Maths I

Lecture 13

$f : \mathbb{R}^2 \to \mathbb{R}$

$$f_x(a,b) = \left.\frac{\partial f}{\partial x}\right|_{(a,b)}$$

$$= \lim_{h \to 0} \frac{f(a+h, b) - f(a,b)}{h}$$

$$f_y(a,b) = \left.\frac{\partial f}{\partial y}\right|_{(a,b)}$$

$$= \lim_{k \to 0} \frac{f(a, b+k) - f(a,b)}{k}$$

Directional derivative

Let $x \in \mathbb{R}^n$ $\qquad x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n$

Derivative of f at x in the direction $v \in \mathbb{R}^n$.

$$\lim_{h \to 0} \frac{f(x+hv) - f(x)}{h}$$

Limaye
Ghorpade

For $\mathbb{R}^2$; $x = (x_1, x_2)$; $v = (1, 0)$

$x + hv = (x_1 + h, x_2)$

Definition :

Let $f: \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a function.
We say that f is differentiable at a point $(x, y) \in \mathbb{R}^2$ if the total increment of f, defined as,

$$[z = f(x, y)]$$

$$\Delta z = f(x+\Delta x, y+\Delta y) - f(x, y)$$

$$= \overset{A}{\boxed{f_x(x, y)}}\,\Delta x + \overset{B}{\boxed{f_y(x, y)}}\,\Delta y$$

$$+ \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

where $\varepsilon_1, \varepsilon_2 \longrightarrow 0$ as $(\Delta x, \Delta y) \to (0, 0)$

If f is differentiable at every (x, y) in $\mathbb{R}^2$, then f is said to be a differentiable function.

Differentiability of f at (x,y) implies

$\rightarrow$ $f_x(x,y)$ and $f_y(x,y)$ exist.

$\rightarrow$ f is continuous at (x,y).

<u>Theorem</u>:

Let $z = f(x,y)$ be a function from $\mathbb{R}^2$ to $\mathbb{R}$ with continuous partial derivatives at a point $(x,y) \in \mathbb{R}^2$. Then $f(x,y)$ is differentiable at (x,y).

<u>Proof</u>:

$$\Delta z = f(x+\Delta x, y+\Delta y) - f(x,y)$$

$$= \underline{f(x+\Delta x, y+\Delta y) - f(x, y+\Delta y)}$$
$$+ \underline{f(x, y+\Delta y) - f(x,y)}$$

Note: $f(x, y+\Delta y) - f(x,y) = \Delta y \cdot \dfrac{\partial f}{\partial y}(x, \bar{y})$

where $\bar{y}$ lies between y & $y+\Delta y$

Similarly,

$$f(x+\Delta x, y+\Delta y) - f(x, y+\Delta y)$$

$$= \Delta x \frac{\partial f}{\partial x}(\bar{x}, y+\Delta y)$$

where $\bar{x}$ is between x & $x+\Delta x$

$\therefore$ the total increment in z can be written as

$$\Delta z = f(x+\Delta x, y+\Delta y) - f(x,y)$$

$$= \Delta x \frac{\partial f}{\partial x}(\bar{x}, y+\Delta y) + \Delta y \frac{\partial f}{\partial y}(x, \bar{y})$$

Continuity of partial derivatives

$$\Rightarrow \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{\partial f}{\partial x}(\bar{x}, y+\Delta y) = \frac{\partial f}{\partial x}(x,y)$$

$$\& \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{\partial f}{\partial y}(x, \bar{y}) = \frac{\partial f}{\partial y}(x,y)$$

$(*)$

Rewriting eq^{ns} in (A)

$$\frac{\partial f}{\partial x}(\bar{x}, y+\Delta y) = \frac{\partial f}{\partial x}(x,y) + \varepsilon_1$$

and $$\frac{\partial f}{\partial y}(x, \bar{y}) = \frac{\partial f}{\partial y}(x,y) + \varepsilon_2$$

with the condition that

$$\varepsilon_1, \varepsilon_2 \to 0 \quad \text{as} \quad \underbrace{(\Delta x, \Delta y) \to (0,0)}$$

$$\Updownarrow$$

$$\sqrt{(\Delta x)^2 + (\Delta y^2)} \to 0$$

$$\varepsilon_1 \Delta x \to 0$$
$$\varepsilon_2 \Delta y \to 0$$

$$\boxed{\frac{\varepsilon_1 \Delta x}{\sqrt{(\Delta x)^2 + (\Delta y^2)}}} \to 0$$

as $(\Delta x, \Delta y) \to (0,0)$

$$\Delta z = f(x+\Delta x, y+\Delta y) - f(x,y)$$

$$= \Delta x \frac{\partial f}{\partial x}(x,y) + \Delta y \frac{\partial f}{\partial y}(x,y) + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

with $\varepsilon_1, \varepsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0,0)$ ∎