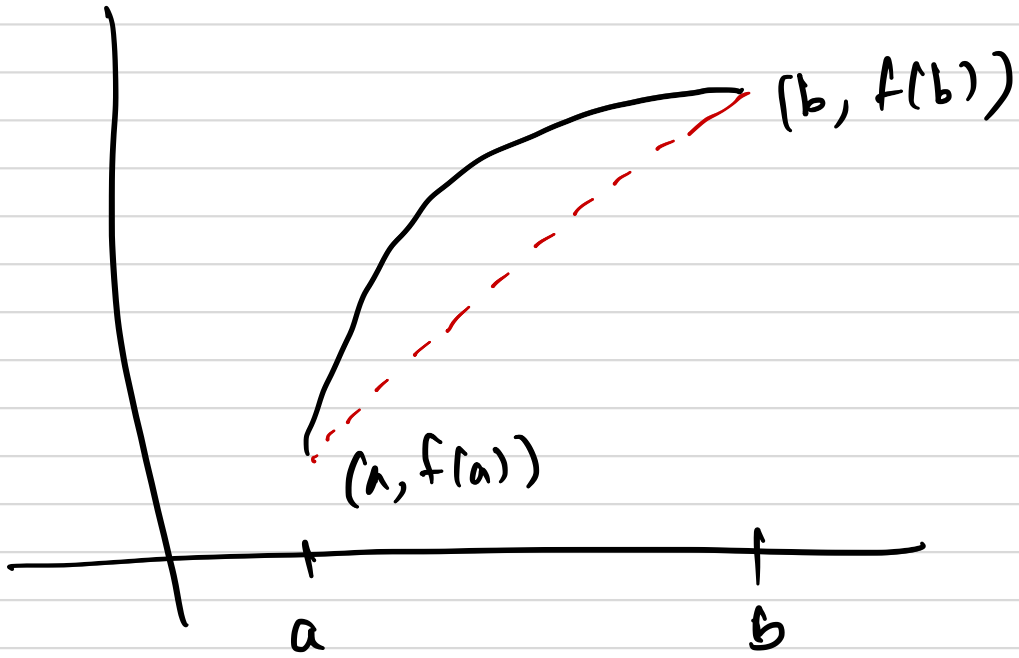
Maths 1

Lecture 3

$f(a) \neq f(b)$ in Rolle's theorem



Lagrange Mean Value Theorem

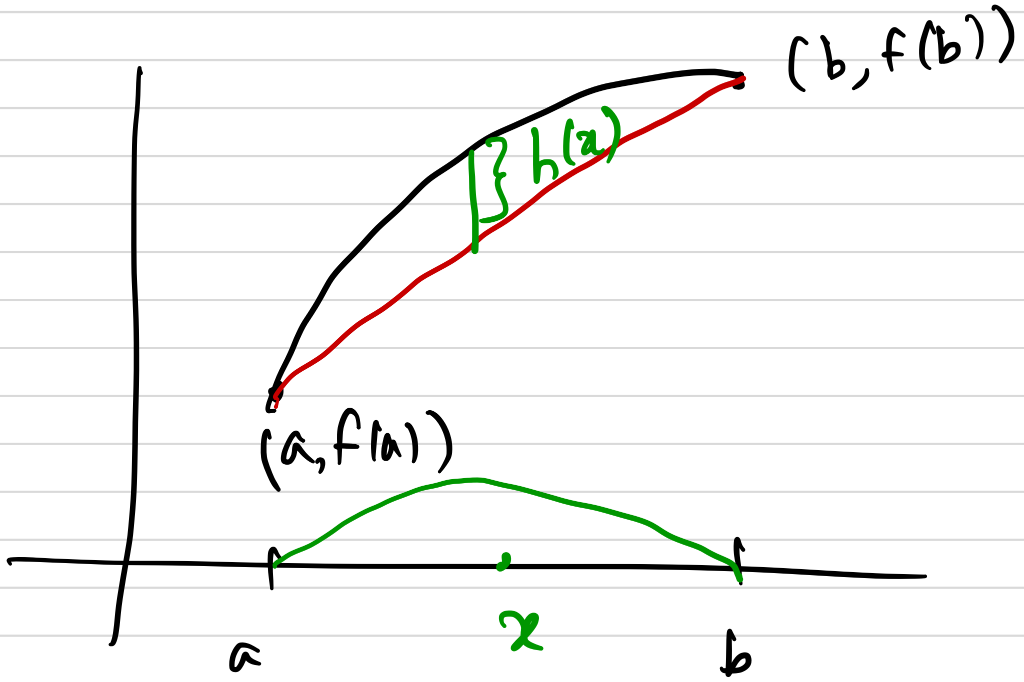
Suppose $y = f(x)$ is continuous over a closed interval $[a,b]$ and differentiable over (a,b). Then there exists at least one point $c \in (a,b)$ such that

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

Proof:

The function whose graph is the line joining points $(a, f(a))$, $(b, f(b))$ is given by

$$g(x) = f(a) + \frac{f(b) - f(a)}{b - a}(x - a)$$



Define $h(x) = f(x) - g(x)$

$$= f(x) - f(a) - \frac{f(b) - f(a)}{b - a}(x - a)$$

Note: $h(a) = h(b) = 0$

$$\left[\, h'(x) = f'(x) - \frac{f(b) - f(a)}{b - a} \,\right]$$

Since $f(x)$ is continuous on $[a,b]$ and differentiable on (a,b) and $g(x)$ is a polynomial function on $[a,b]$; it is clear that $h(x)$ is continuous on $[a,b]$ and differentiable on (a,b)

Thus $h(x)$ satisfies the properties of Rolle's theorem.

$\Rightarrow \exists$ a point $c \in (a,b)$ such that

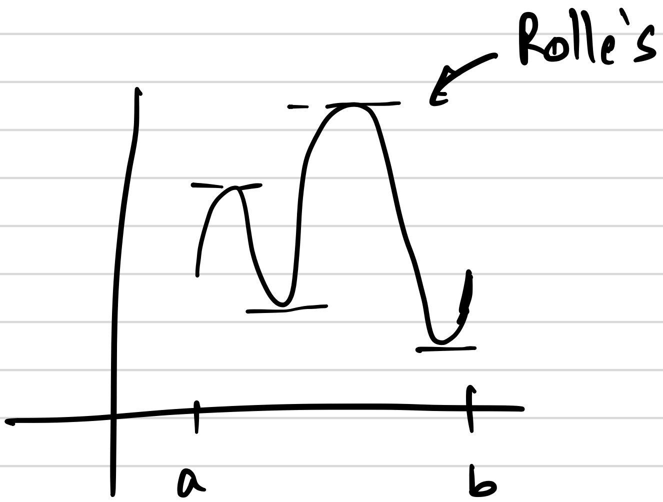
$$h'(c) = 0$$

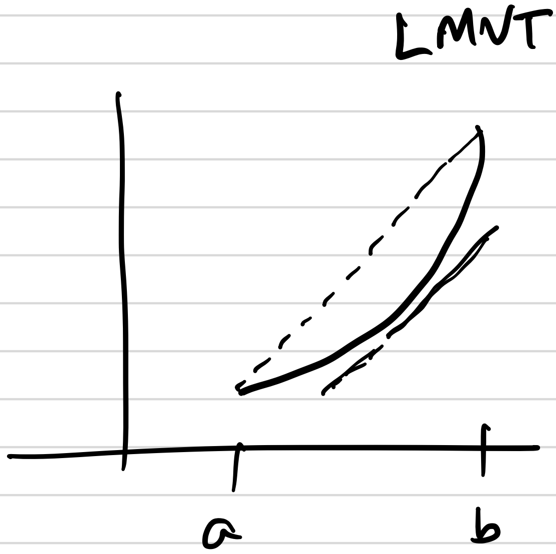
$$h'(c) = f'(c) - \frac{f(b)-f(a)}{b-a} = 0$$

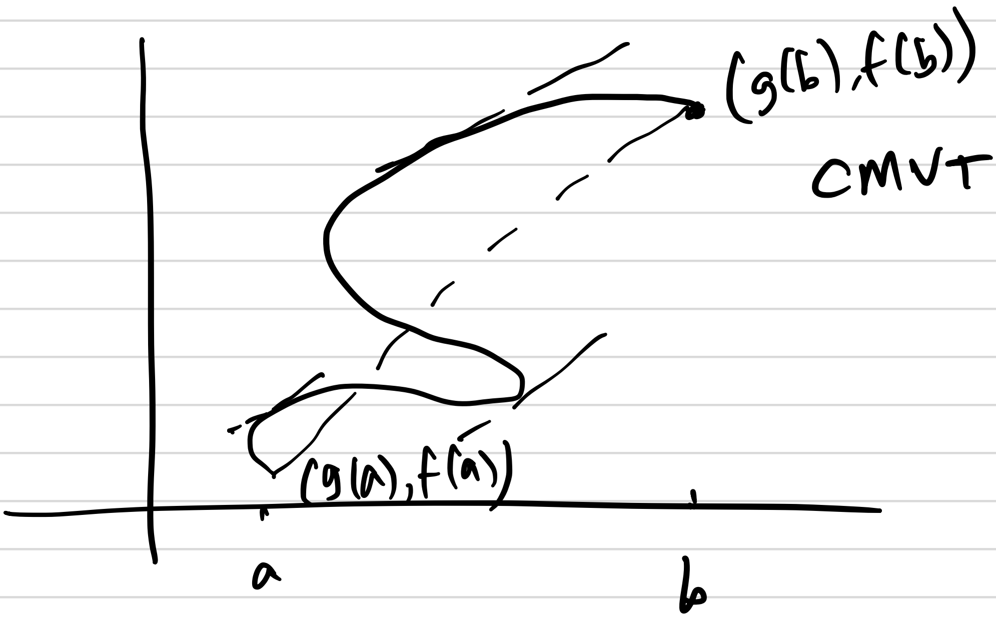
$$\Rightarrow \quad f'(c) = \frac{f(b)-f(a)}{b-a}$$

$\boxtimes$

Cauchy's mean value theorem (CMVT)







Suppose functions f and g are continuous on $[a,b]$ and differentiable on (a,b) and also $g'(x) \neq 0$ on (a,b).

Then there exists $c \in (a,b)$ s.t.

$$\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$$

Proof: Note that $g(a) \neq g(b)$

--- (by Rolle's theorem)

On contrary, suppose $g(a) = g(b)$.
Then g satisfies conditions of Rolle's theorem on $[a,b]$.
$\Rightarrow \exists$ a point $c \in (a,b)$ such that

$$g'(c) = 0$$

which is contradictory to the assumption $g'(x) \neq 0 \quad \forall x \in (a,b)$.

Construct

$$F(x) = f(x) - f(a) - \frac{f(b) - f(a)}{g(b) - g(a)} [g(x) - g(a)]$$

$F(b) = F(a) = 0$.

Thus F satisfies conditions of Rolle's theorem on $[a,b]$. Therefore, there exists a point $c \in (a,b)$ s.t.

$$F'(c) = 0$$

$$\Rightarrow \quad f'(c) - \frac{f(b)-f(a)}{g(b)-g(a)}\left[g'(c)\right] = 0$$

$$\Rightarrow \quad \frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$$

$\blacksquare$

Mathematical consequences of MVTs.

$\boxed{\text{Corollary 1:}}$ If $f'(x) = 0$ at each point $x \in (a,b)$, then $f(x) = K \;\; \forall x \in (a,b)$ where K is a constant.

Proof: Choose $x_1, x_2 \in (a,b)$
s.t. $x_1 < x_2$

Then f satisfies conditions of LMVT on $[x_1, x_2]$; then $\exists$ a point $c \in [x_1, x_2]$ such that

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(c) = 0$$

Corollary 2: If $f'(x) = g'(x)$ at each point x in an open (a,b), then there exists a constant K such that $f(x) = g(x) + K$ on (a,b).

Indeterminate forms in limits and L'Hôpital's rule

Theorem Suppose $f(a) = g(a) = 0$ and f and g are differentiable on an open interval I containing the point a. Further, $g'(x) \neq 0$ on I if $x \neq a$.

Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

assuming that the limit on the right hand side exists.