Lecturer: Yongshan Ding *Scribe: Jonathan Baker*

Lecture 03: Understanding Qubits

October 9, 2018

1 Four Basis Rules for Quantum Computation

1.1 Rule 1 (The Superposition Rule)

First, we define formally a quantum state for a single qubit, namely $|\psi\rangle$, as a vector with two amplitudes $\alpha, \beta \in \mathbb{C}$, the set of complex numbers, denoted as

$$|\psi\rangle = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

such that $|\alpha|^2 + |\beta|^2 = 1$, known as the normalization condition. We use the $|\psi\rangle$, known as a "ket" used in Dirac notation, to represent some quantum state. We may also represent a quantum state as a superposition of two basis states $|0\rangle$ and $|1\rangle$, as $|\psi\rangle = \alpha |0\rangle + \beta |1\rangle$. Note that

$$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

1.2 Review of Complex Numbers

A *complex number* $z \in \mathbb{C}$, where $\mathbb{C}$ represents the set of all complex numbers, is a number of the form $a + bi$, where $a, b \in \mathbb{R}$, the set of all real numbers, and i is the imaginary number defined by $i^2 = -1$. If $z = a + bi$, then $Re(z) = a$, the real part of z, and $Im(z) = b$ the imaginary part of z.

A complex number may be represented on a two dimensional plane using polar coordinates. That is, each z can be represented by a pair of real numbers (a, b). We may also represent z using a pair (r, θ) with $r = \sqrt{a^2 + b^2}$ and $\tan\theta = b/a$. By recalling Euler's Formula which states $e^{i\theta} = \cos\theta + i\sin\theta$, we see that any complex $z = re^{i\theta}$.

We now define the *complex conjugate*. Let $z = a + bi$, then $z^* = a - bi$ is the complex conjugate of z. Notice it is always the case $zz^* = (a + bi)(a - bi) = a^2 + b^2 = |r|^2$. Furthermore, we define the *complex conjugate transpose* or *adjoint*. Let $|\psi\rangle = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$ then the complex conjugate transpose of $|\psi\rangle$ is, written as a "bra" $\langle\psi| = \begin{bmatrix} \alpha^* & \beta^* \end{bmatrix}$. With this handy "bra-ket" notation, it is easy to write the *inner product* of two vectors. Let $|\psi_1\rangle, |\psi_2\rangle$ be two

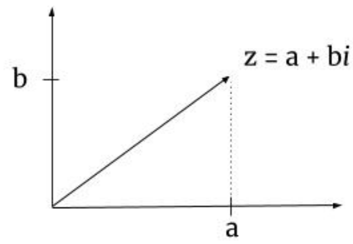

Figure 1: A point $z = a + bi$ in the complex plane.

complex vectors. Then $\langle\psi_1|\cdot|\psi_2\rangle = \langle\psi_1|\psi_2\rangle$ is the inner product of $|\psi_1\rangle$ and $|\psi_2\rangle$. Explicitly, if $|\psi_1\rangle = \begin{bmatrix}\alpha_1\\ \beta_1\end{bmatrix}$ and $|\psi_2\rangle = \begin{bmatrix}\alpha_2\\ \beta_2\end{bmatrix}$, then $\langle\psi_1|\psi_2\rangle = \alpha_1^*\alpha_2 + \beta_1^*\beta_2$.

As two examples of inner products consider first the inner product of a vector with itself, $\langle\psi|\psi\rangle = \alpha^*\alpha + \beta^*\beta = |\alpha|^2 + |\beta|^2 = 1$, by the normalization condition. This gives us an easy way to express this condition, namely $\langle\psi|\psi\rangle = 1$. Next, consider $\langle 0|1\rangle = 0\cdot 1 + 1\cdot 0 = 0$. Recall that $|0\rangle$ and $|1\rangle$ form the computational basis states and from linear algebra we know that they are in fact orthogonal to each other. It turns out that the inner product gives an easy way to check for orthogonality between two vectors. In fact, two states $|\psi_1\rangle$ and $|\psi_2\rangle$ are orthogonal if and only if $\langle\psi_1|\psi_2\rangle = 0$.

A second operation we may perform between two state vectors $|\psi_1\rangle$ and $|\psi_2\rangle$ is the *outer product*. We define this operation so that $|\psi_1\rangle\langle\psi_2| = \begin{bmatrix}\alpha_1\\ \beta_1\end{bmatrix}\begin{bmatrix}\alpha_2^* & \beta_2^*\end{bmatrix} = \begin{bmatrix}\alpha_1\alpha_2^* & \alpha_1\beta_2^*\\ \beta_1\alpha_2^* & \beta_1\beta_2^*\end{bmatrix}$. Unlike the inner product, which transformed two complex vectors to a single value, the outer product takes two complex vectors and produces a matrix. If $\psi_1 = \psi_2$ then we call this matrix $\rho = |\psi\rangle\langle\psi|$ the density matrix of a quantum pure state.

Finally, we define the *tensor product* of two quantum states ψ_1, ψ_2. Formally

$$|\psi_1\rangle\otimes|\psi_2\rangle = \begin{bmatrix}\alpha_1\\ \beta_1\end{bmatrix}\otimes\begin{bmatrix}\alpha_2\\ \beta_2\end{bmatrix} = \begin{bmatrix}\alpha_1\begin{bmatrix}\alpha_2\\ \beta_2\end{bmatrix}\\ \beta_1\begin{bmatrix}\alpha_2\\ \beta_2\end{bmatrix}\end{bmatrix} = \begin{bmatrix}\alpha_1\alpha_2\\ \alpha_1\beta_2\\ \beta_1\alpha_2\\ \beta_1\beta_2\end{bmatrix}$$

2 Basic Rule 2: Composite System Rule

In classical computing, when we move from a single bit system to a system consiting of multiple bits we can continue to consider each bit individually since classical bits cannot be entangled. In quantum computing, because qubits may be entangled, when considering a composite system, one consisting of more than a single qubit, we consider the *joint state* of a multiqubit system. A joint state is represented as the tensor product of the state of each component. For example, the joint state of $|\psi_1\rangle$ with $|\psi_2\rangle$ is represented as $|\psi_1\rangle\otimes|\psi_2\rangle$.

Example 2.1. *We should verify if $|\psi_1\rangle = \alpha_1|0\rangle + \beta_1|1\rangle$ and $|\psi_2\rangle = \alpha_2|0\rangle + \beta_1|1\rangle$ that $|\psi_1\rangle\otimes|\psi_2\rangle$ is indeed a valid quantum state. We computed the tensor of two arbitrary states* *above which gives a* 4×1 *column vector:*

$$\begin{bmatrix} \alpha_1\alpha_2 \\ \alpha_1\beta_2 \\ \beta_1\alpha_2 \\ \beta_1\beta_2 \end{bmatrix}$$

It can be shown that $|\alpha_1\alpha_2|^2 + |\alpha_1\beta_2|^2 + |\beta_1\alpha_2|^2 + |\beta_1\beta_2|^2 = 1$ *if and only if* $|\alpha_1|^2 + |\beta_1|^2 = 1$ *and* $|\alpha_2|^2 + |\beta_2|^2 = 1$*, that is if and only if* $|\psi_1\rangle$ *and* $|\psi_2\rangle$ *are both valid quantum states.*

3 Visualizing Qubits: The Bloch Sphere

In order to visualize how operations on qubits affect the quantum state, we first need a geometric representation of the qubit. Recall we need two complex numbers α, β in order to represent fully the state of a qubit. Each complex number can be specified entirely by two real numbers, a, b. This means to represent the qubit we should need 4 dimensions. However, we are only able to visualize things in at most 3 dimensions. The Bloch Sphere coincides with the so-called principle axes of spin measurement. We will derive a method to visualize a single qubit using the Bloch Sphere, i.e. in three dimensions. First we define a *global phase* for a quantum state as simply some function e^{θ} for some θ on top of the state. A global phase does not affect the measurement outcomes of the state.

First, define $|\psi\rangle = \alpha|0\rangle + \beta 1 = ae^{i\varphi_1}|0\rangle + be^{i\varphi_2}|1\rangle$ for some $a, b, \varphi_1, \varphi_2 \in \mathbb{R}$ recalling each complex α, β can be represented in such a way. The constraints on the coefficients state $|ae^{i\varphi_1}|^2 + |be^{i\varphi_2}|^2 = 1$. Simplifying by noting $|ae^{i\varphi_1}|^2 = a^2 e^{i\varphi_1}(e^{i\varphi_1})^* = a^2 e^{i\varphi_1} e^{-i\varphi_1} = a^2$ and similarly $|be^{i\varphi_2}|^2 = b^2$. Therefore $a^2 + b^2 = 1$. Define $a \equiv \cos(\theta/2)$ and $b \equiv \sin(\theta/2)$ which are valid assignments since $\cos^2(\theta) + \sin^2(\theta) = 1$ for any θ. If we choose a global phase of $e^{-i(\frac{\varphi_1+\varphi_2}{2})}$ then we have

$$\begin{aligned} |\psi\rangle = e^{-i(\frac{\varphi_1+\varphi_2}{2})}|\psi\rangle &= e^{-i(\frac{\varphi_1+\varphi_2}{2})}(ae^{i\varphi_1} + be^{i\varphi_2}) \\ &= ae^{i(\frac{\varphi_1-\varphi_2}{2})} + be^{-i(\frac{\varphi_1-\varphi_2}{2})} \end{aligned}$$

Let $\frac{\varphi_1-\varphi_2}{2} = \varphi$. Then

$$|\psi\rangle = ae^{i\varphi} + be^{-i\varphi} = \cos\left(\frac{\theta}{2}\right)e^{i\varphi}|0\rangle + \sin\left(\frac{\theta}{2}\right)e^{-i\varphi}|1\rangle$$

This equation now has only two unknowns φ and θ. This is enough to represent the qubit in three dimensions using spherical coordinates with a fixed radius $r = 1$. That is the quantum state $|\psi\rangle$ is a vector in $\mathbb{R}^3$ given by $(1, \theta, -\varphi)$. This can be visualized as a point on the Bloch Sphere as in Figure 2.

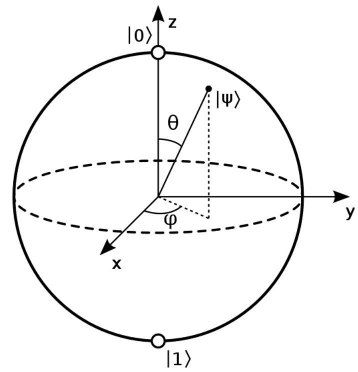

Figure 2: The Bloch Sphere with a state $|\psi\rangle$. Image via Wikipedia.

4 Basic Rule 3: Measurement Rule

When we measure a qubit $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ we observe the classical state $|0\rangle$ with probability $|\alpha|^2$ and the state $|1\rangle$ with probability $|\beta|^2$. The process of measurement is irreversible and probabilistic meaning once measurement has occurred the state collapses into one of two classical states and the original quantum state cannot be recovered.

Example 4.1. *If μ is the measurement operator then $\mu|+\rangle$ gives value $|1\rangle$ with probability $(\frac{1}{\sqrt{2}})^2 = \frac{1}{2}$ and similarly for value $|0\rangle$.*

In the above, we assumed measurement along the z-axis. However, it is possible to measure along a different axis. Observe $\langle 0|\psi\rangle = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \alpha$. Similarly we have $\langle 1|\psi\rangle = \beta$. Therefore, MeasZ, the measure operator along the z-axis, i.e. in the computational basis, gives value $|0\rangle$ with probability $|\langle 0|\psi\rangle|^2$ and value $|1\rangle$ with probability $|\langle 1|\psi\rangle|^2$. In general, suppose we have some orthogonal basis $\{\ |u\rangle,\ |v\rangle\ \}$ so that $\langle u|v\rangle = 1$ (we could simply say they are an orthonormal basis), then we get value $|u\rangle$ with probability $|\langle u|\psi\rangle|^2$ and value $|v\rangle$ with probability $|\langle v|\psi\rangle|^2$.

5 Basic Rule 4: Transforming Qubits

Now that we have the basic objects defined, we would like to manipulate them in order to perform computation. Manipulation of qubits occurs by applying a quantum logic gate. Such a gate is a norm preserving linear transformation. This implies that for a gate transforming $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ to $|\varphi\rangle = \alpha'|0\rangle + \beta'|1\rangle$ we must have $|\alpha|^2 + |\beta|^2 = |\alpha'|^2 + |\beta'|^2 = 1$. This condition means that any quantum logic gate is given by a unitary matrix.