

CHAPTER

# Introduction to Convection

6



**T**hus far we have focused on heat transfer by conduction and have considered convection only to the extent that it provides a possible boundary condition for conduction problems. In Section 1.2.2 we used the term *convection* to describe energy transfer between a surface and a fluid moving over the surface. Convection includes energy transfer by both the bulk fluid motion (advection) and the random motion of fluid molecules (conduction or diffusion).

In our treatment of convection, we have two major objectives. In addition to obtaining an understanding of the physical mechanisms that underlie convection transfer, we wish to develop the means to perform convection transfer calculations. This chapter and the material of Appendix E are devoted primarily to achieving the former objective. Physical origins are discussed, and relevant dimensionless parameters, as well as important analogies, are developed.

A unique feature of this chapter is the manner in which convection mass transfer effects are introduced by analogy to those of convection heat transfer. In mass transfer by convection, gross fluid motion combines with diffusion to promote the transfer of a species for which there exists a concentration gradient. In this text, we focus on convection mass transfer that occurs at the surface of a volatile solid or liquid due to motion of a gas over the surface.

With conceptual foundations established, subsequent chapters are used to develop useful tools for quantifying convection effects. Chapters 7 and 8 present methods for computing the coefficients associated with forced convection in external and internal flow configurations, respectively. Chapter 9 describes methods for determining these coefficients in free convection, and Chapter 10 considers the problem of convection with phase change (boiling and condensation). Chapter 11 develops methods for designing and evaluating the performance of heat exchangers, devices that are widely used in engineering practice to effect heat transfer between fluids.

Accordingly, we begin by developing our understanding of the nature of convection.

## 6.1 The Convection Boundary Layers

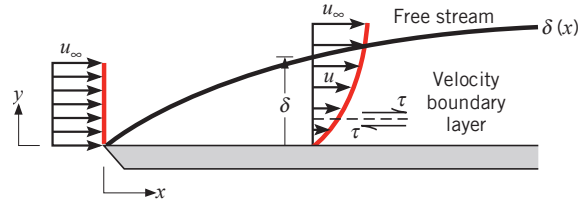
The concept of boundary layers is central to the understanding of convection heat and mass transfer between a surface and a fluid flowing past it. In this section, velocity, thermal, and concentration boundary layers are described, and their relationships to the friction coefficient, convection heat transfer coefficient, and convection mass transfer coefficient are introduced.

### 6.1.1 The Velocity Boundary Layer

To introduce the concept of a boundary layer, consider flow over the flat plate of Figure 6.1. When fluid particles make contact with the surface, their velocity is reduced significantly relative to the fluid velocity upstream of the plate, and for most situations it is valid to assume that the particle velocity is zero at the wall.<sup>1</sup> These particles then act to

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<sup>1</sup>This is an approximation of the situation discussed in Section 3.9, wherein fluid molecules or particles continually collide with and are reflected from the surface. The momentum of an individual fluid particle will change in response to its collision with the surface. This effect may be described by *momentum accommodation coefficients*, as will be discussed in Section 8.8. In this chapter, we assume that nano- and microscale effects are not important, in which case the assumption of zero fluid velocity at the wall is valid.



**FIGURE 6.1** Velocity boundary layer development on a flat plate.

retard the motion of particles in the adjoining fluid layer, which act to retard the motion of particles in the next layer, and so on until, at a distance  $y = \delta$  from the surface, the effect becomes negligible. This retardation of fluid motion is associated with *shear stresses*  $\tau$  acting in planes that are parallel to the fluid velocity (Figure 6.1). With increasing distance  $y$  from the surface, the  $x$  velocity component of the fluid,  $u$ , must then increase until it approaches the free stream value  $u_\infty$ . The subscript  $\infty$  is used to designate conditions in the *free stream* outside the boundary layer.

The quantity  $\delta$  is termed the *boundary layer thickness*, and it is typically defined as the value of  $y$  for which  $u = 0.99u_\infty$ . The *boundary layer velocity profile* refers to the manner in which  $u$  varies with  $y$  through the boundary layer. Accordingly, the fluid flow is characterized by two distinct regions, a thin fluid layer (the boundary layer) in which velocity gradients and shear stresses are large and a region outside the boundary layer in which velocity gradients and shear stresses are negligible. With increasing distance from the leading edge, the effects of viscosity penetrate farther into the free stream and the boundary layer grows ( $\delta$  increases with  $x$ ).

Because it pertains to the fluid velocity, the foregoing boundary layer may be referred to more specifically as the *velocity boundary layer*. It develops whenever there is fluid flow over a surface, and it is of fundamental importance to problems involving convection transport. In fluid mechanics its significance to the engineer stems from its relation to the surface shear stress  $\tau_s$ , and hence to surface frictional effects. For external flows it provides the basis for determining the local *friction coefficient*

$$C_f \equiv \frac{\tau_s}{\rho u_\infty^2 / 2} \quad (6.1)$$

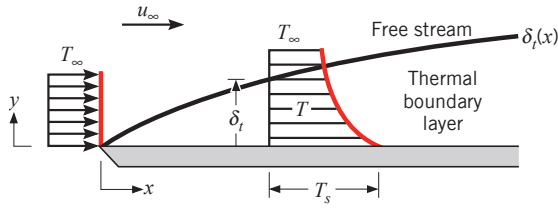
a key dimensionless parameter from which the surface frictional drag may be determined. Assuming a *Newtonian fluid*, the surface shear stress may be evaluated from knowledge of the velocity gradient at the surface

$$\tau_s = \mu \left. \frac{\partial u}{\partial y} \right|_{y=0} \quad (6.2)$$

where  $\mu$  is a fluid property known as the *dynamic viscosity*. In a velocity boundary layer, the velocity gradient at the surface depends on the distance  $x$  from the leading edge of the plate. Therefore, the surface shear stress and friction coefficient also depend on  $x$ .

### 6.1.2 The Thermal Boundary Layer

Just as a velocity boundary layer develops when there is fluid flow over a surface, a *thermal boundary layer* must develop if the fluid free stream and surface temperatures differ. Consider flow over an isothermal flat plate (Figure 6.2). At the leading edge the *temperature*



**FIGURE 6.2** Thermal boundary layer development on an isothermal flat plate.

*profil* is uniform, with  $T(y) = T_\infty$ . However, fluid particles that come into contact with the plate achieve thermal equilibrium at the plate's surface temperature.<sup>2</sup> In turn, these particles exchange energy with those in the adjoining fluid layer, and temperature gradients develop in the fluid. The region of the fluid in which these temperature gradients exist is the thermal boundary layer, and its thickness  $\delta_t$  is typically defined as the value of  $y$  for which the ratio  $[(T_s - T)/(T_s - T_\infty)] = 0.99$ . With increasing distance from the leading edge, the effects of heat transfer penetrate farther into the free stream and the thermal boundary layer grows.

The relation between conditions in this boundary layer and the convection heat transfer coefficient may readily be demonstrated. At any distance  $x$  from the leading edge, the *local* surface heat flux may be obtained by applying Fourier's law to the *fluid* at  $y = 0$ . That is,

$$q_s'' = -k_f \left. \frac{\partial T}{\partial y} \right|_{y=0} \quad (6.3)$$

The subscript  $s$  has been used to emphasize that this is the surface heat flux, but it will be dropped in later sections. This expression is appropriate because, *at the surface, there is no fluid motion and energy transfer occurs only by conduction*. Recalling Newton's law of cooling, we see that

$$q_s'' = h(T_s - T_\infty) \quad (6.4)$$

and combining this with Equation 6.3, we obtain

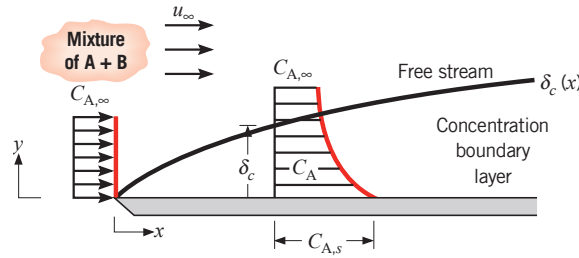
$$h = \frac{-k_f \partial T / \partial y|_{y=0}}{T_s - T_\infty} \quad (6.5)$$

Hence, conditions in the thermal boundary layer, which strongly influence the wall temperature gradient  $\partial T / \partial y|_{y=0}$ , determine the rate of heat transfer across the boundary layer. Since  $(T_s - T_\infty)$  is a constant, independent of  $x$ , while  $\delta_t$  increases with increasing  $x$ , temperature gradients in the boundary layer must decrease with increasing  $x$ . Accordingly, the magnitude of  $\partial T / \partial y|_{y=0}$  decreases with increasing  $x$ , and it follows that  $q_s''$  and  $h$  decrease with increasing  $x$ .

### 6.1.3 The Concentration Boundary Layer

When air moves past the surface of a pool of water, the liquid water will evaporate, and water vapor will be transferred into the airstream. This is an example of convection mass transfer. More generally, consider a *binary mixture* of chemical species A and B that flows

<sup>2</sup>Micro- and nanoscale effects are assumed to be negligible in this chapter. Hence, the thermal accommodation coefficient of Section 3.9 attains a value of unity, in which case the fluid particles achieve thermal equilibrium with the surface of the plate. Micro- and nanoscale effects will be discussed in Section 8.8.



**FIGURE 6.3** Species concentration boundary layer development on a flat plate.

over a surface (Figure 6.3). The molar concentration ( $\text{kmol/m}^3$ ) of species A at the surface is  $C_{A,s}$ , and in the free stream it is  $C_{A,\infty}$ . If  $C_{A,s}$  differs from  $C_{A,\infty}$ , transfer of species A by convection will occur. For example, species A could be a vapor that is transferred into a gas stream (species B) due to *evaporation* at a liquid surface (as in the water example) or due to *sublimation* at a solid surface. In this situation, a *concentration boundary layer* will develop that is similar to the velocity and thermal boundary layers. The concentration boundary layer is the region of the fluid in which concentration gradients exist, and its thickness  $\delta_c$  is typically defined as the value of  $y$  for which  $[(C_{A,s} - C_A)/(C_{A,s} - C_{A,\infty})] = 0.99$ . With increasing distance from the leading edge, the effects of species transfer penetrate farther into the free stream and the concentration boundary layer grows.

Species transfer by convection between the surface and the free stream fluid is determined by conditions in the boundary layer, and we are interested in determining the rate at which this transfer occurs. In particular, we are interested in the molar flux of species A,  $N''_A$  ( $\text{kmol/s} \cdot \text{m}^2$ ). It is helpful to recognize that the molar flux associated with species transfer by *diffusion* is determined by an expression that is analogous to Fourier's law. For the conditions of interest in this chapter, the expression, which is termed *Fick's law*, has the form

$$N''_A = -D_{AB} \frac{\partial C_A}{\partial y} \quad (6.6)^3$$

where  $D_{AB}$  is a property of the binary mixture known as the *binary diffusion coefficient*. At any point corresponding to  $y > 0$  in the concentration boundary layer of Figure 6.3, species transfer is due to both bulk fluid motion (*advection*) and diffusion. However, absent nano- or microscale effects and the influence of species diffusion on the velocity normal to the surface, fluid motion at the surface can be neglected.<sup>4</sup> Accordingly, species transfer at the surface is only by diffusion, and applying Fick's law at  $y = 0$ , the molar flux is

$$N''_{A,s} = -D_{AB} \left. \frac{\partial C_A}{\partial y} \right|_{y=0} \quad (6.7)$$

The subscript  $s$  has been used to emphasize that this is the molar flux at the surface, but it will be dropped in later sections. Analogous to Newton's law of cooling, an equation can be written that relates the molar flux to the concentration difference across the boundary layer, as

$$N''_{A,s} = h_m(C_{A,s} - C_{A,\infty}) \quad (6.8)$$

<sup>3</sup>This expression is an approximation of a more general form of Fick's law of diffusion (Section 14.1.3) when the total molar concentration of the mixture,  $C = C_A + C_B$ , is a constant.

<sup>4</sup>The basis for neglecting the effects of diffusion on bulk fluid motion is considered in Sections 14.2 and 14.3.

where  $h_m$  (m/s) is the *convection mass transfer coefficient*, analogous to the convection heat transfer coefficient. Combining Equations 6.7 and 6.8, it follows that

$$h_m = \frac{-D_{AB} \partial C_A / \partial y|_{y=0}}{C_{A,s} - C_{A,\infty}} \quad (6.9)$$

Therefore, conditions in the concentration boundary layer, which strongly influence the surface concentration gradient  $\partial C_A / \partial y|_{y=0}$ , also influence the convection mass transfer coefficient and hence the rate of species transfer in the boundary layer.

### 6.1.4 Significance of the Boundary Layers

For flow over any surface, there will always exist a velocity boundary layer and hence surface friction. Likewise, a thermal boundary layer, and hence convection heat transfer, will always exist if the surface and free stream temperatures differ. Similarly, a concentration boundary layer and convection mass transfer will exist if the fluid's species concentration at the surface differs from its species concentration in the free stream. The velocity boundary layer is of extent  $\delta(x)$  and is characterized by the presence of velocity gradients and shear stresses. The thermal boundary layer is of extent  $\delta_t(x)$  and is characterized by temperature gradients and heat transfer. Finally, the concentration boundary layer is of extent  $\delta_c(x)$  and is characterized by concentration gradients and species transfer. Situations can arise in which all three boundary layers are present. In such cases, the boundary layers rarely grow at the same rate, and the values of  $\delta$ ,  $\delta_t$ , and  $\delta_c$  at a given location are not the same.

For the engineer, the principal manifestations of the three boundary layers are, respectively, *surface friction*, *convection heat transfer*, and *convection mass transfer*. The key boundary layer parameters are then the *friction coefficient*  $C_f$  and the *heat and mass transfer convection coefficients*  $h$  and  $h_m$ , respectively. We now turn our attention to examining these key parameters, which are central to the analysis of convection heat and mass transfer problems.

## 6.2 Local and Average Convection Coefficients

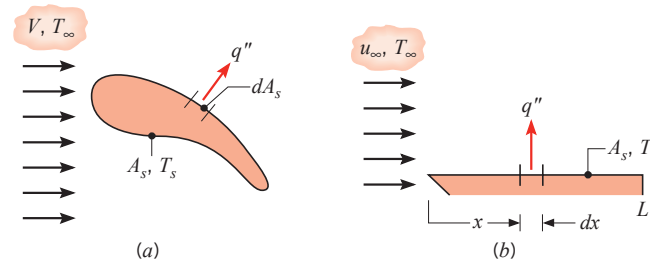
### 6.2.1 Heat Transfer

Consider the conditions of Figure 6.4a. A fluid of velocity  $V$  and temperature  $T_\infty$  flows over a surface of arbitrary shape and of area  $A_s$ . The surface is presumed to be at a uniform temperature,  $T_s$ , and if  $T_s \neq T_\infty$ , we know that convection heat transfer will occur. From Section 6.1.2, we also know that the surface heat flux and convection heat transfer coefficient both vary along the surface. The *total heat transfer rate*  $q$  may be obtained by integrating the local flux over the entire surface. That is,

$$q = \int_{A_s} q'' dA_s \quad (6.10)$$

or, from Equation 6.4,

$$q = (T_s - T_\infty) \int_{A_s} h dA_s \quad (6.11)$$



**FIGURE 6.4** Local and total convection heat transfer. (a) Surface of arbitrary shape. (b) Flat plate.

Defining an *average convection coefficient*  $\bar{h}$  for the entire surface, the total heat transfer rate may also be expressed as

$$q = \bar{h} A_s (T_s - T_\infty) \quad (6.12)$$

Equating Equations 6.11 and 6.12, it follows that the average and local convection coefficients are related by an expression of the form

$$\bar{h} = \frac{1}{A_s} \int_{A_s} h dA_s \quad (6.13)$$

Note that for the special case of flow over a flat plate (Figure 6.4b),  $h$  varies only with the distance  $x$  from the leading edge and Equation 6.13 reduces to

$$\bar{h} = \frac{1}{L} \int_0^L h dx \quad (6.14)$$

### 6.2.2 Mass Transfer

Similar results may be obtained for convection mass transfer. If a fluid of species molar concentration  $C_{A,\infty}$  flows over a surface at which the species concentration is maintained at some uniform value  $C_{A,s} \neq C_{A,\infty}$  (Figure 6.5a), transfer of the species by convection will occur. From Section 6.1.3, we know that the surface molar flux and convection mass transfer coefficient both vary along the surface. The total molar transfer rate for an entire surface,  $N_A$  (kmol/s), may then be expressed as

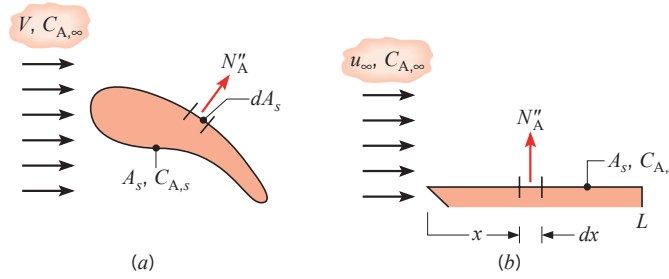
$$N_A = \bar{h}_m A_s (C_{A,s} - C_{A,\infty}) \quad (6.15)$$

where the average and local mass transfer convection coefficients are related by an equation of the form

$$\bar{h}_m = \frac{1}{A_s} \int_{A_s} h_m dA_s \quad (6.16)$$

For the flat plate of Figure 6.5b, it follows that

$$\bar{h}_m = \frac{1}{L} \int_0^L h_m dx \quad (6.17)$$



**FIGURE 6.5** Local and total convection species transfer. (a) Surface of arbitrary shape. (b) Flat plate.

Species transfer may also be expressed as a mass flux,  $n''_A$  ( $\text{kg/s} \cdot \text{m}^2$ ), or as a mass transfer rate,  $n_A$  ( $\text{kg/s}$ ), by multiplying both sides of Equations 6.8 and 6.15, respectively, by the molecular weight  $\mathcal{M}_A$  ( $\text{kg/kmol}$ ) of species A. Accordingly,

$$n''_A = h_m(\rho_{A,s} - \rho_{A,\infty}) \quad (6.18)$$

and

$$n_A = \bar{h}_m A_s(\rho_{A,s} - \rho_{A,\infty}) \quad (6.19)$$

where  $\rho_A$  ( $\text{kg/m}^3$ ) is the mass density of species A.<sup>5</sup> We can also write Fick's law on a mass basis by multiplying Equation 6.7 by  $\mathcal{M}_A$  to yield

$$n''_{A,s} = -D_{AB} \frac{\partial \rho_A}{\partial y} \bigg|_{y=0} \quad (6.20)$$

Furthermore, multiplying the numerator and denominator of Equation 6.9 by  $\mathcal{M}_A$  yields an alternative expression for  $h_m$ :

$$h_m = \frac{-D_{AB} \partial \rho_A / \partial y|_{y=0}}{\rho_{A,s} - \rho_{A,\infty}} \quad (6.21)$$

To perform a convection mass transfer calculation, it is necessary to determine the value of  $C_{A,s}$  or  $\rho_{A,s}$ . Such a determination may be made by assuming thermodynamic equilibrium at the interface between the gas and the liquid or solid phase. One implication of equilibrium is that the temperature of the vapor at the interface is equal to the surface temperature  $T_s$ . A second implication is that the vapor is in a *saturated state*, in which case thermodynamic tables, such as Table A.6 for water, may be used to obtain its density from knowledge of  $T_s$ . To a good approximation, the molar concentration of the vapor at the surface may also be determined from the vapor pressure through application of the equation of state for an ideal gas. That is,

$$C_{A,s} = \frac{p_{\text{sat}}(T_s)}{\mathcal{R}T_s} \quad (6.22)$$

<sup>5</sup>Although the foregoing nomenclature is well suited for characterizing mass transfer processes of interest in this text, there is by no means a standard nomenclature, and it is often difficult to reconcile the results from different publications. A review of the different ways in which driving potentials, fluxes, and convection coefficients may be formulated is provided by Webb [1].



where  $\mathcal{R}$  is the universal gas constant and  $p_{\text{sat}}(T_s)$  is the vapor pressure corresponding to saturation at  $T_s$ . Note that the vapor mass density and molar concentration are related by  $\rho_A = \mathcal{M}_A C_A$ .

### 6.2.3 The Problem of Convection

The local flux and/or the total transfer rate are of paramount importance in any convection problem. These quantities may be determined from the rate equations, Equations 6.4, 6.8, 6.12, and 6.15, which depend on knowledge of the local ( $h$  or  $h_m$ ) and average ( $\bar{h}$  or  $\bar{h}_m$ ) convection coefficients. It is for this reason that determination of these coefficients is viewed as *the problem of convection*. However, the problem is not a simple one, for in addition to depending on numerous *fluid properties* such as density, viscosity, thermal conductivity, and specific heat, the coefficients depend on the *surface geometry* and the *flow conditions*. This multiplicity of independent variables is attributable to the dependence of convection transfer on the boundary layers that develop on the surface.

#### EXAMPLE 6.1

Experimental results for the local heat transfer coefficient  $h_x$  for flow over a flat plate with an extremely rough surface were found to fit the relation

$$h_x(x) = ax^{-0.1}$$

where  $a$  is a coefficient ( $\text{W/m}^{1.9} \cdot \text{K}$ ) and  $x$  (m) is the distance from the leading edge of the plate.

1. Develop an expression for the ratio of the average heat transfer coefficient  $\bar{h}_x$  for a plate of length  $x$  to the local heat transfer coefficient  $h_x$  at  $x$ .
2. Plot the variation of  $h_x$  and  $\bar{h}_x$  as a function of  $x$ .

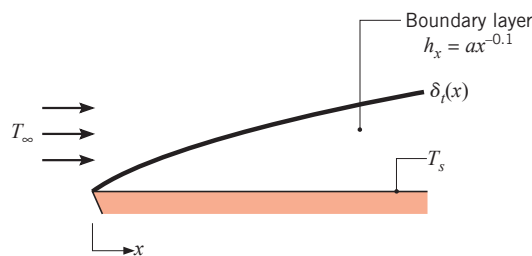
#### SOLUTION

**Known:** Variation of the local heat transfer coefficient,  $h_x(x)$ .

**Find:**

1. The ratio of the average heat transfer coefficient  $\bar{h}(x)$  to the local value  $h_x(x)$ .
2. Plot of the variation of  $h_x$  and  $\bar{h}_x$  with  $x$ .

**Schematic:**



**Analysis:**

1. From Equation 6.14 the average value of the convection heat transfer coefficient over the region from 0 to  $x$  is

$$\bar{h}_x = \bar{h}_x(x) = \frac{1}{x} \int_0^x h_x(x) dx$$

Substituting the expression for the local heat transfer coefficient

$$h_x(x) = ax^{-0.1}$$

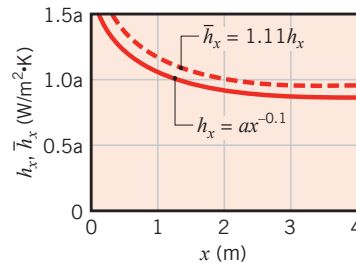
and integrating, we obtain

$$\bar{h}_x = \frac{1}{x} \int_0^x ax^{-0.1} dx = \frac{a}{x} \int_0^x x^{-0.1} dx = \frac{a}{x} \left( \frac{x^{+0.9}}{0.9} \right) = 1.11ax^{-0.1}$$

or

$$\bar{h}_x = 1.11h_x$$

2. The variation of  $h_x$  and  $\bar{h}_x$  with  $x$  is as follows:



**Comments:** Boundary layer development causes both the local and average coefficients to decrease with increasing distance from the leading edge. The average coefficient up to  $x$  must therefore exceed the local value at  $x$ .

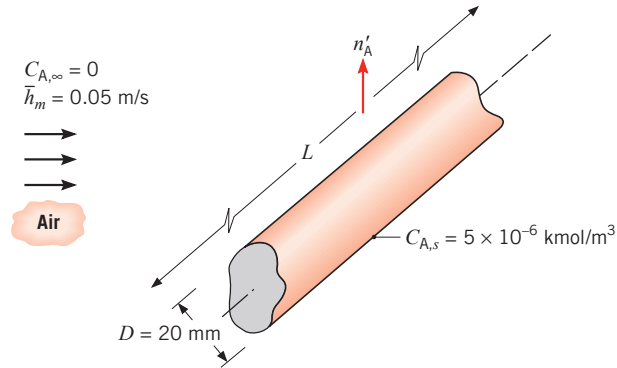
**EXAMPLE 6.2**

A long circular cylinder 20 mm in diameter is fabricated from solid naphthalene, a common moth repellant, and is exposed to an airstream that provides for an average convection mass transfer coefficient of  $\bar{h}_m = 0.05$  m/s. The molar concentration of naphthalene vapor at the cylinder surface is  $5 \times 10^{-6}$  kmol/m<sup>3</sup>, and its molecular weight is 128 kg/kmol. What is the mass sublimation rate per unit length of cylinder?

**SOLUTION**

**Known:** Saturated vapor concentration of naphthalene.

**Find:** Sublimation rate per unit length,  $n'_A$  (kg/s · m).

**Schematic:****Assumptions:**

1. Steady-state conditions.
2. Negligible concentration of naphthalene in free stream of air.

**Analysis:** Naphthalene is transported to the air by convection, and from Equation 6.15, the molar transfer rate for the cylinder is

$$N_A = \bar{h}_m \pi D L (C_{A,s} - C_{A,\infty})$$

With  $C_{A,\infty} = 0$  and  $N'_A = N_A/L$ , it follows that

$$N'_A = (\pi D) \bar{h}_m C_{A,s} = \pi \times 0.02 \text{ m} \times 0.05 \text{ m/s} \times 5 \times 10^{-6} \text{ kmol/m}^3$$

$$N'_A = 1.57 \times 10^{-8} \text{ kmol/s} \cdot \text{m}$$

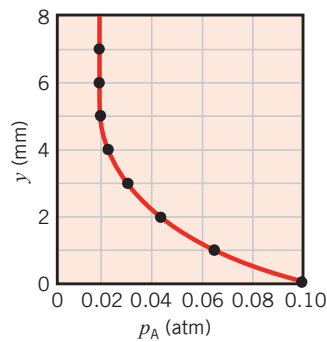
The mass sublimation rate is then

$$n'_A = \mathcal{M}_A N'_A = 128 \text{ kg/kmol} \times 1.57 \times 10^{-8} \text{ kmol/s} \cdot \text{m}$$

$$n'_A = 2.01 \times 10^{-6} \text{ kg/s} \cdot \text{m}$$

**EXAMPLE 6.3**

At some location on the surface of a pan of water, measurements of the partial pressure of water vapor  $p_A$  (atm) are made as a function of the distance  $y$  from the surface, and the results are as follows:



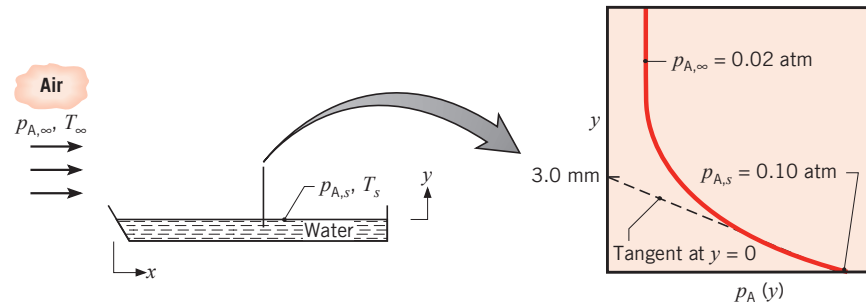
Determine the convection mass transfer coefficient  $h_{m,x}$  at this location.

**SOLUTION**

**Known:** Partial pressure  $p_A$  of water vapor as a function of distance  $y$  at a particular location on the surface of a water layer.

**Find:** Convection mass transfer coefficient at the prescribed location.

**Schematic:**



**Assumptions:**

1. Water vapor may be approximated as an ideal gas.
2. Conditions are isothermal.

**Properties:** Table A.6, saturated vapor (0.1 atm = 0.101 bar):  $T_s = 319$  K. Table A.8, water vapor–air (319 K):  $D_{AB}(319 \text{ K}) = D_{AB}(298 \text{ K}) \times (319 \text{ K}/298 \text{ K})^{3/2} = 0.288 \times 10^{-4} \text{ m}^2/\text{s}$ .

**Analysis:** From Equation 6.21 the local convection mass transfer coefficient is

$$h_{m,x} = \frac{-D_{AB} \partial \rho_A / \partial y|_{y=0}}{\rho_{A,s} - \rho_{A,\infty}}$$

or, approximating the vapor as an ideal gas

$$p_A = \rho_A RT$$

with constant  $T$  (isothermal conditions),

$$h_{m,x} = \frac{-D_{AB} \partial p_A / \partial y|_{y=0}}{p_{A,s} - p_{A,\infty}}$$

From the measured vapor pressure distribution

$$\left. \frac{\partial p_A}{\partial y} \right|_{y=0} = \frac{(0 - 0.1) \text{ atm}}{(0.003 - 0) \text{ m}} = -33.3 \text{ atm/m}$$

Hence

$$h_{m,x} = \frac{-0.288 \times 10^{-4} \text{ m}^2/\text{s} (-33.3 \text{ atm/m})}{(0.1 - 0.02) \text{ atm}} = 0.0120 \text{ m/s} \quad \triangleleft$$

**Comments:** From thermodynamic equilibrium at the liquid–vapor interface, the interfacial temperature,  $T_s = 319$  K, was determined from Table A.6.

## 6.3 Laminar and Turbulent Flow

In the discussion of convection so far, we have not addressed the significance of the *flow conditions*. An essential step in the treatment of any convection problem is to determine whether the boundary layer is *laminar* or *turbulent*. Surface friction and the convection transfer rates depend strongly on which of these conditions exists.

### 6.3.1 Laminar and Turbulent Velocity Boundary Layers

Boundary layer *development* on a flat plate is illustrated in Figure 6.6. In many cases, laminar and turbulent flow conditions both occur, with the laminar section preceding the turbulent section. For either condition, the fluid motion is characterized by velocity components in the  $x$ - and  $y$ -directions. Fluid motion away from the surface is necessitated by the slowing of the fluid near the wall as the boundary layer grows in the  $x$ -direction. Figure 6.6 shows that there are sharp differences between laminar and turbulent flow conditions, as described in the following paragraphs.

In the laminar boundary layer, the fluid flow is highly ordered and it is possible to identify streamlines along which fluid particles move. From Section 6.1.1 we know that the boundary layer thickness grows and that velocity gradients at  $y = 0$  decrease in the streamwise (increasing  $x$ ) direction. From Equation 6.2, we see that the local surface shear stress  $\tau_s$  also decreases with increasing  $x$ . The highly ordered behavior continues until a *transition* zone is reached, across which a conversion from laminar to turbulent conditions occurs. Conditions within the transition zone change with time, with the flow sometimes exhibiting laminar behavior and sometimes exhibiting the characteristics of turbulent flow.

Flow in the fully turbulent boundary layer is, in general, highly irregular and is characterized by random, three-dimensional motion of relatively large parcels of fluid. Mixing within the boundary layer carries high-speed fluid toward the solid surface and transfers slower-moving fluid farther into the free stream. Much of the mixing is promoted by streamwise vortices called *streaks* that are generated intermittently near the flat plate, where they rapidly grow and decay. Recent analytical and experimental studies have suggested that these and other *coherent structures* within the turbulent flow can travel in *waves* at velocities that can exceed  $u_\infty$ , interact nonlinearly, and spawn the chaotic conditions that characterize turbulent flow [2].

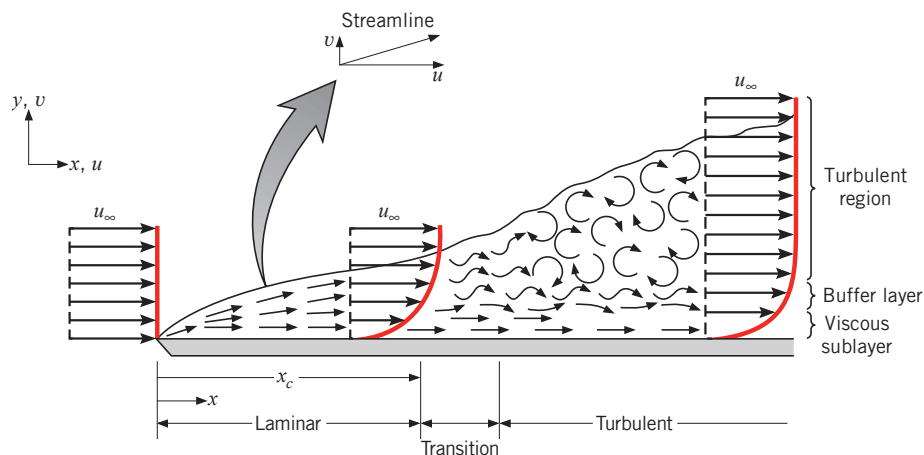


FIGURE 6.6 Velocity boundary layer development on a flat plate.

As a result of the interactions that lead to chaotic flow conditions, velocity and pressure fluctuations occur at any point within the turbulent boundary layer. Three different regions may be delineated within the turbulent boundary layer as a function of distance from the surface. We may speak of a *viscous sublayer* in which transport is dominated by diffusion and the velocity profile is nearly linear. There is an adjoining *buffer layer* in which diffusion and turbulent mixing are comparable, and there is a *turbulent zone* in which transport is dominated by turbulent mixing. A comparison of the laminar and turbulent boundary layer profiles for the  $x$ -component of the velocity, provided in Figure 6.7, shows that the turbulent velocity profile is relatively flat due to the mixing that occurs within the buffer layer and turbulent region, giving rise to large velocity gradients within the viscous sublayer. Hence,  $\tau_s$  is generally larger in the turbulent portion of the boundary layer of Figure 6.6 than in the laminar portion.

The transition from laminar to turbulent flow is ultimately due to *triggering mechanisms*, such as the interaction of unsteady flow structures that develop naturally within the fluid or small disturbances that exist within many typical boundary layers. These disturbances may originate from fluctuations in the free stream, or they may be induced by surface roughness or minute surface vibrations. The onset of turbulence depends on whether the triggering mechanisms are amplified or attenuated in the direction of fluid flow, which in turn depends on a dimensionless grouping of parameters called the *Reynolds number*,

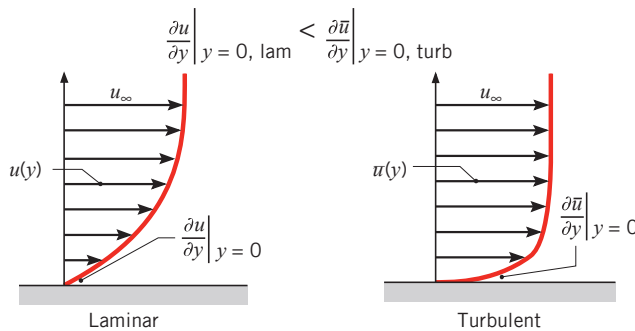
$$Re_x = \frac{\rho u_\infty x}{\mu} \quad (6.23)$$

where, for a flat plate, the characteristic length is  $x$ , the distance from the leading edge. It will be shown later that the Reynolds number represents the ratio of the inertia to viscous forces. If the Reynolds number is small, inertia forces are insignificant relative to viscous forces. The disturbances are then dissipated, and the flow remains laminar. For a large Reynolds number, however, the inertia forces can be sufficient to amplify the triggering mechanisms, and a transition to turbulence occurs.

In determining whether the boundary layer is laminar or turbulent, it is frequently reasonable to assume that transition begins at some location  $x_c$ , as shown in Figure 6.6. This location is determined by the *critical* Reynolds number,  $Re_{x,c}$ . For flow over a flat plate,  $Re_{x,c}$  is known to vary from approximately  $10^5$  to  $3 \times 10^6$ , depending on surface roughness and the turbulence level of the free stream. A representative value of

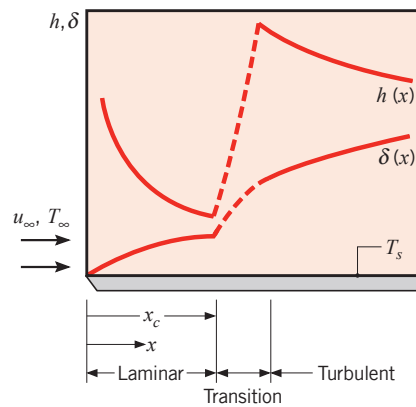
$$Re_{x,c} \equiv \frac{\rho u_\infty x_c}{\mu} = 5 \times 10^5 \quad (6.24)$$

is often assumed for boundary layer calculations and, unless otherwise noted, is used for the calculations of this text that involve a flat plate.



**FIGURE 6.7** Comparison of laminar and turbulent velocity boundary layer profiles for the same free stream velocity.<sup>6</sup>

<sup>6</sup>Since velocity fluctuates with time in turbulent flow, the time-averaged velocity,  $\bar{u}$ , is plotted in Figure 6.7.



**FIGURE 6.8** Variation of velocity boundary layer thickness  $\delta$  and the local heat transfer coefficient  $h$  for flow over an isothermal flat plate.

### 6.3.2 Laminar and Turbulent Thermal and Species Concentration Boundary Layers

Since the velocity distribution determines the advective component of thermal energy or chemical species transfer within the boundary layer, the nature of the flow also has a profound effect on convective heat and mass transfer rates. Similar to the laminar velocity boundary layer, the thermal and species boundary layers grow in the streamwise (increasing  $x$ ) direction, temperature and species concentration gradients in the fluid at  $y = 0$  decrease in the streamwise direction, and, from Equations 6.5 and 6.9, the heat and mass transfer coefficients also decrease with increasing  $x$ .

Just as it induces large velocity gradients at  $y = 0$ , as shown in Figure 6.7, turbulent mixing promotes large temperature and species concentration gradients adjacent to the solid surface as well as a corresponding increase in the heat and mass transfer coefficients across the transition region. These effects are illustrated in Figure 6.8 for the velocity boundary layer thickness  $\delta$  and the local convection heat transfer coefficient  $h$ . Because turbulence induces mixing, which in turn reduces the importance of conduction and diffusion in determining the thermal and species boundary layer thicknesses, *differences* in the thicknesses of the velocity, thermal, and species boundary layers tend to be much smaller in turbulent flow than in laminar flow. As is evident in Equation 6.24, the presence of heat and/or mass transfer can affect the location of the transition from laminar to turbulent flow  $x_c$  since the density and dynamic viscosity of the fluid can depend on the temperature or species concentration.

#### EXAMPLE 6.4

Water flows at a velocity  $u_\infty = 1$  m/s over a flat plate of length  $L = 0.6$  m. Consider two cases, one for which the water temperature is approximately 300 K and the other for an approximate water temperature of 350 K. In the laminar and turbulent regions, experimental measurements show that the local convection coefficients are well described by

$$h_{\text{lam}}(x) = C_{\text{lam}} x^{-0.5} \quad h_{\text{turb}}(x) = C_{\text{turb}} x^{-0.2}$$

where  $x$  has units of m. At 300 K,

$$C_{\text{lam},300} = 395 \text{ W/m}^{1.5} \cdot \text{K} \quad C_{\text{turb},300} = 2330 \text{ W/m}^{1.8} \cdot \text{K}$$

while at 350 K,

$$C_{\text{lam},350} = 477 \text{ W/m}^{1.5} \cdot \text{K} \quad C_{\text{turb},350} = 3600 \text{ W/m}^{1.8} \cdot \text{K}$$

As is evident, the constant  $C$  depends on the nature of the flow as well as the water temperature because of the thermal dependence of various properties of the fluid.

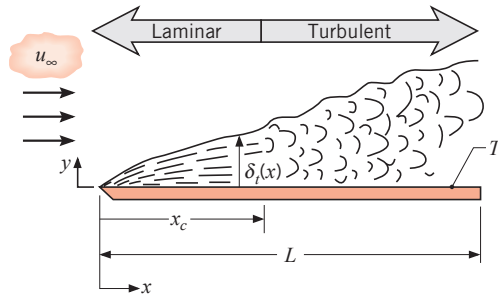
Determine the average convection coefficient,  $\bar{h}$ , over the entire plate for the two water temperatures.

### SOLUTION

**Known:** Water flow over a flat plate, expressions for the dependence of the local convection coefficient with distance from the plate's leading edge  $x$ , and approximate temperature of the water.

**Find:** Average convection coefficient,  $\bar{h}$ .

**Schematic:**



**Assumptions:**

1. Steady-state conditions.
2. Transition occurs at a critical Reynolds number of  $Re_{x,c} = 5 \times 10^5$ .

**Properties:** Table A.6, water ( $\bar{T} \approx 300$  K):  $\rho = v_f^{-1} = 997$  kg/m<sup>3</sup>,  $\mu = 855 \times 10^{-6}$  N·s/m<sup>2</sup>.  
Table A.6 ( $\bar{T} \approx 350$  K):  $\rho = v_f^{-1} = 974$  kg/m<sup>3</sup>,  $\mu = 365 \times 10^{-6}$  N·s/m<sup>2</sup>.

**Analysis:** The local convection coefficient is highly dependent on whether laminar or turbulent conditions exist. Therefore, we first determine the extent to which these conditions exist by finding the location where transition occurs,  $x_c$ . From Equation 6.24, we know that at 300 K,

$$x_c = \frac{Re_{x,c}\mu}{\rho u_\infty} = \frac{5 \times 10^5 \times 855 \times 10^{-6} \text{ N} \cdot \text{s/m}^2}{997 \text{ kg/m}^3 \times 1 \text{ m/s}} = 0.43 \text{ m}$$

while at 350 K,

$$x_c = \frac{Re_{x,c}\mu}{\rho u_\infty} = \frac{5 \times 10^5 \times 365 \times 10^{-6} \text{ N} \cdot \text{s/m}^2}{974 \text{ kg/m}^3 \times 1 \text{ m/s}} = 0.19 \text{ m}$$

From Equation 6.14 we know that

$$\bar{h} = \frac{1}{L} \int_0^L h dx = \frac{1}{L} \left[ \int_0^{x_c} h_{\text{lam}} dx + \int_{x_c}^L h_{\text{turb}} dx \right]$$

or

$$\bar{h} = \frac{1}{L} \left[ \frac{C_{\text{lam}}}{0.5} x^{0.5} \Big|_0^{x_c} + \frac{C_{\text{turb}}}{0.8} x^{0.8} \Big|_{x_c}^L \right]$$



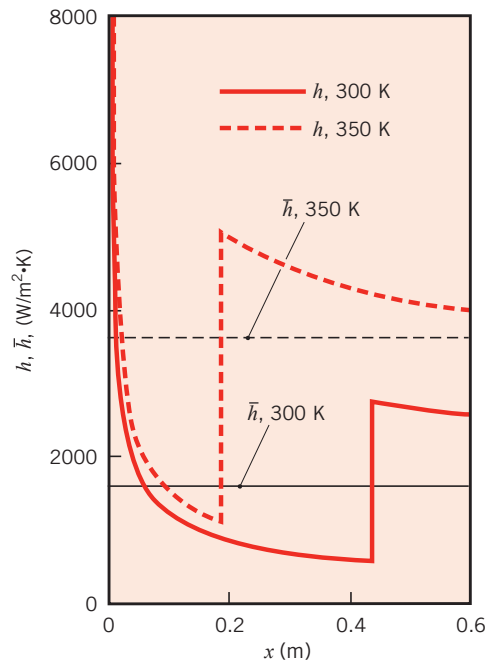
At 300 K,

$$\bar{h} = \frac{1}{0.6 \text{ m}} \left[ \frac{395 \text{ W/m}^{1.5} \cdot \text{K}}{0.5} \times (0.43^{0.5}) \text{ m}^{0.5} + \frac{2330 \text{ W/m}^{1.8} \cdot \text{K}}{0.8} \times (0.6^{0.8} - 0.43^{0.8}) \text{ m}^{0.8} \right] = 1620 \text{ W/m}^2 \cdot \text{K}$$

while at 350 K,

$$\bar{h} = \frac{1}{0.6 \text{ m}} \left[ \frac{477 \text{ W/m}^{1.5} \cdot \text{K}}{0.5} \times (0.19^{0.5}) \text{ m}^{0.5} + \frac{3600 \text{ W/m}^{1.8} \cdot \text{K}}{0.8} \times (0.6^{0.8} - 0.19^{0.8}) \text{ m}^{0.8} \right] = 3710 \text{ W/m}^2 \cdot \text{K}$$

The local and average convection coefficient distributions for the plate are shown in the following figure.



### Comments:

1. The average convection coefficient at  $T \approx 350 \text{ K}$  is over twice as large as the value at  $T \approx 300 \text{ K}$ . This strong temperature dependence is due primarily to the shift of  $x_c$  that is associated with the smaller viscosity of the water at the higher temperature. Careful consideration of the temperature dependence of fluid properties is *crucial* when performing a convection heat transfer analysis.
2. Spatial variations in the local convection coefficient are significant. The largest local convection coefficients occur at the leading edge of the flat plate, where the laminar thermal boundary layer is extremely thin, and just downstream of  $x_c$ , where the turbulent boundary layer is thinnest.

## 6.4 The Boundary Layer Equations

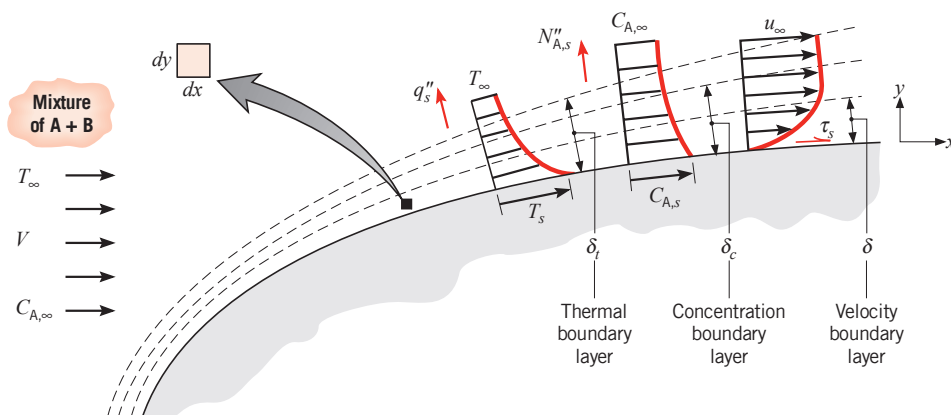
We can improve our understanding of the physical effects that determine boundary layer behavior and further illustrate its relevance to convection transport by considering the equations that govern boundary layer conditions, such as those illustrated in Figure 6.9.

As discussed in Section 6.1, the velocity boundary layer results from the difference between the free stream velocity and the zero velocity at the wall, while the thermal boundary layer results from a difference between the free stream and surface temperatures. The fluid is considered to be a binary mixture of species A and B, and the concentration boundary layer originates from a difference between the free stream and surface concentrations ( $C_{A,\infty} \neq C_{A,s}$ ). Illustration of the relative thicknesses ( $\delta_t > \delta_c > \delta$ ) in Figure 6.9 is arbitrary, for the moment, and the factors that influence relative boundary layer development are discussed later in this chapter.

Our objective in this section is to examine the differential equations that govern the velocity, temperature, and species concentration fields that are applicable to boundary layer flow with heat and species transfer. Section 6.4.1 presents the laminar boundary layer equations, and Appendix F gives the corresponding equations for turbulent conditions. In Section 6.5, these equations will be used to determine important dimensionless parameters associated with convection that will be used extensively in subsequent chapters.

### 6.4.1 Boundary Layer Equations for Laminar Flow

Motion of a fluid in which there are coexisting velocity, temperature, and concentration gradients must comply with several *fundamental laws of nature*. In particular, at each point in the fluid, *conservation of mass, energy, and chemical species*, as well as *Newton's second law of motion*, must be satisfied. Equations representing these requirements are derived by applying the laws to a differential control volume situated in the flow. The resulting equations, in Cartesian coordinates, for the *steady, two-dimensional flow* of an *incompressible fluid* with *constant properties* are given in Appendix E. These equations serve as starting points for our analysis of laminar boundary layers. Note that turbulent flows are inherently unsteady, and the equations governing them are presented in Appendix F.



**FIGURE 6.9** Development of the velocity, thermal, and concentration boundary layers for an arbitrary surface.

These equations are derived in Section 6S.1.

We begin by restricting attention to applications for which *body forces are negligible* ( $X = Y = 0$  in Equations E.2 and E.3), there is *no thermal energy generation in the fluid* ( $\dot{q} = 0$  in Equation E.4), and the *flow is nonreacting* ( $\dot{N}_A = 0$  in Equation E.6). Additional simplifications may be made by invoking approximations pertinent to conditions in the velocity, thermal, and concentration boundary layers. The boundary layer thicknesses are typically very small relative to the size of the object upon which they form, and the  $x$ -direction velocity, temperature, and concentration must change from their surface to their free stream values over these very small distances. Therefore, gradients normal to the object's surface are much larger than those along the surface. As a result, we can neglect terms that represent  $x$ -direction diffusion of momentum, thermal energy, and chemical species, relative to their  $y$ -direction counterparts. That is [3, 4]:

$$\frac{\partial^2 u}{\partial x^2} \ll \frac{\partial^2 u}{\partial y^2} \quad \frac{\partial^2 T}{\partial x^2} \ll \frac{\partial^2 T}{\partial y^2} \quad \frac{\partial^2 C_A}{\partial x^2} \ll \frac{\partial^2 C_A}{\partial y^2} \quad (6.25)$$

By neglecting the  $x$ -direction terms, we are assuming the net shear stress, conduction heat flux, and species diffusion flux in the  $x$ -direction to be negligible. Furthermore, because the boundary layer is so thin, the  $x$ -direction pressure gradient within the boundary layer can be approximated as the free stream pressure gradient:

$$\frac{\partial p}{\partial x} \approx \frac{dp_\infty}{dx} \quad (6.26)$$

The form of  $p_\infty(x)$  depends on the surface geometry and may be obtained from a separate consideration of flow conditions in the free stream where shear stresses are negligible [5]. Hence, the pressure gradient may be treated as a known quantity.

With the foregoing simplifications and approximations, the overall continuity equation is unchanged from Equation E.1:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad (6.27)$$

This equation is an outgrowth of applying conservation of mass to the differential,  $dx \cdot dy \cdot 1$  control volume shown in Figure 6.9. The two terms represent the *net* outflow (outflow minus inflow) of mass in the  $x$ - and  $y$ -directions, the sum of which must be zero for steady flow.

The  $x$ -momentum equation (Equation E.2) reduces to:

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{1}{\rho} \frac{dp_\infty}{dx} + \nu \frac{\partial^2 u}{\partial y^2} \quad (6.28)$$

This equation results from application of Newton's second law of motion in the  $x$ -direction to the  $dx \cdot dy \cdot 1$  differential control volume in the fluid. The left-hand side represents the net rate at which  $x$ -momentum leaves the control volume due to fluid motion across its boundaries. The first term on the right-hand side represents the net pressure force, and the second term represents the net force due to viscous shear stresses.

The energy equation (Equation E.4) reduces to

$$u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \alpha \frac{\partial^2 T}{\partial y^2} + \frac{\nu}{c_p} \left( \frac{\partial u}{\partial y} \right)^2 \quad (6.29)$$

This equation results from application of conservation of energy to the  $dx \cdot dy \cdot 1$  differential control volume in the flowing fluid. Terms on the left-hand side account for the net rate at which thermal energy leaves the control volume due to bulk fluid motion (advection). The first term on the right-hand side accounts for the net inflow of thermal energy due to  $y$ -direction conduction. The last term on the right-hand side is what remains of the viscous dissipation, Equation E.5, when it is acknowledged that, in a boundary layer, the velocity component in the direction along the surface,  $u$ , is much larger than that normal to the surface,  $v$ , and gradients normal to the surface are much larger than those along the surface. In many situations this term may be neglected relative to those that account for advection and conduction. However, aerodynamic heating that accompanies high-speed (especially supersonic) flight is a noteworthy situation in which this term is important.

The species conservation equation (Equation E.6) reduces to

$$u \frac{\partial C_A}{\partial x} + v \frac{\partial C_A}{\partial y} = D_{AB} \frac{\partial^2 C_A}{\partial y^2} \quad (6.30)$$

This equation is obtained by applying conservation of chemical species to the  $dx \cdot dy \cdot 1$  differential control volume in the flow. Terms on the left-hand side account for net transport of species A due to bulk fluid motion (advection), while the right-hand side represents the net inflow due to  $y$ -direction diffusion.

After specifying appropriate boundary conditions, Equations 6.27 through 6.30 may be solved to determine the spatial variations of  $u$ ,  $v$ ,  $T$ , and  $C_A$  in the different laminar boundary layers. For incompressible, constant property flow, Equations 6.27 and 6.28 are *uncoupled* from Equations 6.29 and 6.30. That is, Equations 6.27 and 6.28 may be solved for the *velocity field*,  $u(x, y)$  and  $v(x, y)$ , without consideration of Equations 6.29 and 6.30. From knowledge of  $u(x, y)$ , the velocity gradient  $(\partial u / \partial y)|_{y=0}$  could then be evaluated, and the wall shear stress could be obtained from Equation 6.2. In contrast, through the appearance of  $u$  and  $v$  in Equations 6.29 and 6.30, the temperature and species concentration are *coupled* to the velocity field. Hence  $u(x, y)$  and  $v(x, y)$  must be known before Equations 6.29 and 6.30 may be solved for  $T(x, y)$  and  $C_A(x, y)$ . Once  $T(x, y)$  and  $C_A(x, y)$  have been obtained from such solutions, the convection heat and mass transfer coefficients may be determined from Equations 6.5 and 6.9, respectively. It follows that these coefficients depend strongly on the velocity field.<sup>7</sup>

Because boundary layer solutions generally involve mathematics beyond the scope of this text, our treatment of such solutions will be restricted to the analysis of laminar parallel

<sup>7</sup>Special attention must be given to the effect of species transfer on the velocity boundary layer. Recall that velocity boundary layer development is generally characterized by the existence of zero fluid velocity *at the surface*. This condition pertains to the velocity component  $v$  normal to the surface, as well as to the velocity component  $u$  along the surface. However, if there is simultaneous mass transfer to or from the surface, it is evident that  $v$  can no longer be zero at the surface. Nevertheless, for the mass transfer problems of interest in this text, it is reasonable to assume that  $v = 0$  at the surface, which is equivalent to assuming that mass transfer has a negligible effect on the velocity boundary layer. The assumption is appropriate for many problems involving evaporation or sublimation from gas–liquid or gas–solid interfaces, respectively. It is not appropriate, however, for *mass transfer cooling* problems that involve large surface mass transfer rates [6]. In addition, we note that, with mass transfer, the boundary layer fluid is a binary mixture of species A and B, and its properties should be those of the mixture. However, in all problems in this text,  $C_A \ll C_B$  and it is reasonable to assume that the boundary layer properties (such as  $k$ ,  $\mu$ ,  $c_p$ , etc.) are those of species B.

flow over a flat plate (Section 7.2 and Appendix G). However, other analytical solutions are discussed in advanced texts on convection [7–9], and detailed boundary layer solutions may be obtained by using numerical (finite-difference or finite-element) techniques [10]. It is also essential to recognize that a wide array of situations of engineering relevance involve turbulent convective heat transfer, which is both mathematically and physically more complex than laminar convection. The boundary layer equations for turbulent flow are included in Appendix F.

It is important to stress that we have not developed the laminar boundary layer equations primarily for the purpose of obtaining solutions. Rather, we have been motivated mainly by two other considerations. One motivation has been to obtain an appreciation for the physical processes that occur in boundary layers. These processes affect wall friction as well as energy and species transfer in the boundary layers. A second important motivation arises from the fact that the equations may be used to identify key *boundary layer similarity parameters*, as well as important *analogies* between *momentum*, *heat*, and *mass* transfer that have numerous practical applications. The laminar governing equations will be used for this purpose in Sections 6.5 through 6.7, but the same key parameters and analogies hold true for turbulent conditions as well.

### 6.4.2 Compressible Flow

The equations of the foregoing section and Appendix E are restricted to incompressible flows, that is, for cases where the fluid density can be treated as constant.<sup>8</sup> Flows in which the fluids experience significant density changes as a result of *pressure variations* associated with the fluid motion are deemed to be compressible. The treatment of convection heat transfer associated with *compressible flow* is beyond the scope of this text. Although liquids may nearly always be treated as incompressible, density variations in flowing gases should be considered when the velocity approaches or exceeds the speed of sound. Specifically, a gradual transition from incompressible to compressible flow in gases occurs at a critical Mach number of  $Ma_c \approx 0.3$ , where  $Ma \equiv V/a$  and  $V$  and  $a$  are the gas velocity and speed of sound, respectively [11, 12]. For an ideal gas,  $a = \sqrt{\gamma RT}$  where  $\gamma$  is the ratio of specific heats,  $\gamma \equiv c_p/c_v$ ,  $R$  is the gas constant, and the temperature is expressed in kelvins. As an example, for air at  $T = 300\text{ K}$  and  $p = 1\text{ atm}$ , we may assume ideal gas behavior. The gas constant is  $R \equiv \mathcal{R}/\mathcal{M} = 8315\text{ J/kmol} \cdot \text{K}/28.7\text{ kg/kmol} = 287\text{ J/kg} \cdot \text{K}$  and  $c_v \equiv c_p - R = 1007\text{ J/kg} \cdot \text{K} - 287\text{ J/kg} \cdot \text{K} = 720\text{ J/kg} \cdot \text{K}$ . The ratio of specific heats is therefore  $\gamma = c_p/c_v = 1007\text{ J/kg} \cdot \text{K}/720\text{ J/kg} \cdot \text{K} = 1.4$ , and the speed of sound is  $a = \sqrt{1.4 \times 287\text{ J/kg} \cdot \text{K} \times 300\text{ K}} = 347\text{ m/s}$ . Hence air flowing at 300 K must be treated as being compressible if  $V > 0.3 \times 347\text{ m/s} \approx 100\text{ m/s}$ .

Since the material in Chapters 6 through 9 is restricted to incompressible or *low-speed* flow, it is important to confirm that compressibility effects are not important when utilizing the material to solve a convection heat transfer problem.<sup>9</sup>

<sup>8</sup>Chapter 9 addresses flows that arise due to the variation of density with temperature. These *free convection* flows can nearly always be treated as if the fluid is incompressible but with an extra term in the momentum equation to account for buoyancy forces.

<sup>9</sup>Turbulence and compressibility often coincide, since large velocities can lead to large Reynolds and Mach numbers. It can be shown (Problem 6.26) that, for sufficiently small geometries, any flow that is turbulent is also compressible.

## 6.5 Boundary Layer Similarity: The Normalized Boundary Layer Equations

If we examine Equations 6.28, 6.29, and 6.30 we note a strong similarity. In fact, if the pressure gradient appearing in Equation 6.28 and the viscous dissipation term of Equation 6.29 are negligible, the three equations are of the same form. *Each equation is characterized by advection terms on the left-hand side and a diffusion term on the right-hand side.* This situation describes *low-speed, forced convection flows* which are found in many engineering applications. Implications of this similarity may be developed in a rational manner by first *nondimensionalizing* the governing equations.

### 6.5.1 Boundary Layer Similarity Parameters

The boundary layer equations are *normalized* by first defining dimensionless independent variables of the forms

$$x^* \equiv \frac{x}{L} \quad \text{and} \quad y^* \equiv \frac{y}{L} \quad (6.31)$$

where  $L$  is a *characteristic length* for the surface of interest (e.g., the length of a flat plate). Moreover, dependent dimensionless variables may also be defined as

$$u^* \equiv \frac{u}{V} \quad \text{and} \quad v^* \equiv \frac{v}{V} \quad (6.32)$$

where  $V$  is the velocity upstream of the surface (Figure 6.9), and as

$$T^* \equiv \frac{T - T_s}{T_\infty - T_s} \quad (6.33)$$

$$C_A^* \equiv \frac{C_A - C_{A,s}}{C_{A,\infty} - C_{A,s}} \quad (6.34)$$

The dimensional variables may then be written in terms of the new dimensionless variables (for example, from Equation 6.31  $x \equiv x^*L$  and  $y \equiv y^*L$ ) and the resulting expressions for  $x$ ,  $y$ ,  $u$ ,  $v$ ,  $T$ , and  $C_A$  may be substituted into Equations 6.28, 6.29, and 6.30 to obtain the dimensionless forms of the conservation equations shown in Table 6.1. Note that viscous dissipation has been neglected and that  $p^* \equiv (p_\infty/\rho V^2)$  is a dimensionless pressure. The  $y$ -direction boundary conditions required to solve the equations are also shown in the table.

By normalizing the boundary layer equations, three very important dimensionless *similarity parameters* evolve and are introduced in Table 6.1. They are the Reynolds number,  $Re_L$ ; Prandtl number,  $Pr$ ; and Schmidt number,  $Sc$ . Such similarity parameters are important because they allow us to apply results obtained for a surface experiencing one set of convective conditions to *geometrically similar* surfaces experiencing entirely different conditions. These conditions may vary, for example, with the fluid, the fluid velocity as described by the free stream value  $V$ , and/or the size of the surface as described by the characteristic length  $L$ . As long as the similarity parameters *and* dimensionless boundary conditions are the same for two sets of conditions, the solutions of the differential equations of Table 6.1 for the nondimensional velocity, temperature, and species concentration will be *identical*. This concept will be amplified in the remainder of this section.

**TABLE 6.1** The boundary layer equations and their  $y$ -direction boundary conditions in nondimensional form

Boundary Layer	Conservation Equation	Boundary Conditions		Similarity Parameter(s)
		Wall	Free Stream	
Velocity	$u^* \frac{\partial u^*}{\partial x^*} + v^* \frac{\partial u^*}{\partial y^*} = - \frac{dp^*}{dx^*} + \frac{1}{Re_L} \frac{\partial^2 u^*}{\partial y^{*2}} \quad (6.35)$	$u^*(x^*, 0) = 0$	$u^*(x^*, \infty) = \frac{u_\infty(x^*)}{V}$	$Re_L = \frac{VL}{\nu} \quad (6.41)$
Thermal	$u^* \frac{\partial T^*}{\partial x^*} + v^* \frac{\partial T^*}{\partial y^*} = \frac{1}{Re_L Pr} \frac{\partial^2 T^*}{\partial y^{*2}} \quad (6.36)$	$T^*(x^*, 0) = 0$	$T^*(x^*, \infty) = 1$	$Re_L Pr = \frac{\nu}{\alpha} \quad (6.42)$
Concentration	$u^* \frac{\partial C_A^*}{\partial x^*} + v^* \frac{\partial C_A^*}{\partial y^*} = \frac{1}{Re_L Sc} \frac{\partial^2 C_A^*}{\partial y^{*2}} \quad (6.37)$	$C_A^*(x^*, 0) = 0$	$C_A^*(x^*, \infty) = 1$	$Re_L Sc = \frac{\nu}{D_{AB}} \quad (6.43)$
Convection Similarity Parameters $Re_L$ , $Pr$ , and $Sc$				



### 6.5.2 Functional Form of the Solutions

Equations 6.35 through 6.43 in Table 6.1 are extremely useful from the standpoint of suggesting how important boundary layer results may be simplified and generalized. The momentum equation (6.35) suggests that, although conditions in the velocity boundary layer depend on the fluid properties  $\rho$  and  $\mu$ , the velocity  $V$ , and the length scale  $L$ , this dependence may be simplified by grouping these variables in the form of the Reynolds number. We therefore anticipate that the solution to Equation 6.35 will be of the functional form

$$u^* = f\left(x^*, y^*, Re_L, \frac{dp^*}{dx^*}\right) \quad (6.44)$$

Since the pressure distribution  $p^*(x^*)$  depends on the surface geometry and may be obtained independently by considering flow conditions in the free stream, the appearance of  $dp^*/dx^*$  in Equation 6.44 represents the influence of geometry on the velocity distribution.

From Equations 6.2, 6.31, and 6.32, the shear stress at the surface,  $y^* = 0$ , may be expressed as

$$\tau_s = \mu \left. \frac{\partial u}{\partial y} \right|_{y=0} = \left( \frac{\mu V}{L} \right) \left. \frac{\partial u^*}{\partial y^*} \right|_{y^*=0}$$

and from Equations 6.1 and 6.41, it follows that the friction coefficient is

#### Friction Coefficient

$$C_f = \frac{\tau_s}{\rho V^2/2} = \frac{2}{Re_L} \left. \frac{\partial u^*}{\partial y^*} \right|_{y^*=0} \quad (6.45)$$

From Equation 6.44 we also know that

$$\left. \frac{\partial u^*}{\partial y^*} \right|_{y^*=0} = f\left(x^*, Re_L, \frac{dp^*}{dx^*}\right)$$

Hence, for a prescribed geometry Equation 6.45 may be expressed as

$$C_f = \frac{2}{Re_L} f(x^*, Re_L) \quad (6.46)$$

The significance of this result should not be overlooked. Equation 6.46 states that the friction coefficient, a dimensionless parameter of considerable importance to the engineer, may be expressed exclusively in terms of a dimensionless space coordinate and the Reynolds number. Hence, for a prescribed geometry we expect the function that relates  $C_f$  to  $x^*$  and  $Re_L$  to be *universally* applicable. That is, we expect it to apply to different fluids and over a wide range of values for  $V$  and  $L$ .

Similar results may be obtained for the convection coefficients of heat and mass transfer. Intuitively, we might anticipate that  $h$  depends on the fluid properties ( $k$ ,  $c_p$ ,  $\mu$ , and  $\rho$ ), the fluid velocity  $V$ , the length scale  $L$ , and the surface geometry. However, Equation 6.36 suggests the manner in which this dependence may be simplified. In particular, the solution to this equation may be expressed in the form

$$T^* = f\left(x^*, y^*, Re_L, Pr, \frac{dp^*}{dx^*}\right) \quad (6.47)$$



where the dependence on  $dp^*/dx^*$  originates from the influence of the geometry on the fluid motion ( $u^*$  and  $v^*$ ), which, in turn, affects the thermal conditions. Once again the term  $dp^*/dx^*$  represents the effect of surface geometry. From the definition of the convection coefficient, Equation 6.5, and the dimensionless variables, Equations 6.31 and 6.33, we also obtain

$$h = -\frac{k_f(T_\infty - T_s)}{L} \frac{\partial T^*}{\partial y^*} \bigg|_{y^*=0} = +\frac{k_f}{L} \frac{\partial T^*}{\partial y^*} \bigg|_{y^*=0}$$

This expression suggests defining an important dependent dimensionless parameter termed the Nusselt number.

**Nusselt Number:**

$$Nu \equiv \frac{hL}{k_f} = + \frac{\partial T^*}{\partial y^*} \bigg|_{y^*=0} \quad (6.48)$$

This parameter is equal to the dimensionless temperature gradient at the surface, and it provides a measure of the convection heat transfer occurring at the surface. From Equation 6.47 it follows that, *for a prescribed geometry*,

$$Nu = f(x^*, Re_L, Pr) \quad (6.49)$$

The Nusselt number is to the thermal boundary layer what the friction coefficient is to the velocity boundary layer. Equation 6.49 implies that for a given geometry, the Nusselt number must be some *universal function* of  $x^*$ ,  $Re_L$ , and  $Pr$ . If this function were known, it could be used to compute the value of  $Nu$  for different fluids and for different values of  $V$  and  $L$ . From knowledge of  $Nu$ , the local convection coefficient  $h$  may be found and the *local* heat flux may then be computed from Equation 6.4. Moreover, since the *average* heat transfer coefficient is obtained by integrating over the surface of the body, it must be independent of the spatial variable  $x^*$ . Hence the functional dependence of the *average* Nusselt number is

$$\overline{Nu} = \frac{\overline{h}L}{k_f} = f(Re_L, Pr) \quad (6.50)$$

Similarly, it may be argued that, for mass transfer in a gas flow over an evaporating liquid or a sublimating solid, the convection mass transfer coefficient  $h_m$  depends on the properties  $D_{AB}$ ,  $\rho$ , and  $\mu$ , the velocity  $V$ , and the characteristic length  $L$ . However, Equation 6.37 suggests that this dependence may be simplified. The solution to this equation must be of the form

$$C_A^* = f\left(x^*, y^*, Re_L, Sc, \frac{dp^*}{dx^*}\right) \quad (6.51)$$

where the dependence on  $dp^*/dx^*$  again originates from the influence of the fluid motion. From the definition of the convection coefficient, Equation 6.9, and the dimensionless variables, Equations 6.31 and 6.34, we know that

$$h_m = -\frac{D_{AB}}{L} \frac{(C_{A,\infty} - C_{A,s})}{(C_{A,s} - C_{A,\infty})} \frac{\partial C_A^*}{\partial y^*} \bigg|_{y^*=0} = +\frac{D_{AB}}{L} \frac{\partial C_A^*}{\partial y^*} \bigg|_{y^*=0}$$

Hence we may define a dependent dimensionless parameter termed the Sherwood number ( $Sh$ ).

**Sherwood Number:**

$$Sh \equiv \frac{h_m L}{D_{AB}} = + \left. \frac{\partial C_A^*}{\partial y^*} \right|_{y^*=0} \quad (6.52)$$

This parameter is equal to the dimensionless concentration gradient at the surface, and it provides a measure of the convection mass transfer occurring at the surface. From Equation 6.51 it follows that, *for a prescribed geometry*,

$$Sh = f(x^*, Re_L, Sc) \quad (6.53)$$

The Sherwood number is to the concentration boundary layer what the Nusselt number is to the thermal boundary layer, and Equation 6.53 implies that it must be a universal function of  $x^*$ ,  $Re_L$ , and  $Sc$ . As for the Nusselt number, it is also possible to work with an *average* Sherwood number that depends on only  $Re_L$  and  $Sc$ .

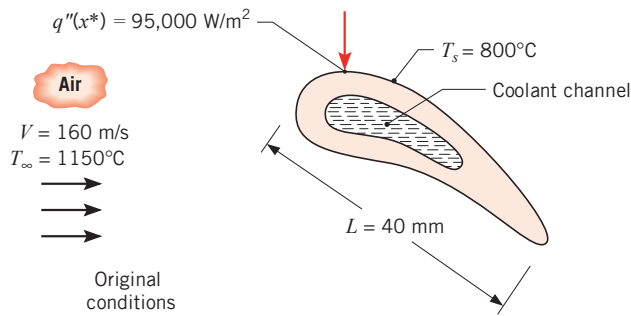
$$\overline{Sh} = \frac{\overline{h}_m L}{D_{AB}} = f(Re_L, Sc) \quad (6.54)$$

From the foregoing development we have obtained the relevant dimensionless parameters for low-speed, forced-convection boundary layers. We have done so by nondimensionalizing the differential equations that describe the physical processes occurring within the boundary layers. An alternative approach could have involved the use of dimensional analysis in the form of the Buckingham pi theorem [5]. However, the success of that method depends on one's ability to select, largely from intuition, the various parameters that influence a problem. For example, knowing beforehand that  $\overline{h} = f(k, c_p, \rho, \mu, V, L)$ , one could use the Buckingham pi theorem to obtain Equation 6.50. However, having begun with the differential form of the conservation equations, we have eliminated the guesswork and have established the similarity parameters in a rigorous fashion.

The importance of an expression such as Equation 6.50 should be fully appreciated. It states that values of the average heat transfer coefficient  $\overline{h}$ , whether obtained theoretically, experimentally, or numerically, can be completely represented in terms of only three dimensionless groups, instead of the original seven dimensional parameters. The convenience and power afforded by such simplification will become evident in Chapters 7 through 10. Moreover, once the form of the functional dependence of Equation 6.50 has been obtained for a particular surface geometry, let us say from laboratory measurements, it is known to be *universally* applicable. By this we mean that it may be applied for different fluids, velocities, and length scales, as long as the assumptions implicit in the originating boundary layer equations remain valid (e.g., negligible viscous dissipation and body forces).

### EXAMPLE 6.5

Experimental tests using air as the working fluid are conducted on a portion of the turbine blade shown in the sketch. The heat flux to the blade at a particular point ( $x^*$ ) on the surface is measured to be  $q'' = 95,000 \text{ W/m}^2$ . To maintain a steady-state surface temperature of  $800^\circ\text{C}$ , heat transferred to the blade is removed by circulating a coolant inside the blade.



1. Determine the heat flux to the blade at  $x^*$  if its temperature is reduced to  $T_{s,1} = 700^\circ\text{C}$  by increasing the coolant flow.
2. Determine the heat flux at the same dimensionless location  $x^*$  for a similar turbine blade having a chord length of  $L = 80\text{ mm}$ , when the blade operates in an airflow at  $T_\infty = 1150^\circ\text{C}$  and  $V = 80\text{ m/s}$ , with  $T_s = 800^\circ\text{C}$ .

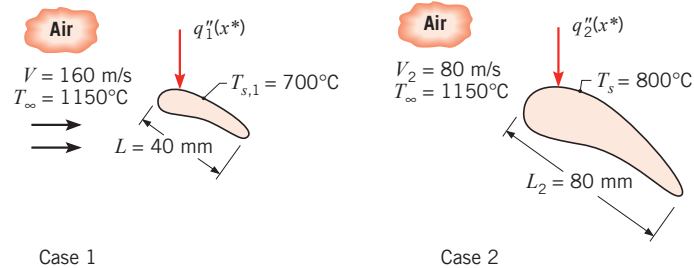
**SOLUTION**

**Known:** Operating conditions of an internally cooled turbine blade.

**Find:**

1. Heat flux to the blade at a point  $x^*$  when the surface temperature is reduced.
2. Heat flux at the same dimensionless location to a larger turbine blade of the same shape with reduced air velocity.

**Schematic:**



**Assumptions:**

1. Steady-state, incompressible flow.
2. Constant air properties.

**Analysis:**

1. When the surface temperature is  $800^\circ\text{C}$ , the local convection heat transfer coefficient between the surface and the air at  $x^*$  can be obtained from Newton's law of cooling:

$$q'' = h(T_\infty - T_s)$$

Thus,

$$h = \frac{q''}{(T_\infty - T_s)}$$

We proceed without calculating the value for now. From Equation 6.49, it follows that, for the prescribed geometry,

$$Nu = \frac{hL}{k} = f(x^*, Re_L, Pr)$$

Hence, since there is no change in  $x^*$ ,  $Re_L$ , or  $Pr$  associated with a change in  $T$ , for constant properties, the local Nusselt number is unchanged. Moreover, since  $L$  and  $k$  are unchanged, the local convection coefficient remains the same. Thus, when the surface temperature is reduced to  $700^\circ\text{C}$ , the heat flux may be obtained from Newton's law of cooling, using the same local convection coefficient:

$$\begin{aligned} q_1'' &= h(T_\infty - T_{s,1}) = \frac{q''}{(T_\infty - T_s)}(T_\infty - T_{s,1}) = \frac{95,000 \text{ W/m}^2}{(1150 - 800)^\circ\text{C}} (1150 - 700)^\circ\text{C} \\ &= 122,000 \text{ W/m}^2 \end{aligned} \quad \triangleleft$$

2. To determine the heat flux at  $x^*$  associated with the larger blade and the reduced airflow (case 2), we first note that, although  $L$  has increased by a factor of 2, the velocity has decreased by the same factor and the Reynolds number has not changed. That is,

$$Re_{L,2} = \frac{V_2 L_2}{\nu} = \frac{VL}{\nu} = Re_L$$

Accordingly, since  $x^*$  and  $Pr$  are also unchanged, the local Nusselt number remains the same.

$$Nu_2 = Nu$$

Because the characteristic length is different, however, the local convection coefficient changes, where

$$\frac{h_2 L_2}{k} = \frac{hL}{k} \quad \text{or} \quad h_2 = h \frac{L}{L_2} = \frac{q''}{(T_\infty - T_s)} \frac{L}{L_2}$$

The heat flux at  $x^*$  is then

$$\begin{aligned} q_2'' &= h_2(T_\infty - T_s) = q'' \frac{(T_\infty - T_s)}{(T_\infty - T_s)} \frac{L}{L_2} \\ q_2'' &= 95,000 \text{ W/m}^2 \times \frac{0.04 \text{ m}}{0.08 \text{ m}} = 47,500 \text{ W/m}^2 \end{aligned} \quad \triangleleft$$

### Comments:

1. If the Reynolds numbers for the two situations of part 2 differed, that is,  $Re_{L,2} \neq Re_L$ , the local heat flux  $q_2''$  could be obtained only if the particular functional dependence of Equation 6.49 were known. Such forms are provided for many different shapes in subsequent chapters.
2. Air temperatures in the boundary layer range from the blade surface temperature  $T_s$  to the ambient value  $T_\infty$ . Hence, as will become evident in Section 7.1 representative air properties could be evaluated at arithmetic mean or *film* temperatures of  $T_{f,1} = (T_{s,1} + T_\infty)/2 = (700^\circ\text{C} + 1150^\circ\text{C})/2 = 925^\circ\text{C}$  and  $T_{f,2} = (800^\circ\text{C} + 1150^\circ\text{C})/2 = 975^\circ\text{C}$ , respectively. Based on properties corresponding to these film temperatures, the

Reynolds (and Nusselt) numbers for the two cases would be slightly different. However, the difference would not be large enough to significantly change the calculated value of the local heat flux for case 2.

3. At  $T = 1150^\circ\text{C} = 1423\text{ K}$ ,  $c_v \equiv c_p - R = 1167\text{ J/kg}\cdot\text{K} - 287\text{ J/kg}\cdot\text{K} = 880\text{ J/kg}\cdot\text{K}$ , and the specific heat ratio is  $\gamma = c_p/c_v = 1167\text{ J/kg}\cdot\text{K}/880\text{ J/kg}\cdot\text{K} = 1.33$ . Assuming the air behaves as an ideal gas, the speed of sound in the air is  $a = \sqrt{\gamma RT} = \sqrt{1.33 \times 287\text{ J/kg}\cdot\text{K} \times 1423\text{ K}} = 736\text{ m/s}$ . Therefore  $Ma = V/a = 0.22$  and  $0.11$  for cases 1 and 2, respectively. Hence the flow is incompressible in both cases. If the flow were to be compressible for either case, the Nusselt number would also depend on the Mach number, and the two cases would not be similar.

### EXAMPLE 6.6

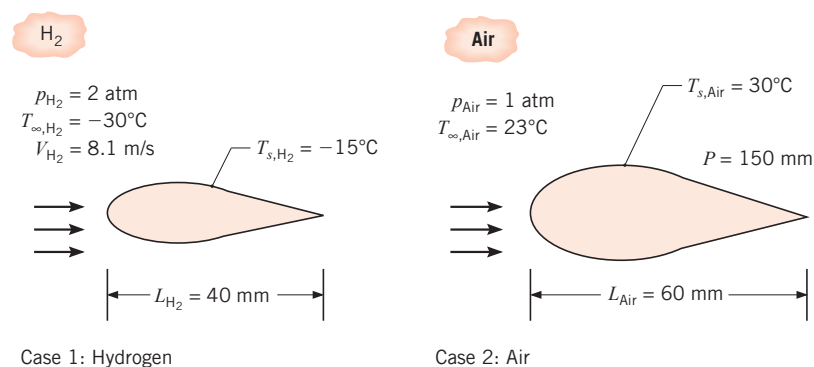
Consider convective cooling of a two-dimensional streamlined strut of characteristic length  $L_{\text{H}_2} = 40\text{ mm}$ . The strut is exposed to hydrogen flowing at  $p_{\text{H}_2} = 2\text{ atm}$ ,  $V_{\text{H}_2} = 8.1\text{ m/s}$ , and  $T_{\infty, \text{H}_2} = -30^\circ\text{C}$ . Of interest is the value of the average heat transfer coefficient  $\bar{h}_{\text{H}_2}$ , when the surface temperature is  $T_{s, \text{H}_2} = -15^\circ\text{C}$ . Rather than conducting expensive experiments involving pressurized hydrogen, an engineer proposes to take advantage of similarity by performing wind tunnel experiments using air at atmospheric pressure with  $T_{\infty, \text{Air}} = 23^\circ\text{C}$ . A geometrically similar strut of characteristic length  $L_{\text{Air}} = 60\text{ mm}$  and perimeter  $P = 150\text{ mm}$  is placed in the wind tunnel. Measurements reveal a surface temperature of  $T_{s, \text{Air}} = 30^\circ\text{C}$  when the heat loss per unit object length (into the page) is  $q'_{\text{Air}} = 50\text{ W/m}$ . Determine the required air velocity in the wind tunnel experiment  $V_{\text{Air}}$  and the average convective heat transfer coefficient in the hydrogen  $\bar{h}_{\text{H}_2}$ .

### SOLUTION

**Known:** Flow across a strut. Hydrogen pressure, velocity, and temperature. Air temperature and pressure, as well as heat loss per unit length. Surface temperatures of the strut in hydrogen and in air.

**Find:** Air velocity and average convective heat transfer coefficient for the strut that is exposed to hydrogen.

**Schematic:**



**Assumptions:**

1. Steady-state, incompressible boundary layer behavior.
2. Ideal gas behavior.
3. Constant properties.
4. Negligible viscous dissipation.

**Properties:** Table A.4, air ( $p = 1 \text{ atm}$ ,  $T_f = (23^\circ\text{C} + 30^\circ\text{C})/2 = 26.5^\circ\text{C} \approx 300 \text{ K}$ ):  $Pr = 0.707$ ,  $\nu = 15.89 \times 10^{-6} \text{ m}^2/\text{s}$ ,  $k = 26.3 \times 10^{-3} \text{ W/m} \cdot \text{K}$ . Table A.4 hydrogen ( $p = 1 \text{ atm}$ ,  $T_f = -22.5^\circ\text{C} \approx 250 \text{ K}$ ):  $Pr = 0.707$ ,  $\nu = 81.4 \times 10^{-6} \text{ m}^2/\text{s}$ ,  $k = 157 \times 10^{-3} \text{ W/m} \cdot \text{K}$ .

The properties  $k$ ,  $Pr$ ,  $c_p$ , and  $\mu$  may be assumed to be independent of pressure to an excellent approximation. However, for a gas, the kinematic viscosity  $\nu = \mu/\rho$  will vary with pressure through its dependence on density. From the ideal gas law,  $\rho = p/RT$ , it follows that the ratio of kinematic viscosities for a gas at the same temperature but at different pressures,  $p_1$  and  $p_2$ , is  $(\nu_1/\nu_2) = (p_2/p_1)$ . Hence, the kinematic viscosity of hydrogen at 250 K and 2 atm is  $\nu_{\text{H}_2} = 81.4 \times 10^{-6} \text{ m}^2/\text{s} \times 1 \text{ atm}/2 \text{ atm} = 40.7 \times 10^{-6} \text{ m}^2/\text{s}$ . Since  $Pr$  is independent of pressure,  $Pr_{\text{H}_2}(p = 2 \text{ atm}, T_f = -22.5^\circ\text{C}) = Pr_{\text{Air}}(p = 1 \text{ atm}, T_f = 26.5^\circ\text{C}) = 0.707$ .

**Analysis:** From Equation 6.50, we know that the average Nusselt numbers are related to the Reynolds and Prandtl numbers by the functional dependence

$$\overline{Nu}_{\text{H}_2} = \frac{\bar{h}_{\text{H}_2} L_{\text{H}_2}}{k_{\text{H}_2}} = f(Re_{L, \text{H}_2}, Pr_{\text{H}_2}) \quad \text{and} \quad \overline{Nu}_{\text{Air}} = \frac{\bar{h}_{\text{Air}} L_{\text{Air}}}{k_{\text{Air}}} = f(Re_{L, \text{Air}}, Pr_{\text{Air}})$$

Since  $Pr_{\text{H}_2} = Pr_{\text{Air}}$ , similarity exists if  $Re_{L, \text{Air}} = Re_{L, \text{H}_2}$ , in which case the average Nusselt numbers for the air and hydrogen will be identical,  $\overline{Nu}_{\text{Air}} = \overline{Nu}_{\text{H}_2}$ . Equating the Reynolds numbers for the hydrogen and air yields the expression

$$\begin{aligned} V_{\text{Air}} &= \frac{Re_{L, \text{Air}} \nu_{\text{Air}}}{L_{\text{Air}}} = \frac{Re_{L, \text{H}_2} \nu_{\text{Air}}}{L_{\text{Air}}} = \frac{V_{\text{H}_2} L_{\text{H}_2} \nu_{\text{Air}}}{\nu_{\text{H}_2} L_{\text{Air}}} \\ &= \frac{8.1 \text{ m/s} \times 0.04 \text{ m} \times 15.89 \times 10^{-6} \text{ m}^2/\text{s}}{40.7 \times 10^{-6} \text{ m}^2/\text{s} \times 0.06 \text{ m}} = 2.10 \text{ m/s} \end{aligned} \quad \triangleleft$$

With  $Re_{L, \text{Air}} = Re_{L, \text{H}_2}$  and  $Pr_{\text{Air}} = Pr_{\text{H}_2}$ , we may equate the Nusselt numbers for the hydrogen and air, and incorporate Newton's law of cooling. Doing so gives

$$\begin{aligned} \bar{h}_{\text{H}_2} &= \bar{h}_{\text{Air}} \frac{L_{\text{Air}} k_{\text{H}_2}}{L_{\text{H}_2} k_{\text{Air}}} = \frac{q'_{\text{Air}}}{P(T_{s, \text{Air}} - T_{\infty, \text{Air}})} \times \frac{L_{\text{Air}} k_{\text{H}_2}}{L_{\text{H}_2} k_{\text{Air}}} \\ &= \frac{50 \text{ W/m}}{150 \times 10^{-3} \text{ m} \times (30 - 23)^\circ\text{C}} \times \frac{0.06 \text{ m} \times 0.157 \text{ W/m} \cdot \text{K}}{0.04 \text{ m} \times 0.0263 \text{ W/m} \cdot \text{K}} = 426 \text{ W/m}^2 \cdot \text{K} \end{aligned} \quad \triangleleft$$

**Comments:**

1. The fluid properties are evaluated at the arithmetic mean of the free stream and surface temperatures. As will become evident in Section 7.1, the temperature dependence of fluid properties is often accounted for by evaluating properties at the film temperature,  $T_f = (T_s + T_\infty)/2$ .
2. Experiments involving pressurized hydrogen can be relatively expensive because care must be taken to prevent leakage of this small-molecule, flammable gas.

## 6.6 Physical Interpretation of the Dimensionless Parameters

All of the foregoing dimensionless parameters have physical interpretations that relate to conditions in the flow, not only for boundary layers but also for other flow types, such as the internal flows we will see in Chapter 8. Consider the *Reynolds number*  $Re_L$  (Equation 6.41), which may be interpreted as the *ratio of inertia to viscous forces* in a region of characteristic dimension  $L$ . Inertia forces are associated with an increase in the momentum of a moving fluid. From Equation 6.28, it is evident that these forces (per unit mass) are of the form  $u\partial u/\partial x$ , in which case an order-of-magnitude approximation gives  $F_I \approx V^2/L$ . Similarly, the net shear force (per unit mass) is found on the right-hand side of Equation 6.28 as  $\nu(\partial^2 u/\partial y^2)$  and may be approximated as  $F_s \approx \nu V/L^2$ . Therefore, the ratio of forces is

$$\frac{F_I}{F_s} \approx \frac{\rho V^2/L}{\mu V/L^2} = \frac{\rho V L}{\mu} = Re_L$$

We therefore expect inertia forces to dominate for large values of  $Re_L$  and viscous forces to dominate for small values of  $Re_L$ .

There are several important implications of this result. Recall from Section 6.3.1 that the Reynolds number determines the existence of laminar or turbulent flow. We should also expect the magnitude of the Reynolds number to influence the velocity boundary layer thickness  $\delta$ . With increasing  $Re_L$  at a fixed location on a surface, we expect viscous forces to become less influential relative to inertia forces. Hence the effects of viscosity do not penetrate as far into the free stream, and the value of  $\delta$  diminishes.

The *Prandtl number* is defined as the ratio of the kinematic viscosity, also referred to as the momentum diffusivity,  $\nu$ , to the thermal diffusivity  $\alpha$ . It is therefore a fluid property. The Prandtl number provides a *measure of the relative effectiveness of momentum and energy transport by diffusion in the velocity and thermal boundary layers*, respectively. From Table A.4 we see that the Prandtl number of gases is near unity, in which case energy and momentum transfer by diffusion are comparable. In a liquid metal (Table A.7),  $Pr \ll 1$  and the energy diffusion rate greatly exceeds the momentum diffusion rate. The opposite is true for oils (Table A.5), for which  $Pr \gg 1$ . From this interpretation it follows that the value of  $Pr$  strongly influences the relative growth of the velocity and thermal boundary layers. In fact for laminar boundary layers (in which transport by diffusion is *not* overshadowed by turbulent mixing), it is reasonable to expect that

$$\frac{\delta}{\delta_t} \approx Pr^n \quad (6.55)$$

where  $n$  is a positive exponent. Hence for a gas  $\delta_t \approx \delta$ ; for a liquid metal  $\delta_t \gg \delta$ ; for an oil  $\delta_t \ll \delta$ .

Similarly, the *Schmidt number*, which is defined in Equation 6.43, is a fluid property and provides a *measure of the relative effectiveness of momentum and mass transport by diffusion in the velocity and concentration boundary layers*, respectively. For convection mass transfer in laminar flows, it therefore determines the relative velocity and concentration boundary layer thicknesses, where

$$\frac{\delta}{\delta_c} \approx Sc^n \quad (6.56)$$



Another dimensionless fluid property, which is related to  $Pr$  and  $Sc$ , is the *Lewis number* ( $Le$ ). It is defined as

$$Le = \frac{\alpha}{D_{AB}} = \frac{Sc}{Pr} \quad (6.57)$$

and is relevant to any situation involving simultaneous heat and mass transfer by convection. From Equations 6.55 through 6.57, it then follows that

$$\frac{\delta_t}{\delta_c} \approx Le^n \quad (6.58)$$

The Lewis number is therefore a measure of the relative thermal and concentration boundary layer thicknesses. For most applications it is reasonable to assume a value of  $n = 1/3$  in Equations 6.55, 6.56, and 6.58.

Table 6.2 lists the dimensionless groups that appear frequently in heat and mass transfer. The list includes groups already considered, as well as those yet to be introduced for special conditions. As a new group is confronted, its definition and interpretation should be committed to memory. Note that the *Grashof number* provides a measure of the ratio of buoyancy forces to viscous forces in the velocity boundary layer. Its role in free convection (Chapter 9) is much the same as that of the Reynolds number in forced convection. The *Eckert number* provides a measure of the kinetic energy of the flow relative to the enthalpy difference across the thermal boundary layer. It plays an important role in high-speed flows for which viscous dissipation is significant. Note also that, although similar in form, the Nusselt and Biot numbers differ in both definition and interpretation. Whereas the Nusselt number is defined in terms of the thermal conductivity of the fluid, the Biot number is based on the solid thermal conductivity, Equation 5.9.

**TABLE 6.2** Selected dimensionless groups of heat and mass transfer

Group	Definition	Interpretation
Biot number ( $Bi$ )	$\frac{hL}{k_s}$	Ratio of the internal thermal resistance of a solid to the boundary layer thermal resistance
Mass transfer Biot number ( $Bi_m$ )	$\frac{h_m L}{D_{AB}}$	Ratio of the internal species transfer resistance to the boundary layer species transfer resistance
Bond number ( $Bo$ )	$\frac{g(\rho_l - \rho_v)L^2}{\sigma}$	Ratio of gravitational and surface tension forces
Coefficient of friction ( $C_f$ )	$\frac{\tau_s}{\rho V^2/2}$	Dimensionless surface shear stress
Eckert number ( $Ec$ )	$\frac{V^2}{c_p(T_s - T_\infty)}$	Kinetic energy of the flow relative to the boundary layer enthalpy difference
Fourier number ( $Fo$ )	$\frac{\alpha t}{L^2}$	Ratio of the heat conduction rate to the rate of thermal energy storage in a solid. Dimensionless time
Mass transfer Fourier number ( $Fo_m$ )	$\frac{D_{AB} t}{L^2}$	Ratio of the species diffusion rate to the rate of species storage. Dimensionless time
Friction factor ( $f$ )	$\frac{\Delta p}{(L/D)(\rho u_m^2/2)}$	Dimensionless pressure drop for internal flow



**TABLE 6.2** Continued

Group	Definitio	Interpretation
Grashof number ( $Gr_L$ )	$\frac{g\beta(T_s - T_\infty)L^3}{\nu^2}$	Measure of the ratio of buoyancy forces to viscous forces
Colburn $j$ factor ( $j_H$ )	$St Pr^{2/3}$	Dimensionless heat transfer coefficient
Colburn $j$ factor ( $j_m$ )	$St_m Sc^{2/3}$	Dimensionless mass transfer coefficient
Jakob number ( $Ja$ )	$\frac{c_p(T_s - T_{sat})}{h_{fg}}$	Ratio of sensible to latent energy absorbed during liquid–vapor phase change
Lewis number ( $Le$ )	$\frac{\alpha}{D_{AB}}$	Ratio of the thermal and mass diffusivities
Mach number ( $Ma$ )	$\frac{V}{a}$	Ratio of velocity to speed of sound
Nusselt number ( $Nu_L$ )	$\frac{hL}{k_f}$	Ratio of convection to pure conduction heat transfer
Peclet number ( $Pe_L$ )	$\frac{VL}{\alpha} = Re_L Pr$	Ratio of advection to conduction heat transfer rates
Prandtl number ( $Pr$ )	$\frac{c_p\mu}{k} = \frac{\nu}{\alpha}$	Ratio of the momentum and thermal diffusivities
Reynolds number ( $Re_L$ )	$\frac{VL}{\nu}$	Ratio of the inertia and viscous forces
Schmidt number ( $Sc$ )	$\frac{\nu}{D_{AB}}$	Ratio of the momentum and mass diffusivities
Sherwood number ( $Sh_L$ )	$\frac{h_m L}{D_{AB}}$	Dimensionless concentration gradient at the surface
Stanton number ( $St$ )	$\frac{h}{\rho V c_p} = \frac{Nu_L}{Re_L Pr}$	Modified Nusselt number
Mass transfer Stanton number ( $St_m$ )	$\frac{h_m}{V} = \frac{Sh_L}{Re_L Sc}$	Modified Sherwood number
Weber number ( $We$ )	$\frac{\rho V^2 L}{\sigma}$	Ratio of inertia to surface tension forces

## 6.7 Boundary Layer Analogies

As engineers, our interest in boundary layer behavior is directed principally toward the dimensionless parameters  $C_f$ ,  $Nu$ , and  $Sh$ . From knowledge of these parameters, we may compute the wall shear stress and the convection heat and mass transfer rates. It is therefore understandable that expressions that relate  $C_f$ ,  $Nu$ , and  $Sh$  to each other can be useful tools in convection analysis. Such expressions are available in the form of *boundary layer analogies*.

### 6.7.1 The Heat and Mass Transfer Analogy

If two or more processes are governed by dimensionless equations of the same form, the processes are said to be *analogous*. Clearly, then, from Equations 6.36 and 6.37 and the boundary conditions, Equations 6.39 and 6.40, of Table 6.1, convection heat and mass transfer are analogous. Each of the differential equations is composed of advection and diffusion terms of the same form. Moreover, as shown in Equations 6.36 and 6.37, each equation is related to the velocity field through  $Re_L$ , and the parameters  $Pr$  and  $Sc$  assume analogous roles. One implication of this analogy is that dimensionless relations that govern thermal boundary layer behavior must be of the same form as those that govern the concentration boundary layer. Hence the boundary layer temperature and concentration profiles must also be of the same functional form if the applied boundary conditions are analogous.

Recalling the discussion of Section 6.5.2, the features of which are summarized in Table 6.3, an important result of the heat and mass transfer analogy may be obtained. From the foregoing paragraph, it follows that Equation 6.47 must be of the same functional form as Equation 6.51. From Equations 6.48 and 6.52, it then follows that the dimensionless temperature and concentration gradients evaluated at the surface, and therefore the values of  $Nu$  and  $Sh$ , are analogous. Similarly, expressions for the average Nusselt and Sherwood numbers, Equations 6.50 and 6.54, respectively, are also of the same form. *Accordingly, heat and mass transfer relations for a particular geometry are interchangeable.* If, for example, one has performed a set of heat transfer experiments to determine the functional form of Equation 6.49 for a particular surface geometry, the results may be used for convection mass transfer involving the same geometry, simply by replacing  $Nu$  with  $Sh$  and  $Pr$  with  $Sc$ .

The analogy may also be used to directly relate the two convection coefficients. In subsequent chapters we will find that  $Nu$  and  $Sh$  are generally proportional to  $Pr^n$  and  $Sc^n$ , respectively, where  $n$  is a positive exponent less than 1. Anticipating this dependence, we use Equations 6.49 and 6.53 to obtain

$$Nu = f(x^*, Re_L)Pr^n \quad \text{and} \quad Sh = f(x^*, Re_L)Sc^n$$

in which case, with equivalent functions,  $f(x^*, Re_L)$ ,

$$\frac{Nu}{Pr^n} = \frac{Sh}{Sc^n} \quad (6.59)$$

**TABLE 6.3** Functional relations pertinent to the boundary layer analogies

Fluid Flow	Heat Transfer	Mass Transfer
$u^* = f\left(x^*, y^*, Re_L, \frac{dp^*}{dx^*}\right) \quad (6.44)$	$T^* = f\left(x^*, y^*, Re_L, Pr, \frac{dp^*}{dx^*}\right) \quad (6.47)$	$C_A^* = f\left(x^*, y^*, Re_L, Sc, \frac{dp^*}{dx^*}\right) \quad (6.51)$
$C_f = \frac{2}{Re_L} \frac{\partial u^*}{\partial y^*} \bigg _{y^*=0} \quad (6.45)$	$Nu = \frac{hL}{k} = + \frac{\partial T^*}{\partial y^*} \bigg _{y^*=0} \quad (6.48)$	$Sh = \frac{h_m L}{D_{AB}} = + \frac{\partial C_A^*}{\partial y^*} \bigg _{y^*=0} \quad (6.52)$
$C_f = \frac{2}{Re_L} f(x^*, Re_L) \quad (6.46)$	$Nu = f(x^*, Re_L, Pr) \quad (6.49)$	$Sh = f(x^*, Re_L, Sc) \quad (6.53)$
	$\overline{Nu} = f(Re_L, Pr) \quad (6.50)$	$\overline{Sh} = f(Re_L, Sc) \quad (6.54)$

Substituting from Equations 6.48 and 6.52 we then obtain

$$\frac{hL/k}{Pr^n} = \frac{h_m L/D_{AB}}{Sc^n}$$

or, from Equation 6.57,

$$\frac{h}{h_m} = \frac{k}{D_{AB} Le^n} = \rho c_p Le^{1-n} \quad (6.60)$$

This result may often be used to determine one convection coefficient, for example,  $h_m$ , from knowledge of the other coefficient. The same relation may be applied to the average coefficients  $\bar{h}$  and  $\bar{h}_m$ , and it may be used in turbulent, as well as laminar, flow. For most applications it is reasonable to assume a value of  $n = \frac{1}{3}$ .

### EXAMPLE 6.7

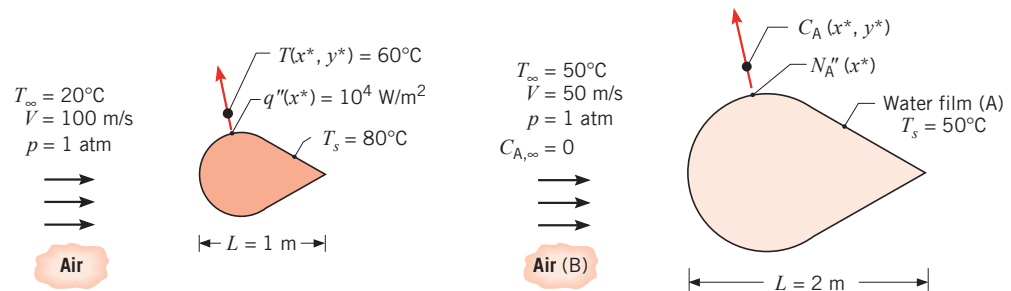
A solid of arbitrary shape is suspended in atmospheric air having a free stream temperature and velocity of 20°C and 100 m/s, respectively. The solid has a characteristic length of 1 m, and its surface is maintained at 80°C. Under these conditions measurements of the convection heat flux at a particular point ( $x^*$ ) on the surface and of the temperature in the boundary layer above this point ( $x^*, y^*$ ) reveal values of  $10^4$  W/m<sup>2</sup> and 60°C, respectively. A mass transfer operation is to be effected for a second solid having the same shape but a characteristic length of 2 m. In particular, a thin film of water on the solid is to be evaporated in dry atmospheric air having a free stream velocity of 50 m/s, with the air and the solid both at a temperature of 50°C. What are the molar concentration and the species molar flux of the water vapor at a location ( $x^*, y^*$ ) corresponding to the point at which the temperature and heat flux measurements were made in the first case?

### SOLUTION

**Known:** A boundary layer temperature and heat flux at a location on a solid in an airstream of prescribed temperature and velocity.

**Find:** Water vapor concentration and flux associated with the same location on a larger surface of the same shape.

### Schematic:



Case 1: heat transfer

Case 2: mass transfer

**Assumptions:**

1. Steady-state, two-dimensional, incompressible boundary layer behavior; constant properties.
2. Boundary layer approximations are valid.
3. Negligible viscous dissipation.
4. Mole fraction of water vapor in concentration boundary layer is much less than unity.

**Properties:** Table A.4, air (50°C):  $\nu = 18.2 \times 10^{-6} \text{ m}^2/\text{s}$ ,  $k = 28 \times 10^{-3} \text{ W/m}\cdot\text{K}$ ,  $Pr = 0.70$ . Table A.6, saturated water vapor (50°C):  $\rho_{A,\text{sat}} = v_g^{-1} = 0.082 \text{ kg/m}^3$ . Table A.8, water vapor–air (50°C):  $D_{AB} \approx 0.26 \times 10^{-4} \text{ m}^2/\text{s}$ .

**Analysis:** The desired molar concentration and flux may be determined by invoking the analogy between heat and mass transfer. From Equations 6.47 and 6.51, we know that

$$T^* \equiv \frac{T - T_s}{T_\infty - T_s} = f\left(x^*, y^*, Re_L, Pr, \frac{dp^*}{dx^*}\right)$$

and

$$C_A^* \equiv \frac{C_A - C_{A,s}}{C_{A,\infty} - C_{A,s}} = f\left(x^*, y^*, Re_L, Sc, \frac{dp^*}{dx^*}\right)$$

However, for case 1

$$Re_{L,1} = \frac{V_1 L_1}{\nu} = \frac{100 \text{ m/s} \times 1 \text{ m}}{18.2 \times 10^{-6} \text{ m}^2/\text{s}} = 5.5 \times 10^6, \quad Pr = 0.70$$

while for case 2

$$Re_{L,2} = \frac{V_2 L_2}{\nu} = \frac{50 \text{ m/s} \times 2 \text{ m}}{18.2 \times 10^{-6} \text{ m}^2/\text{s}} = 5.5 \times 10^6$$

$$Sc = \frac{\nu}{D_{AB}} = \frac{18.2 \times 10^{-6} \text{ m}^2/\text{s}}{26 \times 10^{-6} \text{ m}^2/\text{s}} = 0.70$$

Since  $Re_{L,1} = Re_{L,2}$ ,  $Pr = Sc$ ,  $x_1^* = x_2^*$ ,  $y_1^* = y_2^*$ , and the surface geometries are the same, it follows that the temperature and concentration distributions have the same functional form. Hence

$$\frac{C_A(x^*, y^*) - C_{A,s}}{C_{A,\infty} - C_{A,s}} = \frac{T(x^*, y^*) - T_s}{T_\infty - T_s} = \frac{60 - 80}{20 - 80} = 0.33$$

or, with  $C_{A,\infty} = 0$ ,

$$C_A(x^*, y^*) = C_{A,s}(1 - 0.33) = 0.67C_{A,s}$$

With

$$C_{A,s} = C_{A,\text{sat}}(50^\circ\text{C}) = \frac{\rho_{A,\text{sat}}}{\mathcal{M}_A} = \frac{0.082 \text{ kg/m}^3}{18 \text{ kg/kmol}} = 0.0046 \text{ kmol/m}^3$$

it follows that

$$C_A(x^*, y^*) = 0.67 (0.0046 \text{ kmol/m}^3) = 0.0031 \text{ kmol/m}^3 \quad \triangleleft$$

The molar flux may be obtained from Equation 6.8

$$N_A''(x^*) = h_m(C_{A,s} - C_{A,\infty})$$

with  $h_m$  evaluated from the analogy. From Equations 6.49 and 6.53 we know that, since  $x_1^* = x_2^*$ ,  $Re_{L,1} = Re_{L,2}$ , and  $Pr = Sc$ , the corresponding functional forms are equivalent. Hence

$$Sh = \frac{h_m L_2}{D_{AB}} = Nu = \frac{h L_1}{k}$$

With  $h = q''/(T_s - T_\infty)$  from Newton's law of cooling,

$$h_m = \frac{L_1}{L_2} \times \frac{D_{AB}}{k} \times \frac{q''}{(T_s - T_\infty)} = \frac{1}{2} \times \frac{0.26 \times 10^{-4} \text{ m}^2/\text{s}}{0.028 \text{ W/m} \cdot \text{K}} \times \frac{10^4 \text{ W/m}^2}{(80 - 20)^\circ\text{C}}$$

$$h_m = 0.077 \text{ m/s}$$

Hence

$$N_A''(x^*) = 0.077 \text{ m/s} (0.0046 - 0.0) \text{ kmol/m}^3$$

or

$$N_A''(x^*) = 3.54 \times 10^{-4} \text{ kmol/s} \cdot \text{m}^2 \quad \triangleleft$$

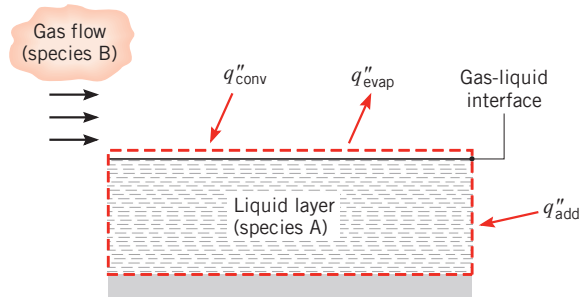
### Comments:

1. Recognize that, since the mole fraction of water vapor in the concentration boundary layer is small, the kinematic viscosity of air ( $\nu_B$ ) may be used to evaluate  $Re_{L,2}$ .
2. Air properties for case 1 are evaluated at the film temperature,  $T_f = (T_s + T_\infty)/2 = (80^\circ\text{C} + 20^\circ\text{C})/2 = 50^\circ\text{C}$ .

## 6.7.2 Evaporative Cooling

An important application of the heat and mass transfer analogy is to the process of *evaporative cooling*, which occurs when a gas flows over a liquid (Figure 6.10). Evaporation occurs from the liquid surface, and the energy associated with the phase change is the latent heat of vaporization of the liquid. Evaporation occurs when liquid molecules near the surface experience collisions that increase their energy above that needed to overcome the surface binding energy. The energy required to sustain the evaporation must come from the internal energy of the liquid, which would then experience a reduction in temperature (the cooling effect). However, if steady-state conditions are to be maintained, the latent energy lost by the liquid because of evaporation must be replenished by energy transfer to the liquid from its surroundings. Neglecting radiation effects, this transfer may be due to the convection of sensible energy from the gas or to heat addition by other means, as, for example, by an electrical heater submerged in the liquid. Applying conservation of energy to a control surface about the liquid (Equation 1.12c), it follows that, for a unit surface area,

$$q''_{\text{conv}} + q''_{\text{add}} = q''_{\text{evap}} \quad (6.61)$$



**FIGURE 6.10** Latent and sensible heat exchange at a gas–liquid interface.

where  $q''_{\text{evap}}$  may be approximated as the product of the evaporative mass flux and the latent heat of vaporization

$$q''_{\text{evap}} = n''_A h_{fg} \quad (6.62)$$

If there is no heat addition by other means, Equation 6.61 reduces to a balance between convection heat transfer from the gas and the evaporative heat loss from the liquid. Substituting from Equations 6.4, 6.18, and 6.62, Equation 6.61 may then be expressed as

$$h(T_\infty - T_s) = h_{fg} h_m [\rho_{A,\text{sat}}(T_s) - \rho_{A,\infty}] \quad (6.63)$$

where the vapor density at the surface is that associated with saturated conditions at  $T_s$ . Hence the magnitude of the cooling effect may be expressed as

$$T_\infty - T_s = h_{fg} \left( \frac{h_m}{h} \right) [\rho_{A,\text{sat}}(T_s) - \rho_{A,\infty}] \quad (6.64)$$

Substituting for  $(h_m/h)$  from Equation 6.60 and for the vapor densities from the ideal gas law, the cooling effect may also be expressed as

$$(T_\infty - T_s) = \frac{\mathcal{M}_A h_{fg}}{\mathcal{R} \rho c_p Le^{2/3}} \left[ \frac{p_{A,\text{sat}}(T_s)}{T_s} - \frac{p_{A,\infty}}{T_\infty} \right] \quad (6.65)$$

In the interest of accuracy, the gas (species B) properties  $\rho$ ,  $c_p$ , and  $Le$  should be evaluated at the arithmetic mean, or film, temperature of the thermal boundary layer,  $T_{\text{am}} = T_f = (T_s + T_\infty)/2$ , while the latent heat of vaporization of species A,  $h_{fg}$ , should be evaluated at the surface temperature  $T_s$ . A representative value of  $n = \frac{1}{3}$  has been assumed for the  $Pr$  and  $Sc$  exponent of Equation 6.60.

Numerous environmental and industrial applications of the foregoing results arise for situations in which the gas is *air* and the liquid is *water*.

### EXAMPLE 6.8

A container, which is wrapped in a fabric that is continually moistened with a highly volatile liquid, may be used to keep beverages cool in hot, arid regions. Suppose that the container is placed in dry ambient air at 40°C, with heat and mass transfer between the wetting agent and the air occurring by forced convection. The wetting agent is known to have a molecular weight of 200 kg/kmol and a latent heat of vaporization of 100 kJ/kg. Its saturated vapor pressure for the prescribed conditions is approximately 5000 N/m<sup>2</sup>, and the

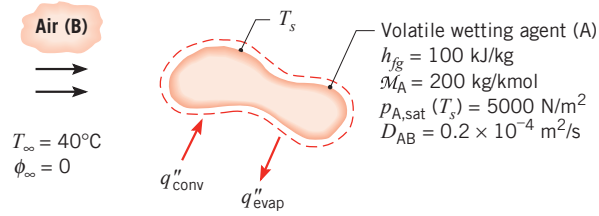
diffusion coefficient of the vapor in air is  $0.2 \times 10^{-4} \text{ m}^2/\text{s}$ . What is the steady-state temperature of the beverage?

### SOLUTION

**Known:** Properties of wetting agent used to evaporatively cool a beverage container.

**Find:** Steady-state temperature of beverage.

**Schematic:**



**Assumptions:**

1. Heat and mass transfer analogy is applicable.
2. Vapor displays ideal gas behavior.
3. Radiation effects are negligible.
4. Air properties may be evaluated at a mean boundary layer temperature assumed to be  $T_f = 300 \text{ K}$ .

**Properties:** Table A.4, air (300 K):  $\rho = 1.16 \text{ kg/m}^3$ ,  $c_p = 1.007 \text{ kJ/kg} \cdot \text{K}$ ,  $\alpha = 22.5 \times 10^{-6} \text{ m}^2/\text{s}$ .

**Analysis:** Subject to the foregoing assumptions, the evaporative cooling effect is given by Equation 6.65.

$$(T_\infty - T_s) = \frac{\mathcal{M}_A h_{fg}}{\mathcal{R} \rho c_p Le^{2/3}} \left[ \frac{p_{A,sat}(T_s)}{T_s} - \frac{p_{A,\infty}}{T_\infty} \right]$$

Setting  $p_{A,\infty} = 0$  and rearranging, it follows that

$$T_s^2 - T_\infty T_s + B = 0$$

where the coefficient  $B$  is

$$B = \frac{\mathcal{M}_A h_{fg} p_{A,sat}}{\mathcal{R} \rho c_p Le^{2/3}}$$

or

$$\begin{aligned} B &= [200 \text{ kg/kmol} \times 100 \text{ kJ/kg} \times 5000 \text{ N/m}^2 \times 10^{-3} \text{ kJ/N} \cdot \text{m}] \\ &\div \left[ 8.315 \text{ kJ/kmol} \cdot \text{K} \times 1.16 \text{ kg/m}^3 \times 1.007 \text{ kJ/kg} \cdot \text{K} \right. \\ &\quad \left. \times \left( \frac{22.5 \times 10^{-6} \text{ m}^2/\text{s}}{20 \times 10^{-6} \text{ m}^2/\text{s}} \right)^{2/3} \right] = 9518 \text{ K}^2 \end{aligned}$$

Hence

$$T_s = \frac{T_\infty \pm \sqrt{T_\infty^2 - 4B}}{2} = \frac{313 \text{ K} \pm \sqrt{(313)^2 - 4(9518)} \text{ K}}{2}$$

Rejecting the minus sign on physical grounds ( $T_s$  must equal  $T_\infty$  if there is no evaporation, in which case  $p_{A,\text{sat}} = 0$  and  $B = 0$ ), it follows that

$$T_s = 278.9 \text{ K} = 5.9^\circ\text{C}$$

**Comments:** The result is independent of the shape of the container as long as the heat and mass transfer analogy may be used.

### 6.7.3 The Reynolds Analogy

A second boundary layer analogy may be obtained by noting from Table 6.1 that, for  $dp^*/dx^* = 0$  and  $Pr = Sc = 1$ , the boundary layer equations, Equations 6.35 through 6.37, are of precisely the same form. For a flat plate parallel to the incoming flow, we have  $dp^*/dx^* = 0$  and there is no variation in the free stream velocity outside the boundary layer. With  $u_\infty = V$ , Equations 6.38 through 6.40 also have the same form. Hence the functional forms of the solutions for  $u^*$ ,  $T^*$ , and  $C_A^*$ , Equations 6.44, 6.47, and 6.51, must be equivalent. From Equations 6.45, 6.48, and 6.52, it follows that

$$C_f \frac{Re_L}{2} = Nu = Sh \quad (6.66)$$

Replacing  $Nu$  and  $Sh$  by the *Stanton number* ( $St$ ) and the *mass transfer Stanton number* ( $St_m$ ), respectively,

$$St \equiv \frac{h}{\rho V c_p} = \frac{Nu}{Re Pr} \quad (6.67)$$

$$St_m \equiv \frac{h_m}{V} = \frac{Sh}{Re Sc} \quad (6.68)$$

Equation 6.66 may also be expressed in the form

$$\frac{C_f}{2} = St = St_m \quad (6.69)$$

Equation 6.69 is known as the *Reynolds analogy*, and it relates the key engineering parameters of the velocity, thermal, and concentration boundary layers. If the velocity parameter is known, the analogy may be used to obtain the other parameters, and vice versa. However, there are restrictions associated with using this result. In addition to relying on the validity of the boundary layer approximations, the accuracy of Equation 6.69 depends on having  $Pr$  and  $Sc \approx 1$  and  $dp^*/dx^* \approx 0$ . However, it has been shown that the analogy may be applied over a wide range of  $Pr$  and  $Sc$ , if certain



corrections are added. In particular the *modified Reynolds*, or *Chilton–Colburn, analogies* [13, 14], have the form

$$\frac{C_f}{2} = St Pr^{2/3} \equiv j_H \quad 0.6 < Pr < 60 \quad (6.70)$$

$$\frac{C_f}{2} = St_m Sc^{2/3} \equiv j_m \quad 0.6 < Sc < 3000 \quad (6.71)$$

where  $j_H$  and  $j_m$  are the *Colburn  $j$  factors* for heat and mass transfer, respectively. For laminar flow Equations 6.70 and 6.71 are only appropriate when  $dp^*/dx^* < 0$ , but in turbulent flow, conditions are less sensitive to the effect of pressure gradients and these equations remain approximately valid. If the analogy is applicable at every point on a surface, it may be applied to the surface average coefficients.

## 6.8 Summary

In this chapter we have considered several fundamental issues related to convection transport phenomena. In the process, however, you should not lose sight of what remains *the problem of convection*. Our primary objective is still one of developing means to determine the convection coefficients  $h$  and  $h_m$ . Although these coefficients may be obtained by solving the boundary layer equations, it is only for simple flow situations that such solutions are readily effected. The more practical approach frequently involves calculating  $h$  and  $h_m$  from empirical relations of the form given by Equations 6.49 and 6.53. The particular form of these equations is obtained by *correlating* measured convection heat and mass transfer results in terms of appropriate dimensionless groups. It is this approach that is emphasized in the chapters that follow.

To test your comprehension of the material, you should challenge yourself with appropriate questions.

- What is the difference between a *local* convection heat transfer coefficient and an *average* coefficient? What are their units? What is the difference between local and average convection coefficients for species transfer? What are their units?
- What are the forms of *Newton's law of cooling* for a *heat flux* and a *heat rate*? What are the analogous forms for convection mass transfer, expressed in molar and mass units?
- Provide some examples for which species transfer by convection is pertinent.
- What is *Fick's law*?
- What are the *velocity*, *thermal*, and *concentration boundary layers*? Under what conditions do they develop?
- What quantities change with location in a *velocity boundary layer*? A *thermal boundary layer*? A *concentration boundary layer*?
- Recognizing that convection heat (mass) transfer is strongly influenced by conditions associated with fluid flow over a surface, how is it that we may determine the convection heat (species) flux by applying Fourier's (Fick's) law to the fluid at the surface?

- Do we expect heat and mass transfer to change with transition from a laminar to a turbulent boundary layer? If so, how?
- What laws of nature are embodied by the *convection transfer equations*?
- What physical processes are represented by the terms of the  $x$ -momentum equation (6.28)? By the energy equation (6.29)? By the species conservation equation (6.30)?
- What special approximations may be made for conditions within *thin* velocity, thermal, and concentration boundary layers?
- What is the *film* temperature and how is it used?
- How is the *Reynolds number* defined? What is its physical interpretation? What role is played by the *critical Reynolds number*?
- What is the definition of the *Prandtl number*? How does its value affect relative growth of the velocity and thermal boundary layers for laminar flow over a surface? What are representative room-temperature values of the Prandtl number for a liquid metal, a gas, water, and an oil?
- What is the definition of the *Schmidt number*? The *Lewis number*? What are their physical interpretations, and how do they influence relative velocity, thermal, and concentration boundary layer development for laminar flow over a surface?
- What is the *coefficient of friction*? The *Nusselt number*? The *Sherwood number*? For flow over a prescribed geometry, what are the independent parameters that determine local and average values of these quantities?
- Under what conditions may velocity, thermal, and concentration boundary layers be termed *analogous*? What is the physical basis of analogous behavior?
- What important boundary layer parameters are linked by the *heat and mass transfer analogy*?
- What is the physical basis of the *evaporative cooling effect*? Have you ever experienced the effect?
- What important boundary layer parameters are linked by the *Reynolds analogy*?
- What physical features distinguish a turbulent flow from a laminar flow?

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