Maths I

Lecture 15

Remember :

Let z be a function defined as $z = f(x,y)$ on some domain D of $\mathbb{R}^2$.

$$\Delta z = f(x+\Delta x, y+\Delta y) - f(x,y)$$

$$= \frac{\partial f}{\partial x}(x,y)\,\Delta x + \frac{\partial f}{\partial y}(x,y)\,\Delta y$$

$$+ \gamma_1 \Delta x + \gamma_2 \Delta y$$

Derivative of composite functions.

one variable case:

$$y = f(x) = f(\varphi(t))$$

$$x = \varphi(t)$$

$$\boxed{\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}}$$

Let us assume that

$$z = f(u, v)$$

and further, u & v are functions of x & y given as

$$u = \varphi(x, y), \quad v = \psi(x, y)$$

This means that z is a composite function of x & y.

$$z = f(\varphi(x, y), \psi(x, y))$$

$$\boxed{\frac{\partial z}{\partial x} \quad \text{and} \quad \frac{\partial z}{\partial y}}$$

$$\Delta z = \frac{\partial f}{\partial u} \Delta u + \frac{\partial f}{\partial v} \Delta v + \eta_1 \Delta_x u + \eta_2 \Delta_x v$$

increments in u & v

$$\frac{\Delta z}{\Delta x} = \frac{\partial f}{\partial y} \frac{\Delta u}{\Delta x} + \frac{\partial f}{\partial v} \frac{\Delta v}{\Delta x} + \eta_1 \cancel{\frac{\Delta u}{\Delta x}} + \eta_2 \cancel{\frac{\Delta v}{\Delta x}}$$

$$\frac{\partial z}{\partial x} \qquad \frac{\partial u}{\partial x} \qquad \frac{\partial v}{\partial x} \qquad 0$$

$\lim_{\Delta x \to 0}$ on both sides

$$\frac{\partial z}{\partial x} = \boxed{\frac{\partial f}{\partial u}\frac{\partial u}{\partial x}} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x}$$

Similarly,

$$\frac{\partial z}{\partial y} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial y}$$

$$z = f(u, v)$$

$$u = \varphi(x, y), \qquad v = \varphi(x, y)$$

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case (ii)

$$z = f(w_1, w_2 w_3)$$

$$w_1 = \varphi_1(t), \quad w_2 = \varphi_2(t), \quad w_3 = \varphi_3(t)$$

$$\frac{dz}{dt} = \frac{\partial z}{\partial w_1}\frac{dw_1}{dt} + \frac{\partial z}{\partial w_2}\frac{dw_2}{dt} + \frac{\partial z}{\partial w_3}\frac{dw_3}{dt}$$

$$z = f\left(\boxed{x, y}\right)$$

$$x = \varphi(w_1, w_2, w_3), \qquad y = \psi(w_1, w_2, w_3)$$

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$$\frac{\partial z}{\partial w_1} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial w_1} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial w_1}$$

$$\frac{\partial z}{\partial w_2} =$$

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$$\frac{\partial z}{\partial w_3} =$$

$$z = f(x)$$

where $x = \varphi(w_1, w_2, w_3, w_4)$

$$\frac{\partial z}{\partial w_3} = \frac{dz}{dx}\frac{\partial x}{\partial w_3}$$

Ex:

$$z = \ln(u^2 + v) = f(u, v)$$

$$u = e^{x+y^2} = \phi(x, y), \quad v = x^2 + y = \psi(x, y)$$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x}$$

$$\frac{\partial z}{\partial u} = \frac{1}{u^2+v} \cdot 2u$$

$$\frac{\partial z}{\partial v} = \frac{1}{u^2+v}$$

$$\frac{\partial u}{\partial x} = e^{x+y^2}$$

$$\frac{\partial v}{\partial x} = 2x$$

$$\frac{\partial z}{\partial x} = \frac{2u}{u^2+v} e^{x+y^2} + \frac{1}{u^2+v} 2x$$

$$= \frac{2e^{x+y^2}}{\left(e^{x+y^2}\right)^2 + x^2+y} e^{x+y^2} + \frac{2x}{\left(e^{x+y^2}\right)^2 + x^2+y}$$

$$z = f(x_1, x_2, \cdots, x_n)$$

$$x_i = \varphi_i(t_1, t_2, \cdots, t_m)$$

n terms

$$\frac{\partial z}{\partial t_i} = \underline{\frac{\partial z}{\partial x_j} \frac{\partial x_j}{\partial t_i}}_{j}$$

n terms

m partial derivaties

$i = 1 - m$

The derivative of a function defined implicitely.

Let y be a function of x defined implicitely as

$$F(x,y) = 0$$

<u>Thm</u>: Let a continuous function y of x be defined implicitely by the equation

$$F(x,y) = 0$$

where $F(x,y)$, $F_x(x,y)$ and $F_y(x,y)$ are continuous functions in some domain D of $\mathbb{R}^2$ containing the point (x,y). Further at this point $F_y(x,y) \neq 0$.

Then
$$\frac{dy}{dx} = -\frac{F_x(x,y)}{F_y(x,y)}$$

Proof: We know

$$F(x, y) = 0$$

Increase independent variable x to $x + \Delta x$.

Then there corresponding change in y denoted as $y + \Delta y$.

$$F(x+\Delta x, y+\Delta y) = 0$$

$$\therefore \quad F(x+\Delta x, y+\Delta y) - F(x, y) = 0$$

$$F(x+\Delta x, y+\Delta y) - F(x, y)$$

$$= \frac{\partial F}{\partial x} \Delta x + \frac{\partial F}{\partial y} \Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y$$

where $\varepsilon_1, \varepsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0, 0)$.

$$\Rightarrow \frac{\partial F}{\partial x} \Delta x + \frac{\partial F}{\partial y} \Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y = 0$$

$$\Rightarrow \quad \frac{\Delta y}{\Delta x} = - \frac{\frac{\partial F}{\partial x} + \varepsilon_1}{\frac{\partial F}{\partial y} + \varepsilon_2}$$

take $\Delta x \to 0$

∎

$$\underbrace{2x}_{F_x}\frac{dx}{dx} + \underbrace{2y}_{F_y}\frac{dy}{dx} = 0$$

Ex: $x^2 + y^2 - 1 = 0$

where y is an implicit f^n of x.

Let z be a function of x, y defined implicitely as

$$F(x, y, z) = 0$$

$$\frac{\partial z}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}} \qquad \frac{\partial z}{\partial y} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}}$$

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z}, \quad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z}$$

Ex: $e^z + x^2 y + z + 5 = 0$

defines z as am implicit f^n of x & y.

z_x, z_y

$$z_x = -\frac{2xy}{e^z+1}, \quad z_y = \frac{-x^2}{e^z+1}$$