Maths I

Lecture 17

Notation:

$$\boxed{\left(h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y} \right)} f$$

$$= h \frac{\partial f}{\partial x} + k \frac{\partial f}{\partial y}$$

$$\left(h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y} \right)^2 = h^2 \frac{\partial^2}{\partial x^2} + 2hk \frac{\partial^2}{\partial x \partial y} + k^2 \frac{\partial^2}{\partial y^2}$$

$$\left(h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y} \right)^2 f = h^2 \frac{\partial^2 f}{\partial x^2} + 2hk \frac{\partial^2 f}{\partial x \partial y} + k^2 \frac{\partial^2 f}{\partial y^2}$$

$$\frac{d^n}{dt^n} F(t) = \left(h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y} \right)^n f(x,y)$$

$$= h^n \frac{\partial^n}{\partial x^n} + \binom{n}{1} h^{n-1} k \frac{\partial^n}{\partial x^{n-1} \partial y} + \binom{n}{2} h^{n-2} k^2 \frac{\partial^n}{\partial x^{n-2} \partial y^2}$$

Taylor's formula at point (a,b) for $f(x,y)$.

Suppose $f(x,y)$ and its partial derivatives through order $(n+1)$ are continuous throughout a open region R containing point (a,b). Then in R

$$f(a+h, b+k) = f(a,b) + \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)f\Big|_{(a,b)} + \frac{1}{2!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f\Big|_{(a,b)} + \frac{1}{3!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^3 f\Big|_{(a,b)} + \cdots + \frac{1}{n!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^n f\Big|_{(a,b)}$$

Approximation upto order n' derivative

$$+ \frac{1}{(n+1)!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^{n+1} f\Big|_{(a+ch,\, b+ck)}$$

Remainder

Taylor's formula at $(0,0)$

$$f(x,y) = f(0,0) + x f_x + y f_y$$

$$+ \frac{1}{2!}\left(x^2 f_{xx} + 2xy f_{xy} + y^2 f_{yy}\right)$$

$$+ \frac{1}{3!}\left(x^3 f_{xxx} + 3x^2 y f_{xxy} + 3xy^2 f_{xyy} + y^3 f_{yyy}\right)$$

$$+ \cdots +$$

$$+ \frac{1}{n!}\left(x^n \frac{\partial^n f}{\partial x^n} + n x^{n-1} y \frac{\partial^n f}{\partial x^{n-1}\partial y} + \cdots + y^n \frac{\partial^n f}{\partial y^n}\right)$$

remainder

$$+ \frac{1}{(n+1)!}\left(x^{n+1} \frac{\partial^{n+1} f}{\partial x^{n+1}} + (n+1) x^n y \frac{\partial^{n+1} f}{\partial x^n \partial y} + \cdots + y^{n+1} \frac{\partial^{n+1} f}{\partial y^{n+1}}\right)\Bigg|_{(cx,cy)}$$

Ex: Find a quadratic approximation to $f(x,y) = \sin x \sin y$ near the origin. How accurate is this approximation if $|x| \leq 0.1$, $|y| \leq 0.1$?

Soln:

$$f(x,y) = f(0,0) + (x f_x + y f_y) + \frac{1}{2!}(x^2 f_{xx} + 2xy f_{xy} + y^2 f_{yy}) + \frac{1}{3!}\left(x^3 f_{xxx} + 3x^2 y f_{xxy} + 3xy^2 f_{xyy} + y^3 f_{yyy}\right)\Big|_{(cx,cy)}$$

$f(0,0) = 0$

$f_x(0,0) = \cos x \sin y \Big|_{(0,0)} = 0$

$f_y(0,0) = 0$

$f_{xx}(0,0) = 0$, $f_{xy}(0,0) = 1$, $f_{yy}(0,0) = 0$

$f_{xxx} = -\cos x \sin y$

$f_{xxy} = -\sin x \cos y$

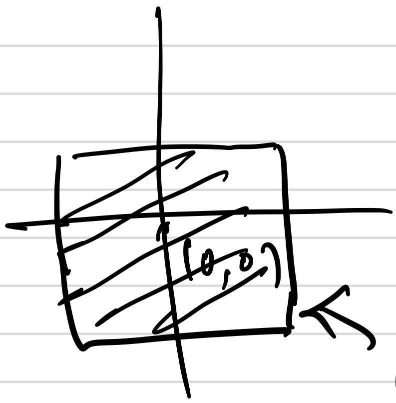
$f_{xyy} = -\cos x \sin y$

$f_{yyy} = -\sin x \cos y$

$\sin x . \sin y = 0 + 0 + 0 + \frac{1}{2!} \left(x^2 . 0 + 2xy + y^2 . 0 \right)$

$+$ remainder

$\sin x . \sin y \approx xy$



$|x| \leq 0.1$

$|y| \leq 0.1$

Remainder

$$= \frac{1}{6}\left(x^3 f_{xxx} + 3x^2y f_{xxy} + 3xy^2 f_{xyy} + y^3 f_{yyy}\right)\Big|_{(cx,cy)}$$

|Remainder|

$$\leq \frac{1}{6}\left[|x^3| + 3x^2|y| + 3|x|y^2 + |y^3|\right]$$

$$\left(\text{when} \quad \underset{10^{-1}}{|x| < 0.1}, \quad \underset{10^{-1}}{|y| < 0.1}\right)$$

|Remainder|

$$\leq \frac{1}{6}\left[10^{-3} + 3\,10^{-3} + 3\,10^{-3} + 10^{-3}\right]$$

$$= \frac{8}{6} \times 10^{-3}$$

Ex: Use Taylor's formula to find quadratic approximation of

$f(x,y) = e^{x} \sin y$

at the origin. Estimate the error in approximation if $|x| \leq 0.1$ and $|y| \leq 0.1$.

Further, compute approximate value of $e^{-0.2} \sin 0.1 \simeq 0.08$

$e^{x} \sin y \simeq y + xy$

$|\text{Remainder}| < \frac{4}{3} 10^{-3} e^{0.1}$

Find cubic approximation of

$$f(x,y) = \frac{1}{1-x-y}$$

at $(0,0)$.