Constraint Satisfaction Problems

Chapter 6

Outline

◇ CSP examples

◇ Backtracking search for CSPs

◇ Problem structure and problem decomposition

◇ Local search for CSPs

Constraint satisfaction problems (CSPs)

Standard search problem:

state is a "black box"—any old data structure
that supports goal test, eval, successor

CSP:

state is defined by *variables* X_i with *values* from *domain* D_i

goal test is a set of *constraints* specifying
allowable combinations of values for subsets of variables

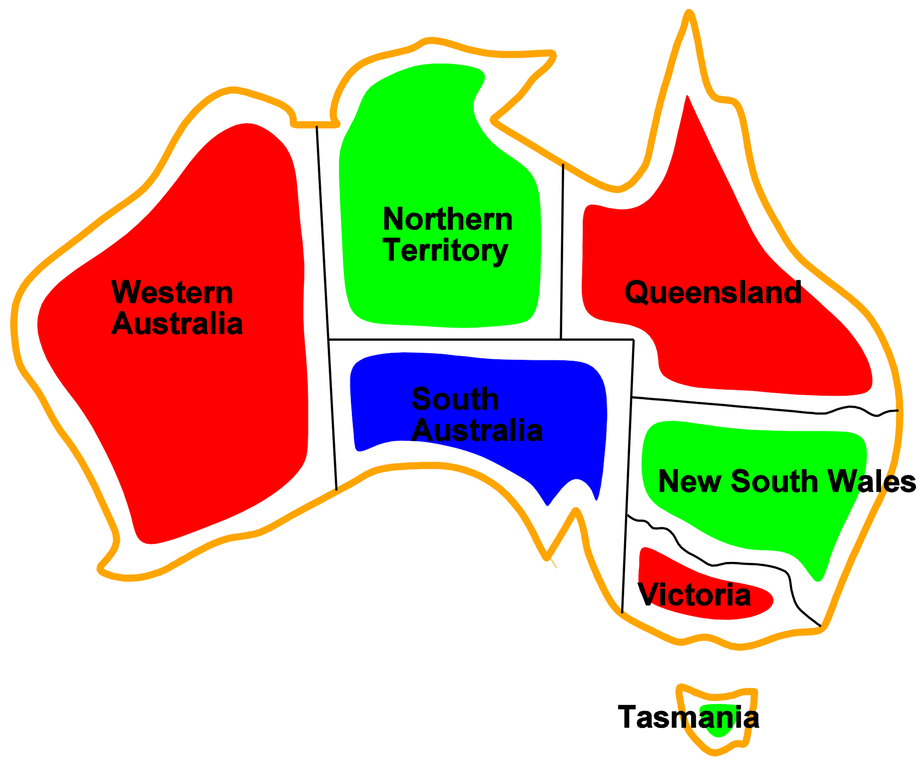
Simple example of a *formal representation language*

Allows useful *general-purpose* algorithms with more power
than standard search algorithms

Example: Map-Coloring

$Variables\ WA,\ NT,\ Q,\ NSW,\ V,\ SA,\ T$
$Domains\ D_i = \{red, green, blue\}$
$Constraints$: adjacent regions must have different colors
e.g., $WA \neq NT$ (if the language allows this), or
$(WA, NT) \in \{(red, green), (red, blue), (green, red), (green, blue), \ldots\}$

Example: Map-Coloring contd.

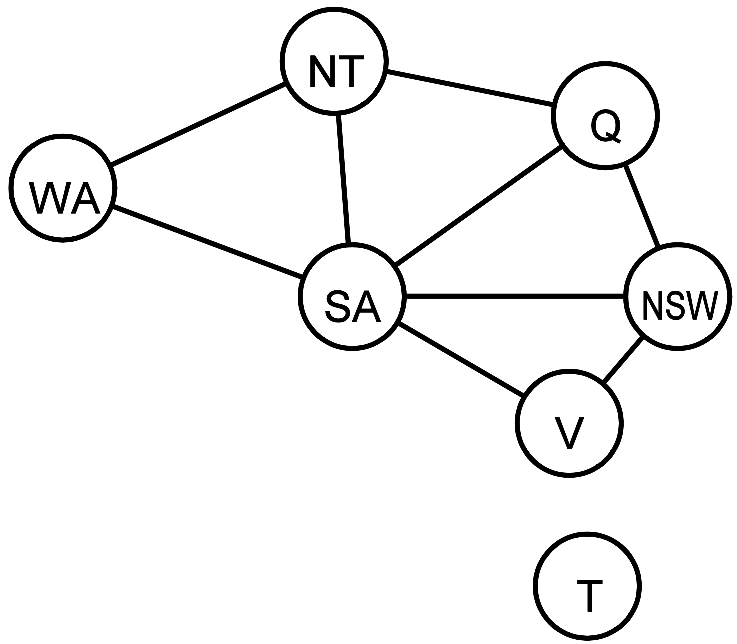


Solutions are assignments satisfying all constraints, e.g.,
$\{WA = red, NT = green, Q = red, NSW = green,$
$V = red, SA = blue, T = green\}$

Constraint graph

Binary CSP: each constraint relates at most two variables

Constraint graph: nodes are variables, arcs show constraints



General-purpose CSP algorithms use the graph structure
to speed up search. E.g., Tasmania is an independent subproblem!

Varieties of CSPs

Discrete variables

finite domains; size $d \Rightarrow O(d^n)$ complete assignments

where n is the number of variables in the CSP

- ♢ e.g., Boolean CSPs, incl. Boolean satisfiability (NP-complete)

infinite domains (integers, strings, etc.)

- ♢ e.g., job scheduling, variables are start/end days for each job
- ♢ need a *constraint language*, e.g., $StartJob_1 + 5 \leq StartJob_3$
- ♢ *linear* constraints solvable, *nonlinear* undecidable

Continuous variables

- ♢ e.g., start/end times for Hubble Telescope observations
- ♢ linear constraints solvable in polynomial time
  by linear programming (LP) methods

Varieties of constraints

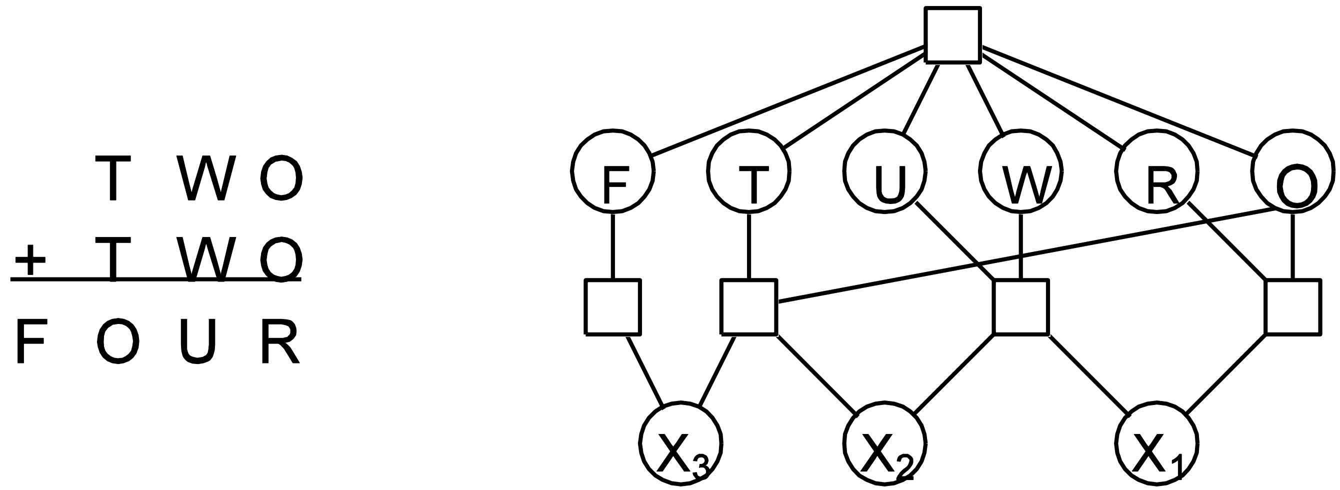
$Unary$ constraints involve a single variable,
e.g., $SA \neq green$

$Binary$ constraints involve pairs of variables,
e.g., $SA \neq WA$

$Higher\text{-}order$ constraints involve 3 or more variables,
e.g., cryptarithmetic column constraints

$Preferences$ (soft constraints), e.g., red is better than $green$
often representable by a cost for each variable assignment
$\rightarrow$ constrained optimization problems

Example: Cryptarithmetic



(a) (b)

Variables: $F\ T\ U\ W\ R\ O\ X_1\ X_2\ X_3$
Domains: $\{0,1,2,3,4,5,6,7,8,9\}$
Constraints
$\quad alldiff(F,T,U,W,R,O)$
$\quad O+O=R+10\cdot X_1$, etc.

Real-world CSPs

Assignment problems
    e.g., who teaches what class

Timetabling problems
    e.g., which class is offered when and where?

Hardware configuration

Spreadsheets

Transportation scheduling

Factory scheduling

Floorplanning

Notice that many real-world problems involve real-valued variables

Standard search formulation (incremental)

Let's start with the straightforward, dumb approach, then fix it

States are defined by the values assigned so far

◇ *Initial state*: the empty assignment, $\emptyset$

◇ *Successor function*: assign a value to an unassigned variable
that does not conflict with current assignment.
$\Rightarrow$ fail if no legal assignments (not fixable!)

◇ *Goal test*: the current assignment is complete

1) This is the same for all CSPs!
2) Every solution appears at depth n with n variables
$\Rightarrow$ use depth-first search
3) $b = (n - \ell)d$ at depth ℓ, hence $n!d^n$ leaves!!!!
4) Path is irrelevant, so can also use complete-state formulation

Backtracking search

Variable assignments are $commutative$, i.e.,

$[WA = red$ then $NT = green]$ same as $[NT = green$ then $WA = red]$

Only need to consider assignments to a single variable at each node

$\Rightarrow \quad b = d$ and there are d^n leaves

Depth-first search for CSPs with single-variable assignments
is called $backtracking$ search

Backtracking search is the basic uninformed algorithm for CSPs

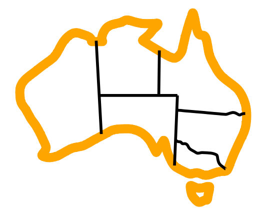
Can solve n-queens for $n \approx 25$

Backtracking search

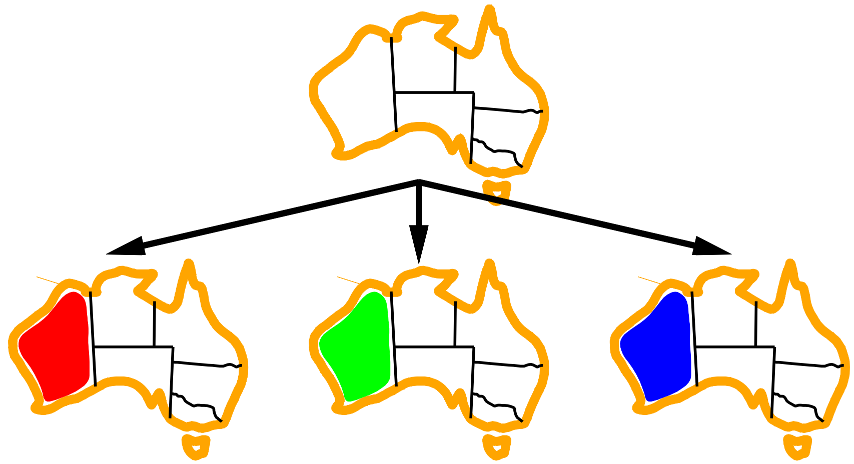
```
function BACKTRACKING-SEARCH(csp) returns solution/failure
   return RECURSIVE-BACKTRACKING({ }, csp)

function RECURSIVE-BACKTRACKING(assignment, csp) returns soln/failure
   if assignment is complete then return assignment
   var ← SELECT-UNASSIGNED-VARIABLE(VARIABLES[csp], assignment, csp)
   for each value in ORDER-DOMAIN-VALUES(var, assignment, csp) do
       if value is consistent with assignment given CONSTRAINTS[csp] then
           add {var = value} to assignment
           result ← RECURSIVE-BACKTRACKING(assignment, csp)
           if result ≠ failure then return result
           remove {var = value} from assignment
   return failure
```

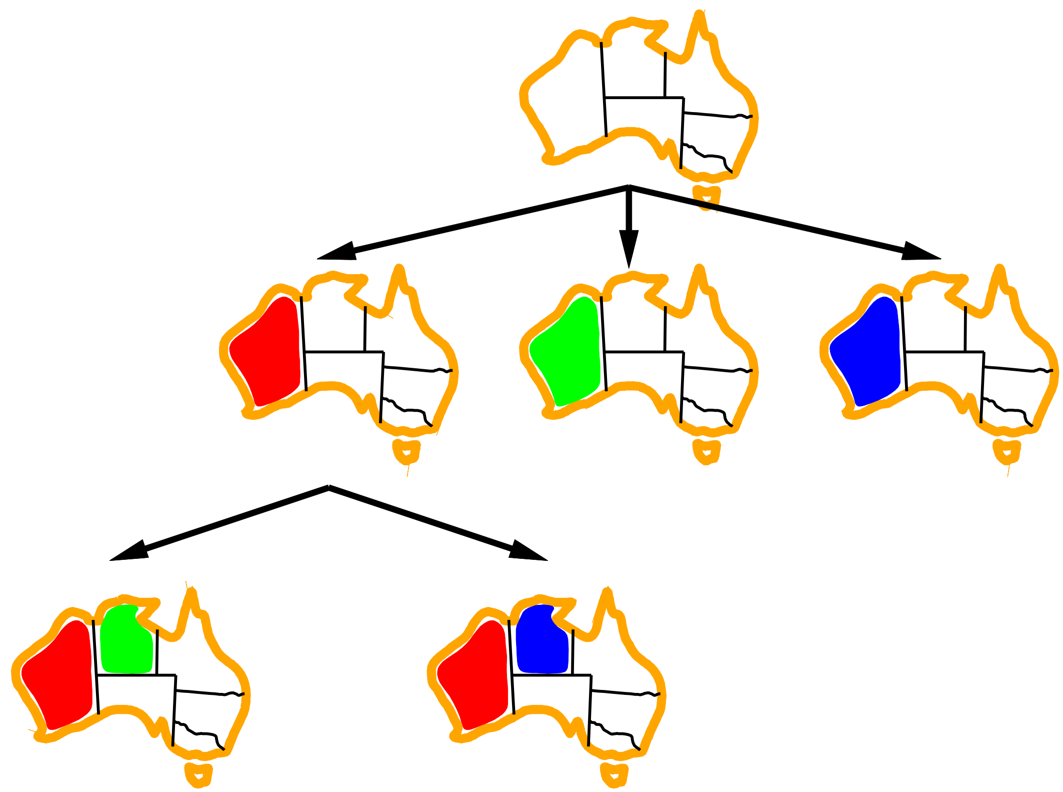
Backtracking example



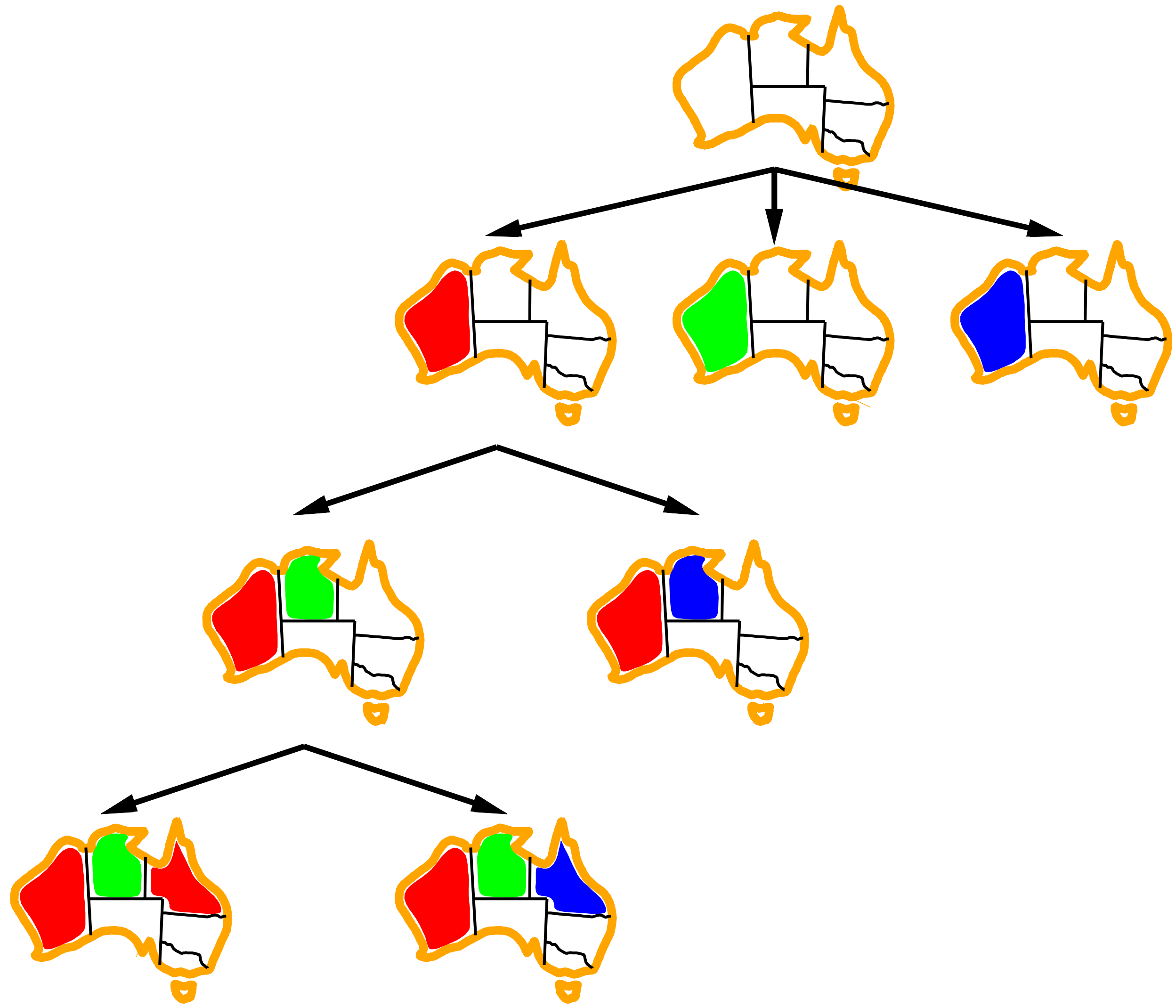
Backtracking example



Backtracking example



Backtracking example



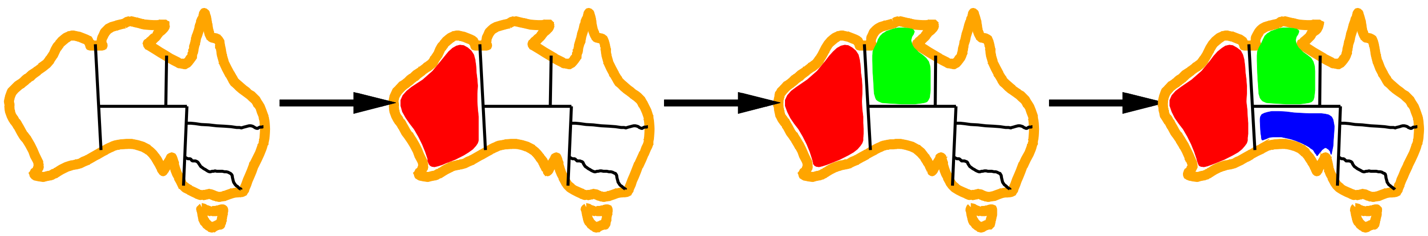
Improving backtracking efficiency

General-purpose methods can give huge gains in speed:

1. Which variable should be assigned next?
2. In what order should its values be tried?
3. Can we detect inevitable failure early?
4. Can we take advantage of problem structure?

Minimum remaining values

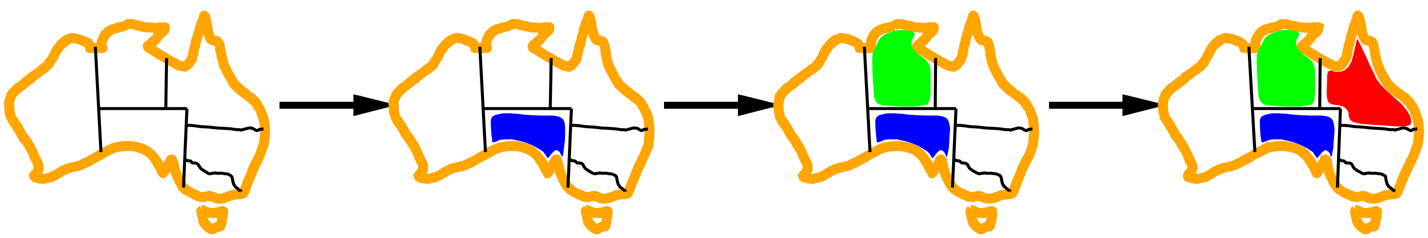
Minimum remaining values (MRV):
 choose the variable with the fewest legal values



Degree heuristic

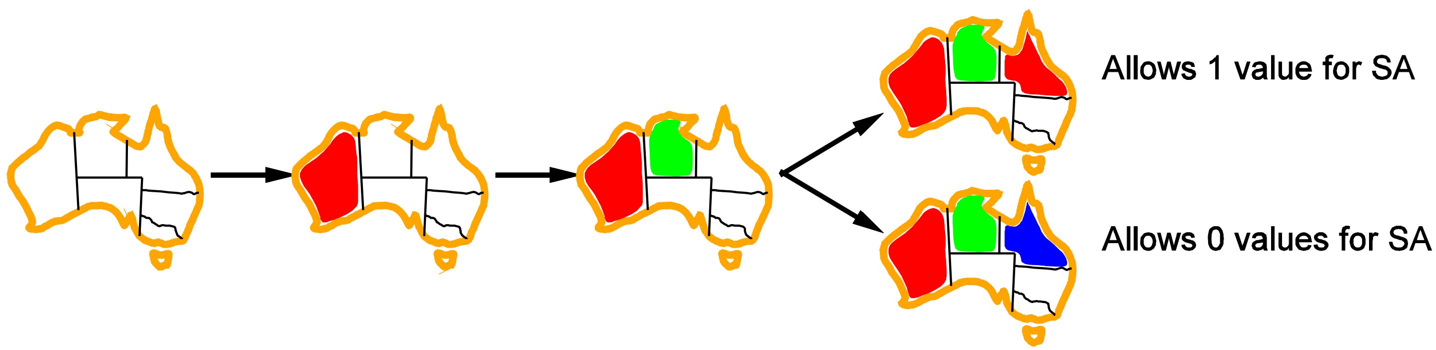
Tie-breaker among MRV variables

Degree heuristic:
choose the variable with the most constraints on remaining variables



Least constraining value

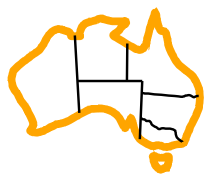
Given a variable, choose the least constraining value:
the one that rules out the fewest values in the remaining variables



Combining these heuristics makes 1000 queens feasible

Forward checking

Idea: Keep track of remaining legal values for unassigned variables
Terminate search when any variable has no legal values



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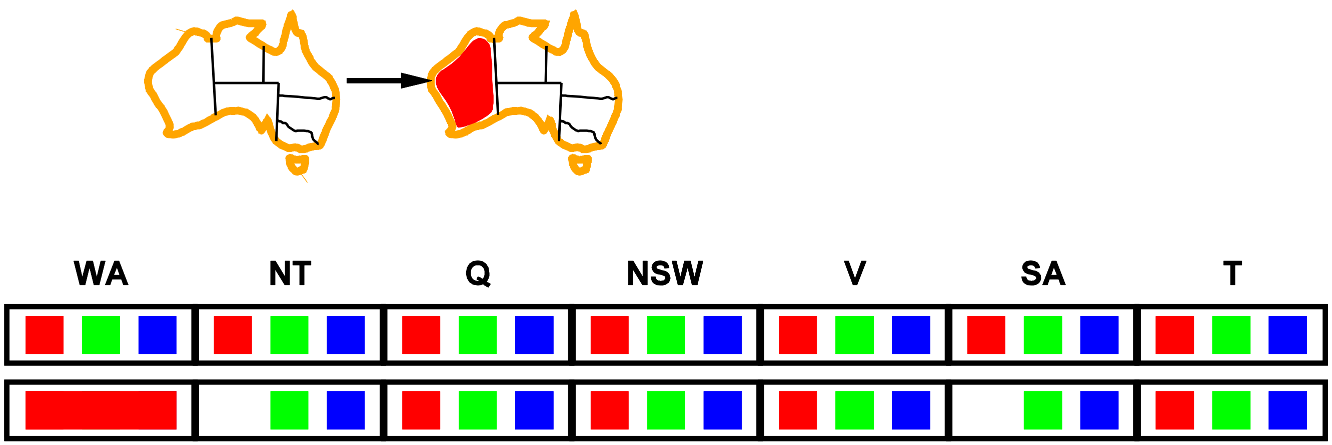

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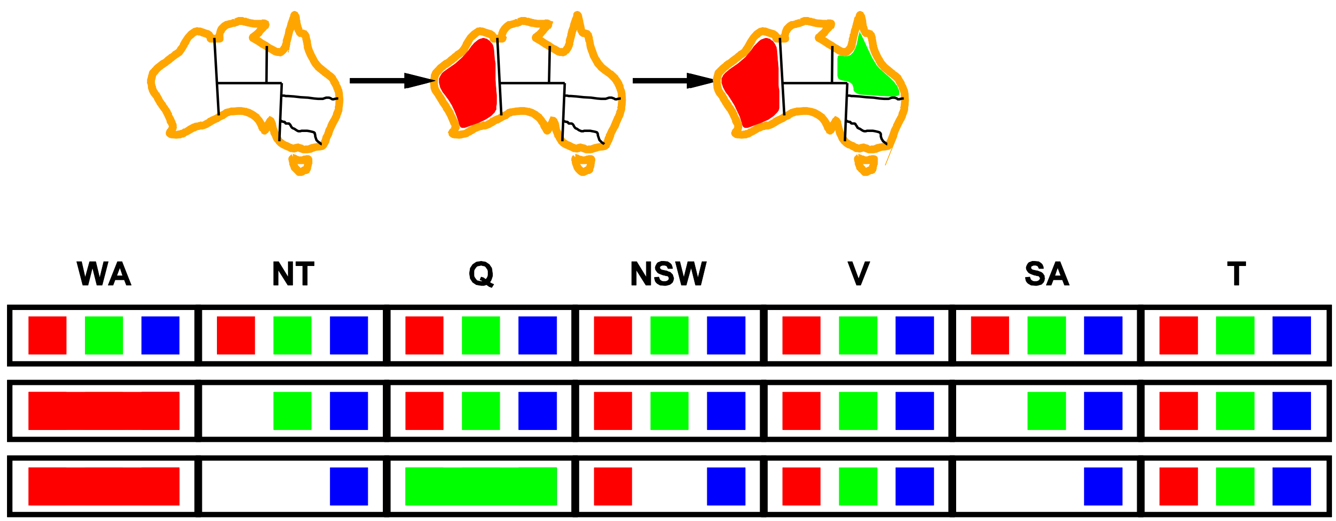

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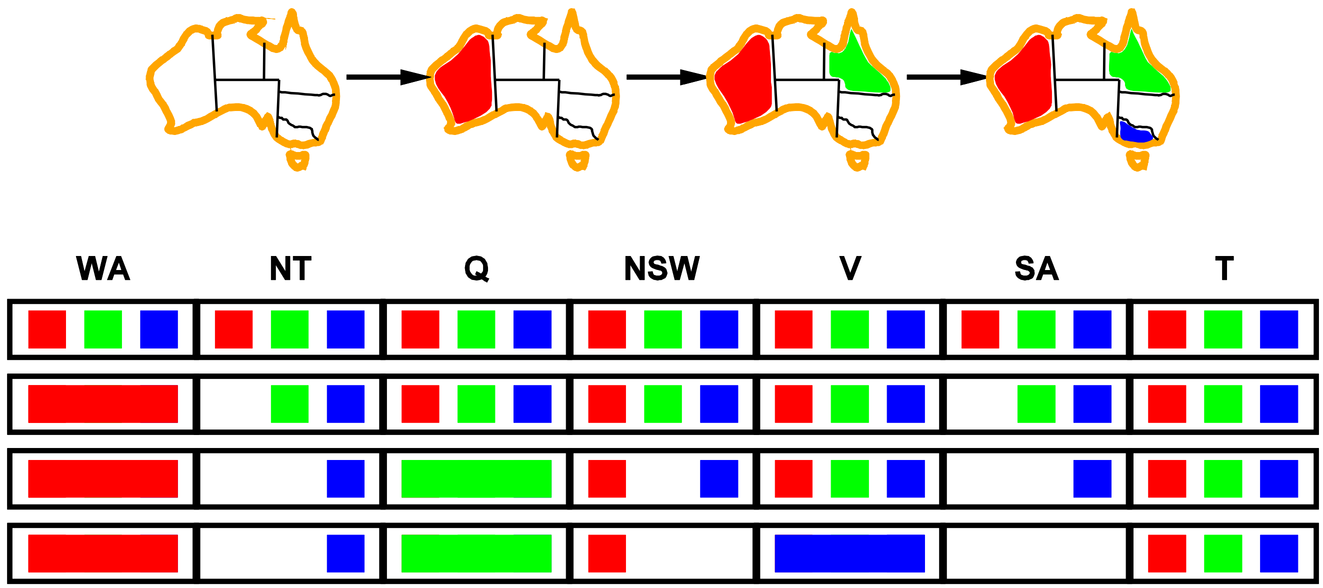
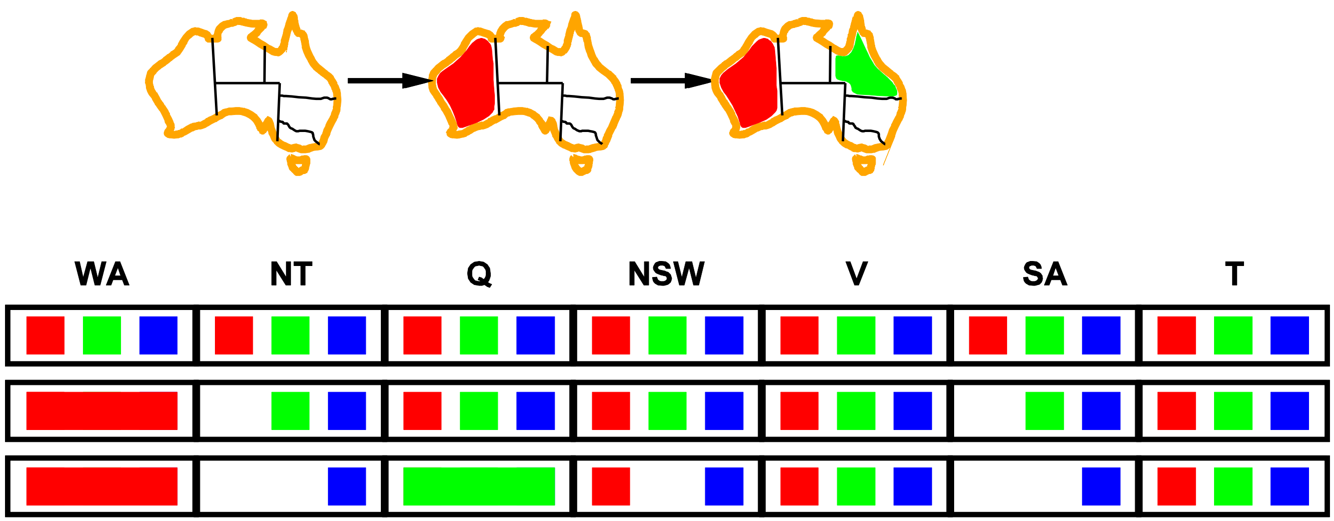


Constraint propagation

Forward checking propagates information from assigned to unassigned variables, but doesn't provide early detection for all failures:



NT and SA cannot both be blue!

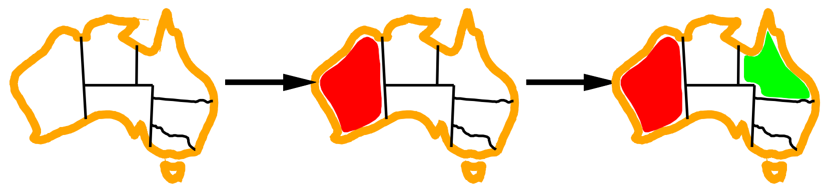
$Constraint\ propagation$ repeatedly enforces constraints locally

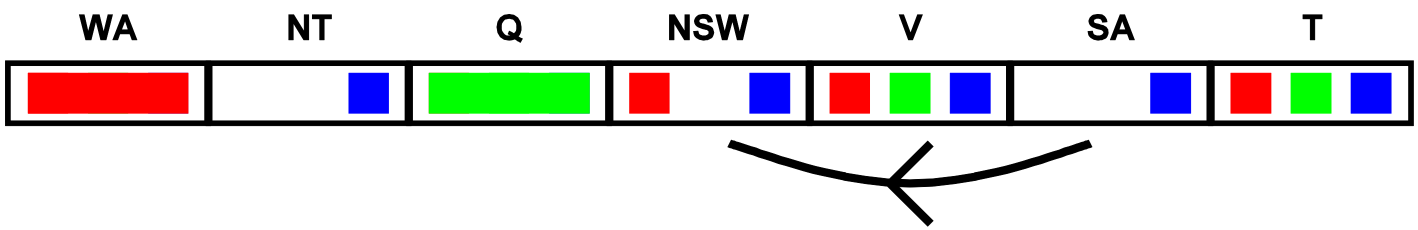
Arc consistency

Simplest form of propagation makes each arc *consistent*

$X \rightarrow Y$ is arc consistent iff

for *every* value x of X there is *at least one value* y of Y
that satisfies the constraint between X and Y

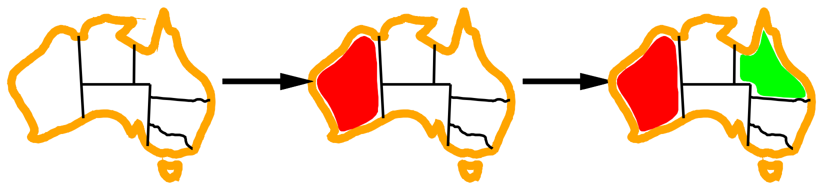


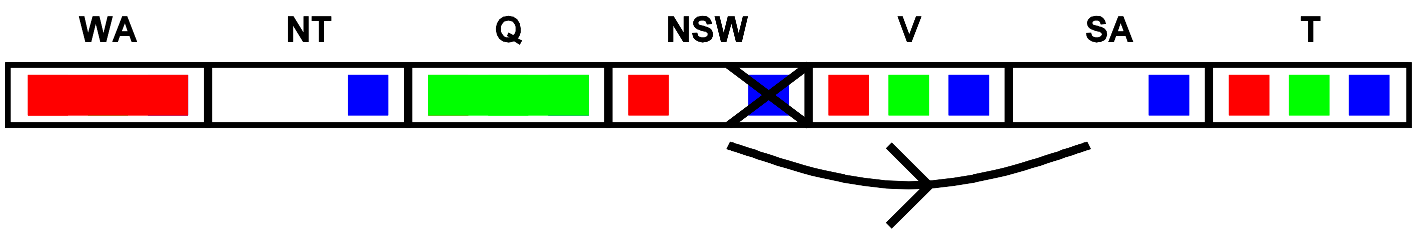


Arc consistency

Simplest form of propagation makes each arc *consistent*

$X \to Y$ is consistent iff
for *every* value x of X there is *some* allowed y





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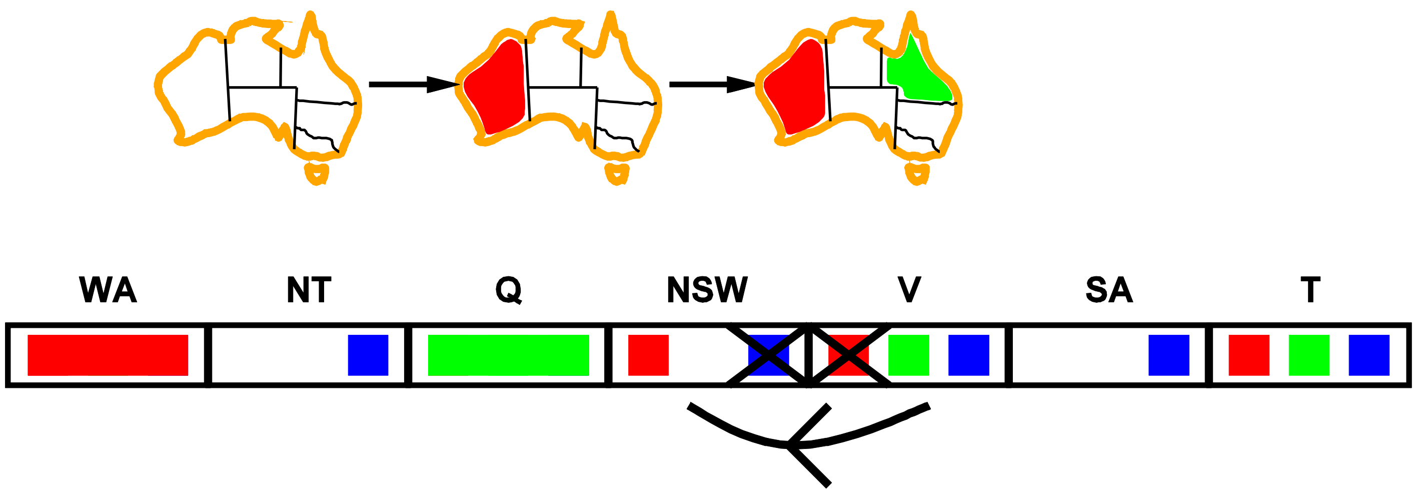

If X loses a value, neighbors of X need to be rechecked

Arc consistency

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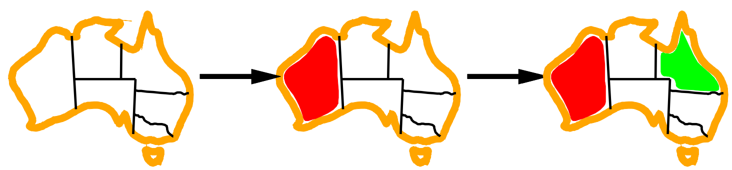
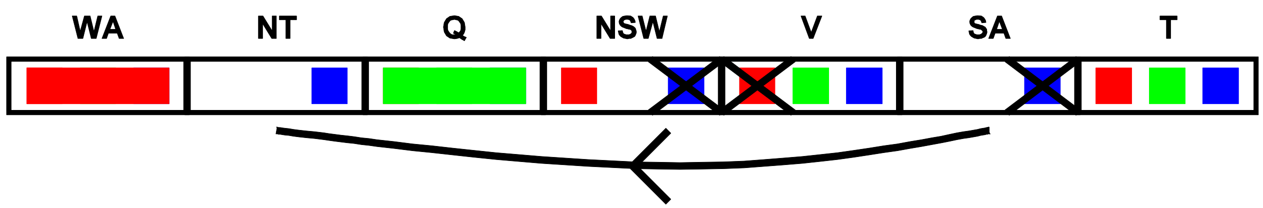




If X loses a value, neighbors of X need to be rechecked

Arc consistency detects failure earlier than forward checking

Can be run as a preprocessor or after each assignment

Arc consistency algorithm

function AC-3(*csp*) **returns** the CSP, possibly with reduced domains
  inputs: *csp*, a binary CSP with variables $\{X_1,\ X_2,\ \ldots,\ X_n\}$
  local variables: *queue*, a queue of arcs, initially all the arcs in *csp*

  while *queue* is not empty **do**
    $(X_i,\ X_j) \leftarrow$ Remove-First(*queue*)
    if Remove-Inconsistent-Values($X_i,\ X_j$) **then**
      for each X_k **in** Neighbors[X_i] **do**
        add $(X_k,\ X_i)$ to *queue*

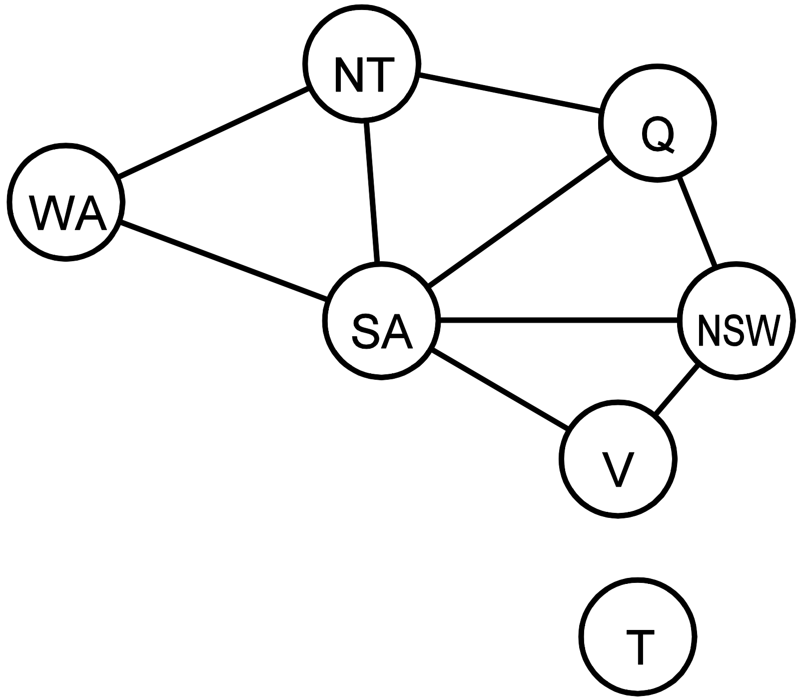
---

function Remove-Inconsistent-Values($X_i,\ X_j$) **returns** true iff succeeds
  removed ← *false*
  for each x **in** Domain[X_i] **do**
    if no value y in Domain[X_j] allows (x,y) to satisfy the constraint $X_i \leftrightarrow X_j$
      then delete x from Domain[X_i]; *removed* ← *true*
  return *removed*

$O(n^2d^3)$, can be reduced to $O(n^2d^2)$ (but detecting *all* is NP-hard)

Alternative approach: exploit structure of the constraint graph
to find independent subproblems ...

Problem structure



Tasmania and mainland are *independent subproblems*

Identifiable as *connected components* of constraint graph

Problem structure contd.

Suppose each subproblem has c variables out of n total

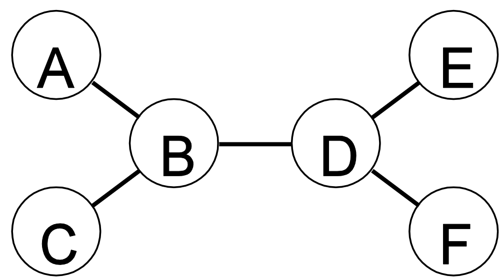
Worst-case solution cost is $n/c \cdot d^c$, *linear* in n

E.g., $n = 80$, $d = 2$, $c = 20$
2^{80} = 4 billion years at 10 million nodes/sec
$4 \cdot 2^{20}$ = 0.4 seconds at 10 million nodes/sec

Unfortunately, completely independent subproblems are rare in practice

However, there are other graph strucures that are easy to solve
e.g. when the constraint graph is a tree

Tree-structured CSPs



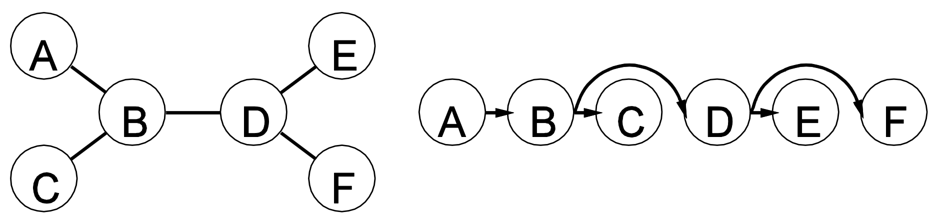
Theorem: if the constraint graph has no loops, the CSP can be solved in $O(n\, d^2)$ time

Compare to general CSPs, where worst-case time is $O(d^n)$

This property also applies to logical and probabilistic reasoning:
an important example of the relation between syntactic restrictions
and the complexity of reasoning.

Algorithm for tree-structured CSPs

1. Choose a variable as root, order variables from root to leaves such that every node's parent precedes it in the ordering

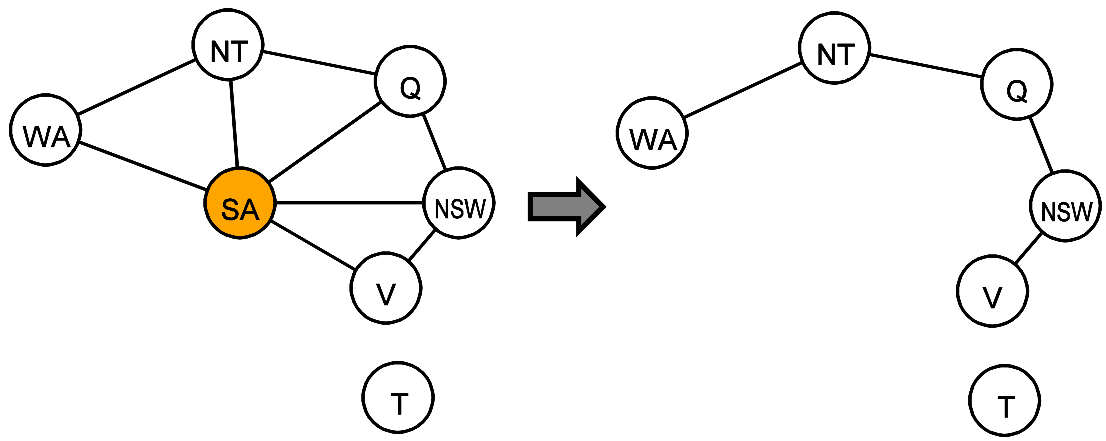


2. For j from n down to 2, apply $\textsc{MakeArcConsistent}(\text{Parent}(X_j), X_j)$

3. For j from 1 to n, assign X_j consistently with $\text{Parent}(X_j)$

Nearly tree-structured CSPs

Conditioning: instantiate a variable, prune its neighbors' domains



Cutset conditioning: instantiate (in all ways) a set of variables
such that the remaining constraint graph is a tree

Cutset: set of variables that can be deleted so constraint graph forms a tree

Cutset size $c \Rightarrow$ runtime $O(d^c \cdot (n-c)d^2)$, very fast for small c

Iterative algorithms for CSPs - Local search

Recall hill-climbing search from Week 3

Hill-climbing typically works with
"complete" states, i.e., all variables assigned

Local search then tries to change one variable assignment at a time

To apply to CSPs:
  allow states with unsatisfied constraints(*variable selection*)
  operators *reassign* variable values (*value selection*)

Variable selection: randomly select any conflicted variable

Value selection by *min-conflicts* heuristic:
  choose value that violates the fewest constraints
  i.e., hillclimb with $h(n) =$ total number of violated constraints

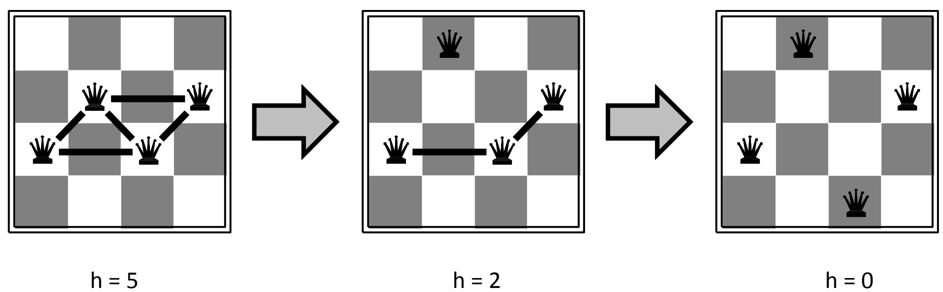
Example: 4-Queens

$States$: 4 queens in 4 columns ($4^4 = 256$ states)

$Operators$: move queen in column

$Goal\ test$: no attacks

$Evaluation$: $h(n) =$ number of attacks

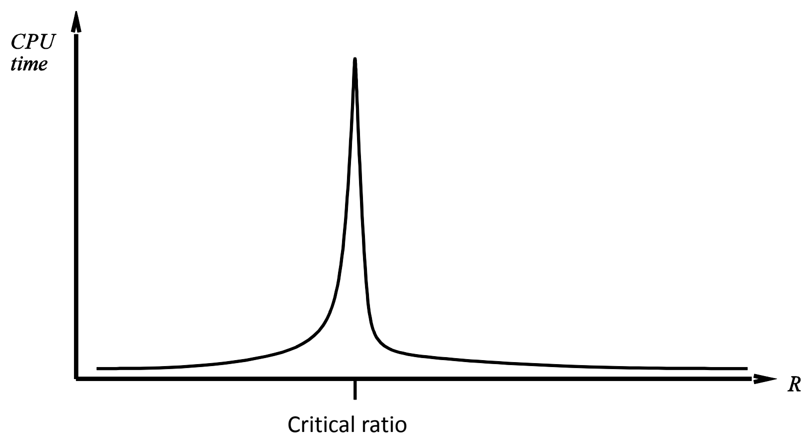


Performance of min-conflicts

Given random initial state, can solve n-queens in almost constant time for arbitrary n with high probability (e.g., n = 10,000,000)

The same appears to be true for any randomly-generated CSP *except* in a narrow range of the ratio

$$R = \frac{\text{number of constraints}}{\text{number of variables}}$$



Summary

CSPs are a special kind of problem:
states defined by values of a fixed set of variables
goal test defined by *constraints* on variable values

Backtracking = depth-first search with one variable assigned per node

Variable ordering and value selection heuristics help significantly

Forward checking prevents assignments that guarantee later failure

Constraint propagation (e.g., arc consistency) does additional work
to constrain values and detect inconsistencies

The CSP representation allows analysis of problem structure

Tree-structured CSPs can be solved in linear time

Iterative min-conflicts is usually effective in practice

Summary

Examples of skills expected:

♢ Model a given problem as a CSP

♢ Demonstrate operation of CSP search algorithms

♢ Discuss and evaluate the properties of different constraint satisfaction techniques