

Maths 1

Lecture 4



Application of CMVT

Theorem: L'Hôpital rule:

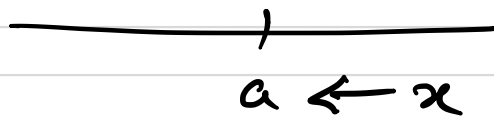
Suppose that $f(a) = g(a) = 0$ and f & g are differentiable functions on an open interval I containing the point a . Further $g'(x) \neq 0$ on I whenever $x \neq a$. Then

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$$

assuming that the limit on the right hand side exists.

Proof:

Consider the case $x \rightarrow a^+$ (RHL)
 $x > a$

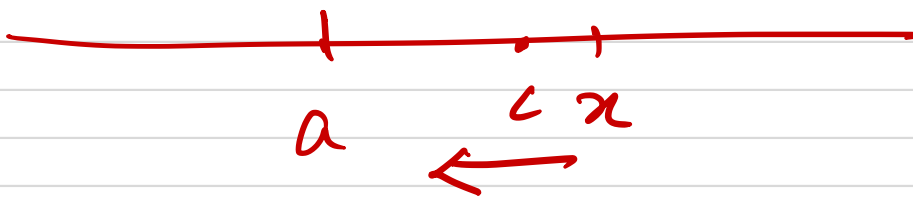


Note that (a, x) ; $g'(x) \neq 0$ and hence f & g satisfy the conditions }
CMVT.

Then \exists a point $c \in (a, x)$ such that

$$\frac{f'(c)}{g'(c)} = \frac{f(x) - f(a)}{g(x) - g(a)}$$

$$\frac{f'(c)}{g'(c)} = \frac{f(x)}{g(x)} \quad \because f(a) = g(a) = 0$$



As x approaches a , c also approaches a

Since $a < c < x$.

Therefore $\lim_{x \rightarrow a^+} \frac{f(x)}{g(x)} = \lim_{c \rightarrow a^+} \frac{f'(c)}{g'(c)}$ ★

$$\lim_{x \rightarrow a^+} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a^+} \frac{f'(x)}{g'(x)} \quad \star$$

Similarly, in case of left hand limit
CMVT for functions f & g on $[a, x]$
will prove that

$$\lim_{x \rightarrow a-} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a-} \frac{f'(x)}{g'(x)}$$

□

Problems :

$$\lim_{\theta \rightarrow 0} \frac{\cos \theta - 1}{e^{\theta} - \theta - 1}$$

$$\lim_{x \rightarrow \infty} (\ln x)^{1/x}$$

$$\lim_{\theta \rightarrow 0} \frac{3^{\sin \theta} - 1}{\theta}$$

$$\lim_{x \rightarrow \infty} (\ln x)^{1/x} = L$$

$$\ln \left(\lim_{x \rightarrow \infty} (\ln x)^{1/x} \right) = \ln L$$

$$\downarrow$$

$$\lim_{x \rightarrow \infty} \ln \left((\ln x)^{1/x} \right) = \ln L$$

$$f \left(\lim_{x \rightarrow a} \textcircled{L} \right) = \lim_{x \rightarrow a} f(x)$$

Limits & Continuity of functions of 1 variable.

$$f : [a, b] \rightarrow \mathbb{R}$$

$$\lim_{x \rightarrow c} f(x) = L \quad \text{--- limit}$$

$$\lim_{x \rightarrow c} f(x) = f(c) \quad \text{--- Continuity at } c.$$

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$$\lim_{\substack{h \rightarrow 0 \\ x \rightarrow c}} f(c+h) = f\left[\lim_{\substack{h \rightarrow 0 \\ x \rightarrow c}} c+h\right]$$

Diagram illustrating the relationship between the limit of a function and the function of a limit. The left side shows the limit of $f(c+h)$ as $h \rightarrow 0$ (with $x \rightarrow c$ indicated). The right side shows the function f applied to the limit of $c+h$ as $h \rightarrow 0$ (with $x \rightarrow c$ indicated). Arrows indicate the mapping from h to x and from the limit expression to the function argument.

$$\lim_{h \rightarrow 0} f(c+h) = \lim_{x \rightarrow c} f(x)$$

Diagram illustrating the relationship between the limit of a function and the function of a limit. The left side shows the limit of $f(c+h)$ as $h \rightarrow 0$. The right side shows the limit of $f(x)$ as $x \rightarrow c$. An arrow indicates the mapping from h to x .

$$f(c) = f\left(\lim_{h \rightarrow 0} c+h\right)$$

Derivatives of a function of 1 variable.

$$f: (a, b) \rightarrow \mathbb{R} \quad ; \quad c \in (a, b)$$

$$\lim_{h \rightarrow 0} \frac{f(c+h) - f(c)}{h} = f'(c)$$

$$f(c+h) = \boxed{f(c) + h \ell} + \phi(h)$$

where ℓ is a constant and $\phi(h)$ is such that

$$\lim_{h \rightarrow 0} \frac{\phi(h)}{h} = 0$$

$$f(c+h) - f(c) = h \ell + \phi(h)$$

$$\frac{f(c+h) - f(c)}{h} = \ell + \frac{\phi(h)}{h}$$