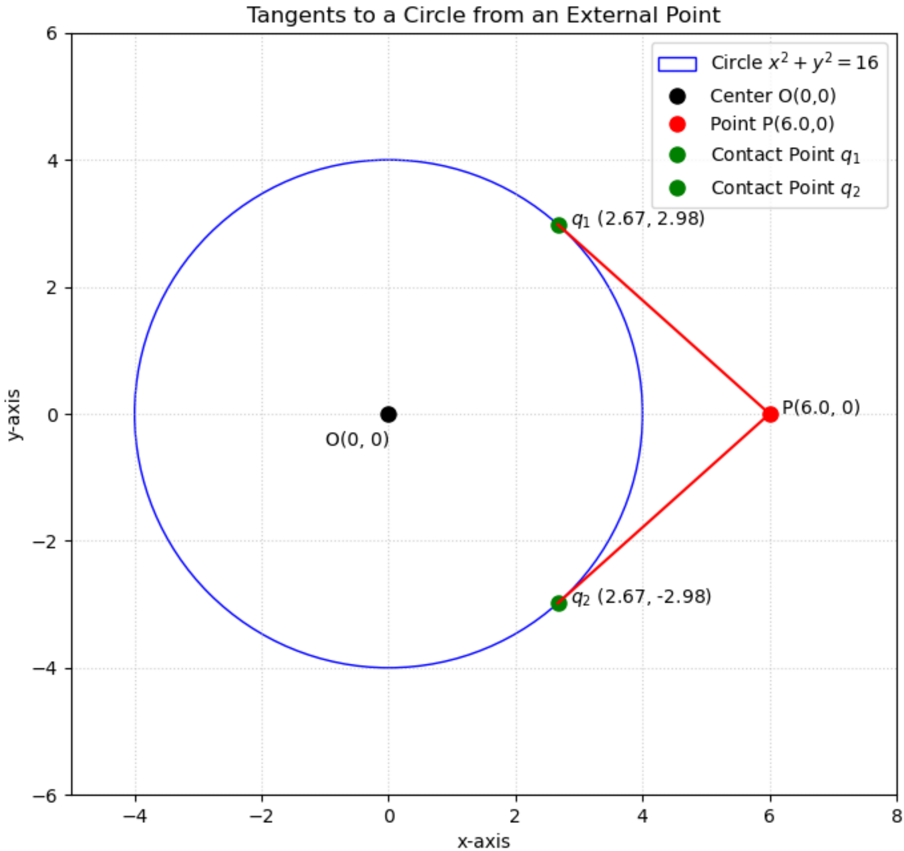
The equations of the pair of tangents are:

$$\boxed{2x - \sqrt{5}y = 12 \quad \text{and} \quad 2x + \sqrt{5}y = 12} \tag{54}$$

Verification - Tangents pass through P:

$$2(6) - \sqrt{5}(0) = 12 \quad \checkmark \tag{55}$$

$$2(6) + \sqrt{5}(0) = 12 \quad \checkmark \tag{56}$$


Tangents to a Circle from an External Point
Circle $x^2 + y^2 = 16$
Center O(0,0)
Point P(6.0,0)
Contact Point q_1
Contact Point q_2
q_1 (2.67, 2.98)
q_2 (2.67, -2.98)
P(6.0, 0)
O(0, 0)
x-axis
y-axis