BAN 602: Quantitative Fundamentals

Spring, 2020 Lecture Slides – Week 2

CAL STATE EAST BAY

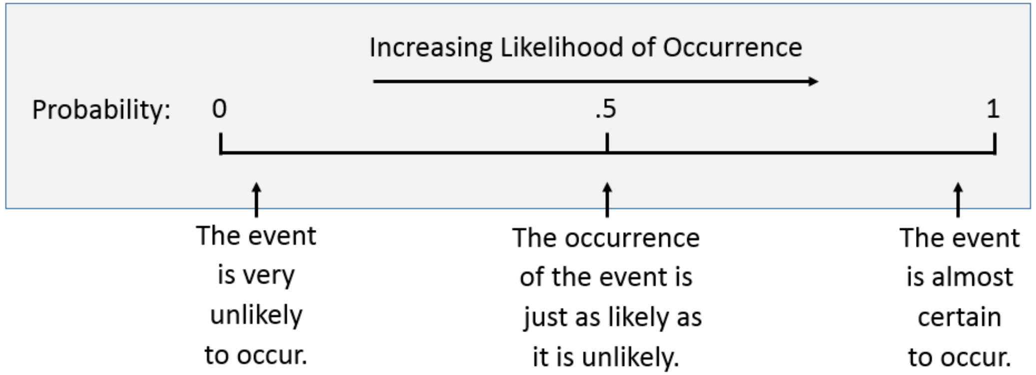
Agenda

- Introduction to Probability
- Discrete Probability Distributions
- Continuous Probability Distributions

Uncertainties and Probability

- Managers often base their decisions on an analysis of uncertainties such as the following:
  - What are the chances that the sales will decrease if we increase prices?
  - What is the likelihood a new assembly method will increase productivity?
  - What are the odds that a new investment will be profitable?

- Probability is a numerical measure of the likelihood that an event will occur.
  - Probability values are always assigned on a scale from 0 to 1.
  - A probability near zero indicates an event is quite unlikely to occur.
  - A probability near one indicates an event is almost certain to occur.



Random Experiment and Its Sample Space

- A Random experiment is a process that generates well-defined experimental outcomes.
- The sample space for an experiment is the set of all experimental outcomes.
- An experimental outcome is also called a sample point.

Experiment	Experimental Outcomes
Toss a coin	Head, tail
Inspect a part	Defective, non-defective
Conduct a sale call	Purchase, no purchase
Roll a die	1, 2, 3, 4, 5, 6
Play a football game	Win, lost, tie

Random Experiment and Its Sample Space

Example: Bradley Investments

Bradley has invested in two stocks, Markley Oil and Collins Mining. Bradley has determined that the possible outcomes of these investments three months from now are as follows:

Investment Gain or Loss in 3 Months (in $1000s):

Markey Oil	Collins Mining
10	8
5	-2
0	EMPTY CELL
-20	EMPTY CELL

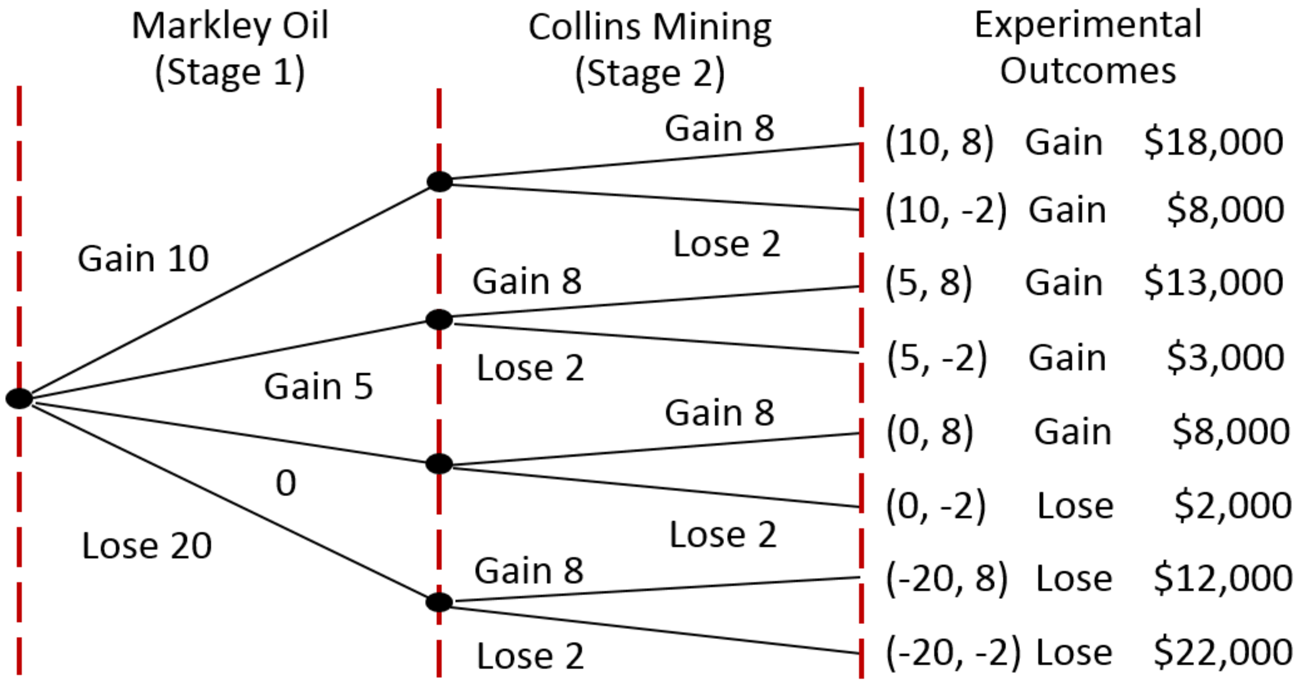
A Counting Rule for Multiple-Step Experiments

- If an experiment consists of a sequence of k steps in which there are n_1 possible results for the first step, n_2 possible results for the second step, and so on, then the total number of experimental outcomes is given by $(n_1)(n_2) \ldots (n_k)$.
- A helpful graphical representation of a multiple-step experiment is a tree diagram.

Markley Oil: $n_1 = 4$

Collins Mining: $n_2 = 2$

Total number of experimental outcomes: $(4)(2) = 8$.



CAL STATE EAST BAY

Counting Rules for Combinations and Permutations

- Number of Combinations of N Objects Taken n at a Time
- A second useful counting rule enables us to count the number of experimental outcomes when n objects are to be selected from a set of N objects.

$$C_n^N = \binom{N}{n} = \frac{N!}{n!(N-n)!}$$

where: $N! = N(N-1)(N-2)\ldots(2)(1)$

$n! = n(n-1)(n-2)\ldots(2)(1)$

$0! = 1$

- Number of Permutations of N Objects Taken n at a Time
- A third useful counting rule enables us to count the number of experimental outcomes when n objects are to be selected from a set of N objects, where the order of selection is important.

$$P_n^N = n!\binom{N}{n} = \frac{N!}{(N-n)!}$$

where: $N! = N(N-1)(N-2)\ldots(2)(1)$

$n! = n(n-1)(n-2)\ldots(2)(1)$

$0! = 1$

Assigning Probabilities

Basic Requirements for Assigning Probabilities

1. The probability assigned to each experimental outcome must be between 0 and 1, inclusively.

$$0 \leq P(E_i) \leq 1 \text{ for all } i$$

where E_i is the i th experimental outcome and $P(E_i)$ is its probability

2. The sum of the probabilities for all experimental outcomes must equal 1.

$$P(E_1) + P(E_2) + \ldots + P(E_n) = 1$$

where n is the number of experimental outcomes.

Methods of Assigning Probabilities:

- Classical Method
  Assigning probabilities based on the assumption of equally likely outcomes
- Relative Frequency Method
  Assigning probabilities based on experimental or historical data
- Subjective Method
  Assigning probability based on judgment.

Examples

Classical Method: If an experiment has n possible outcomes, the classical method would assign a probability of $1/n$ to each outcome. Experiment: Rolling a die

Sample Space: $S = \{1, 2, 3, 4, 5, 6\}$ Probabilities: Each sample point has a 1/6 chance of occurring

Relative Frequency Method: Example: Lucas Tool Rental

Lucas Tool Rental would like to assign probabilities to the number of car polishers it rents each day. Office records show the following frequencies of daily rental for the last 40 days. Each probability assignment is given by dividing the frequency (number of days) by the total frequency (total number of days).

Number of Polishers Rented	Number of Days	Probability
0	4	.10 = 4/40
1	6	.15
2	18	.45
3	10	.25
4	2	.05
sum	40	1.00

Examples

- When economic conditions or a company's circumstances change rapidly it might be inappropriate to assign probabilities based solely on historical data.
- We can use any data available as well as our experience and intuition, but ultimately a probability value should express our degree of belief that the experimental outcome will occur.
- The best probability estimates often are obtained by combining the estimates from the classical or relative frequency approach with the subjective estimate. Example: An analyst made the following probability estimates:

Experimental Outcome	Net Gain or Loss	Probability
(10,8)	$18,000 Gain	0.20
(10, –2)	$8,000 Gain	0.08
(5, 8)	$13,000 Gain	0.16
(5, –2)	$3,000 Gain	0.26
(0, 8)	$8,000 Gain	0.10
(0, –2)	$2,000 Loss	0.12
(–20, 8)	$12,000 Loss	0.02
(–20, –2)	$22,000 Loss	0.06
		Sum equals 1.00

Events and Their Probabilities

- An event is a collection of sample points.
- The probability of any event is equal to the sum of the probabilities of the sample points in the event.
- If we can identify all the sample points of an experimental and assign a probability to each, we can compute the probability of an event.

Event M = Markley Oil is Profitable

$$M = \{ (10, 8), (10, –2), (5, 8), (5, –2) \}$$

$$P(M) = \mathrm{P}(10, 8) + \mathrm{P}(10, –2) + \mathrm{P}(5, 8) + \mathrm{P}(5, –2)$$

$$= 0.20 + 0.08 + 0.16 + 0.26$$

$$= 0.70$$

Event C = Collins Mining is Profitable

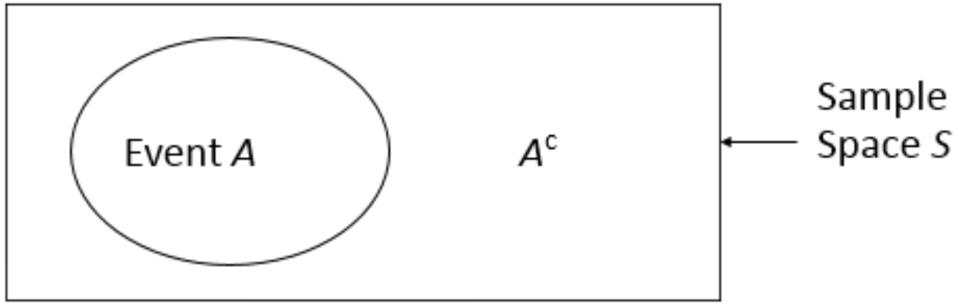
$$C = \{ (10, 8), (5, 8), (0, 8), (–20, 8) \}$$

$$P(C) = P(10, 8) + P(5, 8) + P(0, 8) + P(–20, 8)$$

$$= 0.20 + 0.16 + 0.10 + 0.02 = 0.48$$

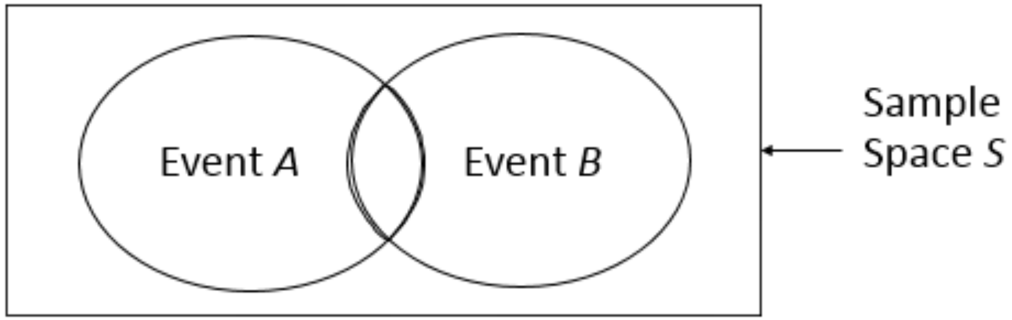
Complement and Union

- The complement of event A is defined to be the event consisting of all sample points that are *not* in A.
- The complement of A is denoted by A^C.



Venn Diagram

- The union of events A and B is the event containing all sample points that are in A and B or both. The union of events A and B is denoted by $A \cup B$.



Venn Diagram

Union of Two Events

Example: Bradley Investments

Event M = Markley Oil is Profitable

Event C = Collins Mining is Profitable

$M \cup C$ = Markley Oil is Profitable OR Collins Mining is Profitable (or both are profitable).

$M \cup C = \{ (10, 8), (10, -2), (5, 8), (5, -2), (0, 8), (-20, 8) \}$

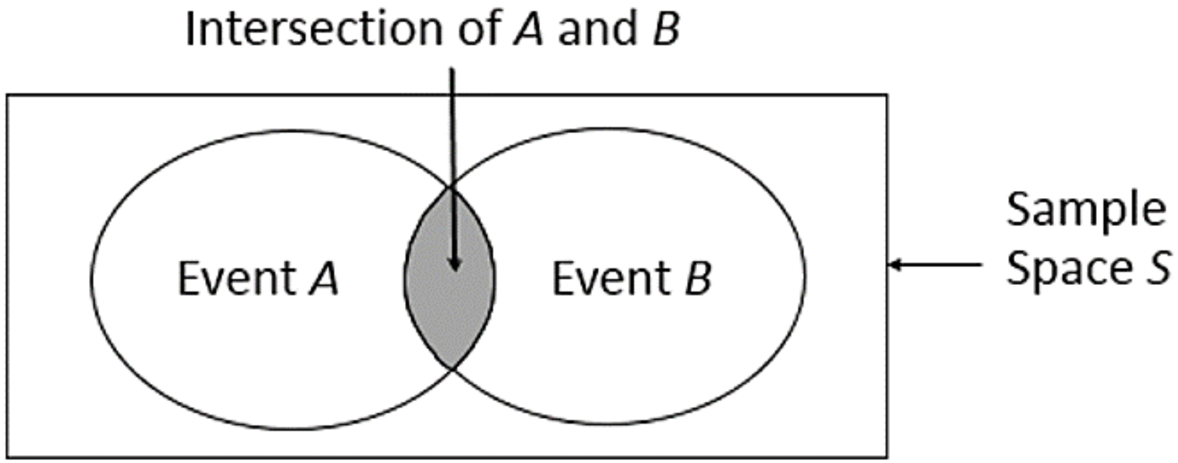
$P(M \cup C) = P(10, 8) + P(10, -2) + P(5, 8) + P(5, -2) + P(0, 8) + P(-20, 8)$

$= 0.20 + 0.08 + 0.16 + 0.26 + 0.10 + 0.02$

$= 0.82$

Intersection of Two Events

- The intersection of events A and B is the set of all sample points that are in both *A* and *B*



Venn Diagram

$M \cap C$ = Markley Oil is Profitable AND Collins Mining is Profitable.

$M \cap C = \{ (10, 8), (5, 8) \}$

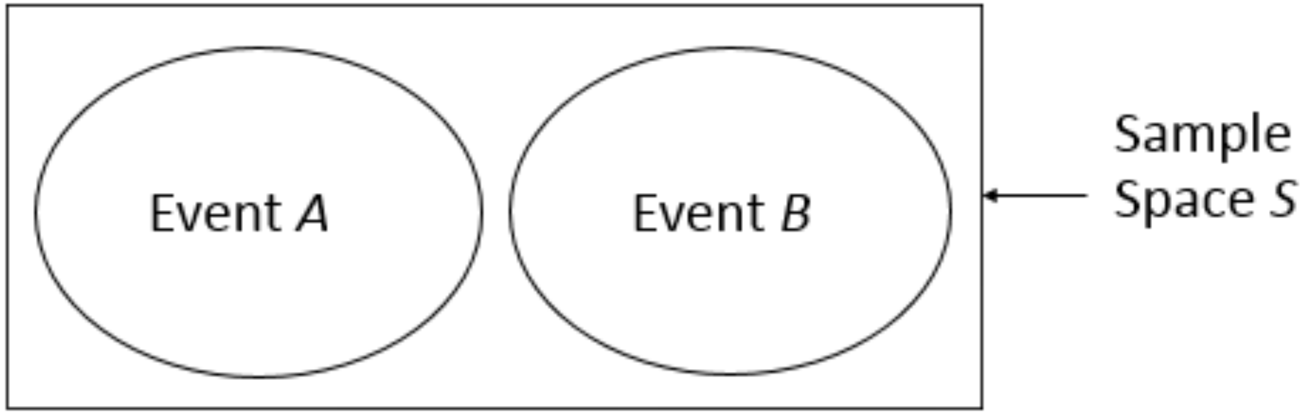
$P(M \cap C) = P(10, 8) + P(5, 8)$

$= 0.20 + 0.16$

$= 0.36$

Mutually Exclusive Events

- Two events are said to be mutually exclusive if the event have no sample points in common
- Two events are mutually exclusive if, when one event occurs, the other cannot occur.
- If events A and B are mutually exclusive, then $P(A \cap B) = 0$.
- The addition law for mutually exclusive events is $P(A \cup B) = P(A) + P(B)$.



Venn Diagram

Conditional Probability

- The probability of an event given that another event has occurred is called a conditional probability.
- The conditional probability of A given B has already occurred denoted by $P(A \mid B)$.
- A conditional probability is computed as follows: $P(A|B) = \dfrac{P(A \cap B)}{P(B)}$

Example:

Event M = Markley Oil Profitable

Event C = Collins Mining Profitable

$P(C|M)$ = Collins Mining Profitable given Markley Oil Profitable

We know: $P(M \cap C) = .36$, $P(M) = .70$

Thus: $P(C|M) = \dfrac{P(C \cap M)}{P(M)} = \dfrac{.36}{.70} = .5143$

Multiplication Law

- The multiplication law provides a way to compute the probability of the intersection of two events.
- The law is written as:

$$P(\mathrm{A} \cap \mathrm{B}) = P(\mathrm{B})P(\mathrm{B}|\mathrm{A}) \quad \text{or} \quad P(\mathrm{A} \cap \mathrm{B}) = P(\mathrm{A})P(\mathrm{A}|\mathrm{B})$$

Example:

Event M = Markley Oil is Profitable

Event C = Collins Mining is Profitable

We know that $P(M) = 0.70$ and $P(C|M) = 0.5143$

$$\begin{aligned} P(M \cap C) &= P(M)\cdot\ P(M|C) \\ &= (0.70)(0.5143) \\ &= 0.36 \end{aligned}$$

Note: This is the same result we obtained earlier.

Joint Probability Table

- Joint probabilities appear in the body of the table
- Marginal probabilities appear in the margins of the table.

Markley Oil	Collins mining Profitable (C)	Collins mining not Profitable (C^C)	Total
Profitable (M)	.36	.34	.70
Not Profitable (M^C)	.12	.18	.30
Total	.48	.52	1.00

Independent Events

- If the probability of event *A* is not changed by the existence of event *B*, we would say that events *A* and *B* and are <u>independent</u>.
- Two events *A* and *B* are independent if: $P(A|B) = P(A)$ or $P(B|A) = P(B)$
- The multiplication law also can be used as a test to see if two events are independent.
- The law is written as: $P(A \cap B) = P(A)P(B)$

Event M = Markley Oil is Profitable

Event C = Collins Mining is Profitable

Are events M and C independent?

Does $P(M \cap C) = P(M)P(C)$?

We know that $P(M \cap C) = 0.36$, $P(M) = 0.70$ and $P(C) = 0.48$

But $P(M)P(C) = (0.7)(0.48) = 0.34$, which does not equal 0.36.

Therefore M and C are <u>not</u> independent.

CAL STATE EAST BAY

Mutual Exclusiveness and Independence

- Do not confuse the notion of mutually exclusive events with that of independent events.

- Two events with nonzero probability cannot both mutually exclusive and independent.

- If one mutually exclusive event is known to occur, the other cannot; occur thus, the probability of the other even occurring is reduced to zero (and therefore dependent).

- Two events that are not mutually exclusive, might or night not be independent.

Bayes' Theorem

- Often we begin probability analysis with initial or prior probabilities.
- Then, from a sample, special report, or a product test we obtain additional information.
- Given this information, we calculate revised or posterior probabilities.

- Bayes' theorem provides the means for revising the prior probabilities.

Prior Probabilities → New Information → Application of Bayes' Theorem → Posterior Probabilities

Example: A proposed shopping center will provide strong competition for downtown businesses like L. S. Clothiers. If the shopping center is built, the owner of L. S. Clothiers feels it would be best to relocate to the shopping center. The shopping center cannot be built unless a zoning change is approved by the town council. The planning board must first make a recommendation, for or against the zoning change, to the council.

Let: A_1 = town council approves the zoning change, A_2 = town council disapproves the change

Using subjective judgment: $P(A_1) = 0.7$ and $P(A_2) = 0.3$

New Information

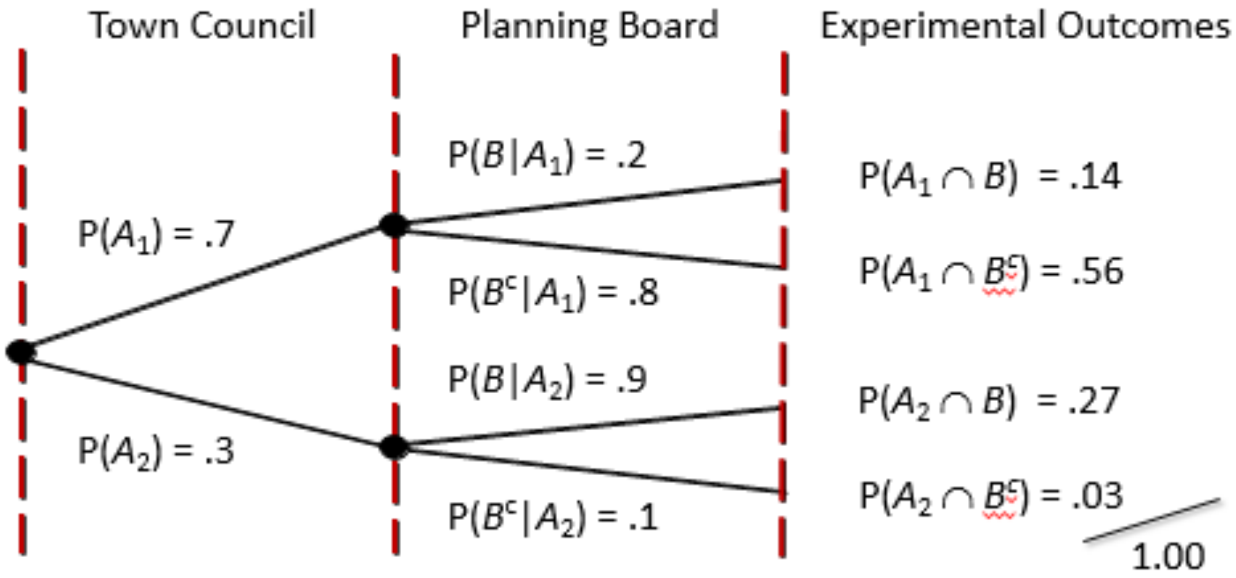
The planning board has recommended <u>against</u> the zoning change. Let B denote the event of a negative recommendation by the planning board.

Given that B has occurred, should L. S. Clothiers revise the probabilities that the town council will approve or disapprove the zoning change?

Past history with the planning board and the town council indicates the following:

Hence: $P(B|A_1) = 0.2$ and $P(B|A_2) = 0.9$

$P(B^c|A_1) = 0.8$ and $P(B^c|A_2) = 0.1$



Bayes' Theorem

- To find the posterior probability that event A_i will occur given that event B has occurred, we apply Bayes' theorem.

$$P(A_i|B) = \frac{P(A_i)P(B|A_i)}{P(A_1)P(B|A_1) + P(A_2)P(B|A_2) + \cdots + P(A_n)P(B|A_n)}$$

- Bayes' theorem is applicable when the events for which we want to compute posterior probabilities are mutually exclusive and their union is the entire sample space.

Example: L. S. Clothiers

Given the planning board's recommendation not to approve the zoning change, we revise the prior probabilities:

$$P(A_1|B) = \frac{P(A_1)P(B|A_1)}{P(A_1)P(B|A_1) + P(A_2)P(B|A_2)}$$

$$= \frac{(0.7)(0.2)}{(0.7)(0.2)+(0.3)(0.9)}$$

$$= 0.34$$

The planning board's recommendation is good news for L. S. Clothiers. The posterior probability of the town council approving the zoning change is 0.34 compared to a prior probability of 0.70.

Bayes' Theorem: Tabular Approach

Step 1:

Prepare the following three columns:

Column 1 - The mutually exclusive events for which posterior probabilities are desired.

Column 2 - The prior probabilities for the events.

Column 3 - The conditional probabilities of the new information *given* each event.

(1)	(2)	(3)	(4)	(5)
Events	Prior Probabilities	Conditional Probability	.	.
A_i	$P(A_i)$	$P(B\|A_i)$	.	.
A_1	.7	.2	.	.
A_2	.3	.9	.	.
.	1.0	.	.	.

Bayes' Theorem: Tabular Approach

Step 2:

Prepare the fourth column

<u>Column 4</u>: Compute the joint probabilities for each event and the new information *B* by using the multiplication law. Multiply the probabilities in column 2 by the corresponding conditional probabilities in column 3. That is, $P(A_i \cap B) = P(A_i)P(B|A_i)$.

(1)	(2)	(3)	(4)	(5)
Events	Prior Probabilities	Conditional Probability	Joint Probabilities	.
A_i	$P(A_i)$	$P(B\|A_i)$	$P(A_i \cap B)$	.
A_1	.7	.2	.14 = .7(.2)	.
A_2	.3	.9	.27	.
.	1.0	.	.	.

We see that there is a 0.14 probability of the town council approving the zoning change and a negative recommendation by the planning board.

There is a 0.27 probability of the town council disapproving the zoning change and a negative recommendation by the planning board.

Bayes' Theorem: Tabular Approach

Step 3:

Sum the joint probabilities in Column 4. The sum is the probability of the new information, $P(B)$. The sum 0.14 + 0.27 shows an overall probability of 0.41 of a negative recommendation by the planning board.

(1)	(2)	(3)	(4)	(5)
Events A_i	Prior Probabilities $P(A_i)$	Conditional Probability $P(B \mid A_i)$	Joint Probabilities $P(A_i \cap B)$	
A_1	.7	.2	.14 = .7 (.2)	
A_2	.3	.9	.27	
.	1.0	.	P(B) = .41	

CAL STATE EAST BAY

Bayes' Theorem: Tabular Approach

Step 4:

Prepare the fifth column: Column 5

Compute the posterior probabilities using the basic relationship of conditional probability.

(1)	(2)	(3)	(4)	(5)
Events	Prior Probabilities	Conditional Probability	Joint Probabilities	Posterior Probabilities
A_i	$P(A_i)$	$P(B\|A_i)$	$P(A_i \cap B)$	$P(A_i\|B)$
A_1	.7	.2	.14 = .7(.2)	.3415 = .14/.41
A_2	.3	.9	.27	.6585
	1.0		P(B) = .41	1.0000

CAL STATE EAST BAY

Random Variables

- A random variable is a numerical description of the outcome of an experiment.
- A discrete random variable may assume either a finite number of values or an infinite sequence of values.
- A continuous random variable may assume any numerical value in an interval or collection of intervals.

Discrete Random Variable with a Finite Number of Values:

Example: JSL Appliances

Let x = number of TVs sold at the store in one day, where x can take on 5 values (0, 1, 2, 3, 4)

We can count the TVs sold, and there is a finite upper limit on the number that might be sold (which is the number of TVs in stock).

Discrete Random Variable with a Infinite Number of Values:

Example: JSL Appliances

Let x = number of customers arriving in one day, where x can take on the values 0, 1, 2, . . .

We can count the customers arriving, but there is no finite upper limit on the number that might arrive.

Random Variables

Illustration	Random Variable x	Type
Family size	x = Number of dependents reported on tax return	Discrete
Distance from home to stores on a highway	x = Distance in miles from home to the store site	Continuous
Own dog or cat	x = 1 if own no pet; = 2 if own dog(s) only; = 3 if own cat(s) only; = 4 if own dog(s) and cat(s)	Discrete

Discrete Probability Distributions

- The probability distribution for a random variable describes how probabilities are distributed over the values of the random variable.

- We can describe a discrete probability distribution with a table, graph, or formula.

Types of discrete probability distributions:

- First type: uses the rules of assigning probabilities to experimental outcomes to determine probabilities for each value of the random variable.

- Second type: uses a special mathematical formula to compute the probabilities for each value of the random variable.
- The probability distribution is defined by a probability function, denoted by $f(x)$, that provides the probability for each value of the random variable.

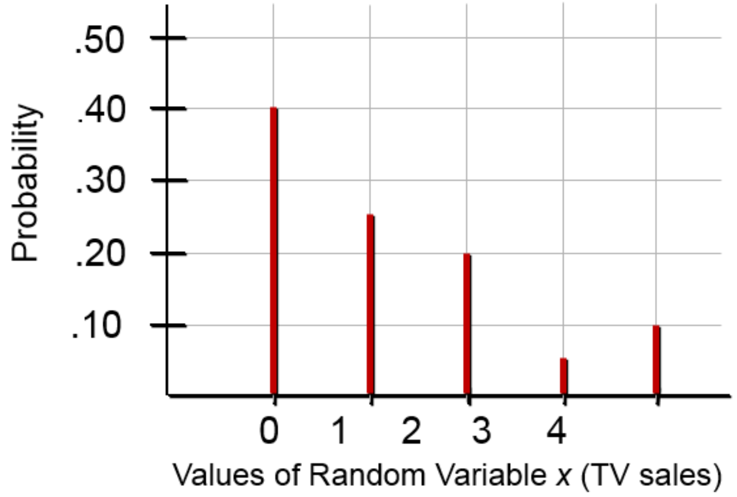
- The required conditions for a discrete probability function are: $f(x) \geq 0$ and $\sum f(x) = 1$
- There are three methods for assigning probabilities to random variables: classical method, subjective method, and relative frequency method.
- The use of the relative frequency method to develop discrete probability distributions leads to what is called an empirical discrete distribution.

Discrete Probability Distributions

Example: JSL Appliances

Using past data on TV sales, a tabular representation of the probability distribution for sales was

Units Sold	Number of Days	x	$f(x)$
0	80	0	.40 = 80/200
1	50	1	0.25
2	40	2	0.20
3	10	3	0.05
4	20	4	0.10
.	200	.	1.00



Graphical representation of probability distribution

- In addition to tables and graphs, a formula that gives the probability function, $f(x)$, for every value of x is often used to describe the probability distributions.
- Several discrete probability distributions specified by formulas are the discrete-uniform, binomial, Poisson, and hypergeometric distributions.

Discrete Probability Distributions

- The discrete uniform probability distribution is the simplest example of a discrete probability distribution given by a formula.
- The discrete uniform probability function is

$$f(x) = 1/n$$

where: n = the number of values the random variable may assume

- The values of the random variable are equally likely.

Expected Value

- The expected value, or mean, of a random variable is a measure of its central location.

$$E(x) = \mu = \sum xf(x)$$

- The expected value is a weighted average of the values the random variable may assume. The weights are the probabilities.
- The expected value does not have to be a value the random variable can assume.
- The variance summarizes the variability in the values of a random variable

$$Var(x) = \sigma^2 = \Sigma(x - \mu)^2 f(x)$$

- The variance is a weighted average of the squared deviations of a random variable from its mean. The weights are the probabilities.

- The standard deviation, σ , is defined as the positive square root of the variance.

Expected Value and Variance

Example: JSL Appliances

x	$f(x)$	$xf(x)$
0	.40	.00
1	.25	.25
2	.20	.40
3	.05	.15
4	.10	.40

$E(x) = 1.20$ = expected number of TVs sold in a day

X	$x-\mu$	$(x-\mu)^2$	f(x)	$(x-\mu)^2$ f(x)
0	–1.2	1.44	.40	.576
1	–0.2	0.04	.25	.010
2	0.8	0.64	.20	.128
3	1.8	3.24	.05	.162
4	2.8	7.84	.10	.784
Empty cell	Empty cell	Empty cell	Empty cell	Variance of daily sales $= \sigma^2 = 1.660$

Standard deviation of daily sales = 1.2884 TVs

CAL STATE EAST BAY

Bivariate Distributions

A <u>bivariate probability distribution</u> is a probability distribution involving two random variables.

Here are the daily sales at the DiCarlo Motors (x) automobile dealership in Saratoga, New York and DiCarlo (y), another dealership in Geneva, New York. The table shows the number of cars sold at each of the dealerships over a 300-day period.

	Saratoga Dealership						
Geneva Dealership	**0**	**1**	**2**	**3**	**4**	**5**	**Total**
0	21	30	24	9	2	0	86
1	21	36	33	18	2	1	111
2	9	42	9	12	3	2	77
3	3	9	6	3	5	0	26
Total	54	117	72	42	12	3	300

We can now divide all of the frequencies by the number of observations (300) to develop a bivariate empirical discrete probability distribution for automobile sales at the two DiCarlo dealerships.

	Saratoga Dealership						
Geneva Dealership	**0**	**1**	**2**	**3**	**4**	**5**	**Total**
0	.0700	.1000	.0800	.0300	.0067	.0000	.2867
1	.0700	.1200	.1100	.0600	.0067	.0033	.3700
2	.0300	.1400	.0300	.0400	.0100	.0067	.2567
3	.0100	.0300	.0200	.0100	.0167	.0000	.0867
Total	.18	.39	.24	.14	.04	.01	1.0000

Bivariate Distributions

The table below shows the expected value for the mean total sales and the standard deviation of total sales for these two dealerships.

s	$f(s)$	$sf(s)$	$s - E(s)$	$(s - E(s))^2$	$(s - E(s))^2 f(s)$
0	.0700	.0000	–2.6433	6.9872	.4891
1	.1700	.1700	–1.6433	2.7005	.4591
2	.2300	.4600	–.6433	.4139	.0952
3	.2900	.8700	.3567	.1272	.0369
4	.1267	.5067	1.3567	1.8405	.2331
5	.0667	.3333	2.3567	5.5539	.3703
6	.0233	.1400	3.3567	11.2672	.2629
7	.0233	.1633	4.3567	18.9805	.4429
8	.0000	.0000	5.3567	28.6939	.0000
N/A	N/A	$E(s) = 2.6433$	N/A	N/A	$Var(s) = 2.3895$

Covariance

The covariance and/or correlation coefficient are good measures of association between two random variables.

Covariance = $\sigma_{xy} = [Var(x+y) - Var(x) - Var(y)]/2.$

$= (2.3895 - 0.8696 - 1.25)/2$

$= 0.1350$

A covariance of .1350 indicates that daily sales at DiCarlo's two dealerships have a positive relationship.
To get a better sense of the strength of the relationship we can compute the correlation coefficient.

Correlation = $\rho_{xy} = \frac{\sigma_{xy}}{\sigma_x \sigma_y}$

$$\rho_{xy} = \frac{0.1350}{(0.9325)(1.1180)} = 0.1295$$

The correlation coefficient of .1295 indicates there is a weak positive relationship between the random variables representing daily sales at the two DiCarlo dealerships. If the correlation coefficient had equaled zero, we would have concluded that daily sales at the two dealerships were independent.

Binomial Probability Distribution

Four Properties of a Binomial Experiment

1. The experiment consists of a sequence of n identical trials.
2. Two outcomes, success and failure, are possible on each trial.
3. The trials are independent.
4. The probability of a success, denoted by p, does not change from trial to trial. (This is referred to as the stationarity assumption.)

- Our interest is in the number of successes occurring in the n trials.
- Let x denote the number of successes occurring in the n trials.
- Binomial Probability Function: $f(x) = \frac{n!}{x!\,(n-x)!} p^x (1-p)^{(n-x)}$

  where:

  x = the number of successes

  p = the probability of a success on one trial, n = the number of trials

  $f(x)$ = the probability of x successes in n trials, and $n! = n(n-1)(n-2) \ldots\ldots (2)(1)$

CAL STATE EAST BAY

Binomial Probability Distribution

Binomial Probability Function

$$f(x) = \underbrace{\frac{n!}{x!\,(n-x)!}}_{\text{Number of experimental outcomes providing exactly } x \text{ successes in } n \text{ trials}} \; \underbrace{p^x (1-p)^{(n-x)}}_{\text{Probability of a particular sequence of trial outcomes with } x \text{ successes in } n \text{ trials}}$$

Example: Evans Electronics

Evans Electronics is concerned about a low retention rate for its employees. In recent years, management has seen a turnover of 10% of the hourly employees annually.

Thus, for any hourly employee chosen at random, management estimates a probability of 0.1 that the person will not be with the company next year.
Choosing 3 hourly employees at random, what is the probability that 1 of them will leave the company this year?

- The probability of the first employee leaving and the second and third employees staying, denoted (S, F, F), is given by: $p(1 - p)(1 - p)$
- With a 0.10 probability of an employee leaving on any one trial, the probability of an employee leaving on the first trial and not on the second and third trials is given by: $(0.10)(0.90)(0.90) = (0.10)(0.90)^2 = 0.081$

Binomial Probability Distribution

Two other experimental outcomes result in one success and two failures. The probabilities for all three experimental outcomes involving one success follow.

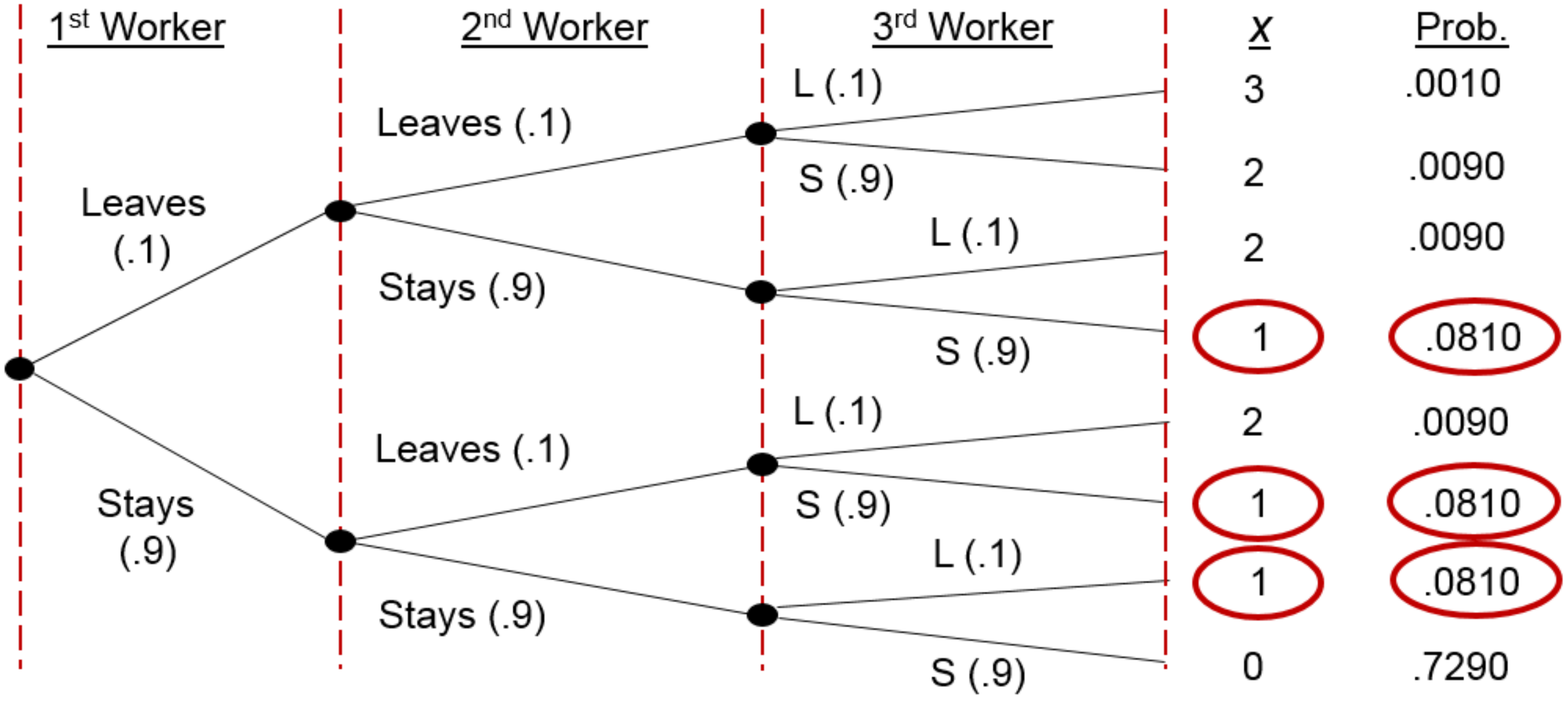
Experimental Outcome	Probability of Experimental Outcome
(S, F, F)	$p(1 - p)(1 - p) = (.1)(.9)(.9) = .081$
(F, S, F)	$(1 - p)p(1 - p) = (.9)(.1)(.9) = .081$
(F, F, S)	$(1 - p)(1 - p)p = (.9)(.9)(.1) = \underline{.081}$
	Total = .243

Using the probability function with $p = 0.10$, $n = 3$, and $x = 1$

$$f(x) = \frac{n!}{x!\,(n-x)!} p^x (1-p)^{(n-x)}$$

$$f(1) = \frac{3!}{1!\,(3-1)!} (0.1)^1 (0.9)^2 = 0.243$$

Binomial Probability Distribution



Binomial Probability Distribution

- The expected value is: $E(x) = \mu = np$
- The variance is: $Var(x) = \sigma^2 = np(1 - p)$
- The standard deviation is: $\sigma = \sqrt{np(1-p)}$

Example: Evans Electronics

- The expected value is: $E(x) = np = 3(.1) = .3$ employees out of 3
- The variance is: $Var(x) = np(1 - p) = 3(.1)(.9) = .27$
- The standard deviation is: $\sigma = \sqrt{3(.1)(.9)} = .52$ employees

Poisson Probability Distribution

- A Poisson distributed random variable is often useful in estimating the number of occurrences over a specified interval of time or space.

- It is a discrete random variable that may assume an infinite sequence of values (x = 0, 1, 2, . . .).
- Examples of Poisson distributed random variables:
  - number of knotholes in 14 linear feet of pine board
  - number of vehicles arriving at a toll booth in one hour
  - Bell Labs used the Poisson distribution to model the arrival of phone calls.

Two Properties of a Poisson Experiment:

- The probability of an occurrence is the same for any two intervals of equal length.
- The occurrence or nonoccurrence in any interval is independent of the occurrence or nonoccurrence in any other interval.

Poisson Probability Function: $$f(x) = \frac{\mu^x e^{-\mu}}{x!}$$

where:

x = the number of occurrences in an interval, $f(x)$ = the probability of x occurrences in an interval

μ = mean number of occurrences in an interval, $e = 2.71828$, $x! = x(x-1)(x-2) \ldots (2)(1)$

Poisson Probability Distribution

Poisson Probability Function–

- Because there is no stated upper limit for the number of occurrences, the probability function $f(x)$ is applicable for values $x = 0, 1, 2, \ldots$ without limit.
- In practical applications, x will eventually become large enough so that $f(x)$ is approximately zero and the probability of any larger values of x becomes negligible.

Example: Mercy Hospital

Patients arrive at the emergency room of Mercy Hospital at the average rate of 6 per hour on weekend evenings.

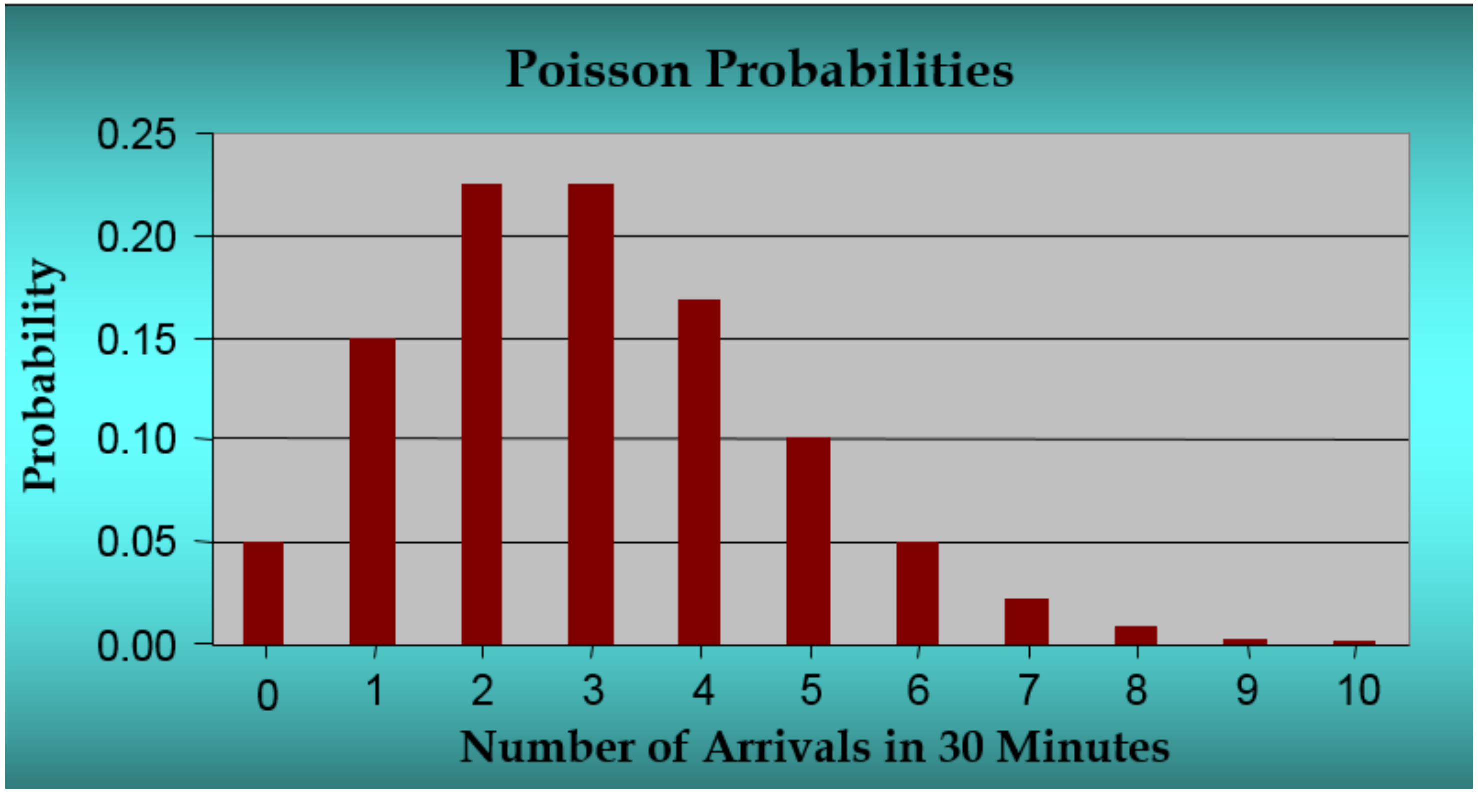
What is the probability of 4 arrivals in 30 minutes on a weekend evening?

Using the probability function with μ = 6/hour = 3/half-hour and x = 4

$$f(x) = \frac{3^4(2.71828)^{-3}}{4!} = 0.1680$$

Poisson Probability Distribution

Example: Mercy Hospital



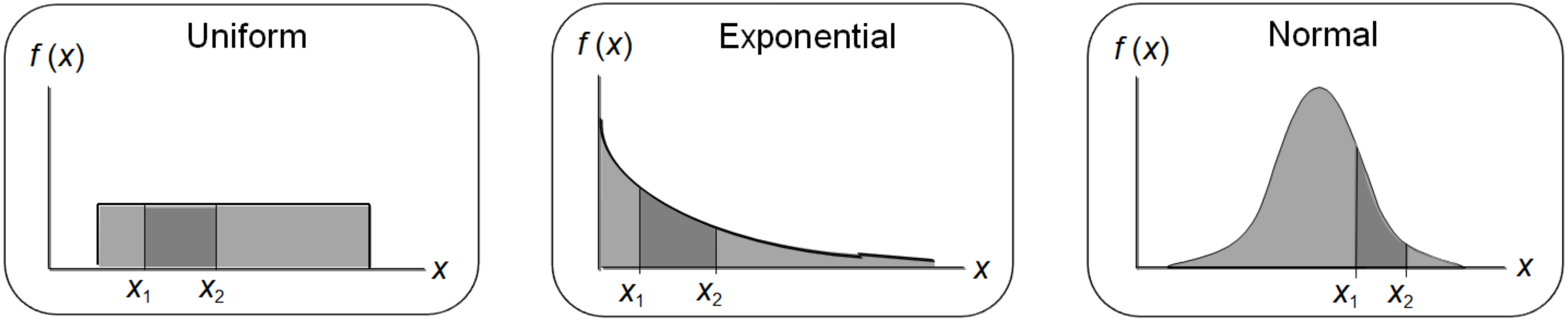
- A property of the Poisson distribution is that the mean and variance are equal: $\mu = \sigma^2$

Example: Mercy Hospital

Variance for Number of Arrivals during 30-Minute periods: $\mu = \sigma^2 = 3$

Continuous Probability Distributions

- A continuous random variable can assume any value in an interval on the real line or in a collection of intervals.
- It is not possible to talk about the probability of the random variable assuming a particular value.
- Instead, we talk about the probability of the random variable assuming a value within a given interval.
- The probability of the random variable assuming a value within some given interval from x_1 to x_2 is defined to be the area under the graph of the probability density function between x_1 and x_2.



Uniform Probability Distribution

- A random variable is uniformly distributed whenever the probability is proportional to the interval's length.
- The uniform probability density function is:

$$f(x) = 1/(b - a) \quad \text{for } a \le x \le b$$
$$= 0 \quad \text{elsewhere}$$

where: a = smallest value the variable can assume

b = largest value the variable can assume

- The expected value of x is $E(x) = (a + b)/2$.
- The variance of x is $Var(x) = (b - a)^2/12$.

Uniform Probability Distribution

Example: Slater's Buffet

Slater's customers are charged for the amount of salad they take. Sampling suggests that the amount of salad taken is uniformly distributed between 5 ounces and 15 ounces.

The uniform probability density function is

$$f(x) = 1/10 \quad \text{for } 5 \le x \le 15$$
$$= 0 \quad \text{elsewhere}$$

where:

x = salad plate filling weight

Uniform Probability Distribution

What is the probability that a customer will take between 12 and 15 ounces of salad?

- The expected value of x is $E(x) = (a + b)/2 = (5 + 15)/2 = 10$
- The variance of x is $\mathrm{Var}(x) = (b - a)^2/12 = (15 - 5)^2/12 = 8.33$

Area as a Measure of Probability

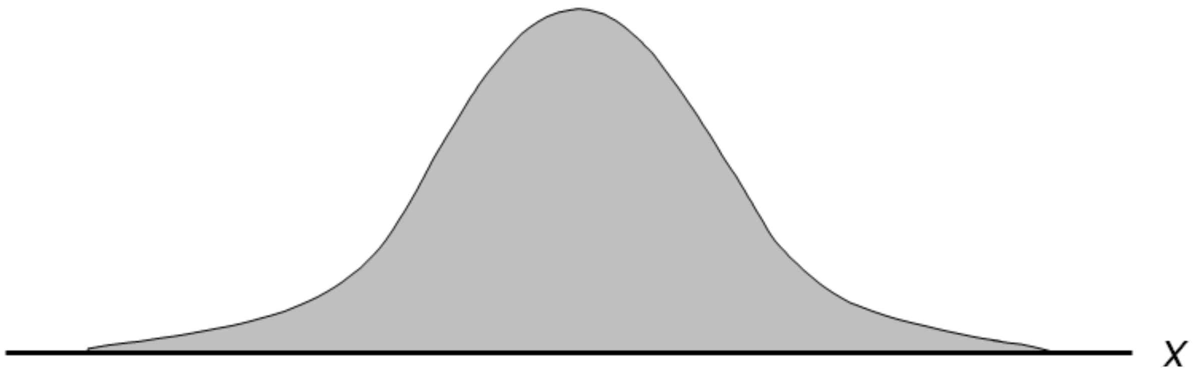
- The area under the graph of $f(x)$ and probability are identical.

- This is valid for all continuous random variables.

- The probability that x takes on a value between some lower value x_1 and some higher value x_2 can be found by computing the area under the graph of $f(x)$ over the interval from x_1 to x_2.

Normal Probability Distribution

- The normal probability distribution is the most important distribution for describing a continuous random variable.
- It is widely used in statistical inference.
- It has been used in a wide variety of applications:
  - Heights of people
  - Amounts of rainfall
  - Test scores
  - Scientific measurements, etc.



- Abraham de Moivre, a French mathematician, published *The Doctrine of Chances* in 1733. He derived the normal distribution.

Normal Probability Density Function

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

where:

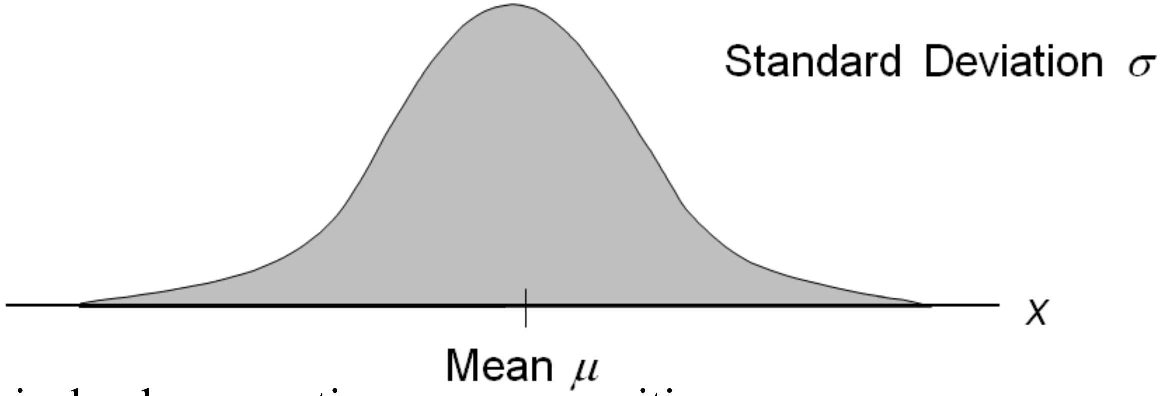
μ = mean, σ = standard deviation,

π = 3.14159, e = 2.71828

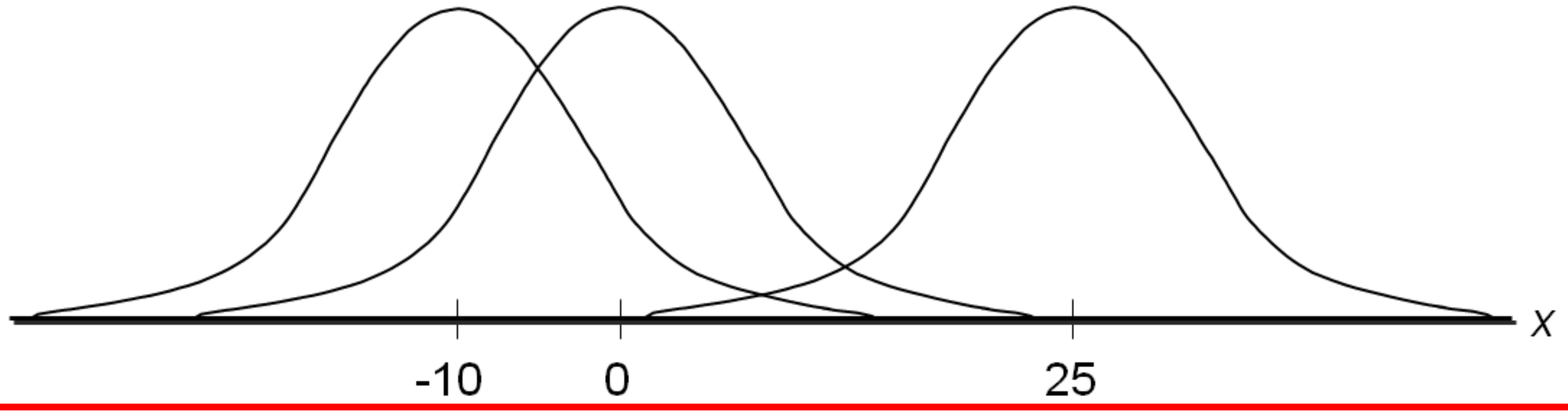
Normal Probability Distribution

The entire family of normal probability distributions is defined by its mean μ and its standard deviation σ.

The highest point on the normal curve is at the mean, which is also the median and mode.

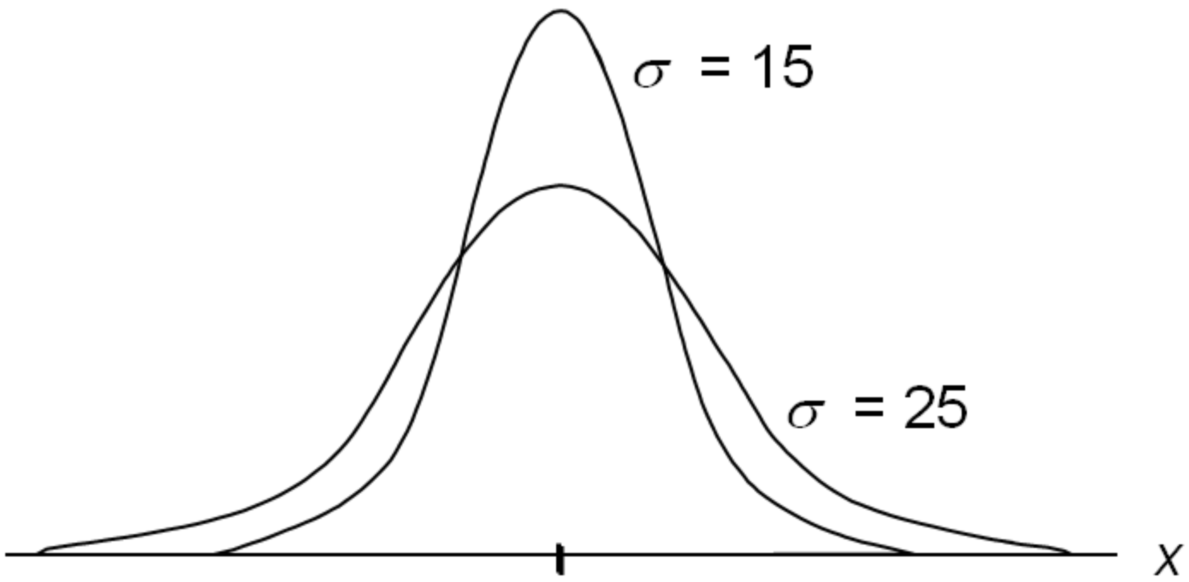


The mean can be any numerical value: negative, zero, or positive.

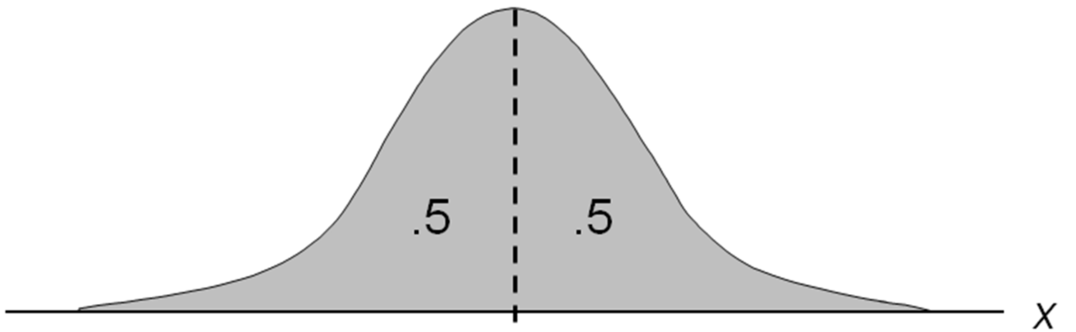


Normal Probability Distribution

- The standard deviation determines the width of the curve: larger values result in wider, flatter curves.



- Probabilities for the normal random variable are given by areas under the curve. The total area under the curve is 1 (0.5 to the left of the mean and 0.5 to the right).



CAL STATE EAST BAY

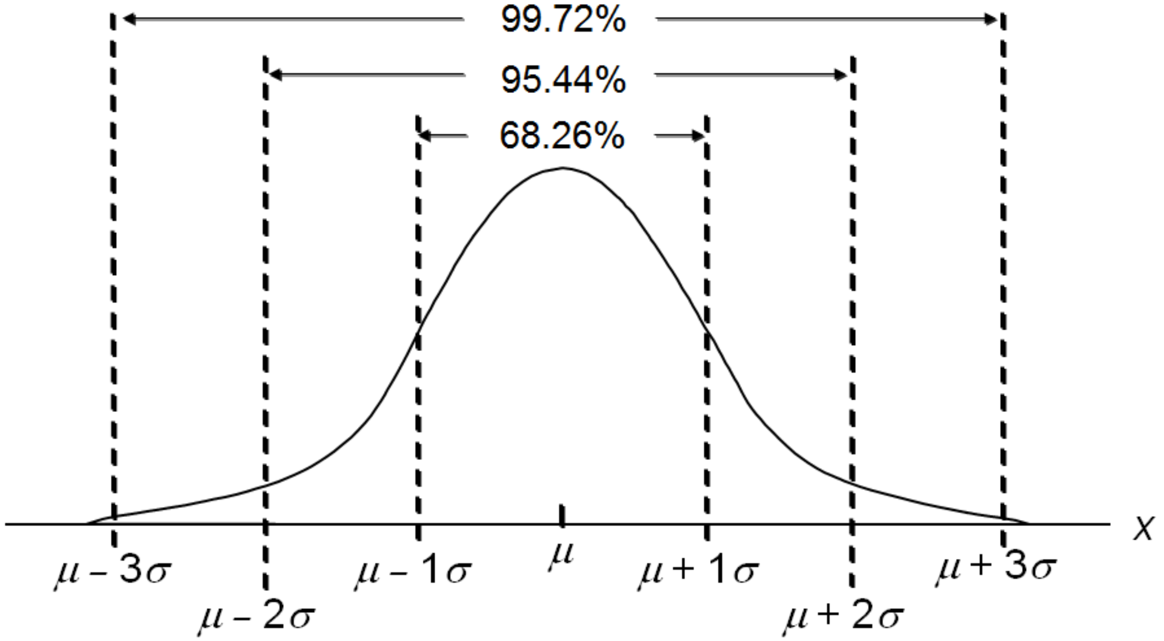
Normal Probability Distribution

Empirical Rule

68.26% of values of a normal random variable are within ± 1 standard deviation of its mean.

95.44% of values of a normal random variable are within ± 2 standard deviations of its mean.

99.72% of values of a normal random variable are within ± 3 standard deviations of its mean.



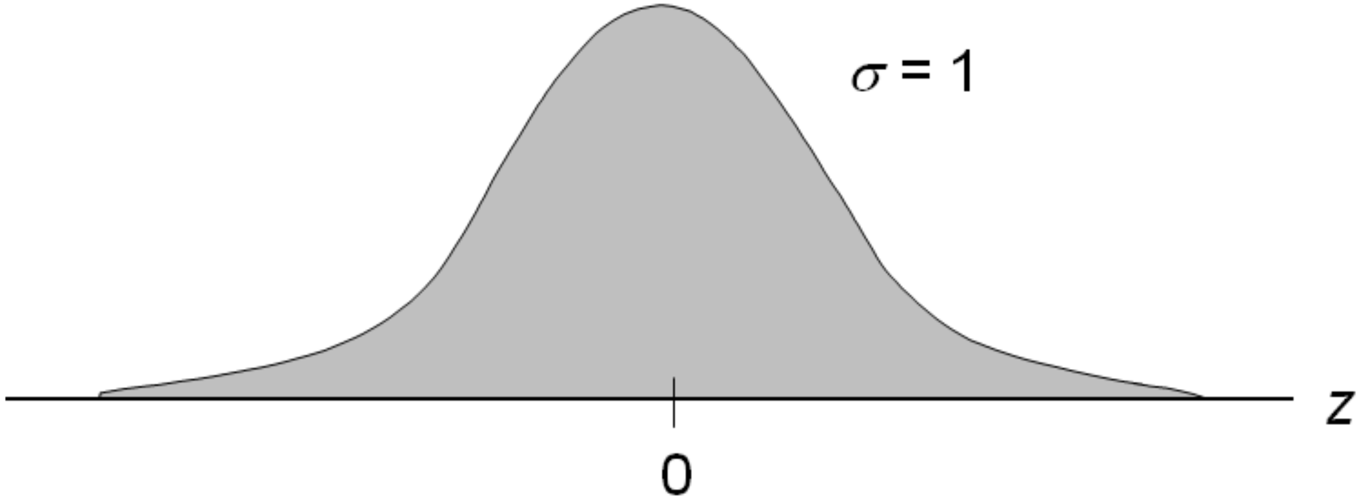
CAL STATE EAST BAY

Standard Normal Probability Distribution

- A random variable having a normal distribution with a mean of 0 and a standard deviation of 1 is said to have a standard normal probability distribution.
- The letter z is used to designate the standard normal random variable.
- Converting to the Standard Normal Distribution:

$$z = \frac{x - \mu}{\sigma}$$

- We can think of z as a measure of the number of standard deviations x is from μ.



Standard Normal Probability Distribution

Example: Pep Zone

Pep Zone sells auto parts and supplies including a popular multi-grade motor oil. When the stock of this oil drops to 20 gallons, a replenishment order is placed.

The store manager is concerned that sales are being lost due to stockouts while waiting for a replenishment order.

It has been determined that demand during replenishment lead-time is normally distributed with a mean of 15 gallons and a standard deviation of 6 gallons.

The manager would like to know the probability of a stockout during replenishment lead-time. In other words, what is the probability that demand during lead-time will exceed 20 gallons?

Solving for the Stockout Probability

Step 1: Convert *x* to the standard normal distribution.

$$z = \frac{(x-\mu)}{\sigma}$$

$$z = \frac{(20-15)}{6}$$

$$z = 0.83$$

CAL STATE EAST BAY

Standard Normal Probability Distribution

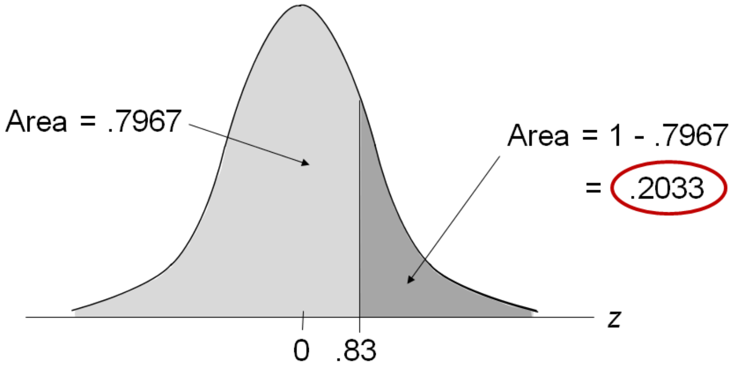
Step 2: Find the area under the standard normal curve to the left of z $P(z \leq 0.83) = 0.7967$

Cumulative Probability Table for the Standard Normal Distribution

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
.	.	.	.	.	.	.	.	.	.	.
.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
.8	.7881	.7910	.7939	.7967	.7795	.8023	.8051	.8078	.8106	.8133
.9	.8129	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
.	.	.	.	.	.	.	.	.	.	.

Step 3: Compute the area under the standard normal curve right of $z = 0.83$.

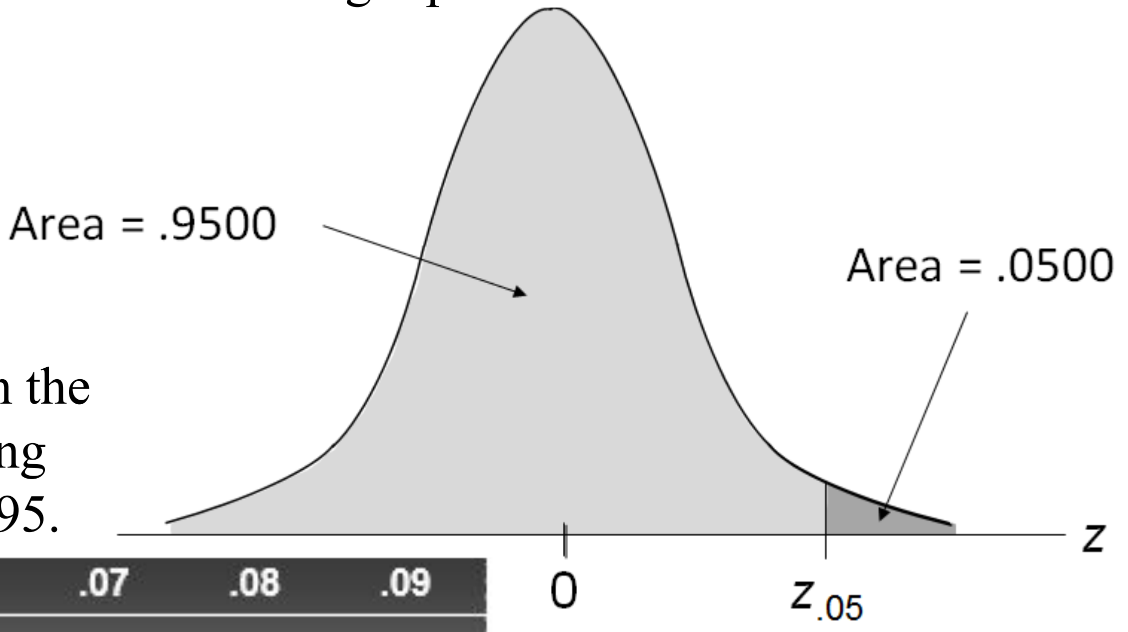
$$P(z > 0.83) = 1 - P(z \leq 0.83)$$
$$= 1 - 0.7967$$
$$= 0.2033$$



Standard Normal Probability Distribution

If the manager of Pep Zone wants the probability of a stockout during replenishment lead-time to be no more than .05, what should the reorder point be?

(Hint: Given a probability, we can use the standard normal table in an inverse fashion to find the corresponding z value.)



Solving for the Reorder Point

Step 1: Find the z-value that cuts off an area of .05 in the right tail of the standard normal distribution by looking up the complement of the right tail area 1 – 0.05 = 0.95.

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
.	.	.	.	.	.	.	.	.	.	.
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
.	.	.	.	.	.	.	.	.	.	.

Standard Normal Probability Distribution

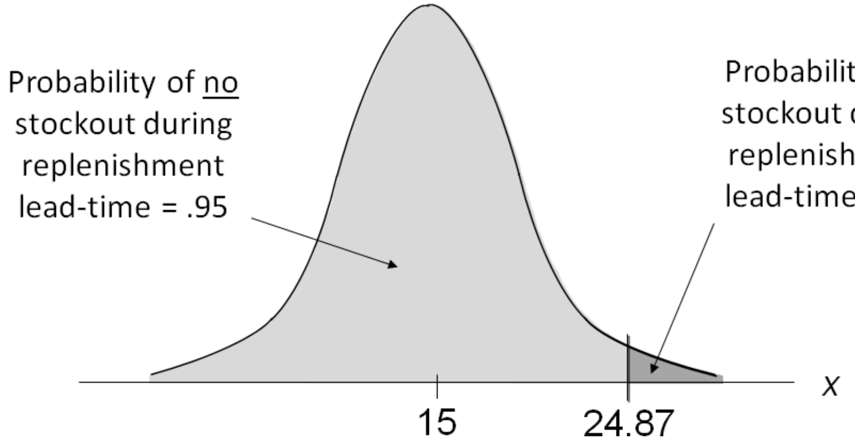
Solving for the Reorder Point

Step 2: Convert $z_{.05}$ to the corresponding value of x:

$$\begin{aligned} x &= \mu + z_{0.05}\sigma \\ &= 15 + 1.645(6) \\ &= 24.87 \end{aligned}$$

which we round to 25.

A reorder point of 25 gallons will place the probability of a stockout during lead time at (slightly less than) 0.05.



- By raising the reorder point from 20 gallons to 25 gallons on hand, the probability of a stockout decreases from about .20 to .05.
- This is a significant decrease in the chance that Pep Zone will be out of stock and unable to meet a customer's desire to make a purchase.

Exponential Probability Distribution

- The exponential probability distribution is useful in describing the time it takes to complete a task.
- The exponential random variables can be used to describe:
  - Time between vehicle arrivals at a toll booth
  - Time required to complete a questionnaire
  - Distance between major defects in a highway
- In waiting line applications, the exponential distribution is often used for service time.
- A property of the exponential distribution is that the mean and standard deviation are equal.
- The exponential distribution is skewed to the right. Its skewness measure is 2.
- Density Function $f(x) = \frac{1}{\mu} e^{-x/\mu}$ for $x \geq 0$ where μ = expected value or mean and e = 2.71828
- Cumulative Probabilities:

$$P(x \leq x_0) = 1 - e^{-x_0/\mu}$$

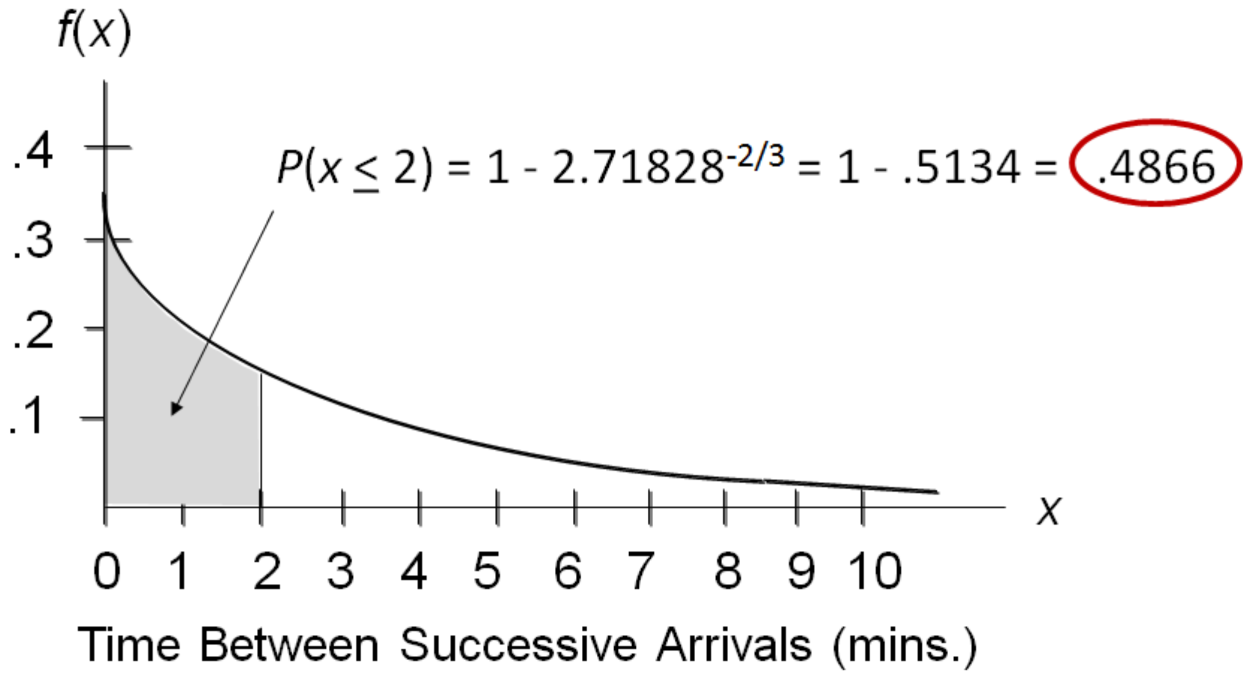
where:

x_0 = some specific value of x

Exponential Probability Distribution

Example: Al's Full-Service Pump

The time between arrivals of cars at Al's full-service gas pump follows an exponential probability distribution with a mean time between arrivals of 3 minutes. Al would like to know the probability that the time between two successive arrivals will be 2 minutes or less.



Relationship between the Poisson and Exponential Distributions

The Poisson distribution provides an appropriate description of the number of occurrences per interval.

The exponential distribution provides an appropriate description of the length of the interval between occurrences.

CAL STATE EAST BAY