Maths I

Lecture 5

Taylor's theorem

Let us consider a function $f(x)$ such that the derivatives upto order $(n+1)$ exist in some interval around a point $x=a$.

Objective: To find a polynomial $P_n(x)$ of degree atmost n such that

$$\left.\begin{aligned} P_n(a) &= f(a) \\ P_n'(a) &= f'(a) \\ P_n''(a) &= f''(a) \\ &\vdots \\ P_n^{(n)}(a) &= f^{(n)}(a) \end{aligned}\right\} \quad (*)$$

Consider the polynomial $P_n(x)$ to be written as:

$$P_n(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots \\ \cdots + c_n(x-a)^n$$

where $c_0, c_1, \ldots, c_n$ are to be determined.

$$P_n(x) = C_0 + C_1(x-a) + C_2(x-a)^2 + \cdots + C_n(x-a)^n$$

$$P_n'(x) = C_1 + 2C_2(x-a) + 3C_3(x-a)^2 + \cdots$$

$$\cdots + nC_n(x-a)^{n-1}$$

$$P_n''(x) = 2C_2 + 3.2\, C_3(x-a) + \cdots$$

$$\cdots + n(n-1)\, C_n(x-a)^{n-2}$$

$$\vdots$$

$$P_{(n)}^{n}(x) = n!\, C_n$$

From (*) it is clear that

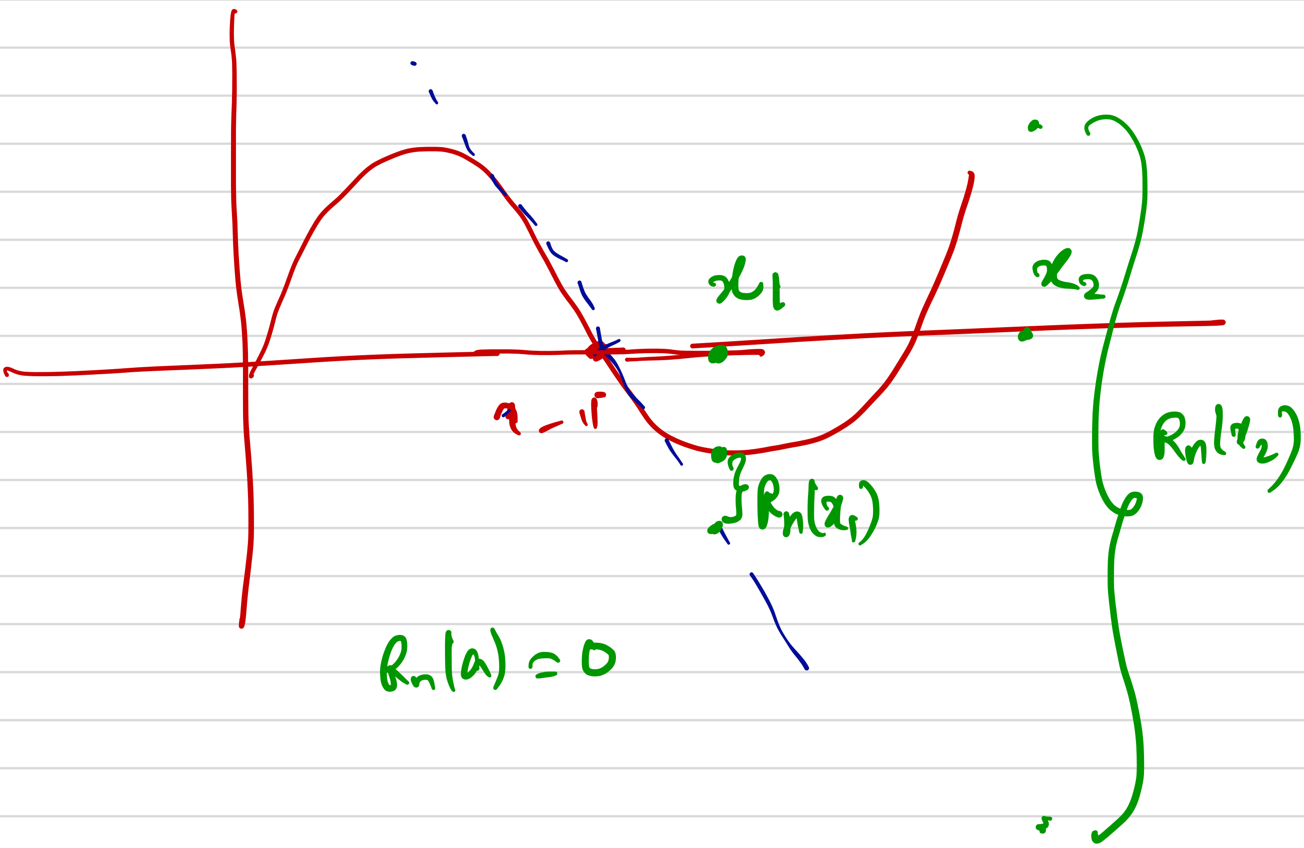
$$\left.\begin{aligned} f(a) &= C_0 \\ f'(a) &= C_1 \\ f''(a) &= 2!\, C_2 \\ f'''(a) &= 3!\, C_3 \\ &\vdots \\ f^{(n)}(a) &= n!\, C_n \end{aligned}\right\} \Rightarrow \begin{aligned} C_0 &= f(a) \\ C_1 &= f'(a) \\ C_2 &= \frac{f''(a)}{2!} \\ C_3 &= \frac{f'''(a)}{3!} \\ &\vdots \\ C_n &= \frac{f^{(n)}(a)}{n!} \end{aligned}$$

Using these expressions for c_i's

$$P_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2$$

$$+ \frac{f'''(a)}{3!}(x-a)^3 + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

This polynomial $P_n(x)$ is called as Taylor polynomial of degree n.



Taylor's theorem.

If f is a function such that $f', f'', \dots, f^{(n)}$ are continuous on the closed interval $[a,b]$ and $f^{(n)}$ is differentiable on (a,b), then there exists a number $c \in (a,b)$ such that

$$f(b) = f(a) + f'(a)(b-a) + \frac{f''(a)}{2!}(b-a)^2$$

$$+ \dots + \frac{f^{(n)}(a)}{n!}(b-a)^n$$

$$+ \frac{f^{(n+1)}(c)}{(n+1)!}(b-a)^{n+1}$$

Proof: The Taylor polynomial at $x=a$

$$P_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2$$

$$+ \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Define $\phi_n(x) = P_n(x) + k(x-a)^{n+1}$

for some constant k.

Clearly, $y = \phi_n(x)$ agree with $y = f(x)$ at $x = a$ along with its first n derivatives.

We choose the constant k such that $\phi_n(b) = f(b)$.

$$f(b) = \phi_n(b) = P_n(b) + k(b-a)^{n+1}$$

$$\Rightarrow \quad k = \frac{f(b) - P_n(b)}{(b-a)^{n+1}}$$

Now define

$$F(x) = f(x) - \phi_n(x)$$

which measures the difference between $f(x)$ and the approximating function $\phi_n(x)$ on $[a,b]$.

Note that $F(a) = F(b) = 0$ and

F is differentiable, then by Rolle's theorem, there exists $c_1 \in (a,b)$ such that

$$F'(c_1) = 0$$

Note that $F'(a) = F'(c_1) = 0$

Apply Rolle's theorem, $\exists\ c_2 \in (a, c_1)$ s.t. $F''(c_2) = 0$

$\vdots$

$c_n \in (a, c_{n-1})$ such that

$$F^{(n)}(c_n) = 0$$

Applying Rolle's theorem once more

$\exists\ c_{n+1} = c \in (a, c_n)$ s.t.

$$\boxed{F^{(n+1)}(c) = 0}$$

$$\left(F^{(n+1)}(x) = f^{(n+1)}(x) - (n+1)!\,K\right)$$

$$\Rightarrow \quad K = \frac{f^{(n+1)}(c)}{(n+1)!}$$

$$f(b) = P_n(b) + \boxed{\frac{f^{(n+1)}(c)}{(n+1)!}(b-a)^{n+1}}$$

Ex