Maths I

Lecture 16

Second order partial derivatives.

$f(x,y)$: given function.

$\frac{\partial f}{\partial x} = f_x$ $\qquad$ $\frac{\partial f}{\partial y} = f_y$: first order

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = f_{xx}$$

$$\boxed{\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)} = \frac{\partial^2 f}{\partial y \partial x} = f_{xy}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} = f_{yx} = (f_y)_x$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = f_{yy}$$

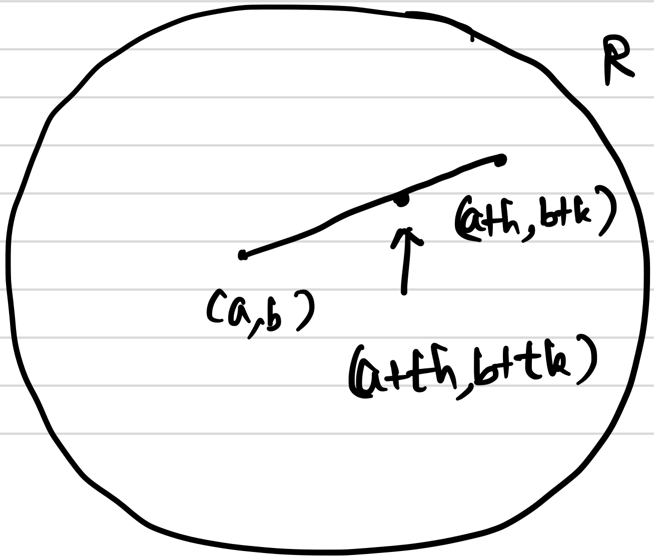
The mixed derivative theorem

If $f(x,y)$ and its partial derivatives f_x, f_y, f_{xy}, f_{yx} are defined throughout an open region containing the point (a,b) and are all continuous at (a,b) then

$$f_{yx}(a,b) = f_{xy}(a,b)$$

Taylor's formula for two variables.

Let $f(x,y)$ have continuous partial derivatives in an open region R containing the point (a,b).



$t \in [0,1]$

$$F(t) = f(\overset{x}{\overset{\shortparallel}{a+th}},\ \overset{y}{\overset{\shortparallel}{b+tk}}) \qquad 0 \le t \le 1.$$

$$F'(t) = \frac{dF}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

$$= f_x\, h + f_y\, k$$

$$= h f_x + k f_y$$

Suppose f_x & f_y also have continuous partial derivatives:

$$F''(t) = \frac{d^2F}{dt^2} = \frac{d}{dt}\left(h f_x + k f_y\right)$$

$$= h \frac{d}{dt} f_x + k \frac{d}{dt} f_y$$

$$= h\left[f_{xx} \cdot h + f_{xy} \cdot k\right]$$

$$+ k\left[f_{yx} \cdot h + f_{yy} \cdot k\right]$$

$$= h^2 f_{xx} + hk f_{xy} + kh f_{yx} + k^2 f_{yy}$$

$f_x(x,y) = f_x(a+th, b+tk)$

$0 \le t \le 1$

$$\frac{d}{dt}(f_x)^{g} = \frac{\partial g}{\partial x}\frac{dx}{dt} + \frac{\partial g}{\partial y}\frac{dy}{dt}$$

$$= h\,(f_x)_x + k\,(f_x)_y$$

$$= h f_{xx} + k f_{xy}$$

Since f_x & f_y have continuous partial derivatives, in particular, $f, f_x, f_y, f_{xy}, f_{yx}$ are continuous in R.

$\therefore \quad f_{xy} = f_{yx}$ in R.

$$F''(t) = h^2 f_{xx} + 2hk f_{xy} + k^2 f_{yy}$$

Note that

F and F' are continuous on $[0,1]$

F' is differentiable on $(0,1)$

$$F(1) = F(0) + F'(0)(1-0) + F''(c)\frac{(1-0)^2}{2!}$$

$$= \underbrace{F(0) + F'(0)}_{} + \frac{F''(c)}{2}$$

linear approximation of F with remainder

where $c \in (0,1)$

However, $F(1) = f(a + 1\cdot h,\ b + 1\cdot k)$

$$= f(a+h, b+k)$$

$\Rightarrow$

$$\boxed{f(a+h, b+k) = \underbrace{f(a,b) + hf_x(a,b) + kf_y(a,b)}_{\text{linear approximation}} + \underbrace{\frac{1}{2}\left(h^2 f_{xx} + 2hk f_{xy} + k^2 f_{yy}\right)\Big|_{(a+ch,\ b+ck)}}_{\text{remainder}}}$$

Let us assume that

$f_{xx}, f_{xy}, f_{yx}, f_{yy}$ are continuous

and have continuous partial derivatives.

in R.

$$F'''(t) = \frac{d^3}{dt^3} F(t)$$

$$= \frac{d}{dt}\left(h^2 f_{xx} + 2hk f_{xy} + k^2 f_{yy}\right)$$

$$= h^3 f_{xxx} + 3h^2k f_{xxy}$$

$$+ 3hk^2 f_{xyy} + k^3 f_{yyy}$$