Maths I

Lecture 10

Limits :

A function $f(x,y)$ is said to have limit L as (x,y) approaches the pt. (x_0, y_0) and is written as

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L$$

if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that for all (x,y) in the domain of $f(x,y)$

$$|f(x,y) - L| < \varepsilon$$

whenever $0 < \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta$

Note: $\varepsilon - \delta$ definition.

1. Guess the limit L.
2. Choose arbitrary $\varepsilon > 0$ (challenge)
3. Find a corresponding $\delta > 0$ (solve the challenge)

If $\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L$

then $\forall\ \varepsilon > 0$, there exists a $\delta > 0$

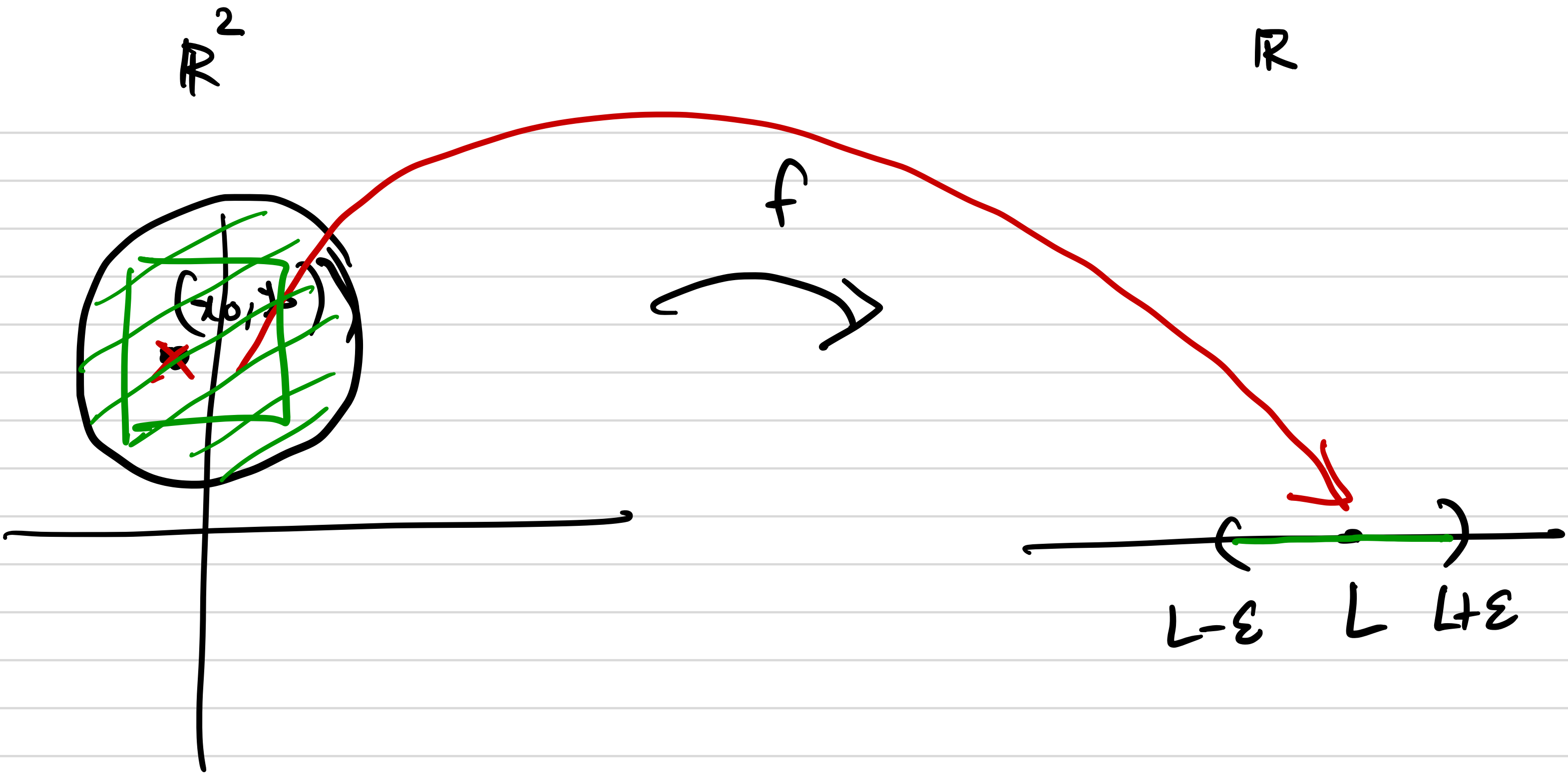
$|f(x,y) - L| < \varepsilon$ whenever

$$0 < \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta$$

Conversely, if $\lim_{(x,y)\to(x_0,y_0)} f(x,y) \neq L$,

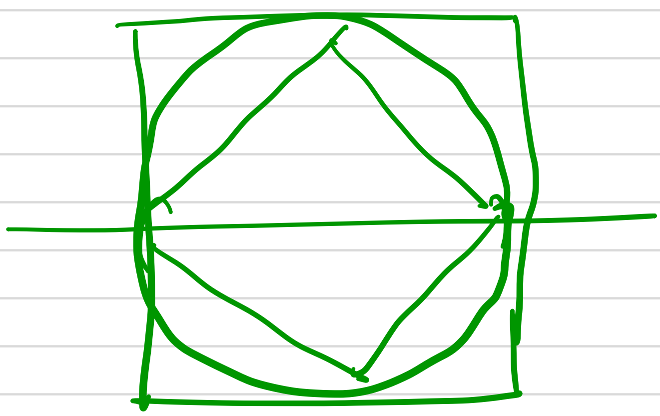
then there will an $\varepsilon > 0$ for which we can not find δ s.t

$$\left[\begin{array}{l} |f(x,y) - L| < \varepsilon \quad \text{and} \\ 0 < \boxed{\sqrt{(x-x_0)^2 + (y-y_0)^2}} < \delta \end{array}\right]$$



$$f : \mathbb{R}^2 \longrightarrow \mathbb{R}$$

$$\lim_{(x,y) \to (x_0, y_0)} f(x,y) = L$$



Properties of Limits.

Let f & g be functions from $\mathbb{R}^2$ to $\mathbb{R}$ such that

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L$$

$$\lim_{(x,y)\to(x_0,y_0)} g(x,y) = M$$

1. Sum
2. Difference
3. Product
4. Quotient
5. Power
6. Root

Ex: $$\lim_{(x,y)\to(0,1)} \frac{x - xy + 3}{x^2y + 5xy - y^3} = -3$$

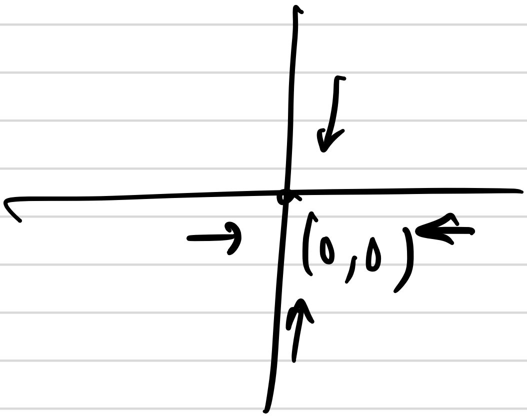
$$\lim_{(x,y)\to(0,0)} x^2+y^2 = 0$$

Ex: $\displaystyle \lim_{(x,y)\to(0,0)} \frac{x^2 - xy}{\sqrt{x} - \sqrt{y}} = 0$

Ex: Find $\displaystyle \lim_{(x,y)\to(0,0)} \frac{4xy^2}{x^2+y^2}$

if it exists.

Guess $L = 0$.



Let $\varepsilon > 0$ be given.

We want to find $\delta > 0$ such that

$$\left| \frac{4xy^2}{x^2+y^2} - 0 \right| < \varepsilon \quad \text{whenever}$$

$$0 < \sqrt{x^2+y^2} < \delta$$

$$\frac{4|x|y^2}{x^2+y^2} < \varepsilon \quad \text{whenever} \quad 0 < \sqrt{x^2+y^2} < \delta \quad \Big\} \ \ast$$

$$\boxed{\begin{aligned} &\underline{\text{Notice:}} \\ &y^2 \leq x^2+y^2 \\ &\frac{4|x|y^2}{x^2+y^2} \leq \frac{4|x|(x^2+y^2)}{x^2+y^2} \leq 4|x| \\ &\qquad \leq 4\sqrt{x^2+y^2} < \varepsilon \end{aligned}} \ \ast$$

Choose $\delta = \varepsilon/4$.

whenever

$$0 < \sqrt{x^2+y^2} < \delta.$$

$$\sqrt{x^2+y^2} < \varepsilon/4$$

$$4\sqrt{x^2+y^2} < \varepsilon.$$

This proves: $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^2} = 0$

Method 2:

$x = r\cos\theta$

$y = r\sin\theta$

$\boxed{(x,y) \to (0,0)}$

$\Updownarrow$

$\boxed{\begin{array}{c} r \to 0 \\ \theta \ \cancel{\text{any}}\ \text{value} \\ \in [0, 2\pi) \end{array}}$

$$\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^2}$$

$$= \lim_{\substack{r\to 0 \\ \theta \text{ any value}}} \frac{r\cos\theta \, r^2\sin^2\theta}{r^2\cos^2\theta + r^2\sin^2\theta}$$

$$= \lim_{\substack{r\to 0 \\ \theta \text{ any value}}} r\cos\theta\sin^2\theta$$

$= 0$ Because ($\cos\theta\sin^2\theta$ is a bounded function)

Ex:

Show $$\lim_{(x,y)\to(0,0)} \frac{2x^2y}{x^4+y^2}$$

does not exist.