Maths 1

Lecture 4

Application of CMVT

Theorem: L'Höpital rule:

Suppose that $f(a) = g(a) = 0$ and f & g are differentiable functions on an open interval I containing the point a. Further $g'(x) \neq 0$ on I whenever $x \neq a$. Then

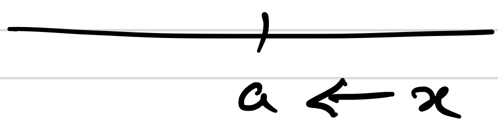
$$\lim_{x \to a} \frac{f(x)}{g(x)} = \underline{\lim_{x \to a} \frac{f'(x)}{g'(x)}}$$

assuming that the limit on the right hand side exists.

Proof:

Consider the case $x \to a^+$ (RHL)

$x > a$

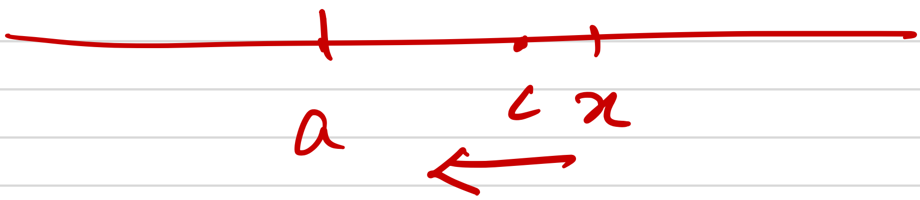


Note that (a, x) ; $g'(x) \neq 0$ and hence f & g satisfy the conditions of CMVT.

Then $\exists$ a point $c \in (a, x)$ such that

$$\frac{f'(c)}{g'(c)} = \frac{f(x) - f(a)}{g(x) - g(a)}$$

$$\frac{f'(c)}{g'(c)} = \frac{f(x)}{g(x)} \qquad \because f(a) = g(a) = 0$$



As x apporaches a, c also approaches a since $a < c < x$.

Therefore $$\lim_{x \to a^+} \frac{f(x)}{g(x)} = \lim_{c \to a^+} \frac{f'(c)}{g'(c)} \quad \star$$

$$\lim_{x \to a^+} \frac{f(x)}{g(x)} = \lim_{x \to a^+} \frac{f'(x)}{g'(x)} \quad \star$$

Similarly, in case of left hand limit CMVT for functions f & g on $[x, a]$ will prove that

$$\lim_{x \to a-} \frac{f(x)}{g(x)} = \lim_{x \to a-} \frac{f'(x)}{g'(x)}$$

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Problems:

$$\lim_{\theta \to 0} \frac{\cos\theta - 1}{e^{\theta} - \theta - 1}$$

$$\lim_{x \to \infty} (\ln x)^{1/x}$$

$$\lim_{\theta \to 0} \frac{3^{\sin\theta} - 1}{\theta}$$

$$\lim_{x\to\infty} (\ln x)^{1/x} = L$$

$$\ln\left(\lim_{x\to\infty} (\ln x)^{1/x}\right) = \ln L$$

$$\lim_{x\to\infty} \ln (\ln x)^{1/x} = \ln L$$

$$f\left(\lim_{x\to a} \bigcirc\right) = \lim_{x\to a} f(\bigcirc)$$

Limits & Continuity of functions of 1 variable.

$$f : [a,b] \longrightarrow \mathbb{R}$$

$$\lim_{x \to c} f(x) = l \qquad \text{— limit}$$

$$\lim_{x \to c} f(x) = f(c) \qquad \text{— Continuity at } c.$$

$$\lim_{x \to c} f(x) \overset{\text{"}}{=} \lim_{h \to 0} f(c+h) = f\left[\lim_{h \to 0} c+h\right]$$

$x \to \infty$

$(\ln x)^{1/x}$

$\ln$

$x \to \infty$

$\ln$

$$\lim_{h \to 0} f(c+h) = \lim_{x \to c} f(x)$$

$$\lim_{h \to 0} f(c+h) = f(c) = f\left(\lim_{h \to 0} c+h\right)$$

Derivatives of a function of 1 variable.

$f : (a,b) \longrightarrow \mathbb{R}$; $c \in (a,b)$

$$\lim_{h \to 0} \frac{f(c+h) - f(c)}{h} = f'(c)$$

$$f(c+h) = \boxed{f(c) + h\,l} + \phi(h)$$

where l is a constant and $\phi(h)$ is such that

$$\lim_{h \to 0} \frac{\phi(h)}{h} = 0$$

$$f(c+h) - f(c) = h\,l + \phi(h)$$

$$\frac{f(c+h) - f(c)}{h} = l + \frac{\phi(h)}{h}$$