THE UNIVERSITY OF MELBOURNE

Discrete & Continuous Data

Semester 1, 2021

Kris Ehinger

Continuous attributes

- Naïve Bayes (as discussed last lecture) assumes nominal data
  - What happens if we have continuous data?

Continuous attributes

- How to compute probabilities $P(x_i | c_j)$?

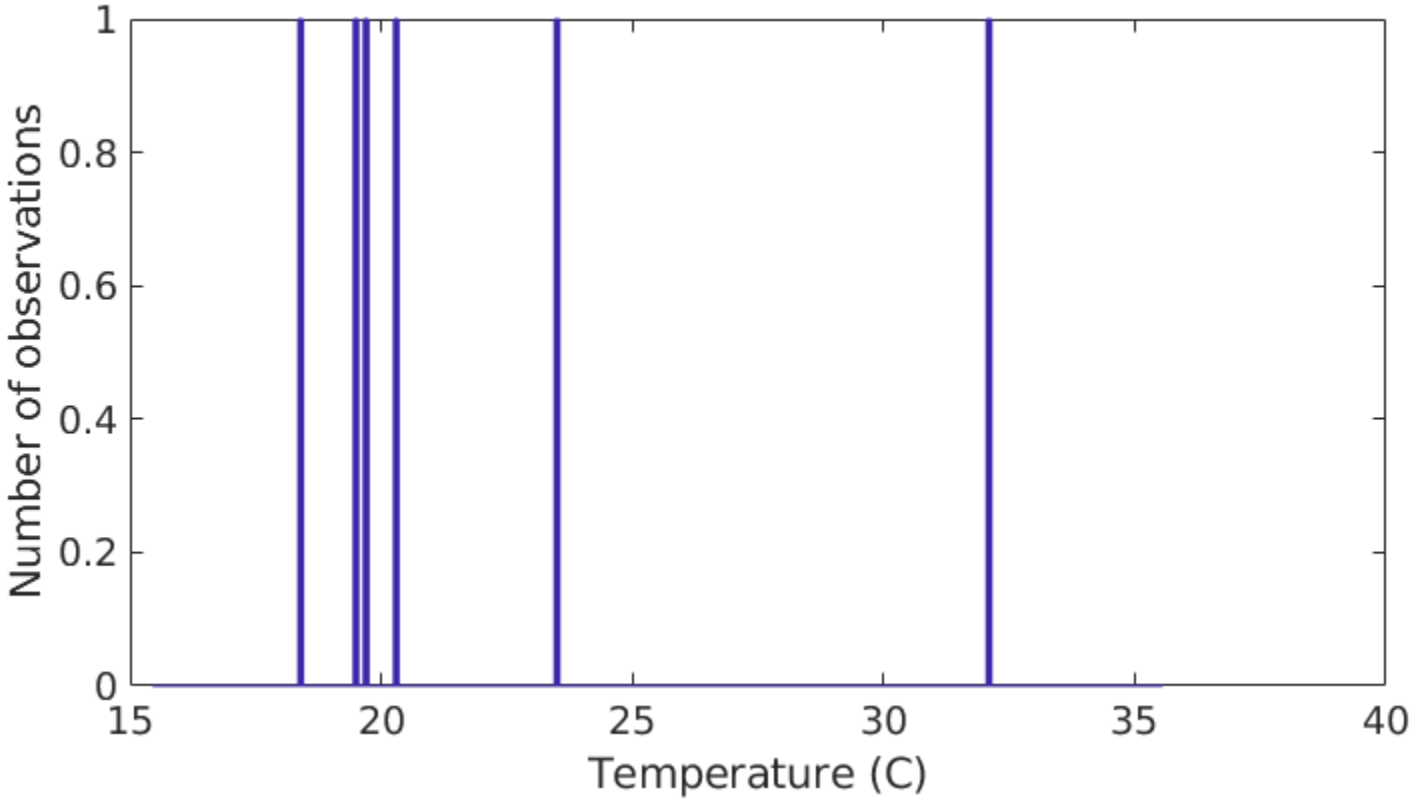
Wind	Temp	Rain?
north	32.1	no
east	18.4	**yes**
east	19.5	no
north	23.5	no
north	20.3	**yes**
east	19.7	**yes**

$P(\text{temp} = 32.1 | \text{Rain} = \text{no}) = 1/3$
$P(\text{temp} = 19.5 | \text{Rain} = \text{no}) = 1/3$
$P(\text{temp} = 23.5 | \text{Rain} = \text{no}) = 1/3$
$P(\text{temp} = 18.4 | \text{Rain} = \text{yes}) = 1/3$
$P(\text{temp} = 20.3 | \text{Rain} = \text{yes}) = 1/3$
$P(\text{temp} = 19.7 | \text{Rain} = \text{yes}) = 1/3$

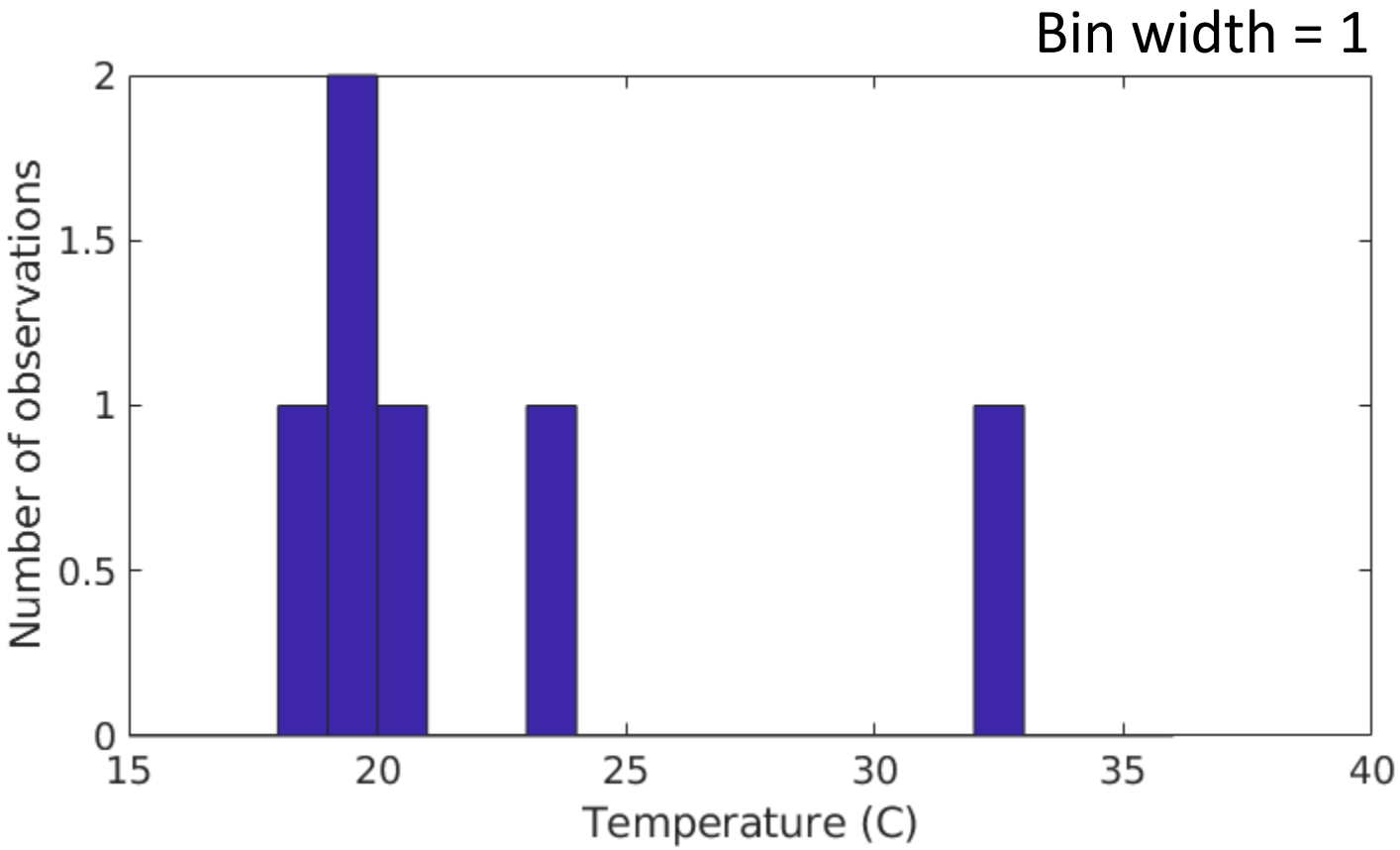
This doesn't look right…

Continuous attributes

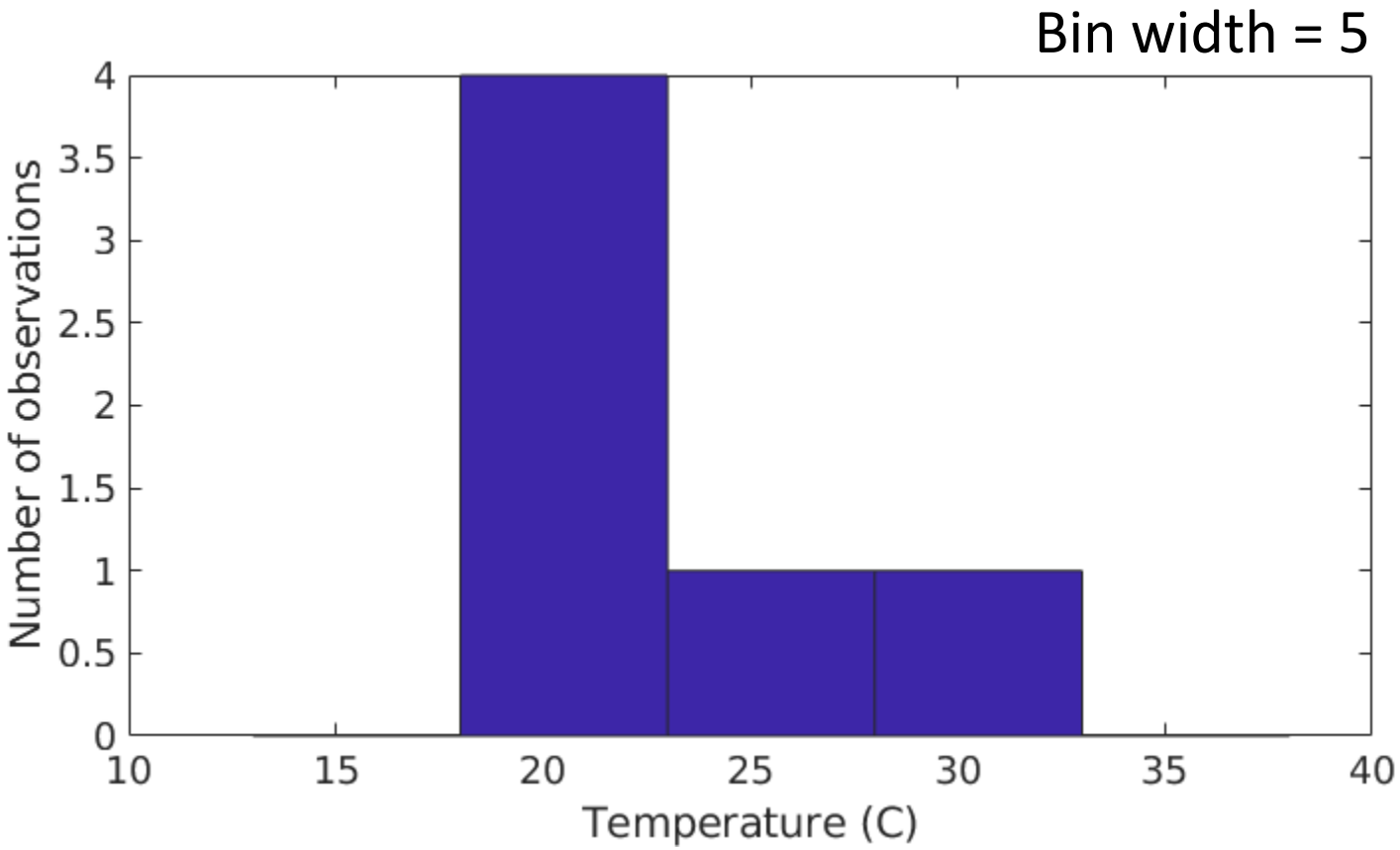
How the model perceives the training data:



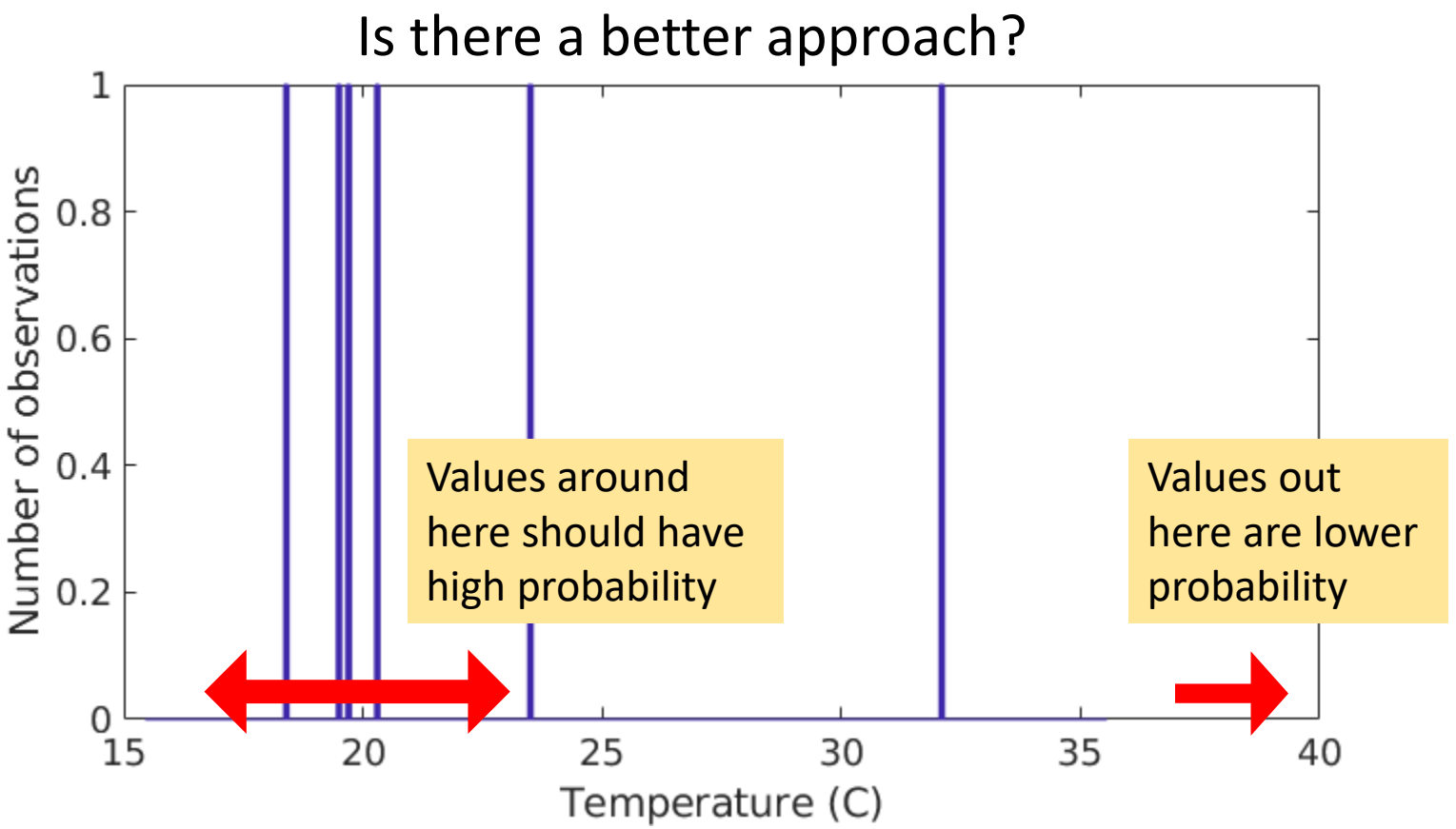
Continuous attributes



Continuous attributes



Continuous attributes



Outline

- Converting data types
  - Nominal -> numeric
  - Numeric -> nominal
- Naïve Bayes with continuous data
  - Gaussian naïve Bayes
  - Kernel Density Estimation (KDE)
  - KDE naïve Bayes

Converting data types

Data types

- The input to a machine learning system consists of instances, which have:
  - Attributes
  - Class labels (if supervised)
- Attributes and class labels can be:
  - Nominal / categorical
  - Ordinal
  - Continuous / numeric

Attribute types

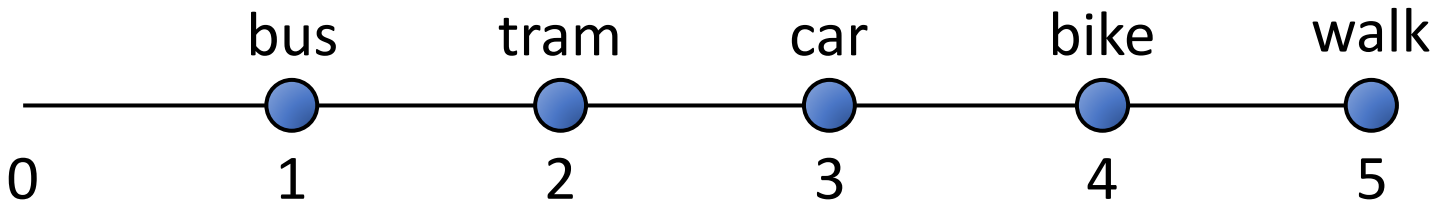
- Machine learning algorithms typically assume attributes have a particular data type
- Algorithms that assume nominal attributes:
  - Naïve Bayes (as described in last lecture)
  - Decision trees
- Algorithms that assume numeric attributes:
  - Support vector machines (SVM)
  - Perceptron, neural network

Attribute types

- What if we have attributes of the wrong type for a given model?
- Options:
  - Discard those attributes
  - Convert the attributes to match the model
  - Change the model assumptions to match the data

Nominal -> numeric

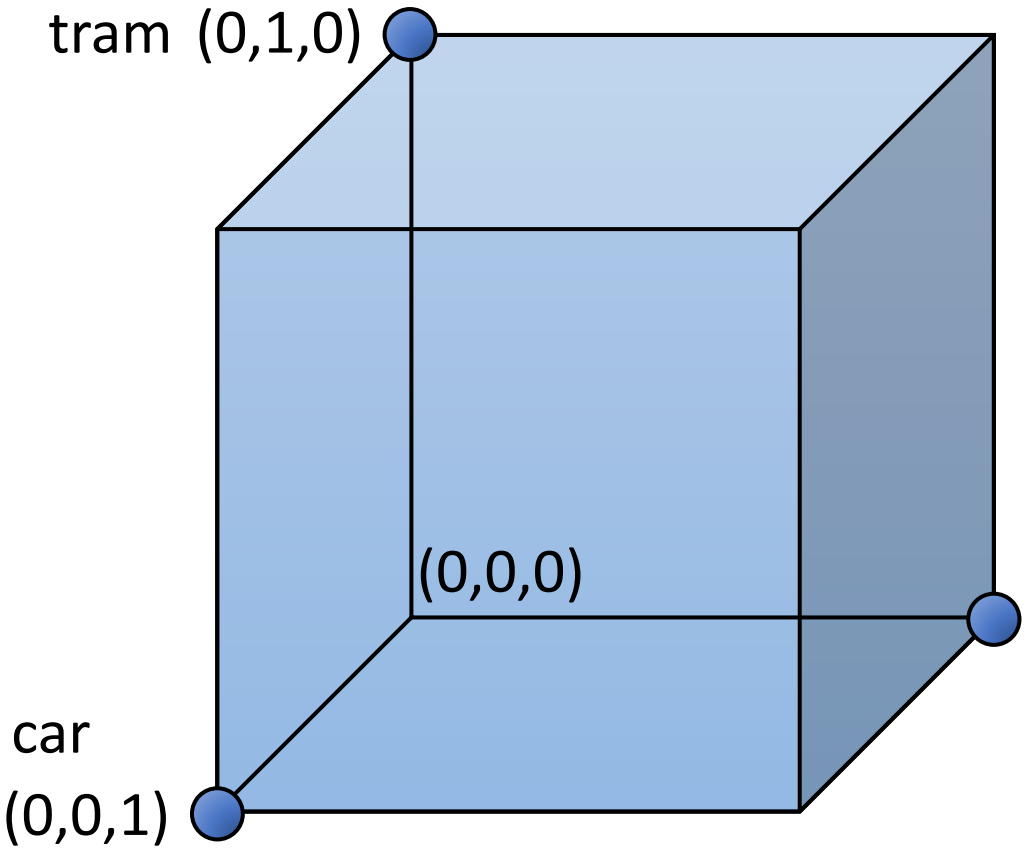
- Option 1: Convert category names to numbers
  - Attribute: mode of transport
  - Nominal: “bus,” “tram,” “car,” “bike,” “walk”
  - Numeric: 0, 1, 2, 3, 4
- Problem: creates an artificial ordering when no order exists, makes some categories seem more/less similar to each other



Nominal -> numeric

- Option 2: One-hot encoding
  - Attribute with *m* possible values -> *m* boolean attributes
  - “bus” = [1, 0, 0, 0, 0]
  - “tram” = [0, 1, 0, 0, 0]
  - “car” = [0, 0, 1, 0, 0]
  - “bike” = [0, 0, 0, 1, 0]
  - “walk” = [0, 0, 0, 0, 1]
- Best way to represent nominal values in a continuous space, but increases dimensionality of the space

Visualisation of option 2



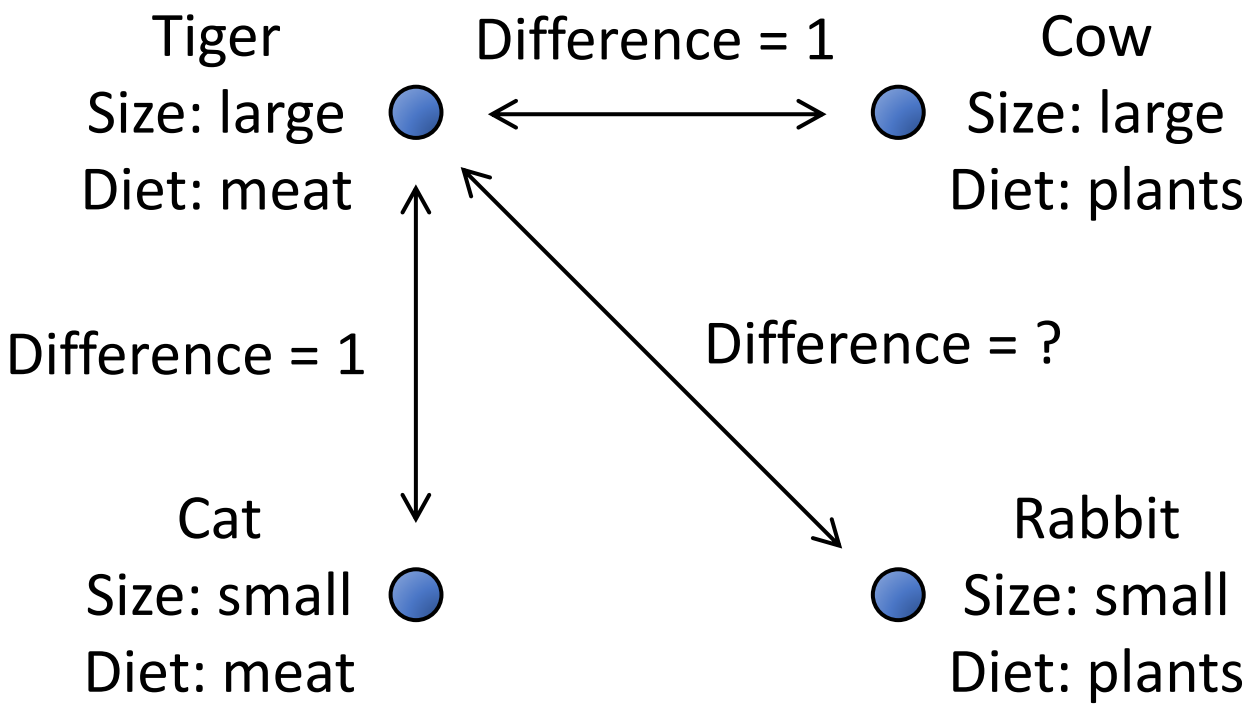
Better encoding:

All categories equal distance apart; any subset is equally easy to group

Computing distances

- How to compute distances between nominal attributes?
- Consider these boolean attributes for an animal classifier:
  - Size: large/small
  - Diet: meat/plants

Computing distances (difference)



Computing distances

- How to compute distances between nominal attributes?
  - Euclidean distance
    - If A and B differ on N attributes, $dist_E(A, B) = \sqrt{N}$

$$dist_E(A, B) = \sqrt{\sum_i (a_i - b_i)^2}$$

  - Hamming distance
    - If A and B differ on N attributes, $dist_H(A, B) = N$

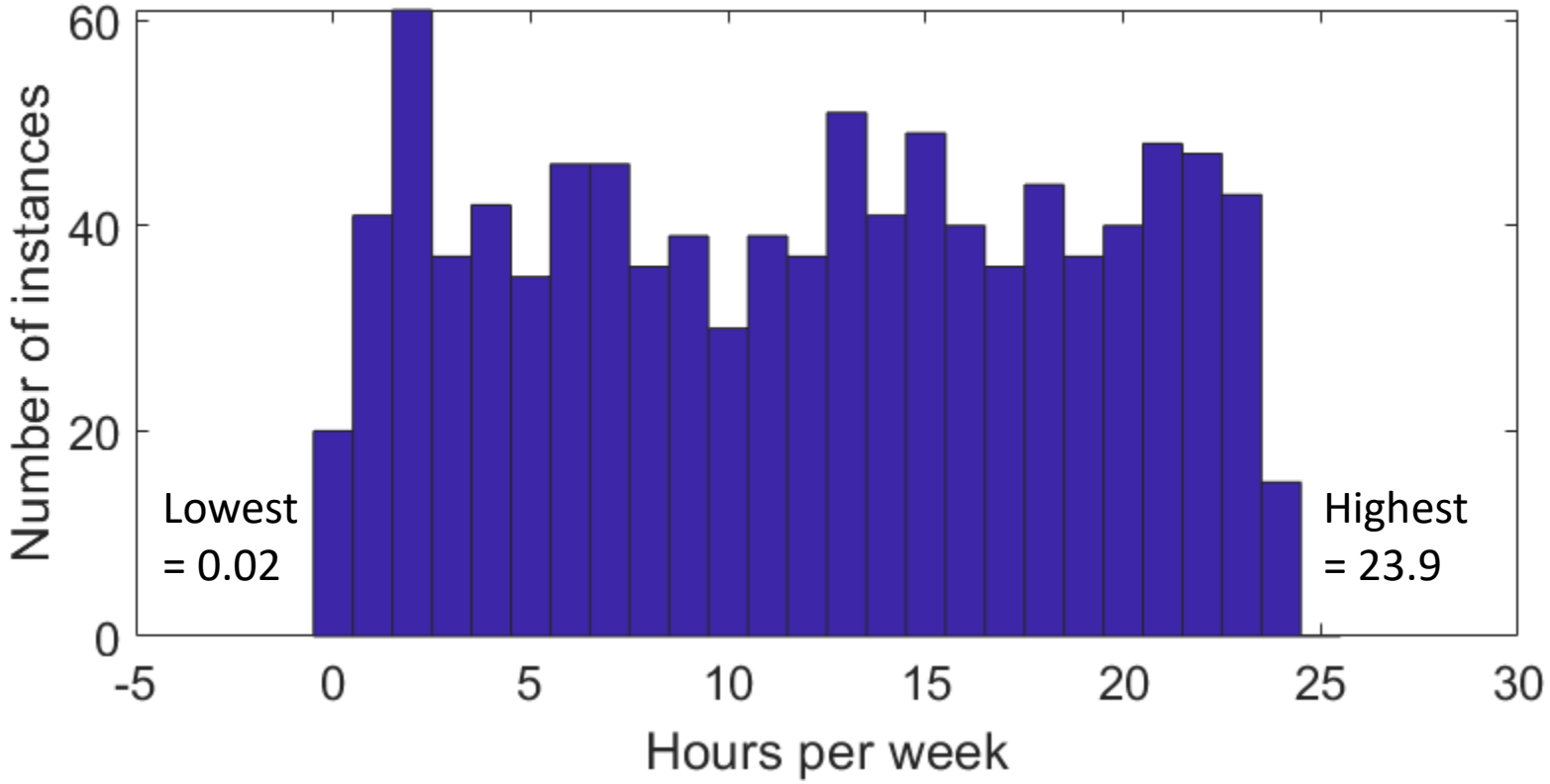
$$dist_H(A, B) = \sum_i \begin{cases} 0, & a_i == b_i \\ 1, & otherwise \end{cases}$$

Numeric -> nominal

- **Discretisation** is the translation of continuous numeric attributes to discrete nominal attributes
  - Example: map temperatures to “hot,” “mild,” “cool”
- Discretisation is generally a two-step process:
  - Decide how many nominal values (= intervals) onto which you will map the numeric values
  - Decide where to place the boundaries for these intervals

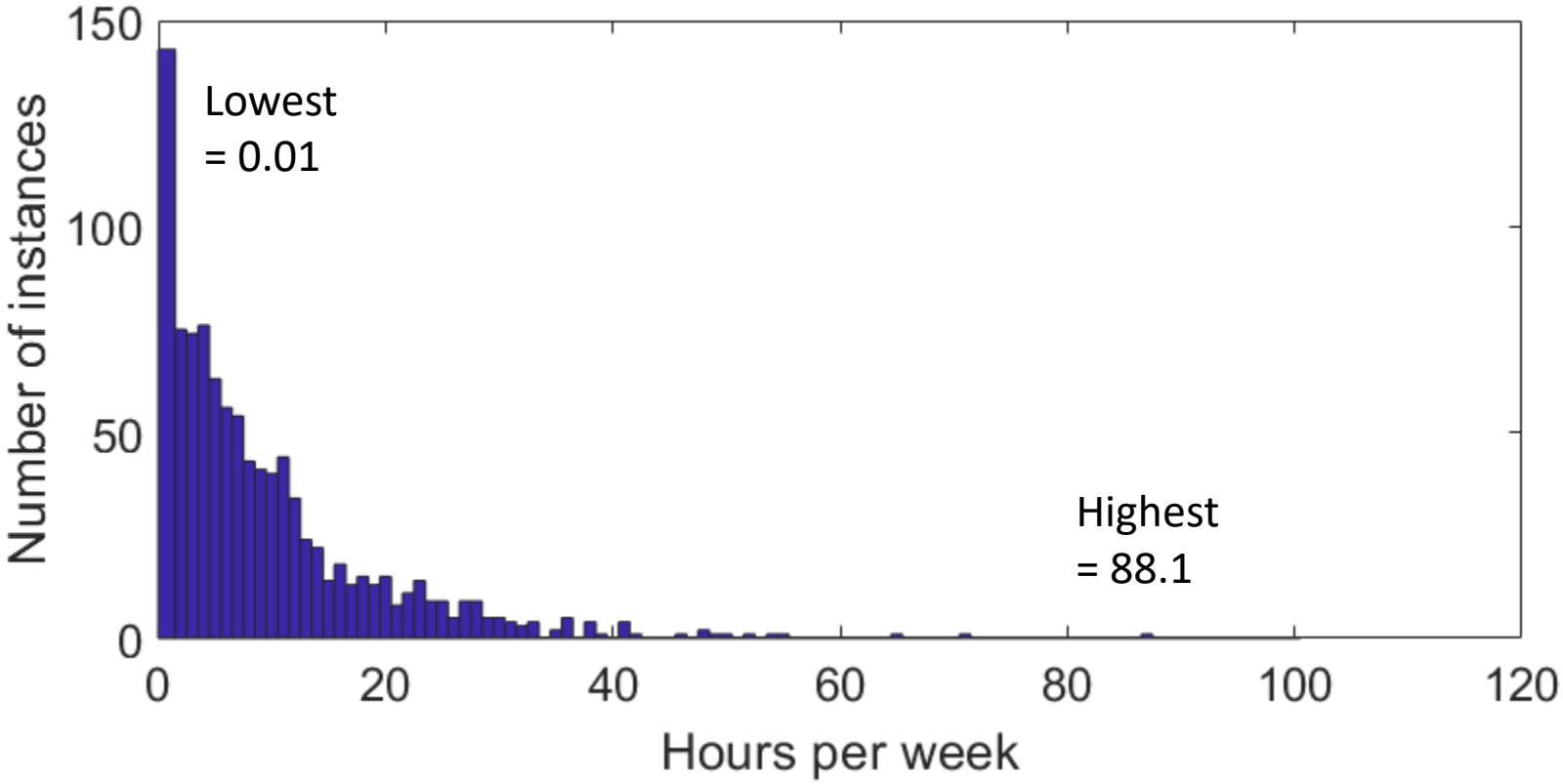
Discretisation example 1

You measured app usage (hours/week) from 1000 users. How would you recode this attribute into 3 levels: low, medium, high?



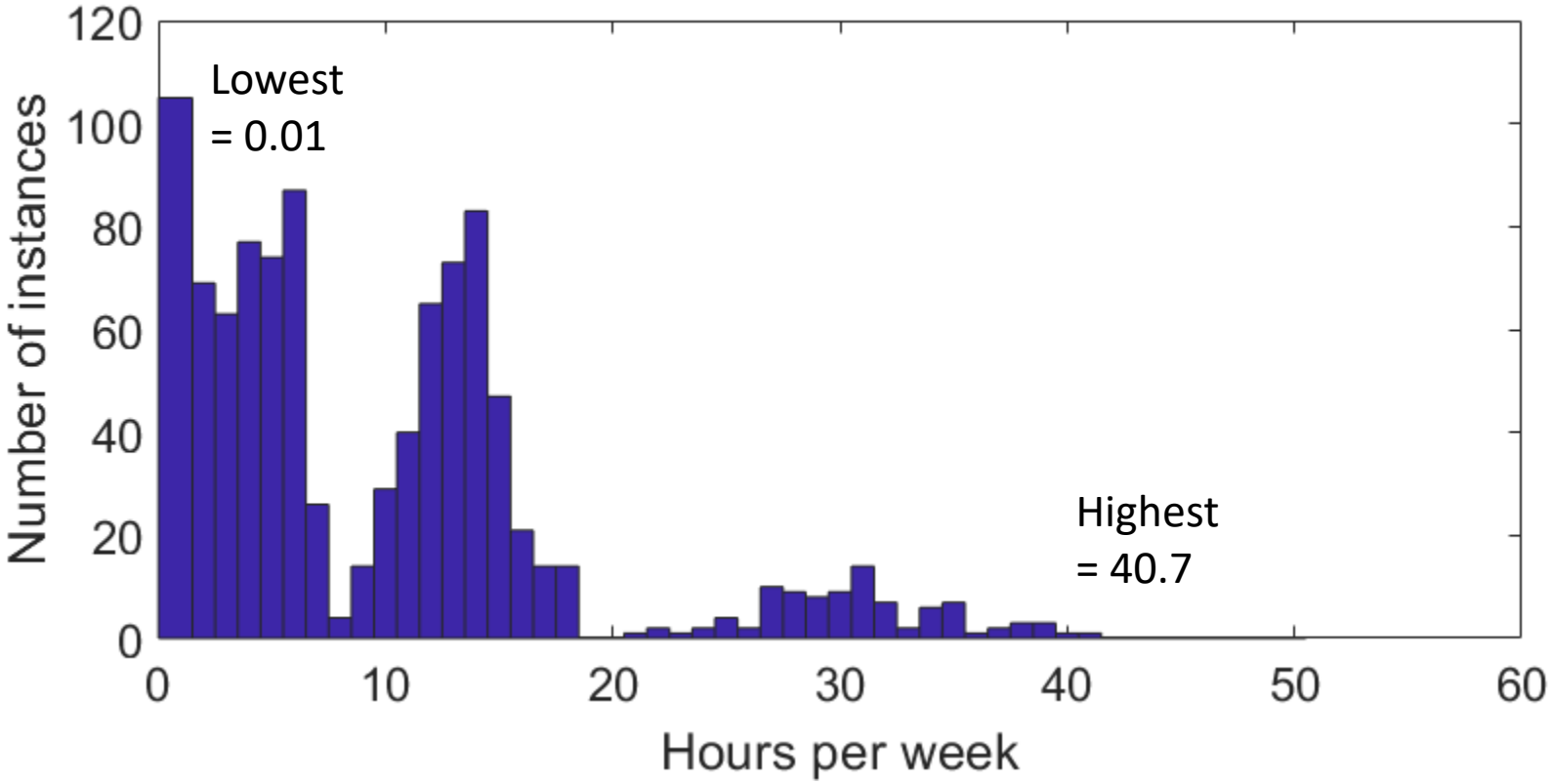
Discretisation example 2

You measured app usage (hours/week) from 1000 users. How would you recode this attribute into 3 levels: low, medium, high?



Discretisation example 3

You measured app usage (hours/week) from 1000 users. How would you recode this attribute into 3 levels: low, medium, high?



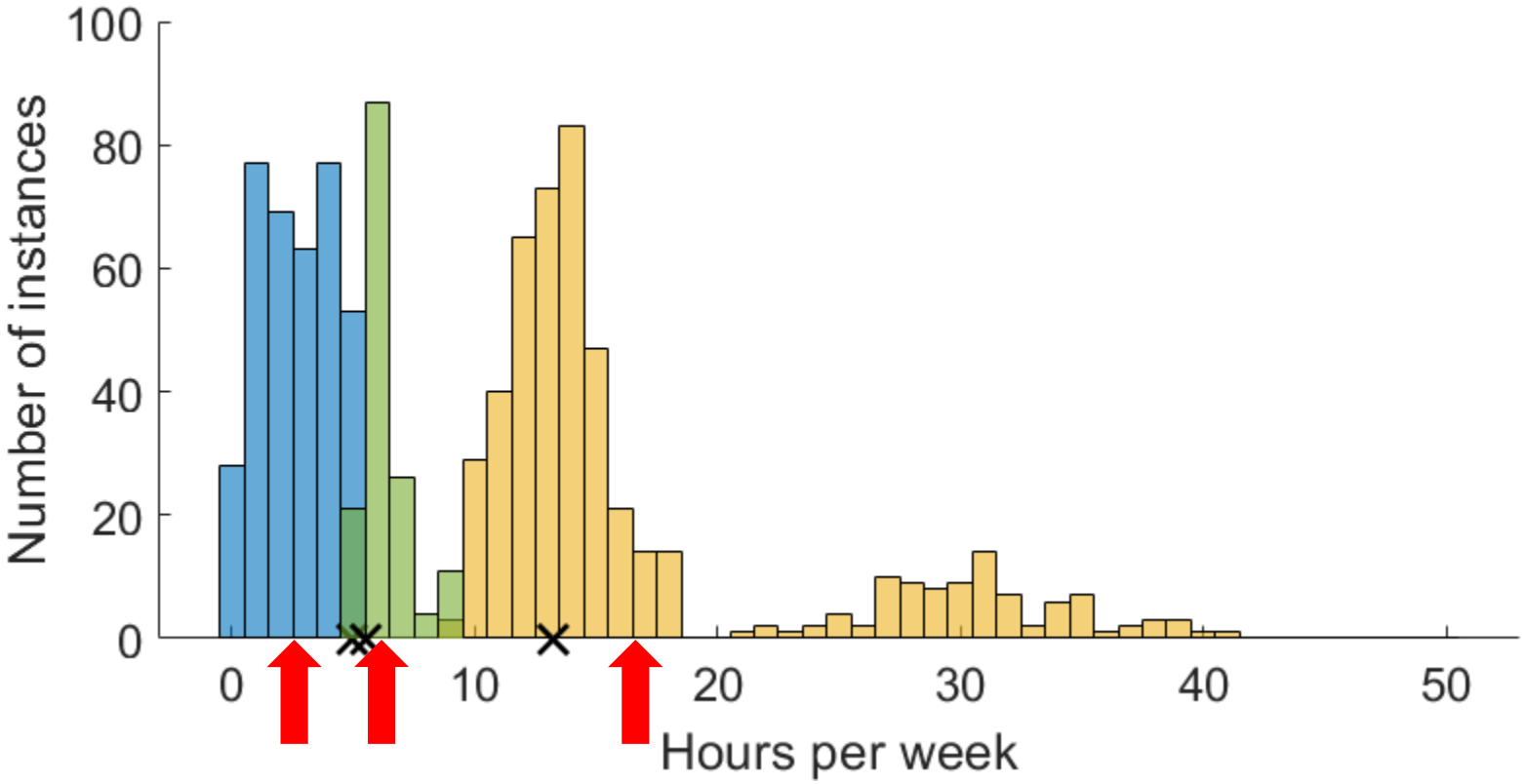
Discretisation options

- **Equal-width** discretisation – find min/max of range, partition into n bins of width (max-min)/n
- **Equal-frequency** discretisation – sort values, find breakpoints that produce n bins with (approximately) equal numbers of items
- Disadvantages:
  - Arbitrary group boundaries
  - Equal-width is sensitive to outliers, equal-frequency is sensitive to sample bias
  - User must choose n

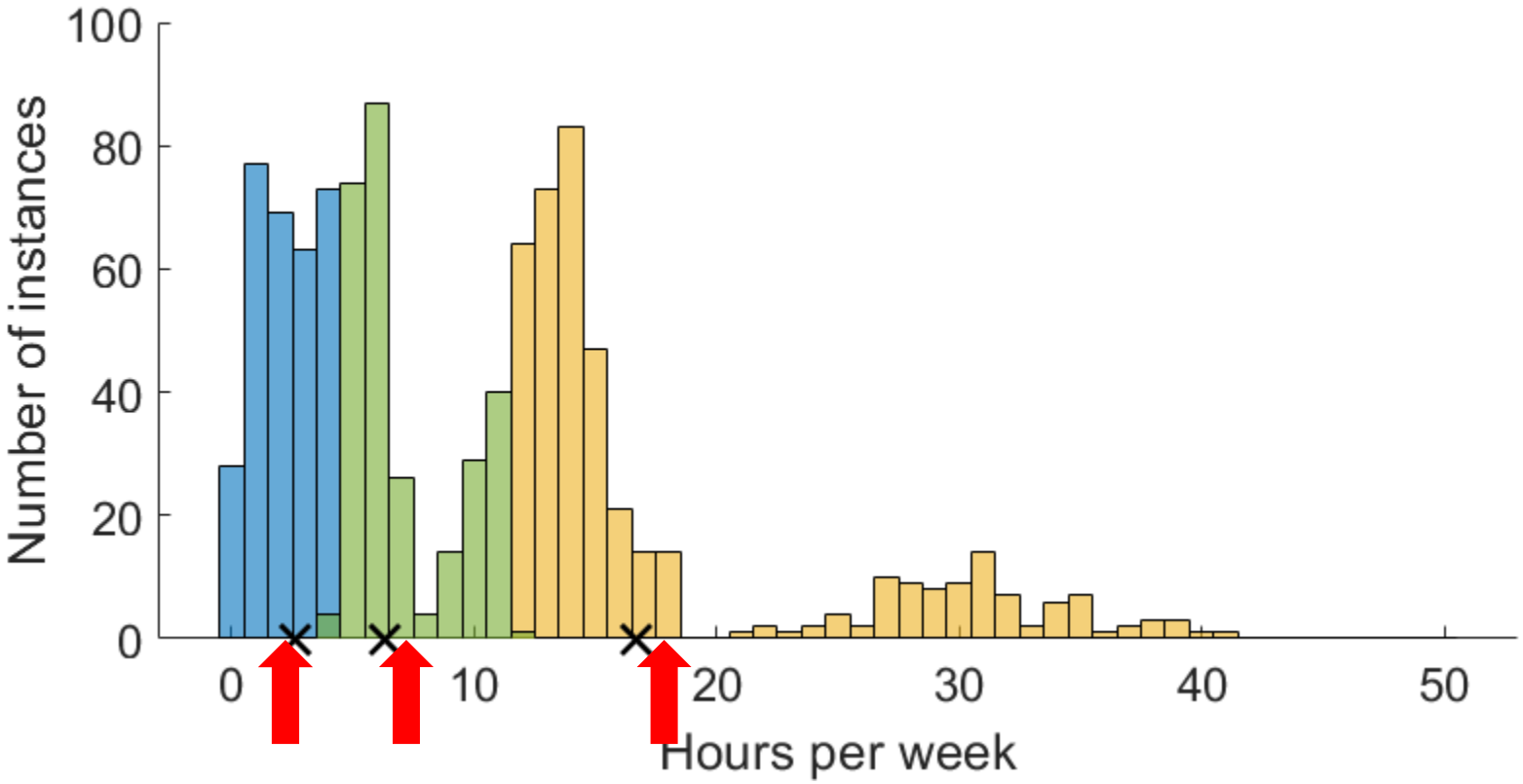
Discretisation options

- Use some clustering method to discover natural breakpoints within your data (e.g., k-means clustering)
- Disadvantages
  - More complicated than equal-width or -frequency
  - If data doesn't have natural "groups," k-means result is the same as equal-width discretisation
  - Sensitive to outliers
  - User must choose n

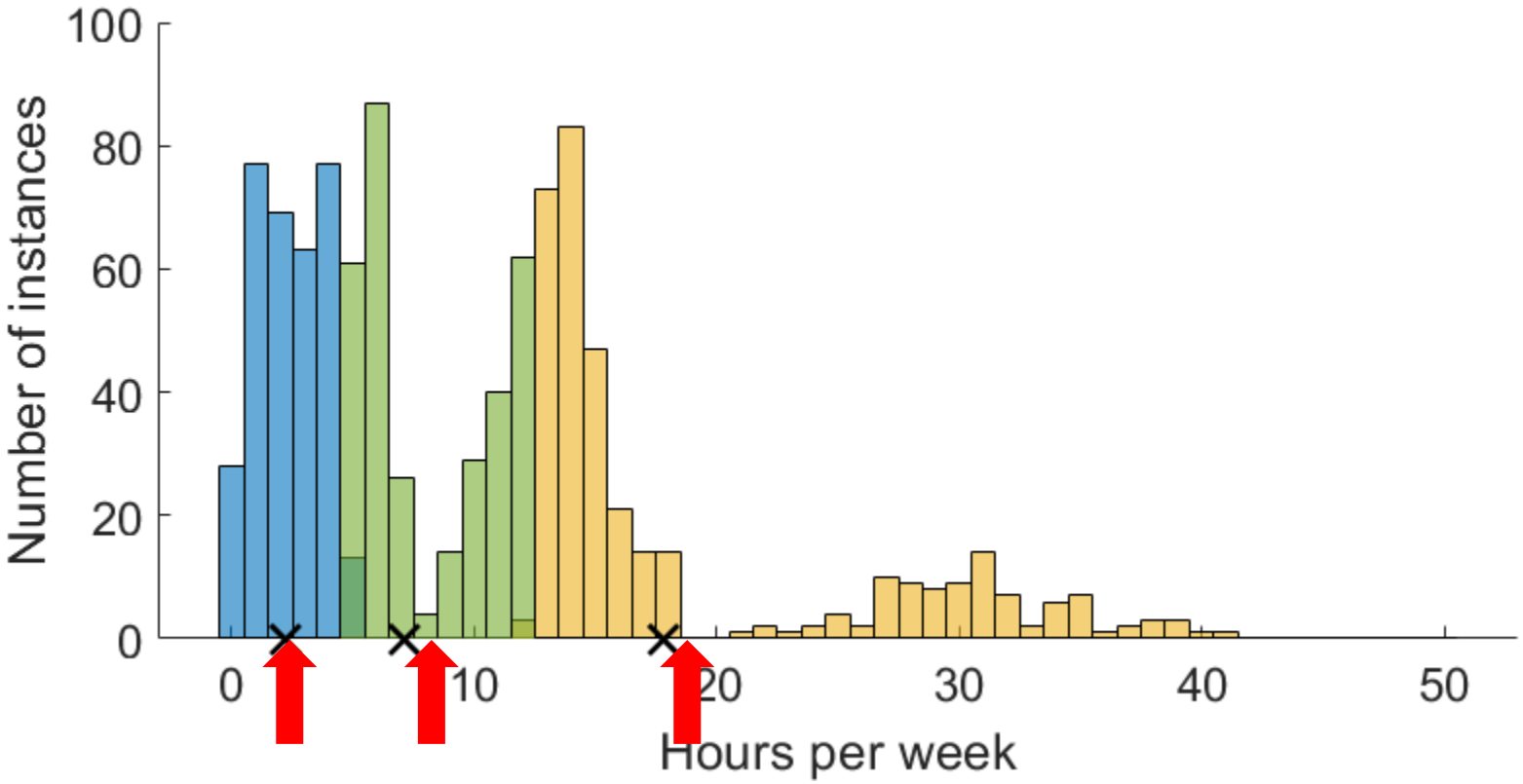
Discretisation: K-means clustering



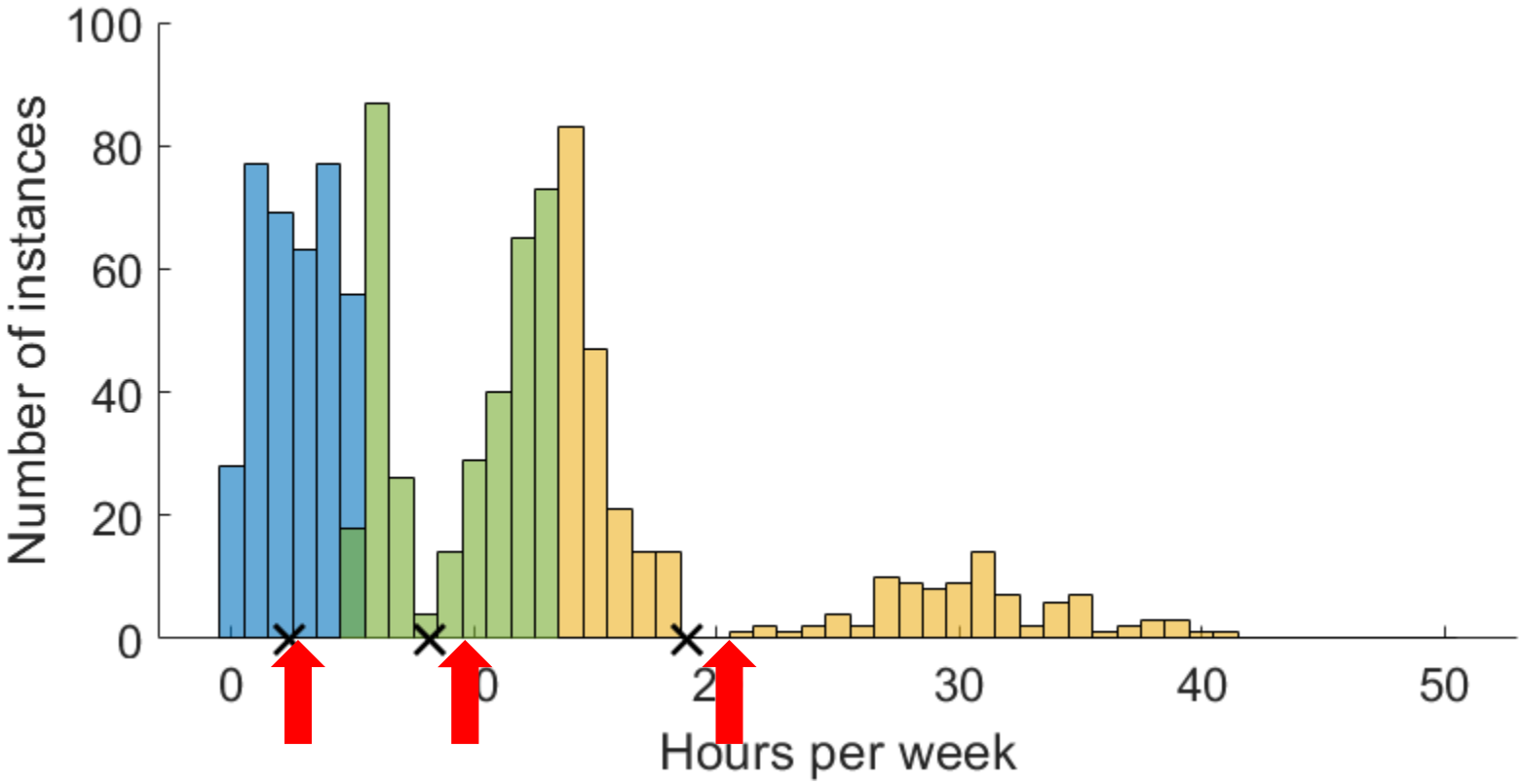
Discretisation: K-means clustering



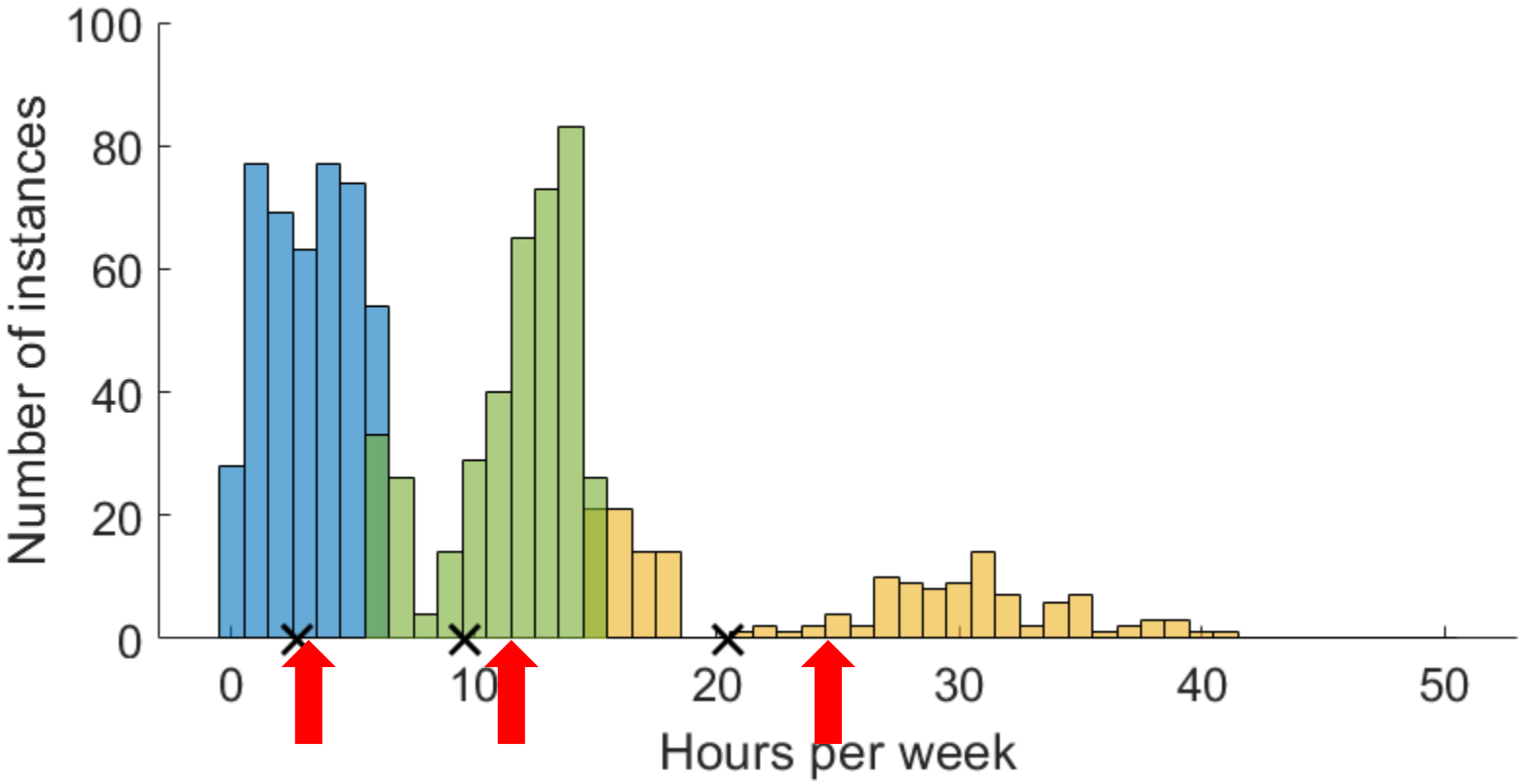
Discretisation: K-means clustering



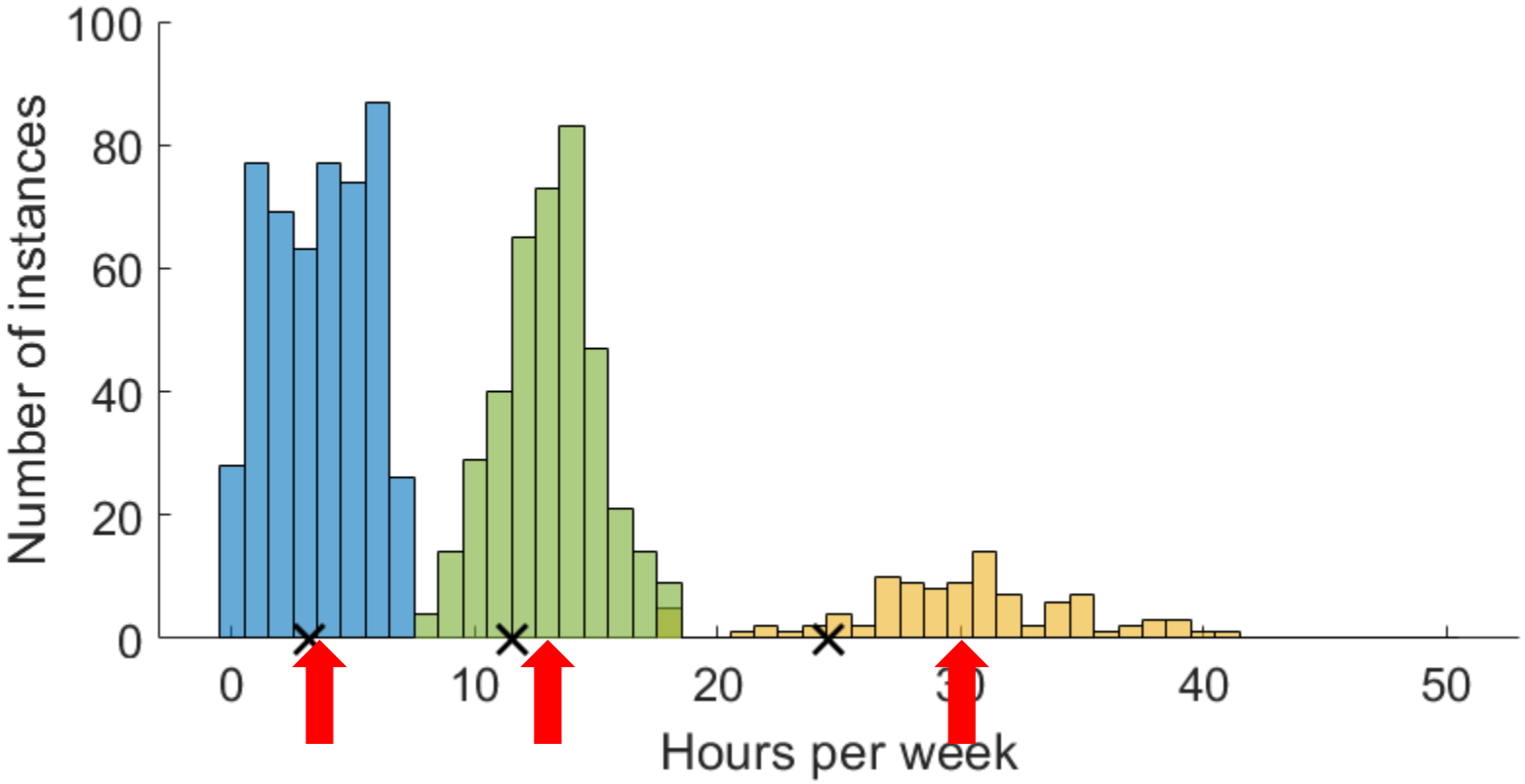
Discretisation: K-means clustering



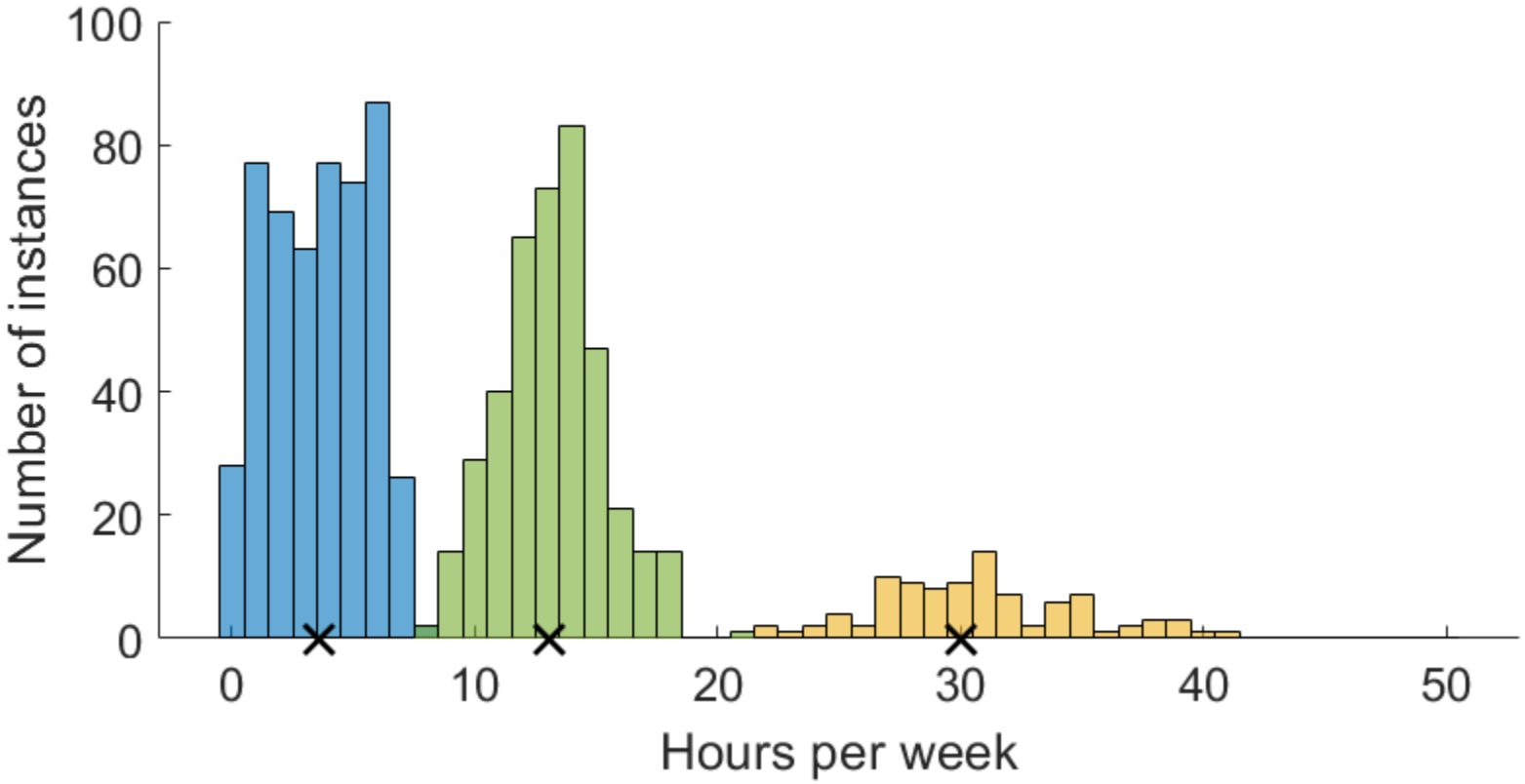
Discretisation: K-means clustering



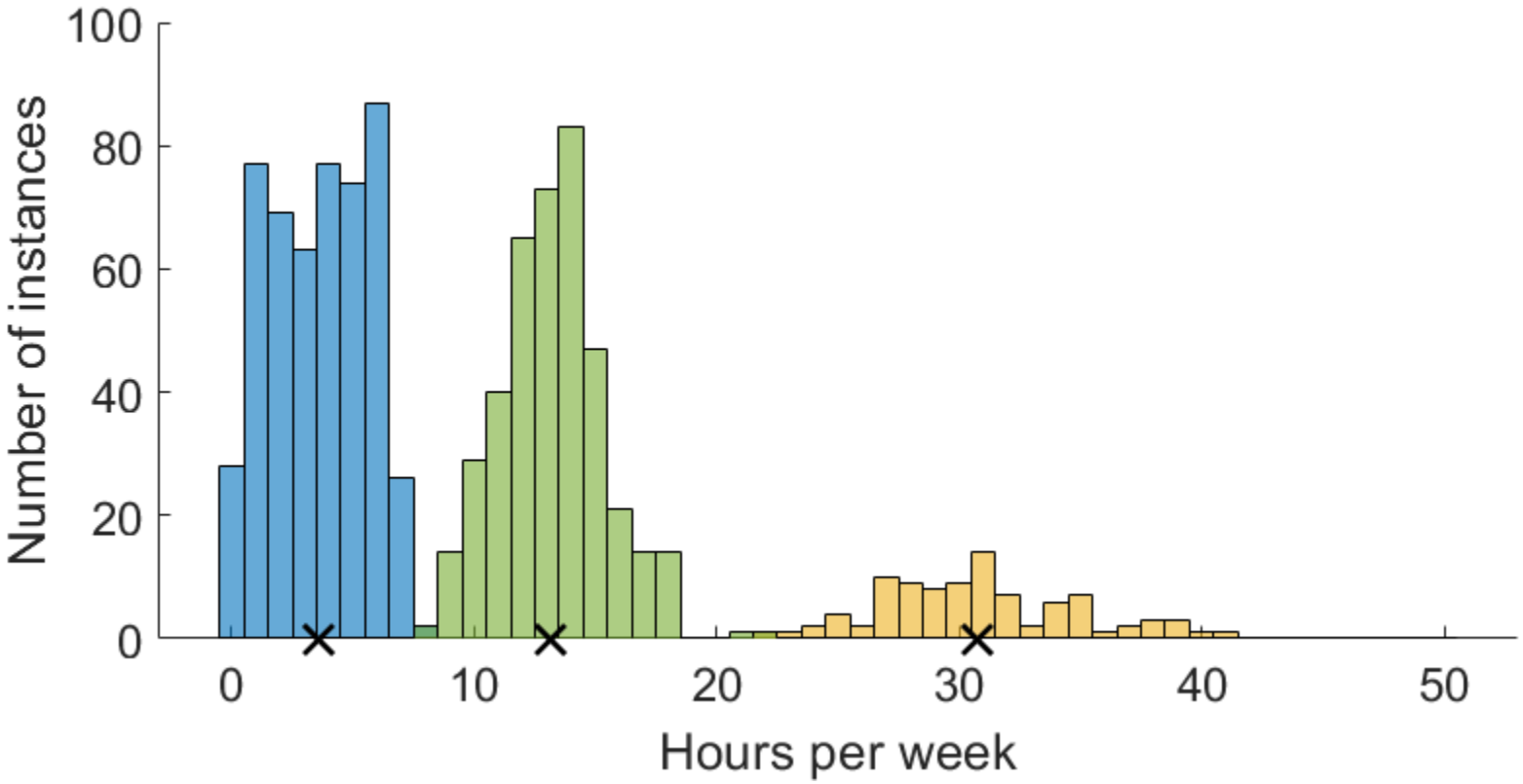
Discretisation: K-means clustering



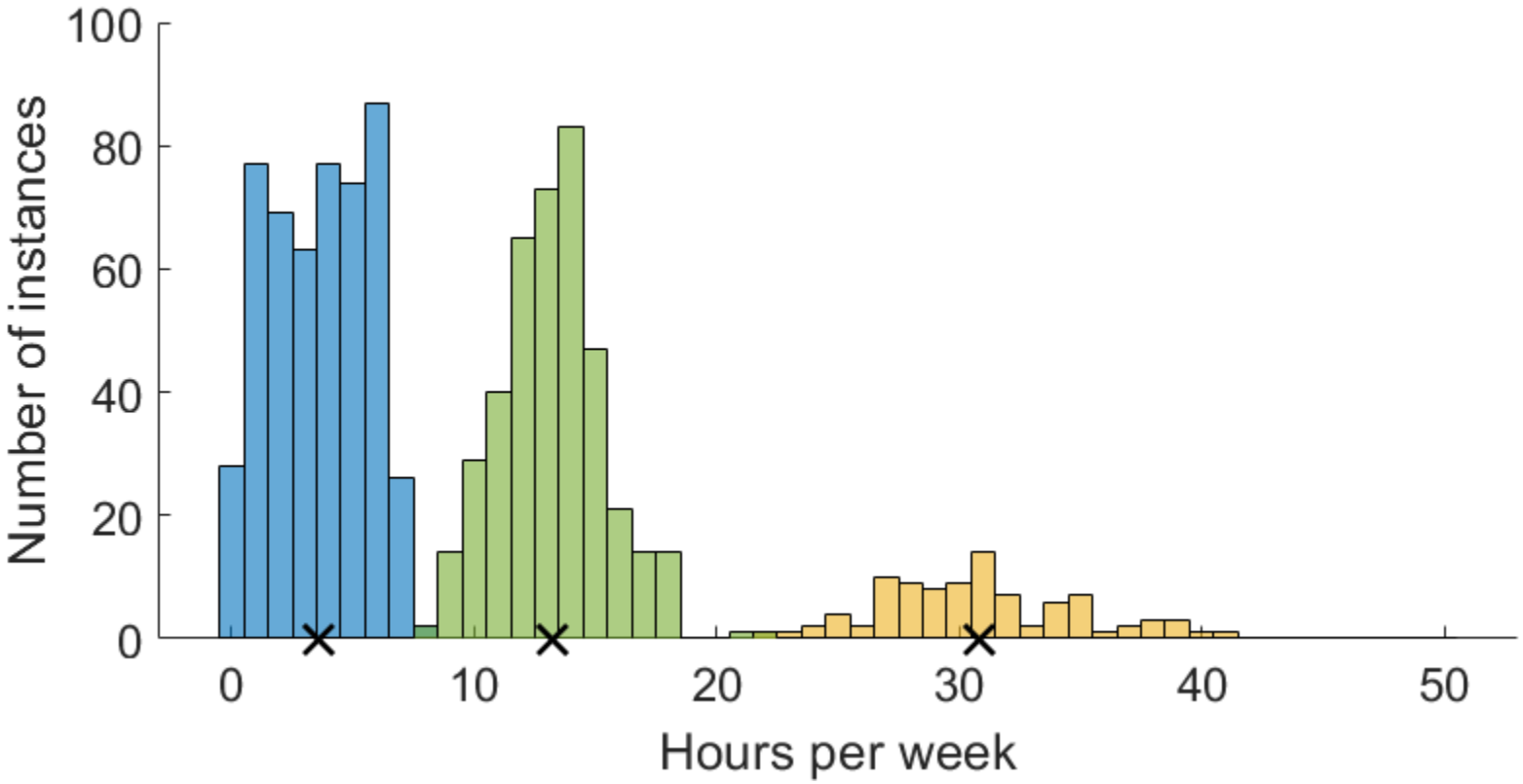
Discretisation: K-means clustering



Discretisation: K-means clustering



Discretisation: K-means clustering



Supervised discretisation

- Group values into class-contiguous intervals
  - Sort values, and identify breakpoints in class membership

    1 | 2 | 4 5 7 | 8 9 | 9 11
  - Reposition breakpoints if the numeric value is the same

    1 | 2 | 4 5 7 | 8 9 9 | 11
  - Set the breakpoints midway between the neighbouring values

    1 | 2 | 4 5 7 | 8 9 9 | 11

    1.5 3 7.5 10

Supervised discretisation

- Supervised discretisation may help you find "groups" that are most relevant for your classification task
- Disadvantages
  - Arbitrary group boundaries
  - Arbitrary number of groups (tends to produce too many groups)
  - Tends to **overfit** training set

Discretisation summary

- Various options; each has advantages and disadvantages
- Discretisation means throwing out some information
  - This can help simplify data to make it easier for the model to learn
  - But you can potentially lose details that would have been useful

Naïve Bayes with continuous data

Naïve Bayes

- In naïve Bayes, we choose the class c_j that maximizes:

$$P(c_j)\prod_i P(x_i|c_j)$$

Prior probability of class c_j — Likelihood of feature x_i in class c_j

- $P(x_i|c_j)$ can be computed differently for different data types:
  - If x_i is a nominal attribute with multiple levels, count number of times each level occurs in class c_j
  - If x_i is a numeric attribute, compute its **probability distribution**

Probability density function

- A popular pdf for continuous data is the **Gaussian distribution** (or **normal distribution**)
- The probability of observing value x from a variable with mean (expected value) μ and standard deviation σ:

$$f(x) = \phi_\sigma(x-\mu) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

ϕ_σ is a Gaussian distribution with mean = 0 and standard deviation = σ

- Important properties of this distribution:
  - Symmetric about the mean μ
  - Area under the curve sums to 1

Probability density function

- Where do the **mean** μ and **standard deviation** σ come from?
- These are descriptive statistics that can be estimated from our data

Mean of N samples of an attribute X:

$$\mu_x = \sum_{i=1}^{N} \frac{x_i}{N}$$

Standard deviation of N samples of an attribute X:

$$\sigma_x = \sqrt{\frac{\sum_{i=1}^{N} (x_i - \mu)^2}{N - 1}}$$

Why Gaussian?

- In practice, the normal distribution is a reasonable approximation for many events
  - Including binomial, Poisson, chi-square, and Student's t-distributions
  - E.g., height, weight, shoe size, reaction time …

Example: Gaussian naïve Bayes

- Compute probability distribution for each class:

Wind	Temp	Rain?
north	32.1	no
east	18.4	**yes**
east	19.5	no
north	23.5	no
north	20.3	**yes**
east	19.7	**yes**

When rain=no, temp is [32.1, 19.5, 23.5]

Mean:

$$\mu_{no} = (32.1 + 19.5 + 23.5)/3 = 25.0$$

Standard deviation:

$$\sigma_{no} = \sqrt{\frac{(32.1 - 25.0)^2 + (19.5 - 25.0)^2 + (23.5 - 25.0)^2}{3 - 1}} = 6.4$$

Example: Gaussian naïve Bayes

- Compute probability distribution for each class:

Wind	Temp	Rain?
north	32.1	no
east	18.4	**yes**
east	19.5	no
north	23.5	no
north	20.3	**yes**
east	19.7	**yes**

When rain=yes, temp is [18.4, 20.3, 19.7]

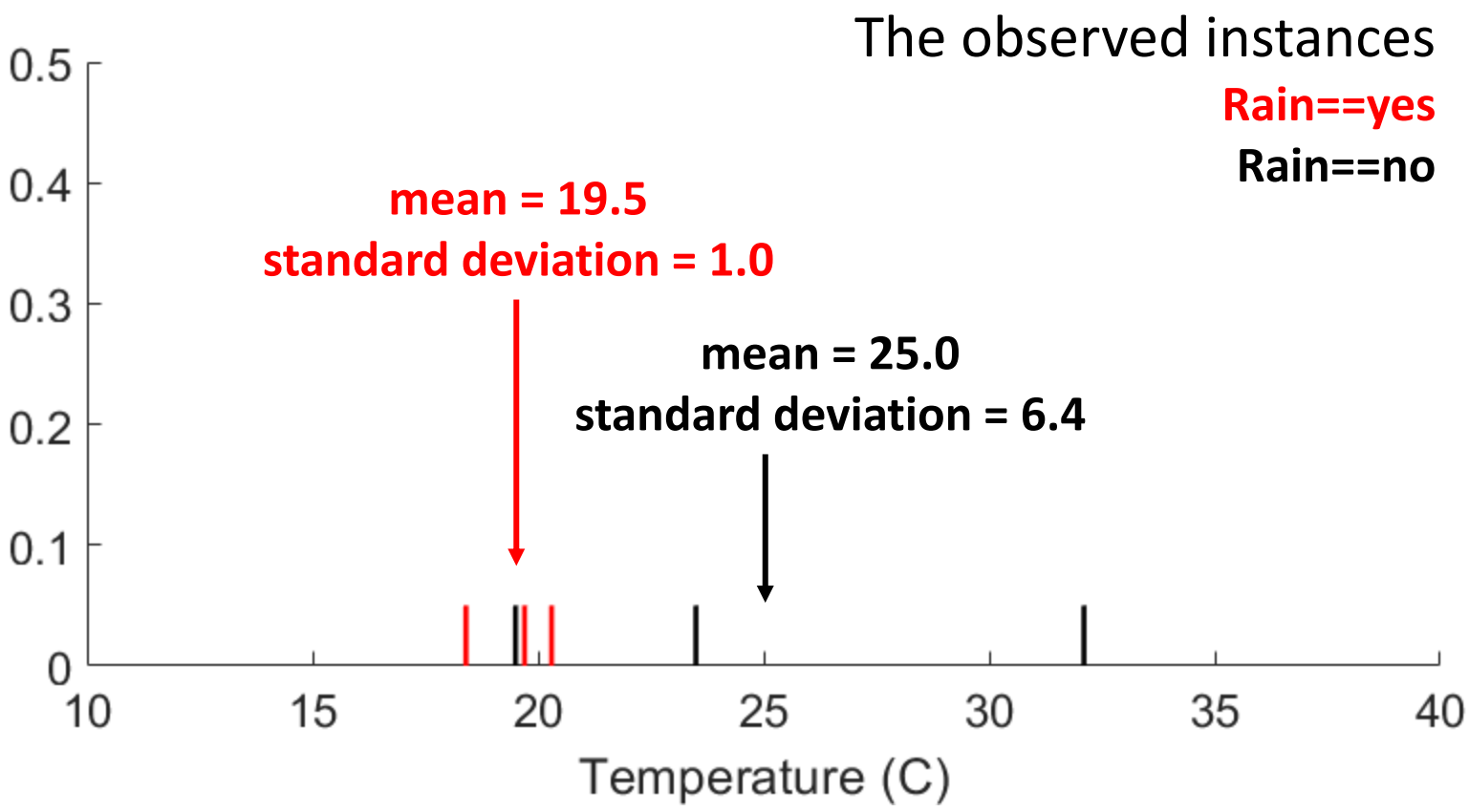
Mean:

$$\mu_{yes} = (18.4 + 20.3 + 19.7)/3 = 19.5$$

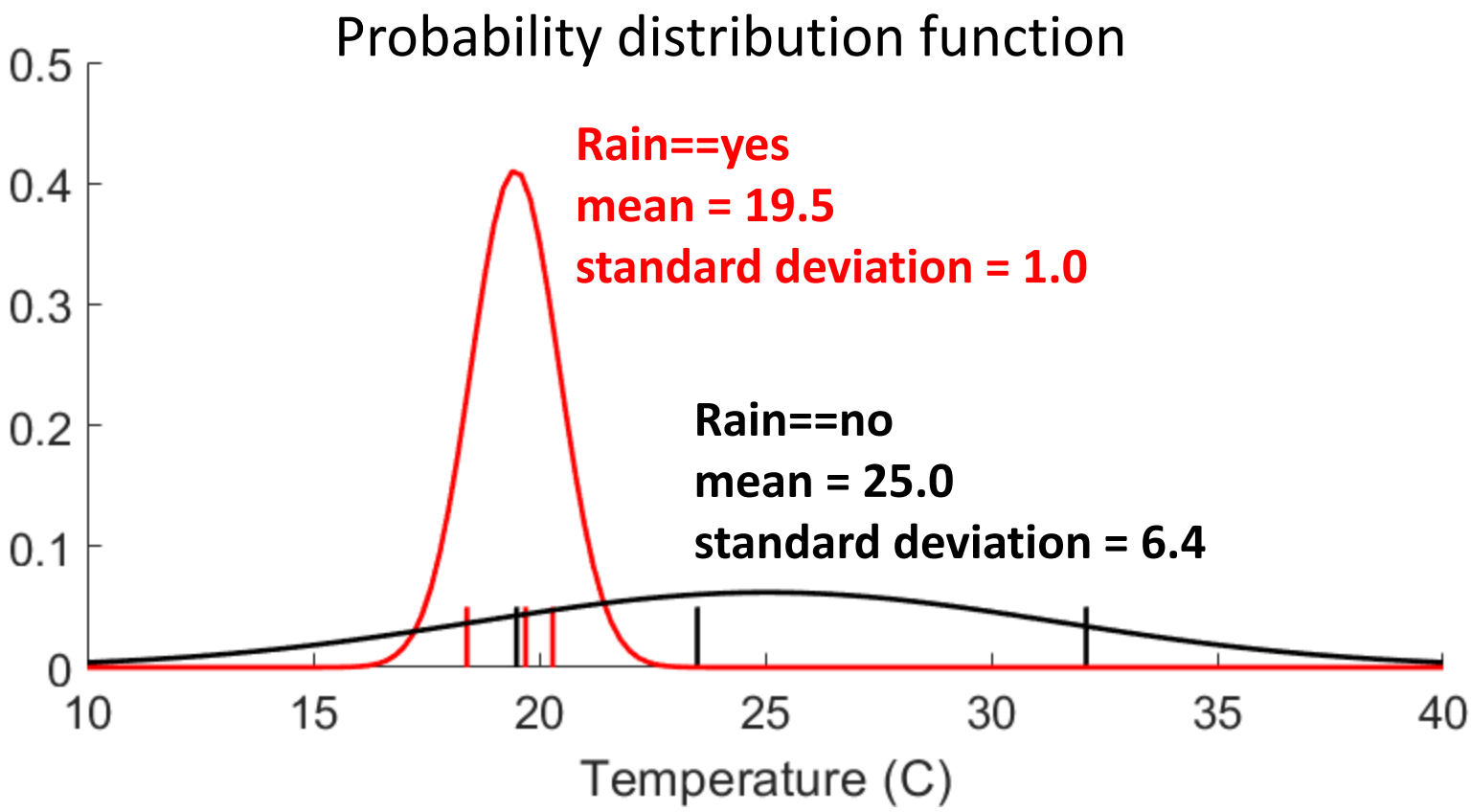
Standard deviation:

$$\sigma_{yes} = \sqrt{\frac{(18.4 - 19.5)^2 + (20.3 - 19.5)^2 + (19.7 - 19.5)^2}{3 - 1}} = 1.0$$

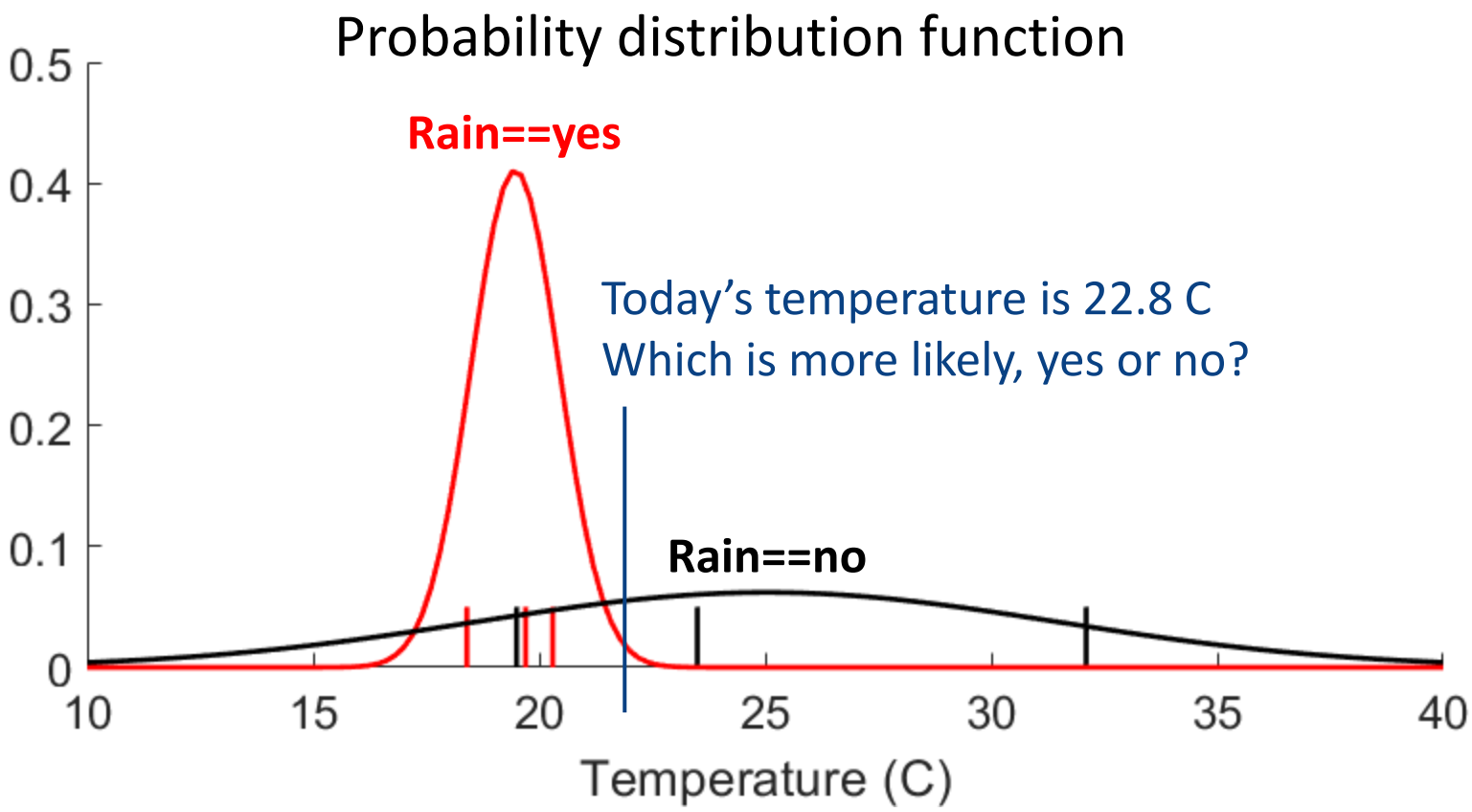
Example: Gaussian naïve Bayes



Example: Gaussian naïve Bayes



Example: Gaussian naïve Bayes



Example: Gaussian naïve Bayes

- Compute the probability of the test instance by substituting the μ and σ for each class:

$$P(temp = 22.8|rain = no) = P(rain == no)\phi_{\sigma_{no}}(22.8 - \mu_{no})$$

$$= (0.5)\frac{1}{6.4\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{22.8-25.0}{6.4}\right)^2} = \mathbf{0.0292}$$

$$P(temp = 22.8|rain = yes) = P(rain == yes)\phi_{\sigma_{yes}}\left(22.8 - \mu_{yes}\right)$$

$$= (0.5)\frac{1}{1.0\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{22.8-19.5}{1.0}\right)^2} = \mathbf{0.0006}$$

Gaussian Naïve Bayes

- In naïve Bayes, we choose the class c_j that maximizes:

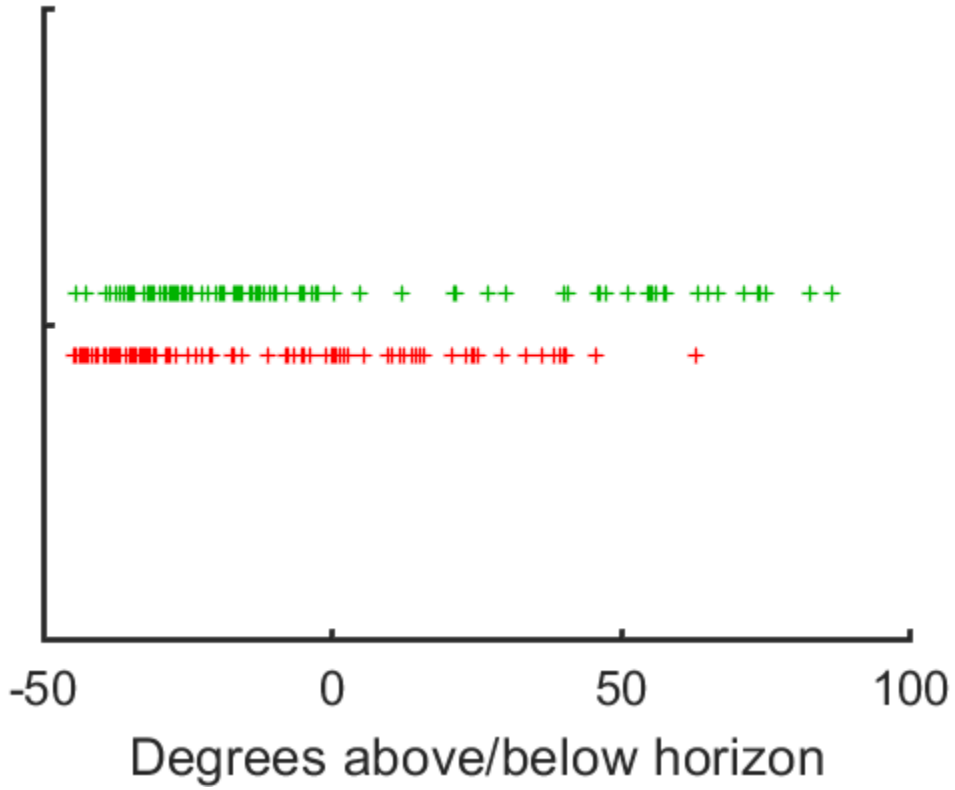
$$P(c_j)\prod_i P(x_i|c_j)$$

Prior probability of class c_j — Likelihood of feature x_i in class c_j

- If x_i is a numeric attribute, $P(x_i|c_j)$ can be represented with a Gaussian distribution
- But not all real-world data follows a Gaussian distribution…

Example: Detect boundaries

- Problem: classify boundary type from elevation



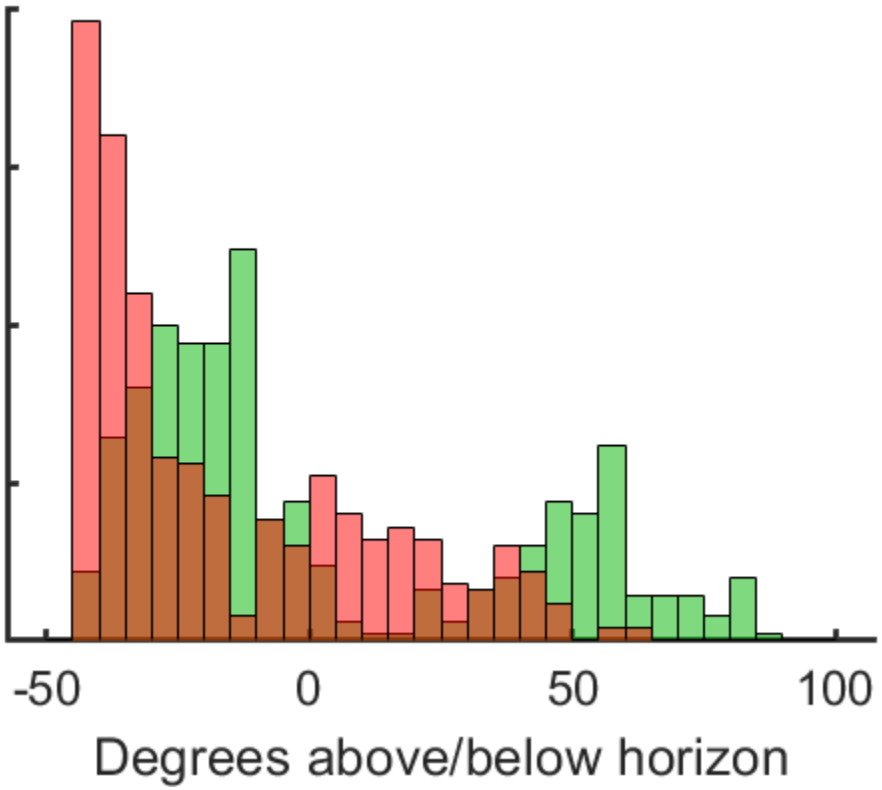
Object boundary

Not object boundary

Image: SYNS dataset (https://syns.soton.ac.uk/)

Example: Detect boundaries

- Are these distributions Gaussian?



Object boundary

Not object boundary

Image: SYNS dataset (https://syns.soton.ac.uk/)

Kernel density estimation (KDE)

- Kernel density estimate of the distribution:

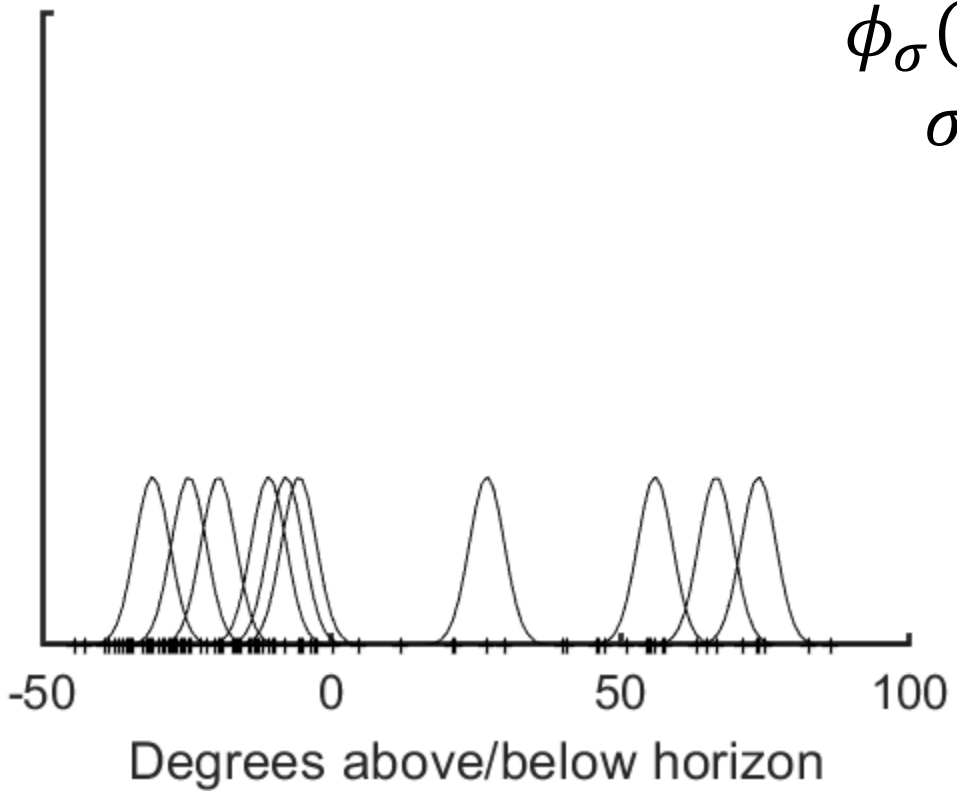
$$\hat{f}(x) = \frac{1}{N}\sum_{i=1}^{N} \phi_\sigma(x - x_i)$$

  where ϕ_σ is the Gaussian distribution with mean of 0 and standard deviation of σ

- Learn the probability distribution from the data instead of assuming a specific distribution (e.g., Gaussian)

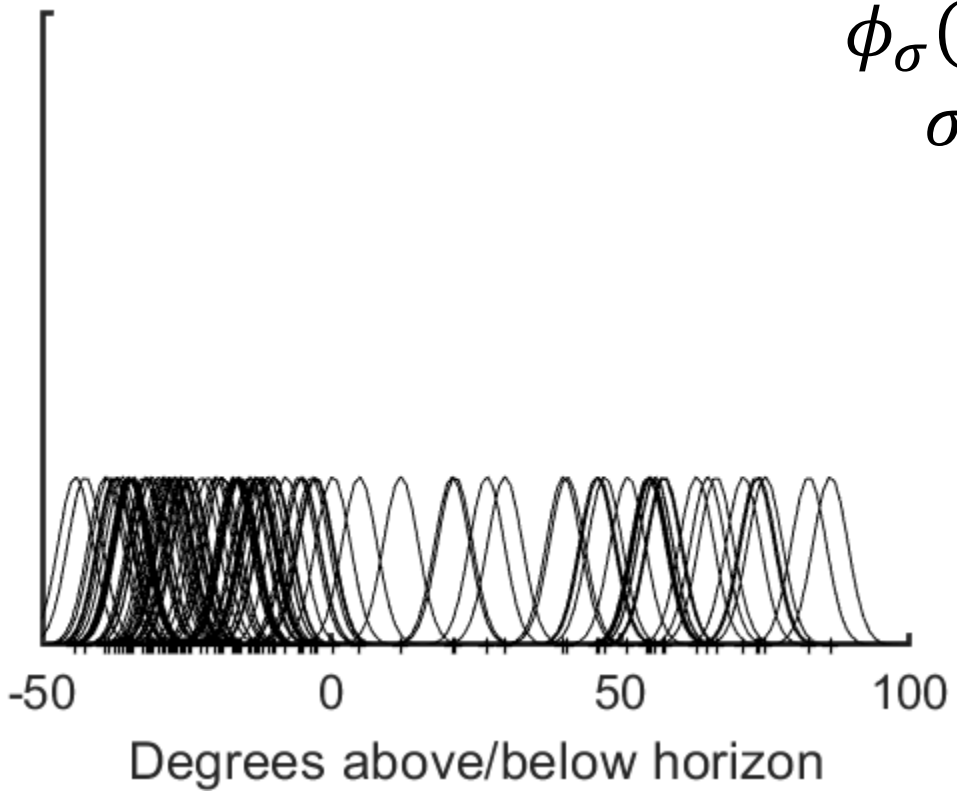
Kernel density estimation (KDE)

$$\phi_\sigma(x - x_i)$$
$$\sigma = 3$$



Kernel density estimation (KDE)

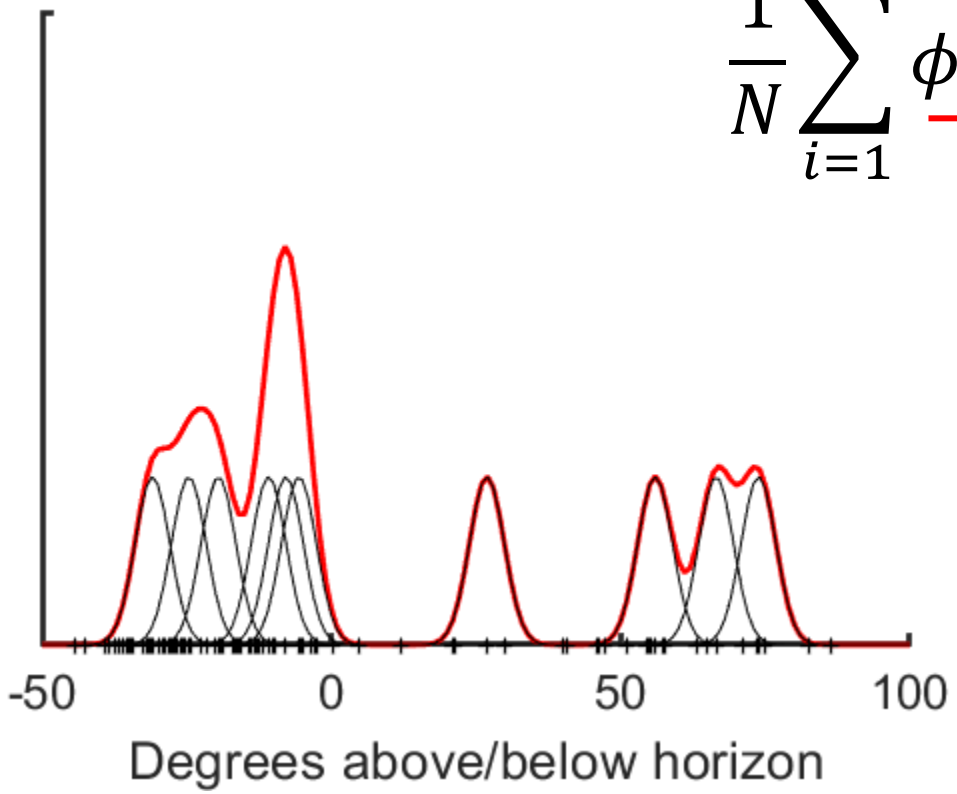
$$\phi_\sigma(x - x_i)$$
$$\sigma = 3$$



Kernel density estimation (KDE)

$$\frac{1}{N}\sum_{i=1}^{N}\phi_\sigma(x - x_i)$$

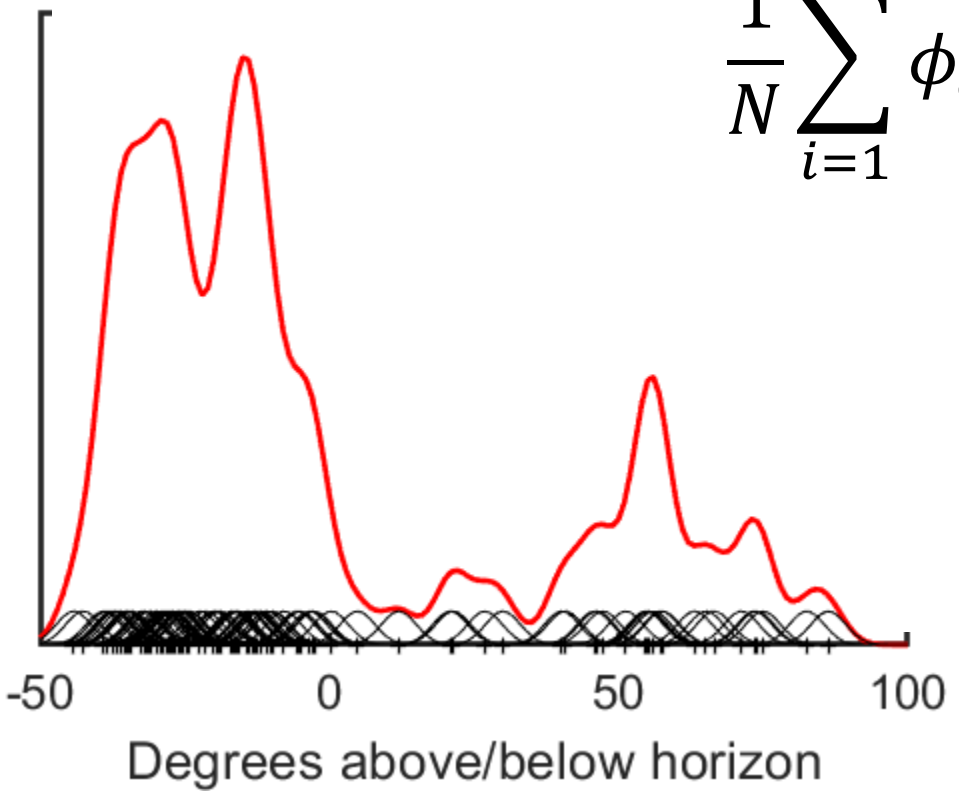
$$\sigma = 3$$



Kernel density estimation (KDE)

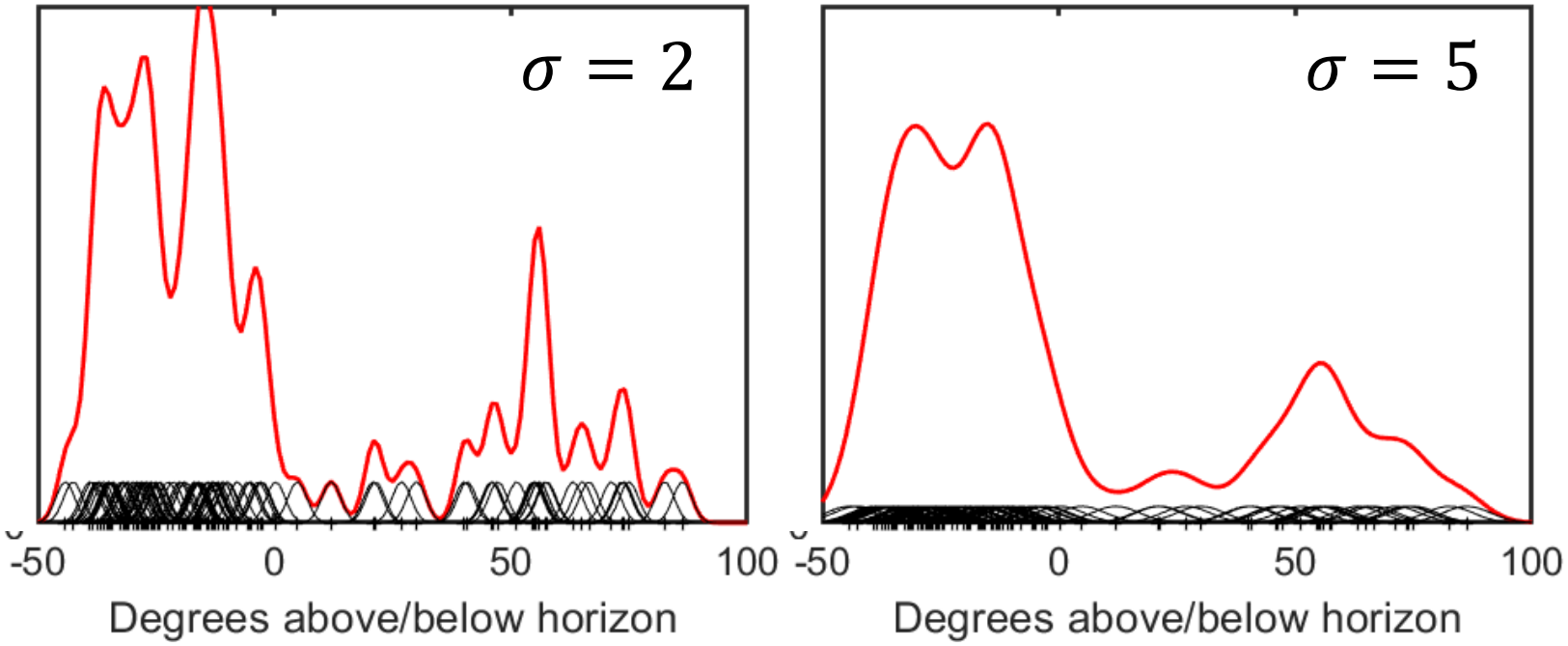
$$\frac{1}{N}\sum_{i=1}^{N}\phi_\sigma(x - x_i)$$

$$\sigma = 3$$

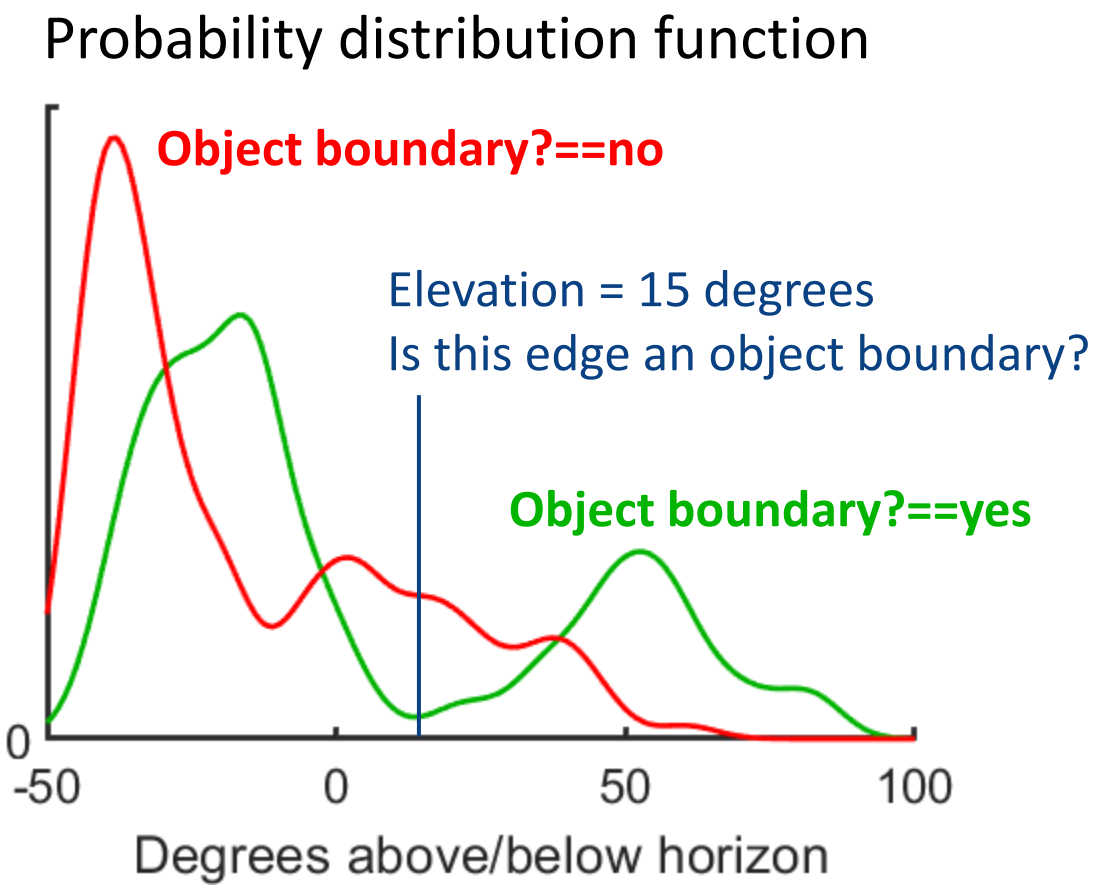


KDE: Kernel bandwidth

- KDE has one parameter: the standard deviation of the Gaussian σ (= “kernel bandwidth”)
- Parameter must be chosen by user



Example: KDE naïve Bayes



KDE Naïve Bayes

- Which class is more likely for test item x_{test}?
- We only have one feature x (elevation), so choose the class c_j that maximizes $P(c_j)P(x_{test}|c_j)$
- Compute likelihood using KDE:

$$P(x_{test}|c_{yes}) = \frac{1}{N}\sum_{n=1}^{N}\phi_{\sigma_{yes}}(x_{test} - x_n)$$

x_1,… x_n = elevations of the N "yes" examples in the training set

$$P(x_{test}|c_{no}) = \frac{1}{M}\sum_{m=1}^{M}\phi_{\sigma_{no}}(x_{test} - x_m)$$

x_1,… x_m = elevations of the M "no" examples in the training set

Example: KDE naïve Bayes

- Compute the probability of the test instance using KDE:

$$P(x = 15|yes) = P(yes)\frac{1}{N}\sum_{n=1}^{N}\phi_{\sigma_{yes}}(x_{test} - x_n)$$

$$= (0.266)\left(\frac{1}{266}\right)\sum_{n=1}^{266}\phi_5(15 - x_n) = \mathbf{0.00029}$$

$$P(x = 15|no) = P(no)\frac{1}{M}\sum_{m=1}^{M}\phi_{\sigma_{no}}(x_{test} - x_m)$$

$$= (0.734)\left(\frac{1}{734}\right)\sum_{n=1}^{734}\phi_5(15 - x_m) = \mathbf{0.00497}$$

KDE summary

- Advantages:
  - Model arbitrary probability distributions
  - No assumptions about the shape of the distribution (e.g., Gaussian)
- Disadvantages:
  - Need to choose a kernel bandwidth
  - Need many parameters to represent the PDF; slow to compute probability at new points

Naïve Bayes summary

- Naïve Bayes works with various data types
  - Different ways to compute $P(x_i | c_j)$ for different types of attributes
- Mix numeric and nominal attributes in one model – no need to convert data types
- Optimal combination of attributes
  - Assuming conditional independence, and correct estimates of the probability distributions

Types of naïve Bayes

- **Multivariate:** attributes are nominal and can take any of a fixed number of values
- Binomial (or Bernoulli): attributes are binary
  - Special case of multivariate
- Multinomial: attributes are natural numbers corresponding to frequency
  - Probability $P(a_k = m|c_j) \approx P(a_k = 1|c_j)^m/(m!)$
- **Gaussian:** attributes are numeric, and we can assume they come from a Gaussian distribution
- **Kernel density estimation:** attributes are numeric, and come from an arbitrary distribution