Maths I

Lecture 11

Limit of a function of 2- variables.

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L$$

For every $\varepsilon > 0$, $\exists$ a $\delta > 0$ such that

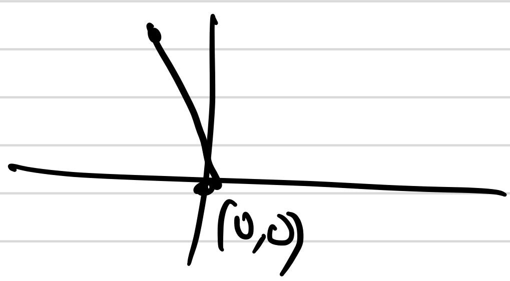
$$|f(x,y) - L| < \varepsilon$$

whenever $0 < \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta$

Show that

$$\lim_{(x,y)\to(0,0)} \frac{2x^2 y}{x^4 + y^2}$$ does not exist.

Consider the path $y = mx$ to approach $(0,0)$



$$\lim_{(x,y)\to(0,0)} \frac{2x^2y}{x^4+y^2}$$

$$= \lim_{x\to 0} \frac{2x^2(mx)}{x^4+(mx)^2} \quad \text{on } \quad y = mx$$

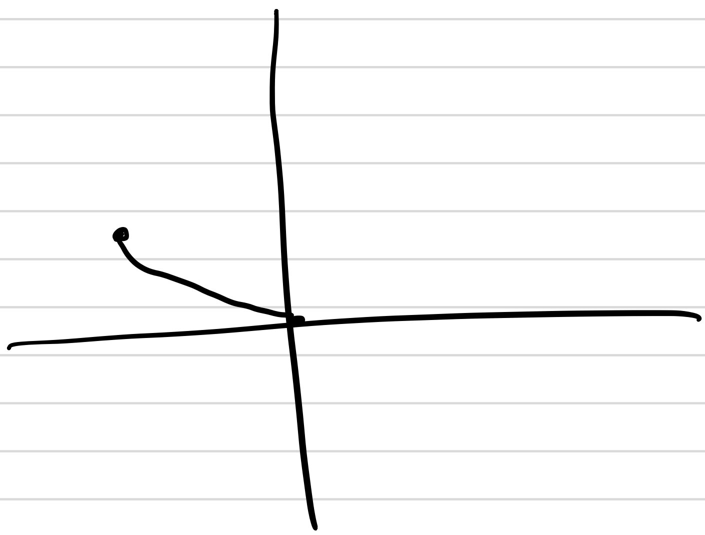
$$= \lim_{x\to 0} \frac{2x^3m}{x^4+m^2x^2}$$

$$= \lim_{x\to 0} \frac{2xm}{x^2+m^2}$$

$$= 0$$

except for $m = 0$

Choose $y = mx^2$



$$\lim_{(x,y)\to(0,0)} \frac{2x^2y}{x^4+y^2}$$

$$= \lim_{x\to 0} \frac{2x^2(mx^2)}{x^4+m^2x^4}$$

$$= \lim_{x\to 0} \frac{(2m)\cancel{x^4}}{(1+m^2)\cancel{x^4}} = \frac{2m}{1+m^2}$$

$\Rightarrow$ the limit depends on the path.

$\therefore$ limit does NOT exist.

Ex: $$\lim_{(x,y)\to(0,0)} \sin(x+y) = 0$$

Let $x = r\cos\theta$, $y = r\sin\theta$
$r > 0$, $\theta \in [0, 2\pi)$

$$\lim_{(x,y)\to(0,0)} \sin(x+y)$$

$$= \lim_{\substack{r\to 0 \\ \theta \text{ any value}}} \sin\left(r(\cos\theta + \sin\theta)\right)$$

Since $\cos\theta + \sin\theta$ is bounded

$r(\cos\theta + \sin\theta) \to 0$ as $r \to 0$

$$= 0$$

Use ε-δ definition.

To prove: Given $\varepsilon > 0$, find a corresponding $\delta > 0$ s.t.

$|\sin(x+y) - 0| < \varepsilon$ whenever
$0 < \sqrt{x^2 + y^2} < \delta$

observe:

$$\left|\sin(x+y)\right| \le |x|+|y| \quad \star$$

$$\le \sqrt{x^2+y^2} + \sqrt{x^2+y^2}$$

$$= 2\sqrt{x^2+y^2}$$

Choose $\delta = \varepsilon/2$.

Ex: $\lim_{(x,y)\to(0,0)} \sin\left(\frac{x}{y}\right) + \sin\left(\frac{y}{x}\right)$

Ex: $\lim_{(x,y)\to(0,0)} \frac{x^3 y^2}{x^6 + y^2}$

$y = mx^{3/2}$

Definition :

A function $f(x,y)$ is continuous at a point (x_0, y_0) if

1) f is defined at (x_0, y_0)

2) $\lim_{(x,y)\to(x_0,y_0)} f(x,y)$ existc.

3) $\lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0, y_0)$

Ex:

$$f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} & (x,y) \neq (0,0) \\ \cancel{0} \ A & (x,y) = (0,0) \end{cases}$$

Discuss continuity of f at $(0,0)$.

Limit does not exist at $(0,0)$

$\therefore$ f is not continuous at $(0,0)$

Ex:

$$f(x,y) = \begin{cases} \dfrac{xy^2}{x^2+y^2} & (x,y) \neq (0,0) \\ A & (x,y) = (0,0) \end{cases}$$