Maths 1

Lecture 7

Taylor's formula.

If f has all its derivatives in an open interval I containing point a, then for every integer n, and $x \in I$,

$$\boxed{f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n(x)}$$

where $R_n(x) = \dfrac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$

for some c is between a & x.

If $a = 0$, Maclaurin's formula

If $R_n(x) \to 0$ as $n \to \infty$ for all $x \in I$, we say that the Taylor series generated by f at $x = a$ is convergent to f on I.

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

Ex: Show that Taylor series generated by $f(x) = e^x$ at $a=0$ converges to $f(x)$ for every real number x.

Soln:

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + R_n(x)$$

for any $x \in \mathbb{R}$ and $n \in \mathbb{N}$

$$R_n(x) = \frac{e^c}{(n+1)!} x^{n+1}$$ where c is between 0 & x.

observe: e^x is an increasing function.

e^c lies between $e^0 = 1$ and e^x.

When $x = 0$, $R_n(x) = 0$

when $x > 0 \Rightarrow c > 0$ and $e^c < e^x$

$$|R_n(x)| \leq \frac{|x|^{n+1}}{(n+1)!}$$ for $x \leq 0$

$e^c < 1$

and $|R_n(x)| \le e^x \dfrac{x^{n+1}}{(n+1)!}$ | for $x>0$ $e^c < e^x$

Since for any $x \in \mathbb{R}$

$$\frac{x^{n+1}}{(n+1)!} \to 0 \quad \text{as} \quad n \to \infty$$

$$R_n(x) \to 0 \quad \text{as} \quad n \to \infty$$

$\Rightarrow$ Taylor's series of $f(x) = e^x$ at $a=0$ is convergent.

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

In particular, for $x=1$,

$$e = 1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{n!} + R_n(1)$$

Ex: For what values of x can we replace $\sin x$ by $x - \frac{x^3}{3!}$ with an error of magnitude not greater than 3×10^{-4}?

Solution:

$$\sin x = x - \frac{x^3}{3!} + R_3(x)$$

where $R_3(x) = \frac{\sin c}{4!} x^4$

c is between 0 & x.

$$|R_3(x)| \leq \frac{x^4}{4!} = 3 \times 10^{-4}$$

$$x^4 \leq 72 \times 10^{-4}$$

$$x \leq \sqrt[4]{72 \times 10^{-4}}$$

$$\boxed{|x| < 0.2912}$$

Ex: Using Maclurin's formula approximate function $f(x) = \sqrt{1+x}$ as a linear polynomial. Estimate the error when $|x| < 0.01$

Ex: Estimate the error if

$$P_4(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24}$$

is used to estimate the value of e^x at $x = 1/2$.

Ex: Write Maclaurin formula for the function $f(x) = (\sin x)^2$ for $n = 3$ and explicitely write the form of the remainder.

$$f(x) = 1 + \frac{x}{2} + R_1(x)$$

$$|R_1(x)| = \left| -\frac{1}{8} \times x^2 \times (1+c)^{-3/2} \right|$$

$$c \in (0, x)$$

for $|x| < 0.01$

$$|R_1(x)| \leq \frac{10^{-4}}{8}$$