Support Equalities Among Ribbon Schur Functions

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Introduction

In [1], McNamara proved that two *skew diagrams* can have the same *Schur support* only if they have the same number of $k \times \ell$ rectangles as subdiagrams. It follows that two *connected ribbons* α and β can have the same Schur support only if one is obtained by permuting row lengths of the other (i.e. $\beta = \alpha_\pi$ for some permutation π). We give a necessary and a sufficient condition for an m-rowed ribbon α to have the same Schur support as every permutation α_π, for $\pi \in S_m$. We conjecture that our necessary condition is also sufficient.

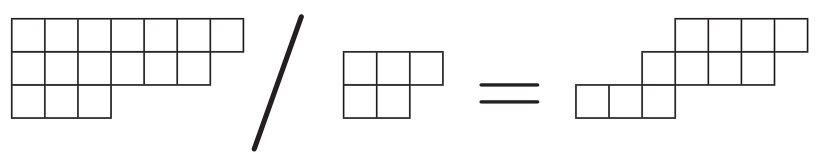
Preliminaries

A **Young diagram** of a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)$ is a collection of rows of left aligned boxes, where row i has length λ_i.

A filling of a Young diagram is **semistandard** if the entries increase weakly across rows and strictly down columns. The **content** of a filling records the number of times each integer is used in the filling. For example, a SSYT of shape $\lambda = (8, 7, 3)$ and content $\nu = (5, 4, 3, 3, 3)$ is:

1	1	1	1	1	2	2	4
2	2	3	3	4	5	5	
3	4	5					

A **skew diagram** λ/μ is obtained by removing μ from the top-left corner of λ, where λ and μ are ordinary ("straight") diagrams. (So λ/μ is only defined when $\mu_i \le \lambda_i$ for all i.) For example, $\lambda/\mu = (7, 6, 3)/(3, 2)$ has diagram



The **Schur function** of a skew partition λ/μ is defined as

$$s_{\lambda/\mu}(x_1, x_2, x_3, \ldots) = \sum_{\substack{T\text{ : skew SSYT of}\\ \text{shape } \lambda/\mu}} x^T = \sum_{\substack{T\text{ : skew SSYT of}\\ \text{shape } \lambda/\mu}} x_1^{t_1} x_2^{t_2} x_3^{t_3} \cdots$$

where t_i is the number of occurrences of i in T.

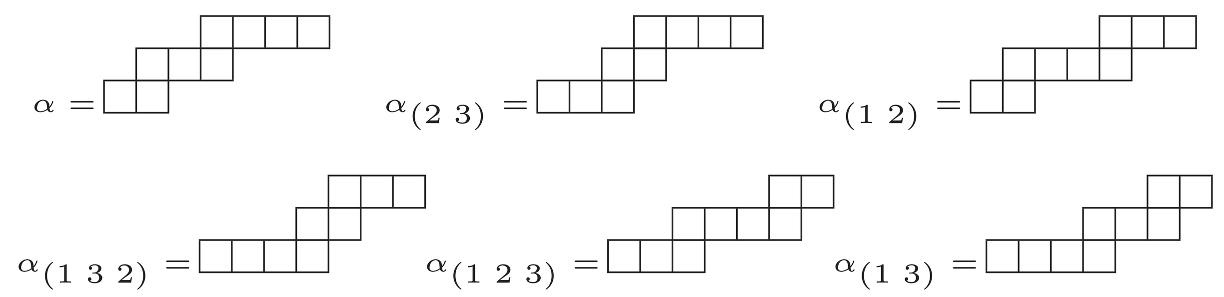
A **ribbon** is a skew partition which does not contain a 2×2 block as a subdiagram. Notice that connected ribbons are fully determined by their row lengths and as such can be represented by integer tuples.

We can express $s_{\lambda/\mu} = \sum_\nu c^\lambda_{\mu,\nu} s_\nu$ with integers $c^\lambda_{\mu,\nu} \ge 0$. We define the **Schur support** of a skew shape λ/μ as

$$[\lambda/\mu] = \{\nu \mid c^\lambda_{\mu,\nu} > 0\}.$$

Question

Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m)$ be a ribbon. Let α_π denote a ribbon formed by applying the permutation $\pi \in S_m$ to the row lengths of α.



A ribbon $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m)$ is said to have *full equivalence class* if for all permutations $\pi \in S_m$, we have $[\alpha] = [\alpha_\pi]$.

Question: Which connected ribbons have full equivalence class?

Sufficient Condition

Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m)$ be a ribbon. If all triples $(\alpha_j, \alpha_k, \alpha_\ell)$ with $1 \le j, k, \ell \le m$ satisfy the **strict** triangle inequality ($\alpha_j < \alpha_k + \alpha_\ell$), then α has full equivalence class.

Proof Sketch

Littlewood Richardson (LR) Rule: [2] Let D be a skew shape. A partition $\lambda = (\lambda_1, \ldots, \lambda_m)$ is in the support of s_D iff there is a valid LR-filling of D with content λ. A filling of D is an LR-filling if:

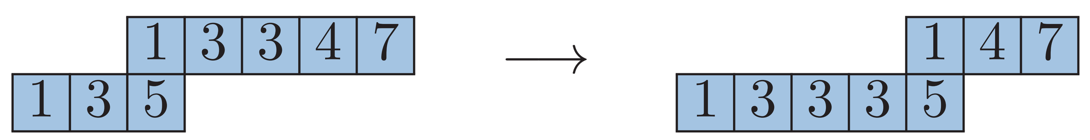
- The tableau is semistandard.
- Every *initial reverse reading word* is *Yamanouchi*: $\#i$'s $\ge \#(i+1)$'s

Reverse Reading Word: 1,1,2,2,1,3,2

		1	1
	1	2	2
2	3		

This is Yamanouchi and semistandard, and hence is a valid LR-filling. The content of the filling is $(3, 3, 1)$, so $(3, 3, 1)$ is in the support of the ribbon $(2, 3, 2)$.

R-Matrix Algorithm: [3] Gives us a way to swap two rows while preserving the Yamanouchi property in the whole tableau and semistandardness *within the two swapped rows*.



When all strict triangle inequalities hold, we can swap any rows i and $i+1$ using this algorithm to get (after some additional work) an LR-filling of $\alpha_{(i\ i+1)}$ of the same content as the original LR-filling for α. Since transpositions generate S_m, this shows that α has full equivalence class.

Necessary Condition

Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m)$ be a ribbon, where $\alpha_1 \ge \alpha_2 \ge \cdots \ge \alpha_m$. If α has full equivalence class, then $N_j < \sum_{i=j+1}^m \alpha_i - (m - j - 2)$ for all $j \le m - 2$, where

$$N_j = \max\left\{ k \,\middle|\, \sum_{i \le j, \alpha_i < k} (k - \alpha_i) < m - j - 2 \right\}$$

Conjecture: This necessary condition is sufficient as well.

Note: A weaker but simpler version of our necessary condition is: $\alpha_i < \sum_{k=i+1}^m \alpha_k$ for all $1 \le i \le m - 2$.

Proof Sketch

- If the j^{th} necessary inequality is not satisfied for a ribbon α, we can use the LR-Rule to show that $[\alpha_{(j\ j+1)}] \ne [\alpha]$.
- In this case, if we fill the i^{th} row of $\alpha_{(j\ j+1)}$ with i's for all $i \le j$ and then use as many j's as possible for the rest of the filling, there will be no LR-filling of α of the same content. In short, row j is too long relative to the rows below it for α to have full equivalence class.

Future Work

- Prove that the necessary condition is also sufficient. (Data from small cases supports this conjecture.)
- Investigate non-full equivalence classes of ribbons.
- Extend the results to generic skew shapes.

References

[1] Peter R. W. McNamara, Necessary conditions for Schur-positivity, *Journal of Algebraic Combinatorics* 28(4): 495–507, 2008

[2] D. E. Littlewood and A. R. Richardson, Group characters and algebra, *Phil. Trans. A* **233**, (1934), 99–141.

[3] R. Inoue, A. Kuniba, and T. Takagi. Integrable structure of box-ball systems: crystal, Bethe ansatz, ultradiscretization and tropical geometry, *J. Phys. A: Math. Theor.* **45** 7 (2012) 073001.

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