Cusp Invariants: Dense or Knot?

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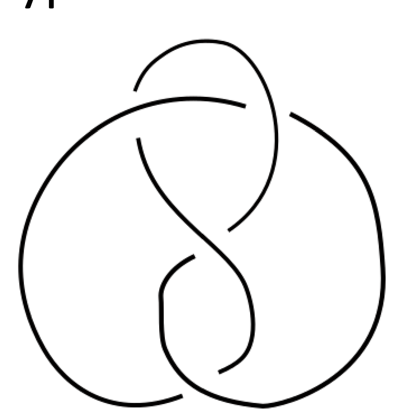
Abstract

A fascinating and fruitful way to study hyperbolic knots is to look at invariants found from their cusps. We show that cusp densities of knots are dense in the interval [0, 0.68…], and that cusp crossing densities of two component links are dense in the interval [0, 1.69..].

Definitions

Hyperbolic Knot: a non-self-intersecting closed circle embedded in $\mathbf{S}^3$ that has a complement that can be given a hyperbolic metric.



Volume=2.02988…

Volume=10.149…

Hyperbolic Volume (V): the volume of knot and link complements in $\mathbf{S}^3$.

Cusp of a hyperbolic knot: the intersection of a tubular neighborhood about a knot with the knot's complement.

Cusp Volume = $\sqrt{3}$

Cusp Volume (C_V): volume of the cusp in the complement.

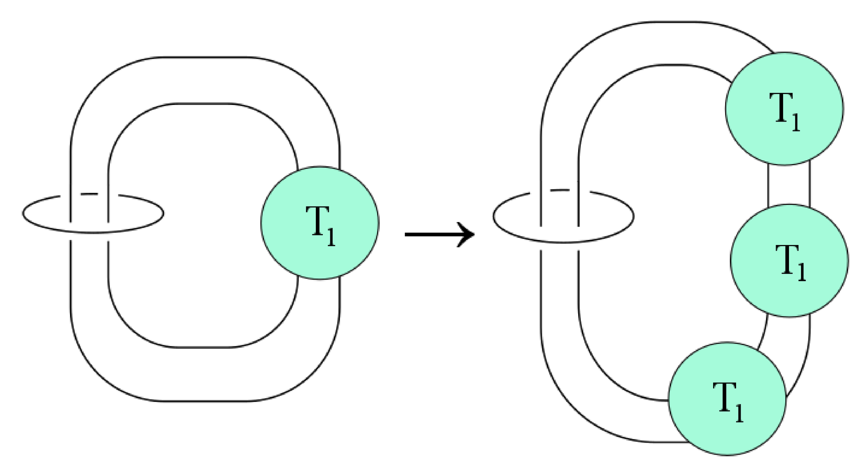
Cusp Invariants

$$Cusp\ Density = \frac{Cusp\ Volume}{Volume}$$

$$Cusp\ Crossing\ Density = \frac{Cusp\ Volume}{Crossing\ Number}$$

Constructions

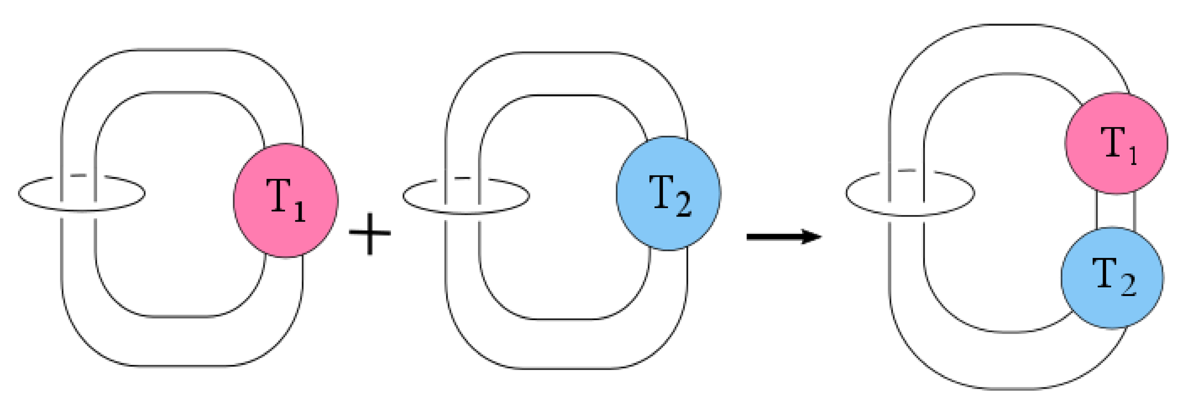
Cyclic Covers:



$$Volume = nV$$

$$Cusp\ Volume = nC_V$$

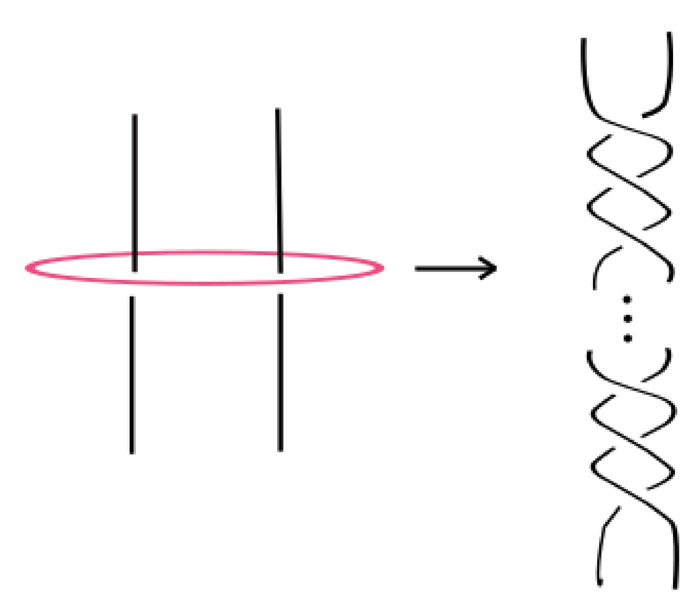
Belted Sum of Tangles



$$V = V_1 + V_2$$

$$C_V = C_1 + C_2\left(\frac{m_1}{m_2}\right)^2$$

Dehn Filling



$$\lim_{n\to\infty} V_n = V$$

$$\lim_{n\to\infty} C_n = C_V$$

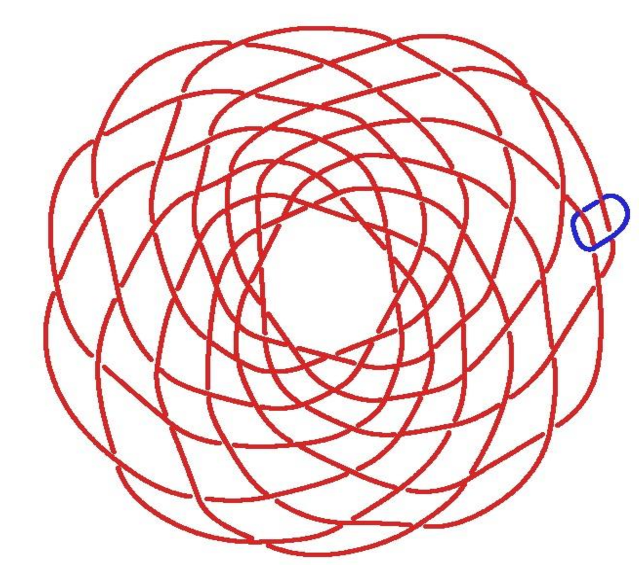
Results

Theorem: Knots have cusp density dense in the interval [0, 0.6826…]

Theorem: Links have cusp density dense in the interval [0, 0.853…]

Theorem: Two component links have cusp crossing density dense in the interval [0, dcc] where dcc= $\frac{total\ cusp\ volume}{crossing\ number-4}$.

Example: The flower link below has dcc=1.69…



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Acknowledgements

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