Spin^c structures : m

$$\widehat{CFK}(S^3,K) = \bigoplus_{\substack{m \in \mathrm{Spin}^c(K_0) \\ \cong \mathbb{Z}}} \widehat{CFK}(S^3,K,m)$$

$$H_*(K_0 = S^3_0(K)) = \mathbb{Z} = H^2(S^3_0(K)) \cong \mathrm{Spin}^c(K_0)$$

characterized by

$\underline{s}(x) \in \mathrm{Spin}^c(K_0)$ is m

when $\langle c_1(\underline{s}(x)), [\hat{F}] \rangle = 2m$

where $\hat{F}$ is a capped off Seifert surface of K.

Define $\sigma[x,i,j] = s(x) + (i-j)\, PD[\mu]$

$CFK^\infty = \bigoplus CFK^\infty(S^3,K,m)$
$\quad = \langle [x,i,j] \mid \sigma = m \rangle$

Now for $[m] \in \mathbb{Z}/p \cong Spin^c(S^3_{-p}(K))$

This is characterized by saying $s_w(y) = [m]$ if it is the restriction of a $Spin^c$ structure τ on the surgery cobordism W_{-p} such that
$\langle c_1(s_w(\psi)), [\hat{S}] \rangle - p = 2m \mod 2p$ $\square$

<u>Defining $\Phi: CFK^\infty \to CF^\infty$</u>

Pick a Heegaard triple $(\Sigma, \alpha, \beta, \gamma)$ associated to surgery data of K_{-p}

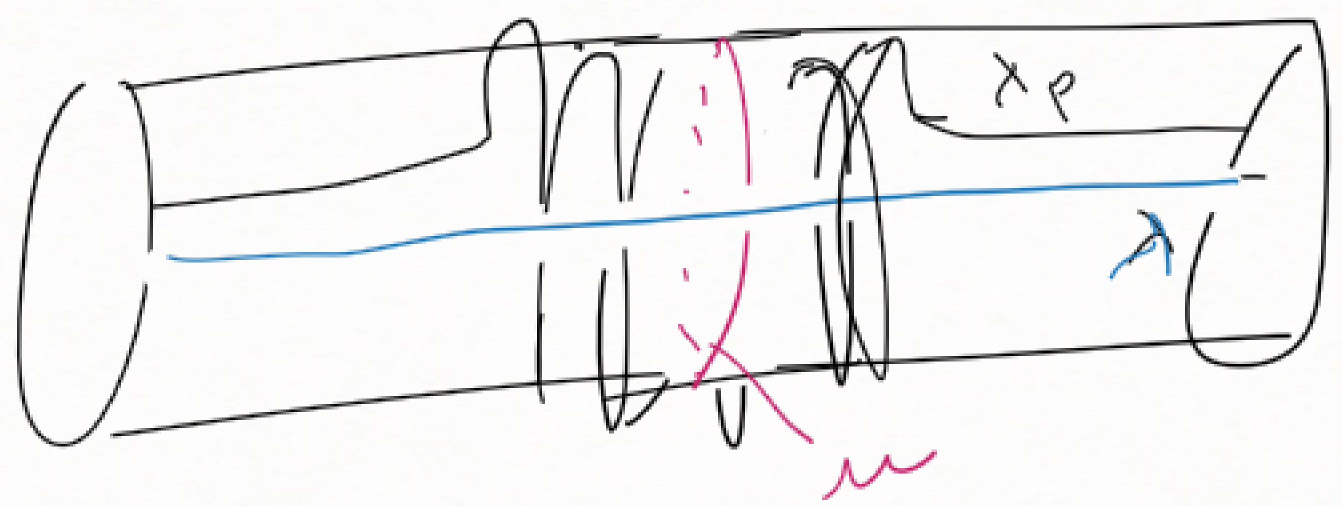
i.e $(\Sigma, \alpha, \beta, w, z)$ is for CFK,

(Σ, α, β) is for S^3 (or Y)

(Σ, α, γ) is for $S^3_{-p}(K)$ (or $Y_{-p}(K)$)

or $\gamma = (\beta \setminus \underset{\mu}{\underset{\shortparallel}{\beta_g}}) \cup \underset{\lambda_p}{\underset{\shortparallel}{(\lambda - p\mu)}}$

Choose λ_p to follow λ closely and wind tightly around μ ~$p/2$ times on the left & right.

i.e 

Then def $\Phi : CFK^\infty \longrightarrow CF^\infty$

$$\Phi([x,i,j]) = \sum_{\substack{y \in T_\alpha \cap T_\gamma \\ \psi \in \pi_2(x, \Theta_{\beta\gamma}, y) \\ \mu(\psi) = 0 \\ n_w(\psi) - n_z(\psi) = i - j}} \# \mathcal{M}(\psi) [y, i - n_w(\psi)] \quad \longrightarrow \text{①}$$

[Diagram].

Proof: $\underline{\Phi \text{ fits in the diagram}}$:

if $i < 0$ or $j < 0$ and $\exists\ \psi \in \pi_2(x, \Theta_{\beta\gamma}, y)$

then ① $\Rightarrow$ $i - n_w(\psi) = j - n_z(\psi) < 0$

so $\Phi : CFK^{i<0 \text{ or } j<0} \longrightarrow CF^-$

Sim $\Phi : CFK^{i \geq 0, j \geq 0} \longrightarrow CF^+$ $\boxtimes$

Φ fits w.r.t spin^c structures:

Lemma:

$$\langle c_1(s(x)), [\hat{F}]\rangle + 2(n_w(\psi) - n_z(\psi)) = \langle c_1(s_w(\psi)), [S]\rangle - P$$

Prove it for $n_w - n_z = 0$

then use $\psi = \phi\psi_0\phi'$ $\square$

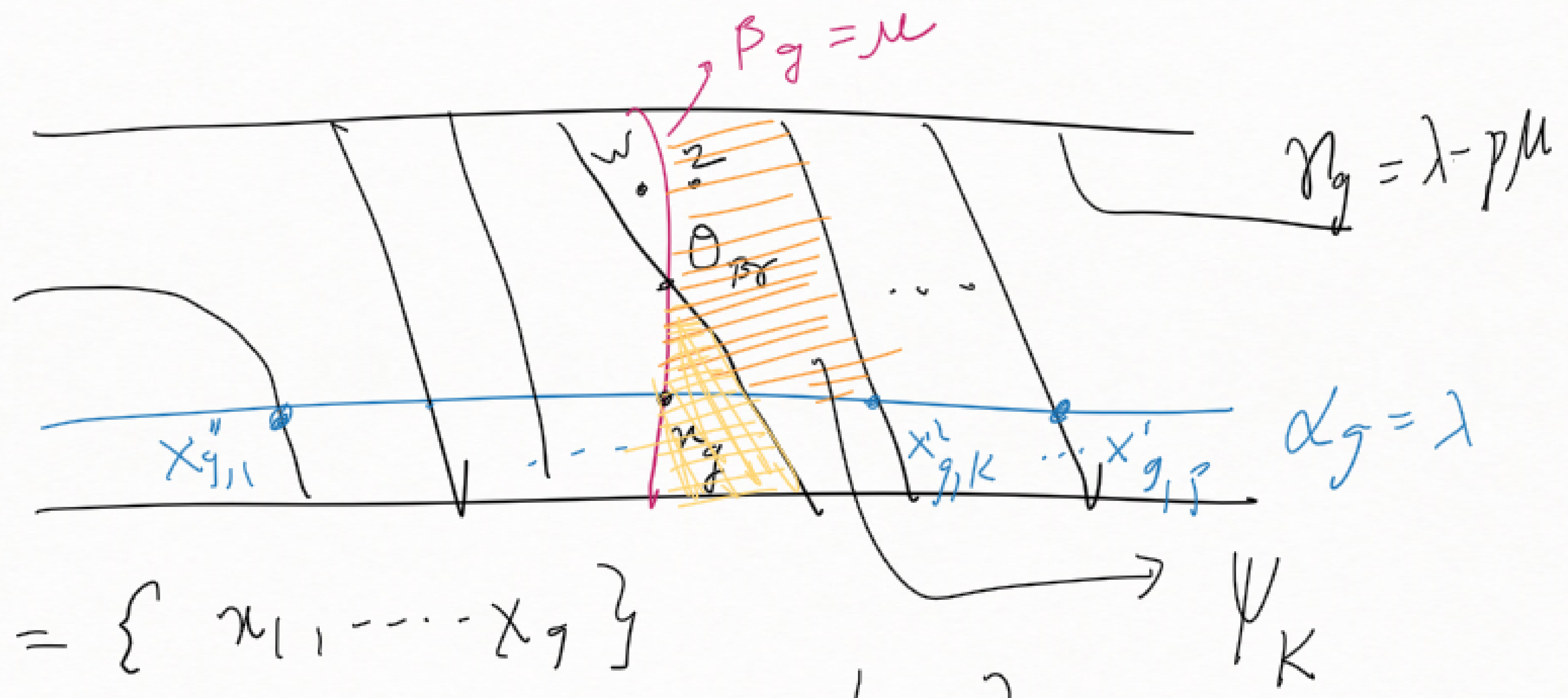
We want $RHS = 2m$

$$RHS = \langle c_1 (s(x)), [\hat{F}]\rangle + 2(i-j)$$

$$= \langle c_1(s(x)) + (i-j)PD[M], [\hat{F}]\rangle$$

$$= \langle tm \quad , [\hat{F}]\rangle = 2m \quad \square$$

We need Φ to be an iso.
We first define Φ_0. Eventually
$\Phi = \Phi_0 +$ lower order terms and
hence Φ_0 being an iso $\Rightarrow \Phi$ is iso.



$X = \{x_1, \cdots x_g\}$
$X'_K = \{x'_1, \cdots x'_{g-1}, x'_{g,K}\}$
x_i' unique closest to x_i

Define ψ_k as $\in \pi_2(x, \Theta_{\beta\gamma}, x'_k)$

$n_w(\psi_k) - n_z(\psi_k) \in \{-p/2, \ldots, p/2\}$

$\mu(\psi_k) = 0$ by RMT & $\# \mathcal{M}(\psi_k) = \pm 1$

Given $[x, i, j]$, for large enough p, $\exists$ unique k such that $n_w(\psi_k) - n_z(\psi_k) = i - j$

i.e $\underbrace{[\overset{x'}{\overset{\shortparallel}{x'_k}}, i - n_w(\psi_k), j - n_z(\psi_k)]}_{\overset{\shortparallel}{\Phi_0([x, i, j])}} \in \Phi(x)$

Define $\Phi_0([x, i, j])$.

Φ_0 is injective:

Given $(x', i - n_w(\psi))$, we first find x as closest pt to x'

Then $\exists !$ small $\Delta \in \pi_2(x, \theta_{\beta\gamma}, x')$

thus $n_w(\psi)$ is known $\Rightarrow$ we get i.

j is determined by the spin^c structure

Φ_0 is surjective

Φ is an isomorphism

We choose the winding region small then so that any ψ other than ψ^0 as earlier cannot be fully supported in the region and hence must be lower in the area filtration $\left(F(x) = -A(D(\psi)) - F(y)\right)$

Absolute graded version:

$K \subset Y$

$$HF^+_\ell(Y_p(K), [m]) = \begin{cases} H_{\ell-n}\left(CFK^{\{i\geq 0,\, j\geq -m\}}(Y, K, 0)\right) & |m| \leq g \\ HF^+_{\ell-n}(Y) & \text{oth} \end{cases}$$

$$n = \frac{p - (p+2m)^2}{4p}$$

When $|m| > g$ $\quad HF^+(K_0, m) = 0$

(adjunction inequality)

then exact

Surg. Δ

$$\begin{tikzcd} & & \widehat{HF}(S^3) \arrow[dll] & \\ \bigoplus_{m' \equiv m} HF^+(K_0, m') \arrow[rrr] & & & HF(K_p, [m]) \arrow[ul] \end{tikzcd}$$

making the seq iso without gradings

when $|m| \leq g$,

$$HF^+(K_{-p}, [m]) \simeq H_*(CFK^{i\geq 0 \text{ or } j\geq 0}(S^3, K, m))$$

$$H_*(\ldots^{i\geq 0 \text{ or } j\geq -m}(S^3, K, 0))$$

given by

$$(x, i, j) \mapsto [x, i, j-m]$$

Now need to check gradings

Note the Δ's in def of Φ are as subset of Δ's in surgery cob.

$$W_{-p} : S^3 \longrightarrow K_{-p}$$

δ's shift degree by

$$\frac{c_1(sw(\psi))^2 - 2\chi(W) - 3\sigma(W)}{4} = \frac{c_1(sw(\psi))^2 + 1}{4}$$

$\langle c_1(sw(\psi)), [S]\rangle - p = 2m \mod p$

$\Rightarrow c_1(sw(\psi)) = -\frac{1}{p}(2m+p)^2$

(Proof

Thus shift degree no seq.).

<u>Applications</u>: L-space knot: $\exists p$ $S^3_p(K)$ is L-space

$\Rightarrow \Delta_K(T)$

$$= (-1)^k + \sum_{j=1}^{k} (-1)^{k-j} \left(T^{n_j} + T^{-n_j}\right)$$

$$0 < n_1 < \cdots < n_k$$