Maths I

Lecture 14

Differentiability of function of 2-variables.

Let $f: \mathbb{R}^2 \to \mathbb{R}$ be a function. We say f is differentiable at a point $(a,b) \in \mathbb{R}^2$ if

$$f(a+h, b+k) - f(a,b) = hf_x(a,b) + kf_y(a,b) + \varepsilon_1 h + \varepsilon_2 k$$

where $\varepsilon_1, \varepsilon_2 \to 0$ as $(h,k) \to (0,0)$

We say f is differentiable everywhere if it is differentiable at every point $(a,b) \in \mathbb{R}^2$.

We proved 2 results.

1) $f: \mathbb{R}^2 \to \mathbb{R}$ is differentiable at (a,b), then f is continuous at (a,b).

2) If $f: \mathbb{R}^2 \to \mathbb{R}$ is such that it has continuous partial derivatives at (a,b), then f is differentiable at (a,b).

Result:

$f : \mathbb{R}^2 \to \mathbb{R}$ be differentiable at (a,b), if and only if

$$\lim_{(h,k)\to(0,0)} \frac{f(a+h,b+k) - f(a,b) - hf_x(a,b) - kf_y(a,b)}{\sqrt{h^2+k^2}}$$

$$= 0$$

Proof:

$f : \mathbb{R}^2 \to \mathbb{R}$ such that f is differentiable at (a,b), then

$$\boxed{\begin{aligned} f(a+h,b+k) - f(a,b) &= hf_x(a,b) + kf_y(a,b) \\ &\quad + \varepsilon_1 h + \varepsilon_2 k \end{aligned}}$$

where $\varepsilon_1, \varepsilon_2 \to 0$ as $(h,k) \to (0,0)$

$$\Updownarrow$$

$$h^2+k^2 \to 0$$

$$\Rightarrow \frac{f(a+h, b+k) - f(a,b) - hf_x(a,b) - kf_y(a,b)}{\sqrt{h^2+k^2}}$$

$$= \frac{h}{\sqrt{h^2+k^2}} \varepsilon_1 + \frac{k}{\sqrt{h^2+k^2}} \varepsilon_2$$

where $\varepsilon_1, \varepsilon_2 \to 0$ as $(h,k) \to (0,0)$

Now observe that

$$\frac{|h|}{\sqrt{h^2+k^2}} \leq 1 \quad \text{and} \quad \frac{|k|}{\sqrt{h^2+k^2}} \leq 1$$

$$\Rightarrow \lim_{(h,k)\to(0,0)} \frac{f(a+h, b+k) - \boxed{f(a,b)} - \boxed{hf_x(a,b) - kf_y(a,b)}}{\sqrt{h^2+k^2}}$$

$$= 0$$

$$\lim_{h \to 0} h^2 e^{\frac{1}{h^2}} \qquad \lim_{\substack{r \to 0 \\ \theta \text{ any value}}} r(\cos\theta + \sin\theta)$$

$$= 0$$

Ex: Discuss differentiability of f at $(0,0)$:

$$f(x,y) = (x^2+y^2)\sin\left(\frac{1}{x^2+y^2}\right) \quad (x,y) \neq (0,0)$$

$$= 0 \quad (x,y) = (0,0)$$

Soln:

$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{h^2\sin\left(\frac{1}{h^2}\right) - 0}{h}$$

$$= \lim_{h\to 0} h\sin\left(\frac{1}{h^2}\right)$$

$$= 0$$ since $\sin\left(\frac{1}{h^2}\right)$ is a bounded function.

$$f_y(0,0) = 0$$

$$\lim_{(h,k)\to(0,0)} \frac{f(h,k)-f(0,0)-hf_x(0,0)-kf_y(0,0)}{\sqrt{h^2+k^2}}$$

$$= \lim_{(h,k)\to(0,0)} \frac{(h^2+k^2)\sin\left(\frac{1}{h^2+k^2}\right)-0-0-0}{\sqrt{h^2+k^2}}$$

$$= \lim_{\substack{r\to 0 \\ \theta \text{ any value}}} r \sin\left(\frac{1}{r^2}\right)$$

$= 0$ since $\left|\sin\left(\frac{1}{r^2}\right)\right| \leq 1$ for any r.

$\Rightarrow f$ is differentiable at $(0,0)$.

Ex: Let $f(x,y) = \begin{cases} \dfrac{xy^2}{x^2+y^4} & (x,y)\neq(0,0) \\ 0 & (x,y)=(0,0) \end{cases}$

Discuss differentiability of f at $(0,0)$.

Ex:

$$f(x,y) = \begin{cases} \dfrac{x^3-y^3}{x^2+y^2} & x^2+y^2 \neq 0 \\ 0 & x^2+y^2 = 0 \end{cases}$$

At $(0,0)$

$$f_x(0,0) = \lim_{h\to 0} \frac{h^3/h^2}{h}$$

$$= 1$$

$$f_y(0,0) = \lim_{k\to 0} \frac{-k^3/k^2}{k}$$

$$= -1$$

$$\lim_{(h,k)\to(0,0)} \frac{f(h,k) - f(0,0) - hf_x(0,0) - kf_y(0,0)}{\sqrt{h^2+k^2}}$$

$$= \lim_{(h,k)\to(0,0)} \frac{\dfrac{h^3-k^3}{h^2+k^2} - h + k}{\sqrt{h^2+k^2}}$$

$$= \lim_{\substack{r\to 0 \\ \theta \text{ any value}}} \frac{\dfrac{r^3\cos^3\theta - r^3\sin^3\theta}{r^2} - r(\cos\theta - \sin\theta)}{r}$$

The limit dependends on value of θ.

$\therefore$ limit depends on path & hence does not exist.

$\therefore$ $f(x,y)$ is <u>NOT</u> differentiable at $(0,0)$.

<u>Ex:</u>

$$f(x,y) = \sqrt{|xy|}$$