Maths 1

Lecture 12

Partial derivatives

How the function changes w.r.t change in its independent variable(s).

$$z = \underbrace{f(x, y)}_{\substack{\text{free} \\ \text{variables / independent variable}}} \qquad f : \mathbb{R}^2 \longrightarrow \mathbb{R}$$

dependent variable

Partial derivatives:

$$\left.\frac{\partial f}{\partial x}\right|_{(a_1, a_2)} = \lim_{h \to 0} \frac{f(a_1 + h, a_2) - f(a_1, a_2)}{h}$$

Partial derivative of f w.r.t. x at a point (a_1, a_2) in the interior of the domain.

$$\left.\frac{\partial f}{\partial x}\right|_{(a_1, a_2)} = \left. f_x \right|_{(a_1, a_2)}$$

$$\left.\frac{\partial f}{\partial y}\right|_{(a_1,a_2)} = \lim_{k \to 0} \frac{f(a_1, a_2+k) - f(a_1,a_2)}{k}$$

Partial derivative of f w.r.t. y at point (a_1, a_2).

Ex: Find $f_x(0,0)$ and $f_y(0,0)$ for the function

$$\boxed{f(x,y) = \begin{cases} \dfrac{x^3 y}{x^2+y^2} & (x,y) \neq (0,0) \\ = 0 & (x,y) = (0,0) \end{cases}}$$

$$f_x(0,0) = \lim_{h \to 0} \frac{f(0+h, 0) - f(0,0)}{h}$$

$$= \lim_{h \to 0} \frac{f(h,0) - f(0,0)}{h}$$

$$= \lim_{h \to 0} \frac{0-0}{h}$$

$$f_x(0,0) = 0$$

$$f_y(0,0) = \lim_{k\to 0} \frac{f(0,k) - f(0,0)}{k}$$

$$= 0$$

$$f_x(x,0) = \lim_{h\to 0} \frac{f(x+h,0) - f(x,0)}{h}$$

$$= 0$$

$$f_y(0,y) = \lim_{k\to 0} \frac{f(0,k+y) - f(0,y)}{k}$$

$$= 0$$

$$f_x(x,y) = \frac{x^2 y}{(x^2+y^2)^2}(x^2+3y^2) \qquad (x,y) \neq (0,0)$$

$$= 0 \qquad (x,y) = (0,0)$$

f_x is continuous on $\mathbb{R}^2$.

Example:

$$f(x,y) = \frac{xy}{\sqrt{x^2+y^2}} \qquad (x,y) \neq (0,0)$$

$$= 0 \qquad (x,y) = (0,0)$$

$f_x(0,0)$; $f_y(0,0)$

Discuss continuity of f_x & f_y.

$$f_y(0,0) = \lim_{k \to 0} \frac{f(0,k) - f(0,0)}{k}$$

$$= \lim_{k \to 0} \frac{0-0}{k}$$

$$= \underline{0}$$

$$f_y(x,y) = \begin{cases} \dfrac{x^3}{(x^2+y^2)^{3/2}} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

To check continuity of $f_y(x, y)$

Note that for $(x, y) \neq (0, 0)$

$$f_y(x, y) = \frac{x^3}{(x^2+y^2)^{3/2}}$$ is a ratio

of two continuous functions &

hence continuous.

To check continuity of $\frac{\partial f}{\partial y}$ at $(0, 0)$

$$\lim_{(x,y)\to(0,0)} f_y(x, y)$$

$$= \lim_{(x,y)\to(0,0)} \frac{x^3}{(x^2+y^2)^{3/2}}$$

$x = r\cos\theta, \ y = r\sin\theta$

$$= \lim_{\substack{r\to 0 \\ \theta \text{ any value}}} \frac{r^3 \cos^3\theta}{(r^2)^{3/2}}$$

$= \cos^3\theta$ (depends on path)

$\Rightarrow \lim_{(x,y)\to(0,0)} f_y(x,y)$ does not exist.

$\therefore f_y(x,y)$ is **NOT** Continuous at $(0,0)$.

Ex:

$$f(x,y) = \frac{xy}{x^2+y^2} \qquad (x,y) \neq (0,0)$$

$$= 0 \qquad (x,y) = (0,0)$$

$f_x(0,0)$, $f_y(0,0)$

$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h}$$

$$= 0$$

$$f_y(x,y) = \begin{cases} \dfrac{x(x^2-y^2)}{(x^2+y^2)^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

Partial derivatives of Higher order and mixed partial derivatives