BAN 602: Quantitative Fundamentals

Spring, 2020 Lecture Slides – Week 4

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Agenda

- Hypothesis Testing

    - Developing Null and Alternative Hypotheses
    - Type I and Type II Errors
    - Population Mean: σ Known
    - Population Mean: σ Unknown
    - Population Proportions

- Inferences About Means and Proportions with Two Populations

    - Inferences About Differences Between Two Population Means: σ_1 and σ_2 are Known
    - Inferences About Differences Between Two Population Means: σ_1 and σ_2 are Unknown
    - Inferences About Differences Between Two Population Means: Matched Samples
    - Inferences About Differences Between Two Population Proportions

Hypothesis Testing

- Hypothesis testing can be used to determine whether a statement about the value of a population parameter should or should not be rejected.
- The null hypothesis, denoted by H_0, is a tentative assumption about a population parameter.
- The alternative hypothesis, denoted by H_a, is the opposite of what is stated in the null hypothesis.
- The hypothesis testing procedure uses data from a sample to test the two competing statements indicated by H_0 and H_a.
- It is not always obvious how the null and alternative hypotheses should be formulated.
- Care must be taken to structure the hypotheses appropriately so that the test conclusion provides the information the researcher wants.
- The context of the situation is very important in determining how the hypotheses should be stated.
- In some cases it is easier to identify the alternative hypothesis first. In other cases the null is easier.
- Correct hypothesis formulation will take practice.

Developing Null and Alternative Hypotheses

Alternative Hypothesis as a Research Hypothesis:

- Many applications of hypothesis testing involve an attempt to gather evidence in support of a research hypothesis.
- In such cases, it is often best to begin with the alternative hypothesis and make it the conclusion that the researcher hopes to support.
- The conclusion that the research hypothesis is true is made if the sample data provides sufficient evidence to show that the null hypothesis can be rejected.

Example: A new teaching method is developed that is believed to be better than the current method.

Null Hypothesis: The new method is no better than the old method.
Alternative Hypothesis: The new teaching method is better.

Developing Null and Alternative Hypotheses

Example: A new sales force bonus plan is developed in an attempt to increase sales.

Null Hypothesis: The new bonus plan will not increase sales.
Alternative Hypothesis: The new bonus plan will increase sales.

Example: A new drug is developed with the goal of lowering blood pressure more than the existing drug.

Null Hypothesis: The new drug does not lower blood pressure more than the existing drug.
Alternative Hypothesis: The new drug lowers blood pressure more than the existing drug.

Developing Null and Alternative Hypotheses

Null Hypothesis as an Assumption to be Challenged

- We might begin with a belief or assumption that a statement about the value of a population parameter is true.
- We then use a hypothesis test to challenge the assumption and determine if there is statistical evidence to conclude that the assumption is incorrect.
- In these situations, it is helpful to develop the null hypothesis first.

Example: The label on a soft drink bottle states that it contains 67.6 fluid ounces.

Null Hypothesis:	The label is correct. $\mu \geq 67.6$ ounces.
Alternative Hypothesis:	The label is incorrect. $\mu < 67.6$ ounces.

Summary of Forms for Null and Alternative Hypotheses

- The equality part of the hypotheses always appears in the null hypothesis.
- In general, a hypothesis test about the value of a population mean μ must take one of the following three forms (where μ_0 is the hypothesized value of the population mean).

1. One-tailed, lower tail: $H_0: \mu \geq \mu_0 \quad H_a: \mu < \mu_0$
2. One-tailed, upper tail: $H_0: \mu \leq \mu_0 \quad H_a: \mu > \mu_0$
3. Two-tailed: $H_0: \mu = \mu_0 \quad H_a: \mu \neq \mu_0$

Null and Alternative Hypotheses

Example: Metro EMS

A major west coast city provides one of the most comprehensive emergency medical services in the world. Operating in a multiple hospital system with approximately 20 mobile medical units, the service goal is to respond to medical emergencies with a mean time of 12 minutes or less.

The director of medical services wants to formulate a hypothesis test that could use a sample of emergency response times to determine whether or not the service goal of 12 minutes or less is being achieved.

$H_0: \mu \leq 12$	The emergency service is meeting the response goal. No follow-up action is necessary.
$H_a: \mu > 12$	The emergency service is not meeting the response goal. Appropriate follow-up action is necessary.

where μ = the mean response time for the population of medical emergency requests.

Type I & Type II Error

- Because hypothesis tests are based on sample data, we must allow for the possibility of errors.
- A Type I error is rejecting H_0 when it is true.
- The probability of making a Type I error when the null hypothesis is true as an equality is called the level of significance.
- Applications of hypothesis testing that only control for the Type I error are often called significance tests.
- A Type II error is accepting H_0 when it is false.
- It is difficult to control for the probability of making a Type II error.
- Statisticians avoid the risk of making a Type II error by using "do not reject H_0" rather than "accept H_0".

Type I and Type II Errors

Conclusion	Population Condition: H_0 True ($\mu \leq 12$)	Population Condition: H_0 False ($\mu > 12$)
Accept H_0 (Conclude $\mu \leq 12$)	Correct Conclusion	Type II Error
Reject H_0 (Conclude $\mu > 12$)	Type I Error	Correct Conclusion

p-Value Approach to One-Tailed Hypothesis Testing

- The *p*-value is the probability, computed using the test statistic, that measures the support (or lack of support) provided by the sample for the null hypothesis.

- If the *p*-value is less than or equal to the level of significance α, the value of the test statistic is in the rejection region.

- Reject H_0 if the p-value $\leq \alpha$.

- Less than 0.01: Overwhelming evidence to conclude H_a is true.

- Between 0.01 and 0.05: Strong evidence to conclude H_a is true.

- Between .05 and .10: Weak evidence to conclude H_a is true.

- Greater than .10: Insufficient evidence to conclude H_a is true.

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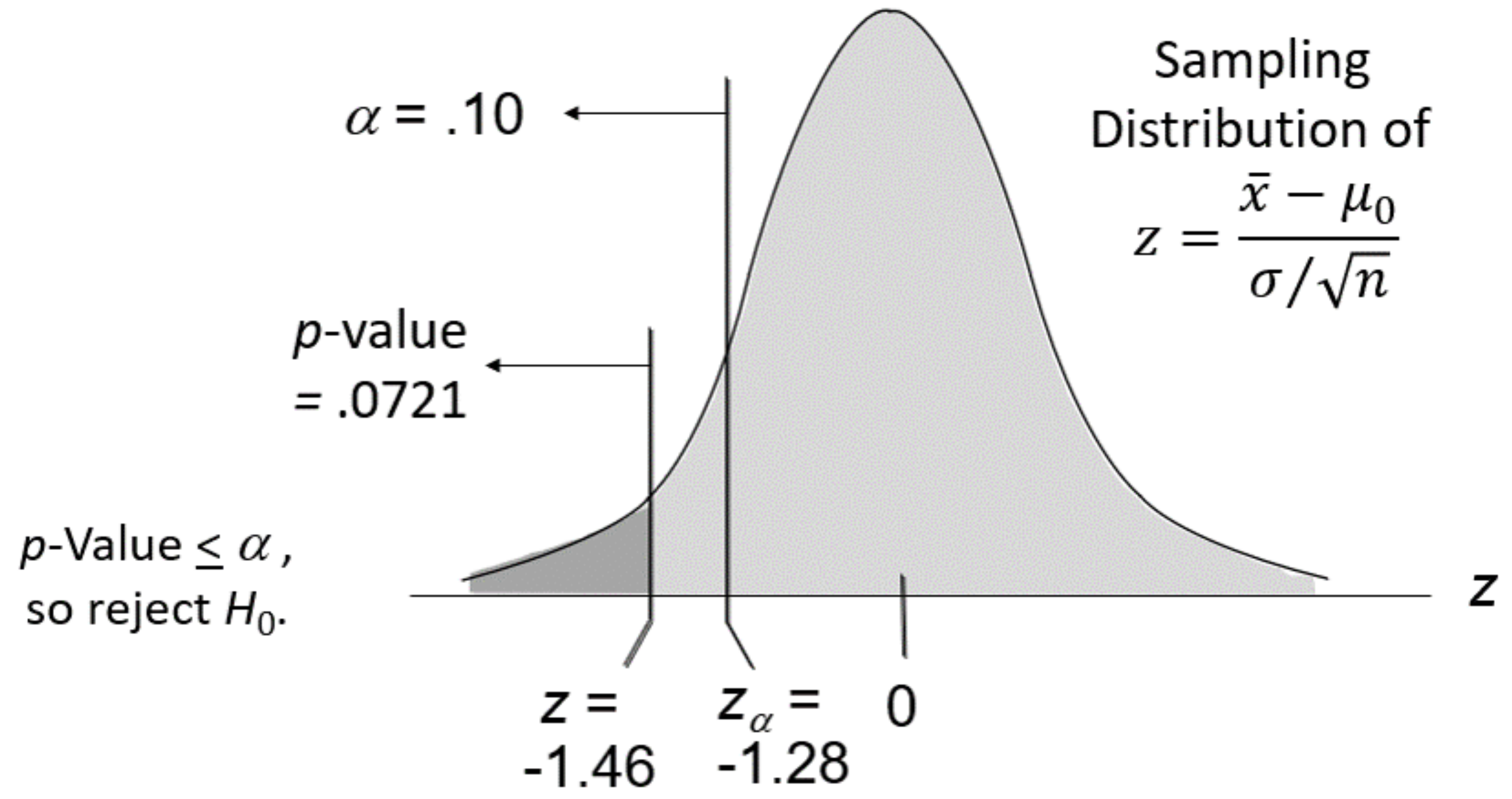
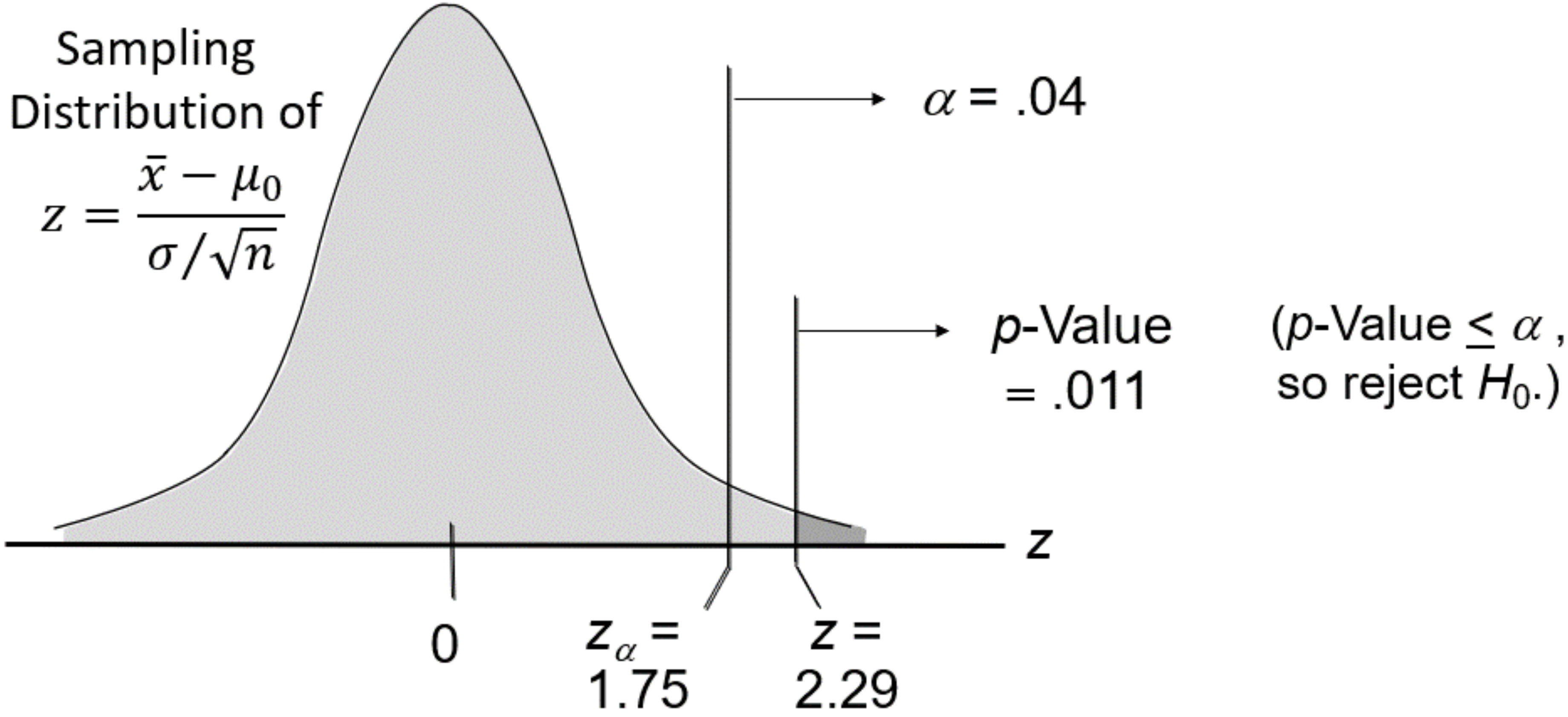
Lower-Tailed Test About a Population Mean: σ Known

<u>p-Value Approach:</u>



Upper-Tailed Test About a Population Mean: σ Known

<u>*p*-Value Approach:</u>



Critical Value Approach to One-Tailed Hypothesis Testing

- The test statistic z has a standard normal probability distribution.
- We can use the standard normal probability distribution table to find the z-value with an area of α in the lower (or upper) tail of the distribution.
- The value of the test statistic that established the boundary of the rejection region is called the critical value for the test.
- The rejection rule is:
  - Lower tail: Reject H_0 if $z \le -z_\alpha$
  - Upper tail: Reject H_0 if $z \ge z_\alpha$

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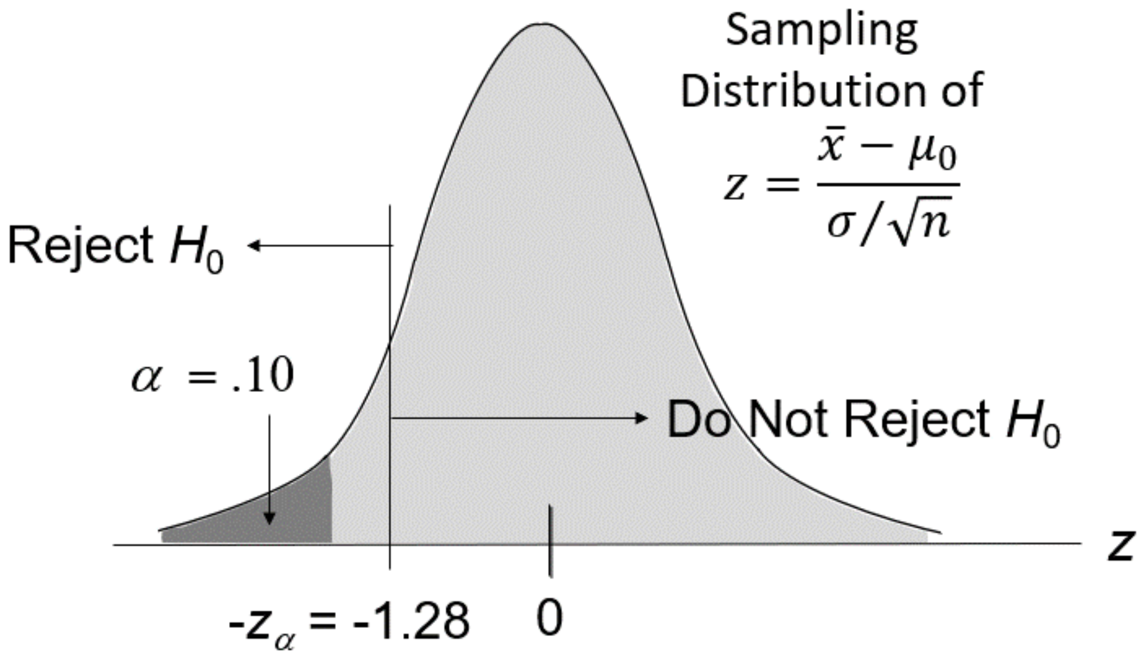
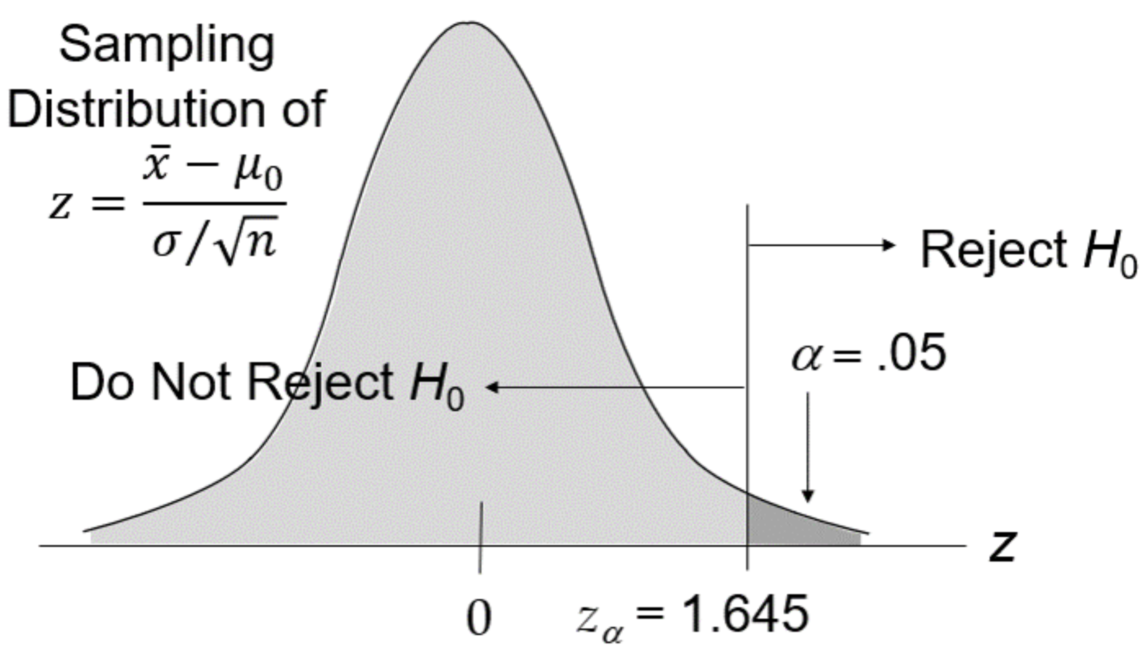
Lower- and Upper-Tailed Test About a Population Mean: σ Known

Lower-Tailed Test:



Upper-Tailed Test:



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Steps of Hypothesis Testing

Step 1. Develop the null and alternative hypotheses.

Step 2. Specify the level of significance α.

Step 3. Collect the sample data and compute the value of the test statistic.

p-Value Approach:

Step 4. Use the value of the test statistic to compute the *p*-value.

Step 5. Reject H_0 if p-value $\leq \alpha$.

Critical Value Approach:

Step 4. Use the level of significance α to determine the critical value and the rejection rule.

Step 5. Use the value of the test statistic and the rejection rule to determine whether to reject H_0.

One-Tailed Tests About a Population Mean: σ Known

Example: Metro EMS

The response times for a random sample of 40 medical emergencies were tabulated. The sample mean is 13.25 minutes. The population standard deviation is believed to be 3.2 minutes.

The EMS director wants to perform a hypothesis test, with a .05 level of significance, to determine whether the service goal of 12 minutes or less is being achieved.

1. Develop the hypotheses. $H_0: \mu \leq 12$

   $H_a: \mu > 12$

2. Specify the level of significance. $\alpha = .05$

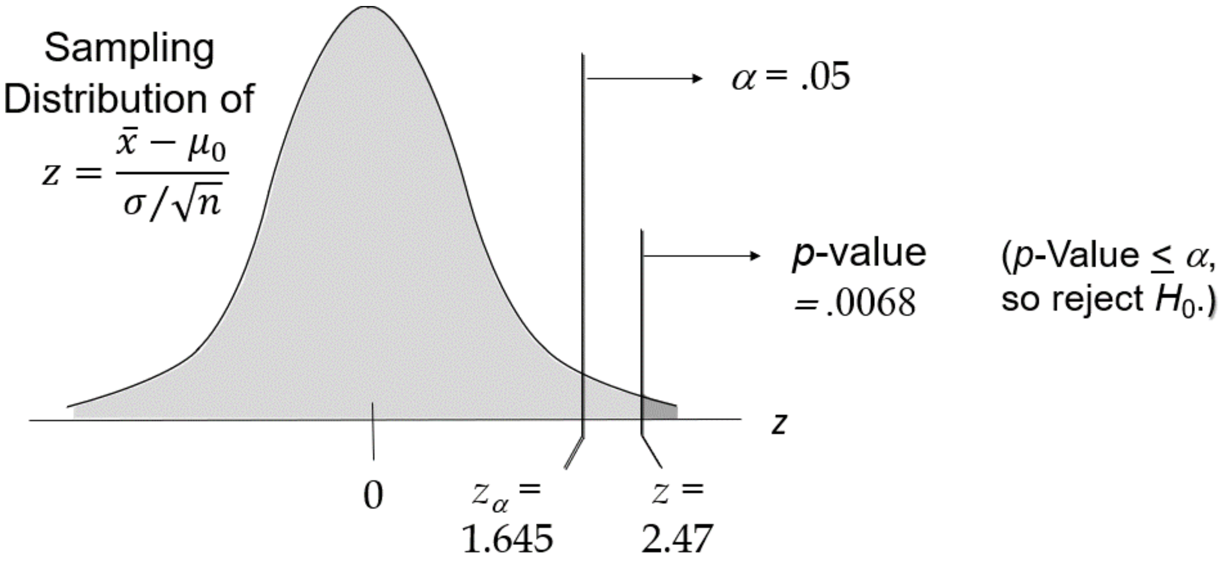
3. Compute the value of the test statistic. $z = \frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}} = \frac{13.25 - 12}{3.2/\sqrt{40}} = 2.47$

One-Tailed Tests About a Population Mean: σ Known

<u>*p* –Value Approach:</u>

4. Compute the *p* –value.

   For z = 2.47, the cumulative probability is 0.9932.
   p-value = 1 – 0.9932 = **0.0068**



5. Determine whether to reject H_0.

   Because *p*-value = 0.0068 ≤ α = 0.05, we reject H_0.
   There is sufficient statistical evidence to infer that Metro EMS is <u>not</u> meeting the response goal of 12 minutes.

One-Tailed Tests About a Population Mean: σ Known

Critical Value Approach:

4. Determine the critical value and the rejection rule.

   For $\alpha = 0.05$, $z_{0.05} = 1.645$. We will reject H_0 if $z \geq 1.645$.

5. Determine whether to reject H_0.

   Because $2.47 \geq 1.645$, we reject H_0.

   There is sufficient statistical evidence to infer that Metro EMS is not meeting the response goal of 12 minutes.

p-Value & Critical-Value Approach to Two-Tailed Hypothesis Testing

p-value approach: Compute the p-value using the following three steps:

1. Compute the value of the test statistic z.

2. If z is in the upper tail ($z > 0$), compute the probability that z is greater than or equal to the value of the test statistic. If z is in the lower tail ($z < 0$), compute the probability that z is less than or equal to the value of the test statistic.

3. Double the tail area obtained in step 2 to obtain the p-value.

The rejection rule: Reject H_0 if the p-value $\leq \alpha$.

Critical value approach: The critical values will occur in both the lower and upper tails of the standard normal curve.

Use the standard normal probability distribution table to find $z_{\alpha/2}$ (the z-value with an area of $\alpha/2$ in the upper tail of the distribution).

The rejection rule is: Reject H_0 if $z \leq -z_{\alpha/2}$ or if $z \geq z_{\alpha/2}$

Two-Tailed Tests About a Population Mean: σ Known

Example: Glow Toothpaste

The production line for Glow toothpaste is designed to fill tubes with a mean weight of 6 oz. Periodically, a sample of 30 tubes will be selected in order to check the filling process.

Quality assurance procedures call for the continuation of the filling process if the sample results are consistent with the assumption that the mean filling weight for the population of toothpaste tubes is 6 oz.; otherwise the process will be adjusted.

Assume that a sample of 30 toothpaste tubes provides a sample mean of 6.1 oz. The population standard deviation is believed to be 0.2 oz.

Perform a hypothesis test, at the 0.03 level of significance, to help determine whether the filling process should continue operating or be stopped and corrected.

Two-Tailed Tests About a Population Mean: σ Known

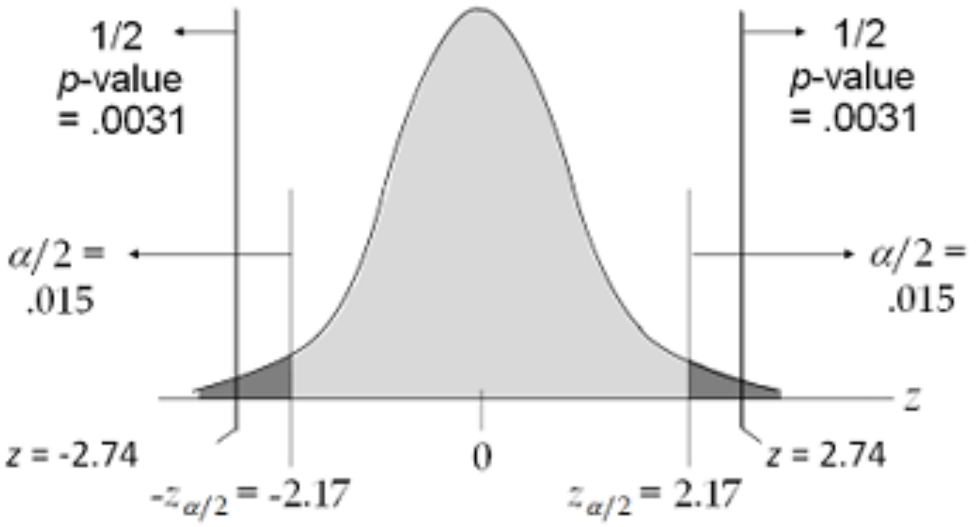
1. Develop the hypotheses. H_0: $\mu = 6$

   H_a: $\mu \neq 6$

2. Specify the level of significance. $\alpha = 0.03$

3. Compute the value of the test statistic. $z = \frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}} = \frac{6.1-6}{2/\sqrt{30}} = 2.74$

<u>*p* –Value Approach:</u>



4. Compute the *p* –value.

   For z = 2.74, the cumulative probability is 0.9969.

   p-value = 2(1 – 0.9969) = **0.0062**

5. Determine whether to reject H_0.

   Because *p*-value = 0.0062 ≤ α = 0.03, we reject H_0.

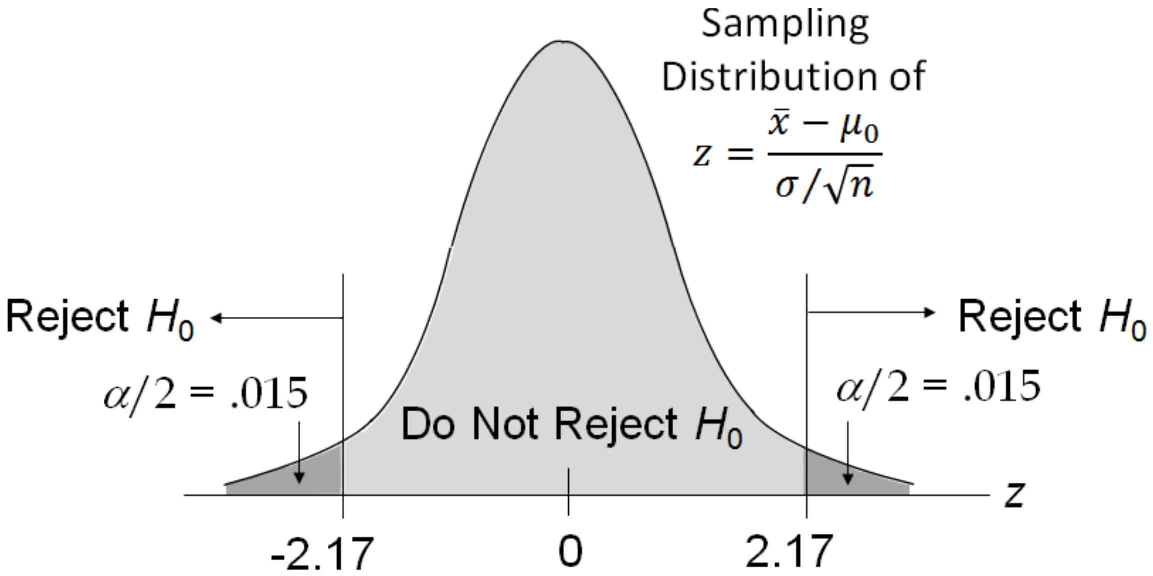
   There is sufficient statistical evidence to infer that the alternative hypothesis is true (i.e. the mean filling weight is not 6 ounces).

Two-Tailed Tests About a Population Mean: σ Known

<u>Critical Value Approach:</u>

4. Determine the critical value and the rejection rule.

For $\alpha/2 = 0.03/2 = 0.015$, $z_{0.015} = 2.17$.
We will reject H_0 if $z \leq -2.17$ or if $z \geq 2.17$



5. Determine whether to reject H_0.
   Because $2.74 \geq 2.17$, we reject H_0.

There is sufficient statistical evidence to infer that the alternative hypothesis is true (i.e. the mean filling weight is not 6 ounces).

Confidence Interval Approach to Two-Tailed Tests For Population Mean

- Select a simple random sample from the population and use the value of the sample mean $\bar{x}$ to develop the confidence interval for the population mean μ. (Confidence intervals are covered in Chapter 8.)

- If the confidence interval contains the hypothesized value μ_0, do not reject H_0. Otherwise, reject H_0. (Actually, H_0 should be rejected if μ_0 happens to be equal to one of the end points of the confidence interval.)

The 97% confidence interval for μ is

$$\bar{x} \pm z_{\alpha/2} \frac{\sigma}{\sqrt{n}}$$

$$6.1 \pm 2.17 \frac{0.2}{\sqrt{30}}$$

$$6.1 \pm 0.07924$$

$$(6.02076, 6.17924)$$

Because the hypothesized value for the population mean, $\mu_0 = 6$, is not in this interval, the hypothesis-testing conclusion is that the null hypothesis, H_0: $\mu = 6$, can be rejected.

Tests About a Population Mean: σ Unknown

Test Statistic: $$t = \frac{\bar{x} - \mu_0}{s/\sqrt{n}}$$

This test statistic has a t distribution with $n - 1$ degrees of freedom.

Rejection Rule: p-value approach

Reject H_0 if p-value $\leq \alpha$

Rejection Rule: Critical value approach

- Lower-tail: Reject H_0 if $t \leq -t_\alpha$
- Upper-tail: Reject H_0 if $t \geq t_\alpha$
- Two-tail: Reject H_0 if $t \leq -t_{\alpha/2}$ or if $t \geq t_{\alpha/2}$

p -Values and the t Distribution

- The format of the t distribution table provided in most statistics textbooks does not have sufficient detail to determine the exact p-value for a hypothesis test.
- However, we can still use the t distribution table to identify a range for the p-value.
- An advantage of computer software packages is that the computer output will provide the p-value for the t distribution.

Example: Highway Patrol - One-Tailed Test About a Population Mean: σ Unknown

A State Highway Patrol periodically samples vehicle speeds at various locations on a particular roadway. The sample of vehicle speeds is used to test the hypothesis H_0: $\mu \leq 65$.

The locations where H_0 is rejected are deemed the best locations for radar traps. At Location F, a sample of 64 vehicles shows a mean speed of 66.2 mph with a standard deviation of 4.2 mph. Use $\alpha = 0.05$ to test the hypothesis.

One-Tailed Test About a Population Mean: σ Unknown

1. Develop the hypotheses. H_0: $\mu \leq 65$
   H_a: $\mu > 65$

2. Specify the level of significance. $\alpha = .05$

3. Compute the value of the test statistic. $t = \frac{\bar{x} - \mu_0}{s/\sqrt{n}} = \frac{66.2 - 65}{4.2/\sqrt{64}} = 2.286$

<u>*p* –Value Approach:</u>

4. Compute the *p* –value.

   For $t = 2.286$, the *p*-value must be greater than 0.01 (for $t = 2.387$), but less than 0.025 (for $t = 1.998$).

$$0.01 < p\text{-value} < 0.025$$

5. Determine whether to reject H_0.

   Because *p*-value < $\alpha = 0.05$, we reject H_0. We are at least 95% confident that the mean speed of vehicles at Location F is greater than 65 mph.

One-Tailed Test About a Population Mean: σ Unknown

Critical Value Approach:

4. Determine the critical value and the rejection rule.

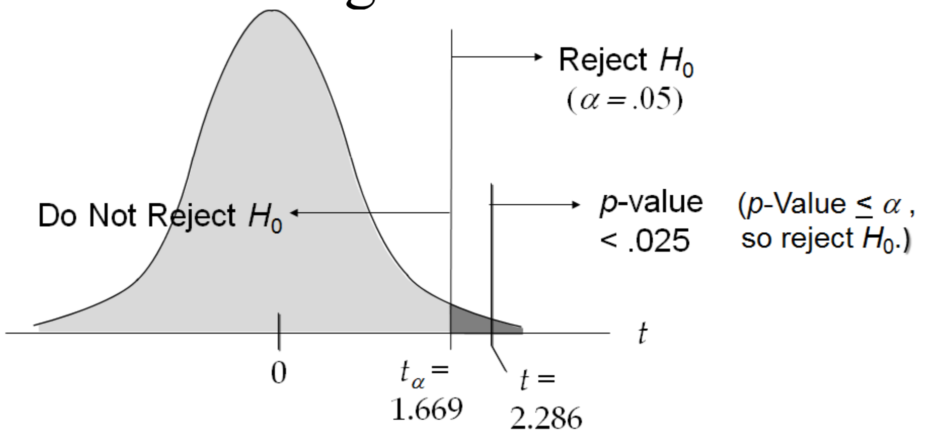
   For $\alpha = 0.05$ and df = 64 – 1 = 63, $t_{0.05} = 1.669$.

   We will reject H_0 if $t \geq 1.669$.

5. Determine whether to reject H_0.

   Because $2.286 \geq 1.669$, we reject H_0.

   We are at least 95% confident that the mean speed of vehicles at Location F is greater than 65 mph. Location F is a good candidate for a radar trap.



A Summary of Forms for Null & Alternative: Population Proportion

- The equality part of the hypotheses always appears in the null hypothesis.
- In general, a hypothesis test about the value of a population proportion p must take one of the following three forms (where p_0 is the hypothesized value of the population proportion).

1. One-tailed, lower tail: $H_0: p \geq p_0$ $H_a: p < p_0$
2. One-tailed, upper tail: $H_0: p \leq p_0$ $H_a: p > p_0$
3. Two-tailed: $H_0: p = p_0$ $H_a: p \neq p_0$

Tests About a Population Proportion

Test Statistic:

$$z = \frac{\bar{p} - p_0}{\sqrt{\frac{p_0(1 - p_0)}{n}}}$$

Assuming $np \geq 5$ and $n(1 - p) \geq 5$.

Rejection Rule: p –Value Approach

Reject H_0 if p –value $\leq \alpha$

Rejection Rule: Critical Value Approach

- Lower-tail: Reject H_0 if $z \leq -z_\alpha$
- Upper-tail: Reject H_0 if $z \geq z_\alpha$
- Two-tail: Reject H_0 if $z \leq -z_\alpha$ or if $z \geq z_\alpha$

Two-Tailed Test About a Population Proportion

<u>Example:</u> National Safety Council (NSC)

For a Christmas and New Year's week, the National Safety Council estimated that 500 people would be killed and 25,000 injured on the nation's roads. The NSC claimed that 50% of the accidents would be caused by drunk driving. A sample of 120 accidents showed that 67 were caused by drunk driving. Use these data to test the NSC's claim with α = 0.05.

1. Determine the hypotheses. $H_0: p = .5$ and $H_a: p \neq .5$

2. Specify the level of significance. $\alpha = .05$

3. Compute the value of the test statistic.

$$\sigma_{\bar{p}} = \sqrt{\frac{p_0(1-p_0)}{n}} = \sqrt{\frac{.5(1-.5)}{120}} = .045644$$

$$z = \frac{\bar{p}-p_0}{\sigma_{\bar{p}}} = \frac{\left(\frac{67}{120}\right)-.5}{.045644} = 1.28$$

Two-Tailed Test About a Population Proportion

p –Value Approach:

4. Compute the *p* –value.

   For $z = 1.28$, the cumulative probability = 0.8997.

   p-value = 2(1 – 0.8997) = **0.2006**.

5. Determine whether to reject H_0.

   Because p-value = 0.2006 > α = 0.05, we cannot reject H_0.

   We do not have convincing evidence that the true proportion of accidents that would be caused by drunk driving is different than 50%.

Critical Value Approach:

4. Determine the critical value and the rejection rule.

   For $\alpha/2 = 0.05/2 = 0.025$, $z_{0.025} = 1.96$.

   We will reject H_0 if $z \leq -1.96$ or if $z \geq 1.96$.

5. Determine whether to reject H_0.

   Because 1.278 is not less than −1.96 and is not greater than 1.96, we cannot reject H_0.

   We do not have convincing evidence that the true proportion of accidents that would be caused by drunk driving is different than 50%.

Estimating the Difference Between Two Population Means

- μ_1 = the mean of population 1 and μ_2 = the mean of population 2.
- The difference between the two population means is $\mu_1 - \mu_2$.

To estimate $\mu_1 - \mu_2$, we will select a simple random sample of size n_1 from population 1 and a simple random sample of size n_2 from population 2.

- $\bar{x}_1$ = the mean of sample 1 and $\bar{x}_2$ = the mean of sample 2.
- The point estimator of the difference between the means of the populations 1 and 2 is $\bar{x}_1 - \bar{x}_2$.

Sampling Distribution of $\bar{x}_1 - \bar{x}_2$

- Mean/Expected value: $E(\bar{x}_1 - \bar{x}_2) = \mu_1 - \mu_2$
- Standard Deviation (Standard Error): $\sigma_{\bar{x}_1 - \bar{x}_2} = \sqrt{\frac{(\sigma_1)^2}{n_1} + \frac{(\sigma_2)^2}{n_2}}$

Where: σ_1 = standard deviation of population 1

σ_2 = standard deviation of population 2

n_1 = sample size from population 1

n_2 = sample size from population 2

Interval Estimate

$$(\bar{x}_1 - \bar{x}_2) \pm z_{\alpha/2} \sqrt{\frac{(\sigma_1)^2}{n_1} + \frac{(\sigma_2)^2}{n_2}}$$

Where $1 - \alpha$ is the confidence coefficient.

Interval Estimation of $\mu_1 - \mu_2$ when σ_1 and σ_2 are Known

Example: Par, Inc. is a manufacturer of golf equipment and has developed a new golf ball that has been designed to provide "extra distance."

In a test of driving distance using a mechanical driving device, a sample of Par golf balls was compared with a sample of golf balls made by Rap, Ltd., a competitor.

	Sample # 1 Par, Inc.	Sample # 2 Rap, Ltd.
Sample Size	120 balls	80 balls
Sample Mean	295 yards	278 yards

Based on data from previous driving distance tests, the two population standard deviations are known with σ_1 = 15 yards and σ_2 = 20 yards.

Let us develop a 95% confidence interval estimate of the difference between the mean driving distances of the two brands of golf ball.

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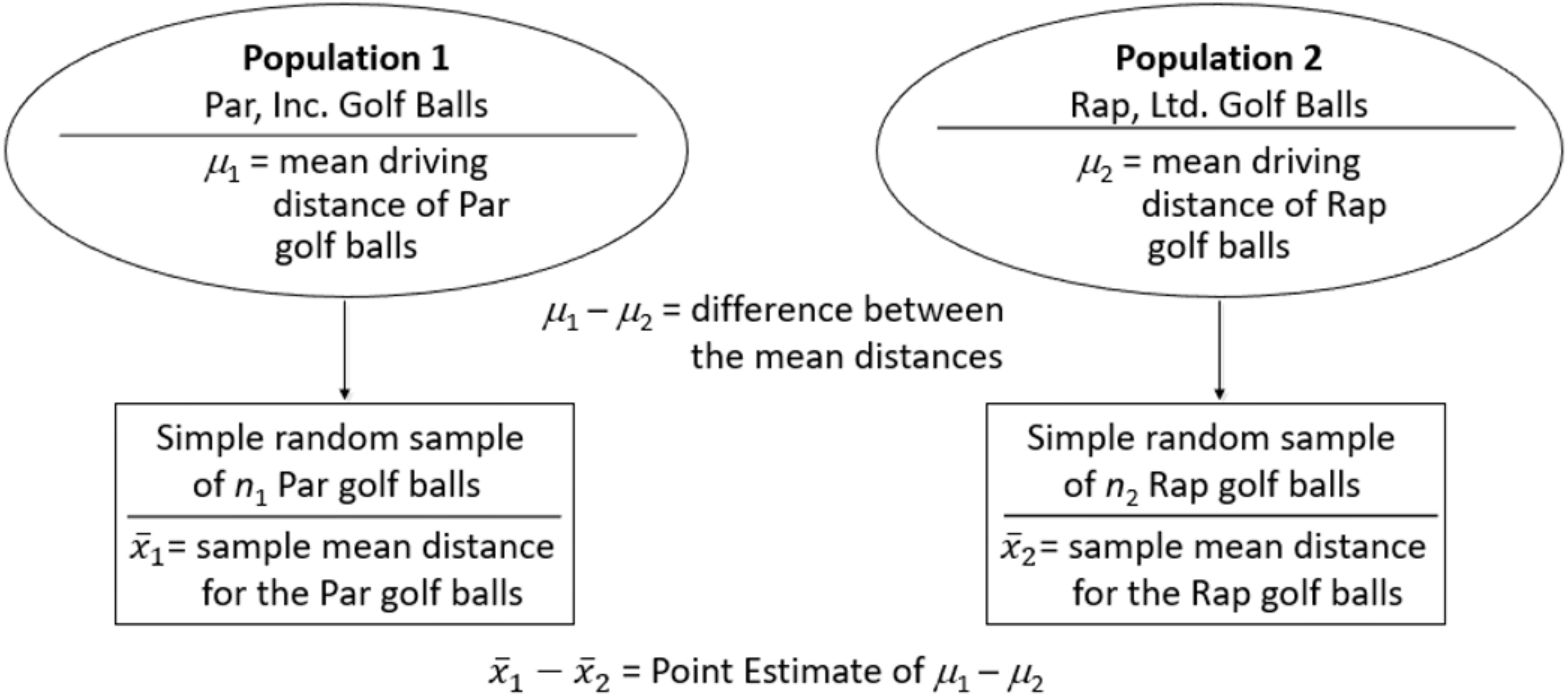
Estimating the Difference Between Two Population Means



Interval Estimation of $\mu_1 - \mu_2$ when σ_1 and σ_2 are Known

$$(\bar{x}_1 - \bar{x}_2) \pm z_{\alpha/2}\sqrt{\frac{(\sigma_1)^2}{n_1} + \frac{(\sigma_2)^2}{n_2}}$$

$$(295 - 278) \pm 1.96\sqrt{\frac{(15)^2}{120} + \frac{(20)^2}{80}}$$

$$17 \pm 5.14$$

$$11.86 \text{ to } 22.14$$

We are 95% confident that the difference between the mean driving distances of Par, Inc. balls and Rap, Ltd. balls is 11.86 to 22.14 yards.

Hypothesis Tests About $\mu_1 - \mu_2$ when σ_1 and σ_2 are Known

A hypothesis test about the value of the difference in two population means $\mu_1 - \mu_2$ must take one of the following three forms (where D_0 is the hypothesized difference in the population means).

1. One-tailed, lower tail: $H_0: \mu_1 - \mu_2 \geq D_0$ $\quad H_a: \mu_1 - \mu_2 < D_0$
2. One-tailed, upper tail: $H_0: \mu_1 - \mu_2 \leq D_0$ $\quad H_a: \mu_1 - \mu_2 > D_0$
3. Two-tailed: $H_0: \mu_1 - \mu_2 = D_0$ $\quad H_a: \mu_1 - \mu_2 \neq D_0$

Test Statistic: $$z = \frac{(\bar{x}_1 - \bar{x}_2) - D_0}{\sqrt{\frac{(\sigma_1)^2}{n_1} + \frac{(\sigma_2)^2}{n_2}}}$$

Hypothesis Tests About $\mu_1 - \mu_2$ when σ_1 and σ_2 are Known

Example: Par, Inc.

Can we conclude, using α = 0.01, that the mean driving distance of Par, Inc. golf balls is greater than the mean driving distance of Rap, Ltd. golf balls?

1. Develop the hypotheses.

$$H_0: \mu_1 - \mu_2 \leq 0$$

$$H_a: \mu_1 - \mu_2 > 0$$

μ_1 = the mean distance for the population of Par, Inc. golf balls

μ_2 = the mean distance for the population of Rap, Ltd. golf balls

2. Specify the level of significance. $\alpha = 0.01$

3. Compute the value of the test statistic.

$$z = \frac{(\bar{x}_1 - \bar{x}_2) - D_0}{\sqrt{\frac{(\sigma_1)^2}{n_1} + \frac{(\sigma_2)^2}{n_2}}} = \frac{(295 - 278) - 0}{\sqrt{\frac{(15)^2}{120} + \frac{(20)^2}{80}}} = \frac{17}{2.62} = \mathbf{6.49}$$

Hypothesis Tests About $\mu_1 - \mu_2$ when σ_1 and σ_2 are Known

p –Value Approach:

4. Compute the *p* –value.

   For $z = 6.49$, the p-value $< \mathbf{0.0001}$

5. Determine whether to reject H_0.

   Because p-value $< 0.0001 \leq \alpha = 0.01$, we reject H_0.

At the 0.01 level of significance, the sample evidence indicates the mean driving distance of Par, Inc. golf balls is greater than the mean driving distance of Rap, Ltd. golf balls.

Critical Value Approach:

4. Determine the critical value and the rejection rule.

   For $\alpha = 0.01$, $z_{0.01} = 2.33$. We will reject H_0 if $z \geq 2.33$.

5. Determine whether to reject H_0.

   Because $6.49 \geq 2.33$, we reject H_0.

The sample evidence indicates the mean driving distance of Par, Inc. golf balls is greater than the mean driving distance of Rap, Ltd. golf balls.

Interval Estimation of $\mu_1 - \mu_2$ when σ_1 and σ_2 are Unknown

When σ_1 and σ_2 are unknown we will:

1. Use the sample standard deviations, s_1 and s_2, as estimates of σ_1 and σ_2 and
2. Replace $z_{\alpha/2}$ with $t_{\alpha/2}$.

Interval Estimate:

$$(\bar{x}_1 - \bar{x}_2) \pm t_{\alpha/2}\sqrt{\frac{(s_1)^2}{n_1} + \frac{(s_2)^2}{n_2}}$$

Where the degrees of freedom for $t_{\alpha/2}$ are:

$$df = \frac{\left(\frac{(s_1)^2}{n_1} + \frac{(s_2)^2}{n_2}\right)^2}{\left(\frac{1}{n_1 - 1}\right)\left(\frac{(s_1)^2}{n_1}\right)^2 + \left(\frac{1}{n_2 - 1}\right)\left(\frac{(s_2)^2}{n_2}\right)^2}$$

Interval Estimation of $\mu_1 - \mu_2$ when σ_1 and σ_2 are Unknown

Example: Specific Motors of Detroit has developed a new Automobile known as the M car. 24 M cars and 28 J cars (from Japan) were road tested to compare miles-per-gallon (mpg) performance.

Empty cell	Sample #1: M Cars	Sample #2: J Cars
Sample Size	24 cars	28 cars
Sample Mean	29.8 miles per gallon	27.3 miles per gallon
Sample Std. Dev.	2.56 miles per gallon	1.81 miles per gallon

Let us develop a 90% confidence interval estimate of the difference between the mpg performances of the two models of automobile.

Let,

μ_1 = the mean miles per gallon for the population of M cars.

μ_2 = the mean miles per gallon for the population of J cars.

Interval Estimation of $\mu_1 - \mu_2$ when σ_1 and σ_2 are Unknown

The degrees of freedom for $t_{\alpha/2}$ are

$$df = \frac{\left(\frac{(2.56)^2}{24} + \frac{(1.81)^2}{28}\right)^2}{\left(\frac{1}{24-1}\right)\left(\frac{(2.56)^2}{24}\right)^2 + \left(\frac{1}{28-1}\right)\left(\frac{(1.81)^2}{28}\right)^2} = 40.59 = \mathbf{41}$$

With $\alpha/2 = 0.05$ and $df = 41$, $t_{\alpha/2} = 1.683$

$$(\bar{x}_1 - \bar{x}_2) \pm t_{\alpha/2}\sqrt{\frac{(s_1)^2}{n_1} + \frac{(s_2)^2}{n_2}}$$

$$(29.8 - 27.3) \pm 1.683\sqrt{\frac{(2.56)^2}{24} + \frac{(1.81)^2}{28}}$$

$$2.5 \pm 1.051$$

1.449 to 3.551 mpg

We are 90% confident that the difference between the miles-per-gallon performances of M cars and J cars is 1.449 to 3.551 mpg.

Hypothesis Tests About $\mu_1 - \mu_2$ when σ_1 and σ_2 are Unknown

A hypothesis test about the value of the difference in two population means $\mu_1 - \mu_2$ must take one of the following three forms (where D_0 is the hypothesized difference in the population means).

1. One-tailed, lower tail: $H_0: \mu_1 - \mu_2 \geq D_0$ $\quad H_a: \mu_1 - \mu_2 < D_0$
2. One-tailed, upper tail: $H_0: \mu_1 - \mu_2 \leq D_0$ $\quad H_a: \mu_1 - \mu_2 > D_0$
3. Two-tailed: $H_0: \mu_1 - \mu_2 = D_0$ $\quad H_a: \mu_1 - \mu_2 \neq D_0$

Test Statistic: $t = \dfrac{(\bar{x}_1 - \bar{x}_2) - D_0}{\sqrt{\dfrac{(s_1)^2}{n_1} + \dfrac{(s_2)^2}{n_2}}}$

<u>Example</u>: Specific Motors: Can we conclude, using a .05 level of significance, that the miles-per-gallon (*mpg*) performance of M cars is greater than the miles-per-gallon performance of J cars?

Hypothesis Tests About $\mu_1 - \mu_2$ when σ_1 and σ_2 are Unknown

1. Develop the hypotheses. $H_0: \mu_1 - \mu_2 \le 0$

   $H_a: \mu_1 - \mu_2 > 0$

   μ_1 = the mean mpg for the population of M cars

   μ_2 = the mean mpg for the population of J cars

2. Specify the level of significance. $\alpha = 0.05$

3. Compute the value of the test statistic $t = \dfrac{(\bar{x}_1 - \bar{x}_2) - D_0}{\sqrt{\dfrac{(s_1)^2}{n_1} + \dfrac{(s_2)^2}{n_2}}} = \dfrac{(29.8 - 27.3) - 0}{\sqrt{\dfrac{(2.56)^2}{24} + \dfrac{(1.81)^2}{28}}} = \mathbf{4.003}$

The degrees of freedom for t_α are
$$df = \frac{\left[\frac{(2.56)^2}{24} + \frac{(1.81)^2}{28}\right]^2}{\frac{1}{24-1}\left[\frac{(2.56)^2}{24}\right]^2 + \frac{1}{28-1}\left[\frac{(1.81)^2}{24}\right]^2} = 40.59 = 41$$

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Hypothesis Tests About $\mu_1 - \mu_2$ when σ_1 and σ_2 are Unknown

p –Value Approach:

4. Compute the *p*-value. For $t = 4.003$ and df = 41 the *p*-value **< 0.005**
5. Determine whether to reject H_0.

   Because *p*-value $\le \alpha = 0.05$, we reject H_0.

   At the 0.05 level of significance, the sample evidence indicates that the miles-per-gallon (*mpg*) performance of M cars is greater than the miles-per-gallon performance of J cars.

Critical Value Approach:

4. Determine the critical value and the rejection rule.

   For $\alpha = 0.05$ and df = 41, $t_{0.05} = 1.683$. We will reject H_0 if $t \ge 1.683$.
5. Determine whether to reject H_0.

   Because $4.003 \ge 1.683$, we reject H_0.

   We are at least 95% confident that the miles-per-gallon (*mpg*) performance of M cars is greater than the miles-per-gallon performance of J cars.

Inferences About Difference Between Two Population Means: Matched Samples

- With a matched-sample design each sampled item provides a pair of data values.
- This design often leads to a smaller sampling error than the independent-sample design because variation between sampled items is eliminated as a source of sampling error.

Example: Express Deliveries

A Chicago-based firm has documents that must be quickly distributed to district offices throughout the U.S. The firm must decide between two delivery services, UPX (United Parcel Express) and INTEX (International Express), to transport its documents.

In testing the delivery times of the two services, the firm sent two reports to a random sample of its district offices with one report carried by UPX and the other report carried by INTEX. Do the data on the next slide indicate a difference in mean delivery times for the two services? Use a 0.05 level of significance.

Inferences About Difference Between Two Population Means: Matched Samples

Here are the data:

The mean of the differences is

$$\bar{d} = \frac{\sum d_i}{n} = \frac{(7+6+\cdots+5)}{10} = 2.7$$

The standard deviation of the differences is

$$s_d = \sqrt{\frac{\sum(d_i-\bar{d})^2}{n-1}} = \sqrt{\frac{76.1}{9}} = 2.9$$

	Delivery Time (Hours)		
District Office	UPX	INTEX	Difference
Seattle	32	25	7
Los Angeles	30	24	6
Boston	19	15	4
Cleveland	16	15	1
New York	15	13	2
Houston	18	15	3
Atlanta	14	15	-1
St. Louis	10	8	2
Milwaukee	7	9	-2
Denver	16	11	5

Inferences About Difference Between Two Population Means: Matched Samples

1. Develop the hypotheses.

$H_0: \mu_d = 0$

$H_a: \mu_d \neq 0$

μ_d = the mean of the difference values for the two delivery services for the population of district offices.

2. Specify the level of significance. $\alpha = 0.05$

3. Compute the value of the test statistic. $t = \dfrac{\bar{d} - \mu_d}{s_d/\sqrt{n}} = \dfrac{2.7 - 0}{2.9/\sqrt{10}} = \mathbf{2.94}$

Inferences About Difference Between Two Population Means: Matched Samples

p –Value Approach:

4. Compute the *p*-value. For $t = 2.94$ and $df = 9$ the *p*-value is between 0.02 and 0.01.

   Note: This is a two-tailed test, so we doubled the upper-tail areas of 0.005 and 0.01.

5. Determine whether to reject H_0. Because *p*-value $\leq \alpha = 0.05$, we reject H_0.

At the 0.05 level of significance, the sample evidence indicates that there is a difference in mean delivery times for the two services.

Critical Value Approach:

4. Determine the critical value and the rejection rule.

   For $\alpha = 0.05$ and $df = 9$, $t_{0.025} = 2.262$. We will reject H_0 if $t \geq 2.262$.

5. Determine whether to reject H_0. Because $2.94 \geq 2.262$, we reject H_0.

We are at least 95% confident that there is a difference in mean delivery times for the two services.

Sampling Distribution of $\bar{p}_1 - \bar{p}_2$

- Mean/Expected value: $E(\bar{p}_1 - \bar{p}_2) = p_1 - p_2$
- Standard Deviation (Standard Error): $$\sigma_{\bar{p}_1 - \bar{p}_2} = \sqrt{\frac{p_1(1-p_1)}{n_1} + \frac{p_2(1-p_2)}{n_2}}$$

where: p_1 = proportion for population 1

p_2 = proportion for population 2

n_1 = sample size from population 1

n_2 = sample size from population 2

If the sample sizes are large, the sampling distribution of $\bar{p}_1 - \bar{p}_2$ can be approximated by a normal probability distribution.

- The sample sizes are sufficiently large if all of these conditions are met:

$$n_1 p_1 \geq 5 \text{ and } n_1(1 - p_1) \geq 5$$

$$n_2 p_2 \geq 5 \text{ and } n_2(1 - p_2) \geq 5$$

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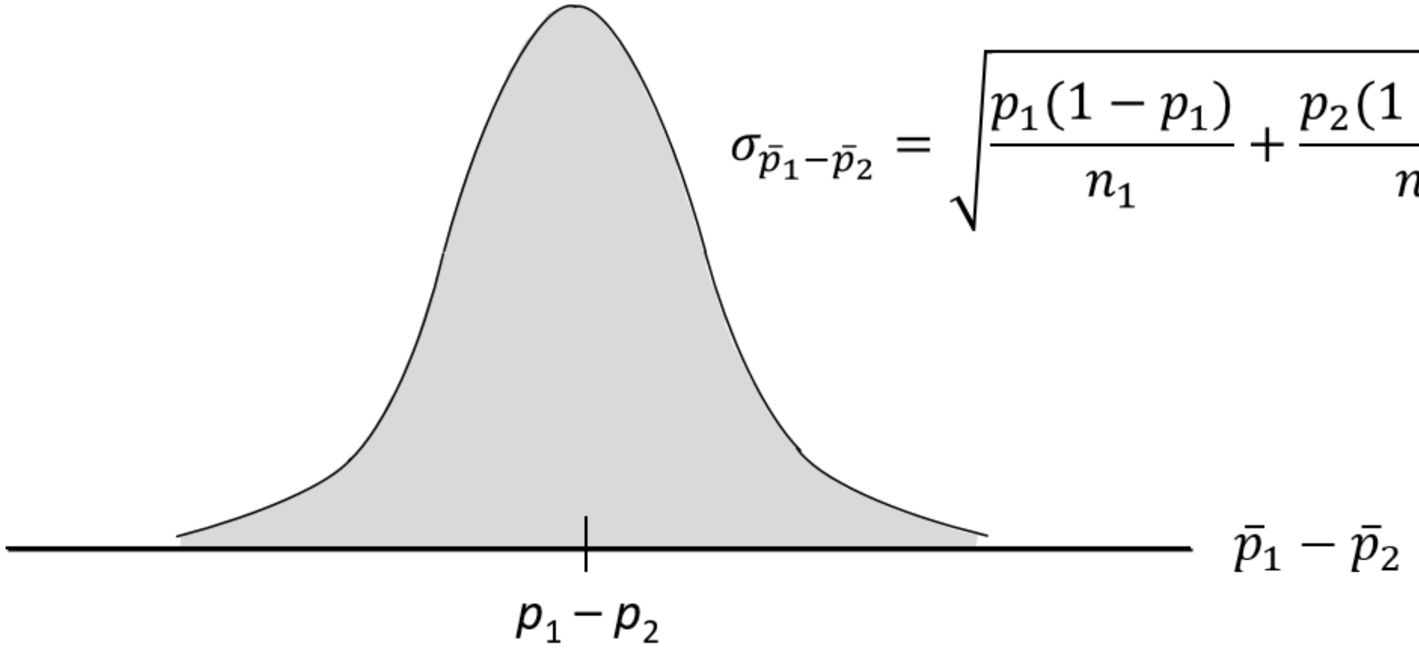
Sampling Distribution of $\bar{p}_1 - \bar{p}_2$



$$\sigma_{\bar{p}_1 - \bar{p}_2} = \sqrt{\frac{p_1(1-p_1)}{n_1} + \frac{p_2(1-p_2)}{n_2}}$$

Interval Estimate: $$\bar{p}_1 - \bar{p}_2 \pm z_{\alpha/2}\sqrt{\frac{\bar{p}_1(1-\bar{p}_1)}{n_1} + \frac{\bar{p}_2(1-\bar{p}_2)}{n_2}}$$

Interval Estimation of $p_1 - p_2$

<u>Example</u>: Market Research Associates is conducting research to evaluate the effectiveness of a client's new advertising campaign. Before the new campaign began, a telephone survey of 150 households in the test market area showed 60 households "aware" of the client's product. The new campaign has been initiated with TV and newspaper advertisements running for three weeks. A survey conducted immediately after the new campaign showed 120 of 250 households "aware" of the client's product. Does the data support the position that the advertising campaign has provided an increased awareness of the client's product?

$$\bar{p}_1 - \bar{p}_2 \pm z_{\alpha/2}\sqrt{\frac{\bar{p}_1(1-\bar{p}_1)}{n_1} + \frac{\bar{p}_2(1-\bar{p}_2)}{n_2}}$$

$$(0.48 - 0.40) \pm 1.96\sqrt{\frac{0.48\,(0.52)}{250} + \frac{0.40\,(0.60)}{150}}$$

$$0.08 \pm 0.10$$

Hence, the 95% confidence interval for the difference in before and after awareness of the product is –0.02 to 0.18.

Hypothesis Tests about $p_1 - p_2$

Hypotheses:

1. One-tailed, lower tail: $H_0: p_1 - p_2 \geq 0$ $H_a: p_1 - p_2 < 0$

2. One-tailed, upper tail: $H_0: p_1 - p_2 \leq 0$ $H_a: p_1 - p_2 > 0$

3. Two-tailed: $H_0: p_1 - p_2 = 0$ $H_a: p_1 - p_2 \neq 0$

Note: We will focus on tests involving no difference between the two population proportions.

Standard error of $\bar{p}_1 - \bar{p}_2$ when $p_1 = p_2 = p$: $\sigma_{\bar{p}_1 - \bar{p}_2} = \sqrt{p(1-p)\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}$

Pooled estimator of p when $p_1 = p_2 = p$: $\bar{p} = \frac{n_1\bar{p}_1 + n_2\bar{p}_2}{n_1 + n_2}$

Hypothesis Tests about $p_1 - p_2$

Test Statistic: $$z = \frac{(\bar{p}_1 - \bar{p}_2)}{\sqrt{\bar{p}(1-\bar{p})\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}}$$

Example: Market Research Associates

Can we conclude, using a 0.05 level of significance, that the proportion of households aware of the client's product increased after the new advertising campaign?

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Hypothesis Tests about $p_1 - p_2$

Example: Market Research Associates

Can we conclude, using a 0.05 level of significance, that the proportion of households aware of the client's product increased after the new advertising campaign?

1. Develop the hypotheses.

$$H_0: p_1 - p_2 \leq 0$$

$$H_a: p_1 - p_2 > 0$$

p_1 = the proportion of the population of households that are "aware" of the product after the new campaign.

p_2 = the proportion of the population of households that are "aware" of the product before the new campaign.

Hypothesis Tests about $p_1 - p_2$

2. Specify the level of significance. $\alpha = 0.05$

3. Compute the value of the test statistic.

$$\bar{p} = \frac{250(.48)+150(.40)}{250+150} = \frac{180}{400} = .45$$

$$s_{\bar{p}_1-\bar{p}_2} = \sqrt{.45(.55)\left(\frac{1}{250}+\frac{1}{150}\right)} = .0514$$

$$z = \frac{(.48-.40)}{.0514} = \frac{.08}{.0514} = 1.56$$

Hypothesis Tests about $p_1 - p_2$

p –Value Approach:

4. Compute the p –value. For $z = 1.56$, the p-value = .0594

5. Determine whether to reject H_0.

   Because p-value > $\alpha = 0.05$, we cannot reject H_0.

   We cannot conclude that the proportion of households aware of the client's product increased after the new campaign.

Critical Value Approach:

4. Determine the critical value and rejection rule.

   For $\alpha = .05$, $z_{.05} = 1.645$, Reject H_0 if $z \geq 1.645$

5. Determine whether to reject H_0. Because $1.56 < 1.645$, we cannot reject H_0.

   We cannot conclude that the proportion of households aware of the client's product increased after the new campaign.