Maths I

Lecture 18

Extreme values and saddle points.

Definitions. Let $f(x,y)$ be defined on a region R containing the point (a,b). Then

1) $f(a,b)$ is a local maximum value of f if $f(a,b) \geq f(x,y)$ for all domain points (x,y) in an open disk centered at (a,b).

2) $f(a,b)$ is a local minimum value of f if $f(a,b) \leq f(x,y)$ for all domain points (x,y) in an open disk centered at (a,b).

Theorem : First derivative test.

If $f(x,y)$ has a local maximum or minimum value at an interior point (a,b) of its domain and if the first partial derivatives exist at (a,b),

then $f_x(a,b) = 0 = f_y(a,b)$

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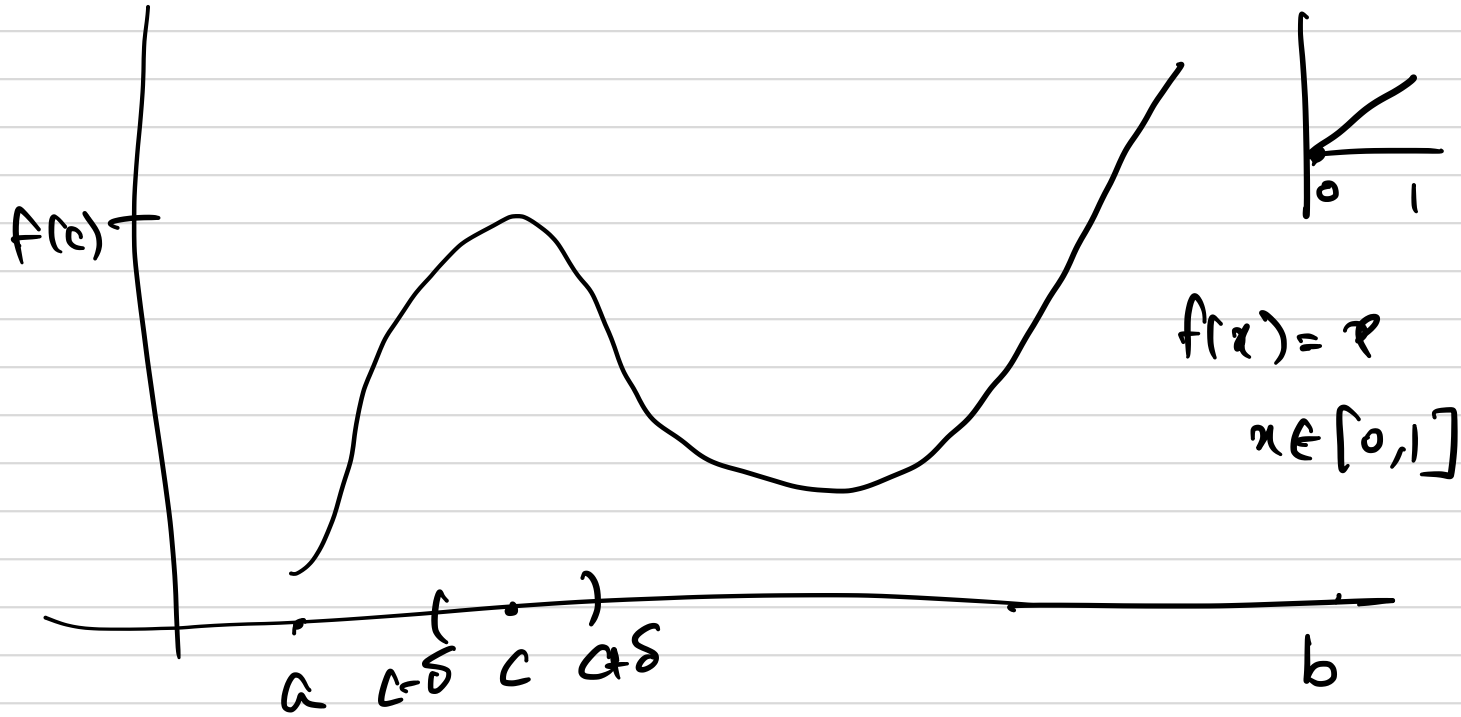
case of 1 variable calculus.



$f(x) = x$

$x \in [0,1]$

$(c-\delta, c+\delta)$ $\quad \forall x \in (c-\delta, c+\delta)$

$f(x) \leq f(c)$

Proof: If f has a local extremum (local maximum or local minimum) at (a,b), then define the function

$$g(x) = f(x, b).$$

Then $g(x)$ has a local extremum at $x = a$. $g'(a) = 0 \Rightarrow f_x(a,b) = 0$.

Similarly, we can prove that

$f_y(a,b) = 0$ ∎

Definition: An interior point of the domain of the function $f(x,y)$ where both f_x and f_y are zero, or where one or both of f_x & f_y do not exist is called a critical point of f.

<u>Definition</u>: A differentiable function $f(x,y)$ has a saddle point at a critical point (a,b) if in every open disk centered at (a,b) there are domain points (x,y) for which $f(x,y) > f(a,b)$ and domain points (x,y) for which $f(x,y) < f(a,b)$.

The corresponding point $(a,b,f(a,b))$ on the surface $z = f(x,y)$ is called as a saddle point.

Ex: $f(x,y) = x^2 + y^2 - 4y + 9$

critical point : $(0,2)$

Ex: $f(x,y) = 10xye^{-(x^2+y^2)}$

$f_x = 10ye^{-(x^2+y^2)} + 10xye^{-(x^2+y^2)}(-2x)$

$= (10y - 20x^2y)\, e^{-(x^2+y^2)}$

$f_y =$

$f_x = 0 \Rightarrow 10(y - 2x^2y)\, e^{-(x^2+y^2)} = 0$

$\Rightarrow y(1-2x^2) = 0$

Critical points:

$(0,0)$, $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$, $\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$, $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$,

$\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$