Linear statistical models
Random vectors

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Random vectors

We must still do some more groundwork in order to analyse our linear models. Once we have done this, the theoretical results come out quite easily!

Normally we think of matrices and vectors as a bunch of numbers. However, they can also be a bunch of random variables.

We can then extend the traditional concepts of expectation, variance, etc. to random vectors.

Expectation

Traditionally random variables are denoted with capital letters. However we will denote them by lowercase according to linear algebra notation.

We define the expectation of a random vector $\mathbf{y}$ as follows:

$$\text{If} \quad \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_k \end{bmatrix}, \quad \text{then} \quad E[\mathbf{y}] = \begin{bmatrix} E[y_1] \\ E[y_2] \\ \vdots \\ E[y_k] \end{bmatrix}.$$

Expectation properties

- If $\mathbf{a}$ is a vector of constants, then $E[\mathbf{a}] = \mathbf{a}$.
- If $\mathbf{a}$ is a vector of constants, then $E[\mathbf{a}^T\mathbf{y}] = \mathbf{a}^T E[\mathbf{y}]$.
- If A is a matrix of constants, then $E[A\mathbf{y}] = AE[\mathbf{y}]$.

Expectation properties

Example. Let

$$A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}, \quad \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

and assume that $E[y_1] = 10$ and $E[y_2] = 20$.

Then

$$AE[\mathbf{y}] = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} 10 \\ 20 \end{bmatrix} = \begin{bmatrix} 80 \\ 90 \end{bmatrix}.$$

Expectation properties

On the other hand,

$$
\begin{aligned}
E[A\mathbf{y}] &= E\left[\begin{array}{r} 2y_1 + 3y_2 \\ y_1 + 4y_2 \end{array}\right] \\
&= \left[\begin{array}{r} E[2y_1 + 3y_2] \\ E[y_1 + 4y_2] \end{array}\right] \\
&= \left[\begin{array}{r} 2E[y_1] + 3E[y_2] \\ E[y_1] + 4E[y_2] \end{array}\right] \\
&= \left[\begin{array}{c} 80 \\ 90 \end{array}\right] = AE[\mathbf{y}].
\end{aligned}
$$

Variance

Defining the variance of a random vector is slightly trickier. We want to not just include the variance of the variables themselves, but also how the variables affect each other.

Recall that the variance of a random variable Y with mean μ is defined to be $E[(Y - \mu)^2]$.

We define the variance (or *covariance matrix*) of the random vector $\mathbf{y}$ to be

$$\text{var } \mathbf{y} = E[(\mathbf{y} - \boldsymbol{\mu})(\mathbf{y} - \boldsymbol{\mu})^T]$$

where $\boldsymbol{\mu} = E[\mathbf{y}]$.

Variance

The diagonal elements of the covariance matrix are the variances of the elements of $\mathbf{y}$:

$$[\mathsf{var}\ \mathbf{y}]_{ii} = \mathsf{var}\ y_i, \qquad i = 1, 2, \ldots, k.$$

The off-diagonal elements are the covariances of the elements:

$$[\mathsf{var}\ \mathbf{y}]_{ij} = \mathsf{cov}(y_i, y_j) = E[(y_i - \mu_i)(y_j - \mu_j)].$$

This means that all covariance matrices are symmetric.

Variance properties

Suppose that $\mathbf{y}$ is a random vector with var $\mathbf{y} = V$. Then:

- If $\mathbf{a}$ is a vector of constants, then var $\mathbf{a}^T\mathbf{y} = \mathbf{a}^T V \mathbf{a}$.
- If A is a matrix of constants, then var $A\mathbf{y} = AVA^T$.
- V is positive semidefinite.

Variance properties

Example. Let

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

be a random vector with var $y_i = \sigma^2$ for all i, and the elements of $\mathbf{y}$ are independent.

Then the covariance matrix of $\mathbf{y}$ is

$$\text{var } \mathbf{y} = \begin{bmatrix} \sigma^2 & 0 & 0 \\ 0 & \sigma^2 & 0 \\ 0 & 0 & \sigma^2 \end{bmatrix} = \sigma^2 I.$$

Variance properties

Example. Assume that X is a matrix of full rank (with more rows than columns), which implies that X^TX is nonsingular. Let

$$\mathbf{z} = (X^TX)^{-1}X^T\mathbf{y} = A\mathbf{y}.$$

Then

$$\begin{aligned}\text{var}\ \mathbf{z} &= AVA^T = [(X^TX)^{-1}X^T]\sigma^2 I[(X^TX)^{-1}X^T]^T \\ &= (X^TX)^{-1}X^T(X^T)^T[(X^TX)^{-1}]^T\sigma^2 \\ &= (X^TX)^{-1}X^TX[(X^TX)^T]^{-1}\sigma^2 \\ &= (X^TX)^{-1}\sigma^2.\end{aligned}$$

We will be using this quite a bit later on!

Matrix square root

The square root of a matrix A is a matrix B such that $B^2 = A$. In general the square root is not unique.

If A is symmetric and positive semidefinite, there is a unique symmetric positive semidefinite square root, called the principal root, denoted $A^{1/2}$.

Suppose that P diagonalises A, that is $P^T AP = \Lambda$. Then

$$\begin{aligned} A &= P\Lambda P^T \\ &= P\Lambda^{1/2}\Lambda^{1/2}P^T \\ &= P\Lambda^{1/2}P^T P\Lambda^{1/2}P^T \\ A^{1/2} &= P\Lambda^{1/2}P^T. \end{aligned}$$

Multivariate normal distribution

Definition 3.1

Let $\mathbf{z}$ be a $k \times 1$ vector of independent standard normal random variables, A an $n \times k$ matrix, and $\mathbf{b}$ an $n \times 1$ vector. We say that

$$\mathbf{x} = A\mathbf{z} + \mathbf{b}$$

has (an n-dimensional) multivariate normal distribution, with mean $\boldsymbol{\mu} = \mathbf{b}$ and covariance matrix $\Sigma = AA^T$.

We write $\mathbf{x} \sim MVN(\boldsymbol{\mu}, \Sigma)$ or just $\mathbf{x} \sim N(\boldsymbol{\mu}, \Sigma)$.

Multivariate normal distribution

For any $\boldsymbol{\mu}$ and any symmetric positive semidefinite matrix Σ, let $\mathbf{z}$ be a vector of independent standard normals. Then

$$\boldsymbol{\mu} + \Sigma^{1/2}\mathbf{z} \sim MVN(\boldsymbol{\mu}, \Sigma).$$

If $\mathbf{x} \sim MVN(\boldsymbol{\mu}, \Sigma)$ and Σ is $k \times k$ positive definite, then $\mathbf{x}$ has the density

$$f(\mathbf{x}) = \frac{1}{(2\pi)^{k/2}|\Sigma|^{1/2}} e^{-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^T \Sigma^{-1}(\mathbf{x}-\boldsymbol{\mu})}.$$

Compare this with the univariate normal density

$$f(x) = \frac{1}{(2\pi)^{1/2}\sigma} e^{-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}}.$$

Multivariate normal distribution

Any linear combination of multivariate normals results in another multivariate normal: if $\mathbf{x} \sim MVN(\boldsymbol{\mu}, \Sigma)$ is $k \times 1$, A is $n \times k$, and $\mathbf{b}$ is $n \times 1$, then

$$\mathbf{y} = A\mathbf{x} + \mathbf{b} \sim MVN(A\boldsymbol{\mu} + \mathbf{b}, A\Sigma A^T).$$

To see why, put $\mathbf{x} = \Sigma^{1/2}\mathbf{z} + \boldsymbol{\mu}$, then

$$\mathbf{y} = A\Sigma^{1/2}\mathbf{z} + A\boldsymbol{\mu} + \mathbf{b}.$$

Multivariate normal distribution

If $\mathbf{z} = (z_1, z_2)^T$ is multivariate normal, then z_1 and z_2 are independent if and only if they are uncorrelated.

In general, if z_1 and z_2 are normal random variables, $\mathbf{z} = (z_1, z_2)^T$ does not have to be multivariate normal. Moreover, z_1 and z_2 can be uncorrelated but not independent.

For example, suppose that $z_1 \sim N(0, 1)$ and $u \sim U(-1, 1)$, then $z_2 = z_1 \text{sign}(u) \sim N(0, 1)$, but $\mathbf{z} = (z_1, z_2)^T$ is *not* multivariate normal. (Consider its support.) Moreover z_1 and z_2 are uncorrelated, but clearly dependent.

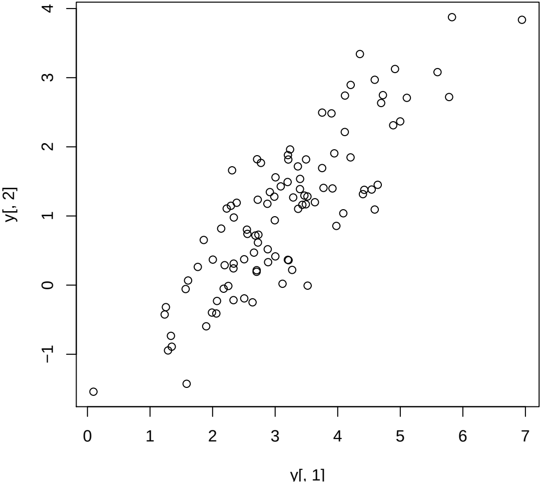
Multivariate normals in R

To generate a sample of size 100 with distribution

$$MVN\left(\left[\begin{array}{c} 3 \\ 1 \end{array}\right], \left[\begin{array}{rr} 1 & 0.8 \\ 0.8 & 1 \end{array}\right]\right)$$

```
> library(MASS)
> a <- matrix(c(3, 1), 2, 1)
> V <- matrix(c(1, .8, .8, 1), 2, 2)
> y <- mvrnorm(100, mu = a, Sigma = V)
> plot(y[,1], y[,2])
```

Multivariate normals in R

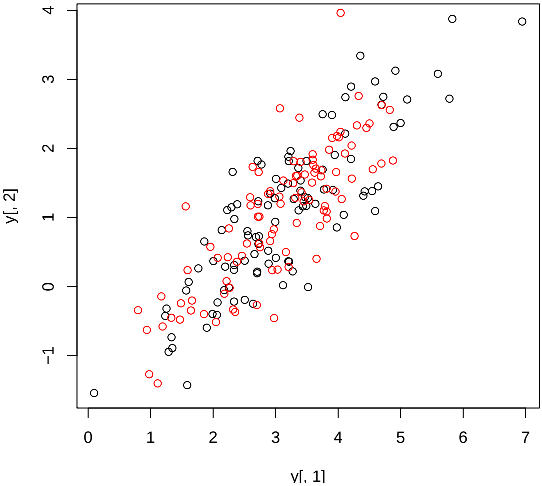


Multivariate normals in R

Alternatively, starting with standard normals:

```
> P <- eigen(V)$vectors
> sqrtV <- P %*% diag(sqrt(eigen(V)$values)) %*% t(P)
> z <- matrix(rnorm(200), 2, 100)
> y_new <- sqrtV %*% z + rep(a, 100)
> points(y_new[1,], y_new[2,], col = "red")
```

Multivariate normals in R



Random quadratic forms

We have seen that a matrix induces a quadratic form (multivariate function).

What happens when the variables in a quadratic form are random variables?

The form becomes a scalar function of random variables, so it is itself a random variable.

Random quadratic forms will pop up regularly in our theory of linear models. To analyse our models, we want to know their distributions.

Random quadratic forms

Theorem 3.2

Let $\mathbf{y}$ be a random vector with $E[\mathbf{y}] = \boldsymbol{\mu}$ and var $\mathbf{y} = V$, and let A be a matrix of constants. Then

$$E[\mathbf{y}^T A\mathbf{y}] = tr(AV) + \boldsymbol{\mu}^T A\boldsymbol{\mu}.$$

Proof. We denote the (i, j)th element of the matrix V by σ_{ij}. For $i \neq j$,

$$\sigma_{ij} = \text{cov}(y_i, y_j) = E[y_i y_j] - \mu_i \mu_j.$$

Also

$$\sigma_{ii} = \text{var } y_i = E[y_i^2] - \mu_i^2.$$

Random quadratic forms

$$
\begin{aligned}
E[\mathbf{y}^T A\mathbf{y}] &= E\left[\sum_{i=1}^{k}\sum_{j=1}^{k} a_{ij} y_i y_j\right] \\
&= \sum_{i=1}^{k}\sum_{j=1}^{k} a_{ij} E[y_i y_j] \\
&= \sum_{i=1}^{k}\sum_{j=1}^{k} a_{ij}(\sigma_{ij} + \mu_i\mu_j) \\
&= \sum_{i=1}^{k}\sum_{j=1}^{k} a_{ij}\sigma_{ji} + \sum_{i=1}^{k}\sum_{j=1}^{k} \mu_i a_{ij}\mu_j \\
&= \sum_{i=1}^{k}[AV]_{ii} + \boldsymbol{\mu}^T A\boldsymbol{\mu} \\
&= tr(AV) + \boldsymbol{\mu}^T A\boldsymbol{\mu}.
\end{aligned}
$$

Random quadratic forms

Example. Let $\mathbf{y}$ be a 2×1 random vector with

$$\boldsymbol{\mu} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \quad V = \begin{bmatrix} 2 & 1 \\ 1 & 5 \end{bmatrix}.$$

Let

$$A = \begin{bmatrix} 4 & 1 \\ 1 & 2 \end{bmatrix}.$$

Consider the quadratic form

$$\mathbf{y}^T A \mathbf{y} = 4y_1^2 + 2y_1 y_2 + 2y_2^2.$$

Random quadratic forms

The expectation of this form is

$$E[\mathbf{y}^T A\mathbf{y}] = 4E[y_1^2] + 2E[y_1 y_2] + 2E[y_2^2].$$

From the given covariance matrix,

$$\begin{aligned} 2 &= \mathsf{var}\ y_1 = E[y_1^2] - E[y_1]^2 = E[y_1^2] - 1 \\ 5 &= \mathsf{var}\ y_2 = E[y_2^2] - E[y_2]^2 = E[y_2^2] - 9 \end{aligned}$$

so $E[y_1^2] = 3$ and $E[y_2^2] = 14$. Also

$$1 = \mathsf{cov}(y_1, y_2) = E[y_1 y_2] - E[y_1]E[y_2] = E[y_1 y_2] - 3$$

so $E[y_1 y_2] = 4$.

Random quadratic forms

This gives

$$E[\mathbf{y}^T A\mathbf{y}] = 4 \times 3 + 2 \times 4 + 2 \times 14 = 48.$$

But from Theorem 3.2,

$$\begin{aligned} E[\mathbf{y}^T A\mathbf{y}] &= tr(AV) + \boldsymbol{\mu}^T A\boldsymbol{\mu} \\ &= tr\left(\begin{bmatrix} 4 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & 5 \end{bmatrix}\right) + \begin{bmatrix} 1 & 3 \end{bmatrix}\begin{bmatrix} 4 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \end{bmatrix} \\ &= tr\left(\begin{bmatrix} 9 & 9 \\ 4 & 11 \end{bmatrix}\right) + \begin{bmatrix} 1 & 3 \end{bmatrix}\begin{bmatrix} 7 \\ 7 \end{bmatrix} \\ &= 9 + 11 + 7 + 21 = 48. \end{aligned}$$

Random quadratic forms

```
> mu <- c(1,3)
> V <- matrix(c(2,1,1,5),2,2)
> y <- t(mvrnorm(100, mu = mu, Sigma = V))
> A <- matrix(c(4,1,1,2),2,2)
> sum(diag(A%*%V)) + t(mu)%*%A%*%mu

     [,1]
[1,]   48

> quadform <- function(y, A) t(y) %*% A %*% y
> mean(apply(y, 2, quadform, A = A))

[1] 46.26975
```

Noncentral χ^2 distribution

Definition 3.3

Let $\mathbf{y} \sim MVN(\boldsymbol{\mu}, I)$ be a $k \times 1$ random vector. Then

$$x = \mathbf{y}^T\mathbf{y} = \sum_{i=1}^{k} y_i^2$$

has a *noncentral* χ^2 *distribution* with k degrees of freedom and noncentrality parameter $\lambda = \frac{1}{2}\boldsymbol{\mu}^T\boldsymbol{\mu}$. We write $x \sim \chi^2_{k,\lambda}$.

Warning: some authors (including R!) define λ to be $\boldsymbol{\mu}^T\boldsymbol{\mu}$.

Note that the distribution of x depends on $\boldsymbol{\mu}$ only through λ.

Noncentral χ^2 distribution

Example. Let

$$\mathbf{y} \sim MVN\left(\begin{bmatrix} 4 \\ 2 \\ -2 \end{bmatrix}, I_3 \right).$$

Then $y_1^2 + y_2^2 + y_3^2$ has a noncentral χ^2 distribution with 3 degrees of freedom and noncentrality parameter

$$\lambda = \frac{1}{2}\boldsymbol{\mu}^T\boldsymbol{\mu} = \frac{1}{2}\begin{bmatrix} 4 & 2 & -2 \end{bmatrix}\begin{bmatrix} 4 \\ 2 \\ -2 \end{bmatrix} = 12.$$

Noncentral χ^2 distribution

Suppose $\mathbf{y} \sim MVN(\boldsymbol{\mu}, I_k)$ and $x = \mathbf{y}^T\mathbf{y} \sim \chi^2_{k,\lambda}$. Then from Theorem 3.2,

$$E[x] = tr(I_k) + \boldsymbol{\mu}^T\boldsymbol{\mu} = k + 2\lambda.$$

The noncentrality parameter $\lambda = \frac{1}{2}\boldsymbol{\mu}^T\boldsymbol{\mu}$ is zero if and only if $\boldsymbol{\mu} = \mathbf{0}$, in which case x is just the sum of k independent standard normals.

In this case, we already know that x has an ordinary (central) χ^2 distribution with k degrees of freedom.

Noncentral χ^2 distribution

The density of the noncentral χ^2 distribution is complicated; one expression is

$$f(x; k, \lambda) = \sum_{i=0}^{\infty} \frac{e^{-\lambda}\lambda^i}{i!} g(x; k + 2i)$$

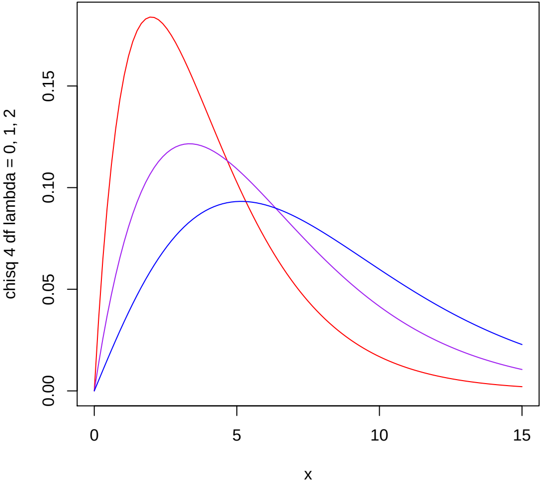
where g is the density of the χ^2 distribution

$$g(x; k) = \frac{1}{2^{k/2}\Gamma\left(\frac{k}{2}\right)} x^{k/2-1} e^{-x/2}.$$

It can also be shown that if $x \sim \chi^2_{k,\lambda}$, then

$$\text{var } x = 2k + 8\lambda.$$

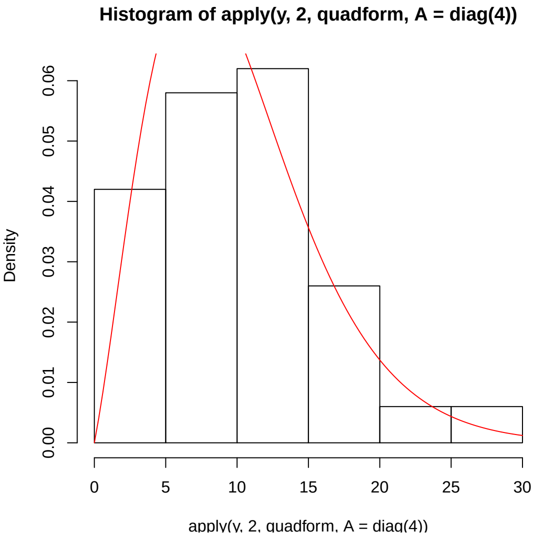
Noncentral χ^2 distribution



Generating a noncentral χ^2

```
> y <- t(mvrnorm(100, mu = c(1,2,1,0), Sigma = diag(4)))
> hist(apply(y,2,quadform,A=diag(4)), freq=FALSE)
> curve(dchisq(x,4,6), add=TRUE, col="red")
```

Generating a noncentral χ^2



Adding noncentral χ^2s

Theorem 3.4

Let $X^2_{k_1,\lambda_1}, \ldots, X^2_{k_n,\lambda_n}$ be a collection of n independent noncentral χ^2 random variables, with $X^2_{k_i,\lambda_i} \sim \chi^2_{k_i,\lambda_i}$. Then

$$\sum_{i=1}^{n} X^2_{k_i,\lambda_i}$$

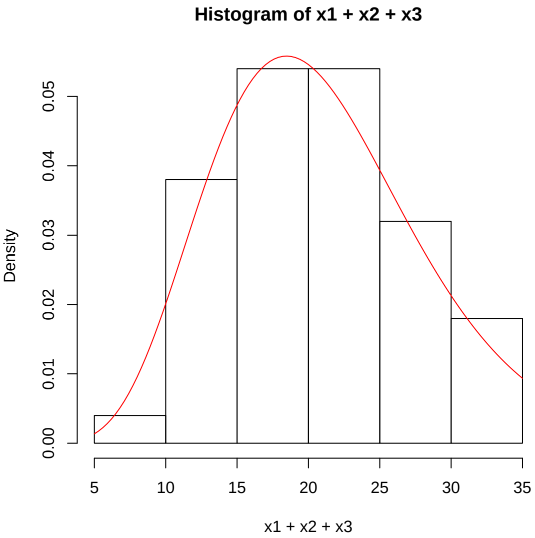
has a noncentral χ^2 distribution with $\sum_{i=1}^{n} k_i$ degrees of freedom and noncentrality parameter $\sum_{i=1}^{n} \lambda_i$.

If we set $\lambda_i = 0$ for all i, we get the result that the sum of independent χ^2 variables is another χ^2 variable.

Adding noncentral χ^2s

```
> x1 <- rchisq(100,3,0)
> x2 <- rchisq(100,6,2)
> x3 <- rchisq(100,5,5)
> hist(x1+x2+x3,freq=FALSE)
> curve(dchisq(x,14,7),add=TRUE,col='red')
```

Adding noncentral χ^2s



Distribution of quadratic forms

Theorem 3.5

Let $\mathbf{y} \sim MVN(\boldsymbol{\mu}, I)$ *be a* $n \times 1$ *random vector and let* A *be a* $n \times n$ *symmetric matrix. Then* $\mathbf{y}^T A \mathbf{y}$ *has a noncentral* χ^2 *distribution with* k *degrees of freedom and noncentrality parameter* $\lambda = \frac{1}{2}\boldsymbol{\mu}^T A \boldsymbol{\mu}$ *if and only if* A *is idempotent and has rank* k.

Distribution of quadratic forms

Proof. ($\Leftarrow$) Assume that A is idempotent and has rank k. Because it is symmetric, it can be diagonalised. Let the (orthogonal) diagonalising matrix be P.

Since A is symmetric and idempotent, all its eigenvalues are either 0 or 1. Moreover, we know that the sum of the eigenvalues is

$$tr(A) = r(A) = k.$$

Therefore, A must have k eigenvalues of 1 and $n - k$ eigenvalues of 0.

Distribution of quadratic forms

Now arrange the columns of P so that all the 1 eigenvalues are first. Then we can partition the diagonalisation of A as

$$P^T A P = \left[\begin{array}{c|c} I_k & 0 \\ \hline 0 & 0 \end{array}\right].$$

Now define the random vector $\mathbf{z} = P^T \mathbf{y} \sim MVN(P^T \boldsymbol{\mu}, I_n)$. Partition the vectors/matrices as

$$\mathbf{z} = \left[\begin{array}{c} \mathbf{z}_1 \\ \hline \mathbf{z}_2 \end{array}\right], \quad P = \left[\begin{array}{c|c} P_1 & P_2 \end{array}\right]$$

where $\mathbf{z}_1$ is $k \times 1$ and P_1 is $n \times k$.

Distribution of quadratic forms

Then $\mathbf{z}_1 = P_1^T \mathbf{y} \sim MVN(P_1^T \boldsymbol{\mu}, I_k)$ and

$$
\begin{aligned}
\mathbf{y}^T A \mathbf{y} &= (P\mathbf{z})^T A (P\mathbf{z}) = \mathbf{z}^T P^T A P \mathbf{z} \\
&= \left[\begin{array}{c|c} \mathbf{z}_1^T & \mathbf{z}_2^T \end{array}\right] \left[\begin{array}{c|c} I_k & 0 \\ \hline 0 & 0 \end{array}\right] \left[\begin{array}{c} \mathbf{z}_1 \\ \hline \mathbf{z}_2 \end{array}\right] \\
&= \left[\begin{array}{c|c} \mathbf{z}_1^T & \mathbf{z}_2^T \end{array}\right] \left[\begin{array}{c} \mathbf{z}_1 \\ \hline 0 \end{array}\right] \\
&= \mathbf{z}_1^T \mathbf{z}_1.
\end{aligned}
$$

Distribution of quadratic forms

Therefore, $\mathbf{y}^T A\mathbf{y} = \mathbf{z}_1^T\mathbf{z}_1$ has a noncentral χ^2 distribution with k degrees of freedom and noncentrality parameter

$$\lambda = \frac{1}{2}\boldsymbol{\mu}^T P_1 P_1^T \boldsymbol{\mu}.$$

Since

$$\begin{aligned} A &= \left[\begin{array}{c|c} P_1 & P_2 \end{array}\right] \left[\begin{array}{c|c} I & 0 \\ \hline 0 & 0 \end{array}\right] \left[\begin{array}{c} P_1^T \\ \hline P_2^T \end{array}\right] \\ &= \left[\begin{array}{c|c} P_1 & P_2 \end{array}\right] \left[\begin{array}{c} P_1^T \\ \hline 0 \end{array}\right] \\ &= P_1 P_1^T, \end{aligned}$$

we have

$$\lambda = \frac{1}{2}\boldsymbol{\mu}^T A\boldsymbol{\mu}.$$

Distribution of quadratic forms

Corollary 3.6

Let $\mathbf{y} \sim MVN(\mathbf{0}, I_n)$ be a $n \times 1$ random vector and let A be a $n \times n$ symmetric matrix. Then $\mathbf{y}^T A\mathbf{y}$ has a (ordinary) χ^2 distribution with k degrees of freedom if and only if A is idempotent and has rank k.

Corollary 3.7

Let $\mathbf{y} \sim MVN(\boldsymbol{\mu}, \sigma^2 I_n)$ be a $n \times 1$ random vector and let A be a $n \times n$ symmetric matrix. Then $\frac{1}{\sigma^2}\mathbf{y}^T A\mathbf{y}$ has a noncentral χ^2 distribution with k degrees of freedom and noncentrality parameter $\lambda = \frac{1}{2\sigma^2}\boldsymbol{\mu}^T A\boldsymbol{\mu}$ if and only if A is idempotent and has rank k.

Distribution of quadratic forms

Example. Let y_1 and y_2 be independent normal random variables with means 3 and -2 respectively and variance 1. Let

$$A = \frac{1}{2}\left[\begin{array}{cc} 1 & 1 \\ 1 & 1 \end{array}\right].$$

It is easy to verify that A is symmetric and idempotent, and has rank 1.

Distribution of quadratic forms

Therefore

$$\mathbf{y}^T A\mathbf{y} = \frac{1}{2}\begin{bmatrix} y_1 & y_2 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \frac{1}{2}y_1^2 + y_1 y_2 + \frac{1}{2}y_2^2$$

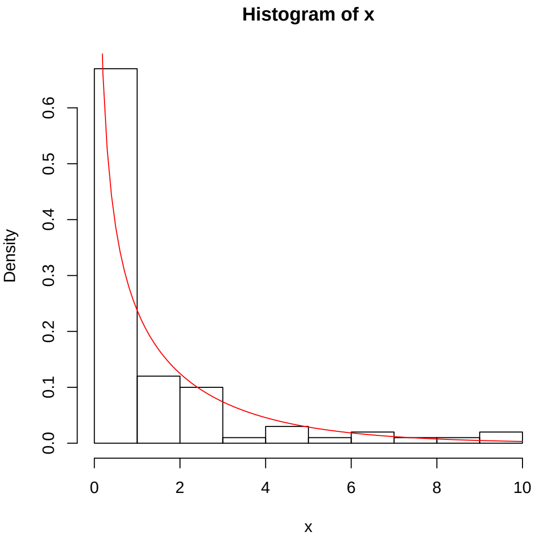
has a noncentral χ^2 distribution with 1 degree of freedom and noncentrality parameter

$$\lambda = \frac{1}{4}\begin{bmatrix} 3 & -2 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ -2 \end{bmatrix} = \frac{1}{4}.$$

Distribution of quadratic forms

```
> y1 <- rnorm(100,3,1)
> y2 <- rnorm(100,-2,1)
> x <- y1^2/2+y1*y2+y2^2/2
> hist(x,freq=FALSE)
> curve(dchisq(x,1,1/2),add=T,col='red')
```

Distribution of quadratic forms



Distribution of quadratic forms

What happens if $\mathbf{y}$ does not have variance I?

Theorem 3.8

Let $\mathbf{y} \sim MVN(\boldsymbol{\mu}, V)$ be a $n \times 1$ random vector, and let A be a $n \times n$ symmetric matrix. Then $\mathbf{y}^T A \mathbf{y}$ has a noncentral χ^2 distribution with k degrees of freedom and noncentrality parameter $\lambda = \frac{1}{2}\boldsymbol{\mu}^T A \boldsymbol{\mu}$ if and only if AV is idempotent and has rank k.

Distribution of quadratic forms

Corollary 3.9

Let $\mathbf{y} \sim MVN(\mathbf{0}, V)$ *be a* $n \times 1$ *random vector and let* A *be a* $n \times n$ *symmetric matrix. Then* $\mathbf{y}^T A\mathbf{y}$ *has a (ordinary)* χ^2 *distribution with* k *degrees of freedom if and only if* AV *is idempotent and has rank* k*.*

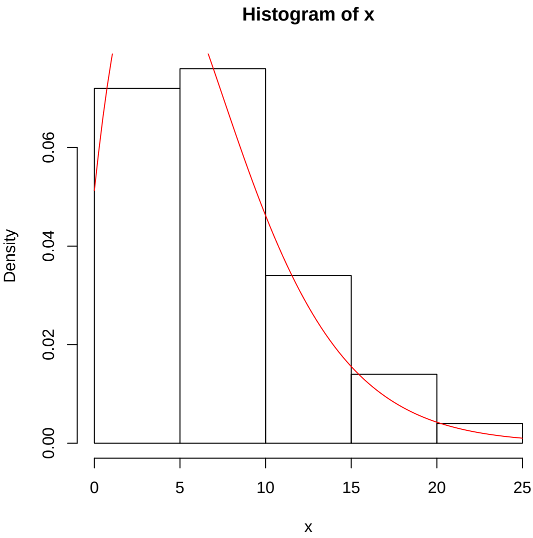
Corollary 3.10

Let $\mathbf{y} \sim MVN(\boldsymbol{\mu}, V)$ *be a* $n \times 1$ *random vector where the variance* V *is nonsingular. Then* $\mathbf{y}^T V^{-1}\mathbf{y}$ *has a noncentral* χ^2 *distribution with* n *degrees of freedom and noncentrality parameter* $\lambda = \frac{1}{2}\boldsymbol{\mu}^T V^{-1}\boldsymbol{\mu}$*.*

R example: noncentral χ^2

```
> y <- t(mvrnorm(100, mu=a, Sigma=V))
> x <- apply(y, 2, quadform, A = solve(V))
> (lambda <- t(a) %*% solve(V) %*% a / 2)

         [,1]
[1,] 2.277778

> mean(x)

[1] 7.496516

> 2 + 2*lambda

         [,1]
[1,] 6.555556

> hist(x, freq=F)
> curve(dchisq(x, 2, 2*lambda), add = TRUE, col='red')
```

R example: noncentral χ^2



Independence of quadratic forms

Sometimes we will want to know when two quadratic forms are independent. The next theorem tells us when this happens.

Theorem 3.11

Let $\mathbf{y} \sim MVN(\boldsymbol{\mu}, V)$ *be a* $n \times 1$ *random vector with nonsingular variance* V*, and let* A *and* B *be symmetric* $n \times n$ *matrices. Then* $\mathbf{y}^T A\mathbf{y}$ *and* $\mathbf{y}^T B\mathbf{y}$ *are independent if and only if*

$$AVB = 0.$$

Independence of quadratic forms

Proof. ($\Leftarrow$) Suppose that $AVB = 0$. Since V is symmetric and positive definite we have $V = C^2$ for some C, thus $ACCB = 0$.

Let

$$R = CAC, \quad S = CBC,$$

then

$$RS = CACCBC = 0.$$

Independence of quadratic forms

Because A, B, and C are symmetric, R and S are also symmetric. Thus

$$SR = S^T R^T = (RS)^T = 0 = RS.$$

By Theorem 2.4, we can find an orthogonal matrix P which diagonalises R and S simultaneously.

Since C is nonsingular, $r(R) = r(CAC) = r(A)$.

Independence of quadratic forms

Thus

$$P^T RP = \left[\begin{array}{c|c} D_1 & 0 \\ \hline 0 & 0 \end{array}\right]$$

where D_1 has dimension $r(A) \times r(A)$.

Because $RS = 0$, it can be shown that

$$P^T SP = \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & D_2 \end{array}\right]$$

where the partition has the same dimensions.

Independence of quadratic forms

Now define

$$\mathbf{z} = P^T C^{-1} \mathbf{y}.$$

Then $\mathbf{z}$ is multivariate normal with

$$E[\mathbf{z}] = P^T C^{-1} \boldsymbol{\mu}$$

and

$$\text{var } \mathbf{z} = P^T C^{-1} V C^{-1} P = P^T P = I.$$

Thus the elements of $\mathbf{z}$ are independent.

Independence of quadratic forms

Partition $\mathbf{z}$ into

$$\mathbf{z} = \left[\begin{array}{c} \mathbf{z}_1 \\ \hline \mathbf{z}_2 \end{array}\right]$$

where $\mathbf{z}_1$ has dimension $r(A) \times 1$.

By rewriting our equations, we see that

$$\begin{aligned} \mathbf{y} &= CP\mathbf{z} \\ A &= C^{-1}RC^{-1} \\ B &= C^{-1}SC^{-1} \end{aligned}$$

Independence of quadratic forms

So

$$
\begin{aligned}
\mathbf{y}^T A \mathbf{y} &= \mathbf{z}^T P^T C C^{-1} R C^{-1} C P \mathbf{z} \\
&= \mathbf{z}^T P^T R P \mathbf{z} \\
&= \left[\begin{array}{c|c} \mathbf{z}_1^T & \mathbf{z}_2^T \end{array}\right] \left[\begin{array}{c|c} D_1 & 0 \\ \hline 0 & 0 \end{array}\right] \left[\begin{array}{c} \mathbf{z}_1 \\ \hline \mathbf{z}_2 \end{array}\right] \\
&= \mathbf{z}_1^T D_1 \mathbf{z}_1
\end{aligned}
$$

and similarly

$$\mathbf{y}^T B \mathbf{y} = \mathbf{z}_2^T D_2 \mathbf{z}_2.$$

But $\mathbf{z}_1$ and $\mathbf{z}_2$ are mutually independent of each other, since all elements of $\mathbf{z}$ are independent. Therefore $\mathbf{y}^T A \mathbf{y}$ and $\mathbf{y}^T B \mathbf{y}$ are independent.

Independence of quadratic forms

Example. Let y_1 and y_2 follow a multivariate normal distribution with covariance matrix

$$V = \begin{bmatrix} a & b \\ b & c \end{bmatrix}.$$

Consider the symmetric matrices

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Then

$$\mathbf{y}^T A \mathbf{y} = y_1^2, \quad \mathbf{y}^T B \mathbf{y} = y_2^2.$$

Independence of quadratic forms

These quadratic forms will be independent if and only if

$$\begin{aligned} AVB &= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & b \\ 0 & c \end{bmatrix} \\ &= \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} \end{aligned}$$

is the 0 matrix.

But this means $b = 0$, i.e. y_1 and y_2 have zero covariance.

We've just "shown" that for multivariate random normals, uncorrelatedness is independence.

Independence of quadratic forms

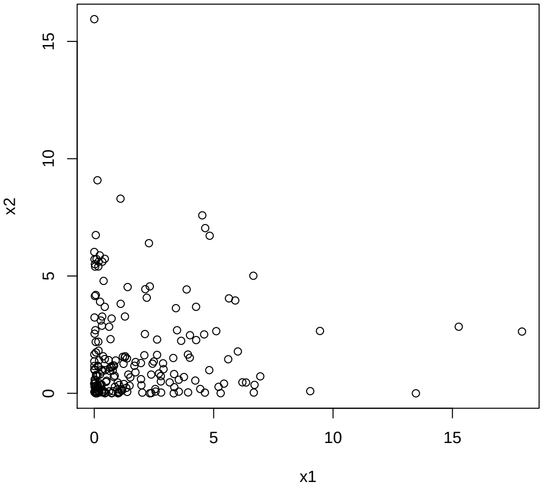
Corollary 3.12

Let $\mathbf{y} \sim MVN(\boldsymbol{\mu}, \sigma^2 I)$ be a random vector and let A and B be symmetric matrices. Then $\mathbf{y}^T A\mathbf{y}$ and $\mathbf{y}^T B\mathbf{y}$ are independent if and only if $AB = 0$.

Independence example

```
> A <- matrix(1, 2, 2)
> B <- matrix(c(1,-1,-1,1), 2, 2)
> A %*% B

     [,1] [,2]
[1,]    0    0
[2,]    0    0

> y <- t(mvrnorm(200, c(0, 0), diag(2)))
> x1 <- apply(y, 2, quadform, A = A)
> x2 <- apply(y, 2, quadform, A = B)
> cor(x1, x2)

[1] 0.008079555

> plot(x1, x2)
```

Independence example



Independence of quadratic forms

Next we consider when a quadratic form is independent of a random vector. Firstly, we define a random variable to be independent of a random vector if and only if it is independent of all elements of that vector.

Theorem 3.13

Let $\mathbf{y} \sim MVN(\boldsymbol{\mu}, V)$ be a $n \times 1$ random vector, and let A be a $n \times n$ symmetric matrix and B a $m \times n$ matrix. Then $\mathbf{y}^T A\mathbf{y}$ and $B\mathbf{y}$ are independent if and only if $BVA = 0$.

Lastly, we can combine several of the theorems we have seen before to tell when a group of quadratic forms (more than two) are independent.

Independence of quadratic forms

Theorem 3.14

Let $\mathbf{y} \sim MVN(\boldsymbol{\mu}, I)$ be a random vector, and let $A_1, \ldots, A_m$ be a set of symmetric matrices. If any two of the following statements are true:

- *All A_i are idempotent;*
- *$\sum_{i=1}^m A_i$ is idempotent;*
- *$A_i A_j = 0$ for all $i \neq j$;*

then so is the third, and

- *For all i, $\mathbf{y}^T A_i \mathbf{y}$ has a noncentral χ^2 distribution with $r(A_i)$ d.f. and noncentrality parameter $\lambda_i = \frac{1}{2}\boldsymbol{\mu}^T A_i \boldsymbol{\mu}$;*
- *$\mathbf{y}^T A_i \mathbf{y}$ and $\mathbf{y}^T A_j \mathbf{y}$ are independent for $i \neq j$; and*
- *$\sum_{i=1}^m r(A_i) = r\left(\sum_{i=1}^m A_i\right)$.*

Independence of quadratic forms

When $\sum_i A_i = I$, the previous result is related to the following theorem (which we will not prove):

Theorem 3.15 (Cochran-Fisher Theorem)

Let $\mathbf{y} \sim MVN(\boldsymbol{\mu}, \sigma^2 I)$ *be a* $n \times 1$ *random vector. Decompose the sum of squares of* $\mathbf{y}/\sigma$ *into the quadratic forms*

$$\frac{1}{\sigma^2}\mathbf{y}^T\mathbf{y} = \sum_{i=1}^{m} \frac{1}{\sigma^2}\mathbf{y}^T A_i \mathbf{y}.$$

Then the quadratic forms are independent and have noncentral χ^2 *distributions with parameters* $r(A_i)$ *and* $\frac{1}{2\sigma^2}\boldsymbol{\mu}^T A_i \boldsymbol{\mu}$*, respectively, if and only if*

$$\sum_{i=1}^{m} r(A_i) = n.$$