THE UNIVERSITY OF MELBOURNE

Student ID

Semester 1 Assessment, 2018

School of Mathematics and Statistics

MAST30025 Linear Statistical Models

Writing time: 3 hours

Reading time: 15 minutes

This is NOT an open book exam

This paper consists of 15 pages (including this page)

SOLUTIONS

Authorised materials:

- The only permitted scientific calculator is the Casio FX82.
- Two A4 double-sided handwritten sheets of notes.

Instructions to Students

- You must NOT remove this question paper at the conclusion of the examination.
- You should attempt all questions. Marks for individual questions are shown.
- The total number of marks available is 90.

Instructions to Invigilators

- Students must NOT remove this question paper at the conclusion of the examination.

Question 1 (9 marks)

(a) Let $A_1, \ldots, A_m$ be a set of symmetric idempotent matrices with $A_i A_j = 0$ for $i \neq j$. Show directly that $\sum_{i=1}^m A_i$ is idempotent.

Solution [3 marks]:

$$\begin{aligned}\left(\sum_i A_i\right)^2 &= \sum_i \sum_j A_i A_j \\ &= \sum_i A_i^2 \\ &= \sum_i A_i.\end{aligned}$$

(b) Let $\mathbf{y}$ be a random vector with $E[\mathbf{y}] = \boldsymbol{\mu}$ and var $\mathbf{y} = V$. Show directly that var $\mathbf{c}^T\mathbf{y} = \mathbf{c}^T V \mathbf{c}$.

Solution [3 marks]:

$$\begin{aligned}\text{var } \mathbf{c}^T\mathbf{y} &= E[(\mathbf{c}^T\mathbf{y} - \mathbf{c}^T\boldsymbol{\mu})(\mathbf{c}^T\mathbf{y} - \mathbf{c}^T\boldsymbol{\mu})^T] \\ &= \mathbf{c}^T E[(\mathbf{y} - \boldsymbol{\mu})(\mathbf{y} - \boldsymbol{\mu})^T]\mathbf{c} \\ &= \mathbf{c}^T V \mathbf{c}.\end{aligned}$$

(c) Find a conditional inverse for the matrix

$$A = \begin{bmatrix} 5 & 5 & 0 & 6 \\ 9 & 13 & 2 & 14 \\ -4 & -8 & -2 & -8 \end{bmatrix}.$$

Solution [3 marks]: We see by inspection that $r(A) = 2$. A conditional inverse (there are many others) is

$$A^c = \frac{1}{10}\begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ -9 & 5 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

```
> A <- matrix(c(5,5,0,6,9,13,2,14,-4,-8,-2,-8),3,4,byrow=T)
> library(Matrix)
> rankMatrix(A)[1]

[1] 2

> solve(A[c(1,2),c(1,3)])

     [,1] [,2]
[1,]  0.2  0.0
[2,] -0.9  0.5
```

Question 2 (12 marks)

(a) Let $X \sim \chi^2_{k,\lambda}$. Show that $E[X] = k + 2\lambda$.

Solution [3 marks]: We can write $X = \mathbf{y}^T\mathbf{y}$ for some $\mathbf{y} \sim MVN(\boldsymbol{\mu}, I_k)$ with $\lambda = \frac{1}{2}\boldsymbol{\mu}^T\boldsymbol{\mu}$. Then

$$E[X] = tr(I_k) + \boldsymbol{\mu}^T\boldsymbol{\mu} = k + 2\lambda.$$

(b) Let $\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \sim MVN(\boldsymbol{\mu}, V)$, where

$$\boldsymbol{\mu} = \begin{bmatrix} 4 \\ -1 \end{bmatrix}, \qquad V = \begin{bmatrix} 3 & -1 \\ -1 & 2 \end{bmatrix}.$$

Describe the distribution of $\begin{bmatrix} y_1 + y_2 \\ y_1 - y_2 \end{bmatrix}$.

Solution [3 marks]:

$$\begin{bmatrix} y_1 + y_2 \\ y_1 - y_2 \end{bmatrix} \sim MVN\left(\begin{bmatrix} 3 \\ 5 \end{bmatrix}, \begin{bmatrix} 3 & 1 \\ 1 & 7 \end{bmatrix}\right).$$

```
> A <- matrix(c(1,1,1,-1),2,2)
> mu <- c(4,-1)
> V <- matrix(c(3,-1,-1,2),2,2)
> A %*% mu

     [,1]
[1,]    3
[2,]    5

> A %*% V %*% t(A)

     [,1] [,2]
[1,]    3    1
[2,]    1    7
```

(c) Describe the distribution of $2y_1^2 + 3y_2^2 + 2y_1y_2$.

Solution [4 marks]: $2y_1^2 + 3y_2^2 + 2y_1y_2$ is 5 times a noncentral χ^2 distribution with 2 degrees of freedom and noncentrality parameter 2.7.

```
> B <- matrix(c(2,1,1,3),2,2)
> B %*% V

     [,1] [,2]
[1,]    5    0
[2,]    0    5

> (t(mu) %*% B %*% mu)/2/5
```

```
     [,1]
[1,]  2.7
```

(d) Using your answers to (a) and (c), find $E[2y_1^2 + 3y_2^2 + 2y_1y_2]$.

Solution [2 marks]:

$$E[2y_1^2 + 3y_2^2 + 2y_1y_2] = 5(2 + 2 \times 2.7) = 37.$$

```
> 5*(2+2*2.7)

[1] 37
```

Question 3 (17 marks) In this question, we study a dataset of 50 US states. This dataset contains the variables:

- `Life.Exp`: life expectancy in years (1969–71)
- `Murder`: murder and non-negligent manslaughter rate per 100,000 population (1976)
- `HS.Grad`: percentage of high-school graduates (1970)

We wish to model life expectancy in terms of the other variables. The data are stored in the `statedata` data frame and the following R calculations performed:

```
> X <- cbind(rep(1,50), statedata$Murder, statedata$HS.Grad)
> y <- statedata$Life.Exp
> t(X) %*% X

       [,1]     [,2]      [,3]
[1,]   50.0   368.90   2655.40
[2,]  368.9  3389.49  18878.61
[3,] 2655.4 18878.61 144219.64

> solve(t(X)%*%X)

            [,1]          [,2]          [,3]
[1,]  1.62861066 -0.0377838755 -0.0250403183
[2,] -0.03778388  0.0019656212  0.0004383807
[3,] -0.02504032  0.0004383807  0.0004105962

> t(X)%*%y

          [,1]
[1,]   3543.93
[2,]  25957.51
[3,] 188520.36

> t(y)%*%y

         [,1]
[1,] 251277.1

> qt(0.975,45:50)

[1] 2.014103 2.012896 2.011741 2.010635 2.009575 2.008559

> qf(0.95,1,45:50)

[1] 4.056612 4.051749 4.047100 4.042652 4.038393 4.034310

> qf(0.95,2,45:50)

[1] 3.204317 3.199582 3.195056 3.190727 3.186582 3.182610
```

(a) Calculate the least squares estimates of $\boldsymbol{\beta}$, the parameters of the model.

Solution [2 marks]:

```
> (b <- solve(t(X)%*%X,t(X)%*%y))

              [,1]
[1,] 70.2970840
[2,] -0.2370901
[3,]  0.0438873
```

(b) Calculate the sample variance s^2.

Solution [3 marks]:

```
> (s2 <- (t(y)%*%y - t(y)%*%X%*%b)/(n-p))

          [,1]
[1,] 0.6334118
```

(c) Calculate a 95% confidence interval for the parameter corresponding to `Murder`.

Solution [3 marks]:

```
> b[2] + c(-1,1)*qt(0.975,47)*sqrt(s2*solve(t(X)%*%X)[2,2])

[1] -0.3080748 -0.1661054
```

(d) Calculate a 95% prediction interval for the life expectancy in a state with murder rate 7 per 100,000 population and 50% of high-school graduates.

Solution [4 marks]:

```
> tt <- c(1, 7, 50)
> tt %*% b + c(-1,1)*qt(0.975,47)*sqrt(s2)*sqrt(1 + t(tt) %*% solve(t(X)%*%X) %*% tt)

[1] 69.21062 72.45301
```

(e) Test the relevance of the `HS.Grad` variable at the 5% level.

Solution [5 marks]: Since $7.41 > 4.05$, we reject $H_0 : \beta_{\texttt{HS.Grad}} = 0$. `HS.Grad` is a relevant variable in this model.

```
> b_2 <- solve((t(X)%*%X)[-3,-3], (t(X)%*%y)[-3])
> ssres_2 <- t(y)%*%y - (t(y)%*%X)[-3] %*% b_2
> (Fstat <- (ssres_2 - s2*(n-p))/s2)

         [,1]
[1,] 7.405878
```

Question 4 (14 marks) Consider the full rank linear model $\mathbf{y} = X\boldsymbol{\beta} + \boldsymbol{\varepsilon}$ with p parameters.

(a) Calculate the variance of the residuals $\mathbf{e} = \mathbf{y} - X\mathbf{b}$.

Solution [3 marks]:

$$\begin{aligned}\text{var }\mathbf{e} &= \text{var }(I-H)\mathbf{y} \\ &= (I-H)\sigma^2 I(I-H)^T \\ &= \sigma^2(I-H).\end{aligned}$$

(b) Ridge regression estimates the parameters $\boldsymbol{\beta}$ by minimising $\mathbf{e}^T\mathbf{e}+\lambda\mathbf{b}^T\mathbf{b}$. Derive an expression for the resulting estimators $\mathbf{b}$.

Solution [4 marks]:

$$\begin{aligned}\mathbf{e}^T\mathbf{e}+\lambda\mathbf{b}^T\mathbf{b} &= \mathbf{y}^T\mathbf{y} - 2\mathbf{y}^T X\mathbf{b} + \mathbf{b}^T(X^TX+\lambda I)\mathbf{b} \\ \frac{d}{d\mathbf{b}}\left(\mathbf{e}^T\mathbf{e}+\lambda\mathbf{b}^T\mathbf{b}\right) &= -2X^T\mathbf{y} + 2(X^TX+\lambda I)\mathbf{b} \\ &= 0 \\ \mathbf{b} &= (X^TX+\lambda I)^{-1}X^T\mathbf{y}.\end{aligned}$$

(c) Calculate the expected value of SS_{Reg}, the regression sum of squares.

Solution [3 marks]:

$$\begin{aligned}E[SS_{Reg}] &= E[\mathbf{y}^T H\mathbf{y}] \\ &= tr(H\sigma^2) + \boldsymbol{\beta}^T X^T H X\boldsymbol{\beta} \\ &= p\sigma^2 + \boldsymbol{\beta}^T X^T X\boldsymbol{\beta}.\end{aligned}$$

(d) Consider two nested models $\mathbf{y} = X_2\boldsymbol{\gamma}_2 + \boldsymbol{\varepsilon}_2$ and $\mathbf{y} = X\boldsymbol{\beta} + \boldsymbol{\varepsilon}$, where $\boldsymbol{\beta} = \begin{bmatrix}\boldsymbol{\gamma}_1 \\ \boldsymbol{\gamma}_2\end{bmatrix}$ and $\boldsymbol{\gamma}_i$ contains p_i parameters ($i = 1, 2$). It can be shown that the second model has a larger AIC if and only if

$$\frac{SS_{Res}(\boldsymbol{\gamma}_2)}{SS_{Res}(\boldsymbol{\beta})} \le c,$$

for some constant c. Find c.

Solution [4 marks]:

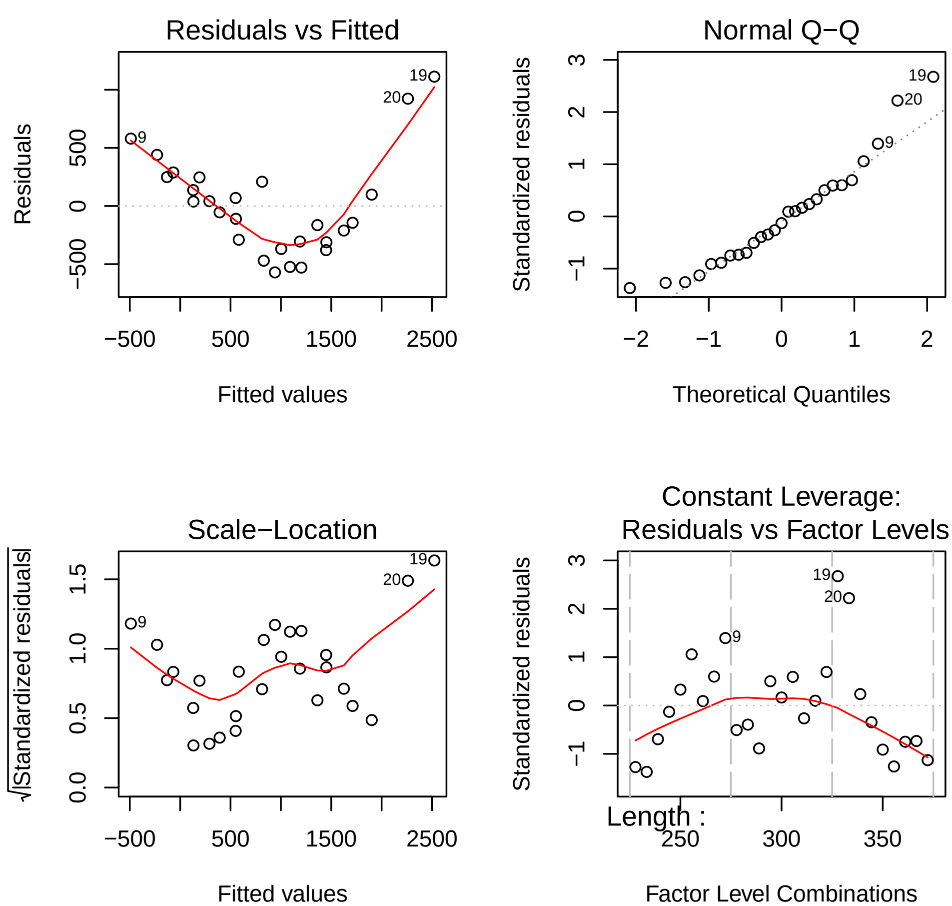
$$\begin{aligned}AIC(\boldsymbol{\gamma}_2) - AIC(\boldsymbol{\beta}) &= n\ln\frac{SS_{Res}(\boldsymbol{\gamma}_2)}{n} + 2p_2 - n\ln\frac{SS_{Res}(\boldsymbol{\beta})}{n} - 2p \\ &= n\ln\frac{SS_{Res}(\boldsymbol{\gamma}_2)}{SS_{Res}(\boldsymbol{\beta})} - 2p_1 \\ &\le 0 \\ \frac{SS_{Res}(\boldsymbol{\gamma}_2)}{SS_{Res}(\boldsymbol{\beta})} &\le e^{2p_1/n} = c.\end{aligned}$$

Question 5 (16 marks) An experiment was conducted to understand the strength of wool as a function of three factors. The variables measured are:

- `Length`: Length of test specimen (200, 300, 350 mm)
- `Amplitude`: Amplitude of loading cycle (8, 9, 10 mm)
- `Load`: Load put on the specimen (40, 45, 50 g)
- `Cycles`: Number of cycles until the specimen fails

One sample was measured for each combination of `Length`, `Amplitude` and `Load` (i.e., 27 samples in total). The data is analysed below.

```
> wool <- read.csv('wool.csv', header=T)
> wool$Length <- factor(wool$Length)
> wool$Amplitude <- factor(wool$Amplitude)
> wool$Load <- factor(wool$Load)
> model1 <- lm(Cycles ~ ., data = wool)
> par(mfrow=c(2,2))
> plot(model1)
```



```
> model2 <- lm(log(Cycles) ~ ., data = wool)
> summary(model2)

Call:
lm(formula = log(Cycles) ~ ., data = wool)

Residuals:
     Min       1Q   Median       3Q      Max
-0.36860 -0.13002  0.00902  0.10129  0.30469

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)  6.48287    0.09644  67.225  < 2e-16 ***
Length300    0.91833    0.08928  10.286 1.97e-09 ***
Length350    1.66477    0.08928  18.646 4.10e-14 ***
Amplitude9  -0.65521    0.08928  -7.339 4.31e-07 ***
Amplitude10 -1.26173    0.08928 -14.132 7.19e-12 ***
Load45      -0.32529    0.08928  -3.643  0.00162 **
Load50      -0.78524    0.08928  -8.795 2.62e-08 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 0.1894 on 20 degrees of freedom
Multiple R-squared:  0.9691,	Adjusted R-squared:  0.9598
F-statistic: 104.5 on 6 and 20 DF,  p-value: 4.979e-14

> anova(model2)

Analysis of Variance Table

Response: log(Cycles)
          Df  Sum Sq Mean Sq F value    Pr(>F)
Length     2 12.5159  6.2579 174.456 2.193e-13 ***
Amplitude  2  7.1674  3.5837  99.905 3.889e-11 ***
Load       2  2.8019  1.4010  39.055 1.239e-07 ***
Residuals 20  0.7174  0.0359
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

> model3 <- lm(log(Cycles) ~ .^2, data = wool)
> anova(model2, model3)

Analysis of Variance Table

Model 1: log(Cycles) ~ Length + Amplitude + Load
Model 2: log(Cycles) ~ (Length + Amplitude + Load)^2
  Res.Df     RSS Df Sum of Sq     F Pr(>F)
1     20 0.71742
2      8 0.16591 12   0.55151 2.216 0.1325
```

```
> model4 <- step(model3)

Start:  AIC=-99.49
log(Cycles) ~ (Length + Amplitude + Load)^2

                   Df Sum of Sq     RSS      AIC
- Amplitude:Load    4   0.01460 0.18051 -105.211
<none>                          0.16591  -99.487
- Length:Load       4   0.13575 0.30167  -91.345
- Length:Amplitude  4   0.40116 0.56707  -74.304

Step:  AIC=-105.21
log(Cycles) ~ Length + Amplitude + Load + Length:Amplitude +
    Length:Load

                   Df Sum of Sq     RSS      AIC
<none>                          0.18051 -105.211
- Length:Load       4   0.13575 0.31626  -98.069
- Length:Amplitude  4   0.40116 0.58167  -81.618

> summary(model4)

Call:
lm(formula = log(Cycles) ~ Length + Amplitude + Load + Length:Amplitude +
    Length:Load, data = wool)

Residuals:
      Min        1Q    Median        3Q       Max
-0.153728 -0.055232 -0.008017  0.067786  0.175706

Coefficients:
                       Estimate Std. Error t value Pr(>|t|)
(Intercept)            6.384806   0.091416  69.843  < 2e-16 ***
Length300              0.913780   0.129282   7.068 1.30e-05 ***
Length350              1.963516   0.129282  15.188 3.37e-09 ***
Amplitude9            -0.449946   0.100142  -4.493 0.000735 ***
Amplitude10           -1.232398   0.100142 -12.307 3.65e-08 ***
Load45                -0.401464   0.100142  -4.009 0.001734 **
Load50                -0.649468   0.100142  -6.485 3.00e-05 ***
Length300:Amplitude9  -0.001114   0.141622  -0.008 0.993851
Length350:Amplitude9  -0.614678   0.141622  -4.340 0.000961 ***
Length300:Amplitude10  0.064964   0.141622   0.459 0.654638
Length350:Amplitude10 -0.152966   0.141622  -1.080 0.301328
Length300:Load45       0.083463   0.141622   0.589 0.566565
Length350:Load45       0.145059   0.141622   1.024 0.325914
Length300:Load50      -0.133655   0.141622  -0.944 0.363913
Length350:Load50      -0.273658   0.141622  -1.932 0.077269 .
---
```

```
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Residual standard error: 0.1226 on 12 degrees of freedom
Multiple R-squared:  0.9922,	Adjusted R-squared:  0.9831
F-statistic: 109.3 on 14 and 12 DF,  p-value: 1.968e-10
```

```
> qt(0.975,20:27)

[1] 2.085963 2.079614 2.073873 2.068658 2.063899 2.059539 2.055529 2.051831
```

(a) Identify the features in the diagnostic plots which support the use of the logarithmic transformation on the `Cycles` variable.

Solution [2 marks]: From the first diagnostic plot, there is a clear non-linear trend where the data are underestimated by the model for both small and large fitted values. From the second plot, there is some evidence of right-skewed errors.

(b) From the additive model, calculate a 95% confidence interval for the average ratio of the number of cycles to failure for 50g loads against 40g loads. (*Hint: The logarithm of the ratio is the difference of the logarithms.*)

Solution [3 marks]:

```
> exp(model2$coef[7] + c(-1,1)*qt(0.975,df=20)*summary(model2)$coef[7,2])

[1] 0.3785228 0.5493613
```

(c) From the additive model, test whether length has an effect on wool strength, at the 5% significance level.

Solution [2 marks]: With a p-value of 2.193×10^{-13}, length clearly has an effect on wool strength.

(d) Calculate the change in AIC if the amplitude variable was removed from the additive model.

Solution [3 marks]: For the additive model, we have $SS_{Res} = 0.7174$ and $r = 7$. If amplitude is removed, we have $SS_{Res} = 0.7174 + 7.1674$ and $r = 5$. Therefore the change in AIC will be

$$27 \ln \frac{0.7174 + 7.1674}{27} + 2 \times 5 - 27 \ln \frac{0.7174}{27} - 2 \times 5 = 60.72.$$

```
> 27*log((0.7174+7.1674)/27) + 2*5 - 27*log(0.7174/27) - 2*7

[1] 60.72058
```

(e) Test for the presence of 2-way interaction between the factors.

Solution [2 marks]: With a p-value of 0.1325, we do not find significant 2-way interaction between the factors.

(f) Is your answer above consistent with the results of the variable selection? Why or why not?

Solution [2 marks]: While stepwise selection does include some interaction terms, they are not selected because they are significant in an F test, but because they lower the AIC. Thus there is no contradiction between the two answers.

(g) Using the model resulting from variable selection, calculate a point estimate for the average number of cycles to failure for a wool specimen of length 350mm, loading cycle amplitude 8mm, with 45g load.

Solution [2 marks]:

```
> exp(sum(model4$coef[c(1,3,6,13)]))

[1] 3267.947
```

Question 6 (14 marks) Consider the general linear model, $\mathbf{y} = X\boldsymbol{\beta} + \boldsymbol{\varepsilon}$. This model may be of full or less than full rank.

(a) Explain the difference between an error and a residual.

Solution [2 marks]: An error is the difference between an observed and expected response, while a residual is the difference between an observed and fitted response.

(b) Define and explain the purpose of the standardised residual of a point.

Solution [2 marks]: The standardised residual of the ith point is defined as

$$z_i = \frac{e_i}{s\sqrt{1 - H_{ii}}},$$

where e_i is the ith residual and H_{ii} is the leverage of the ith point. It is a scaled version of the residual which has been scaled to have variance approximately 1, for easier comparison across residuals.

(c) When is a model with fewer explanatory variables more desirable than a model with more explanatory variables? When is it less desirable?

Solution [2 marks]: A model with fewer explanatory variables is more desirable if and only if the variables left out are not relevant. Reducing the number of variables prevents overfitting, but removing relevant variables hurts the fit.

(d) State the general linear hypothesis and explain how it is tested.

Solution [2 marks]: The general linear hypothesis is the hypothesis $H_0 : C\boldsymbol{\beta} = \boldsymbol{\delta}^*$. It is tested by comparing the statistic

$$\frac{(C\boldsymbol{\beta} - \boldsymbol{\delta}^*)^T[C(X^TX)^{-1}C^T]^{-1}(C\boldsymbol{\beta} - \boldsymbol{\delta}^*)/r(C)}{s^2}$$

against a null F distribution with $r(C)$ and $n - r(X)$ degrees of freedom on the right tail.

(e) Define a treatment contrast and explain its usage.

Solution [2 marks]: A treatment contrast is a linear combination of treatment effects $\sum_i a_i\tau_i$ where $\sum_i a_i = 0$. They are used to estimate treatment effects relative to each other because they are usually estimable (in additive models).

(f) Explain what randomisation is and its use in experimental design.

Solution [2 marks]: Randomisation is allocating treatments randomly to an otherwise homogeneous pool of experimental subjects. It is used to avoid confounding by ensuring that treatment and control groups differ only by the treatment and not by other confounding variables.

(g) Explain what a Latin square is and its use in experimental design.

Solution [2 marks]: A Latin square is an $n \times n$ square which contains the numbers 1 to n such that every number occurs once in each row and column. It is used to design experiments with two confounding factors in a more efficient way than a complete block design.

Question 7 (8 marks) An experiment compares four different mixtures of the components of a rocket propellant; the mixtures contain different proportions of oxidizer, fuel, and binder. To compare the mixtures, five different samples of propellant are prepared for each mixture. Each of five investigators is randomly assigned one sample of each of the four mixtures and is asked to measure the propellant thrust. The data is given below:

Mixture	Investigator					Mixture
	1	2	3	4	5	Total
A	2340	2355	2362	2350	2348	11755
B	2658	2650	2665	2640	2653	13266
C	2449	2458	2432	2437	2445	12221
D	2403	2410	2418	2397	2405	12033
Investigator Total	9850	9873	9877	9824	9851	

(a) What type of experimental design is described above?

Solution [1 marks]: A complete block design (CBD).

(b) Which are the treatment and blocking variables in this experiment?

Solution [1 marks]: Treatment is propellant mixture; blocking is investigator.

(c) Is it better to analyse this data as a complete block design or completely randomised design? Justify your answer.

Solution [2 marks]: It appears that block effects are very small, so it may be better to use a CRD.

(d) A larger experiment is planned with the goal of testing whether mixture D, a newly developed formula, is more effective than industry standard mixtures A and B. This experiment has resources to prepare 100 samples of propellant. Calculate the best number of samples for each mixture. (*Hint: In a completely randomised design with treatment effects τ_i, we have var $\tau_i = \frac{\sigma^2}{n_i}$. To minimise a function $f(\mathbf{x})$ under the constraint $g(\mathbf{x}) = c$, minimise $f(\mathbf{x}, \lambda) = f(\mathbf{x}) + \lambda(g(\mathbf{x}) - c)$.*)

Solution [4 marks]: To test whether mixture D is the most effective, we need to estimate the contrasts $\tau_A - \tau_D$ and $\tau_B - \tau_D$. This requires us to minimise

$$f(n_1, n_2, n_3, n_4, \lambda) = \sigma^2 \left(\frac{1}{n_1} + \frac{1}{n_2} + \frac{2}{n_4} \right) + \lambda \left(\sum_{i=1}^{4} n_i - n \right).$$

This produces the equations

$$
\begin{aligned}
-\frac{\sigma^2}{n_1^2} + \lambda = 0 \\
-\frac{\sigma^2}{n_2^2} + \lambda = 0 \\
-2\frac{\sigma^2}{n_4^2} + \lambda = 0 \\
n_4^2 &= 2n_1^2 = 2n_2^2 \\
n_4 &= \sqrt{2}n_1 = \sqrt{2}n_2 \\
n_1 &= n_2 = \frac{100}{2+\sqrt{2}}
\end{aligned}
$$

This gives $n_1 = n_2 = 29, n_4 = 42$. Obviously $n_3 = 0$.

```
> 100/(2+sqrt(2))

[1] 29.28932

> 100/(1+sqrt(2))

[1] 41.42136
```

End of Exam—Total Available Marks = 90.