

MODERN APPLIED STATISTICS

MAST30027

SHROMANN MAJUMDER

1 General Linear Models

1.1 Binomial Regression

Its main assumptions is that Y_i follows a Binomial Distribution and p_i has a relationship with the design constants and thier respective β parameters through a **Link Function**. take the inverse of g to make g^{-1} and use that for the *bin*'s p_i

$$Y_i \sim \text{bin}(m_i, p_i = g^{-1}(\eta_i = X_i^T \beta))$$

- **Link Function** to link p_i with x_i and β_i

$$g(p_i) = \eta_i = X_i^T \beta = \sum_{j=1}^n \beta_{ij} x_{ij}$$

1. logit:

$$\log \frac{p}{1-p}$$

2. complementary log-log:

$$\log(-\log(1-p))$$

3. probit:

$$\Phi^{-1}(p)$$

- **Log-Likelihood** to estimate β 's values

$$\begin{aligned} l(\beta) &= \sum_{i=1}^n \log \Pr(Y_i = y_i) \\ &= c + \sum_{i=1}^n y_i \log(g^{-1}(\eta_i)) + (m_i - y_i) \log(1 - g^{-1}(\eta_i)) \end{aligned}$$

This has no closed form solution. So numerical search is needed. R uses **optim**, which is a greedy search algorithm. So multiple initial values are tested to avoid getting stuck in local optimums. Theres also **glm**, **predict**.

- **Asymptotic properties MLE** to find CI's of β estimates.

$$\hat{\theta}_{MLE} = \arg \min_{\theta} [l(\theta; y_{\text{observed}}) = f_i(\cdot; \theta) = f(\cdot; x_i, \theta)]$$

MLE's asymptotic properties:

1. Asymptotically Consistent:

$$n \rightarrow \infty, \hat{\theta} \rightarrow \theta^*$$

2. Asymptotically Normal:

$$\hat{\theta} = N(\theta^*, \mathcal{I}(\theta^*)^{-1})$$

Observed Information: depends on the *observed y* I doubt many understand this well, but in summary: the hessian matrix is filled with second-order partial derivatives to describe the curvature of log-likelihood w.r.t θ .

$$\mathcal{J}(\theta) = -H_{l\theta} = -\frac{\partial^2 l(\theta)}{\partial \theta \partial \theta^T}$$

Fischer's Information: depends on the *r.v Y*

$$\mathcal{I} = E[\mathcal{J}(\theta; Y)]$$

Binomial Regression with 2 parameters, has the \mathcal{I} of the form

$$\mathcal{I}(\beta) = \begin{bmatrix} \sum_{i=1}^n m_i p_i (1-p_i) & \sum_{i=1}^n m_i x_i p_i (1-p_i) \\ \sum_{i=1}^n m_i x_i p_i (1-p_i) & \sum_{i=1}^n m_i x_i^2 p_i (1-p_i) \end{bmatrix}$$

3. Asymptotically Efficient: If the above two conditions are met, then $\hat{\theta}_{MLE}$ is asymptotically unbiased estimator with smallest variance $\mathcal{I}(\theta^*)^{-1}$

- **Wald CI for $t^T\theta$** , when calculating CI through Asymptotic Normality We know the following,

$$\hat{\theta} \approx N(\theta^*, \mathcal{I}(\theta^*)^{-1})$$

However, θ^* (the true value of θ) is unknown. So we approximate $\mathcal{I}(\theta^*)^{-1}$ using $\mathcal{I}(\hat{\theta})^{-1}$. Resulting in following statements and the $100(1 - \alpha)\%$ confidence interval.

$$\hat{\theta} \approx N(\theta^*, \mathcal{I}(\hat{\theta})^{-1}) \implies \mathbf{t}^T \hat{\theta} \approx N(\mathbf{t}^T \theta^*, \mathbf{t}^T \mathcal{I}(\hat{\theta})^{-1} \mathbf{t})$$

$$\mathbf{t}^T \hat{\theta} \pm z_\alpha \sqrt{\mathbf{t}^T \mathcal{I}(\hat{\theta})^{-1} \mathbf{t}}, \quad z_\alpha = \Phi^{-1}(1 - \alpha/2)$$

If \mathbf{t} is a standard basis vector for its dimension. Then we can obtain the CI for each individual parameter: θ_1, \dots . In that case, the approximate CI would be

$$\hat{\theta}_i \pm z_\alpha \sqrt{[\mathcal{I}(\hat{\theta})^{-1}]_{i,i}}$$

But if we don't have \mathcal{I} , then just use \mathcal{J} as.

$$\mathcal{I}(\hat{\theta})^{-1} \approx \mathcal{J}(\hat{\theta}; y)^{-1}$$

The steps behind calculating CI's is rather recursive. The way to go about it is to follow the steps below

1. Calculate CI for $\eta = \mathbf{t}^T \hat{\theta}$: (η_l, η_u) , \mathbf{t}^T can be a possible covariate matrix and thus used to calculate the CI, or better known as the confidence region for $\mathbf{t}^T \hat{\theta}$.
2. Calculate CI for $p = g^{-1}(\eta)$: $(g^{-1}(\eta_l), g^{-1}(\eta_u))$, this p is from the $Y_i \sim \text{bin}(m_i, p_i)$

- **log likelihood ratio CI**, is better than *Wald CI* as

1. Wald CI does $2 \times$ CIs to get the CI of p , where 'log likelihood ratio CI' does 1.
2. likelihood ratio CI holds for smaller sample size.

We begin with the following

$$2l(\hat{\theta}) - 2l(\theta^*) \sim \chi_k^2$$

• k is number of columns of θ^* , thus the log likelihood ratio CI is defined as

$$\{\theta : 2l(\hat{\theta}) - 2l(\theta^*) \leq \chi_k^2(1 - \alpha)\}$$

• $\chi_k^2(1 - \alpha)$ is the $100(1 - \alpha)\%$ point for χ_k^2 distribution

- **MLE: regularity conditions**, what we need for MLE to actually work

1. log-likelihood function (l) is smooth, i.e third-order derivatives w.r.t to θ exists and continuous.
2. third-order derivatives of l w.r.t to θ have bounded expectations.
3. support of Y_i does not depend on θ .
4. the domain of θ is finite dimensional and does not depend on Y_i .
5. θ^* is not on the boundary of its domain.