Introduction to Quantum Noise

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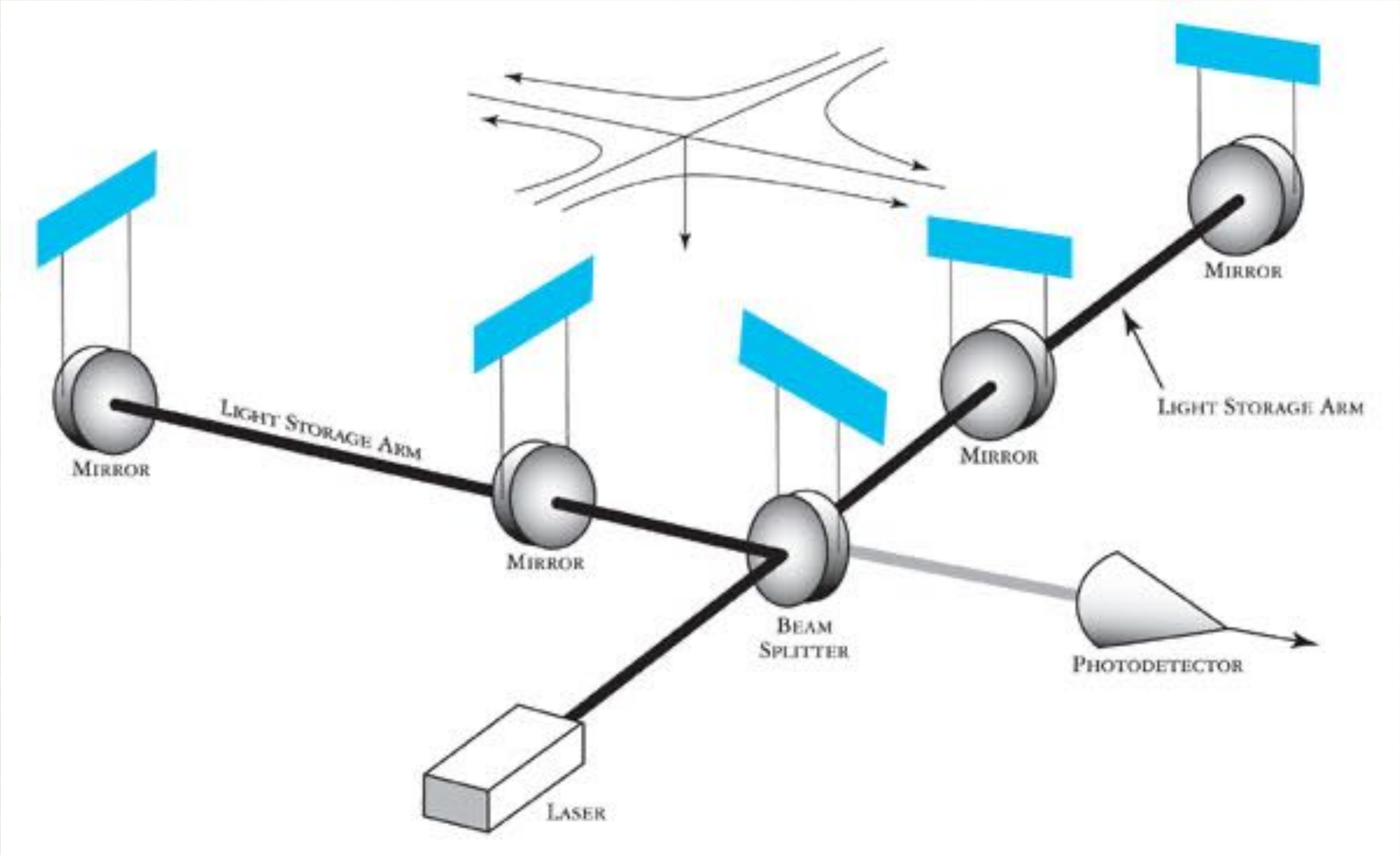
What is quantum?

Spacetime?
Mirrors?
Light?



GW strain predicted by GR is a tiny, but classical, quantity

- (Spacetime) Displacement rms ~10^{-19} m, while the Planck length ~10^{-35} m
- (Mirrors) Measurement of non-commuting observables. Heisenberg Uncertainty Principle (HUP)
  - $[x, p] = i\hbar$
  - $[x(0), x(t)] = \dfrac{4i\hbar t}{m}$
- (Light) Laser light is noisy also because of HUP
  - (1) radiation pressure, (2) shot noise
  - Coherent state $\Longrightarrow$ standard quantum limit
  - Squeezed states $\Longrightarrow$ Heisenberg limit (move the noise to where we don't see it)

AIM

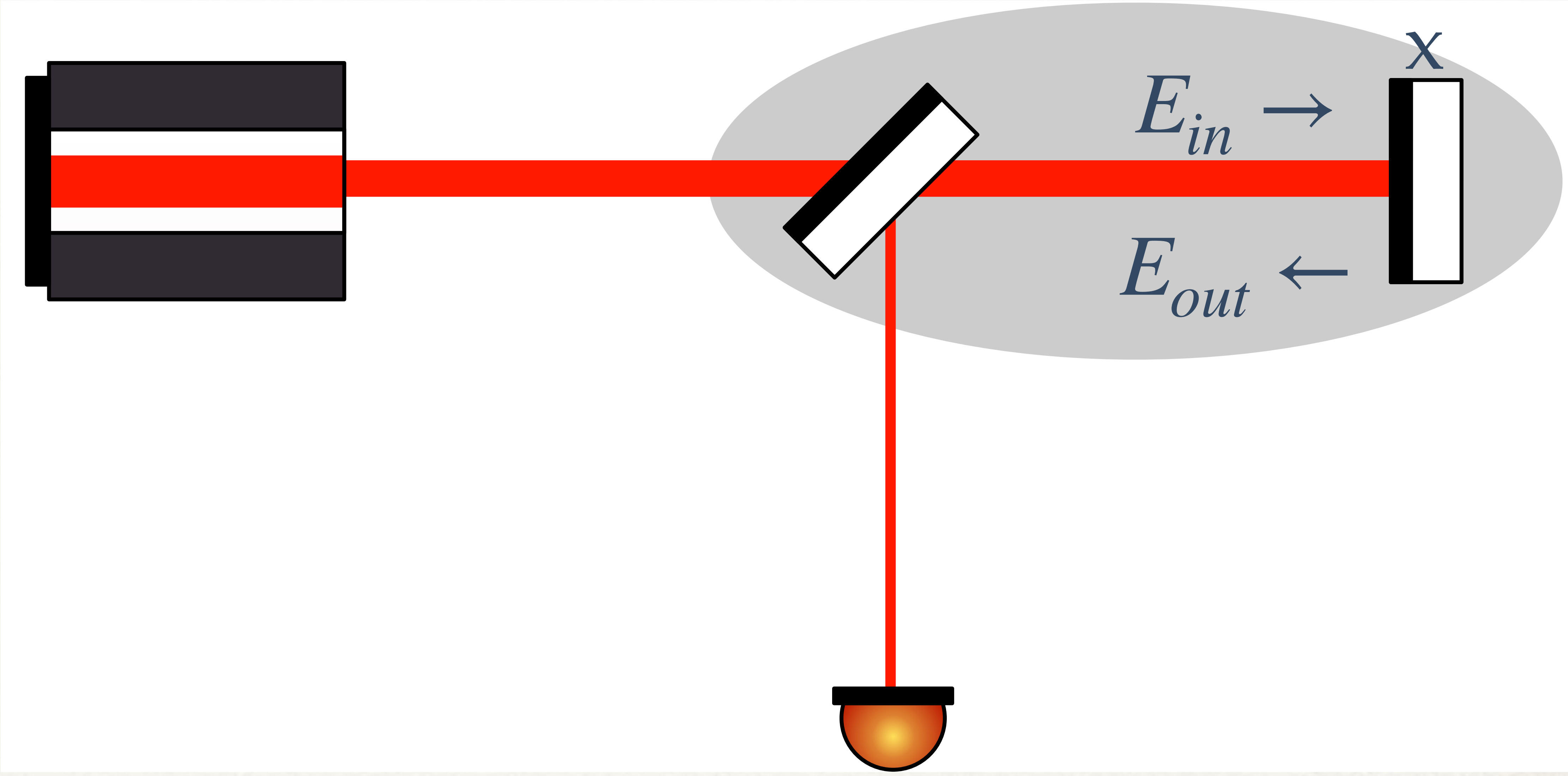
The aim of the talk is to try and answer the following questions:

1. How does quantum mechanics contribute to additional noise in LIGO and other ‘classical’ precision measurements?
   - Basic quantum optics
   - The standard quantum limit (SQL) and where it comes from
   - What is the *fundamental* limit to the precision of these measurements?
2. How can quantum mechanics be used to minimise the noise?
   - Squeezing
   - Different readout schemes [DC, Homodyne, Heterodyne, etc.]
   - Quantum non-demolition measurements and other ideas (briefly)

Quantum Harmonic Oscillator (QHO)

- Test mass: QHO
- Light fields: QHO's

Effective differential arm motion



Sec 1 of Jupyter notebook

Intro to QHO

- $\mathbf{H} = \dfrac{\mathbf{p}^2}{2m} + \dfrac{1}{2}m\omega^2\mathbf{x}^2$ This describes an object trapped in a quadratic potential.
- $[\mathbf{x}, \mathbf{p}] = i\hbar \implies \Delta x \Delta p \geq \hbar/2$
- $\mathbf{x} = \sqrt{\dfrac{\hbar}{2m\omega}}\left(\mathbf{a} + \mathbf{a}^\dagger\right)$, and $\mathbf{p} = \sqrt{\dfrac{\hbar m\omega}{2}}\left(\mathbf{a}^\dagger - \mathbf{a}\right)$
- The Hamiltonian looks much nicer: $\mathbf{H} = \hbar\omega\left(\mathbf{a}^\dagger\mathbf{a} + \dfrac{1}{2}\right)$ where $\mathbf{N} = \mathbf{a}^\dagger\mathbf{a}$

Optical cavity: harmonic potential for the EM field

- $\mathbf{E}_1 \leftrightarrow \mathbf{x}, \quad \mathbf{E}_2 \leftrightarrow \mathbf{p}$: the non-commuting quadrature operators of the EM field

- $\mathbf{E}_1 = \left(\frac{\mathbf{a}+\mathbf{a}^\dagger}{2}\right), \mathbf{E}_2 = \left(i\frac{\mathbf{a}^\dagger-\mathbf{a}}{2}\right)$ in terms of photon creation and annihilation operators $\implies \Delta E_1 \Delta E_2 \geq 1/2$

- $E(t) \sim E_1 \cos \omega t + E_2 \sin \omega t$

- The Hamiltonian is exactly the same for a single mode with frequency ω :
$\mathbf{H} = \hbar\omega\left(\mathbf{a}^\dagger\mathbf{a} + \frac{1}{2}\right)$

Amplitude and phase quadrature

- $\mathbf{E}_1$ is known as the *amplitude* quadrature and $\mathbf{E}_2$ the *phase* quadrature, as affected by amplitude and phase modulation, respectively

- Carrier: $E_0 \cos(\omega_o t)$

- <u>AM</u> with mod. depth δA: $E_0(1 + \delta A \cos(\omega_1 t))\cos(\omega_o t)$
$$= E_0 \cos(\omega_0 t)\left(1 + \frac{\delta A}{2}(e^{i\omega_1 t} + e^{-i\omega_1 t})\right)$$

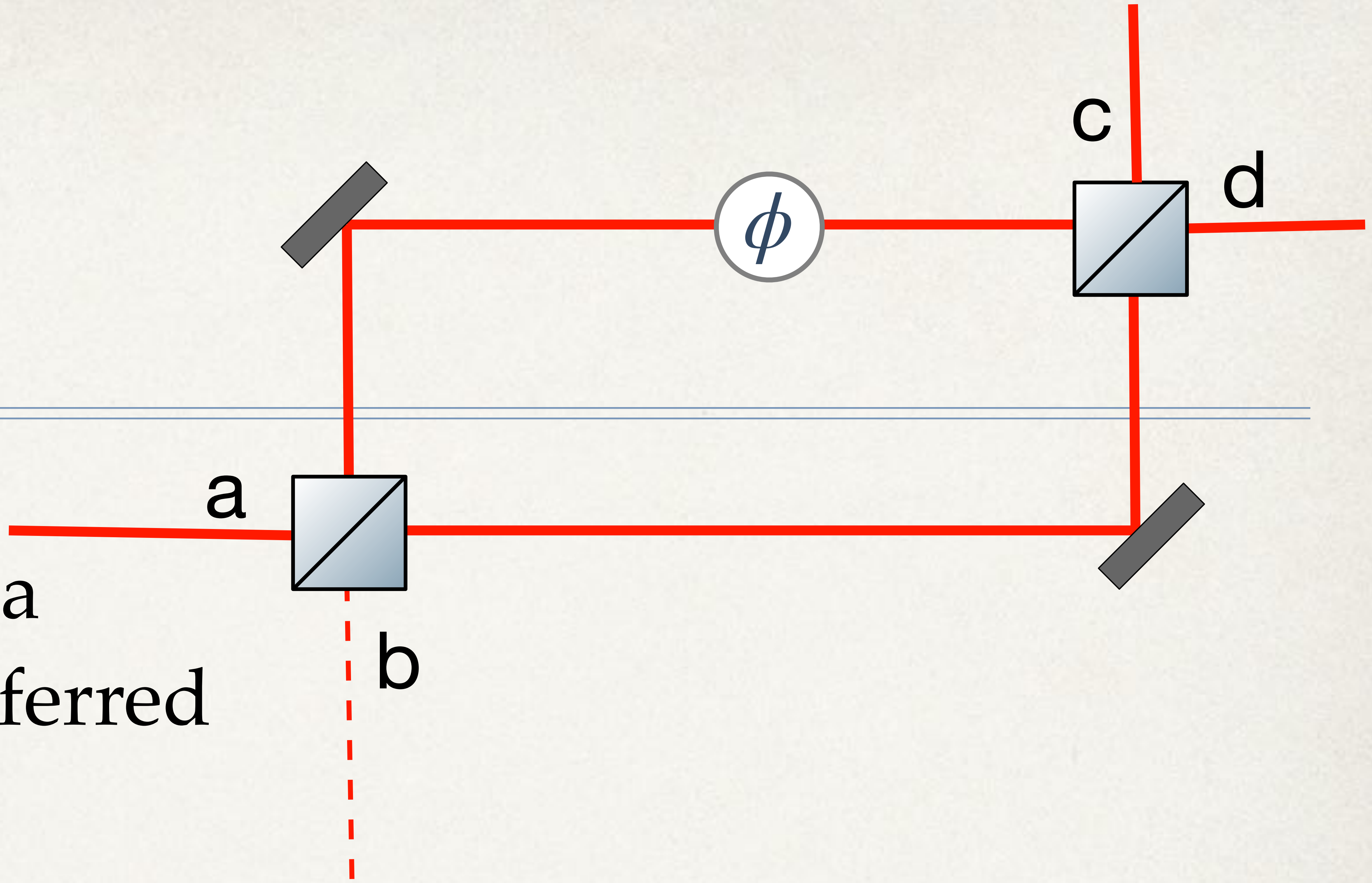
- <u>PM</u> with mod. depth $\delta\phi$: $E_0 \cos(\omega_o t + \delta\phi \cos(\omega_2 t))$
$$\approx E_0 \cos(\omega_0 t) - E_0 \sin(\omega_0 t)\left(1 + \frac{\delta\phi}{2}(e^{i\omega_2 t} + e^{-i\omega_2 t})\right)$$

Some useful optical states

- Photon number eigen-states: {|n>}
- The 'classical' coherent state: $|\alpha\rangle = e^{-|\alpha^2|/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle$
    - For a single-mode l of the EM field, $\langle E(\vec{r}, t)\rangle = i\vec{\epsilon_l}\sqrt{\frac{\hbar\omega_l}{2\epsilon_0 V}}\left(\alpha_l e^{i\vec{k}\cdot\vec{r}-i\omega t} - \alpha_l^* e^{-i\vec{k}\cdot\vec{r}+i\omega t}\right)$
    - $\Delta E_1 \Delta E_2 = 1/2$ and $\Delta E_1 = \Delta E_2 = 1/2$ [Minimum uncertainty possible for classical light]
- Squeezed coherent state:

Number-phase uncertainty

Sec 2 of Jupyter notebook



- Hard to define a phase operator, but phase is still a measurable quantity, whose uncertainty can be inferred indirectly (just like the energy-time uncertainty)

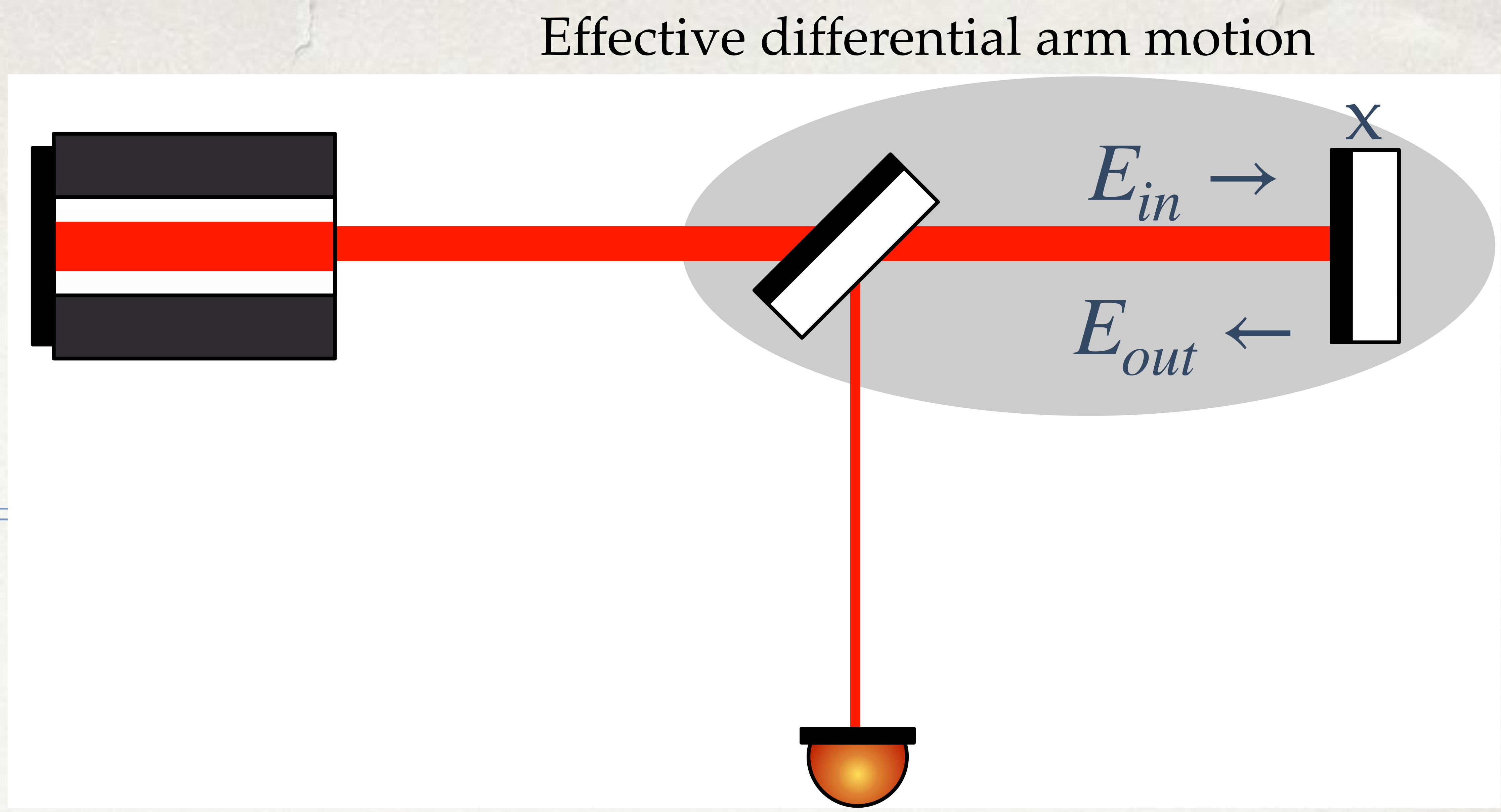
- $$\begin{pmatrix} c \\ d \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} e^{i\phi} & 0 \\ 0 & 1 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \times \begin{pmatrix} a \\ b \end{pmatrix} = e^{i\phi/2} \begin{pmatrix} \cos\phi & i\sin\phi \\ i\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}$$

- $\mathbf{D} = \mathbf{c}^\dagger\mathbf{c} - \mathbf{d}^\dagger\mathbf{d} = \cos\phi \left(\mathbf{a}^\dagger\mathbf{a} - \mathbf{b}^\dagger\mathbf{b}\right) + i\sin\phi \left(\mathbf{a}^\dagger\mathbf{b} - \mathbf{b}^\dagger\mathbf{a}\right)$

- At $\phi = \pi/2$: $\mathbf{D} = i(\mathbf{a}^\dagger\mathbf{b} - \mathbf{b}^\dagger\mathbf{a})$ and $|\mathrm{d}\langle\mathbf{D}\rangle/\mathrm{d}\phi| = |\langle\mathbf{a}^\dagger\mathbf{a} - \mathbf{b}^\dagger\mathbf{b}\rangle|$

- For coherent state input, $\Delta\phi = \dfrac{\Delta D}{|\mathrm{d}\langle\mathbf{D}\rangle/\mathrm{d}\phi|} \sim 1/\sqrt{N}$

Standard Quantum Limit



- $\mathbf{H}_{int} = \mathbf{x}\mathbf{F}_{rad} = \alpha\mathbf{x}\mathbf{E}_{in,1}(t)$
  - $\mathbf{E}_{out,1}(t) = \mathbf{E}_{in,1}(t),$
  - Change in x impacts phase of returning field:

    $$\mathbf{E}_{out,2}(t) = \mathbf{E}_{in,2}(t) + \alpha(t)\int \mathrm{d}t' G_x(t-t')[\alpha(t')\mathbf{E}_{in,1}(t') + F_{GW}(t')]$$

- In the frequency domain, $G_x(\omega)$ is the mechanical TF of the test mass:

$$\mathbf{E}_{out,2}(\omega) = \mathbf{E}_{in,2}(\omega) - \kappa\mathbf{E}_{in,1}(\omega) + \frac{\sqrt{2\kappa}}{h_{SQL}} h_{GW}(\omega)$$

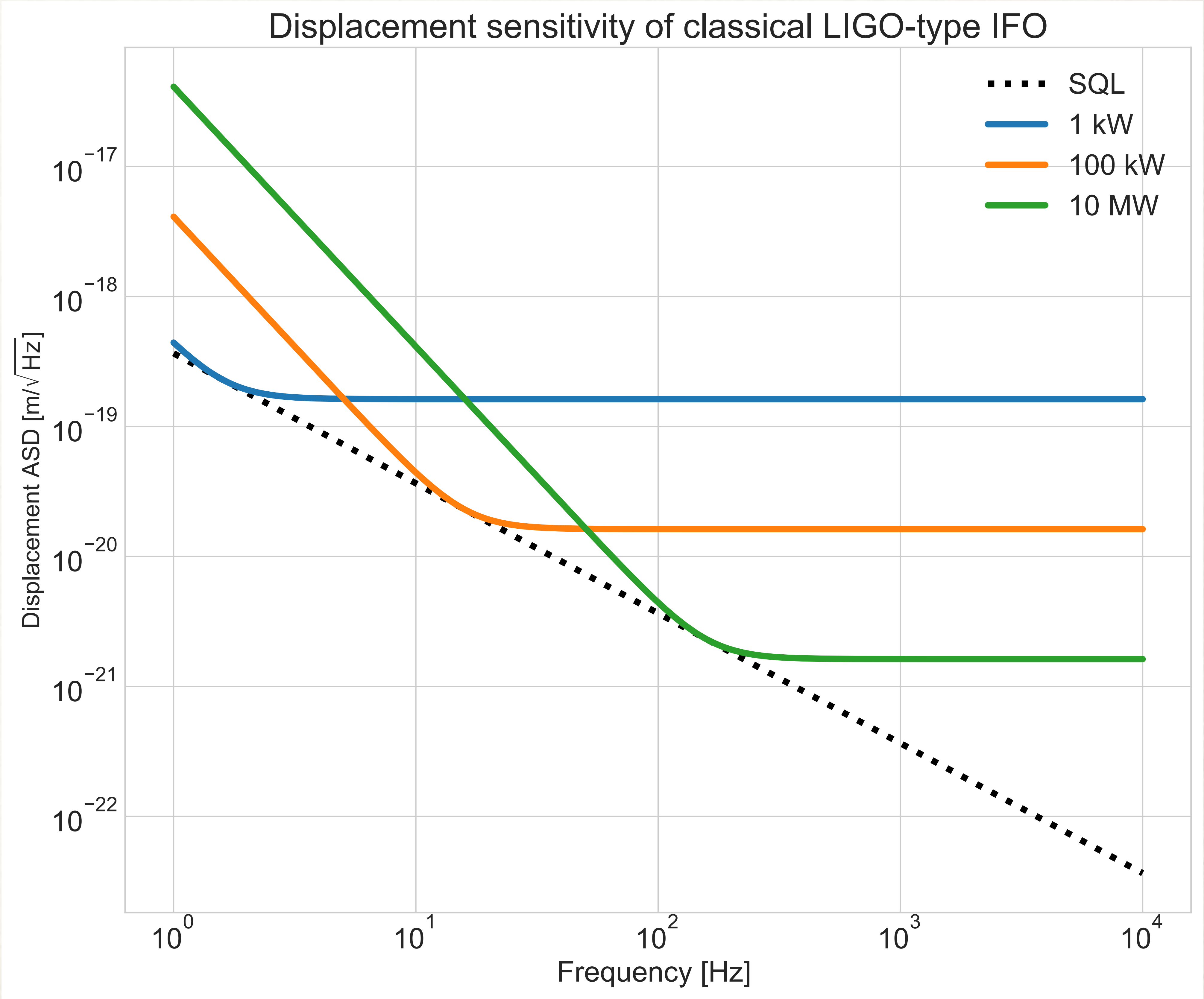
- $\kappa = -\frac{\alpha^2}{\hbar} G_x(\omega) \propto \frac{P}{M\omega^2}$ for a free mass, $h_{SQL}^2 = \frac{2\hbar}{M\omega^2 L^2}$

Understanding the SQL

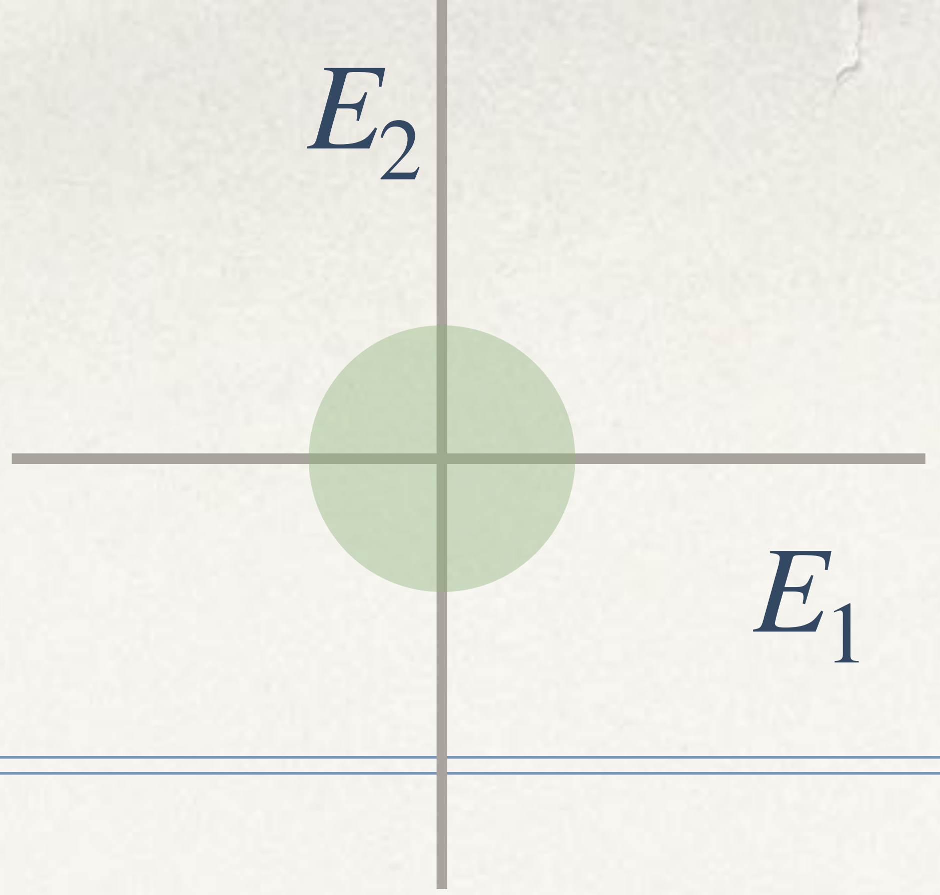
- $[\mathbf{x}(t), \mathbf{x}(t')] = \dfrac{4i\hbar(t'-t)}{m} \implies \Delta x(t)\Delta x(t') \geq \dfrac{2\hbar|t'-t|}{m}$
  (ignore light; include 4 test mass mirrors)

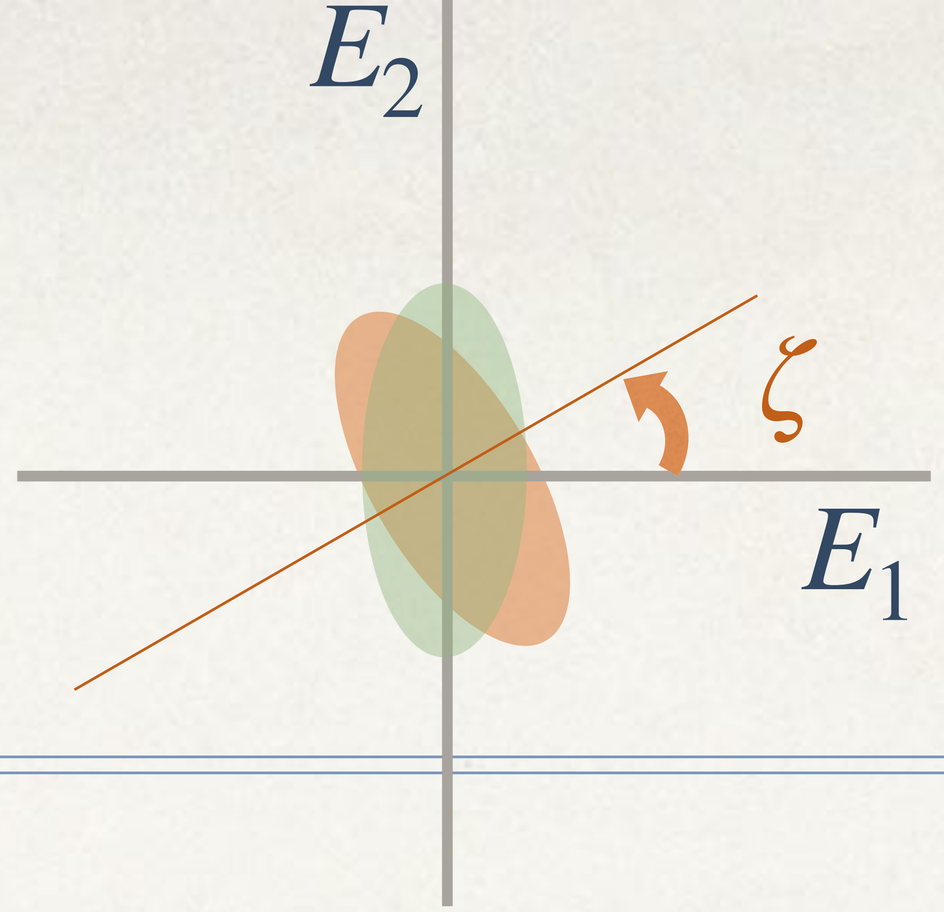
- Minimum classical uncertainty:
  $\Delta x(t) = \Delta x(t') = \sqrt{S_x(\omega_{min})/|t'-t|} \implies$ the SQL, where
  $\omega_{min} \sim 1/|t'-t|$

- $S_h(\omega) = \left(\dfrac{1}{\kappa} + \kappa\right)\dfrac{h^2_{SQL}}{2}$
  - At low ω, $S_h \propto P$
  - At high ω, $S_h \propto 1/P$



Squeezed light





- $\psi_\alpha(E_{1,2}) = \dfrac{1}{\pi^{1/4}} e^{-(E_{1,2}-\sqrt{2}\alpha)^2/2}$

- Single-mode squeeze operator: $\mathbf{S}(r,\zeta) = \exp[re^{-i\zeta}\mathbf{a}^{\dagger 2} - (re^{i\zeta}\mathbf{a}^2)/2]$

- For $\zeta = 0$:
  - $\mathbf{E}_1(r) = \mathbf{S}^\dagger(r)\mathbf{E}_1(0)\mathbf{S}(r) = \mathbf{E}_1(0)e^r$
  - $\mathbf{E}_2(r) = \mathbf{S}^\dagger(r)\mathbf{E}_2(0)\mathbf{S}(r) = \mathbf{E}_2(0)e^{-r}$

- $$\begin{pmatrix} \mathbf{E}_1(r,\zeta) \\ \mathbf{E}_2(r,\zeta) \end{pmatrix} = \begin{pmatrix} \cos\zeta & -\sin\zeta \\ \sin\zeta & \cos\zeta \end{pmatrix} \begin{pmatrix} e^r & 0 \\ 0 & e^{-r} \end{pmatrix} \begin{pmatrix} \cos\zeta & \sin\zeta \\ -\sin\zeta & \cos\zeta \end{pmatrix} \begin{pmatrix} \mathbf{E}_1(0) \\ \mathbf{E}_2(0) \end{pmatrix}$$

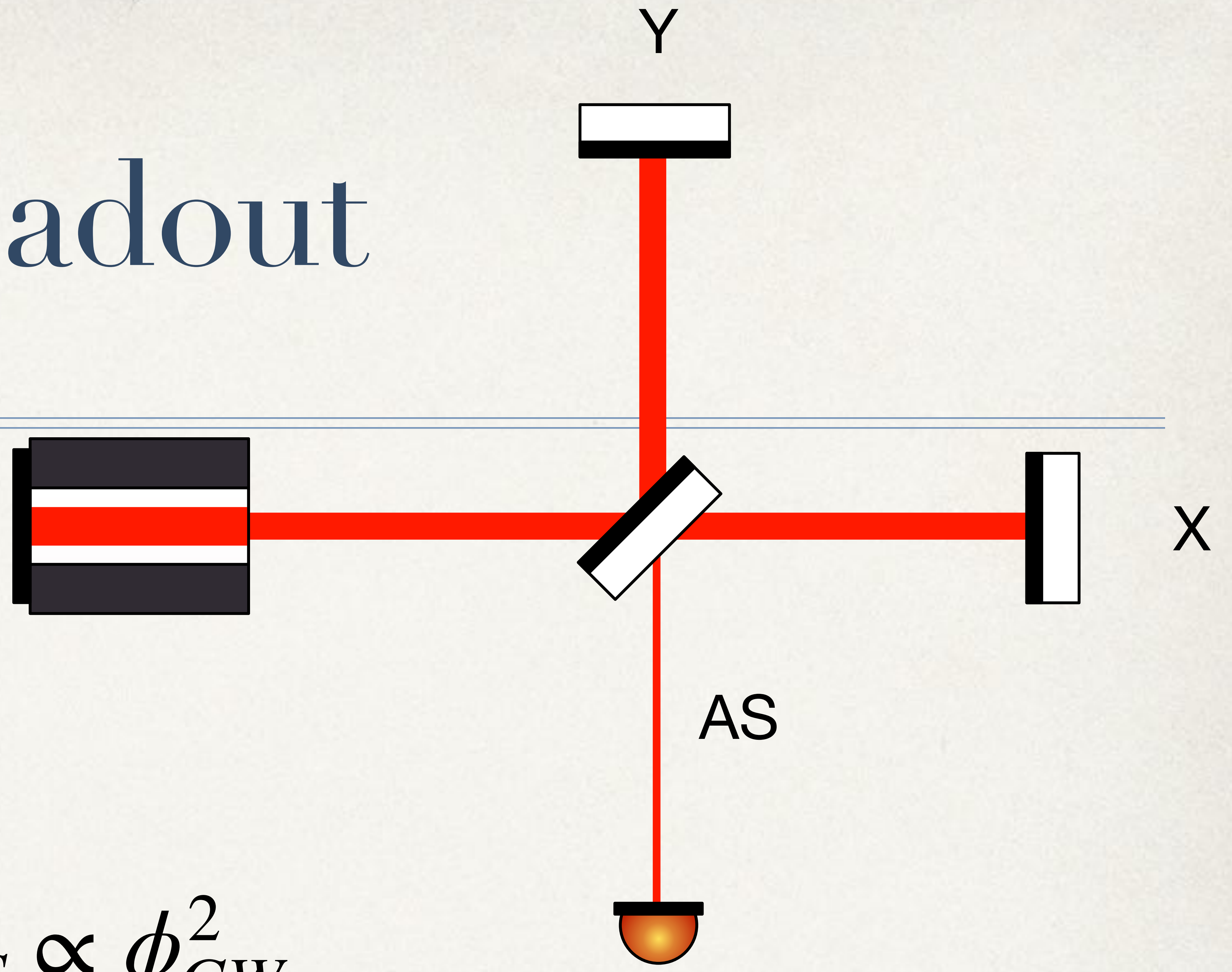
- $S_h^{sqz} = e^{-2r} S_h$ if optimal squeezing angle is selected for each frequency

Build correlations: ponderomotive squeezing

- $S_{xx}(\omega)S_{FF}(\omega) - |S_{xF}(\omega)|^2 \geq \hbar^2$
- $S_{xx} = L^2 S_h$

In progress or will skip

Readout schemes : (1) DC readout



- $E_{AS} = \frac{E_{in}}{2}\left(e^{i\phi_X} - e^{i\phi_Y}\right)$

- If $\phi_X = -\phi_Y = \phi_{GW}$, $E_{AS} \simeq iE_{in}\phi_{GW} \implies P_{AS} \propto \phi_{GW}^2$

- But with a DC offset, i.e, $\phi_X - \phi_Y = 2\phi_{DC} + 2\phi_{GW}$, we have $E_{AS} \simeq iE_{in}(\phi_{GW} + \phi_{DC}) = A_{GW} + A_{DC}$

- O/P now linear $\implies P_{AS} = |A_{DC}|^2 + 2\mathrm{Re}[A_{DC}A_{GW}^*]$

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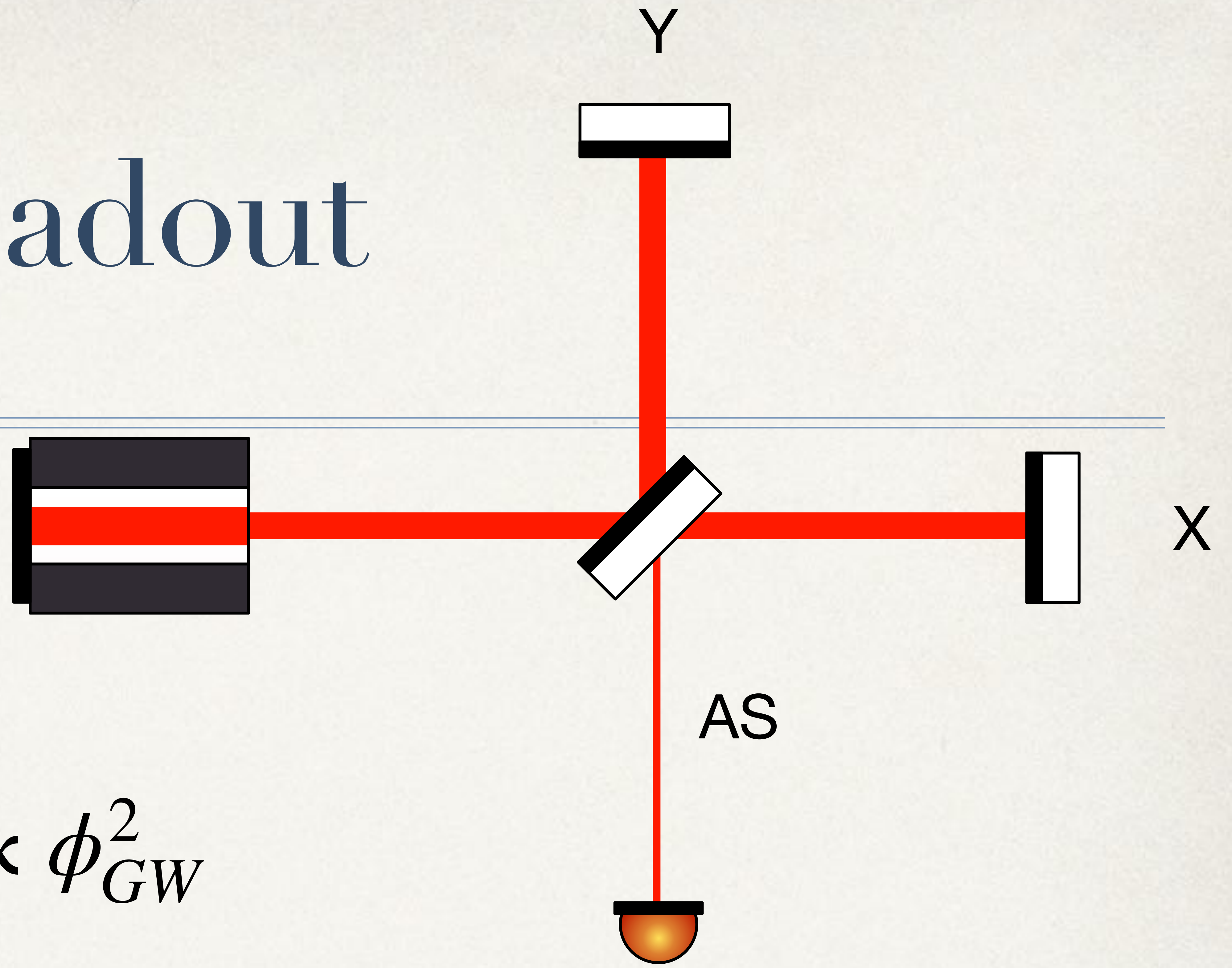

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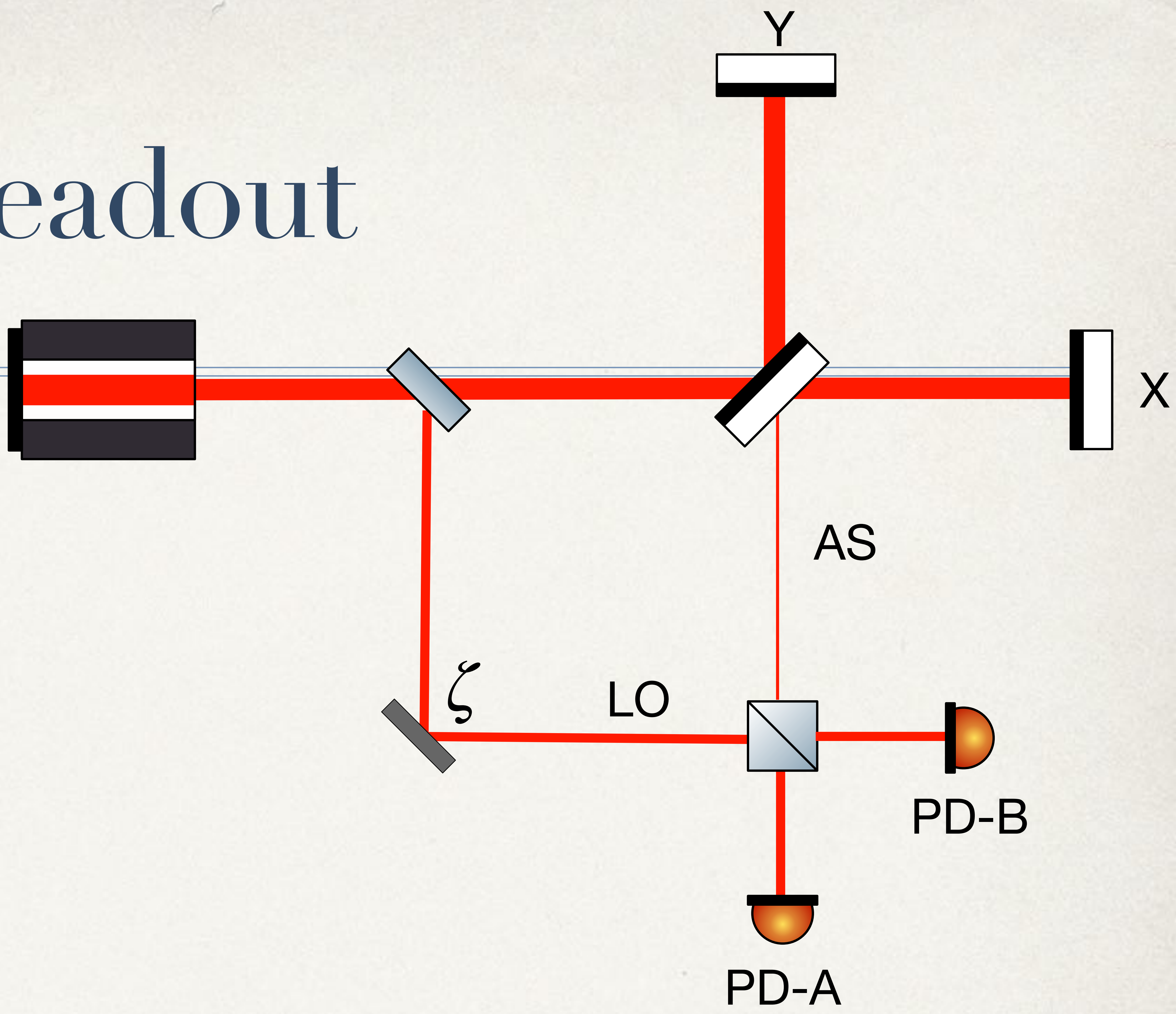
- O/P now linear $\implies P_{AS} = |A_{DC}|^2 + 2\mathrm{Re}[A_{DC}A_{GW}^*]$
  - (Exercise: find ϕ_{DC} where sensitivity is maximum)

(2) Balanced Homodyne Readout



- $$P_A = \frac{P_{LO}}{2} + \mathrm{Re}[E_{LO}A^*_{GW}e^{i\zeta}]$$

$$P_B = \frac{P_{LO}}{2} - \mathrm{Re}[E_{LO}A^*_{GW}e^{i\zeta}]$$

- $$P_A - P_B = 2\mathrm{Re}[e^{i\zeta}E_{LO}A^*_{GW}]$$

- Only the linear GW signal, ζ can be optimized with squeezing for maximum sensitivity

References

- *Macroscopic Quantum Mechanics* Yanbei Chen (2013)
- *The noise in gravitational-wave detectors and other classical-force measurements is not influenced by test-mass quantisation* Braginsky et. al. (2002)
- Thesis of Haixing Miao (2010)
- *Quantum measurement theory in gravitational-wave detectors* Danilishin and Khalili (arxiv 1203.1706)
- *Balanced homodyne readout for quantum limited gravitational wave detectors* Fritschel, Evans and Frolov (2014)