THE UNIVERSITY OF
MELBOURNE

Semester 1 Assessment, 2020

School of Mathematics and Statistics

MAST30025 Linear Statistical Models

This exam consists of 24 pages (including this page)

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- You should attempt all questions.

- There are 7 questions with marks as shown. The total number of marks available is 90.

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Question 1 (10 marks)

(a) [**2 marks**] Give an example of a 3×3 idempotent matrix which is not 0 or I_3.

A simple example is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. $\boxed{2}$

$\boxed{\text{Subtotal 2}}$

(b) [**2 marks**] Show that the matrix $A = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}$ is positive definite.

The eigenvalues of A satisfy the characteristic equation $\det A - \lambda I = (3-\lambda)^2 - 4 = \lambda^2 - 6\lambda + 5 = (\lambda - 1)(\lambda - 5) = 0$, so the eigenvalues are 1 and 5, which are both greater than 0. Hence A is positive definite. $\boxed{2}$

$\boxed{\text{Subtotal 2}}$

(c) [**3 marks**] Show directly that $\frac{\partial}{\partial \mathbf{y}} \mathbf{y}^T A \mathbf{y} = A\mathbf{y} + A^T \mathbf{y}$.

$$
\left[\frac{\partial}{\partial \mathbf{y}} \mathbf{y}^T A \mathbf{y} \right]_i = \frac{\partial}{\partial y_i} \sum_{j,k} y_j A_{jk} y_k
$$

$$
= 2 A_{ii} y_i + \sum_{j \neq i} y_j A_{ji} + \sum_{k \neq i} A_{ik} y_k
$$

$$
= \sum_j y_j A_{ji} + \sum_k A_{ik} y_k
$$

$$
= \left[A^T \mathbf{y} + A\mathbf{y} \right]_i.
$$

$\boxed{3}$

$\boxed{\text{Subtotal 3}}$

(d) [**3 marks**] Show directly that for any $n \times k$ matrix A with $n \geq k$, the matrix $I - A(A^T A)^c A^T$ has a rank of $n - r(A)$. (You may assume that this matrix is idempotent.)

$$
\begin{aligned}
r(I - A(A^T A)^c A^T) &= tr(I - A(A^T A)^c A^T) \\
&= tr(I_n) - tr(A(A^T A)^c A^T) \\
&= n - r(A(A^T A)^c A^T) \\
&= n - r(A).
\end{aligned}
$$

3

Subtotal 3

Question 2 (14 marks)

(a) [**5 marks**] Let $X_1 \sim \chi^2_{k_1, \lambda_1}$ and $X_2 \sim \chi^2_{k_2, \lambda_2}$ be independent. Show directly that $X_1 + X_2 \sim \chi^2_{k_1+k_2, \lambda_1+\lambda_2}$.

We can write $X_i = \mathbf{y}_i^T \mathbf{y}_i$ for some $\mathbf{y}_i \sim MVN(\boldsymbol{\mu}_i, I_{k_i})$, with $\lambda_i = \frac{1}{2}\boldsymbol{\mu}_i^T \boldsymbol{\mu}_i$ and $\mathbf{y}_1$ independent from $\mathbf{y}_2$. Then

$$X_1 + X_2 = \begin{bmatrix} \mathbf{y}_1^T & \mathbf{y}_2^T \end{bmatrix} \begin{bmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \end{bmatrix}$$

and since

$$\begin{bmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \end{bmatrix} \sim MVN\left(\begin{bmatrix} \boldsymbol{\mu}_1 \\ \boldsymbol{\mu}_2 \end{bmatrix}, I_{k_1+k_2} \right),$$

we have the desired result, with noncentrality parameter

$$\frac{1}{2} \begin{bmatrix} \boldsymbol{\mu}_1^T & \boldsymbol{\mu}_2^T \end{bmatrix} \begin{bmatrix} \boldsymbol{\mu}_1 \\ \boldsymbol{\mu}_2 \end{bmatrix} = \lambda_1 + \lambda_2.$$

5

Subtotal 5

(b) [**3 marks**] Let

$$\mathbf{y} \sim MVN\left(\begin{bmatrix} -3 \\ 8 \end{bmatrix}, \begin{bmatrix} 5 & -3 \\ -3 & 2 \end{bmatrix}\right), \qquad A = \begin{bmatrix} 0 & -6 \\ 6 & 7 \end{bmatrix}.$$

Calculate $E[\mathbf{y}^T A \mathbf{y}]$.

```
> A <- matrix(c(0,-6,6,7),2,2,byrow=T)
> V <- matrix(c(5,-3,-3,2),2,2)
> mu <- c(-3,8)
> sum(diag(A%*%V)) + t(mu)%*%A%*%mu

      [,1]
[1,]  462
```

3

Subtotal 3

(c) [**3 marks**] Describe the distribution of $3y_1 - 2y_2$.

$3y_1 - 2y_2 \sim N(A\boldsymbol{\mu}, AVA^T).$

```
> A <- matrix(c(3,-2),1,2)
> A %*% mu

      [,1]
[1,]   -25

> A %*% V %*% t(A)

      [,1]
[1,]    89
```

3

Subtotal 3

(d) [**3 marks**] Find all values of a and b for which $ay_1 + by_2$ is independent of $3y_1 - 2y_2$.

We set

$$A = \begin{bmatrix} 3 & -2 \\ a & b \end{bmatrix}$$

$$\text{var } A\mathbf{y} = AVA^T$$

$$= \begin{bmatrix} 3 & -2 \\ a & b \end{bmatrix} \begin{bmatrix} 5 & -3 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} 3 & a \\ -2 & b \end{bmatrix}$$

$$= \begin{bmatrix} 3 & -2 \\ a & b \end{bmatrix} \begin{bmatrix} ? & 5a - 3b \\ ? & -3a + 2b \end{bmatrix}$$

$$= \begin{bmatrix} ? & 3(5a - 3b) - 2(-3a + 2b) \\ ? & ? \end{bmatrix}$$

$$21a - 13b = 0$$

$$a = \frac{13}{21}b.$$

3

Subtotal 3

Question 3 (18 marks)

The international bank UBS produced a report on prices and earnings in major cities throughout the world. One of the variables that they measured was the price of 1kg of rice, measured in minutes of labour required for a "typical" worker to purchase the rice. This was measured in 2003 (`rice2003`) and again in 2009 (`rice2009`).

We wish to model the 2009 price in terms of the 2003 price, using the linear model $\mathbf{y} = X\boldsymbol{\beta} + \boldsymbol{\varepsilon}$. The following R calculations are performed:

```
> UBS <- read.csv('UBSprices.csv', header=T)
> plot(UBS$rice2003, UBS$rice2009)
```
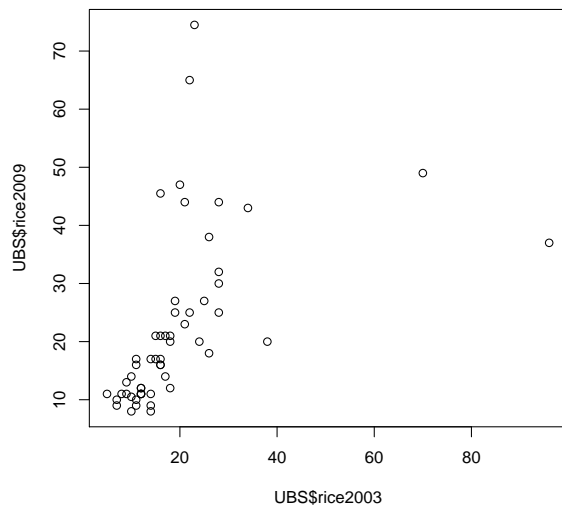


```
> plot(log(UBS$rice2003), log(UBS$rice2009))
```
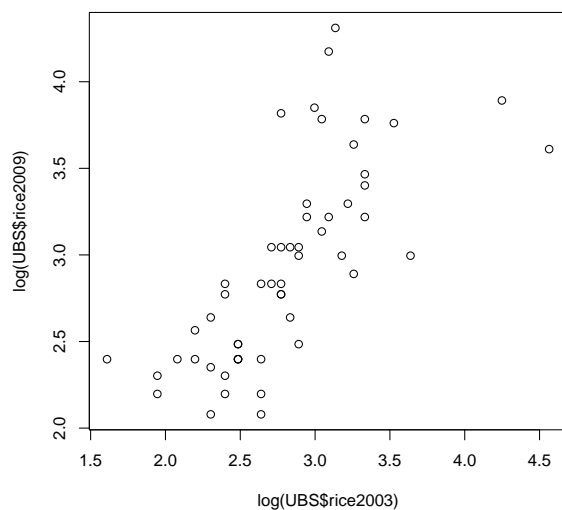
```
> (n <- length(UBS$rice2009))

[1] 54

> X <- cbind(1, log(UBS$rice2003))
> y <- log(UBS$rice2009)
> t(X)%*%X

          [,1]       [,2]
[1,]   54.0000 151.5818
[2,] 151.5818 440.4496

> t(X)%*%y

          [,1]
[1,] 158.3701
[2,] 456.1961

> t(y)%*%y

          [,1]
[1,] 481.9005

> sum(y)

[1] 158.3701

> qt(0.975,50:55)

[1] 2.008559 2.007584 2.006647 2.005746 2.004879 2.004045

> qf(0.95,1,50:55)

[1] 4.034310 4.030393 4.026631 4.023017 4.019541 4.016195

> qf(0.95,2,50:55)

[1] 3.182610 3.178799 3.175141 3.171626 3.168246 3.164993
```

(*Hint: To alleviate rounding error, keep as many digits in internal calculations as possible.*)

(a) [**2 marks**] A logarithmic transformation has been applied to both variables. Give two reasons to justify this transformation.

The transformation is justified due to right-skewness in both variables, as well as some heteroskedasticity in the original plot. $\boxed{2}$

$\boxed{\text{Subtotal 2}}$

(b) [**3 marks**] Calculate the least squares estimates of β.

```
> (b <- solve(t(X)%*%X,t(X)%*%y))

          [,1]
[1,]  0.7470346
[2,]  0.7786571
```

3

Subtotal 3

(c) [**3 marks**] Calculate the sample variance s^2.

```
> (SSRes <- t(y)%*%y - t(y)%*%X%*%b)

          [,1]
[1,]  8.372216

> (s2 <- SSRes/(n-2))

          [,1]
[1,]  0.1610041
```

3

Subtotal 3

(d) [**4 marks**] In 2003, it cost 50 minutes of labour to buy 1kg of rice in the Republic of Linearmodelstan. Calculate (with 95% probability) an interval for the 2009 price of rice (in minutes of labour) in Linearmodelstan.

```
> tt <- c(1, log(50))
> exp(tt%*%b + c(-1,1)*qt(0.975,n-2)*sqrt(s2)*sqrt(1+t(tt)%*%solve

[1]   19.07946 103.30706
```

4

Subtotal 4

(e) [**3 marks**] Test for model relevance at the 5% significance level, using a corrected sum of squares.

Since $56.29 > 4.03$, we reject the null hypothesis of model irrelevance.

```
> (t(y)%*%y - (sum(y)^2/n) - SSRes)/s2

          [,1]
[1,] 56.29408
```

3

Subtotal 3

(f) [**3 marks**] It is claimed that, on average, the price of rice in 2003 is the same as the price of rice in 2009, in terms of labour. This corresponds to a parameter estimate of $\beta = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Determine if this point lies within the joint 95% confidence region for the parameters.

This point does not lie within the joint 95% confidence region.

```
> bst <- c(0,1)
> t(b-bst)%*%t(X)%*%X%*%(b-bst)

          [,1]
[1,] 1.585738

> 2*s2*qf(0.95,2,n-2)

          [,1]
[1,] 1.022422
```

3

Subtotal 3

Question 4 (12 marks)

Consider the full rank linear model $\mathbf{y} = X\boldsymbol{\beta} + \boldsymbol{\varepsilon}$ with p parameters. Now suppose that we transform the design variables x in a linear manner:

$$z_i = \sum_{j=1}^{p} a_{ji} x_j, \qquad i = 1, \ldots, p.$$

(Note that the x variables include the intercept term.) Now consider the linear model $\mathbf{y} = Z\boldsymbol{\beta}_2 + \boldsymbol{\varepsilon}_2$, which also has p parameters.

(a) [**2 marks**] Express the design matrix Z in terms of X, and state a condition under which the second linear model is also full rank.

> We have $Z = XA$. If A is invertible, then $r(Z) = p$ and the model is full rank. [2]
>
> Subtotal 2

(b) [**3 marks**] Calculate the least squares estimators for $\boldsymbol{\beta}_2$ from the second model, and express them in terms of $\mathbf{b}$, the least squares estimators for $\boldsymbol{\beta}$.

> $$\begin{aligned} \mathbf{b}_2 &= (Z^T Z)^{-1} Z^T \mathbf{y} \\ &= (A^T X^T X A)^{-1} A^T X^T \mathbf{y} \\ &= A^{-1} (X^T X)^{-1} (A^T)^{-1} A^T X^T \mathbf{y} \\ &= A^{-1} \mathbf{b}. \end{aligned}$$
>
> [3]
>
> Subtotal 3

(c) [**2 marks**] Consider a subject with design variables $\mathbf{x}^*$ (for the first model). Calculate a point estimate for the average response for this subject, using the second model, and express it in terms of $\mathbf{b}$.

The point estimate is

$$(\mathbf{z}^*)^T \mathbf{b}_2 = (\mathbf{x}^*)^T A A^{-1} \mathbf{b}$$
$$= (\mathbf{x}^*)^T \mathbf{b}.$$

2

Subtotal 2

(d) [**3 marks**] Calculate the sample variance for the second model, and express it in terms of the sample variance for the first model.

$$SS_{Reg}(Z) = \mathbf{y}^T Z \mathbf{b}_2$$
$$= \mathbf{y}^T X A A^{-1} \mathbf{b}$$
$$= SS_{Reg}(X).$$

Since both SS_{Reg} and SS_{Total} are unchanged between models, so is SS_{Res} and therefore s^2. 3

Subtotal 3

(e) [**2 marks**] Briefly discuss the implications of the results you have derived above in the context of fitting a linear model, with particular reference to variable standardisation.

We see that both models have identical predictions for response variables and errors, so they are equivalent. Thus we can linearly transform design variables without changing the model, and in particular can standardise design variables (which is a linear transformation) without changing the model.   2

Subtotal 2

Question 5 (12 marks)

Consider the general linear model, $\mathbf{y} = X\boldsymbol{\beta} + \boldsymbol{\varepsilon}$. This model may be of full or less than full rank.

(a) [**2 marks**] Define the term BLUE (best linear unbiased estimator), and give an example of when one might choose not to use the BLUE.

> The BLUE of a quantity is an estimator of the form $\mathbf{c}^T\mathbf{y}$ which is unbiased, and has the lowest variance among all unbiased linear estimators. An example of when it might not be used is when you want to use a biased estimator, e.g., ridge regression or LASSO estimators. 2
>
> Subtotal 2

(b) [**2 marks**] Describe how the parameters $\boldsymbol{\beta}$ of a linear model may be estimated by the method of maximum likelihood, and relate this to least squares estimation.

> Maximum likelihood estimation chooses the parameters $\boldsymbol{\beta}$ which maximise the likelihood of the responses under the linear model assumption $\mathbf{y} \sim N(X\boldsymbol{\beta}, \sigma^2 I)$. Under this assumption, the MLE estimators are also the least squares estimators. 2
>
> Subtotal 2

(c) [**2 marks**] Define the Cook's distance and explain its purpose.

> The Cook's distance of the ith point is defined as $\frac{1}{k+1}z_i^2\frac{H_{ii}}{1-H_{ii}}$, where z_i is the standardised residual and H_{ii} is the leverage of the ith point. It is used to determine if a point distorts the fit of the model. 2
>
> Subtotal 2

(d) [**2 marks**] Define estimability, and explain its significance for a linear model.

An estimable quantity is one which can be estimated by a linear function of the responses, i.e., $\mathbf{t}^T\boldsymbol{\beta}$ is estimable if there exists $\mathbf{c}$ such that $E[\mathbf{c}^T\mathbf{y}] = \mathbf{t}^T\boldsymbol{\beta}$. Only estimable quantities have true fixed values, and can thus be estimated or tested, in a linear model. $\boxed{2}$

$\boxed{\text{Subtotal 2}}$

(e) [**2 marks**] Define interaction between a categorical and a continuous predictor, and explain how to model it.

Interaction between a categorical and continuous predictor occurs when the effect of the continuous predictor on the response (i.e., the parameter) may vary depending on the level of the categorical predictor. It can be modelled by including one parameter per level of the categorical predictor for the continuous predictor. $\boxed{2}$

$\boxed{\text{Subtotal 2}}$

(f) [**2 marks**] Define single and double blinding, and describe their use in experimental design.

Blinding refers to the practice of ensuring that the sample subjects do not know which treatment they are receiving (single blinding), and those administering also do not know (double blinding). It is used in experiments to ensure that there is no bias due to the placebo effect. $\boxed{2}$

$\boxed{\text{Subtotal 2}}$

Question 6 (16 marks)

Data on 220 agricultural land sales in Minnesota over the period 2002–2011 were collected. The dataset contains the following variables:

- `id`: ID
- `acrePrice`: Sale price, in thousands of dollars per acre
- `region`: One of six major agricultural regions in Minnesota
- `improvements`: Percentage of property value in buildings
- `year`: Year of sale
- `acres`: Size of property
- `tillable`: Percentage of tillable area of the land
- `financing`: Type of financing (title transfer or seller financed)
- `crpPct`: Percentage of land in the US Conservation Reserve Program
- `productivity`: A score measuring the productivity of the land

We wish to model the selling price (`acrePrice`) in terms of the other variables (except `id`). The following R calculations are produced:

```
> ML <- read.csv('ML2.csv', header=T)
> interaction_model <- lm(acrePrice ~ (. - id)^2, data=ML)
> additive_model <- lm(acrePrice ~ . - id, data=ML)
> anova(additive_model, interaction_model)

Analysis of Variance Table

Model 1: acrePrice ~ (id + region + improvements + year + acres + tillable +
    financing + crpPct + productivity) - id
Model 2: acrePrice ~ ((id + region + improvements + year + acres + tillable +
    financing + crpPct + productivity) - id)^2
  Res.Df    RSS Df Sum of Sq      F Pr(>F)
1    207 182.99
2    153 125.15 54    57.845 1.3096 0.1034
```

```
> selected_model <- step(additive_model)

Start:  AIC=-14.52
acrePrice ~ (id + region + improvements + year + acres + tillable +
    financing + crpPct + productivity) - id

                Df Sum of Sq     RSS      AIC
- financing      1      1.135  184.13  -15.159
- improvements   1      1.431  184.42  -14.806
- acres          1      1.582  184.58  -14.626
<none>                         182.99  -14.519
- productivity   1      4.189  187.18  -11.540
- crpPct         1      5.001  187.99  -10.588
- tillable       1      6.770  189.76   -8.527
- region         5     64.123  247.12   41.571
- year           1    140.960  323.95  109.134

Step:  AIC=-15.16
acrePrice ~ region + improvements + year + acres + tillable +
    crpPct + productivity

                Df Sum of Sq     RSS      AIC
- improvements   1      1.509  185.64  -15.363
- acres          1      1.596  185.72  -15.260
<none>                         184.13  -15.159
- productivity   1      4.168  188.30  -12.235
- crpPct         1      5.079  189.21  -11.173
- tillable       1      6.439  190.57   -9.596
- region         5     64.875  249.00   41.245
- year           1    140.494  324.62  107.588

Step:  AIC=-15.36
acrePrice ~ region + year + acres + tillable + crpPct + productivity

                Df Sum of Sq     RSS      AIC
<none>                         185.64  -15.363
- acres          1      1.737  187.37  -15.314
- crpPct         1      4.353  189.99  -12.264
- productivity   1      4.666  190.30  -11.902
- tillable       1      5.163  190.80  -11.328
- region         5     63.368  249.01   39.247
- year           1    143.335  328.97  108.516
```

```
> summary(selected_model)

Call:
lm(formula = acrePrice ~ region + year + acres + tillable + crpPct +
    productivity, data = ML)

Residuals:
    Min      1Q  Median      3Q     Max
-2.1397 -0.5763 -0.1042  0.3114  5.8682

Coefficients:
                    Estimate Std. Error t value Pr(>|t|)
(Intercept)        -6.763e+02  5.338e+01 -12.670  < 2e-16 ***
regionNorthwest    -1.915e+00  2.879e-01  -6.654 2.46e-10 ***
regionSouth Central 1.191e-03  2.376e-01   0.005   0.9960
regionSouth East    5.592e-02  2.887e-01   0.194   0.8466
regionSouth West   -5.216e-01  2.236e-01  -2.332   0.0206 *
regionWest Central -1.064e+00  2.332e-01  -4.565 8.53e-06 ***
year                3.379e-01  2.660e-02  12.703  < 2e-16 ***
acres              -7.921e-04  5.664e-04  -1.398   0.1635
tillable            1.109e-02  4.599e-03   2.411   0.0168 *
crpPct             -9.941e-03  4.490e-03  -2.214   0.0279 *
productivity        1.319e-02  5.753e-03   2.292   0.0229 *
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1


Residual standard error: 0.9425 on 209 degrees of freedom
Multiple R-squared:  0.6094,       Adjusted R-squared:  0.5907
F-statistic: 32.61 on 10 and 209 DF,  p-value: < 2.2e-16

> qt(0.975,209:214)

[1] 1.971379 1.971325 1.971271 1.971217 1.971164 1.971111

> qf(0.95,5,209:214)

[1] 2.257274 2.257066 2.256860 2.256657 2.256455 2.256255

> qf(0.95,6,209:214)

[1] 2.142153 2.141943 2.141736 2.141530 2.141327 2.141125
```
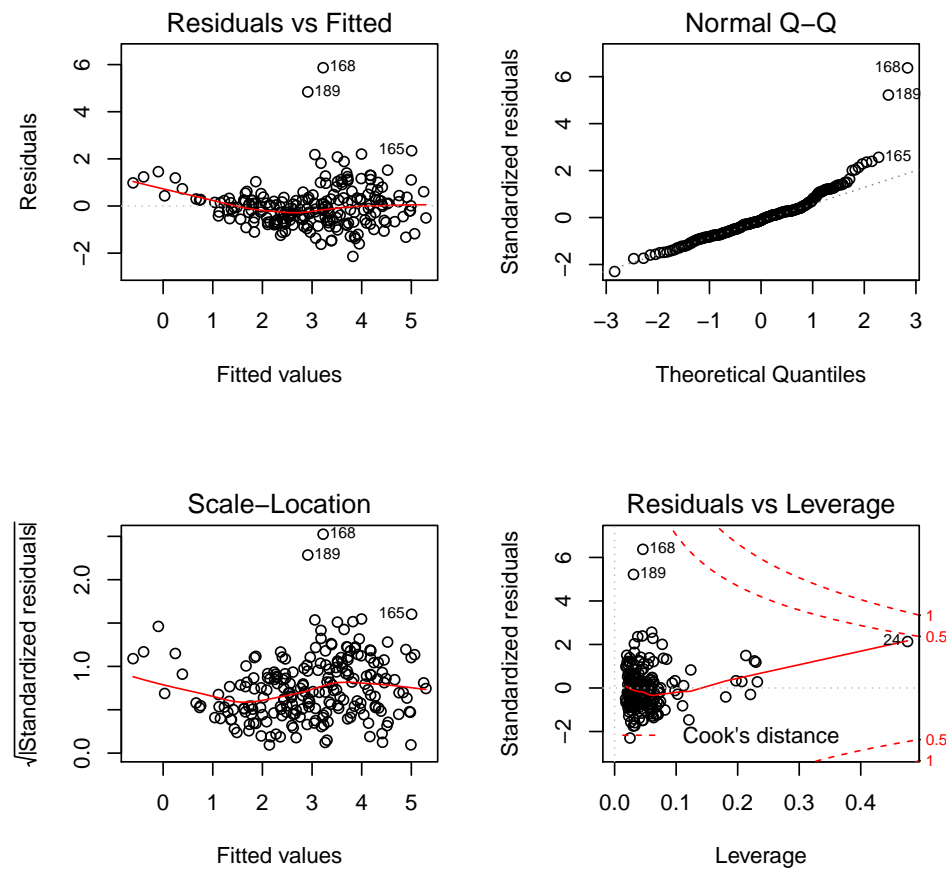
```
> par(mfrow=c(2,2))
> plot(selected_model)
```


(a) [**2 marks**] Interpret the output of the `anova` function.

According to the `anova` function, there is no significant pairwise interaction between the variables. 2

Subtotal 2

(b) [**2 marks**] Identify the variable selection procedure that has been used here.

Stepwise selection with AIC. 2

Subtotal 2

(c) [**3 marks**] From the model `selected_model`, test for the relevance of the `region` variable, at the 5% level. Clearly state your F-statistic and critical value, and interpret your results in the context of the study.

```
> 63.368/5/0.9425^2

[1] 14.26715
```

Since $14.27 > 2.257$, we reject the null hypothesis: region has a clear effect on the price per acre of the land.
3

Subtotal 3

(d) [**2 marks**] Perform one step of backwards elimination on `selected_model`.

Considering that `region` is a relevant variable, we would remove the `acres` variable. 2

Subtotal 2

(e) [**4 marks**] From the diagnostic plots, comment on the suitability of the linear model for this data. If the model is not suitable, suggest how it can be improved.

From the diagnostic plots, we can identify a slight non-linear trend for small fitted values; a slight right-skew in the residual QQ-plot; and a couple of outliers (ids 168 and 189). It might be worth removing the outliers and considering a logarithmic transformation on the response variable. 4

Subtotal 4

(f) [**3 marks**] For a plot of agricultural land in the South West, calculate a 95% confidence interval for the effect of the year on the price per acre in the model `selected_model`.

The confidence interval does not change depending on the region, because there is no interaction in the model.

```
> 0.3379 + c(-1,1)*qt(0.975,209)*0.0266

[1] 0.2854613 0.3903387
```

3

Subtotal 3

Question 7 (8 marks)

(a) [**5 marks**] You wish to perform a study to determine if 3 treatments each produce no effect, using a completely randomised design. To do this, you will test the hypothesis $H_0 : \mu + \tau_1 = \tau_1 - \tau_2 = \tau_2 - \tau_3 = 0$. You are given resources to study 50 sample units. Determine the optimal allocation of the number of units to assign to each treatment. (*Hint: In a completely randomised design with treatment effects* τ_i, *we have* $\text{var}(\mu + \tau_i) = \sigma^2 \frac{1}{n_i}$ *and* $\text{var}(\tau_i - \tau_j) = \sigma^2 \left(\frac{1}{n_i} + \frac{1}{n_j} \right)$. *To minimise a function* $f(\mathbf{x})$ *under the constraint* $g(\mathbf{x}) = c$, *minimise* $f(\mathbf{x}, \lambda) = f(\mathbf{x}) + \lambda(g(\mathbf{x}) - c).$)

We want to minimise

$$f(n_1, n_2, n_3, \lambda) = \sigma^2 \left(\frac{2}{n_1} + \frac{2}{n_2} + \frac{1}{n_3} \right) + \lambda \left(\sum_{i=1}^{3} n_i - n \right).$$

This gives

$$
\begin{aligned}
\frac{\partial f}{\partial n_1} &= -2\frac{\sigma^2}{n_1^2} + \lambda = 0 \\
\frac{\partial f}{\partial n_2} &= -2\frac{\sigma^2}{n_2^2} + \lambda = 0 \\
\frac{\partial f}{\partial n_3} &= -\frac{\sigma^2}{n_3^2} + \lambda = 0 \\
n_1^2 &= 2\frac{\sigma^2}{\lambda} = n_2^2 = 2n_3^2 \\
n_1 &= n_2 = \sqrt{2}n_3.
\end{aligned}
$$

This gives $n_1 = 19, n_2 = 18, n_3 = 13$ — due to rounding, one of n_1 or n_2 has to be larger. $\boxed{5}$

$\boxed{\text{Subtotal 5}}$

(b) [**3 marks**] Compare and contrast blocking and randomisation as tools for eliminating confounding, and discuss their best use in experimental design.

> Both blocking and randomisation will eliminate confounding, but blocking will only eliminate confounding in the blocking variable, while randomisation will eliminate all confounding. They should be used in conjunction to reduce variability from the blocking variable(s). $\boxed{3}$
>
> $\boxed{\text{Subtotal 3}}$

End of Exam—Total Available Marks = 90