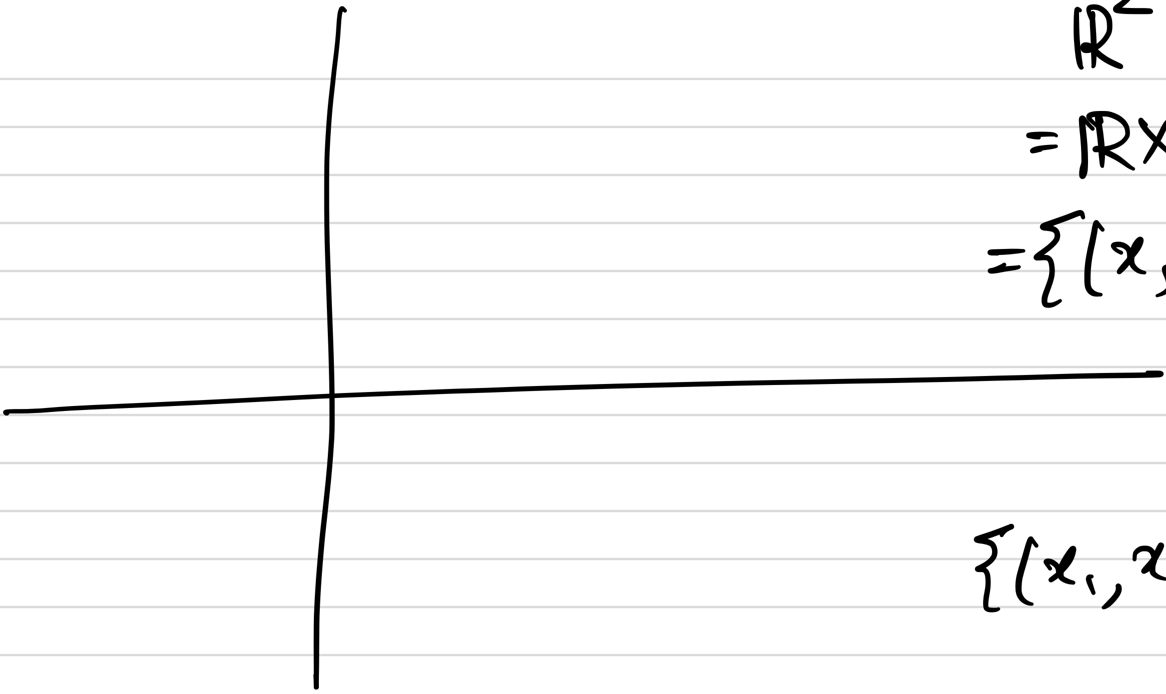
Maths 1

Lecture 8

Calculus of several variables



$\mathbb{R}^2$

$= \mathbb{R} \times \mathbb{R}$

$= \{(x, y) \mid x, y \in \mathbb{R}\}$

$\{(x_1, x_2) \mid x_1, x_2 \in \mathbb{R}\}$

$\mathbb{R}^n$

<u>Definition</u>: Suppose D is a subset of $\mathbb{R}^n$. A real valued function f on D is a rule that assigns a unique real number to every point $x \in D$.

$$f : D \longrightarrow \mathbb{R} \quad ; \quad D \subseteq \mathbb{R}^n$$

$$x = (x_1, x_2, \cdots, x_n) \longmapsto z$$

$$z = f(x) = f(x_1, x_2, \cdots, x_n)$$

Ex:

1) Constant functions.

$D \subseteq \mathbb{R}^n$; $f(x_1, x_2, \dots, x_n) = 0$

2) $D = \mathbb{R}^n$

$f(x_1, \dots, x_n) = x_1$

3) $D = \mathbb{R}^2$

$f(x_1, x_2) = x_1 + x_2$

4) distance functions

i) $D = \mathbb{R}^n$

$f(x_1, x_2, \dots, x_n) = \sqrt{x_1^2 + x_2^2 + \dots + x_n^2}$

ii) $D = \mathbb{R}^n$

$f(x_1, x_2, \dots, x_n) = |x_1| + |x_2| + \dots + |x_n|$

iii) $D = \mathbb{R}^n$

$f(x_1, \dots, x_n) = \left(|x_1|^p + |x_2|^p + \dots + |x_n|^p \right)^{1/p}$

$p \geq 1$

Definition:

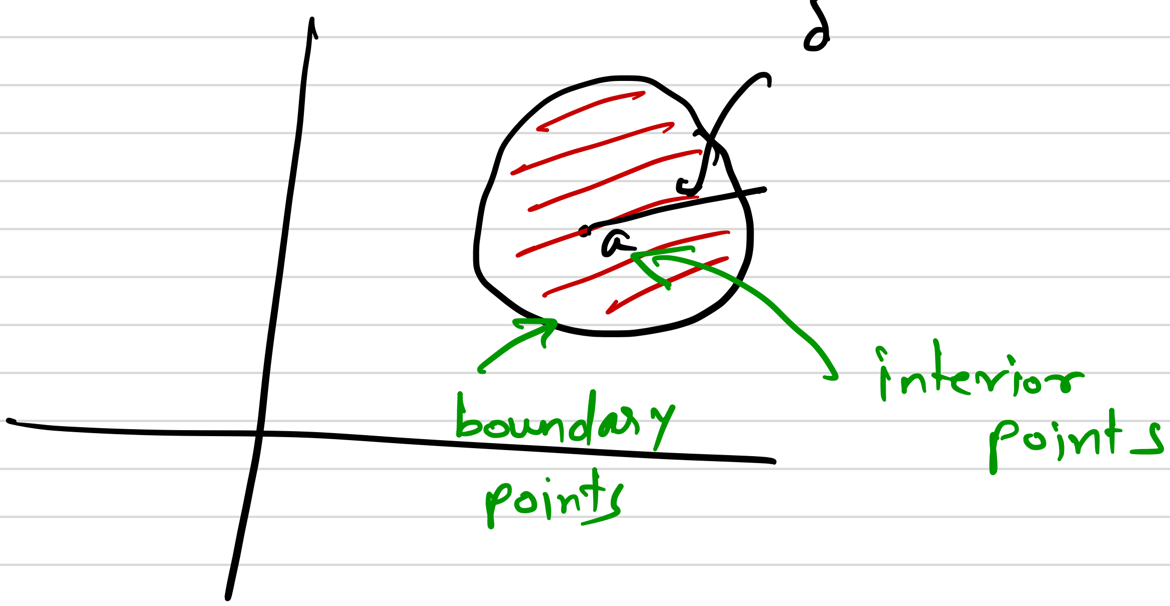
Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a distance function defined as

$$f(x_1, \ldots, x_n) = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$$

Then for a point $a = (a_1, \ldots, a_n) \in \mathbb{R}^n$ open neighbourhood of radius δ is defined as

$$N_\delta(a) = \left\{ x \in \mathbb{R}^n \;\middle|\; \sqrt{(x_1 - a_1)^2 + \cdots + (x_n - a_n)^2} \leq \delta \right\}$$

For $n = 2$



$$N_\delta(a) \subseteq \mathbb{R}^n$$

Definition: Graph of a function

Let $D \subseteq \mathbb{R}^n$ and define a function $f: D \to \mathbb{R}$. Then the graph of the function f is defined as

$$\{(x_1, \dots, x_n, f(x_1, \dots, x_n)) \mid (x_1, \dots, x_n) \in D\}$$

Ex:

$f: D \longrightarrow \mathbb{R}$ where

$D \subseteq \mathbb{R}^2$

$D = N_1(0)$

$$z = f(x, y)$$
$$= \sqrt{1 - x^2 - y^2}$$

Ex:

Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ defined by

$$f(x,y) = x^2 + y^2$$

$$\{(x, y, f(x,y)) \mid (x,y) \in \mathbb{R}^2\} = G(f)$$

Level surfaces:

Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be a function.

for a fixed $c \in \mathbb{R}$, the set

$$L_f(c) = \{(x_1, \dots, x_n) \mid f(x_1, \dots, x_n) = c\}$$

let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$

$$f(x,y) = x^2 + y^2$$

$L_f(-1) = \phi$; $L_f(0) = \{(0,0)\}$

$$L_f(1) = \{(x_1, x_2) \mid x_1^2 + x_2^2 = 1\}$$