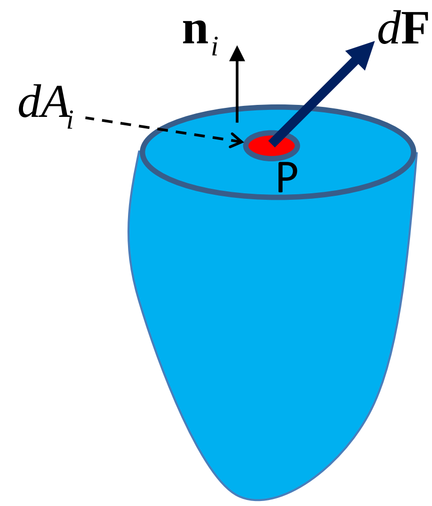
Stress

Stress

•To define the stress at a point P in a solid we need to consider the intensity of force acting on a plane through P, since the intensity of force at P is different, depending on the orientation of the plane.



•Hence we consider an infinitesimal area dA_i on a plane with normal $\mathbf{n}_i$ passing through the point P.

•In the limit $dA_i \to 0$ the intensity of the force acting on the area dA_i becomes uniform.

•Denoting the uniform force intensity at dA_i as $\mathbf{t}_i$, the force vector $d\mathbf{F}$ acting at point P can therefore be represented as:

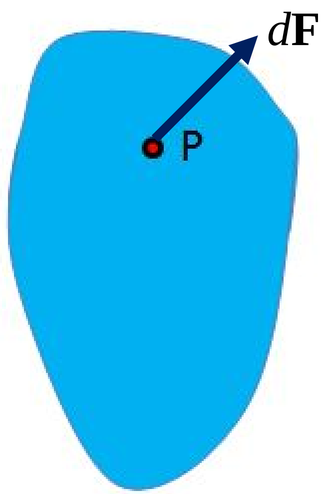
$$d\mathbf{F} = \lim_{dA_i \to 0} \mathbf{t}_i \, dA_i$$

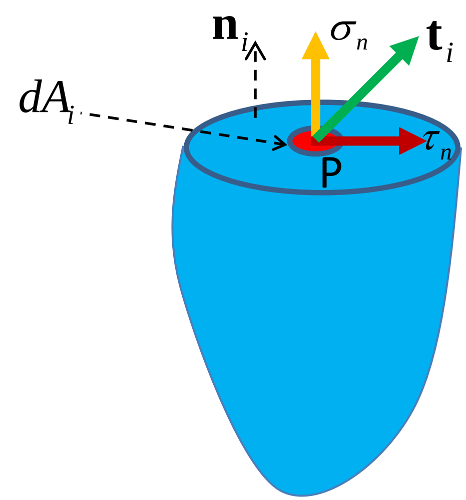
•The uniform force intensity acting over the infinitesimal area dA_i, denoted in mechanics as the traction vector, can then be defined as:

$$\mathbf{t}_i = \lim_{dA_i \to 0} d\mathbf{F}/dA_i$$

Traction

•The traction $\mathbf{t}_i$ can therefore be viewed as the force per unit area, **acting on the plane** with normal $\mathbf{n}_i$, that passes through our point of interest, P.





$$\mathbf{t}_i = \lim_{dA_i \to 0} d\mathbf{F}/dA_i$$

•The traction vector **on the plane** can be decomposed into two components: a component normal to the infinitesimal area dA_i and a component lying in the plane of the infinitesimal area dA_i

•The magnitude of the normal component is known as the normal stress σ_n while the magnitude of the in-plane component is known as the shear stress τ_n

$$\sigma_n = \mathbf{t}_i \cdot \mathbf{n}_i \qquad \tau_n = \sqrt{\mathbf{t}_i \cdot \mathbf{t}_i - (\mathbf{t}_i \cdot \mathbf{n}_i)^2}$$

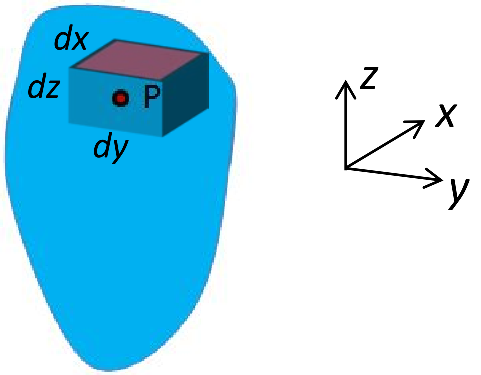
State of stress at a point

• In order to define the entire state of stress at a point, we need information on the traction vectors acting on any three mutually orthogonal planes passing through the point.

•Since these planes can be any three mutually orthogonal planes, we can, without loss of generality, take the three planes to be parallel to the *xy*, *yz* and *zx* planes.

•We therefore consider an infinitesimal cube formed by planes parallel to the *xy*, *yz* & *zx* planes, and separated by infinitesimal distances *dx*, *dy* and *dz*.

•The point at which the stress is to be determined can be assumed to lie at the centre of the infinitesimal cube.



•In the limit, as the volume of the cube approaches zero, we get three mutually orthogonal planes passing through the centre of the cube.

State of stress at a point

- The traction vector on the face parallel to the *xz* plane is split into normal and shear components. We denote the normal stress component aligned with the *y* axis as σ_{yy}

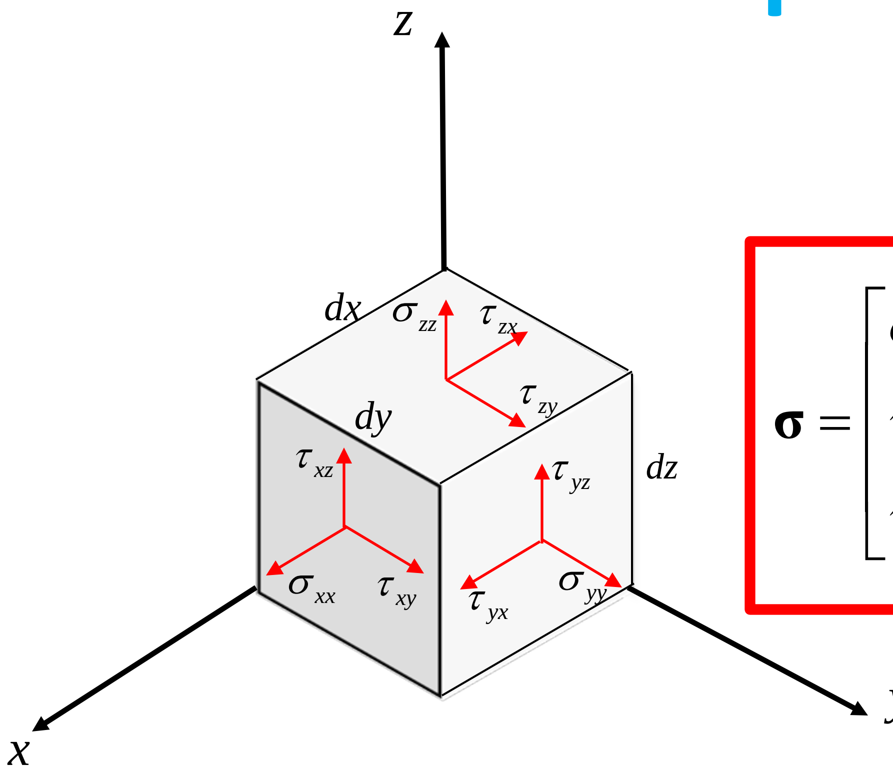
- The shear component is split into two components, one aligned along the *x* axis and denoted as τ_{yx} and the other aligned with the *z* axis and denoted as τ_{yz}

- Similarly the traction vector on the face parallel to the *yz* plane (with normal aligned with the *x* axis) is split into normal and shear stress components denoted as σ_{xx} and τ_{xy}, τ_{xz} respectively.

- Likewise the traction vector on the face parallel to the *xy* plane (with normal aligned with the *z* axis) is split into normal and shear stress components denoted as σ_{zz} and τ_{zx}, τ_{zy} respectively.

- Thus we have three normal $(\sigma_{xx}, \sigma_{yy}, \sigma_{zz})$ and six shear components:
$(\tau_{zx}, \tau_{zy}, \tau_{xy}, \tau_{xz}, \tau_{yz}, \tau_{yx})$

State of stress at a point

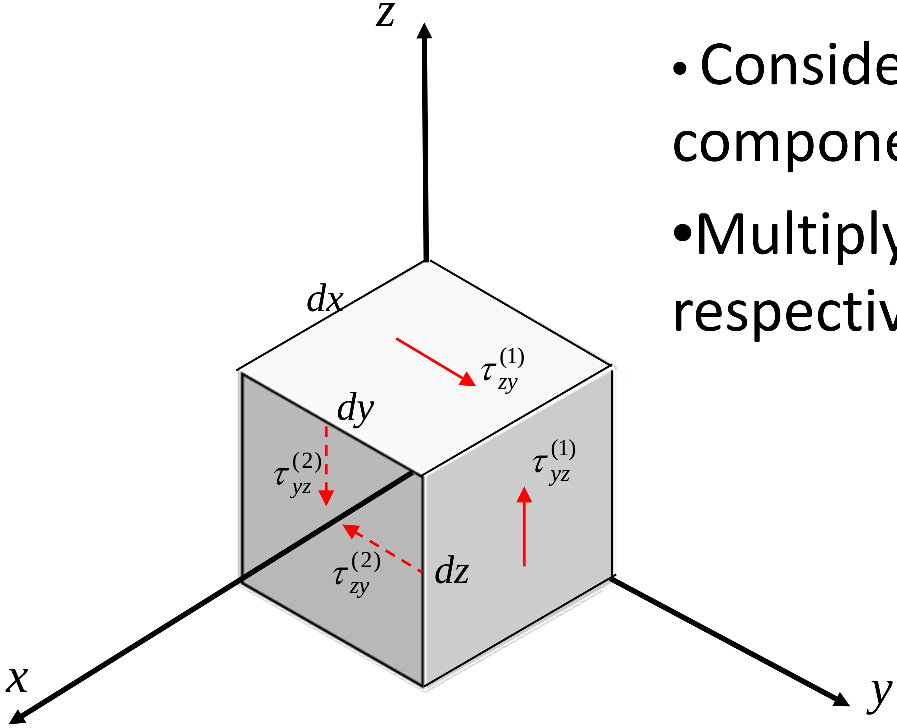


$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix}$$

- These nine components are sufficient to define the state of stress at the point. They comprise one normal stress component and two shear stress components acting on each of the three planes.

- The state of stress at a point is therefore represented by a stress matrix. Each row of the matrix represents the components of the traction vector on one of three mutually orthogonal planes through the point.

Shear stresses on mutually orthogonal planes



- Consider the τ_{yz} and τ_{zy} shear stress components acting on the *xz* and *xy* planes
- Multiplying the stress components by their respective areas, and applying $\sum F_z = 0$:

$$\tau_{yz}^{(1)} dxdz = \tau_{yz}^{(2)} dxdz$$

$$\therefore \tau_{yz}^{(1)} = \tau_{yz}^{(2)} = \tau_{yz}$$

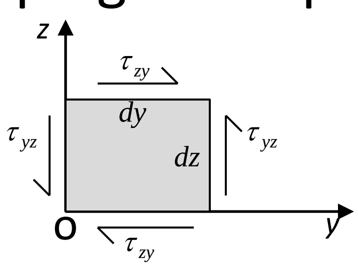
Similarly, from $\sum F_y = 0$

$$\tau_{zy}^{(1)} dxdy = \tau_{zy}^{(2)} dxdy$$

$$\therefore \tau_{zy}^{(1)} = \tau_{zy}^{(2)} = \tau_{zy}$$

- Hence pairs of shear stress components acting on opposite faces are equal.

- Next considering a plane representation of the problem, after dropping the superscripts as redundant.



Taking moments about O,

$$\tau_{zy}(dy\, dx)dz - \tau_{yz}(dz\, dx)dy = 0 \qquad \therefore \boxed{\tau_{zy} = \tau_{yz}}$$

Shear stresses on mutually orthogonal planes

- Similarly it can be shown that:

$$\boxed{\tau_{xz} = \tau_{zx}\ ; \quad \tau_{yx} = \tau_{xy}}$$

- Therefore shear stresses on mutually orthogonal planes of infinitesimal elements are numerically equal.

- Shearing stresses on mutually perpendicular planes only act towards or away from the intersection of such planes

- Hence the arrangement of arrowheads for the shearing stresses must meet at diametrically opposite corners to satisfy equilibrium conditions for the element.

- It is possible to have an element in both force and moment equilibrium only when shearing stresses occur on four sides of an element simultaneously.

Axial Stress

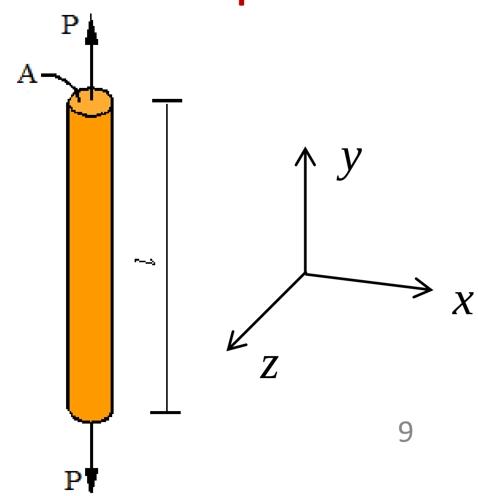
• Now that we know how to define the nine components of stress at a point inside a solid, we want to look at individual stress components in more detail.

•First we will look at the normal or axial stress components.

•To simplify things, we consider a situation where a specimen is loaded in such a way that there is only a single non zero stress component, i.e. the stress matrix has all components zero except for a single diagonal component, say the *y-y* component σ_{yy}

•The simplest example is a rod under uniaxial tension or compression.

$$\text{Stress matrix } (\boldsymbol{\sigma}) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \sigma_{yy} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$



Axial Stress

- If the load P passes through the centroid of a cross section of the rod, whose normal is along the y axis, clearly there is only an axial force acting and no net moment acting on this cross section.

- In order to satisfy force equilibrium, the total force acting at all such cross sections must be equal to the applied load P.

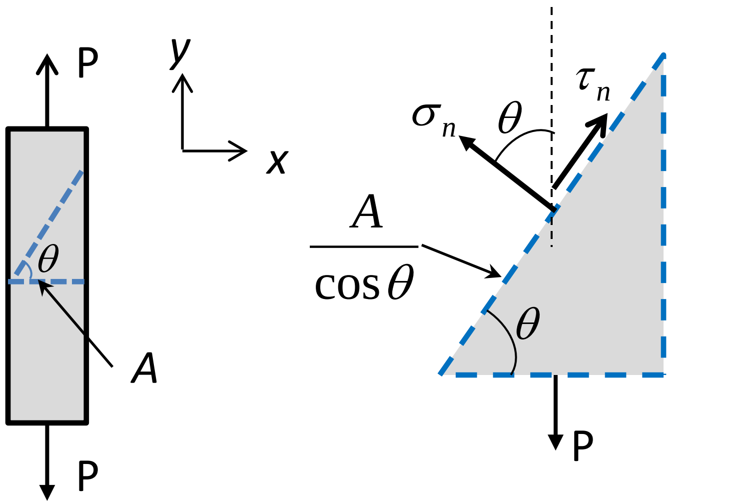
- Let us assume that the rod is prismatic and the material of the rod is perfectly homogeneous.

- Then the average stress component at any such plane section can be obtained by making a cut through the rod, and dividing the force acting by the area A of the section.

- However, the rod also has plane cross sections whose normals are not aligned along the direction of loading, i.e. are not aligned with the y axis.

Average normal (axial) Stress

- Consider a plane section whose normal makes an angle of θ with the y axis:



$$\sum F_x = 0$$

$$\sigma_n \frac{A}{\cos\theta}\sin\theta = \tau_n \frac{A}{\cos\theta}\cos\theta$$

$$\sum F_y = 0$$

$$\sigma_n \frac{A}{\cos\theta}\cos\theta + \tau_n \frac{A}{\cos\theta}\sin\theta = \mathrm{P}$$

$$\therefore \sigma_n \tan\theta = \tau_n$$

$$\sigma_n A + \tau_n A \tan\theta = \mathrm{P}$$

$$\theta = 0^{\mathrm{o}}$$

$$\tau_n = 0$$

$$\sigma_n = \mathrm{P}/A$$

$$\theta \neq 0^{\mathrm{o}}$$

$$\sigma_n A + \sigma_n A \tan^2\theta = \mathrm{P}$$

$$\therefore \sigma_n = \mathrm{P}/[A(1+\tan^2\theta)] = (P/A)\cos^2\theta$$

$$(\tau_n/\tan\theta)A + \tau_n A\tan\theta = \mathrm{P}$$

$$\therefore \tau_n = (\mathrm{P}/\mathrm{A})\cos\theta\sin\theta$$

$$\text{Specifically, for } \theta = 45^{\mathrm{o}}$$

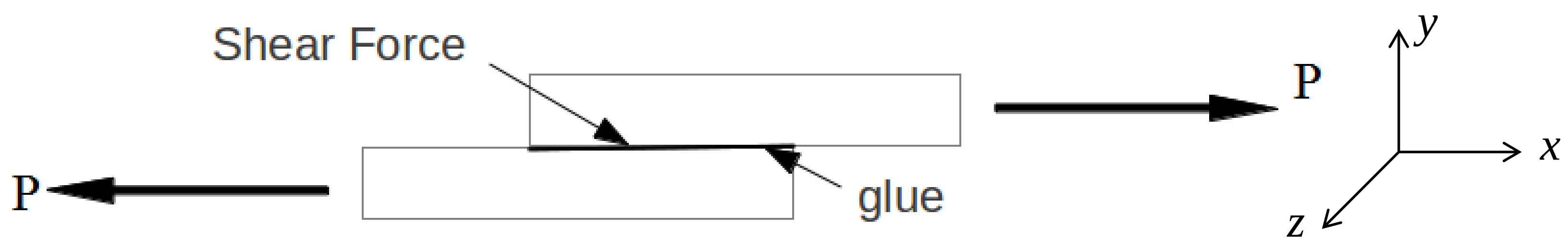
$$\sigma_n = P/(2A)$$

$$\tau_n = P/(2A)$$

Shear Stress

•Next let us consider a simple loading case where instead of normal (axial) stresses, shear stresses predominate.

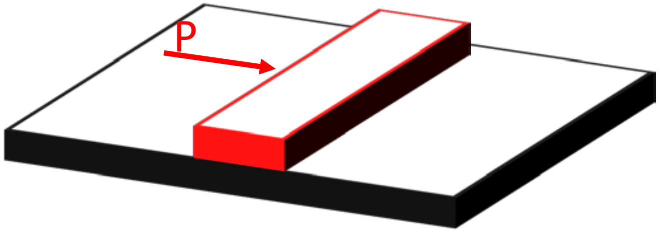
• Such a situation arises when two parts glued together are sought to be separated by application of tangential (sliding) forces.



•This loading gives rise to a state of stress where the only two non zero stress components in the stress matrix are shear stress components:

$$\text{Stress matrix } (\boldsymbol{\sigma}) = \begin{bmatrix} 0 & \tau_{xy} & 0 \\ \tau_{yx} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \tau & 0 \\ \tau & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

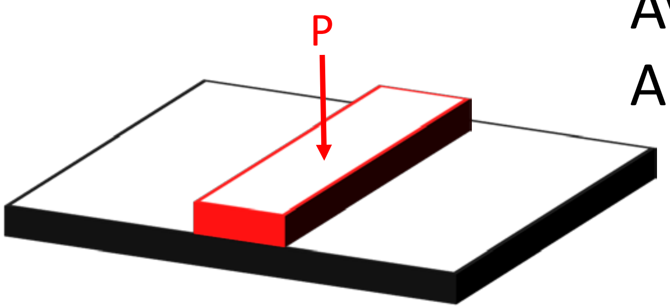
Average Shear and Bearing Stress



Average shear stress: $\tau = \dfrac{\mathrm{P}}{\mathrm{A}}$

A = area of interface

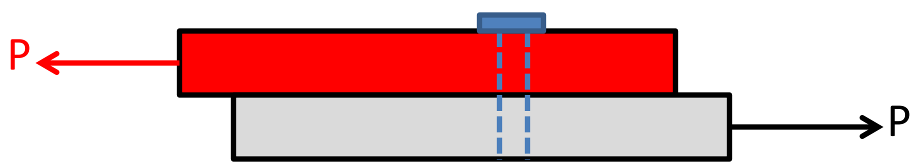
Average bearing stress: Normal contact stress e.g. a block resting on a bigger block.



Average bearing stress: $\sigma_b = \dfrac{\mathrm{P}}{\mathrm{A}}$

A = area of interface

Bolted plates (single lap):



Thickness of plates: t

Diameter of bolts: d

Bearing stress : $\sigma_b = \mathrm{P}/(td)$

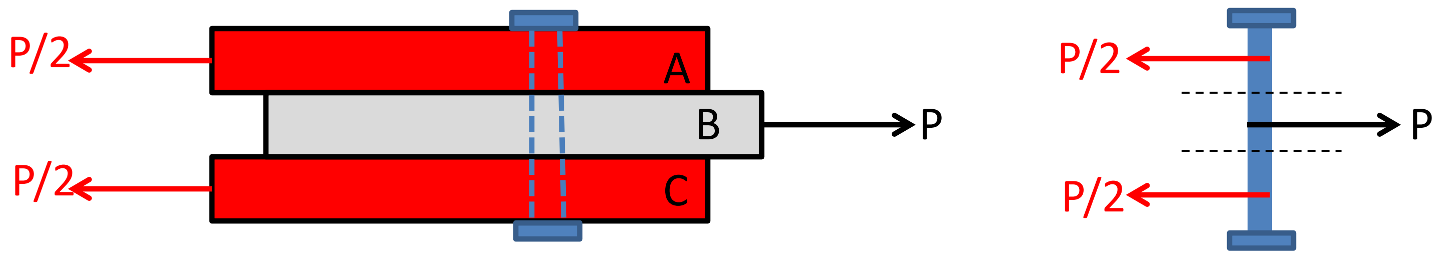
(between bolt & plate)

Shearing stress : $\tau = \mathrm{P}/(\pi d^2/4)$

(in bolt)

Shear Stress and Bearing Stress

Bolted plates (double lap):



$$\text{Bearing stress between bolt and plate B}: \quad \sigma_b = P/(td)$$

$$\text{Bearing stress between bolt and plate A}: \quad \sigma_b = P/(2td)$$

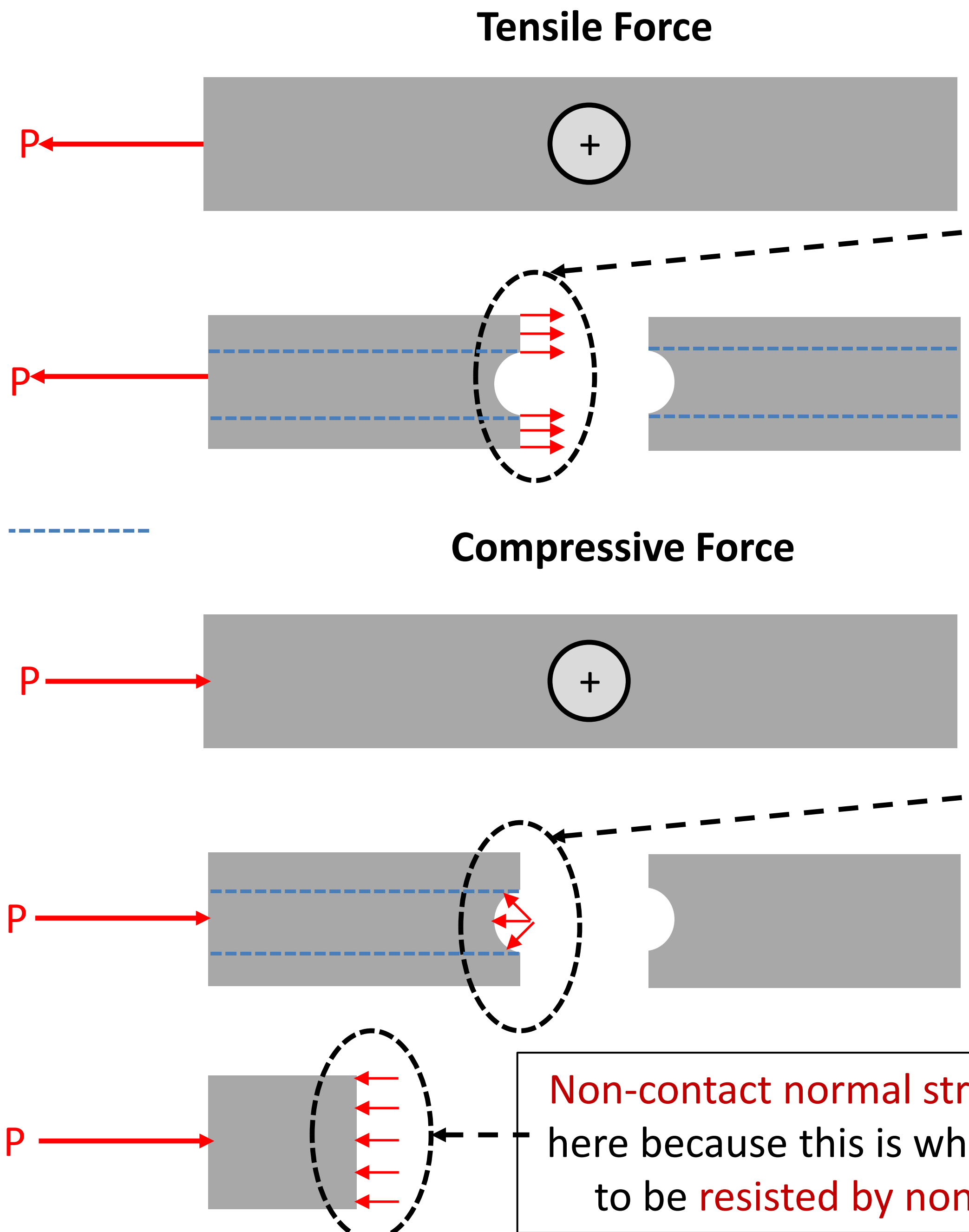
$$\text{Bearing stress between bolt and plate C}: \quad \sigma_b = P/(2td)$$

$$\text{Shearing stress in bolt}: \tau = (P/2)/(\pi d^2/4)$$

•Bearing stress is calculated by dividing the force transmitted by the projected area of the bolt on to the plate

•Shearing stress in bolt is calculated by dividing the force transmitted by the cross sectional area of the bolt

Stress Distbn. in Bolted & Lapped Connections



Non-contact normal stress in the plate is highest here because this is the cross section with the smallest area

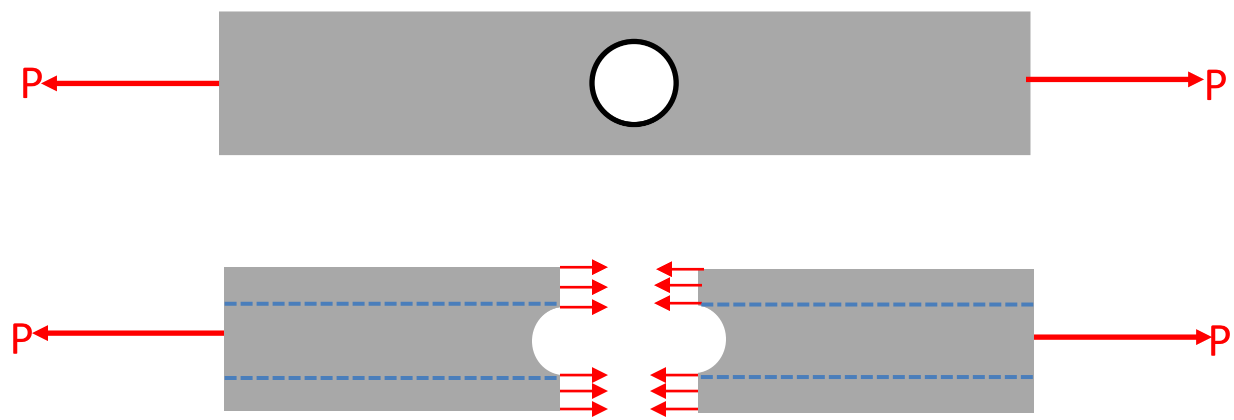
Need to check for shear along dotted blue lines

Non-contact normal stress in the plate is lowest (zero) in this cross section because no force has to be resisted. The entire force P is resisted by contact normal stress

Non-contact normal stress in the plate is highest here because this is where the entire force P has to be resisted by non-contact normal stress

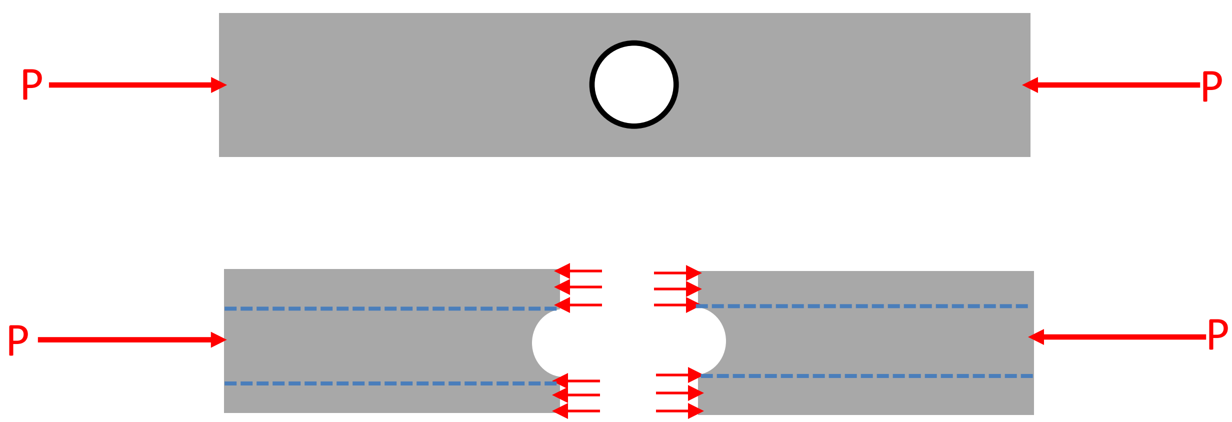
Stress Distribution in Plate with a hole

Tensile Force



Need to check for shear along dotted blue lines

Compressive Force



Ultimate Strength, Allowable Stress, Factor of Safety

- Ultimate Strength: To find the failure load of a structural component, one must know the ultimate strength of the material from which the specimen is made.

- On the other hand, if the failure load is known, the ultimate strength can be obtained by dividing the failure load by the original area of the specimen.

- Allowable Stress: Allowable stresses for design of a structural component are set at values considerably less than the ultimate strength.

- This is because material properties vary, the loading may not be known exactly, or there has been strength reduction due to repeated loading on account of fatigue.

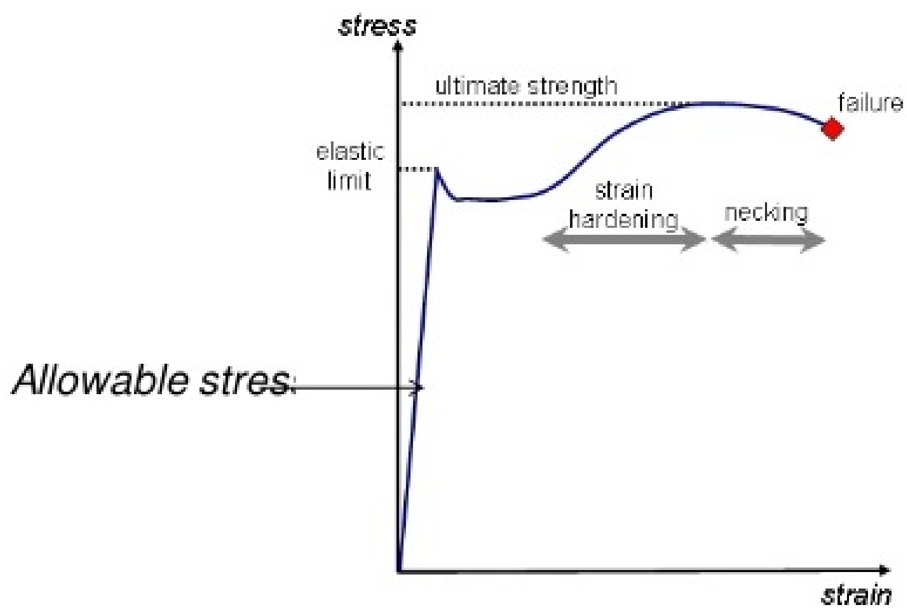
$$\boxed{Factor\ of\ safey = \frac{Ultimate\ strength\ of\ a\ member}{Allowable\ stress\ for\ the\ member}}$$

Allowable Stress Design

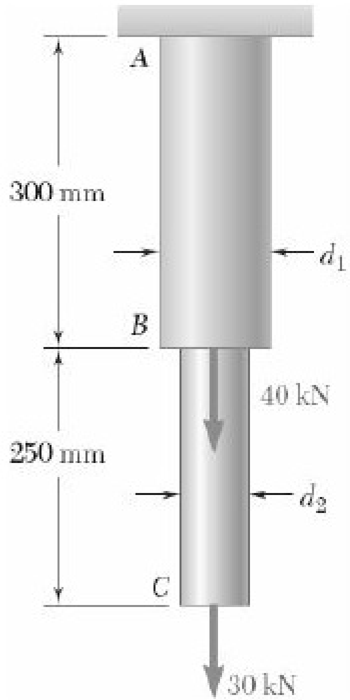
• Structures and machines should be designed for allowable stress levels below the ultimate stress or yield stress.

•If a component is designed to prevent yield, since ductile materials yield before they fail:

$$\textit{Factor of safety} = \frac{\textit{Yield strength of a member}}{\textit{Allowable stress in the member}}$$

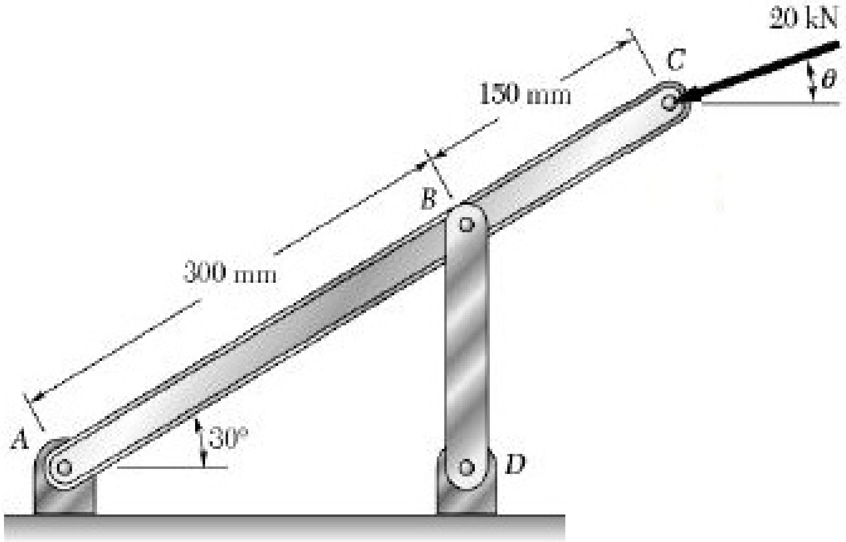


Problem 1



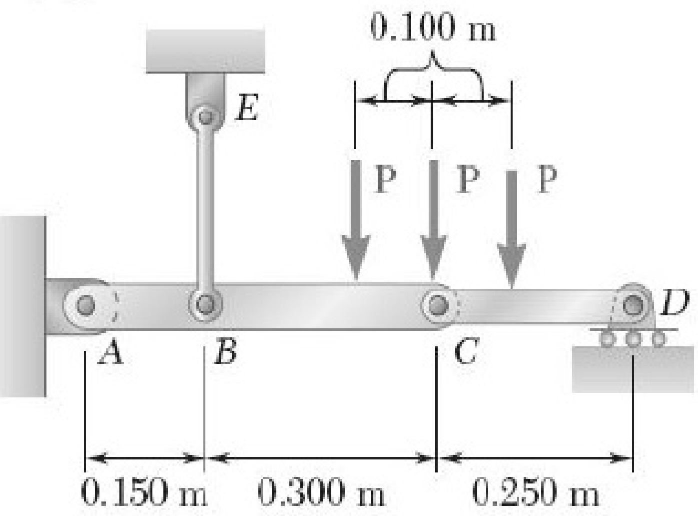
Two solid cylindrical rods AB and BC are welded together at B and loaded as shown. Knowing that the average normal stress must not exceed 175 MPa in rod AB and 150 MPa in rod BC, determine the smallest allowable values of d1 and d2. Neglect the weight of the rods.

Problem 2



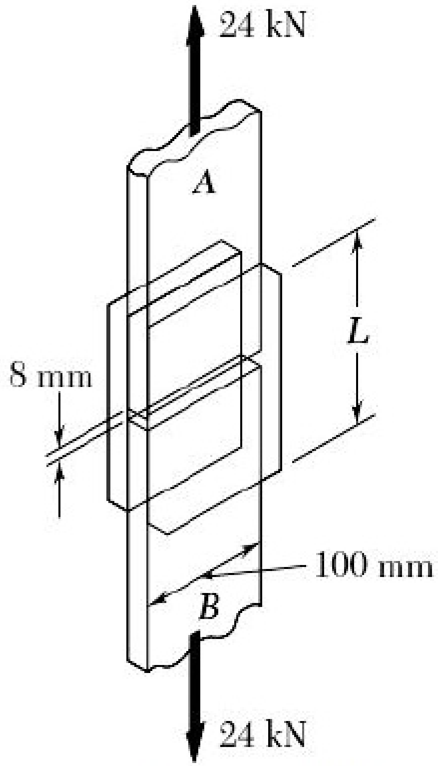
Link *BD* consists of a single bar 30 mm wide and 12 mm thick. Knowing that each pin has a 10 mm diameter, determine the maximum value of the average normal stress in link *BD* if (*a*) $\theta=0^0$ and (b) $\theta=90^0$. *Hint: identify the critical section in both the cases.*

Problem 3



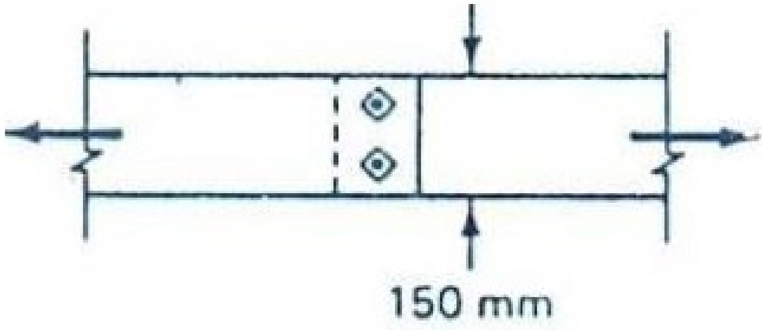
Three forces, each of magnitude P=4 kN, are applied to the structure shown. Determine the cross-sectional area of the uniform portion of rod BE for which the normal stress in that portion is 100 MPa.

Problem 4



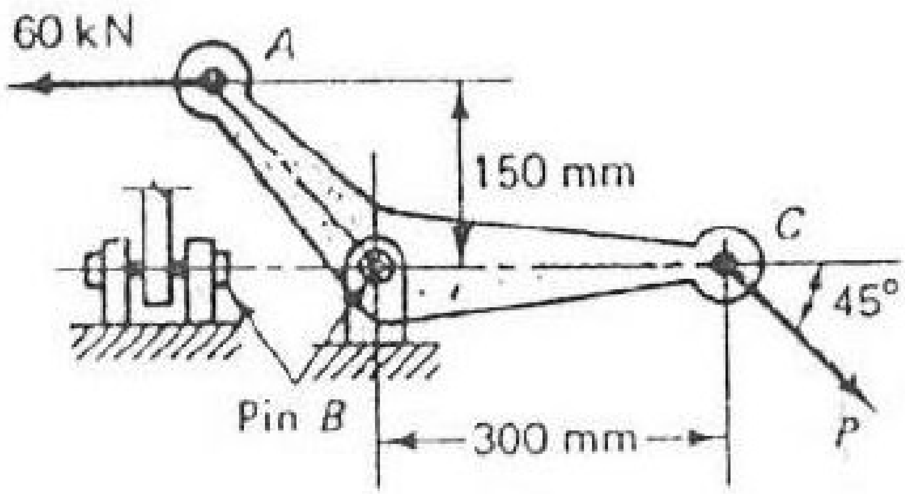
The wooden members *A* and *B* are to be joined by plywood splice plates which will be fully glued on the surfaces in contact. As part of the design of the joint and knowing that the clearance between the ends of the members is to be 8 mm, determine the smallest allowable length *L* if the average shearing stress in the glue is not to exceed 800 kPa.

Problem 5



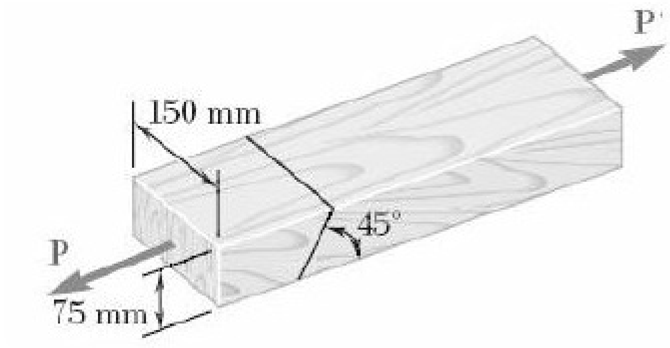
Two 10 mm thick steel plates are fastened together by means of two 20 mm bolts that fit tightly into the holes. If the joint transmits a tensile force of 45 kN, determine (a) average normal stress in the plates at the section where no holes occur; (b) the average normal stress at the critical section; (c) the average shear stress in the bolts and (d) the average bearing stress between the bolts and the plates.

Problem 6



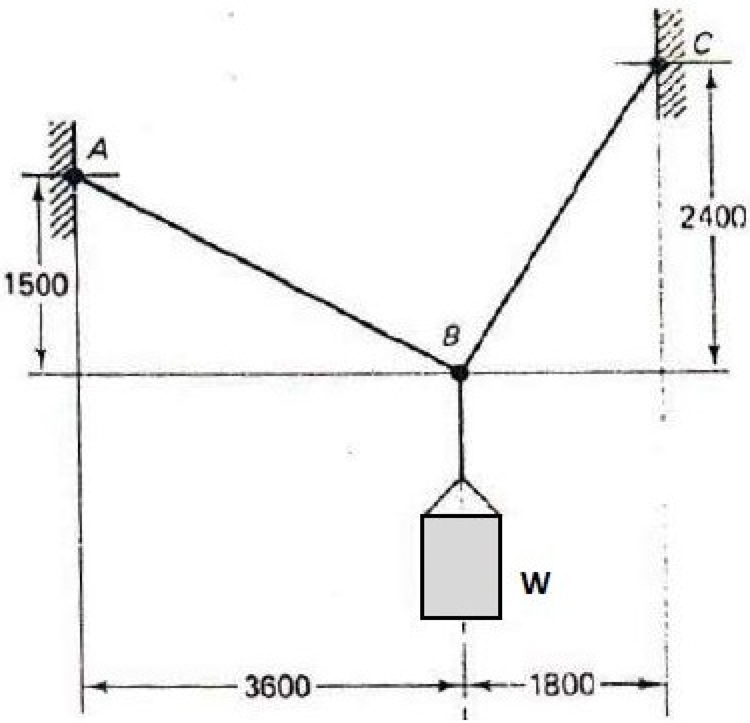
What is the required diameter of pin B for the bell crank mechanism, if an applied force of 60 kN is resisted by a force P at C? The allowable shear stress is 100MPa.

Problem 7



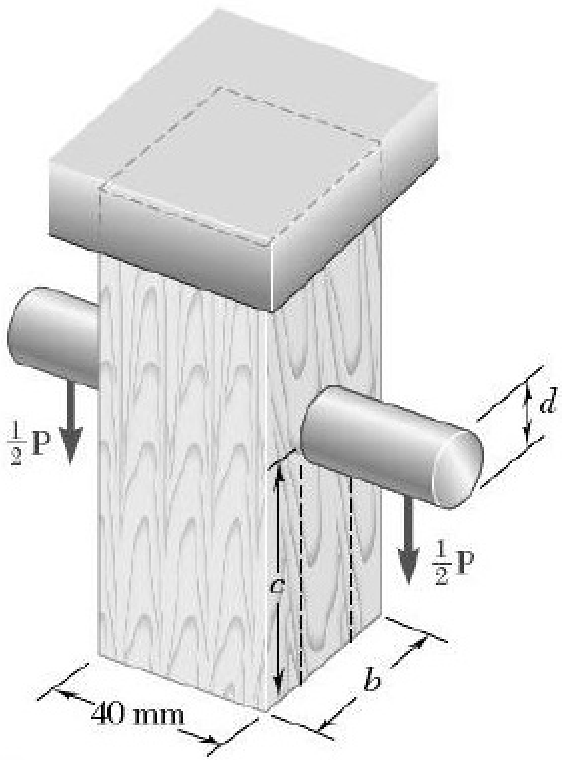
Two wooden members of uniform rectangular cross section are joined by the simple glued scarf splice shown. Knowing that the maximum allowable shearing stress in the glued splice is 620 kPa, determine (a) the largest load P that can be safely applied, (b) the corresponding tensile stress in the splice.

Problem 8



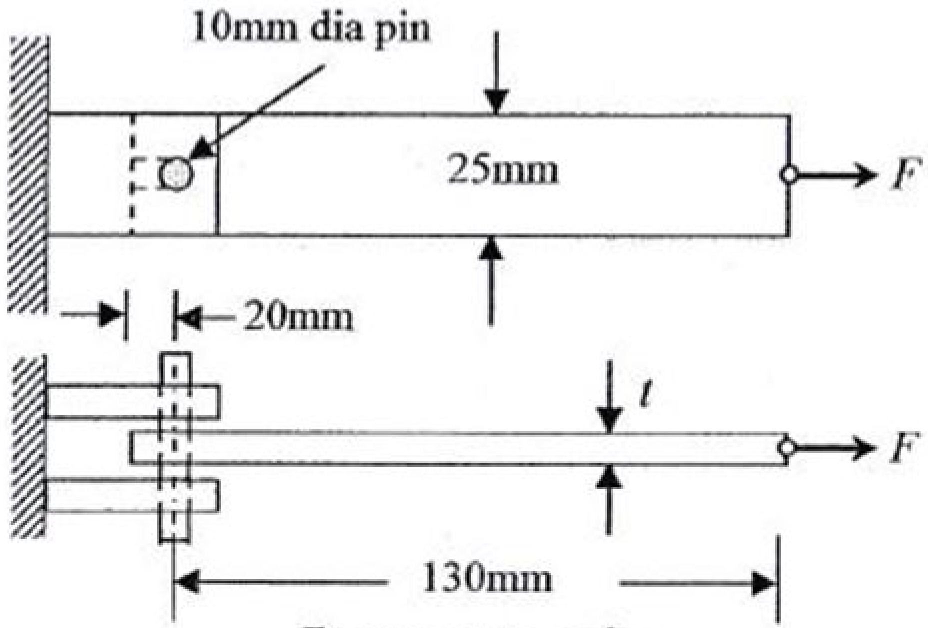
Two high strength steel rods of different diameters are attached at A and C and support a weight W. The ultimate strength of the rods is 800 MPa. Rods AB and BC have cross-sectional areas of 200 mm^2 and 400 mm^2 respectively. If the factor of safety is 2, what weight W can be supported by the wires?

Problem 9



A load P is supported as shown by a steel pin that has been inserted in a short wooden member hanging from the ceiling. The ultimate strength of the wood used is 60 MPa in tension and 7.5 MPa in shear, while the ultimate strength of the steel is 145 MPa in shear. Knowing that b = 40 mm, c = 55 mm, and d = 12 mm, determine the load P if the overall factor of safety is 3.2.

Problem 10



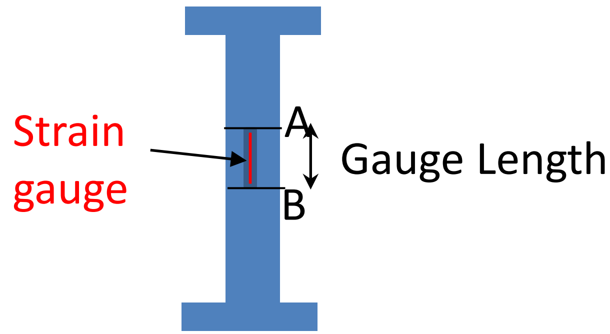
Drawn not to scale

In a material testing machine the axial load F applied at the end of a 150-mm long and 25-mm wide steel bar is increased until the 10mm diameter pin, shown in the figure, breaks. (a) Calculate the ultimate shear stress of the pin material if it breaks when $F = 30$ kN. (b) Find the minimum thickness, t, of the bar required so that it does not get damaged before the pin breaks. Consider the failure of the bar due to (i) tensile stress (ii) bearing stress and (iii) shearing along the horizontal dotted lines shown in the figure. Given: Ultimate tensile stress of steel (σ_u) = 300 MPa, Ultimate bearing stress (σ_b) = 400 MPa and Ultimate shear stress of steel (τ_u) = 180 MPa.

Strain

• Axial strain: In a uniaxial tension/compression test, we can measure the deformation of a specimen subject to axial loading.

•To quantify the deformation we measure the distance between two points on the specimen (say A & B), located a specific distance (gage length) apart in the undeformed specimen.



•However the deformed distance between A & B, is not a true measure of the distortion.

•This is because if two specimens of different lengths, say L_1 and L_2 ($L_1 > L_2$), of the same material, are subjected to the same axial stress, the specimen with length L_1 will show a larger change in gauge length than the specimen with length L_2 .

Strain

•However since the stress in both specimens is the same, and they are made of the same material, infinitesimal (volume or area) elements of both specimens should experience the same amount of stretching in response to the applied stress.

•Clearly the total deformation, as measured from the change in gauge length, cannot capture this vital aspect of the material response.

•Therefore in order to get a specimen length independent measure of the distortion we define the strain (ε) in the specimens as:

$$\text{strain } (\varepsilon) = \frac{change\ in\ length}{original\ length} = \frac{\Delta}{L}$$

•While $(\Delta)_{L_1} \neq (\Delta)_{L_2}$, the ratio $(\Delta)_{L_1}/L_1$ however is equal to $(\Delta)_{L_2}/L_2$.

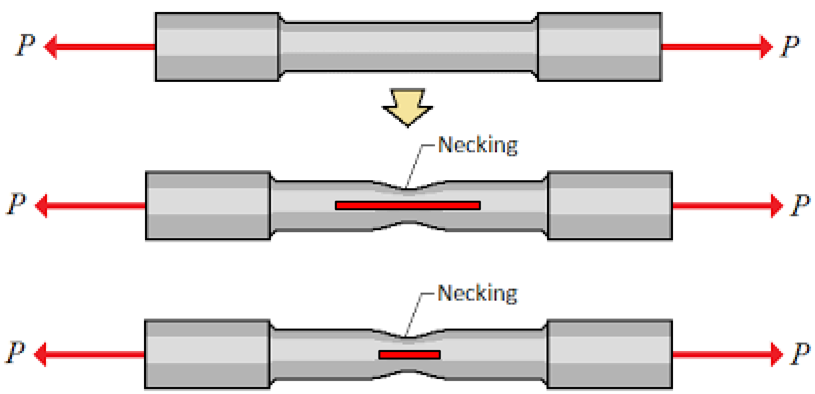
Hence the axial strain in both the specimens are the same.

Strain

•It is important to note however, that in an axially loaded specimen, ε, defined as the change in gauge length divided by the original gauge length, is an accurate measure of the deformation only if the deformation is homogeneous.

•If the deformation is not homogeneous, the measured strain will depend on the gauge length.

•For example in the presence of necking, i.e. localized straining, there may be very large strains in the necking zone. If the gauge length is much larger than the size of the necking zone, this will not be reflected in the measured strain, which will be much smaller.



Incorrect strain: deformation not homogeneous over gauge length

Correct strain: deformation homogeneous over gauge length

Summary formulae for stress and strain

- The strain we have considered till now was calculated based on the original (undeformed) configuration of the specimen.

- Strains/stresses measured on the original configuration are known as engineering strains/engineering stresses. Alternatively, one can measure strains/stresses based on the deformed configuration.

- Strains/stresses measured on the deformed configuration are known as true strains/true stresses.

True normal stress $:\sigma_{true} = \lim_{dA \to 0} \frac{dF}{dA} \cdot \mathbf{n}$ $\quad dA$ is the deformed area

Engineering normal stress $\sigma_{engineering} = \lim_{dA_0 \to 0} \frac{d\mathbf{F}}{dA_0} . \mathbf{n}$ $\quad dA_0$ is the original area

Engineering strain $: \varepsilon_{engineering} = \frac{\Delta}{L_o};$ $\quad L_o$ is the original length

True strain $: \varepsilon_{logarithmic} = \int_{L_0}^{L_1} \frac{dL}{L} = \ln(\frac{L_1}{L_0})$ $\quad L_o$ is the original length,

L_1 is the final length

Also, known as logarithmic strain

Stress-strain relationship

• To reiterate, strain is a more fundamental indicator of the distortion than the total deformation. Similarly, stress is a more fundamental measure of the load acting on the material than the load itself.

•Characterizing material behaviour involves determining the relationship between these two quantities.

$$\sigma = f(\varepsilon)$$

•Stress strain diagrams, obtained experimentally, provide information on the nature of this relationship.

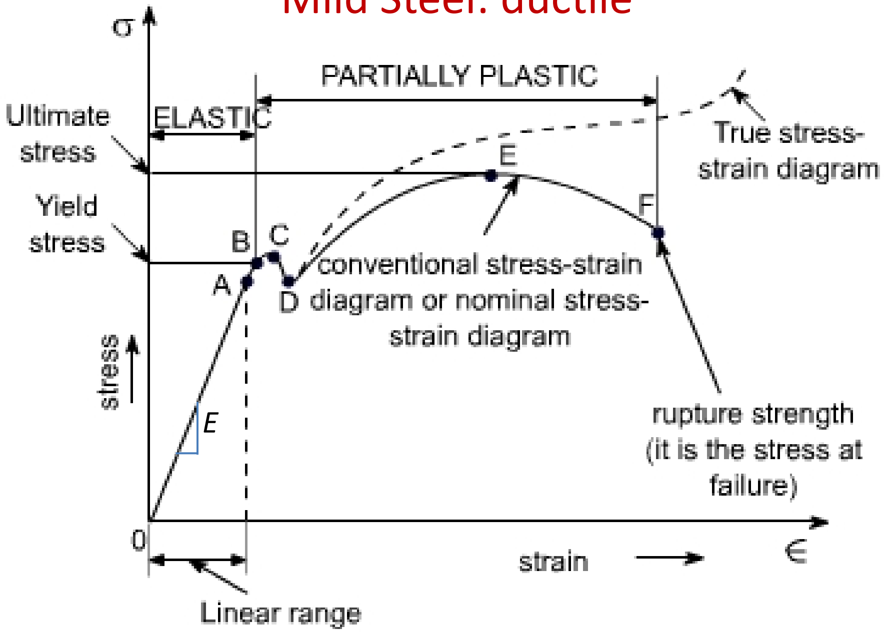
•Most materials have stress strain curves that have an initially linear portion. In the linear region, the stress is related to the strain through a proportionality constant known as the Young's modulus (*E*):

$$\sigma = E\varepsilon$$

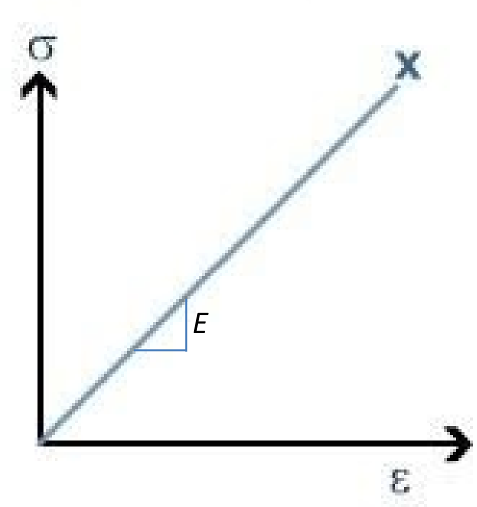
•However when the stresses exceed a certain value, the stresses are no longer proportional to the strains.

Stress-strain relationship

Mild Steel: ductile



Glass: brittle



A: Proportional Limit, B: Elastic Limit/Yield Point, E: Ultimate Strength

X = A = B = E, Proportional Limit = Elastic Limit = Ultimate Strength

- The maximum stress for which the stress is proportional to the strain is known as the proportional limit of the material.

- If the material is loaded beyond the proportional limit, the stress-strain relationship is non-linear.

Stress-strain relationship

- At a certain point beyond the proportional limit, the material reaches yield.

- A material is said to be ductile if it undergoes significant amount of straining beyond its yield point; mild steel is a ductile material.

- If a ductile material is loaded beyond the yield point, it undergoes permanent (plastic) deformations: if fully unloaded it does not recover its original configuration.

- Finally at some strain level beyond the yield point, the material attains its ultimate strength: beyond which the stress strain curve shows a negative slope.

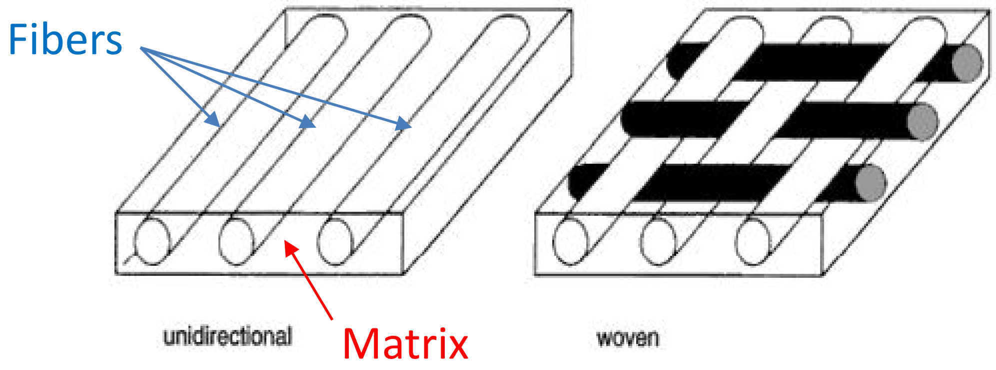
- Some materials attain their ultimate strength at a stress at or just beyond the elastic limit. Such materials are known as brittle materials.

Isotropic and Anisotropic materials

- Isotropic material: An isotropic material is a material whose properties are the same in all directions. Usually such materials have randomly oriented crystalline structures.

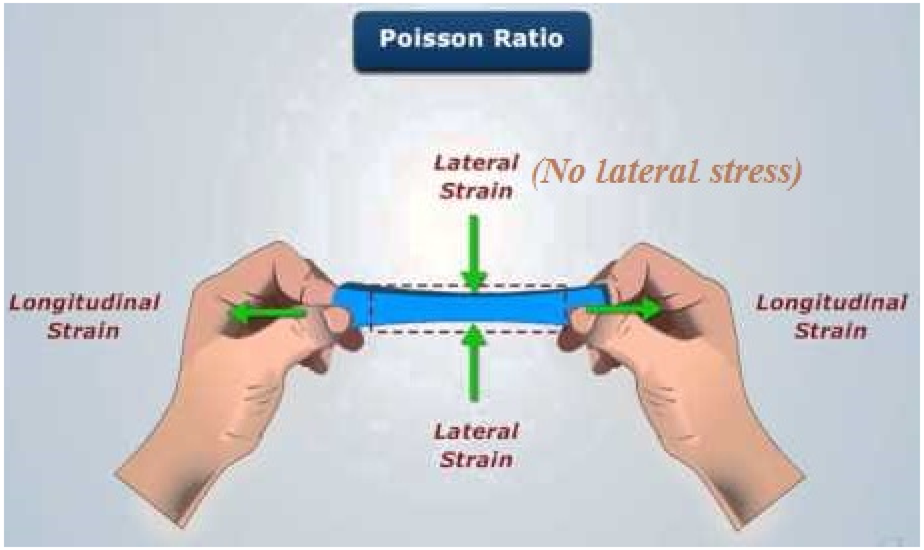
- However in some materials, notably in many composites, the existence of preferred directions result in directional dependence of material properties.

- For example in unidirectional laminates, the Young's modulus in the fibre directions is usually much higher than in the direction orthogonal to the fibers.



Poisson's Effect

- During uniaxial stretching of a specimen a certain amount of transverse contraction takes place.



- On the other hand, if a specimen is subjected to unaxial compressive loading, lateral expansions occur. These lateral deformations are best characterized as lateral strains with the following meaning:

$$lateral\ strain = \frac{change\ in\ length\ in\ lateral\ direction}{gage\ length\ in\ lateral\ direction}$$

- Given that axial expansion leads to lateral contraction and vice versa, the sign for lateral strains is opposite to that for axial strains.

Poisson's Effect

- It has been shown experimentally that within the elastic limit, lateral strains bear a constant relationship to axial strains.

- This constant of proportionality is a material property and is known as the Poisson's ratio.

$$Poisson's\ ratio = \upsilon = -\frac{lateral\ strain}{axial\ strain}$$

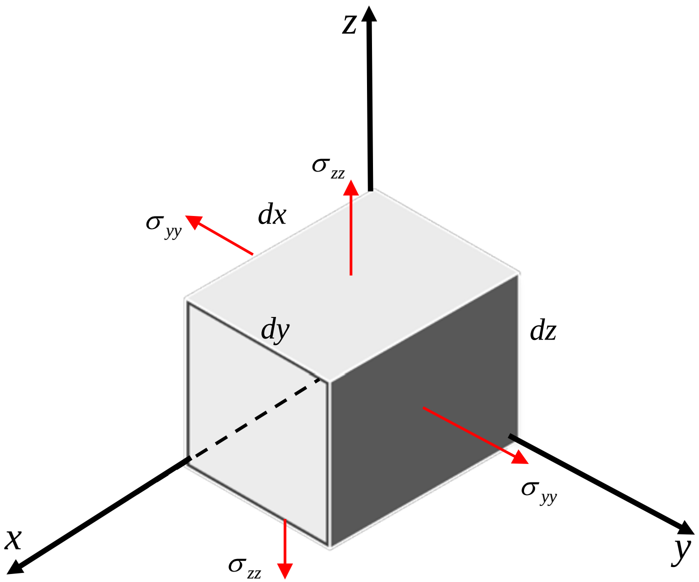
- Poisson's ratio for typical materials: steel: 0.3, concrete: 0.19, rubber: 0.5

- In a specimen subjected to axial loading, the Poisson's effect causes no additional stresses unless the transverse deformation is prevented.

Generalized Hooke's Law

• We next want to investigate behaviour of materials when they are subjected to a generalized state of stress, or multiaxial state of stress, as distinct from an uniaxial state of stress.

•We first consider an infinitesimal element with edges aligned along the global *x*, *y* and *z* axis, subjected to normal stresses in the *x*, *y* and *z* directions. Considering strains in the *x* direction:



$$\text{Due to } \sigma_{xx}: \quad \sigma_{xx}/E$$

$$\text{Due to } \sigma_{yy}: \quad \sigma_{yy}/E$$

$$\text{Due to } \sigma_{zz}: \quad \sigma_{zz}/E$$

$$\therefore \varepsilon_{xx} = \sigma_{xx}/E - \upsilon\sigma_{yy}/E - \upsilon\sigma_{zz}/E$$

Similarly,

$$\varepsilon_{yy} = \sigma_{yy}/E - \upsilon\sigma_{xx}/E - \upsilon\sigma_{zz}/E$$

$$\varepsilon_{zz} = \sigma_{zz}/E - \upsilon\sigma_{xx}/E - \upsilon\sigma_{yy}/E$$

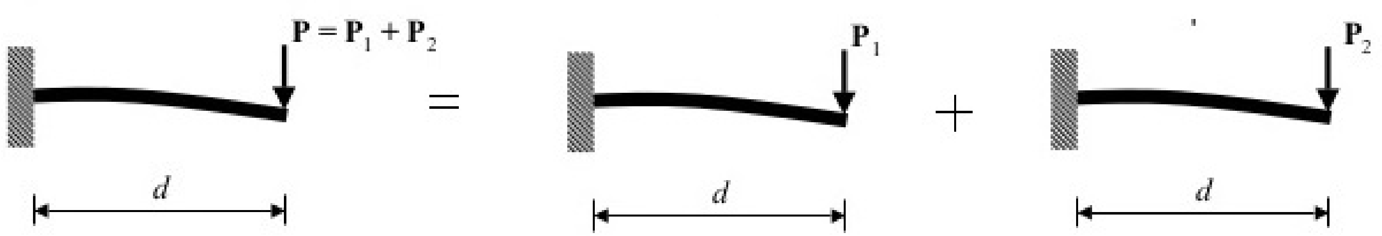
Generalized Hooke's Law

- In the above the following assumptions have been made:

(i) The material is isotropic: Young's modulus and Poisson's ratios are the same in all directions.

(ii) The effects due to different stresses can be linearly superimposed i.e. the resultant stress or strain in a system due to several forces is the algebraic sum of their effects when separately applied.

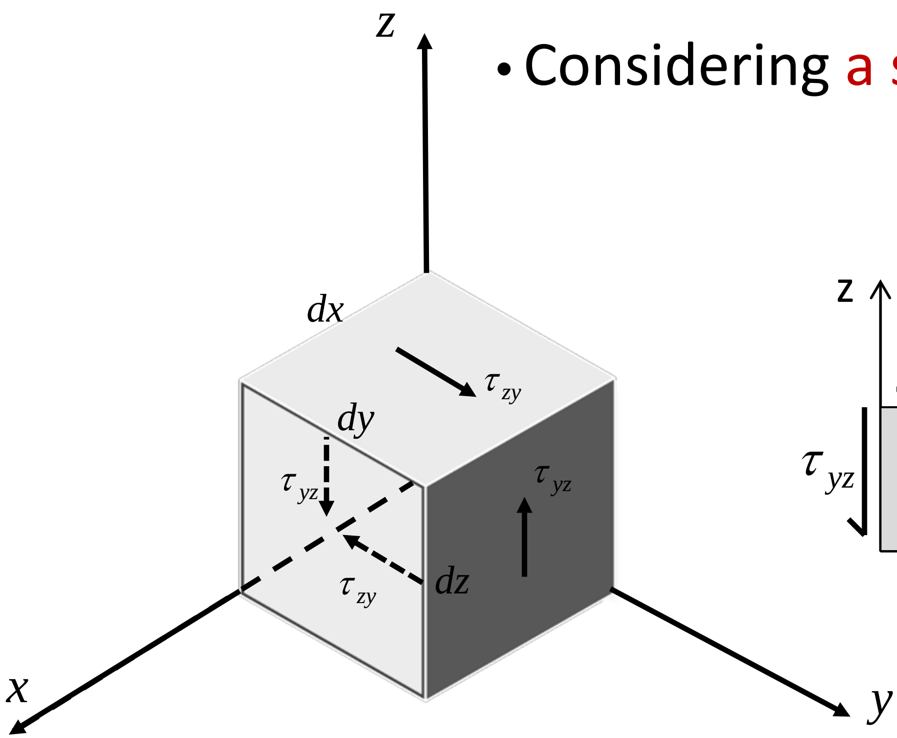
- The linear superposition assumption is true only if each effect is directly and linearly related to the force causing it.



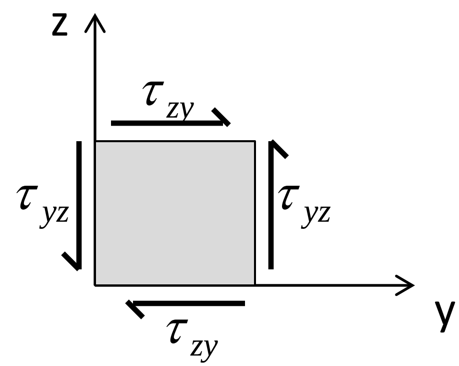
- It is only approximately true when the deflections or deformations due to one force cause a change in the effect of another force.

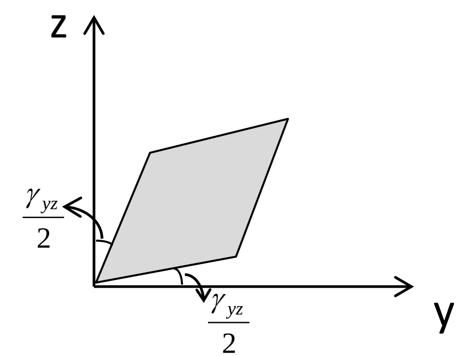
Shear stress – Shear strain

- Recall that shearing stresses occur in pairs and act on mutually perpendicular planes.



- Considering a slice parallel to the *y-z* plane:





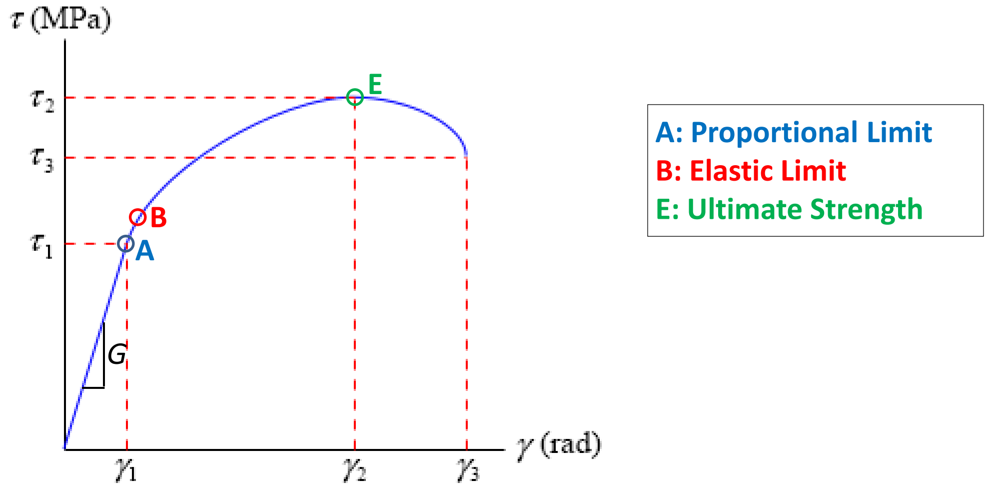
- There is a linear relationship between the shearing stress and the total shearing angle γ, which is the shearing strain:

$$\tau_{yz} = G\gamma_{yz}$$

G is the shear modulus of elasticity

Shear stress-shear strain

$$\text{Similary,} \gamma_{xy} = \tau_{xy}/G \quad \gamma_{zx} = \tau_{zx}/G$$



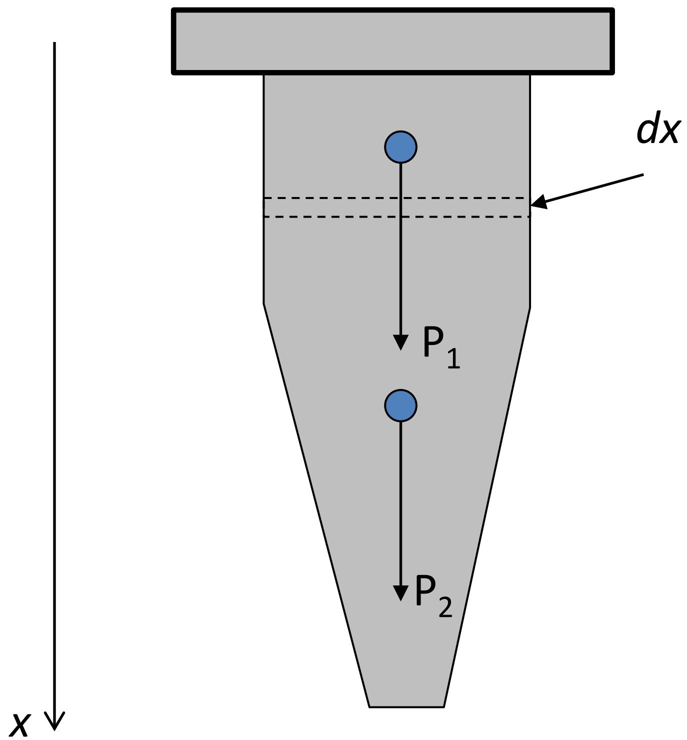
• In a pure shear experiment if we plot the shearing stress vs. shearing strain curve, we will again get points similar to those obtained from the axial stress-axial strain curve: proportional limit, elastic limit/yield point, ultimate shearing stress etc.

•The shear modulus of elasticity *G* is related to the Young's modulus *E* through the Poisson's ratio *v:*

$$G = \frac{E}{2(1+\nu)}$$

Deflection of axially loaded members

$$\varepsilon = \frac{\Delta}{L}$$



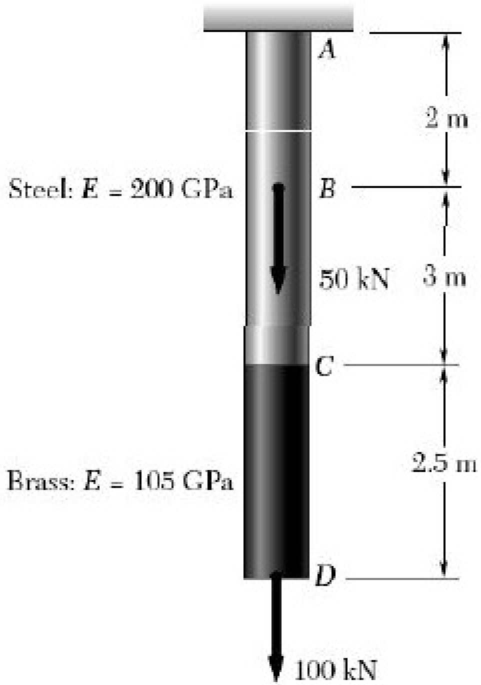
$$\Delta = \varepsilon \, dx = (\sigma / E) dx$$

$$= \frac{P(x)}{A(x)E} dx$$

$$\therefore \text{Total displacement} = \Delta_T$$

$$= \int_0^L \frac{P(x)}{A(x)E} dx$$

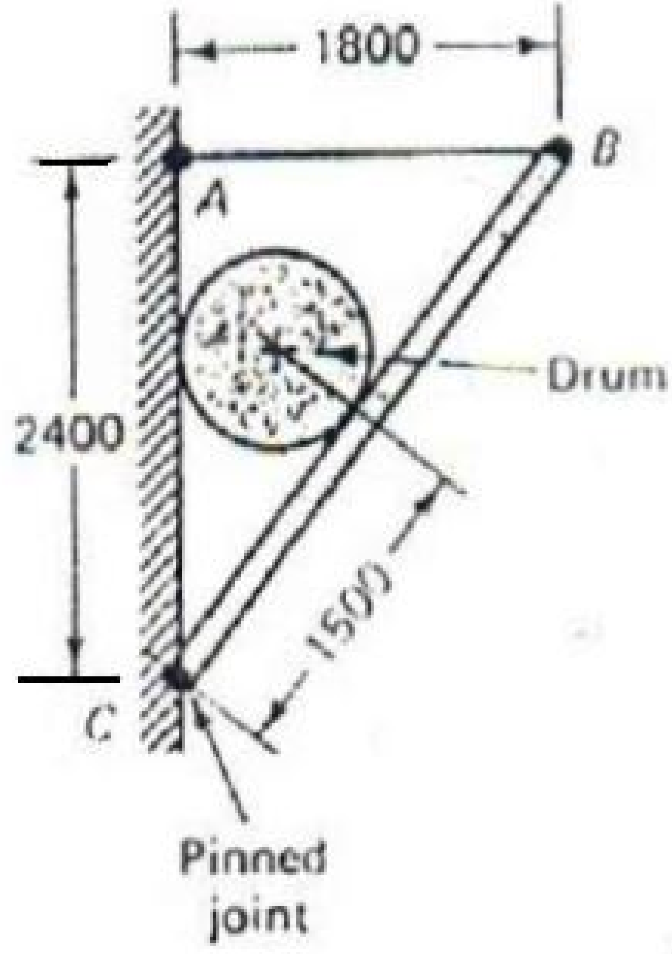
Problem 1



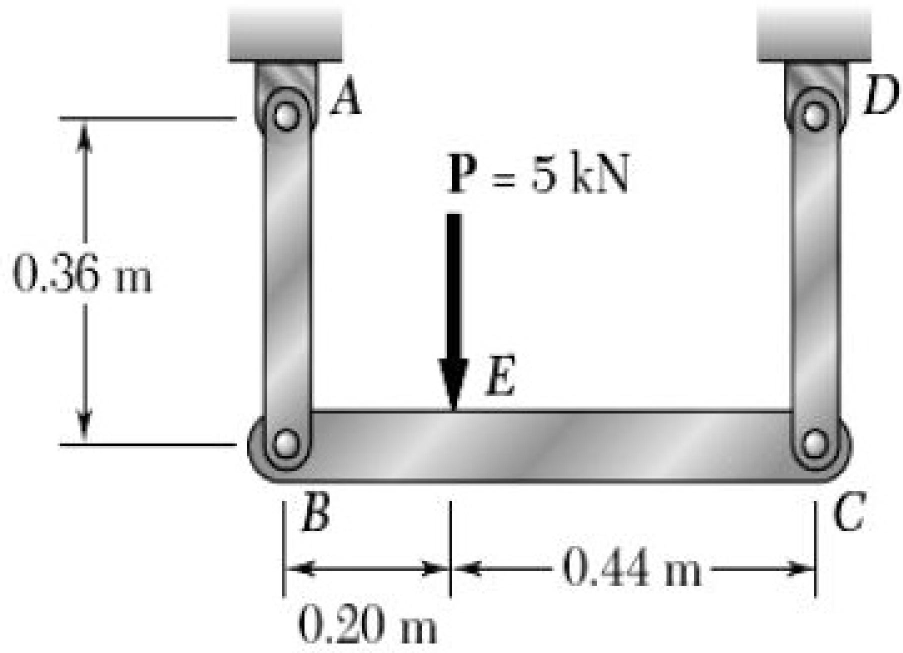
The 36-mm-diameter steel rod ABC and a brass rod CD of the same diameter are joined at point C to form the 7.5-m rod ABCD. For the loading shown, and neglecting the weight of the rod, determine the deflection of points C and D.

Problem 2

A wall bracket is constructed as shown in the figure. All joints may be considered to be pin connected. The steel rod AB has a cross section of 5 mm^2. The member BC is a rigid beam. If a 1 m diameter frictionless drum weighing 459 kg is placed in the position shown, what will be the elongation of the rod AB?

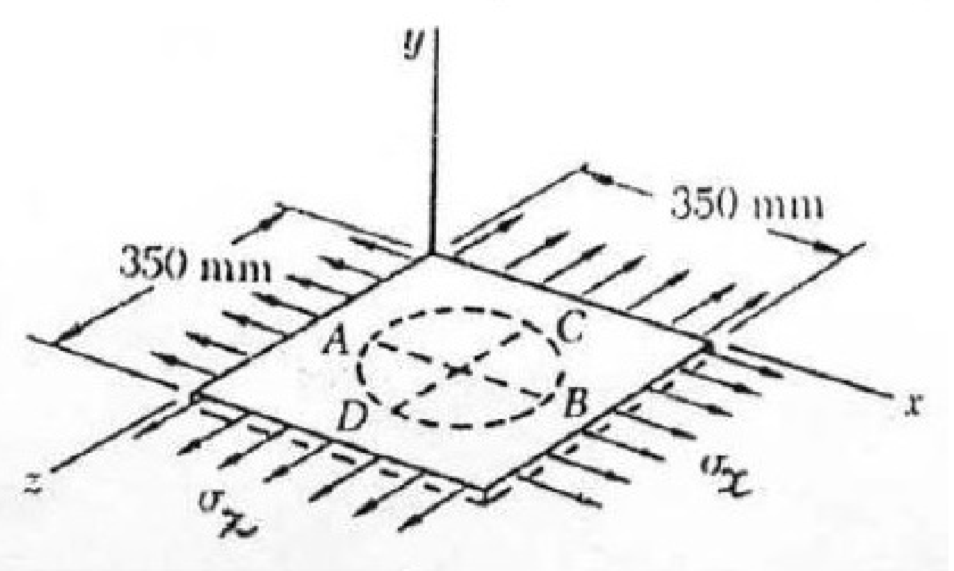


Problem 3



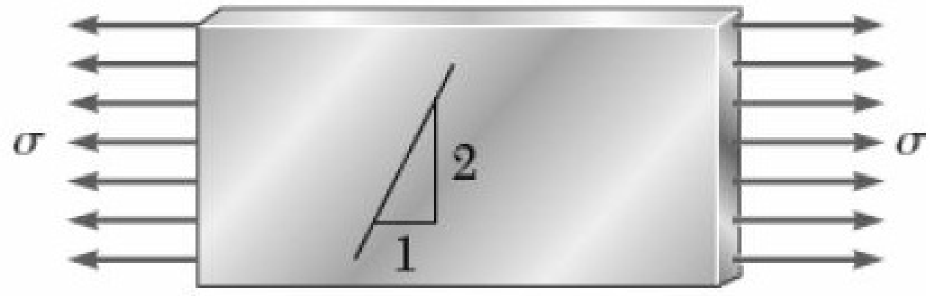
Each of the links *AB* and *CD* is made of aluminum of *E* =75 GPa and has a cross-sectional area of 125 mm^2. Knowing that they support the rigid member *BC;* determine the deflection of point *E*.

Problem 4



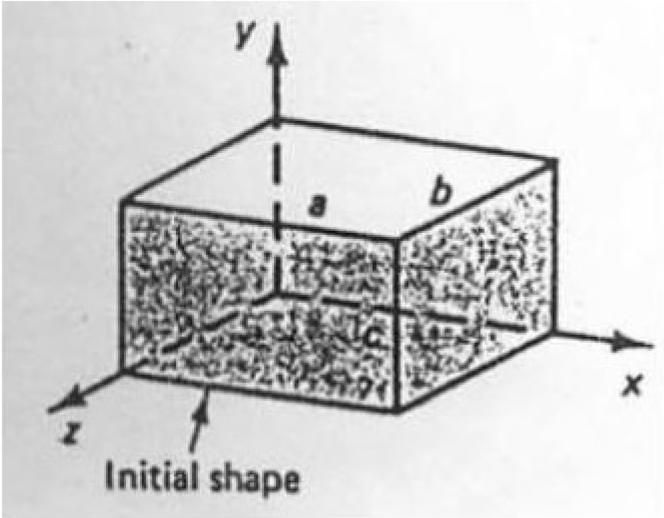
A circle of diameter 200mm is scribed on an unstressed aluminum plate of thickness 18 mm. Forces acting in the plane of the plate later causes normal stresses σ_x=85 MPa and σ_z=150 MPa. For E=70 GPa and v = 0.33, determine the changes in (a) the length of diameter AB, (b) the length of diameter CD, (c) the thickness of the plate and (d) the volume of the plate.

Problem 5



An aluminum plate ($E = 74$ GPa, $v = 0.33$) is subjected to a centric axial load that causes a normal stress σ. Knowing that, before loading, a line of slope 2:1 is scribed on the plate, determine the slope of the line when $\sigma = 125$ MPa.

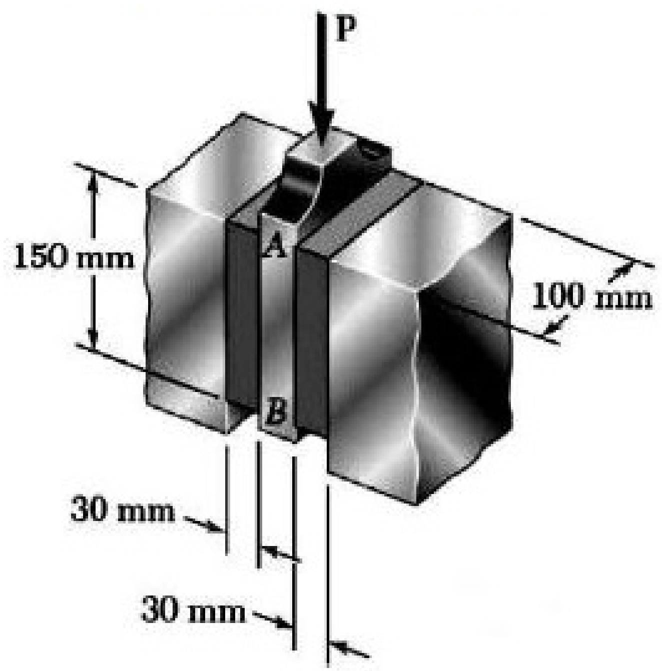
Problem 6



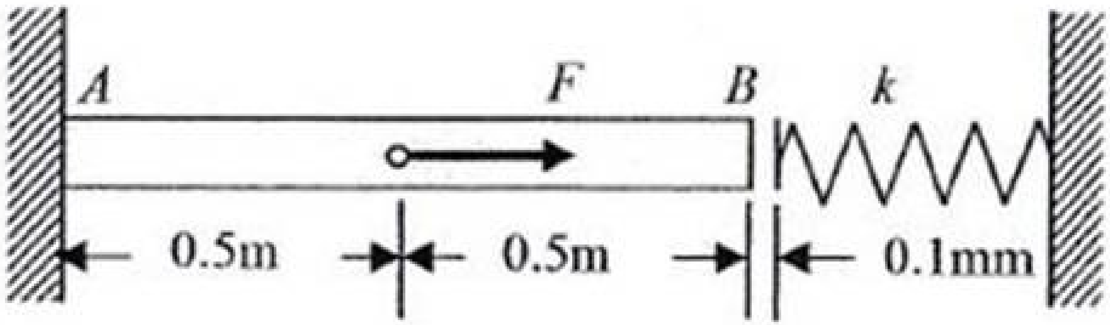
A rectangular steel block has the following dimensions: a=50 mm, b=75 mm and c=100 mm. The faces of this block are subjected to uniformly distributed forces of 180 kN (tension) in the x-direction, 200 kN (tension) in the y-direction and 240 kN (compression) in the z-direction. Determine the magnitude of a single system of forces acting only in the y-direction that would cause the same deformation in the y-direction as the initial forces. Consider v =0.25.

Problem 7

A vibration isolation unit consists of two blocks of rubber bonded to a rigid metal plate AB and to rigid supports as shown in the figure. The Poisson's ratio is $\nu=0.5$ and modulus of elasticity is $E=60$ MPa for rubber. If a force of magnitude P = 40 KN is applied to the plate AB as shown, such that the bonding remains intact at all surfaces, then find the a) deflection of plate AB in the downward direction. b) change in volume of the two rubber blocks.



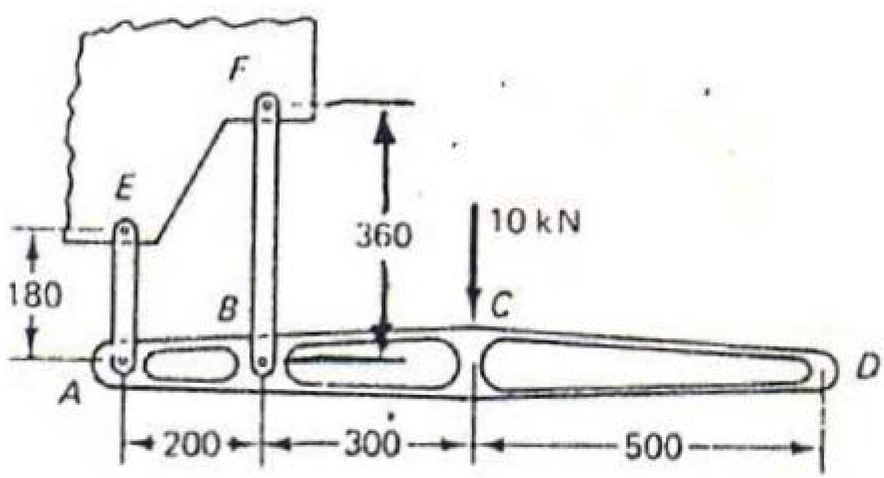
Problem 8



Drawn not to scale

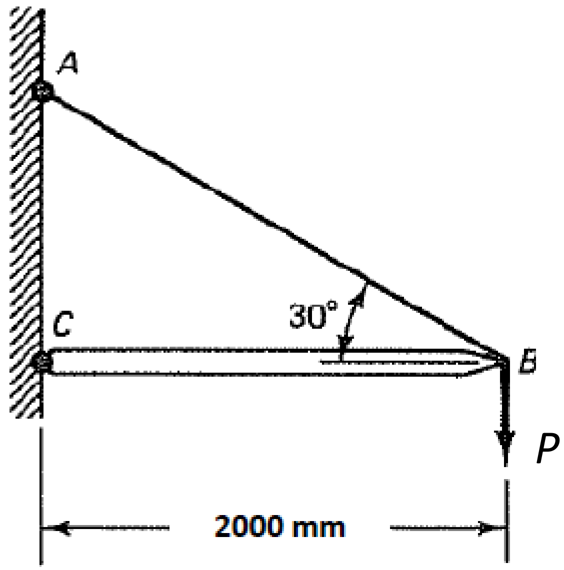
A 1m × 25mm × 10mm bar is pulled by an axial force, F, as shown in the figure. It is found that when $F = 10$ kN the bar just touches the linear spring (spring constant, $k = 10^7$ N/m) whose free end is kept at a distance of 0.1 mm from the bar end. (a) Find out the Young's modulus (E) of the bar material. (b) If the force, F is increased further then find its required value to compress the spring by 0.1 mm (c) Average normal stress at end A and B when the spring is compressed by 0.1 mm.

Problem 9



A rigid machine part AD is suspended by double hangers AE of cross sectional area of 50 mm^2 each and BF of cross sectional area of 100 mm^2 each respectively. The elastic modulus of hanger material is 180GPa and yield stress is 600MPa. Determine the deflection that would occur at D by applying a downward force of 10kN at C. Check hanger stress to assure that an elastic solution is applicable.

Problem 10

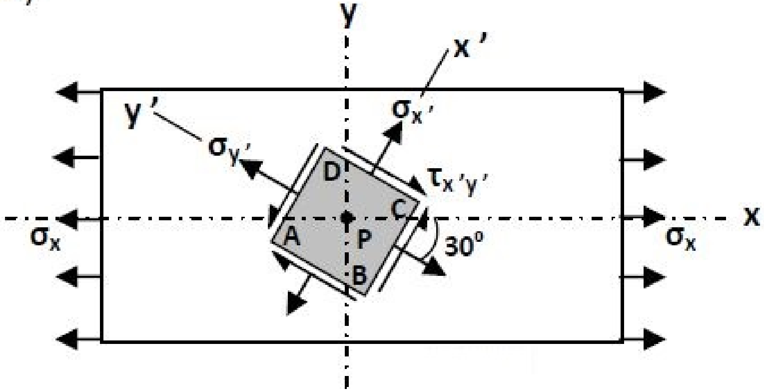


The jib crane shown in the figure has the cable AB of cross-sectional area of 300 mm^2 and the bar BC of cross-sectional area of 320 mm^2. (a) Determine the deflection vector at B caused by the application of a force P=16 kN. (b) Hence, estimate the vertical stiffness of the crane at point B. Take E=200 GPa.

Problem 11

A large plate is subjected to uniform edge stresses (σ_x = 200MPa), as shown in Fig. 5. Before loading, a small square element ABCD of side 100mm was inscribed at an angle on the plate as shown in the figure.

In the loaded condition, determine (i) the normal and shear stresses on the edges of the element, (ii) the changes in the dimensions AB and BC, and (iii) the magnitude of the shearing strain Take E=80GPa and v=0.3. Remember that G=E/2(1+v).



$$\text{(i)}\sigma_{x'} = 50\,\text{MPa}, \quad \sigma_{y'} = 150\text{ MPa}, \quad \tau_{x'y'} = -86.6\text{ MPa}$$

(ii) ΔAB = 0.16875 mm ΔBC=6.25E-3 mm

(iii) 2.8145E-3 radians

Biaxially Stress State: Thin walled Pressure Vessels

Thin walled pressure vessels

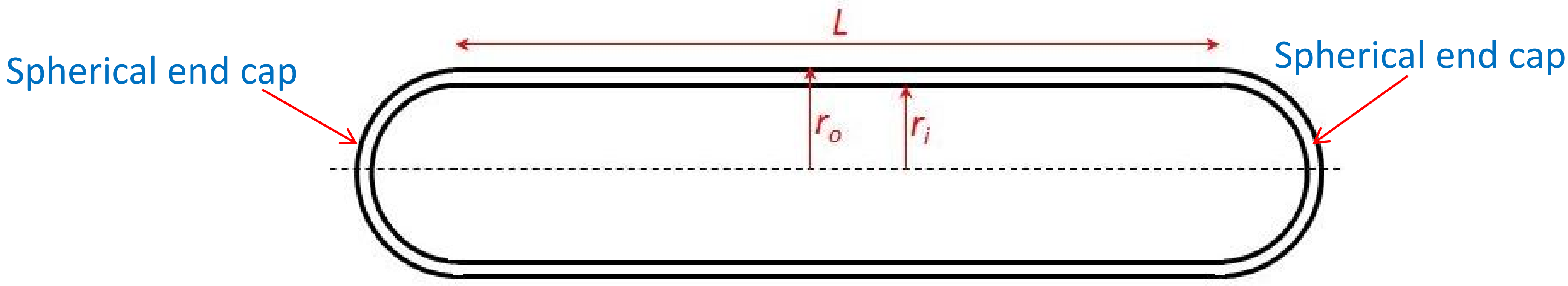
Spherical Pressure Vessels

Cylindrical Pressure Vessels

•Thin wall pressure vessels are widely used in industry for storage and transportation of liquids and gases. They are also used in aerospace and marine vehicles such as rockets and submarines. They can be spherical or cylindrical in shape.

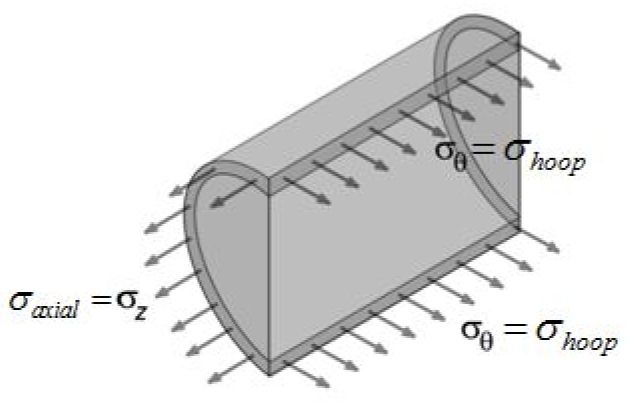
•The walls of a thin walled pressure vessel act as a membrane, i.e. under the internal pressure generated by the containing fluid, the walls stretch but do not bend.

Cylindrical pressure vessels



•We consider a thin walled cylinder of length L and inner radius r_i and outer radius r_o

•Isolate a section by passing two planes perpendicular to the axis of they cylinder, and one plane parallel to the axis.

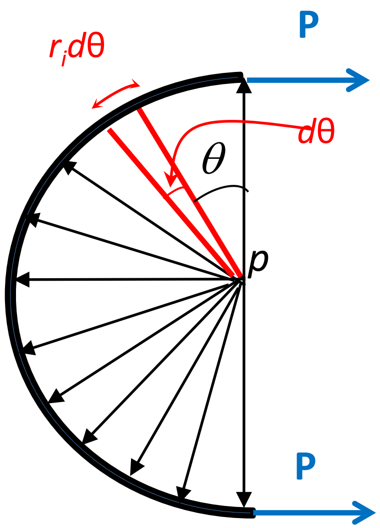


• Stresses act in the axial direction σ_{axial} and in the hoop direction σ_{hoop}

•The section is in equilibrium under the action of hoop stresses, axial stresses and internal pressure.

Hoop stress

•The section cannot shear because of the symmetric geometry and symmetric loading. Hence the hoop and axial stresses are the only non-zero stress components.



Considering equilibrium of this isolated section:

Force due to internal pressure acting on an infinitesimal element $= pr_i d\theta L$

Horizontal component of force $= pr_i d\theta L\, sin\theta$

This force is resisted by the hoop stress in the walls of the cylinder.

Integrating over the inner half of the cut section, the total horizontal force that has to be resisted by the hoop stress is: $\int_0^{\pi} pr_i\, Lsin\theta d\theta = 2pr_i L$

Denoting the average hoop stress in the walls of the cylinder as σ_{hoop}, clearly:

$$2\text{P} = \sigma_{hoop} \times (r_o - r_i) \times L \times 2$$

$$\therefore 2pr_i L = \sigma_{hoop} \times (r_o - r_i) \times L \times 2$$

Hence,

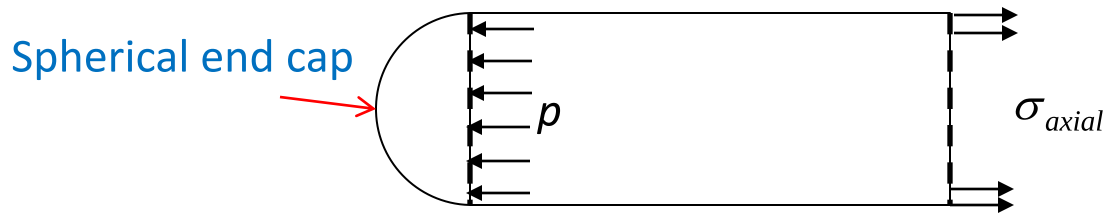
$$\sigma_{hoop} = pr_i/(r_o - r_i) = pr_i/t$$

•The above relation is valid only for thin walled cylinders, since taking the average hoop stress to calculate the total hoop force is correct only if the hoop stress is constant/linear through the thickness.

Longitudinal stress

•For thick walled cylinders, this may not be a good assumption. However even if the thickness is one-tenth of the internal radius, the error, if the above equation is used, is going to be small.

•The other normal stress σ_{axial} acts longitudinally and can be obtained by considering the following section:



Force due to internal pressure $= p\pi r_i^2$

Force developed by longitudinal stress in the walls: $\sigma_{axial} \times \pi(r_o^2 - r_i^2)$

Hence, $$\sigma_{axial} = \frac{p\pi r_i^2}{\pi(r_o^2 - r_i^2)} = \frac{pr_i^2}{(r_o - r_i)(r_o + r_i)} = \frac{pr_i^2}{t(r_o + r_i)}$$

However, $r_o \approx r_i$. Therefore, $\sigma_{axial} = \dfrac{pr_i}{2t}$

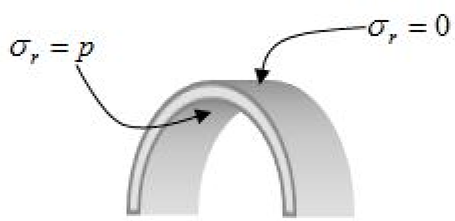
Hence for cylindrical pressure vessels, $\sigma_{axial} = \sigma_{hoop}/2$ i.e. longitudinal stress is half the hoop stress

Radial stress

•For thin walled cylinders, where the thickness is less than one-tenth of the internal radius, we assume that the radial stress σ_r is zero.

•To understand the basis of this assumption, recall that the radial stress is the normal stress acting in the through-thickness direction.

•At the interior walls, the radial stress is equal to the internal pressure, from equilibrium considerations. At the outer wall the radial stress is equal to zero.



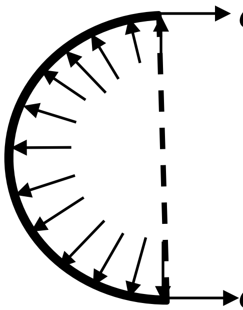
•Thus the radial stress decreases through the wall thickness. However even the maximum value of σ_r i.e. p is much smaller than the hoop stress and axial stress, since:

$$p << \frac{pr_i}{2t}; \quad p << \frac{pr_i}{t} \text{ for thin walled shells, since } \frac{r_i}{t} >> 1$$

•This is the basis for the assumpation that $\sigma_r = 0$

Spherical pressure vessels

- As in case of cylindrical pressure vessels, in spherical special vessels too the section cannot shear because of the symmetric geometry and symmetric loading. Hence the only non-zero stress components are the hoop and axial stresses, radial stresses being zero.



Considering a hemisphere:

$$\sigma_2 \times \pi (r_o^2 - r_i^2) = p \times \pi r_i^2$$

Hence, $$\sigma_2 = \frac{pr_i}{2t}$$

- But because of symmetry, any section that passes through the centre of the sphere yields the same results.

Hence, the hoop stress is equal to te axial stress in this case:

$$\sigma_1 = \sigma_2 = \frac{pr_i}{2t}$$

- Thus a spherical pressure vessel gives rise to an equibiaxial state of stress.

Constitutive Relations in matrix form

Recall the generalized Hooke's law:

$$\varepsilon_{xx} = \sigma_{xx}/E - \nu\sigma_{yy}/E - \nu\sigma_{zz}/E \qquad \gamma_{xy} = \tau_{xy}/G = (2(1+\nu)/E)\tau_{xy}$$

$$\varepsilon_{yy} = \sigma_{yy}/E - \nu\sigma_{xx}/E - \nu\sigma_{zz}/E \qquad \gamma_{xz} = \tau_{xz}/G = (2(1+\nu)/E)\tau_{xz}$$

$$\varepsilon_{zz} = \sigma_{zz}/E - \nu\sigma_{yy}/E - \nu\sigma_{xx}/E \qquad \gamma_{yz} = \tau_{yz}/G = (2(1+\nu)/E)\tau_{yz}$$

In matrix form:

$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{xz} \end{Bmatrix} = \frac{1}{E} \begin{bmatrix} 1 & -\nu & -\nu & 0 & 0 & 0 \\ -\nu & 1 & -\nu & 0 & 0 & 0 \\ -\nu & -\nu & -\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & 2(1+\nu) & 0 & 0 \\ 0 & 0 & 0 & 0 & 2(1+\nu) & 0 \\ 0 & 0 & 0 & 0 & 0 & 2(1+\nu) \end{bmatrix} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \tau_{xy} \\ \tau_{xz} \\ \tau_{xz} \end{Bmatrix}$$

Constitutive Relations in matrix form

Inverting this matrix, we can get the stresses in terms of strains:

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \tau_{xy} \\ \tau_{xz} \\ \tau_{xz} \end{Bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\upsilon & -\nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & (1-2\nu)/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & (1-2\nu)/2 & 0 \\ 0 & 0 & 0 & 0 & 0 & (1-2\nu)/2 \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{xz} \end{Bmatrix} \qquad (*)$$

Denoting $\lambda = E\nu/(1-2\nu)(1+\nu)$ and $\mu = E/2(1+\nu)$, where λ and μ are the Lame$'$ constants:

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \tau_{xy} \\ \tau_{xz} \\ \tau_{xz} \end{Bmatrix} = \begin{bmatrix} \lambda+2\mu & \lambda & \lambda & 0 & 0 & 0 \\ \lambda & \lambda+2\mu & \lambda & 0 & 0 & 0 \\ \lambda & \lambda & \lambda+2\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{xz} \end{Bmatrix} \qquad (**)$$

(**) is a more concise representation of (*)

Strains in cylindrical pressure vessels

From the generalized Hooke's law, in the absence of shear stresses:

$$\begin{Bmatrix} \varepsilon_{axial} \\ \varepsilon_{hoop} \\ \varepsilon_{radial} \end{Bmatrix} = \frac{1}{E} \begin{bmatrix} 1 & -\upsilon & -\upsilon \\ -\upsilon & 1 & -\upsilon \\ -\upsilon & -\upsilon & 1 \end{bmatrix} \begin{Bmatrix} \sigma_{axial} \\ \sigma_{hoop} \\ \sigma_{radial} \end{Bmatrix}$$

But in thin shells, $\sigma_{rr} = 0$ and $\sigma_{axial} = pr_i/2t$ and $\sigma_{hoop} = pr_i/t$

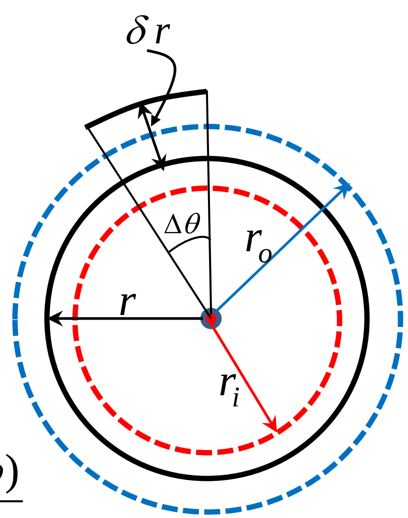
$$\therefore \varepsilon_{radial} = (-\upsilon/E)\, pr_i/2t + (-\upsilon/E)\, pr_i/t = (-3\upsilon/E)\, pr_i/2t$$

$$\varepsilon_{hoop} = (-\upsilon/E)\, pr_i/2t + (1/E)\, pr_i/t = ((2-\upsilon)/E)\, pr_i/2t$$

$$\varepsilon_{axial} = (1/E)\, pr_i/2t + (-\upsilon/E)\, pr_i/t = ((1-2\upsilon)/E)\, pr_i/2t$$

Because of the straining of the walls, the vessel expands.

To determine the amount by which the vessel expands, consider a circumference at average radius r which moves out with a radial displacement of δr:



$$\varepsilon_{hoop} = \frac{(r+\delta r)\Delta\theta - r\Delta\theta}{r\Delta\theta} = \frac{\delta r}{r} \text{ (circumferential strain at mid-radius)}$$

$$\text{The expansion of the sphere} = \delta r = r\varepsilon_{hoop} = r\left[-\frac{\upsilon}{E}\frac{pr_i}{2t} + \frac{1}{E}\frac{pr_i}{t}\right] \approx \frac{pr_i^2}{t}\frac{(2-\upsilon)}{2E}$$

The straining of the walls also results in changes in the circumference and the volume of the pressure vessel. These are calculated next.

Circumferential and Volumetric Strain

$$\text{The increase in circumference} = \delta C = 2\pi r \varepsilon_{hoop} = 2\pi r \frac{pr_i}{t}\frac{(2-\upsilon)}{2E} = \frac{\pi p r_i^2}{Et}(2-\upsilon)$$

$$\text{But } \delta r = \frac{pr_i^2}{2Et}(2-\upsilon) \qquad \therefore \delta C = 2\pi \delta r$$

$$\text{Change in thickness} = \text{original thickness} \times \varepsilon_r = t \times (-\frac{pr_i}{t})\frac{\upsilon}{E}\frac{3}{2}$$

Note that increase in circumference is accompanied by reduction in the thickness.

To determine the change in volume:

$$\text{The original volume of the cylinder} = V_0 = \pi r_i^2 \times L$$

$$\text{New volume} = V_1 = \pi (r+\delta r)^2 (L+\delta L) = \pi r^2 L\,(1+\frac{\delta r}{r})^2 (1+\frac{\delta L}{L})$$

$$\text{Recall, } \frac{\delta r}{r} = \varepsilon_{hoop} \qquad \frac{\delta L}{L} = \varepsilon_{axial} \qquad \therefore V_1 = \pi r^2 L\,(1+\varepsilon_{hoop})^2 (1+\varepsilon_{axial})$$

$$\text{Hence volumetric strain, } \varepsilon_{vol} = \frac{V_1 - V_0}{V_0} = (1+\varepsilon_{hoop})^2 (1+\varepsilon_{axial}) - 1$$

$$= (1 + 2\varepsilon_{hoop} + \varepsilon_{hoop}^2)(1+\varepsilon_{axial}) - 1$$

$$= 1 + 2\varepsilon_{hoop} + \varepsilon_{hoop}^2 + \varepsilon_{axial} + 2\varepsilon_{hoop}\varepsilon_{axial} + \varepsilon_{hoop}^2\varepsilon_{axial} - 1$$

$$\approx 2\varepsilon_{hoop} + \varepsilon_{axial} \quad \text{(neglecting quadratic and cubic terms)}$$

Strains in spherical pressure vessels

From the generalized Hooke's law, in the absence of shear stresses:

$$\begin{Bmatrix} \varepsilon_{axial} \\ \varepsilon_{hoop} \\ \varepsilon_{radial} \end{Bmatrix} = \frac{1}{E}\begin{bmatrix} 1 & -\upsilon & -\upsilon \\ -\upsilon & 1 & -\upsilon \\ -\upsilon & -\upsilon & 1 \end{bmatrix}\begin{Bmatrix} \sigma_{axial} \\ \sigma_{hoop} \\ \sigma_{radial} \end{Bmatrix}$$

But in sphericl shells, $\sigma_{rr} = 0$ and $\sigma_{axial} = pr/2t$ and $\sigma_{hoop} = pr/2t$

$\boxed{\text{Taking } r_i \sim r}$

$$\therefore \varepsilon_{radial} = (-\upsilon/E)(pr/2t) + (-\upsilon/E)(pr/2t) = -(\upsilon/E)(pr/t)$$

$$\varepsilon_{hoop} = \varepsilon_{axial} = (-\upsilon/E)(pr/2t) + (1/E)(pr/2t) = ((1-\upsilon)/E)(pr/2t)$$

Again, using the same argument as for cylindrical shells, $\varepsilon_{hoop} = \dfrac{\delta r}{r}$

$$\therefore \delta r = r\varepsilon_{hoop} = r((1-\upsilon)/E)(pr/2t) = \frac{pr^2}{2tE}(1\text{-}\upsilon)$$

Increase in circumference $= \delta C = 2\pi r \varepsilon_{hoop} = \dfrac{2\pi r^2 p}{2t}\dfrac{(1-\upsilon)}{E}$

Transformation of Stresses

The Stress Matrix in 2D

- We will look at the transformation of 2D stress states. Hence , in what follows, we will focus on the planar components of the stress matrix consisting of stress components that act on the x-y plane:

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_{xx} & \tau_{xy} \\ \tau_{yx} & \sigma_{yy} \end{bmatrix}$$

- Recall, that by rotating the coordinate system for the 2nd moment of area matrix **I** to the principal coordinate system, we could diagonalize it. The diagonal components of **I** then yield the largest and smallest values of the 2nd moment of area about any pair of orthogonal axes.

Largest 2nd moment of area in any system

$$\mathbf{I} = \begin{bmatrix} I_{xx} & I_{xy} \\ I_{yx} & I_{yy} \end{bmatrix} \xrightarrow{\text{Transformation to Principal Coordinate System}} \begin{bmatrix} \boxed{I_{x'x'}} & 0 \\ 0 & \boxed{I_{y'y'}} \end{bmatrix}$$

Smallest 2nd moment of area in any system

The Stress Matrix in 2D

• The 2D stress matrix can similarly be transformed to its principal system. In that case we will get a biaxial state of stress: i.e. only two normal stress components and no shear stress components.

• Also the two surviving normal stress components will be the largest and smallest normal stress components acting on any pair of orthogonal planes passing through the point (whose state of stress we are interested in).

Largest normal component of stress in any system

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_{xx} & \tau_{xy} \\ \tau_{yx} & \sigma_{yy} \end{bmatrix} \xrightarrow{\text{Transformation to Principal Coordinate System of } \boldsymbol{\sigma}} \begin{bmatrix} \boxed{\sigma_{x'x'}} & 0 \\ 0 & \boxed{\sigma_{y'y'}} \end{bmatrix}$$

Smallest normal component of stress in any system

Importance of the Principal Stresses

- The largest principal stress component $\sigma_{x'x'}$ is the algebraically largest normal stress component acting at the point.

- If it is tensile, and its magnitude exceeds the tensile strength of the material, it will initiate crack growth in the material along the plane on which it is acting.

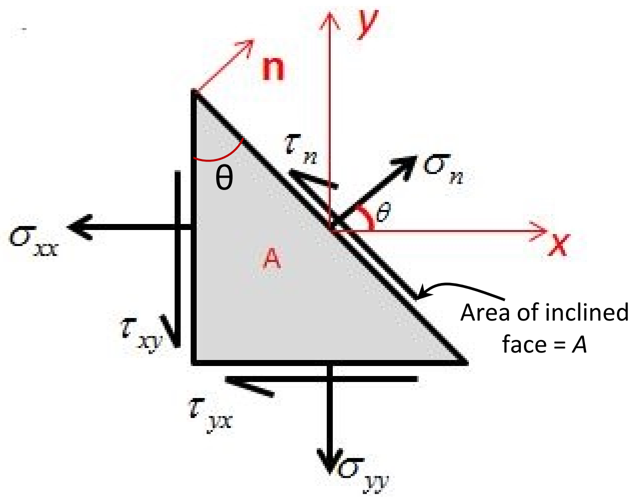
- Similarly, the smallest principal stress component $\sigma_{y'y'}$ is the algebraically smallest normal stress component acting at the point.

- If it is compressive in nature, and its magnitude exceeds the crushing strength of the material, it will result in crushing failure along the plane on which it is acting.

- Thus, to determine if cracking or crushing of a material would occur on account of the stress state generated in the material, it is critical to calculate the principal stresses.

Normal and Shear Stress on an inclined plane

•First we will determine the normal and shear stresses on an inclined plane whose normal makes an angle of θ with the positive x axis. To do this, we consider the stresses acting on an infinitesimal triangular wedge of material centered at point A:



For force equilbrium at A:

$$\sum F_x = 0$$

$$\therefore \sigma_n A\cos\theta - \sigma_{xx} A\cos\theta - \tau_{yx} A\sin\theta - \tau_n A\sin\theta = 0 \qquad (1)$$

$$\sum F_y = 0$$

$$\therefore \sigma_n A\sin\theta - \sigma_{yy} A\sin\theta - \tau_{xy} A\cos\theta + \tau_n A\cos\theta = 0 \qquad (2)$$

$$\boxed{(1)\times\cos\theta + (2)\times\sin\theta:}$$

$$\sigma_n A(\cos^2\theta + \sin^2\theta) - \sigma_{xx} A\cos^2\theta - \tau_{yx} A\sin\theta\cos\theta - \tau_{xy} A\sin\theta\cos\theta - \sigma_{yy} A\sin^2\theta = 0$$

$$\boxed{\begin{aligned} \therefore \sigma_n &= \sigma_{xx}\cos^2\theta + \sigma_{yy}\sin^2\theta + 2\tau_{xy}\sin\theta\cos\theta \\ &= \sigma_{xx}\frac{(1+\cos 2\theta)}{2} + \sigma_{yy}\frac{(1-\cos 2\theta)}{2} + \tau_{xy}\sin 2\theta \\ &= \frac{\sigma_{xx}+\sigma_{yy}}{2} + \frac{(\sigma_{xx}-\sigma_{yy})}{2}\cos 2\theta + \tau_{xy}\sin 2\theta \qquad (3) \end{aligned}}$$

Note: θ is positive if the orientation of **n** with respect to the positive x axis is in the anti-clockwise sense; it is negative if the orientation is in the clockwise sense.

Normal and shear stress on inclined plane

$$\boxed{(2)\times\cos\theta-(1)\times\sin\theta:}$$

$$\sigma_n A\cos\theta\sin\theta-\sigma_n A\sin\theta\cos\theta+\tau_n A(\cos^2\theta+\sin^2\theta)-\tau_{xy}A\cos^2\theta-\sigma_{yy}A\cos\theta\sin\theta+\sigma_{xx}A\cos\theta\sin\theta$$

$$+\tau_{xy}A\sin^2\theta=0$$

$$\boxed{\begin{aligned}\therefore \tau_n &= \tau_{xy}(\cos^2\theta-\sin^2\theta)+(\sigma_{yy}-\sigma_{xx})\sin\theta\cos\theta\\ &= \tau_{xy}\cos 2\theta-\frac{(\sigma_{xx}-\sigma_{yy})}{2}\sin 2\theta \quad (4)\end{aligned}}$$

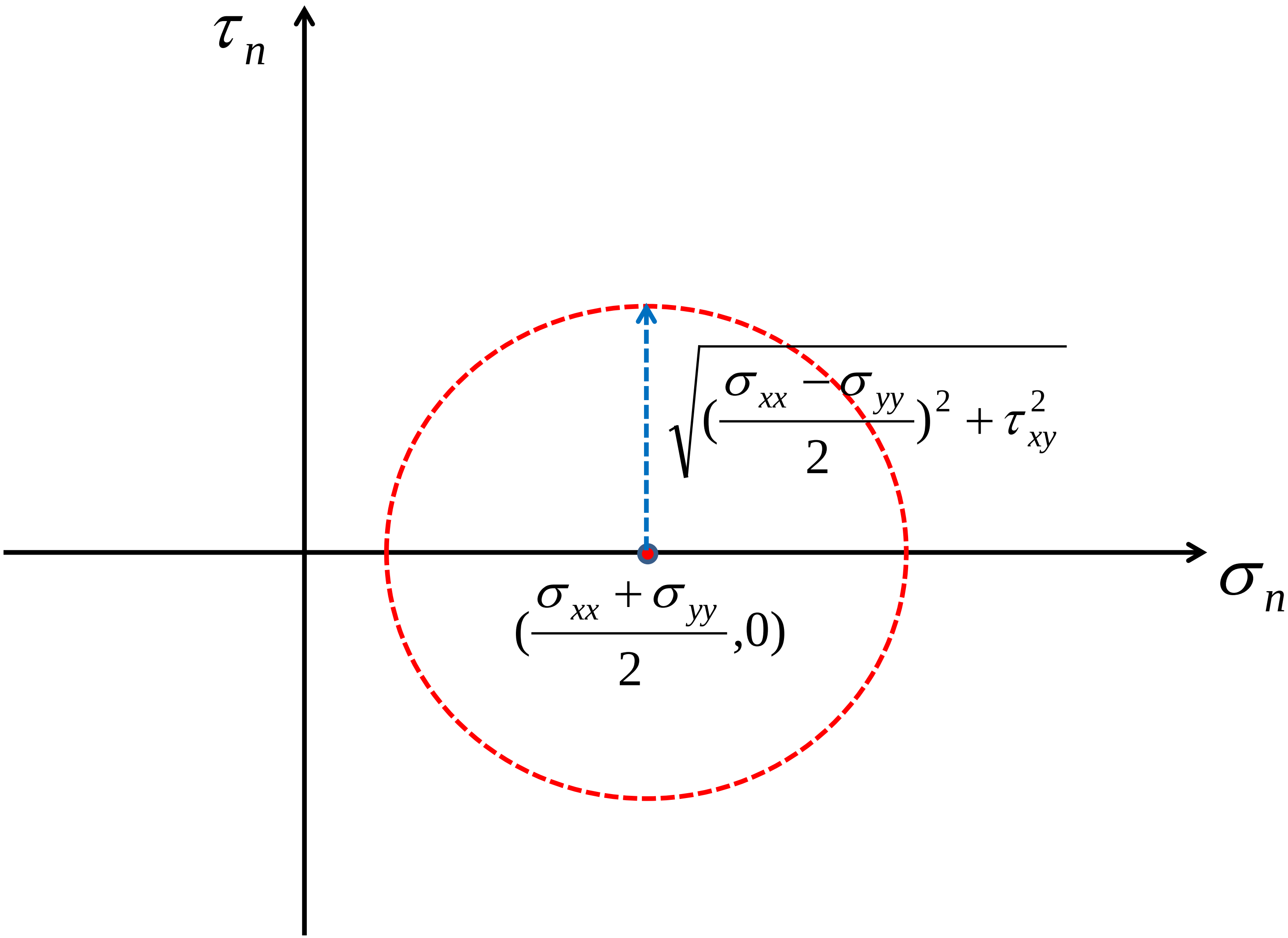
From (3) and (4):

$$\begin{aligned}\left\{\sigma_n-\frac{(\sigma_{xx}+\sigma_{yy})}{2}\right\}^2+\tau_n^2 &= \left\{\frac{(\sigma_{xx}-\sigma_{yy})}{2}\right\}^2\cos^2 2\theta+\tau_{xy}^2\sin^2 2\theta+\tau_{xy}(\sigma_{xx}-\sigma_{yy})\cos 2\theta\sin 2\theta\\ &\quad+\left\{\frac{(\sigma_{xx}-\sigma_{yy})}{2}\right\}^2\sin^2 2\theta+\tau_{xy}^2\cos^2 2\theta-\tau_{xy}(\sigma_{xx}-\sigma_{yy})\cos 2\theta\sin 2\theta\\ &= \left\{\frac{(\sigma_{xx}-\sigma_{yy})}{2}\right\}^2+\tau_{xy}^2 \quad (5)\end{aligned}$$

- (5) represents the equation of the Mohr's circle. Every point on the circle gives the normal and shear stress acting on a particular plane passing through the point A.

The Mohr's Circle

$$\left\{\sigma_n - \frac{(\sigma_{xx}+\sigma_{yy})}{2}\right\}^2 + \tau_n^2 = \left\{\frac{(\sigma_{xx}-\sigma_{yy})}{2}\right\}^2 + \tau_{xy}^2$$



Principal Planes

- Which is the plane for which the normal stress is maximum?

$$\frac{d\sigma_n}{d\theta} = -(\sigma_{xx} - \sigma_{yy})\sin 2\theta + 2\tau_{xy}\cos 2\theta = 0$$

$$\therefore \tan 2\theta = \frac{\tau_{xy}}{(\sigma_{xx} - \sigma_{yy})/2} \Rightarrow \theta_1 = \frac{1}{2}\tan^{-1}\frac{\tau_{xy}}{(\sigma_{xx} - \sigma_{yy})/2}$$

$$\theta_2 = \frac{1}{2}\tan^{-1}\frac{\tau_{xy}}{(\sigma_{xx} - \sigma_{yy})/2} + \frac{\pi}{2}$$

Hence,

$$\sin 2\theta_1 = \frac{\tau_{xy}}{\sqrt{(\frac{\sigma_{xx} - \sigma_{yy}}{2})^2 + \tau_{xy}^2}} \qquad \cos 2\theta_1 = \frac{(\sigma_{xx} - \sigma_{yy})/2}{\sqrt{(\frac{\sigma_{xx} - \sigma_{yy}}{2})^2 + \tau_{xy}^2}}$$

On substituting the above values for $\sin 2\theta_1$ and $\cos 2\theta_1$ in the expression for σ_n (Recall (3)):

$$\sigma_n = \frac{\sigma_{xx} + \sigma_{yy}}{2} + (\frac{\sigma_{xx} - \sigma_{yy}}{2})\cos 2\theta + \tau_{xy}\sin 2\theta \quad \text{we get:}$$

$$\sigma_n = \sigma_1 = \frac{\sigma_{xx} + \sigma_{yy}}{2} + \frac{((\sigma_{xx} - \sigma_{yy})/2)^2}{\sqrt{(\frac{\sigma_{xx} - \sigma_{yy}}{2})^2 + \tau_{xy}^2}} + \frac{\tau_{xy}^2}{\sqrt{(\frac{\sigma_{xx} - \sigma_{yy}}{2})^2 + \tau_{xy}^2}} = \frac{\sigma_{xx} + \sigma_{yy}}{2} + \sqrt{(\frac{\sigma_{xx} - \sigma_{yy}}{2})^2 + \tau_{xy}^2}$$

Principal Planes

If instead we substitute the values for $\sin 2\theta_2$ and $\cos 2\theta_2$ in the expression for σ_n:

$$\sigma_n = \sigma_2 = \frac{\sigma_{xx} + \sigma_{yy}}{2} - \sqrt{\left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right)^2 + \tau_{xy}^2}$$

For the principal planes, the shear stress (Recall (4)):

$$\tau_n = -\frac{\sigma_{xx} - \sigma_{yy}}{2}\sin 2\theta + \tau_{xy}\cos 2\theta = -\frac{\sigma_{xx} - \sigma_{yy}}{2} \times \frac{\tau_{xy}}{\sqrt{\left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right)^2 + \tau_{xy}^2}} + \frac{\tau_{xy}((\sigma_{xx} - \sigma_{yy})/2)}{\sqrt{\left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right)^2 + \tau_{xy}^2}}$$

$$= 0$$

- Thus on the planes where the normal stress is a maximum or a minimum, the shear stress is zero. This makes the stress matrix diagonal

- The shear stress also varies with orientation. If we are interested in the planes on which the shear stress is a maximum or minimum, then:

$$\frac{d}{d\theta}\tau_n = 0 \Rightarrow -(\sigma_{xx} - \sigma_{yy})\cos 2\theta - 2\tau_{xy}\sin 2\theta = 0$$

$$\boxed{\tau_n = \tau_{xy}\cos 2\theta - \frac{(\sigma_{xx} - \sigma_{yy})}{2}\sin 2\theta}$$

$$\therefore \tan 2\theta = -\frac{\sigma_{xx} - \sigma_{yy}}{2\tau_{xy}}$$

Planes of Maximum Shear

Hence, $\theta_1 = \frac{1}{2}\tan^{-1}\frac{\sigma_{yy}-\sigma_{xx}}{2\tau_{xy}}$

$$\theta_2 = \frac{1}{2}\tan^{-1}\frac{\sigma_{yy}-\sigma_{xx}}{2\tau_{xy}} + \frac{\pi}{2}$$

$$\Rightarrow \quad \sin 2\theta_1 = \frac{(\sigma_{yy}-\sigma_{xx})/2}{\sqrt{((\sigma_{yy}-\sigma_{xx})/2)^2 + \tau_{xy}^2}}$$

$$\cos 2\theta_1 = \frac{\tau_{xy}}{\sqrt{((\sigma_{yy}-\sigma_{xx})/2)^2 + \tau_{xy}^2}}$$

The corresponding value of τ_n is:

$$\tau_n = \frac{-\{(\sigma_{xx}-\sigma_{yy})/2\}\times -\{(\sigma_{xx}-\sigma_{yy})/2\}}{\sqrt{(\frac{\sigma_{xx}-\sigma_{yy}}{2})^2 + \tau_{xy}^2}} + \frac{\tau_{xy}^2}{\sqrt{(\frac{\sigma_{xx}-\sigma_{yy}}{2})^2 + \tau_{xy}^2}} = \sqrt{(\frac{\sigma_{xx}-\sigma_{yy}}{2})^2 + \tau_{xy}^2}$$

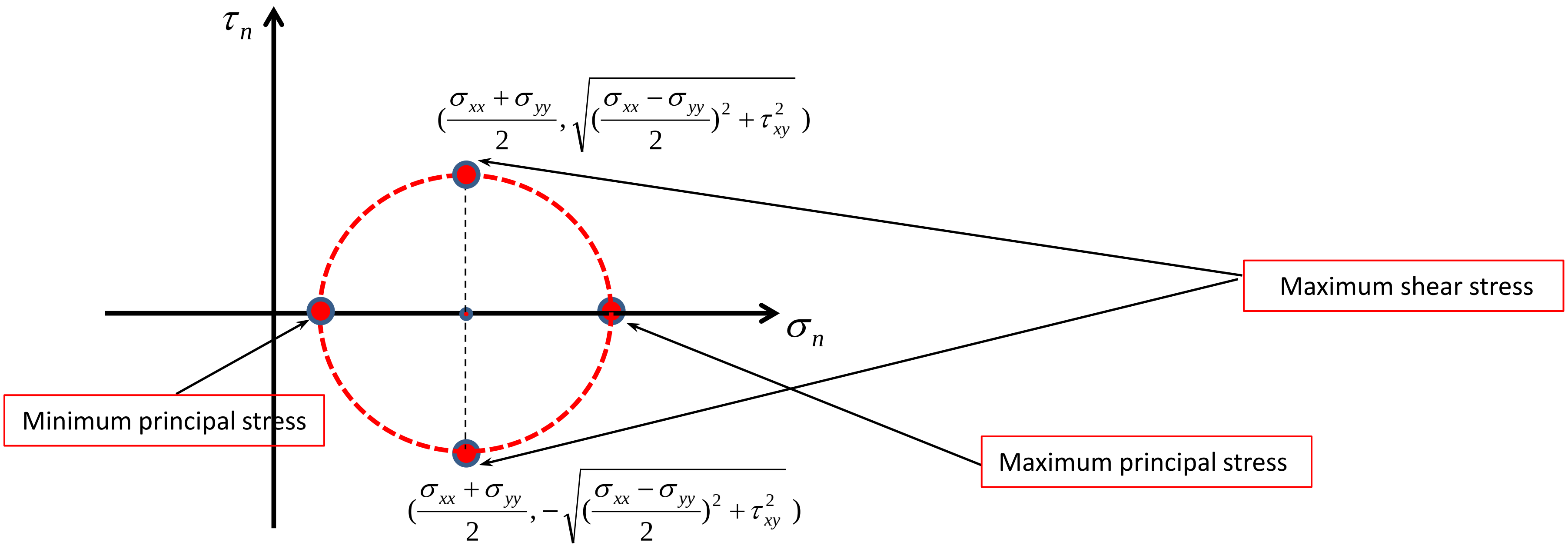
The normal stress on the plane with maximum shear:

$$\sigma_n = \frac{\sigma_{xx}+\sigma_{yy}}{2} + \frac{\sigma_{xx}-\sigma_{yy}}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

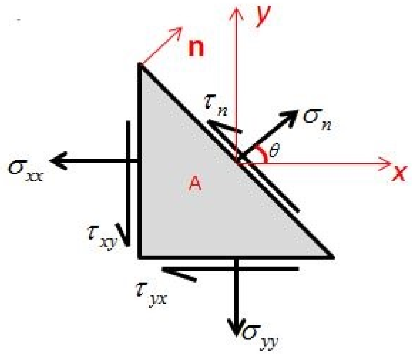
$$= \frac{\sigma_{xx}+\sigma_{yy}}{2} + \frac{\sigma_{xx}-\sigma_{yy}}{2}\frac{\tau_{xy}}{\sqrt{(\frac{\sigma_{xx}-\sigma_{yy}}{2})^2 + \tau_{xy}^2}} + \frac{\tau_{xy}((\sigma_{yy}-\sigma_{xx})/2)}{\sqrt{(\frac{\sigma_{xx}-\sigma_{yy}}{2})^2 + \tau_{xy}^2}} = \frac{\sigma_{xx}+\sigma_{yy}}{2}$$

$$\boxed{\tau_n = \tau_{xy}\, cos2\theta - \frac{(\sigma_{xx} - \sigma_{yy})}{2} sin2\theta}$$

Location of principal stresses and maximum shear stress in Mohr's circle

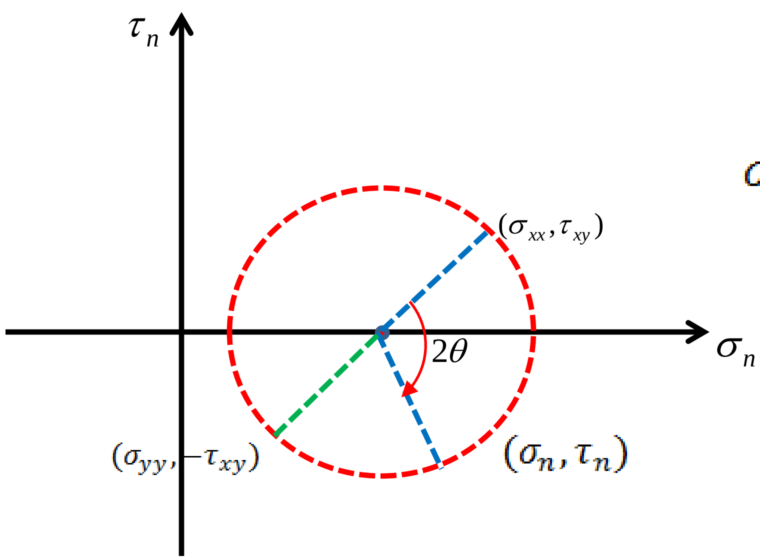


The Mohr's Circle



(σ_{xx}, τ_{xy}): normal and shear stress acting on a plane whose normal is given by the x axis.

•The stress state (normal & shear stresses) on any plane whose normal makes an angle θ with the x axis will subtend an angle of 2θ (clockwise) at the centre of the Mohr's circle with the stress state (normal & shear stress) acting on the plane with normal given by the x axis.



$$\sigma_n = \frac{(\sigma_{xx} + \sigma_{yy})}{2} + \frac{(\sigma_{xx} - \sigma_{yy})}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

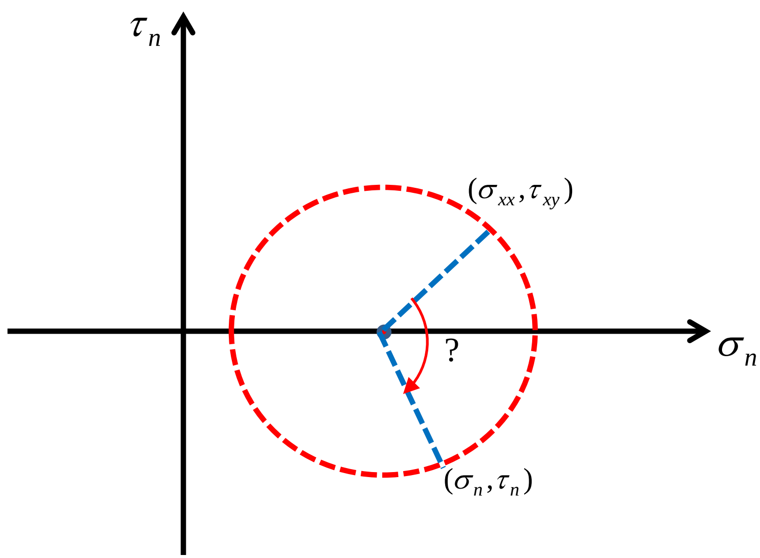
$$\tau_n = -\frac{(\sigma_{xx} - \sigma_{yy})}{2}\sin 2\theta + \tau_{xy}\cos 2\theta$$

Use of the Mohr's circle

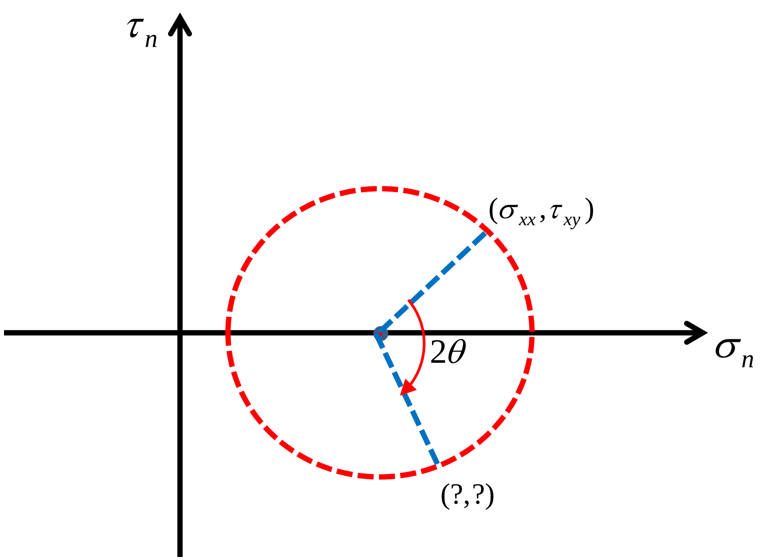
- The Mohr's circle can therefore be used to:

(i) Find the angle of inclination of the normal to a plane with the x-axis, provided the normal and shear stress on the plane is known

(ii) If the angle of inclination of the normal to a plane with the x axis is known, to find the normal and shear stress acting on the plane.



(i)



(ii)

Transformation equations for strain

- Similar to the transformation equations for planar stress components, there are transformation equations for planar strain components as well.

- Suppose the normal and shear strain components in the x-y system, $\varepsilon_{xx}, \varepsilon_{yy}$ and γ_{xy} are known at a point.

- Then one can compute the axial and shear strain components acting on a plane whose normal **n** makes an angle of θ degrees with the positive x axis.

Axial strain along the direction normal to the plane

$$\varepsilon_n = \varepsilon_{xx}\cos^2\theta + \varepsilon_{yy}\sin^2\theta + \gamma_{xy}\sin\theta\cos\theta$$

$$= \varepsilon_{xx}\frac{(1+\cos 2\theta)}{2} + \varepsilon_{yy}\frac{(1-\cos 2\theta)}{2} + \frac{\gamma_{xy}}{2}\sin 2\theta$$

$$= \frac{\varepsilon_{xx}+\varepsilon_{yy}}{2} + \frac{(\varepsilon_{xx}-\varepsilon_{yy})}{2}\cos 2\theta + \frac{\gamma_{xy}}{2}\sin 2\theta$$

Recall: θ is positive if the orientation of **n** with respect to the positive x axis is in the anti-clockwise sense; it is negative if the orientation is in the clockwise sense.

Transformation equations & principal strains

Shear strain in the plane whose normal makes a clockwise angle of θ degree with x axis

$$\gamma_n = \gamma_{xy}(\cos^2\theta - \sin^2\theta) + (\varepsilon_{yy} - \varepsilon_{xx})2\sin\theta\cos\theta$$
$$= \gamma_{xy}\cos 2\theta - (\varepsilon_{xx} - \varepsilon_{yy})\sin 2\theta$$

- Similarly the principal strains are:

$$\varepsilon_1 = \frac{\varepsilon_{xx} + \varepsilon_{yy}}{2} + \sqrt{\left(\frac{\varepsilon_{xx} - \varepsilon_{yy}}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2}$$
$$\varepsilon_2 = \frac{\varepsilon_{xx} + \varepsilon_{yy}}{2} - \sqrt{\left(\frac{\varepsilon_{xx} - \varepsilon_{yy}}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2}$$

- The angle between the positive x axis, and the normal to the plane on which the principal strain ε_1 acts must satisfy the equation:

$$\tan 2\theta = \frac{\gamma_{xy}}{(\varepsilon_{xx} - \varepsilon_{yy})}$$

Maximum shear strain and its orientation

- The maximum shear strain is:

$$\gamma_{\max} == \sqrt{(\varepsilon_{xx} - \varepsilon_{yy})^2 \;\; + \gamma_{xy}^2}$$

- The angle between the positive x axis, and the normal to the plane on which the maximum shear strain $\gamma_{\max}$ acts must satisfy the equation:

$$\tan 2\theta = -\frac{\varepsilon_{xx} - \varepsilon_{yy}}{\gamma_{xy}}$$

- The axial strain acting on the plane in which the shear strain is a maximum is given by:

$$\varepsilon_n = \frac{\varepsilon_{xx} + \varepsilon_{yy}}{2}$$