Cusps of Hyperbolic Knots

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Unknot III, 2016

Hyperbolic Knots

Definition

A *hyperbolic knot or link* is a knot or link whose complement in the 3-sphere is a 3-manifold that admits a complete hyperbolic structure.

This gives us a very useful invariant for hyperbolic knots: Volume (V) of the hyperbolic knot complement.

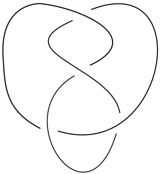
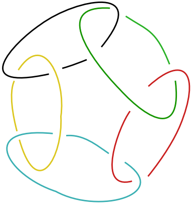


Figure 8 Knot
Volume=2.0298...



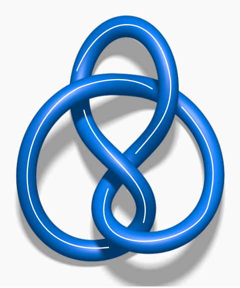
5 Chain
Volume=10.149.....

Cusp

Definition

A *Cusp* of a knot K in S^3 is defined as an open neighborhood of the knot intersected with the knot complement: $S^3 \setminus K$

Thus, a cusp is a subset of the knot complement.



Cusp Volume

The *maximal cusp* is the cusp of a knot expanded as much as possible until the cusp intersects itself.

Cusp Volume

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Definition

The *Cusp Volume* of a **hyperbolic** knot is the volume of the maximal cusp in the hyperbolic knot complement.

Since the cusp is a subset of the knot complement, the cusp volume of a knot is always less than or equal the volume of the knot complement.

Fact

The cusp volume of the figure-8 knot is $\sqrt{3}$

Cusp Volume of Links

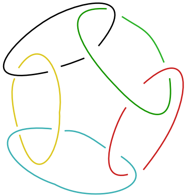
How should we expand cusps in a link to achieve maximal volume?

Cusp Volume of Links

How should we expand cusps in a link to achieve maximal volume?

Fact

The optimal cusp volume of links will be achieved by first, maximizing a particular cusp till it touches itself, then a particular second cusp till it touches itself or the first cusp and so on...



Fact

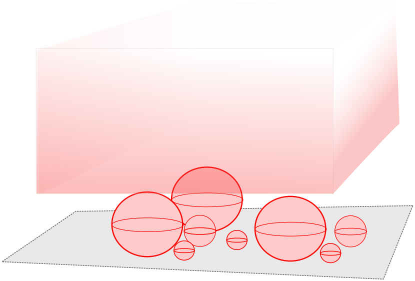
The optimal cusp volume of the minimally twisted 5-chain is $5\sqrt{3}$ *where the individual cusps have volumes:* $4\sqrt{3}, \frac{\sqrt{3}}{4}, \frac{\sqrt{3}}{4}, \frac{\sqrt{3}}{4}, \frac{\sqrt{3}}{4}$ *respectively.*

Cusps in the upper Half space model

How do we visualize cusps? We move to the upper half space covering of the hyperbolic complement.

Fact

Cusps of hyperbolic knots lift to disjoint unions of horoballs in the upper half model of hyperbolic space



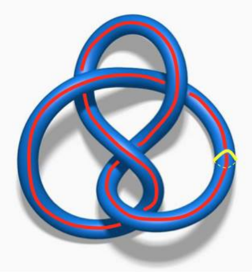
Cusp Diagrams

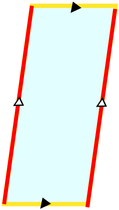
Where do different parts of the cusp go in this horoball diagram?

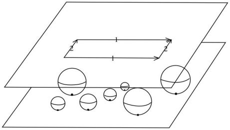
Cusp Diagrams

Where do different parts of the cusp go in this horoball diagram?

We first look at the boundary of the cusp. It is a torus. When we cut along the **meridian** and along the **longitude**, we get a rectangle with the identifications as shown. This corresponds to a fundamental domain on the boundary of the horoball centered at ∞.



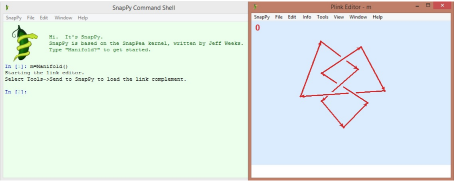




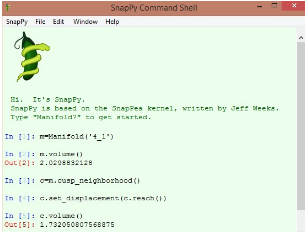
SnapPy - A tool to study hyperbolic manifolds

SnapPy is a program written by Jeff Weeks, Nathan Dunfield, Marc Culler, and Matthias Goerner to study the geometry and topology of hyperbolic manifolds.

We can draw knots and links and calculate hyperbolic volumes, cusp volumes and other invariants.



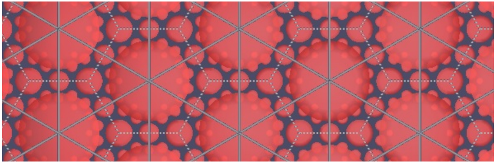
Knot Input



Knot invariants computed

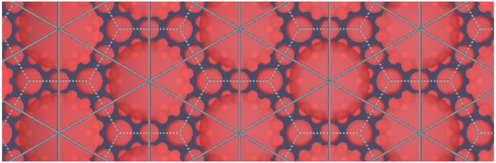
We can also generate horoball diagrams...

View from ball at infinity

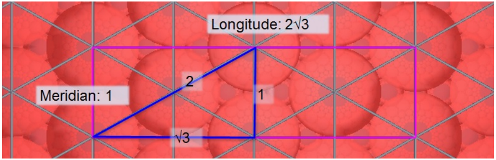


Cusp Diagrams-Example of figure 8

View from ball at infinity



When the cusp is maximized



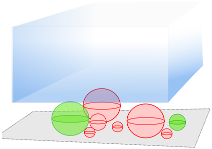
Pictures: Joshua Wakefield

By fundamental properties of hyperbolic geometry,

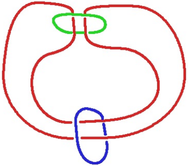
$$\text{cusp volume} = \text{area of cusp}/2 = \frac{2\sqrt{3}}{2} = \sqrt{3}$$

Horoball Diagrams of Links

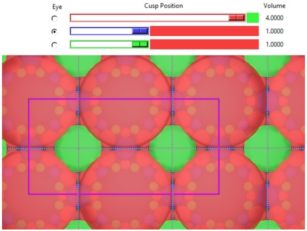
Consider the Borromean Rings. The 3 cusps lift to horoballs in $\mathbb{H}^3$. Each horoball (tangent to $z = 0$ or centered at ∞ is actually centered at the same point at infinity. This gives us a **choice** to view the horoball diagram from a particular horoball centered at infinity in the diagram (the one that looks like space enclosed above a plane).



Horoball Diagram



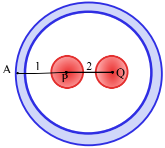
The Borromean Rings



Cusp Diagram looking from the blue cusp at infinity

Links and Twice Punctured Discs

Consider a trivial component around 2 strands. This bounds a twice punctured disc in the complement as shown.



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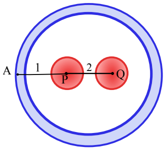
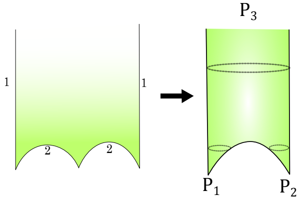


A theorem by Adams states that a twice punctured disc in a hyperbolic 3-manifold always lifts to a totally geodesic surface in the upper half space, as shown.



Twice Punctured Disc

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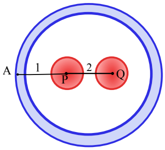

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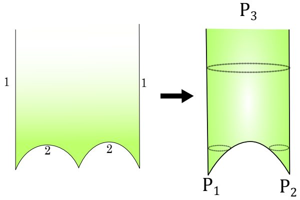

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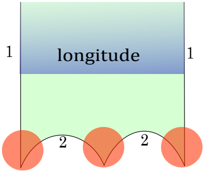
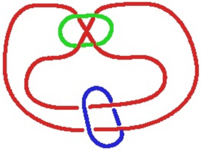
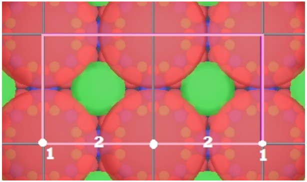


Horoball Diagram

Twice Punctured Discs in horoball Diagrams



Twisted Borromean Rings

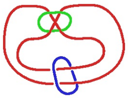


Cusp Diagram of the twice punctured looking from the blue cusp

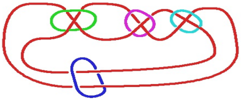
The twice punctured disc, when viewed from the blue cusp at infinity surfaces can be seen as passing through the longitude of the blue cusp in the horoball diagram.

Covers across twice punctured discs

The following construction is taking "n-fold cyclic covers" of a tangle across a twice punctured disc.



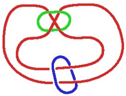
Twisted Borromean Rings



3-fold cover

Covers across twice punctured discs

The following construction is taking "n-fold cyclic covers" of a tangle across a twice punctured disc.



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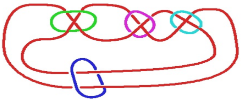


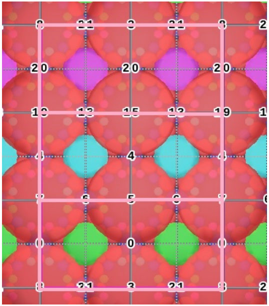
3-fold cover

We know that volume of an $n-$fold cover L_n is n times the volume of the original link L.

$$V(L_n) = nV(L)$$

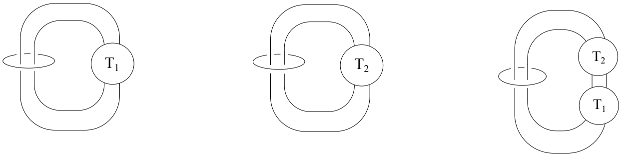
In fact, the same holds for cusp volumes and we can see this as a multiplication of the fundamental domain.

$$C_v(L_n) = nC_v(L)$$



Belted Sums across twice punctured discs

The following construction is taking a belted sum of links cross a twice punctured disc



L_1 L_2 Belted Sum: L_{1+2}

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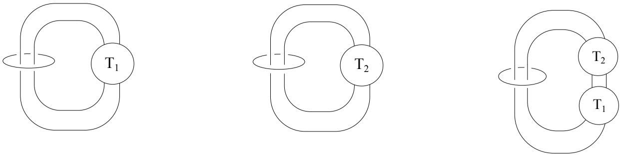


L_1 L_2 Belted Sum: L_{1+2}

$$V(L_1) + V(L_2) = V(L_{1+2})$$

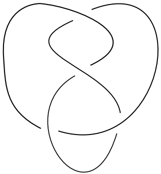
We know that volume of a belted sum is the sum of the volumes of the original 2 links. What happens to cusp volumes and cusp diagrams?

Cusp Invariants

Cusp Invariants

Definition

Volume Density of a knot or link is defined as the ratio: $\frac{V}{C}$ where V is the hyperbolic volume and C is the crossing number (minimum number of crossings in a planar projection)



Volume $= 2.0298..$
Crossing number$=4$
Volume Density$=0.5074..$

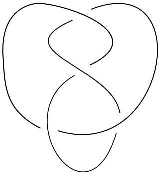
Theorem

(D. Thurston) The maximum possible volume density of hyperbolic links is 3.6638 and (Adams et. al. 2015) the set of volume densities of knots is dense in $[0, 3.6638]$

Cusp Crossing Density

Definition

Cusp Crossing Density of a knot or link is defined as the ratio: $\frac{C_v}{C}$ where C_v is the total cusp volume and C is the crossing number



Cusp Volume $= \sqrt{3}$

Crossing number$=4$

Cusp Crossing Density$=\frac{\sqrt{3}}{4}$

Cusp Crossing Density

Question 1

What are the bounds of cusp crossing densities for knots and links? Which knots or links achieve these bounds?

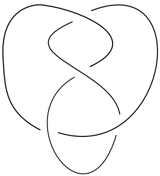
Question 2

Are cusp crossing densities of knots or links *dense* in any interval?

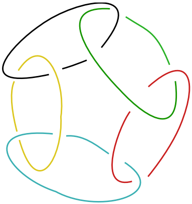
Cusp Density

Definition

Cusp Density (D_c) of a knot or link is the ratio: $\frac{C_v}{V}$ where C_v is the total cusp volume and V is the hyperbolic volume of the complement.



Volume=2.0298...
Cusp Volume=$\sqrt{3}$
Cusp Density=0.853



Volume=10.149...
Cusp Volume = $5\sqrt{3}$
Cusp Density=0.853

Cusp Density

Fact

(Böröczky 1978) The highest cusp density a hyperbolic manifold can have is 0.853..., *the cusp density of the figure-8 knot and the minimally twisted 5-chain.*

Cusp Density

Fact

(Böröczky 1978) The highest cusp density a hyperbolic manifold can have is 0.853..., *the cusp density of the figure-8 knot and the minimally twisted 5-chain.*

Theorem

(SMALL 2016) For any $x \in [0, 0.853...]$, *there exist hyperbolic* ***link*** *complements with cusp density arbitrarily close to* x.

Theorem

(SMALL 2016) For any $x \in [0, 0.6826...]$, *there exist hyperbolic* ***knot*** *complements with cusp density arbitrarily close to* x.

Stay tuned for more on these results...

Acknowledgements

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Science Center

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Unknot III

Prof. Colin Adams

References

- Colin Adams (1985) Thrice-punctured spheres in hyperbolic 3-manifolds Trans. AMS, 287 (2), pp. 645-656
- Colin Adams et. al. (2015) Volume and determinant densities of Hyperbolic Links *http://arxiv.org/pdf/1510.06050.pdf*
- Böröczky, K (1978). Packing of spheres in spaces of constant curvature *Acta Mathematica Academiae Scientiarum Hungaricae* 32: 243-261.
- Marc Culler, Nathan M Dunfield, Matthias Goerner, and Jeffrey R Weeks. SnapPy a computer program for studying the geometry and topology of 3-manifolds. *http://snappy.computop.org*