J. P. Morgan Quant Mentorship Program 2022

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1 Question 1-Estimating the Value of pi using Monte Carlo Simulations

We can go about this in multiple ways, but the most efficient way will be using the area of a circle.
We take a circle of unit radius and a square which has a side 2. The circle touches the square at 4 points. We now scatter points randomly, in this square. When the number of points is large, the ratio of number of points inside the circle to the total number of points will correspond to the ratio of area of the circle to the area of square.
In out case, we have considered a 1x1 square and a quarter of a circle of unit radius. The ratio of area in this case would be:

$$Ratio = \frac{\text{Area of Quarter}}{\text{Area of Square}} = \frac{\text{Number of points inside the circle}}{\text{Total number of points}} = \frac{\pi/4}{1} = \frac{\pi}{4}$$

Hence, we can conclude that estimated value of pi will be:

$$\pi = 4\frac{\text{Number of points inside the circle}}{\text{Total number of points}}$$

While using Monte Carlo Simulations, we take x and y as uniformly distributed variables, ranging from 0 to 1, storing the random values in an array of size N = 10000. We now have a 10000 pairs of points (x,y). The next step is checking if they lie inside the circle. For this, the equation $x^2 + y^2 < 1$ is checked and if it's true, we store it inside another array. We can now easily calculate the number of points inside the circle by the size of array or maintaining a variable while checking the condition. We get the value of pi as:

Figure 1: **Estimated value of pi**

The plot, with scattered points looks something like:

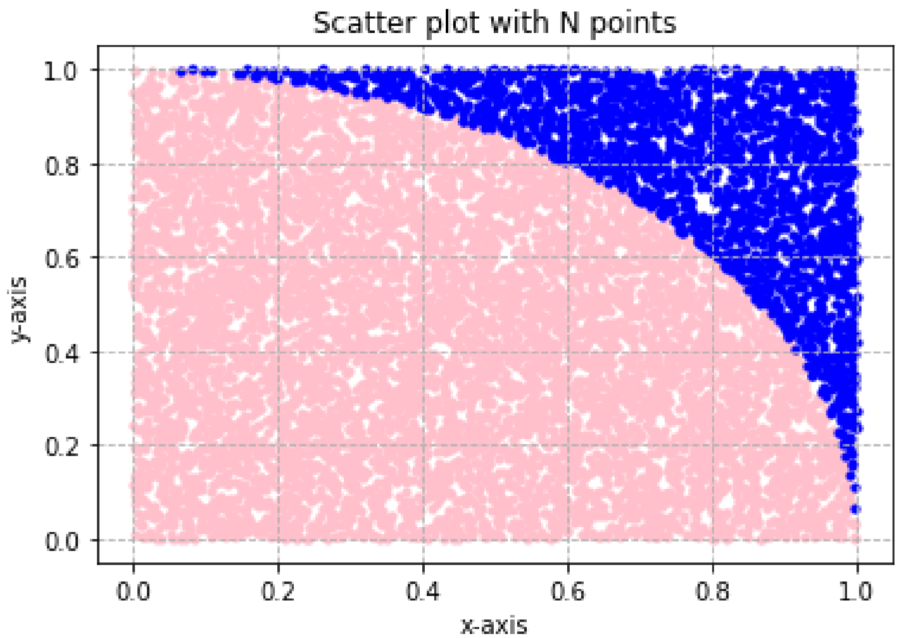
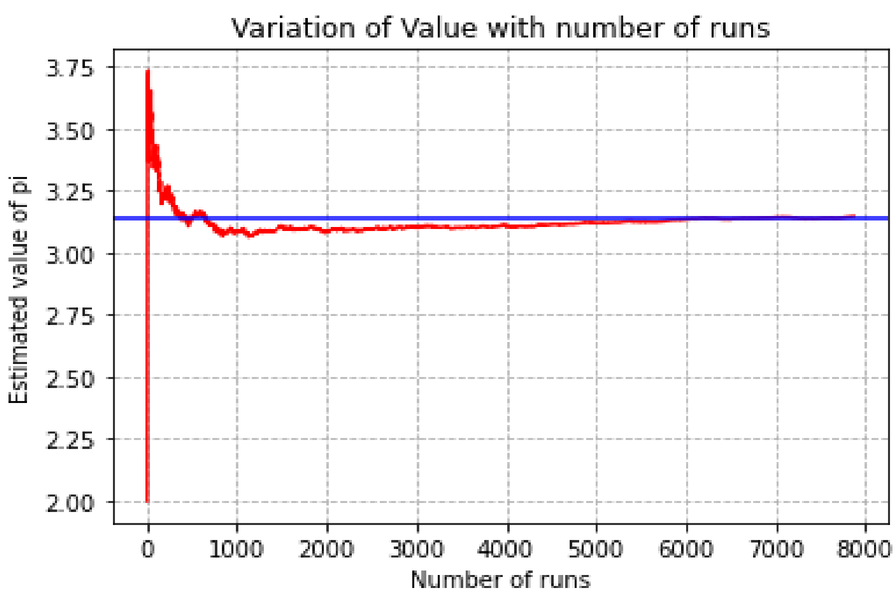


Figure 2: **Monte Carlo estimation of pi**

As the number of runs is increased, the value of pi converges towards the real value, and the plot looks something like:



2 Question 2-Value of Stock Price

We are given that, a company, "XYZ Capital" is currently trading at ₹100, at time t_0. It moves up or down by ₹1 with a probability of $\frac{1}{2}$ each.

2.1 Part A-Expected Value of Stock Price

We know that since the stock price initially is ₹100, it can only go up and down by ₹1 and after 1 minute, the price can be ₹101 or ₹99. The probability and prices can be seen in the following flowchart.
a) The expected value of stock after 1 minute will be:
E[X] = $\frac{1}{2}101+\frac{1}{2}99 = 100$
b) The expected value of stock price after 10 minutes will be:
E[X] = $\frac{1}{2^{10}}2^{10}(100) = 100$
c) The expected value of stock price after 1 hour will be:
E[X] = $\frac{1}{2^{60}}2^{60}(100) = 100$
d) The expected value of stock price after 1 month will be:
E[X] = 100(explanation on next page)
Yes, all the values are same. This can be explained by the flowchart on the next page which explains how the value of stock progresses after each stage. The probability after i^{th} stage, for all prices individually would be $\frac{1}{2^i}$ and the sum of all prices/values will be $2^i(100)$ because at each stage, the price goes up by +1 or down by -1, effectively cancelling the effect of any change in price at all. Hence, the expected value of stock would be the same at any time.

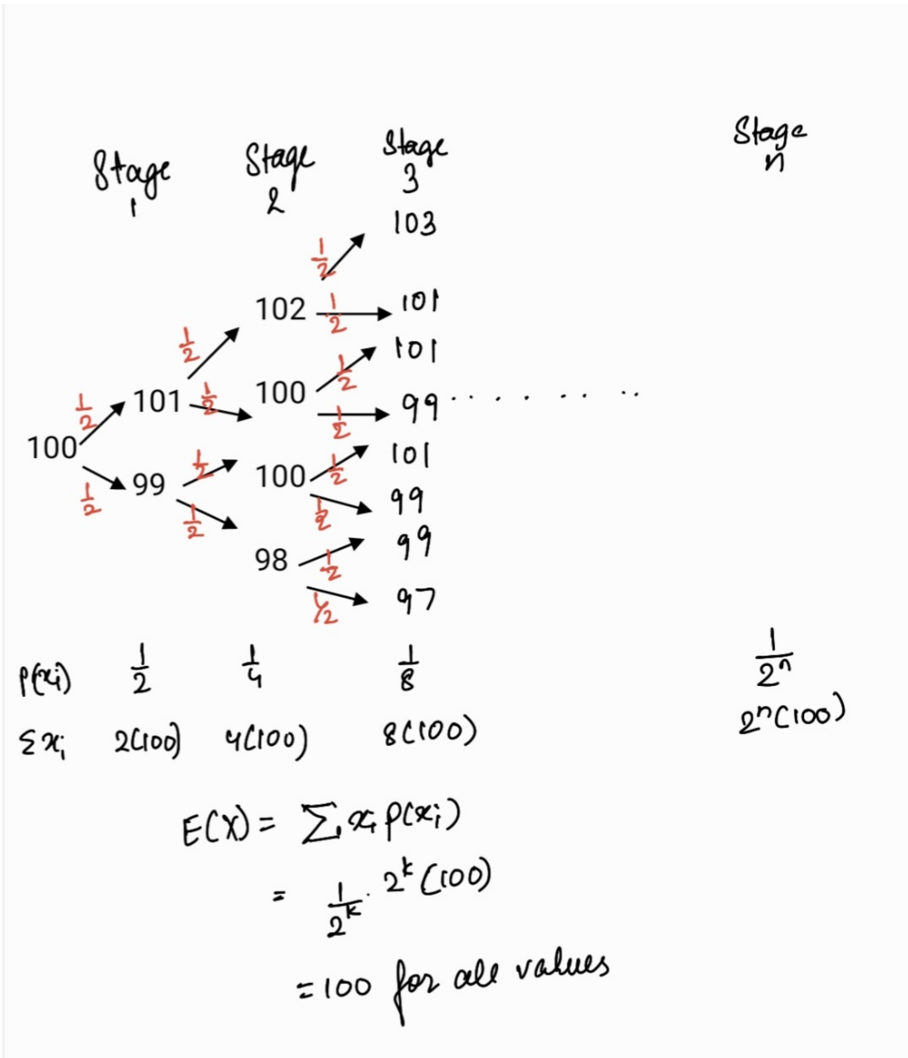


Figure 3: **Expected value of Stock**

2.2 Part B

2.2.1 (i)Solving Analytically

Let's take f(X) as the probability of the value starting with X and reaching 102 before 96.

Therefore, we can write:

$$f(102) = 1$$
$$f(96) = 0$$

We also know that the probability of price going up or down by ₹1 is $\frac{1}{2}$.Hence, we can write:

$$\begin{aligned}
f(x) &= \frac{1}{2}f(x-1) + \frac{1}{2}f(x+1) \\
f(97) &= \frac{1}{2}f(96) + \frac{1}{2}f(98) \\
&= \frac{1}{2}f(98) \\
f(98) &= \frac{1}{2}f(97) + \frac{1}{2}f(99) \\
&= \frac{2}{3}f(99) \\
f(99) &= \frac{1}{2}f(98) + \frac{1}{2}f(100) \\
&= \frac{3}{4}f(100) \\
f(101) &= \frac{1}{2}f(100) + \frac{1}{2}f(102) \\
&= \frac{1}{2}f(100) + \frac{1}{2} \\
f(100) &= \frac{1}{2}f(101) + \frac{1}{2}f(99) \\
&= \frac{1}{4}f(100) + \frac{1}{4} + \frac{3}{8}f(100) \\
&= \frac{2}{3}
\end{aligned}$$

Hence, the probability that it reaches 102 before 96 is $\frac{2}{3}$

2.2.2 ii)Solving using Monte Carlo method

I have taken the number of runs to be 10000. In each run, there are 3 possibilities:

- The series reaches 102 before 96(favourable case)
- The series reaches 96 before 102
- The series continues to infinity without reaching any of these values(negligible)

Answer: $\frac{\text{Number of times Case 1 takes place}}{\text{Total Number of Runs}}$ For this, we make a loop for each run, which increaments the favourable outcomes by 1 if case 1 occurs, or terminates if case 2 occurs.
To decide how many runs gets us to the coverged value, we check after how many runs is the error always less that 0.01 and print that value. After multiple runs, I have concluded that this lies approximately between 1000 and 5000.

```
Estimated value of Probability: 0.6668
```

Figure 4: **Estimated value of Probability**

```
Numbers of runs to get to the converged value: 1775
```

Figure 5: **Number of runs to reach converged value**

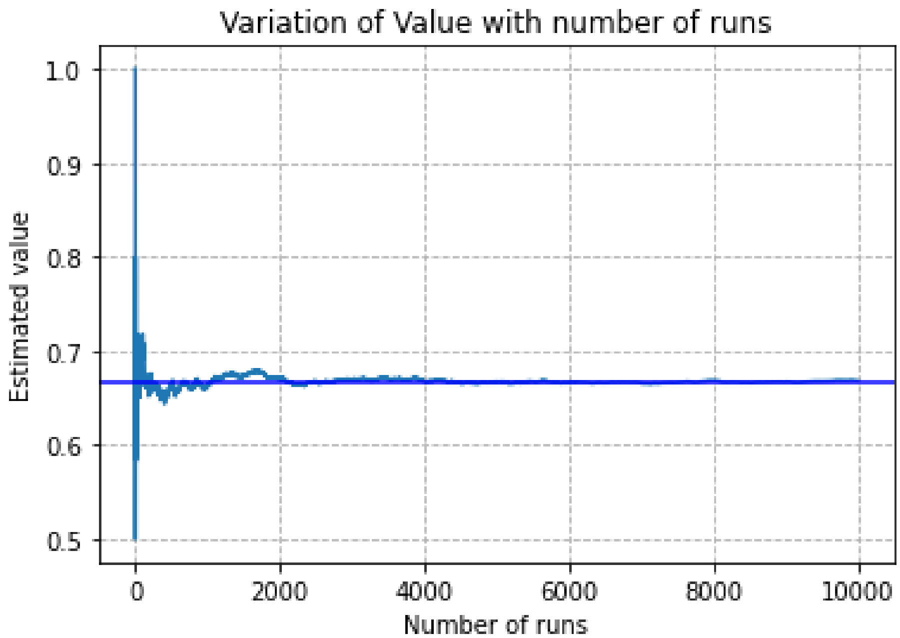


Figure 6: **Value vs Number of runs**

3 Question 3: BSM Pricing

We can write the price of the call option as:

$$
\begin{aligned}
C\left(S_t, t\right) &= N\left(d_1\right) S_t - N\left(d_2\right) K e^{-rT} \\
d_1 &= \frac{1}{\sigma\sqrt{T}}\left[\ln\left(\frac{S_t}{K}\right) + \left(r + \frac{\sigma^2}{2}\right) T\right] \\
d_2 &= d_1 - \sigma\sqrt{T}
\end{aligned}
$$

where,
• $C\left(S_t, t\right)$ - BSM price of a call option at time t and stock price S_t (in S)
• N - CDF of the standard normal distribution
• T = $(T_1 - t)$Time left til maturity (in years)
• S_t - Stock price at time t
• K - Option strike price (in S)
• σ - Stock volatility (absolute, not percent)
• r-annual risk free rate of interest (absolute, not percent)
We are given a company "XYZ Capital" currently trading at ₹200, at time t_0. We have an option contract with:

$$
\begin{aligned}
K &= 180 \\
T & \\
\sigma &= 0.15 \\
r = \mu &= 0.02 \\
S_0 &= 200
\end{aligned}
$$

We use the Black Scholes formula to calculate this in python and use the following function for the normal distribution and use the above function given, plugging in the values of the variables

`scipy.stats.norm`

```
Call option price using BSM: 20.317700916589644
```

Figure 7: **Number of Runs vs Estimated Value of call option**

4 Question 4: Brownian Motion to model Stock Prices

We are now told how the underlying asset moves with time. We also know how to compute the payoff of an Option, in this case, a Call option, given the stock price and the strike price at the given time. We also know how to calculate the present value of a future sum of money, given the rate of interest and the time period. The formulae for the above statements are:

$$S_t = S_{t-1} e^{(\mu - \frac{\sigma^2}{2})\Delta t + \sigma \mathcal{N}(0,1)\sqrt{\Delta t}}$$
$$S_{\Delta t} = S_0 e^{(\mu - \frac{\sigma^2}{2})\Delta t + \sigma \mathcal{N}(0,1)\sqrt{\Delta t}}$$
$$\text{Pay-off of a call option} = \text{Max(0,S-K)}$$
$$\text{Present Value} = \text{(Future Value)} e^{-rT}$$

We want to calculate the Stock Price at a time 30 days after S_0, taking 240 timesteps. Hence we define Δt as:

$$\begin{aligned} \Delta t &= \frac{T}{N} \\ &= \frac{\frac{1}{12}}{240} \\ &= \frac{1}{12\text{x}240} \end{aligned}$$

Other values are same as the above question, μ being ythe risk free interest rate and σ being the volatality. What we essentially do here is, calculate $S_0, S_1, S_2, ..S_{240}$, where S_{240} represents the Stock price at a time 30 days after time t_0. Each time, we update the value of $\mathcal{N}$(0,1) so as to randomize the function as much as we can. I have taken an array to store 240 random values of $\mathcal{N}$(0,1), which gets updated after each run. We have the strike price K = ₹180, for a time period of 30 days. We find the price of the Call option here, and bring it to the present value using the formulae mentioned above.
We vary the number of runs and conclude that as the number of runs increases, the mean value of all the Call option prices(current) converges to the BSM model price.

```
Estimated Value of call option using Brownian Motion: 20.317574923432883
```

Figure 8: **Value of Call option when number of runs is 100000**

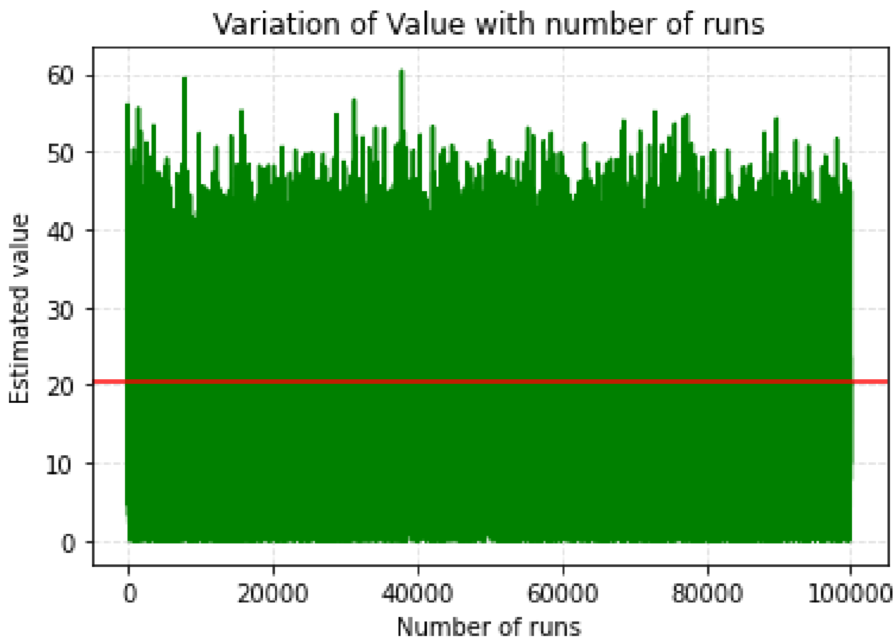


Figure 9: **Number of Runs vs Estimated Value of call option**