5) $Q(x) = \int_x^\infty p_w(x) \cdot dx$

$= \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-x^2/2} dx$ $\boxed{\frac{x}{\sqrt{2}} = t}$

$dx = \sqrt{2} dt$.

$= \frac{1}{2} \cdot \left(\frac{2}{\sqrt{\pi}} \int_{x/\sqrt{2}}^\infty e^{-t^2} dt \right)$

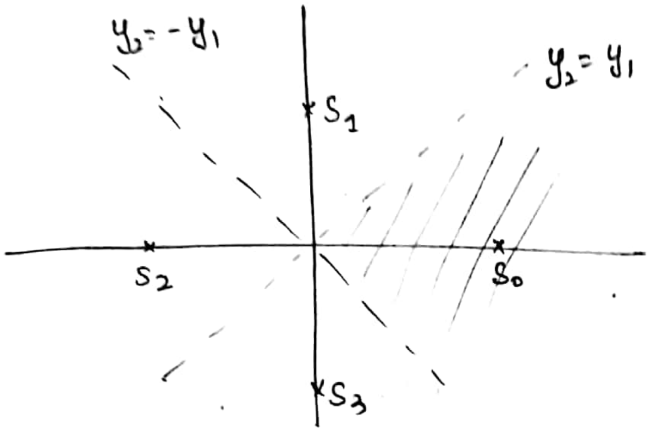
$= \frac{1}{2} erfc(x/\sqrt{2})$ if $erfc = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2} dt$.

Scanned with CamScanner

Problem 12 :

$s \in \{s_0, s_1, s_2, s_3\}$ — QPSK

$$s_0 = \begin{pmatrix} \sqrt{E_s} \\ 0 \end{pmatrix}, \quad s_1 = \begin{pmatrix} 0 \\ \sqrt{E_s} \end{pmatrix}, \quad s_2 = \begin{pmatrix} -\sqrt{E_s} \\ 0 \end{pmatrix}, \quad s_3 = \begin{pmatrix} 0 \\ -\sqrt{E_s} \end{pmatrix}$$



For uniform distribution, MAP $\approx$ ML

MAP for detecting s_0 results in

$$y_2 < y_1 \quad \cap \quad y_2 > -y_1$$
$$\Rightarrow -y_2 < y_1$$

$$\Rightarrow \quad |y_2| < y_1 .$$

Problem 13, 14, 15, 16, 17

$X = n_2 - n_1$, $Y = -n_2 - n_1$ $\qquad E[X] = 0 - 0 = 0$

$n = (n_1, n_2)$ $\qquad E[Y] = -0 - 0 = 0$

$P_r(\hat{S} = S_0 | S = S_0)$

FSK $\Rightarrow$ $S_0 = \begin{pmatrix} \sqrt{E_s} \\ 0 \end{pmatrix}$ $\quad S_1 = \begin{pmatrix} 0 \\ \sqrt{E_s} \end{pmatrix}$

$r_1 = \begin{pmatrix} \sqrt{E_s} + n_1 \\ 0 + n_2 \end{pmatrix}$ for S_0

$r_2 = \begin{pmatrix} n_1 \\ \sqrt{E_s} + n_2 \end{pmatrix}$ for S_1

Correlation co-efficient :-

$$\rho = \frac{E[(x-\mu_x)(y-\mu_y)]}{\sigma_x \sigma_y}$$

X and Y are uncorrelated if $\rho = 0$

$$Cov(n_1, n_2) = E(n_1 n_2) - E(n_1)E(n_2)$$
$$= 0 - 0$$
$$= 0$$

Formula

$$Var(aX + bY) = a^2 var(X) + b^2 var(Y) + 2ab\, Cov(X, Y)$$

$$Var(Y) = Var((-1)n_1 + (-1)n_2) = 1\left(\frac{N_0}{2}\right) + 1\left(\frac{N_0}{2}\right) + 0$$
$$= N_0$$

$$Var(X) = Var((-1)n_1 + 1(n_2)) = (-1)^2\left(\frac{N_0}{2}\right) + (1)^2\left(\frac{N_0}{2}\right) - 0$$
$$= N_0$$

$\therefore$ $X, Y \sim N(0, N_0)$ — (16)

Covariance $(X, Y) = 0$ if X and Y are independent

$$Cov(X, Y) = E[(n_2 - n_1)(-n_2 - n_1)] - E(X)E(Y)$$
$$= E[-n_2^2 + n_1^2 - \cancel{n_1 n_2} + \cancel{n_1 n_2}] - 0$$
$$= E[n_1^2 - n_2^2] = E[n_1^2] - E[n_2^2]$$
$$= \frac{N_0}{2} - \frac{N_0}{2} = 0$$

$Cov(X, Y) = 0 \Rightarrow \rho = 0$ (uncorrelated) — (14)

(15) $E[XY] - E[X]E[Y] = 0$

$E[XY] = 0 \Rightarrow$ X and Y are independent

$$P_{XY}(x, y) = P_X(x) P_Y(y)$$

$$\Pr(\hat{S}=S_0 \mid S=S_0) = \Pr(|y_2| < y_1)$$

$$= \Pr(|n_2| < \sqrt{E_s}+n_1)$$

$$= \Pr((n_2 < \sqrt{E_s}+n_1) \cap (-n_2 < \sqrt{E_s}+n_1))$$

$$= \Pr((\underbrace{n_2-n_1}_{X} < \sqrt{E_s}) \cap (\underbrace{-n_2-n_1}_{Y} < \sqrt{E_s}))$$

X and Y are independent

$$= \Pr(X<\sqrt{E_s})\Pr(Y<\sqrt{E_s}) \qquad \text{using } 15$$

$$\Pr(\hat{S}=S_0 \mid S=S_0) = \Pr(X<\sqrt{E_s}, Y<\sqrt{E_s}) \quad \text{---} \quad \cancel{18} \quad 17$$

Problem 18 :-

X and Y are independent $N \sim (0, N_0)$

$$\Pr(X<\sqrt{E_s}, Y<\sqrt{E_s}) = \Pr(X<\sqrt{E_s})\Pr(Y<\sqrt{E_s})$$

$$= \left(1-Q\left(\frac{\sqrt{E_s}}{\sqrt{N_0}}\right)\right)\left(1-Q\left(\frac{\sqrt{E_s}}{\sqrt{N_0}}\right)\right)$$

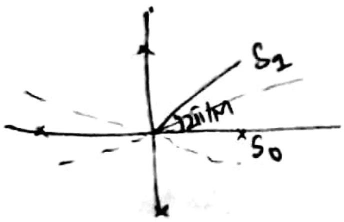
$$= \left(1-Q\left(\sqrt{\frac{E_s}{N_0}}\right)\right)^2.$$

Problem 20:-

4 M-PSK

$$S_i = \begin{pmatrix} \cos\left(\frac{2\pi i}{M}\right) \\ \sin\left(\frac{2\pi i}{M}\right) \end{pmatrix} \qquad i = 0, 1, \ldots, M-1$$

$$y|S_0 = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} \sqrt{E_s}+n_1 \\ n_2 \end{pmatrix} \qquad n_1, n_2 \sim N\left(0, \frac{N_0}{2}\right)$$



$$\Pr(y|S_0) = \Pr\left(\left(\tan^{-1}\left(\frac{y_2}{y_1}\right) \le \frac{\pi}{M}\right) \text{ or } \tan^{-1}\left(\frac{y_2}{y_1}\right) \le -\frac{\pi}{M}\right)$$

$$= P_r\left(\tan^{-1}\left(\frac{n_2}{\sqrt{E_s}+n_1}\right) \le \frac{\pi}{M}\right) P_r\left(-\tan^{-1}\left(\frac{n_2}{\sqrt{E_s}+n_1}\right) \le \frac{\pi}{M}\right)$$

$$= P_r\Big(\underbrace{n_2 - n_1\tan\tfrac{\pi}{M}}_{X} \le \frac{\sqrt{E_s}}{\tan\frac{\pi}{M}}\Big) P_r\Big(\underbrace{n_2 + n_1\tan\tfrac{\pi}{M}}_{Y} \le \sqrt{E_s}\tan\tfrac{\pi}{M}\Big)$$

$E[X]=0$ $\quad E[Y]=0$

$Var(X) = \left(\sec^2\frac{\pi}{M}\right)\left(\frac{N_0}{2}\right)$ $\quad Var(Y) = \left(\sec^2\frac{\pi}{M}\right)\left(\frac{N_0}{2}\right)$

$$= \left(1 - Q\left(\frac{\sqrt{E_s}\tan\frac{\pi}{M}}{\sqrt{\frac{N_0}{2}}\sec\frac{\pi}{M}}\right)\right)^2$$

$$P_r\left(\hat{y}=S_0 \middle| S=S_0\right) = \left(1 - Q\left(\sqrt{\frac{2E_s}{N_0}}\sin\frac{\pi}{M}\right)\right)^2 \longrightarrow \text{correct estimation}$$

Problem 25:-

$\therefore$ Probability of error

$$P_{e|S_0} = P_r\Big(\underbrace{n_2 - n_1\tan\tfrac{\pi}{M}}_{X} \ge \sqrt{E_s}\tan\tfrac{\pi}{M}\Big) + P_r\Big(\underbrace{n_2 + n_1\tan\tfrac{\pi}{M}}_{Y} \ge \sqrt{E_s}\tan\tfrac{\pi}{M}\Big)$$

$Y, X \sim N\left(0, \frac{N_0}{2}\sec^2\frac{\pi}{M}\right)$ (proved above)

$$= 2Q\left(\frac{\sqrt{E_s}\tan\pi/M}{\sqrt{\frac{N_0}{2}}\sec\frac{\pi}{M}}\right)$$

$$P_{e|S_0} = 2Q\left(\sqrt{2\left(\frac{E_s}{N_0}\right)}\sin\frac{\pi}{M}\right)$$

Problem 21:-

$$y|s_0 = \begin{pmatrix} \sqrt{E_s}+n_1 \\ n_2 \end{pmatrix} \qquad R\cos\theta = \sqrt{E_s}+n_1 \Rightarrow n_1 = R\cos\theta - \sqrt{E_s}$$

$$R\sin\theta = n_2$$

$$\cancel{P_r(R,\theta)} = $$

$$P_r(n_1,n_2) \quad \frac{1}{\sqrt{\frac{2\pi N_0}{2}}} e^{\frac{-n_1^2}{2N_0/2}} \quad \frac{1}{\sqrt{\frac{2\pi N_0}{2}}} e^{-\frac{n_2^2}{2N_0/2}}$$

$$n_1^2+n_2^2 = R^2 + E_s - 2R\sqrt{E_s}\cos\theta$$

$$P_r(R,\theta) = \frac{R}{\pi N_0} e^{-\frac{(R^2-2R\sqrt{E_s}\cos\theta + E_s)}{N_0}}$$

Problem 22:-

$$\underset{\alpha\to\infty}{Lt} \int_0^\infty (v-\alpha) e^{-(v-\alpha)^2} dv$$

$$v-\alpha = x$$

$$dv = dx$$

$$\underset{\alpha\to\infty}{Lt} \int_{-\alpha}^{\infty} x e^{-x^2} dx$$

$N \sim (0, \frac{N_0}{2})$

$$x = \frac{k}{\sqrt{2}} \qquad = \underset{\alpha\to\infty}{Lt} \int_{-\sqrt{2}\alpha}^{\infty} \frac{k}{\sqrt{2}} \frac{1}{\sqrt{2}} e^{-k^2/2} dk$$

$$= \frac{\sqrt{\pi}}{\sqrt{2}} \underbrace{\int_{-\infty}^{\infty} \frac{k}{\sqrt{2\pi}} e^{-k^2/2} dk}_{E[k]=0}$$

$$= \frac{\sqrt{\pi}}{\sqrt{2}}(0) = 0.$$

Problem 23:-

$$\int_0^\infty v e^{-[(v-\sqrt{r}\cos\theta)^2 + (\sqrt{r}\sin\theta)^2]} dv$$

$$= \int_0^\infty (v - \sqrt{r}\cos\theta + \sqrt{r}\cos\theta) e^{-((v-\sqrt{r}\cos\theta)^2)} e^{-(\sqrt{r}\sin\theta)^2} dv$$

$$= e^{-r\sin^2\theta}\underbrace{\int_0^\infty (v-\sqrt{r}\cos\theta)e^{-(v-\sqrt{r}\cos\theta)^2}dv}_{\text{as } r\to\infty \;=\; 0} + e^{-r\sin^2\theta}\int_0^\infty \sqrt{r}\cos\theta\, e^{-(v-\sqrt{r}\cos\theta)^2}$$

$$= e^{-r\sin^2\theta}\sqrt{r}\cos\theta\sqrt{\pi}\,.$$

Problem 24:-

$$I = 1 - \sqrt{\frac{r}{\pi}}\int_{-\pi/M}^{\pi/M} e^{-r\sin^2\theta}\cos\theta\, d\theta$$

$$\sqrt{r}\sin\theta = x$$

$$\sqrt{r}\cos\theta\, d\theta = dx$$

$$= 1 - \sqrt{\frac{r}{\pi}}\int_{-\sqrt{r}\sin\frac{\pi}{M}}^{\sqrt{r}\sin\pi/M} e^{-x^2}\frac{dx}{\sqrt{r}} \qquad \boxed{x = \frac{y}{\sqrt{2}}}$$

$$dx = \frac{dy}{\sqrt{2}}$$

$$= 1 - \frac{1}{\sqrt{\pi}}\int_{-\sqrt{2r}\sin\pi/M}^{\sqrt{2r}\sin\pi/M} \frac{1}{\sqrt{2}} e^{-y^2/2}dy$$

$$I = 1 - \int_{-\sqrt{2r}\sin\frac{\pi}{M}}^{\sqrt{2r}\sin\pi/M} \frac{1}{\sqrt{2\pi}} e^{-y^2/2}dy$$

$$I = 1 - \Pr\left(-\sqrt{2r}\sin\frac{\pi}{M} < Y < \sqrt{2r}\sin\frac{\pi}{M}\right)$$

where $Y \sim N(0,1)$

(or)

$$I = Pr\left(Y < -\sqrt{2r}\sin\frac{\pi}{M}\right) + Pr\left(Y > \sqrt{2r}\sin\frac{\pi}{M}\right)$$

$$I = 1 - Q\left(-\sqrt{2r}\sin\frac{\pi}{M}\right) + Q\left(\sqrt{2r}\sin\frac{\pi}{M}\right)$$

$Y \sim N(0,1)$