Maths 1

Lecture 6

Taylor's formula.

If f has derivatives of all orders in an open interval I containing point a, then for each positive integer n and every $x \in I$,

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n(x)$$

where $$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

for some $c \in (a, x)$ or (x, a)

Particular case: $a = 0$ (Maclaurin's formula)

Ex: $f(x) = \sin x \qquad a = \pi$

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2$$

$$+ \frac{f'''(a)}{3!}(x-a)^3 + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$$+ R_n(x)$$

$f(x) = \sin x \qquad f(\pi) = 0$

$f'(x) = \cos x \qquad f'(\pi) = -1$

$f''(x) = -\sin x \qquad f'(\pi) = 0$

$f'''(x) = -\cos x \qquad f'''(\pi) = 1$

$f^{IV}(x) = \sin x \qquad f^{IV}(\pi) = 0$

$f^{V}(x) = \cos x \qquad f^{V}(\pi) = -1$

$$\sin x = 0 - 1(x-\pi) + 0 + \frac{1}{3!}(x-\pi)^3$$

$$- \frac{1}{5!}(x-\pi)^5 + R_5(x)$$

$$\sin x \approx -(x-\pi) + \frac{1}{3!}(x-\pi)^3 - \frac{1}{5!}(x-\pi)^5$$

Similarly for $a=0$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + R_5(x)$$

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Ex: Consider $f(x) = \frac{1}{1-x}$.

Find Maclaurin's formula for $f(x)$ of any degree n.

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + R_n(x)$$

$$f(x) = \frac{1}{1-x} = (1-x)^{-1} \qquad f(0) = 1$$

$$f'(x) = (1-x)^{-2} \qquad f'(0) = 1$$

$$f''(x) = 2(1-x)^{-3} \qquad f''(0) = 2$$

$$f'''(x) = 3!\,(1-x)^{-4} \qquad f'''(0) = 3!$$

$$\vdots \qquad \vdots$$

$$f^{(n)}(x) = n!\,(1-x)^{-n-1} \qquad f^{(n)}(0) = n!$$

Maclaurin's formula

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots + x^n + R_n(x)$$

This formula is valid for an interval I containing 0 and not containing $x = 1$.

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!} x^{n+1}$$

Here, $R_n(x) = \dfrac{x^{n+1}}{(1-c)^{n+2}}$ for c between 0 & x

$$R_n(x) = \frac{x^{n+1}}{(1-c)^{n+2}} \quad \text{for } |x| < 0.1$$

where c is between 0 & x.