Lab 3 (Practice)

Shreya Rao sr3843

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Optimization

The goal of this lab is to write a simple optimization function in **R** which estimates the global minimum of a convex differentiable function $f(x)$. Specifically, consider the function

$$f(x) = \frac{-\log(x)}{1+x}, \quad x > 0,$$

where $\log(x)$ is the natural logarithm of x. We seek to estimate the value of $x > 0$ such that $f(x)$ achieves its global minimum. For example, the global minimum of the function $g(x) = x^2 - 4x + 3$ is at $x = 2$. The minimum of $g(x)$ can easily be computed using the vertex formula for quadratic functions, i.e., $x = -b/(2a) = 4/(2 * 1) = 2$. In most cases, the minimum does not have a closed form solution and must be computed numerically. Hence we seek to estimate the global minimum of $f(x)$ numerically via gradient descent.

Tasks

1) Using **R**, define the function

$$f(x) = \frac{-\log(x)}{1+x}, \quad x > 0.$$

Test the points $f(0)$ and $f(2)$.

```
convex_func <- function(x) {
    return((-log(x))/(1 + x))
}

convex_func(0)
```

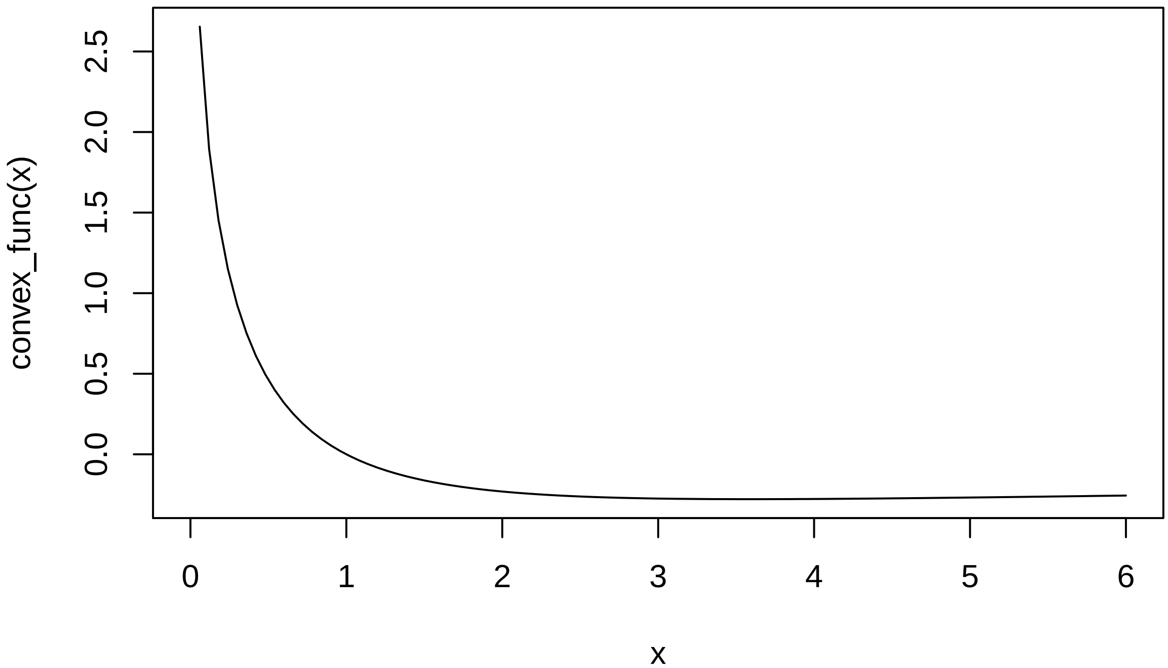
```
## [1] Inf
```

```
convex_func(2)
```

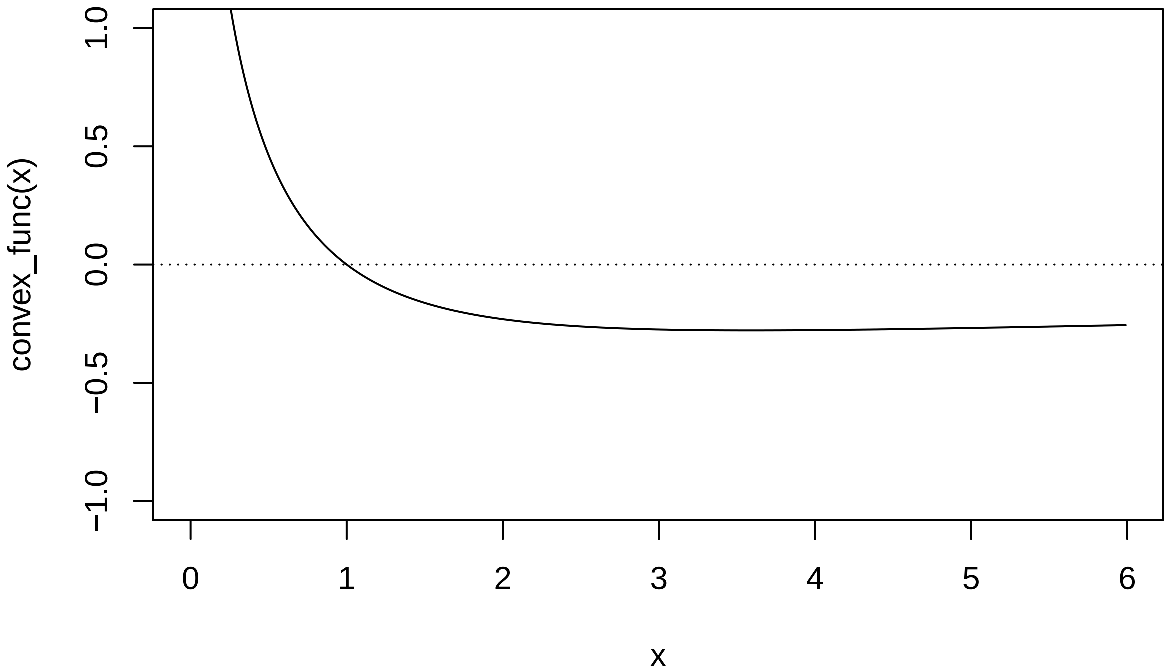
```
## [1] -0.2310491
```

2) Plot the function $f(x)$ over the interval $(0, 6]$.

```r
curve(convex_func, 0, 6)
```




```r
x <- seq(1e-04, 6, by = 0.01)
plot(x, convex_func(x), type = "l", ylim = c(-1, 1))
abline(h = 0, lty = 3)
```



3) By inspection, were do you think global minimum is located at?

Around 3.5

4) Define a **R** function which computes the difference quotient of $f(x)$, i.e., for $h > 0$,

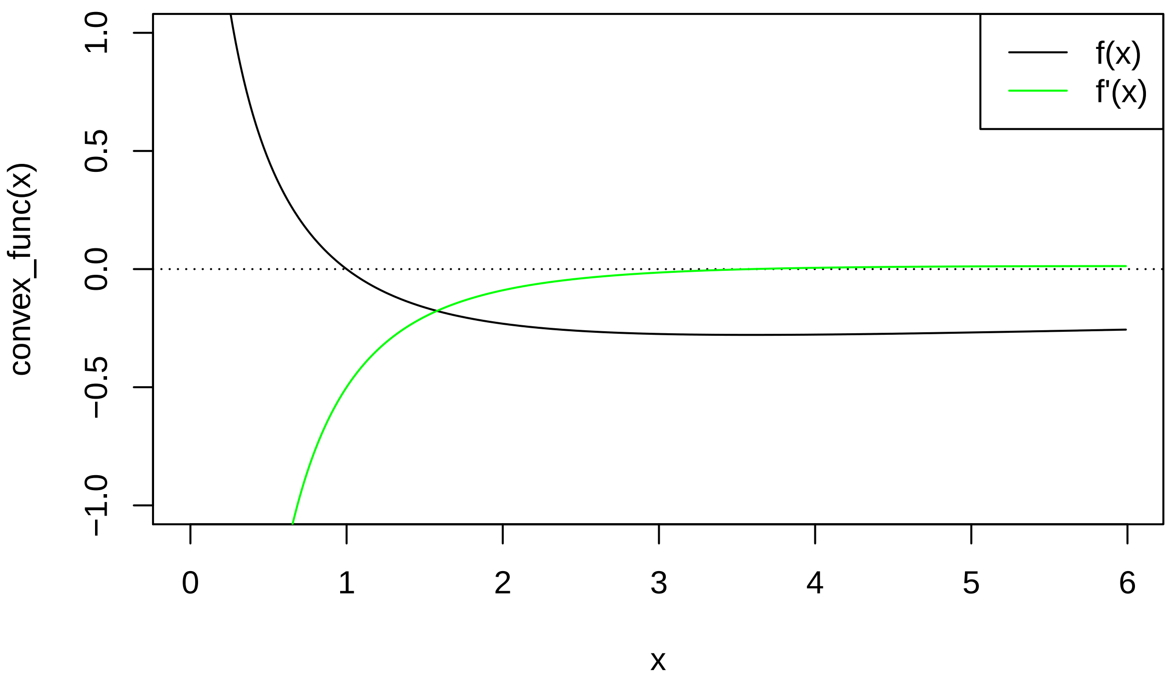
$$\frac{f(x+h) - f(x)}{h}.$$

This function should have two inputs; h and x. Name the difference quotient function **diff.quot**. Note that for small h, this function is the approximate derivative of $f(x)$.

```
diff.quot <- function(x, h) {
    dq <- (convex_func(x + h) - convex_func(x))/h
    return(dq)
}
```

5) Plot both the difference quotient function **diff.quot** and $f(x)$ over the interval $(0, 6]$. Fix $h = .0001$ to construct this plot. Comment on any interesting features.

```
x <- seq(1e-04, 6, by = 0.01)
plot(x, convex_func(x), type = "l", ylim = c(-1, 1))
abline(h = 0, lty = 3)
lines(x, diff.quot(x, h = 1e-04), col = "green")
legend("topright", c("f(x)", "f'(x)"), lty = c(1, 1), col = c(1,
    "green"))
```



6) Write a **R** function named **basic.grad.descent** that runs the basic gradient descent algorithm on the function $f(x)$. The function should have inputs:

1. Initial value **x**
2. Maximum iterations **max.iter** with default 10000.
3. Stopping criterion **stop.deriv** with default 1e-10.
4. Derivative step size **h** with default .0001.
5. Step size **step.scale** with default .5.

The function should have outputs:

1. The value x that yields the minimum of $f(x)$.
2. The minimum value of $f(x)$.
3. The number of iterations the algorithm took to reach the minimum.
4. A logical indicator displaying whether or not the algorithm converged.

7) Check the optimal value using the base **R** function **nlm()**.

```r
basic.grad.descent <- function(x, max.iter = 10000, stop.deriv = 10^(-10),
    h = 1e-04, step.scale = 0.5) {
    iter = 0
    deriv = Inf

    for (i in 1:max.iter) {
        iter = iter + 1
        deriv <- (convex_func(x + h) - convex_func(x))/h
        x = x - deriv * step.scale
        if (abs(deriv) < stop.deriv) {
            (break)()
        }
    }

    fit <- list(x_value = x, min_value = convex_func(x), num_iterations = iter,
        converged = (iter < max.iter))
    return(fit)
}

basic.grad.descent(x = 1)
```

```
## $x_value
## [1] 3.591071
##
## $min_value
## [1] -0.2784645
##
## $num_iterations
## [1] 2201
##
## $converged
## [1] TRUE
```

```r
nlm(convex_func, p = 1)
```

```
## $minimum
## [1] -0.2784645
##
## $estimate
## [1] 3.591119
##
## $gradient
## [1] -1.273731e-08
##
## $code
## [1] 1
##
## $iterations
## [1] 10
```

Hints

I) The main idea of this algorithm is to update x using the relation

$$x_{i+1} = x_i - \text{step.scale} * f'(x_i),$$

where $f'(x_i)$ is approximated by the difference quotient.

II) Build your function using the sample code from slide 50 in lecture notes **ISCDS_Set7**.

III) On slide 50, we were performing a least squares optimization procedure with objective function $SSE(\beta)$. In this lab, the objective function is $f(x)$.

```
x <- seq(from = 0.1, to = 6, 0.01)
plot(x, convex_func(x), type = "l", ylim = c(-1, 1))
abline(h = 0, lty = 3)
lines(x, diff.quot(x, h = 0.001), col = "green")
legend("topright", legend = c("f(x)", "f'(x)"), lty = c(1, 1),
    col = c("black", "green"))
abline(v = basic.grad.descent(x = 1)$x_value, col = "purple")
```