Geometry and Topology: Quick reference

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Part I

Notation

Part II

Themes

Study of properties that describe how a space is assembled, such as connectedness and orientability.

Topology: Studies properties that are preserved through deformations, twistings, and stretchings of objects. Tearing not allowed.

Part III

Euclidean and coordinate geometry

1 Euclidean geometry

See [1]. Deals with R^k, the Euclidian k space, which is described in the vector spaces survey.

Considers metric properties such as distances between points. Perpendiculars, parallels, projections.

Relationship between angles at the intersection of a line with parallels or in a triangle (and other polygons). Similarity and congruence of triangles (and other polygons). Angle bisectors, Medians in a triangle intersect at incenter, circumcenter.

Circle; lines, angles, triangles, chords, arcs in circles. Tangents, Intersection of circles. Ellipsoids, spheres.

Polytopes: an object in R^n whose boundary surfaces are flat. Convex polytopes: polytopes which are also convex sets.

1.1 Area/ volume

The notion of area/ volume in case of euclidean spaces corresponds to the box (Lebesgue) measure over the euclidean space. This is described in the vector spaces survey.

Ellipse, circle. Surface area of n-ball: $\frac{dV_n}{dr}$.

The r-radius $n-1$ hypersphere $S^{n-1} = \{x \in R^n : \|x\| = r\}$: a n-1 dim manifold. Encloses a n-ball with volume $V_n = \frac{\pi^{\frac{n}{2}} r^n}{\Gamma(\frac{n}{2}+1)}$ [**Find proof**].

1.2 Trigonometry

Obvious with the right construction: sin (A+B) = sin A cos B + cos A sin B. cos (A+B) = cos A cos B - sin A sin B. Trigonometry in triangle calculations. $\sin^{-1} x, \cos^{-1} x$.

2 Coordinate geometry

2.1 Vector spaces

(See linear algebra ref). Vector and scalar product of 2 vectors; effect of Left vs right handedness of coordinate system.

2.2 High dimensional objects

Get vector equations from geometric properties. Use linear transformations like scaling, rotation, projection to describe effects.

2.2.1 Hyperplane

Hyperplane $\perp w$ through $0:$ x such that $w^T x = 0$; shift c from 0: $w^T(x-c) = 0$. For halfspace, replace '=' with $\leq$.

2.2.2 Polyhedron

$\{x : Ax \leq b\}$: The intersection of halfspaces.

2.2.3 Simplex/ hypertriangle

n-d triagle. Construct from (d+1) hyperplanes with linearly independent w's.

2.2.4 Hypersphere surface

$x^*x = r$. Sift from 0: $(x-c)^*(x-c) = x^*x - 2c^*x = r'$.

2.2.5 Hyper-ellipse surface E

2.2.5.1 Aligned with std basis

Skewed norm-ball form E aligned with the standard basis: diagonal $\Sigma \succeq 0$. $\left\{x : x^*\Sigma x = r^2\right\}$ is hyper-ellipse aligned with the axes, skewed as per Σ. After rescaling: $\{x : x^*\Sigma x = 1\}$.

Matrix image form Take $\Sigma^{1/2}x = y$, assume $\Sigma \succ 0$. This is $\equiv E = \{My : \|y\| = 1\}$, where $\Sigma^{-1/2} = M \succ 0$.

Radii along major axes $\left\{\sigma_i^{-1/2} e_i\right\} \subset E$. So, radii are: $\left\{\sigma_i^{-1/2}\right\} = \{\sigma_i(M)\}$.

2.2.5.2 Aligned with arbitrary basis

Rotate previous ellipse. Take orthogonal rotator U and apply it to previous ellipses (do $y = U^*x$): major axes of E will then be aligned with U's columns.

Radii along major axes remains the same.

Rotated Skewed norm-ball form $\{x : x^*U\Sigma U^*x = 1\}$.

Matrix image form Take: $E = \{M'y : \|y\| = 1\}$, rotate to get: $M = M'U^* \succ 0$. $E = \{My : \|y\| = 1\}$

2.2.5.3 Shifted away from 0

Just use y = x-c.

Radii along major axes remains the same.

Shifted Rotated Skewed norm-ball form $\{x : (x-c)^*U\Sigma U^*(x-c) = 1\}$.

Using unscaled $M' = U\Sigma U^*$, the equation is : $(x-c)^*M'(x-c) - r^2 = x^*M'x - r^2 - 2x^*M'c = 0$.

Shifted Matrix image form $\{c + My | \|y\| = 1\}$ for $M \succ 0$. This can be reparametrized as: $\left\{x | \left\|M^{-1}x - M^{-1}c\right\| = 1\right\} = \{x : \|Ax + b\| = 1\}$.

2.2.5.4 Volume

Take the general expression for E: $\{c + My | \|y\| = 1\}$. $vol(E) \propto \prod r_i$, where $r_i = \sigma_i(M)$ are the radii along the major axes of the ellipsoid E. [**Find proof**]

So, $vol(E) \propto det(M^{-1}) = det(A^{-1})$.

For 2-D ellipsoids: $vol(E) = \pi \prod r_i$.

2.3 Other coordinate systems

Cylindrical and spherical coordinates: $x = r\cos\theta$.

2.4 Graph drawing

Find axis meeting points, maxima/ minima,
inflection points.

For visualization of functionals over vector spaces, their gradients: see linear algebra ref.

2.5 Manifold

Take any small enough area in a manifold: it resembles a euclidian space of a certain dimension, aka the manifold's dimension. 0 dim manifold: A point. 1 dim manifold: line, arc. 2 dim manifold: sphere surface.

Part IV

Metric spaces and topologies

3 Metric space S

Set with a metric $d : S \times S \to R^{\geq 0}$. Metric obeys non negativity, positive definiteness, symmetry, $\triangle$ inequality. Eg: Euclidian k space: R^k: every point is a vector.

Absolute value: $|x| = d(x, 0)$ for some 0.

3.1 Open ball around p of radius r

Aka r- neighborhood (nbd) of p: $N_r(p) = \{x \in S : d(x,p) < r\}$. Similarly define r-nbd of set of points S. A uniform nbd of S contains some r-nbd of S.

Set of open balls defines a topological space: topology from nbds. Similar topologies for vector spaces, manifolds.

Open ball in R^k is convex.

3.1.1 Interior point p of S

If $\exists r : N_r(p) \subset S$. $(0,0) \in N_2(1,1) \subset R^{\geq 0} \times R^{\geq 0}$ is interior pt. All others are boundary points. Thence defined interior of S: int(S), and boundary of S: bd(S) = cl(S) - int(S). If S has a non-empty interior, it is **solid!**

[0, 1] has an interior wrt R, but not wrt R^2: then every pt is in boundary.

3.1.2 Limit point p of set S

$\forall r : N_r(p)$ contains a pt in S other than itself.

$\forall r : |N_r(p)| = \infty$: Else, can find small r' with $N_{r'}(p) = \{p\}$. So, a finite set has no limit points.

p is the limit of some Cauchy sequence: Keep reducing r and pick $q \neq p \in N_r(p)$ in each step.

Every interior pt is a limit pt, but not vice versa. For $E \subset R$, sup(E) is a limit pt.

If p is a lt pt of E, $\exists$ convergent seq (s_i) in S with $s_i \to p$. Set with 1 limit pt: A convergent sequence in R.

3.1.3 Closure of E

cl(E): E with all its limit pts. Also: cl(E) = S - int(S - E).

3.1.4 Diameter of E

$diam(E) = \sup_{p,q} d(p,q)$. diam(E) = diam(cl(E)): by $\Rightarrow\Leftarrow$, using triangle inequality.

3.2 Sets in S: Topology

3.2.1 Nature of the boundary

3.2.1.1 Open set S

Aka Open space. For every $p \in S$ is an interior point. Diagramatic representation: [] and () in R, dotted an undotted lines in R^2. Eg: dotted dumbbell in R^2.

Open sets S_i: $\cup S_i$ is open. $S = \cap_{i=1}^{k} S_i$ is open: for any $p \in S$, pick r small enough to ensure $\forall i : N_r(p) \in S_i$.

If $S \subset Y \subset X$: S open wrt Y iff $\exists G \subset X$, G open wrt X and $S = G \cap Y$: [**Proof**]: i □f G open wrt Y, $G \cap Y$ open wrt Y; If S open wrt Y, take $\cup_{p \in S} N_r^X(p)$ where r is radius which shows interiorness of p in S.

3.2.1.2 Closed set S

Set with all its limit points. So, finite sets closed. $[n, \infty)$ closed. Same as S with all its boundary points.

$S \subset X$ closed iff S' open (good trick to show closedness). $\cap$ of closed sets S_i is closed: $\cup S_i'$ is open. Similarly, $\cap_{i=1}^{k} S_i$ is closed.

cl(E) is closed: as $(cl(E))'$ is open.

3.2.1.3 Non-oppositeness of Openness and Closedness

Eg: ϕ and R are both open and closed. (0, 1) open wrt R but nor wrt R^2. Half open intervals in R are neither open nor closed.

3.2.1.4 Boundedness of set S

A is bounded if $\exists r, p : A \subset N_r(p)$.

3.2.2 Compactness

3.2.2.1 Open cover of S

Bounded Open sets $\{G_i\}$ with $\cup G_i \supset S$. Subcovers: Subsets of open cover which also cover S.

3.2.2.2 Definition

Every open cover of S has a finite sub cover. In R^d, compactness $\equiv$ closed and bounded.

3.2.2.3 Properties

Finite S is compact. R is not compact: Take $G_n = (n - \frac{2}{3}, n + \frac{2}{3})$, $\{G_{n \in Z}\}$ is an open cover, but no finite or even proper subcover. Similarly, $[n, \infty]$ closed but not compact.

Any compact set S is closed: Any $p \in S'$ is interior pt in S': $\cup_{q \in S} N_r(q) : r = \frac{d(p-q)}{2}$ is an open cover of S, within it is some finite subcover; so $\exists N_{r'}(p) \subset S'$.

Closed subset E of compact set S is compact: Take any open cover of E; add open set E' to it to get open cover of S; some finite subset of this without E' is also open cover of E.

Finite union of compact sets is compact.

If F closed and K compact, $F \cap K \subset K$ compact: $F \cap K$ is closed.

If $\{K_i\}$ is (possibly ∞) set of compact sets and if $\cap$ of every finite subclass $\neq \phi$, $\cap K_i \neq \phi$. Assume $\cap K_i = \phi$; Take K_1; every $p \in K_1$ is $\notin \cap_{i \neq 1} K_i$; so $p \in \cup_{i \neq 1} K_i'$; so finite subset of $\{K_i'\}$ is an open cover of K_1; so some finite $\cap$ of $\{K_i\}$ is ϕ: contradiciton.

So, if $\{K_i\}$ compact, $K_n \supset K_{n+1}$: $\cap_i K_i \neq \phi$. Does not hold for open sets: Take $G_n = (0, n^{-1})$.

If E is an ∞ subset of compact set K, E has a limit pt in K: Else every $p \in K$ would have some $N_r(p) = \{p\}$; $\cup N_r(p)$ is an open cover of E without a finite subcover. Also, if every $E \subset K, |E| = \infty$ has a lt pt in K, K is compact. [**Find proof**]

If $\{K_i\}$ compact, $K_n \supset K_{n+1} \neq \phi$, $\lim_{n\to\infty} diam(K_n) = 0$, then $\cap K_n$ is 1 pt: else $\Rightarrow\Leftarrow$.

3.2.3 Connectedness and completeness

3.2.3.1 Connectedness

A, B separated if $A \cap cl(B) = cl(B) \cap A = \phi$. Eg: (0, 1) and (1, 2) but not (0,1] and (1, 2). S is connected if it is not $\cup$ of separated sets.

$E \subset R$ connected iff it is an interval.

3.2.3.2 Dense set

Contains points in the neighborhood of every point.

3.2.3.3 Completeness of S

Limit of every Cauchy sequence (s_n) wrt metric = some point $s \in S$.

Any closed set in complete metric space S is complete. Also, any compact space is complete.

3.2.4 Sigma algebra of open sets

Aka Borel Sigma algebra. This is the sigma algebra $(X, \mathbf{S})$ formed by the closure wrt $\cup, \cap, \bar{X}$ of all open sets in X. All sets in $\mathbf{S}$ are called Borel sets.

3.3 Covering and packing Number

Let the space have norm $\|\|$, and let C be a set in it.

3.3.1 Covering number $N(\epsilon, C, \|\|)$

ϵ covering F_ϵ: Set of ϵ balls which contains C. Covering number $N(\epsilon, C, \|\|) = \min|F_\epsilon|$.

3.3.1.1 Covering entropy

Aka metric entropy. $\log(N(\epsilon, C, \|\|))$.

3.3.1.2 Total boundedness

If $N(\epsilon, C, \|\|)$ is finite for all ϵ, C is totally bounded. Else, C is non totally bounded: for every n, there is some $\epsilon : N(\epsilon, C, \|\|) > n$.

3.3.1.3 For D dim sphere

$\frac{Vol(sphere(r_1))}{Vol(sphere(r_2))} = (\frac{r_1}{r_2})^D$. Let $vol(B(f', \epsilon)) = k\epsilon^D$. Then,
$k(R+\epsilon)^N \geq N(\epsilon, C, \|\|)k\epsilon^D \geq kR^D$. Thence, $\log(N(\epsilon, C, \|\|)) \approx D\log(\frac{R}{\epsilon})$.

3.3.2 Packing number $M(\epsilon, C, \|\|)$

ϵ packing is a set of points $\{g_i\}$ with $g_i \in C; \|g_i - g_j\| \geq \epsilon$. The maximal ϵ packing: packing number.

3.3.2.1 Relationship with N

$M(2\epsilon, C, \|\|) \leq N(\epsilon, C, \|\|) \leq M(\epsilon, C, \|\|)$. 2nd ineq: For maximal packing $\{g_i\}$, $\forall h \in$C: $\|g_i - h\| \leq \epsilon$. 1st ineq: For maximal 2ϵ packing: Any ϵ ball has ≤ 1 g_i.

3.3.2.2 Use

Often easier to find than covering number; thence can bound covering number.

3.4 R^k: Topological properties

See complex analysis ref.

3.5 Sequence (s_n) in S

For properties of sequences in fields and vector spaces, see complex analysis and linear algebra ref.

3.5.1 Cauchy sequence

After some point, elements get closer as sequence progresses: contraction or Cauchy criterion: $\forall m, n > N : d(p_m, p_n) < \epsilon$ or diameter of tail of seq tends to 0. Limit of sequence may not exist in S. Like convergence without needing a limit.

Any cauchy seq S in compact set X converges: As X compact, S has limit pt in X, also limit of S is unique.

3.5.2 Bounded sequences

Range is bounded.

3.5.3 Convergent sequence

Convergence to limit c: $\forall i > N : d(x_i, c) < \epsilon : x_n \to c$. Divergence. Limit is unique. If $x_n \to c$, every $N_r(c)$ has all but finitely many x_i.

Any convergent sequence is bounded. 1^n convergent but has finite range. If range not 1, it is ∞.

All convergent sequences are cauchy sequences.

Every subsequence of a convergent sequence converges to the same limit. If every subsequence of a sequence converges to the same limit, the sequence is convergent.

Sequence (s_n) in compact S has convergent subsequence: If S compact, every ∞ subset has limit pt p; make seq out of s_i in decreasing $N_r(p)$.

3.5.4 Subsequential limits

Take seq s_n, subsequential limits form closed set E: Take any limit pt p of E, can find subseq limit e close to it, so can find s_n close to it; so p is in E.

3.6 Function across metric spaces: f:X to Y

See algebra ref for general properties of functions. Also ref on analysis of functions over R and C.

3.6.1 Limit of f

$\lim_{x\to p} f(x) = q : \forall\epsilon, \exists\delta : 0 < d(x,p) < \delta \implies d(f(x),q) < \epsilon$: f has a limit at p. q is unique. Visualize as balls in X, f(X).

$\forall(p_n), p_n \to p, f(p_n) \to q \equiv lt_{x\to p} f(x) = q$: show $\implies$ by $\Rightarrow\Leftarrow$. So, can use properties of sequences. So, get $\lim f+g, f(x)g(x), f/g$.

3.6.2 Continuity of f:X to Y

f continuous at $p \in E$ if $\forall\epsilon\exists\delta : d(x,p) < \delta \implies d(f(x),f(p)) < \epsilon$. If f has limit at p, continuity iff $\lim_{x\to p} f(x) = f(p)$: f defined only over p has no limit at p but is continuous. Continuity over $E \subseteq X$.

If f continuous at p, g continuous at f(p), then f(g(x)) continuous at p.

f continuous over X iff $\forall$ open $V \subseteq Y$, $f^{-1}(V)$ open in X: Visualize interior pts, match δ balls in X with ϵ balls in Y.

If f continuous, X compact, then f(X) compact: Take open cover $\{V_i\}$ of f(X); $\left\{f^{-1}(V_i)\right\}$ is open, covers X; so take finite subcover; get $f(X) \subseteq \cup_{i=1}^{k} f(f^{-1}(V_i)) \subseteq \cup_{i=1}^{k} V_i$.

If f continuous, bijection, then f^{-1} is cont: f(V) open iff V is open.

If f continuous, $E \subseteq X$ connected, then f(E) connected: else if f(E) separated into A, B but $f^{-1}(A) \cup f^{-1}(B)$ not separated, $cl(f^{-1}(A)) \cap f^{-1}(B) \neq \phi$ or $cl(f^{-1}(B)) \cap f^{-1}(A) \neq \phi$; then continuity of f violated, so $\Rightarrow\Leftarrow$.

3.6.3 Uniform continuity over X

$\forall p,q \in X \forall\epsilon > 0, \exists\delta :$
$d_x(p,q) < \delta \implies d_y(f(p),f(q)) < \epsilon$. $1/x$ continuous, but not uniformly cont over R: consider points near 0; neither is x^2. A measure of whether gradient gets very big.

If f continuous, X compact, then f uniformly cont: As Y compact: Given ϵ, take $\forall p \in X : g(p)$, radius which guarantees $\epsilon/2$ closedness to $f(p)$; make open cover $\left\{N_{g(p)}\right\}$; get finite subcover; take max $g(p)$; use $\triangle$ ineq to guarantee ϵ closedness anywhere.

Also see the more powerful notion of absolute continuity in the complex analysis survey.

3.6.4 Bounding steepness

Aka Lipschitz continuity/ smoothness. Lipschitz condition: $d(f(x), f(y)) \leq Ld(x, y)$. L is lipshcitz constant. Note that it implies the usual notion of continuity.

But, it does not imply differentiability! When differentiable, there is a relationship with the derivative, see complex analysis ref.

3.6.4.1 A generalization

Holder continuity: Holder condition of order a: $d(f(x), f(y)) \leq Ld(x, y)^a$.

3.7 Sequence of functions (f_n :$\mathbf{X}\to Y$)

Consider the properties of sequence of functions from any set to a metric space, which is described in the survey on basic mathematical structures.

If x is a limit pt of $E \subseteq X$, $lt_{t\to x} f_n(t) = A_n$, then A_n converges, $lt_{t\to x} f(t) = li_{n\to\infty} A_n$. Pf: $d(f(t), A) \leq d(f(t), f_n(t)) + d(f_n(t), A_n) + d(A_n, A)$: make 1st and 3rd terms small by picking large N, make 2nd term small by picking large t.

So, if (f_n) continuous, f continuous: see $lt_{t\to x} f_n(t) = f_n(x)$, get $lt_{t\to x} f(t) = lt_{t\to x} f_n(x) = f(x)$.

4 Point set topology

4.1 Motivation

Coffee cup and donut are geometrically different, but topologically same: isotopes! Can deform one to the other. Generalize notions of convergence, connectedness, continuity.

4.2 The topological space

Set of points or Topological space X. Topology T: Class of some sets of points closed under $\cup, \cap$.

Sets S_i in T are said to be open. S_i' are closed sets. Neighborhood of p is a set $V \supset$ open set $U \ni p$. Similarly define nbd of set S of points. $A \in X$ is dense if any nbd has some $a \in A$.

Spanning set (of sets); its linear span. Basis of topology.

4.3 Topological Morphisms

Every 'object' in Y is a continuous function f :X→ Y, where X and Y are topological spaces. A tea-cup is a function to R^3.

4.3.1 Homotopy

Take 2 objects/ cont functions f, g :X→ Y. Homotopy is continuous function $H : X \times [0, 1] \to Y$, with $H(x, 0) = f; H(x, 1) = g$. Think of second parameter as time, and H as a continuous deformation.

If $H(x, t)$ is also 1:1, H is an isotopy.

4.3.2 Continuous morphism

f(x) neighborhood corresponds to x neighborhood.

4.3.3 Homeomorphism

A bicontinuous fn: $X \to Y$. Respects topological properties.

4.4 Knots

A circular piece of thread. The simple ring or the unknot. The trefoil. Sketching knots. Strands: segments involved in a cross-over.

4.5 The 3 Reidemeister moves

See wikipedia article for figures. Sufficient and necessary to produce any valid deformation possible from a starting configuration.

4.5.1 Knot invariants

Property invariant to the Reidemesiter moves. 3 colorability of strands: Can assign 3 colors to strands such that all 3 colors are used; at each crossing, 3 or 1 colors seen.

Bibliography

[1] Hall and Stevens. *School Geometry*. Macmillian, 1906.