Portfolio Optimization in the Indian Market: A Gram-Schmidt Orthogonalization Approach

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Abstract

This research explores advanced portfolio construction techniques tailored to the unique characteristics of the Indian financial market, focusing on risk distribution strategies that aim to reduce concentration in high-risk assets. By examining four portfolio method, Risk Parity Portfolio (RPP), Risk Budgeting Portfolio (RBP), Principal Component Risk Budgeting Portfolio (PCRBP), and Gram-Schmidt Orthonormalization Portfolio (GSOP the study compares how each approach distributes risk and improves diversification.

The primary objective is to assess the performance of these portfolios across key Indian asset classes, including equities (Sensex and Nifty 50), government bonds, gold, and the INR/USD exchange rate. Each method's ability to balance risk contributions and enhance risk-adjusted returns is evaluated under different market conditions, including periods of high volatility. A notable innovation in this study is the application of Gram-Schmidt orthogonalization, a method that transforms asset returns into independent components, creating uncorrelated risk sources and helping to minimize overlap between assets' risk contributions. This process is compared with traditional risk-based portfolios to highlight how GSOP can offer a more stable risk-return profile.

By conducting this analysis, the study provides valuable insights for portfolio managers and institutional investors interested in building resilient, risk-aware portfolios that adapt well to the complex dynamics of emerging markets like India.

Introduction

As finance and investment management evolve, developing effective portfolio allocation strategies has become essential for both academics and industry professionals. The concept of risk parity/budgeting has emerged as a compelling approach that seeks to balance risk across various asset classes. Traditional methods often struggle with diversification, especially in volatile markets where asset correlations can shift rapidly.

Risk parity/budgeting offers a solution by focusing on each asset's risk contribution rather than just expected returns, ensuring that no single asset dominates the portfolio's risk. This can help mitigate potential losses during market downturns. However, implementing risk parity effectively requires understanding asset interactions, which is where mathematical techniques play a crucial role.

This paper introduces a novel methodology for implementing risk parity/budgeting using the Gram-Schmidt orthogonalization process. This mathematical framework aims to enhance portfolio robustness and minimize concentration risk by creating a more diversified strategy aligned with investors' risk tolerance and goals. We all explore the theoretical foundations of risk parity/budgeting, the principles of the Gram-Schmidt process, and how it can be applied to reduce redundancy and improve risk management in portfolios.

The findings of this study are expected to contribute to the broader discourse on efficient asset allocation techniques. By offering a deeper understanding of risk parity/budgeting and its implementation through the Gram-Schmidt orthogonalization process, we aim to provide valuable insights for both institutional and retail investors seeking to optimize their investment strategies in an increasingly complex financial environment.

Rationale

This study is motivated by the growing need for adaptive risk management strategies within the Indian financial market, where asset volatility and covariation structures can vary significantly due to external economic and market factors. Traditional portfolio methods often fail to account for these shifts, leading to portfolios that are heavily weighted in high-volatility assets, which can increase risk exposure and make portfolios vulnerable to sharp market declines. To address this, the study employs the principles of risk parity/budgeting, which focuses on balancing risk contributions across assets rather than focusing on nominal weights. This approach ensures that no single asset class dominates the portfolio’s risk profile, helping to achieve a more even distribution of risk.

Additionally, this study integrates Gram-Schmidt orthogonalization, a mathematical technique that further enhances diversification by transforming asset returns into uncorrelated components. This de-correlation helps manage overlapping risks and provides a clearer picture of each asset’s unique risk contribution to the portfolio. By applying these advanced techniques specifically to Indian asset classes, this study aims to offer a more robust framework for building portfolios that can withstand market fluctuations.

Overall, this research contributes to the broader field of portfolio management by demonstrating how a combination of risk parity/budgeting and orthogonalization can create a more adaptable investment strategy for emerging markets. The findings are intended to support investors and portfolio managers in developing balanced portfolios with improved risk-adjusted returns, helping them navigate the complexities of the Indian market while reducing vulnerability to market volatility.

Aim and Objective

The primary objective of this research is to construct a risk parity/budgeting portfolio for the Indian market, analyzing the effectiveness of diversification strategies across major asset classes.

- **Evaluate the Volatility and Returns of Portfolio:** Analyze the risk and return characteristics of a portfolio including Sensex, Nifty 50, Government Bonds, Gold Rates, and Currency Exchange Rates. This will highlight how each asset class contributes to the portfolio's overall performance in the Indian market.

- **Compare Risk Parity, Risk Budgeting, and Gram-Schmidt Orthogonalization:** Assess how traditional risk parity and risk budgeting strategies perform compared to the Gram-Schmidt Orthogonalization method. To determine which approach provides better diversification and risk management for Indian investors.

- **Determine the Optimal Order of Orthogonalization:** Identify the most effective sequence of assets for orthogonalization, ensuring that the Gram-Schmidt method balances risk contributions and improves portfolio diversification.

Data Description

The data which we are working on for the study includes crucial financial years from 2014 to 2023. The dataset includes key economic indicators which provide valuable insights about the performance and the dynamics of Indian financial market as well as the minor factors influencing it. The data includes:

1. Sensex and Nifty 50

- Data Source: Bombay Stock Exchange (BSE) and National Stock Exchange (NSE) of India.
- Data Type: Daily returns.
- Period: From January 1, 2014, to December 31, 2023.
- Description: Two of the major market indices of India includes Sensex and Nifty 50. 30 stocks of Sensex are included in BSE, while 50 Stocks of Nifty 50 are included in NSE. The contains the daily returns, providing us knowledge about the overall performance of the Indian market.

2. Bond Exchange Rates

- Data Source: Indian government bonds, specifically the 10-year yield.
- Data Type: Daily yields.
- Period: From January 1, 2014, to December 31, 2023.
- Description: The data consists of the variation in the prices of bonds, which provides the market performance of government bonds, which are affected by the interest rates and macroeconomic conditions.

3. Gold Rates

- Data Source: Commodity exchanges and financial platforms.
- Data Type: Daily closing prices in Indian Rupees (INR).
- Period: From January 1, 2014, to December 31, 2023.
- Description: The data consists of past prices of gold in INR, which is significant asset in India. Prices of gold are usually used as a hedge against inflation and market volatility, and provide understanding about the economic condition as well as investor sentiment.

4. Currency Exchange Rates

- Data Source: Reserve Bank of India (RBI) and forex platforms.
- Data Type: Daily exchange rates (INR/USD).
- Period: From January 1, 2014, to December 31, 2023.
- Description: The data consists of the daily exchange rate of Indian Rupee (INR) and US Dollar (USD). The rate of exchange variates by a lot of factors which includes interest rate, inflation, global market condition, etc.

Data Source:

The data which we used for the study is taken form reputated financial platforms including:

- Reserve Bank of India (RBI)
- National Stock Exchange (NSE)
- Bombay Stock Exchange (BSE)
- Commodity Markets

The data gives a thorough understanding about the dynamics of the market and the and macroeconomic conditions over the specific period of time.

Data Sample:

Date	Sensex	Nifty 50	Bond Rates	Gold Rates	Currency Rates
01-01-2023	59,549.90	17,662.15	7.343	1,962.20	81.74
02-01-2023	58,962.12	17,303.95	7.457	1,853.20	82.64
03-01-2023	58,991.52	17,359.75	7.315	2,004.10	82.16
04-01-2023	61,112.44	18,065.00	7.116	2,018.30	81.72
05-01-2023	62,622.24	18,534.40	6.989	1,982.10	82.68
06-01-2023	64,718.56	19,189.05	7.11	1,929.40	82.09
07-01-2023	66,527.67	19,753.80	7.172	2,009.20	82.24
08-01-2023	64,831.41	19,253.80	7.166	1,965.90	82.7
09-01-2023	65,828.41	19,638.30	7.21	1,885.40	83.03
10-01-2023	63,874.93	19,079.60	7.351	1,994.30	83.26
11-01-2023	66,988.44	20,133.15	7.279	2,038.10	83.36
12-01-2023	72,240.26	21,731.40	7.176	2,071.80	83.19

Exploratory Data Analysis

Volatility

Volatility of an asset is calculated over a sample period of time. The Volatility measures the degree to which the assets return deviate from the mean, and it is calculated by the standard deviation of the returns.

1. Daily Returns Calculation

Firstly, we will calculate the daily returns for each asset, we can either use simple return or logarithmic returns. In finance we usually use simple returns, but one can use either of the methods for the calculation of volatility.

- **Simple Return Formula**:

$$R_{\text{simple}} = \frac{P_t - P_{t-1}}{P_{t-1}}$$

Where:

1. Pt is the price of the asset on day t.
2. Pt−1 is the price of the asset on the previous day.

2. Volatility Calculation (Standard Deviation)

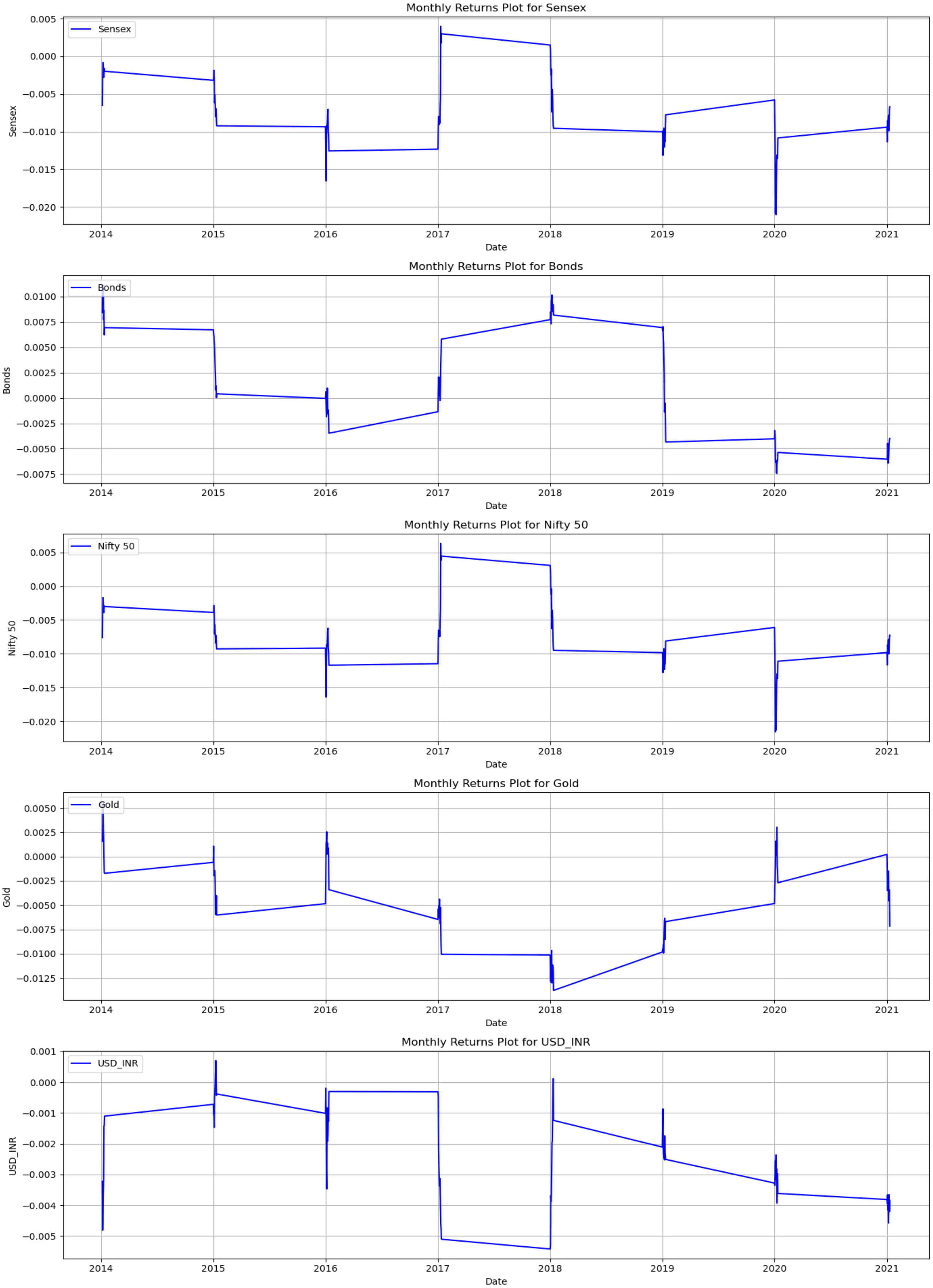
After the calculation of daily returns, the next step is to calculate the standard deviation of the daily returns in order to calculate the volatility of the assets over the sample time. Standard deviation measures the dispersion of returns, it will also record the variation of the return about the mean of daily returns.

- **Daily Volatility Formula**:

$$\sigma_{\text{daily}} = \text{std}(\text{daily returns})$$

Where:

1. σdaily is the daily volatility (standard deviation of daily returns).
2. std(daily returns) is the standard deviation function that calculates the variation in the daily returns from the average daily returns.



Line Chart of Daily Returns From 2014 to 2023

Interpretation:

Generally, Sensex and Nifty 50 i.e., the Stock indices are more volatile, with more uncertainties. As compared to Bonds and Gold. Golds and Bonds are considered safer investments, as the show more stability as compared to stock. The USD/INR exchange rate also shows some volatility but in smaller amounts, as impacted by currency market dynamics.

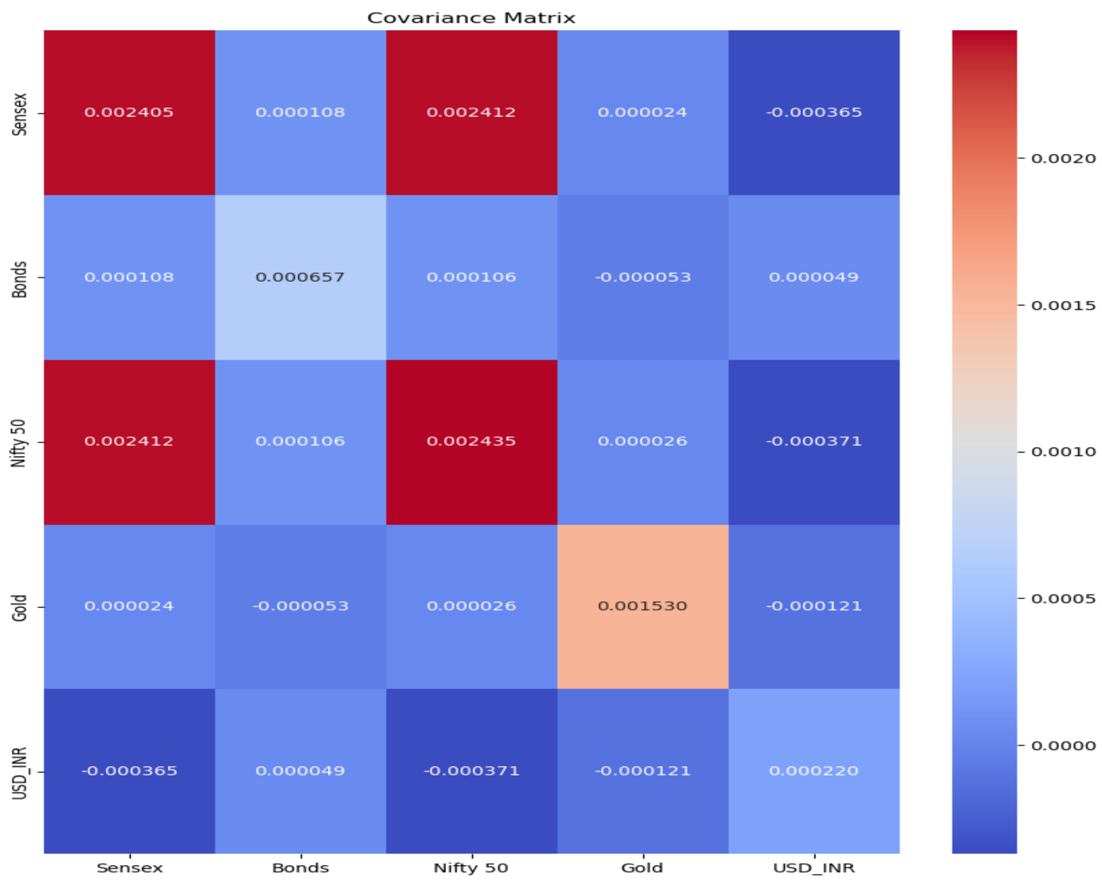
Table Showing Volatility for Different Assets

Financial Variables	Volatility
Sensex	0.04903939
Government Bond	0.02562743
Nifty 50	0.04934217
Gold Rates	0.03911159
Currency Exchange Rates	0.01481852

- Higher Volatility: The most fluctuation is shown by Sensex and Nifty 50 i.e., they have higher risk but the also have the chance of getting higher returns. They're better for investors who are comfortable with some ups and downs in exchange for better rewards.
- Moderate Volatility: Gold prices vary moderately, which offers a mixture of stability as well as risk. It is usually preferred by investors who want to a safer option for uncertain times.
- Lower Volatility: Most stable options are Government Bonds and Exchange Rate, with lower price changes. It is the best investing option for the conservative investors, who want reliable and stable returns with very low risk.

Covariance Analysis

Covariance analysis is used in finance to measure how to assets proceed together. Positive covariance indicates that the two assets tend to fluctuate together. Negative Covariance indicates that if the price of one asset rises when the price of the other asset falls down. It is different from correlation, which tells only the strength of relationship, covariance also tells us about the fluctuation (ups or downs) and the size of that movement. By analysing covariance, an investor can know which assets move similarly and which move differently. This helps in building a balanced portfolio by choosing assets that may reduce overall risk when combined together.



Interpretation:

- Covariance between Sensex, Bonds and Gold have a positive covariance indicating that as the prices of Sensex increases the prices of Bonds & Gold increases. But the covariance between Sensex and USD_INR(Currency) is negative indicating that as the price of Sensex increases the price of USD_INR will fall.

- Covariance between Bonds, Sensex, Nifty 50 & USD_INR have a positive covariance indicating that as the prices of Bonds increases the prices of Sensex, Nifty 50 &USD_INR increases. But the covariance between Bonds and Gold is negative indicating that as the price of Bonds increases the price of Gold will fall.

- The covariance between Nifty 50 and USD_INR is negative indicating that as the price of Nifty 50 increases the price of USD_INR will fall.

- The covariance between Gold and USD_INR is negative indicating that as the price of Gold increases the price of USD_INR will fall.

Methodology

1. Risk Parity Portfolio (RPP)

The aim in RPP is to allocate equal risk contribution for each asset, which will assure that risk is not concentrated in just one or more than one asset. This approach seeks heterogeneity not just by asset count, but by the level of risk.

By allocating each asset an equal portion in the total risk of portfolio, RPP aims to avoid over dependencies on volatile asset and boosts stability. This is very useful in times of market uncertainty when assumptions about expected returns are challenging.

RPP can be calculated by knowing each asset's marginal contribution to portfolio risk and adjusting the weights iteratively until each asset's risk contribution is equal.

2. Risk Budgeting Portfolio (RBP)

The aim of RBP is to assign, each asset a predefined risk budget unlike RPP, that has equal risk contributions. This ensures liberty to prioritize certain assets while still managing overall portfolio risk. In this the investor decides how much of the total risk they want each asset or asset class to contribute based on factors like risk tolerance, investment objectives, or market conditions.

The process of optimisation in RBP adjusts asset weights so that it match these specified risk budgets. For example, more volatile assets like equities may be allocated higher risk budgets for growth-focused investors, while bonds might have smaller risk budgets for stability.

Risk Parity / Risk Budgeting Portfolio Formulas

1.1 Risk Contribution of Each Asset

$$RC_i = w_i \frac{(\partial \mathcal{R}(w))}{\partial w_i}$$

Where:

1. wi is the weight of asset i,
2. R(w) is the risk measure of the portfolio (often portfolio volatility).

1.2 Portfolio Volatility

If the risk measure is portfolio volatility σ(w), then it is calculated as:

$$\sigma(w) = \sqrt{w^{\mathrm{T}} \Sigma w}$$

Where:

1. w is the vector of asset weights,
2. Σ is the covariance matrix of asset returns.

1.3 Marginal Contribution to Volatility

The marginal contribution to portfolio volatility for each asset is given by the derivative of σ(w) with respect to the asset weight wi:

$$\frac{\partial \sigma(w)}{\partial w} = \frac{1}{2}(w^{\mathrm{T}} \Sigma w)^{-1/2}(2\Sigma w)$$

1.4 Optimization Problem for Risk Parity

The objective in RPP is to obtain the optimal weights w^* that equalize the risk contributions according to a specified **risk budget** , formulating this as an optimization problem:

$$w^* = \arg\min_{w} f(w; c)$$

Where:

c=(c1,…,cn) is a vector of risk budgets for each asset, i.e. the desired risk contribution for each asset in proportion to the total risk.

Objective Function

The objective function f(w;c) is to minimize :

$$f(w; c) = \sum_{i=1}^{n} \left(w_i \cdot \frac{\partial \mathcal{R}(w)}{\partial w_i} - c_i \mathcal{R}(w) \right)^2$$

The above function computes the squared deviation of each asset’s actual risk contribution from its target risk budget ci. The optimization will minimizes these deviations, to achieve a portfolio where each asset's risk contribution aligns with the predetermined budget.

Practical Applications:

RPP and RBP are used mostly for creating diversified portfolios that reduce dependence on return forecasts, which makes them popular among institutional investors looking more vigorous risk management.

These approaches permit investors to adapt to changing market conditions. For illustration, risk budgets in RBP can be adjusted periodically to reflect new market shifts in risk tolerance.

Advantages and Limitations:

Advantages: Both RPP and RBP are highly adaptable and can be costumed to indicate risk control, giving stability across different conditions of the market.

Limitations: Allocating risk budget in RBP can be judgemental and difficult. Moreover, both the methods rely past data for the estimation of covariance, which may not always capture future volatilities or correlations.

3. Principal Component Risk Budgeting Portfolio

In this approach PCA(Principal component Analysis) is used to determine the independent source of risk and optimize portfolio based on these risk components. The objective of PCRBP is to make a portfolio in which the value of risk is diversified across uncorrelated components rather than each asset individually, using the structure of RPP or RBP.

3.1 Principal Component Analysis (PCA):

The covariance matrix Σ of n assets. PCA decomposes Σ into eigenvalues and eigenvectors:

$$E'\Sigma E = \Lambda$$

Where:

$\Lambda = \mathrm{diag}(\lambda 1, \dots, \lambda n)$ is a diagonal matrix with eigenvalues λi

$E=(e1, \dots, en)$ is the matrix of eigenvectors associated with these eigenvalues.

Each eigenvector defines a principal component, which represents uncorrelated portfolio derived form the original assets, with the variance of the i-th component being λi.

3.2 Weights in Principal Components:

The weight in terms of principal components can be derived by inverting the eigenvector matrix and applying it to the asset weights:

$$\tilde{w}=E^{-1}w$$

Where:

w=(w1,…,wn) represents the vector of asset weights.

3.3 Total Portfolio Variance:

The principal component are uncorrelated, the total portfolio variance Var(Rp) can be expressed as:

$$Var\,(R_p)=\sum_{i=1}^{n}\tilde{w}_i^2\lambda_i$$

In this, each term is the variance contribution of i-th principal component to the total portfolio variance.

3.4 Proportion of Component Variance to Total Portfolio Variance:

The proportion of each principal component's variance relative to the total portfolio variance is given by:

$$\frac{\tilde{w}_i^2\lambda_i}{Var\,(R_p)}$$

3.5 Effective Number of Bets (ENB):

Introduction to the concept of the **Effective Number of Bets (ENB)** is to measure diversification. The ENB can be calculated as:

$$N_{Ent}=\exp\,(-\sum_{i=1}^{n}p_i\ln p_i)$$

Where:

$$p_i=\frac{\tilde{w}_i^2\lambda_i}{Var\,(R_p)}$$

The proportion of each component’s variance contribution.

4. Gram-Schmidt Orthonormalization Process

The PCRBP (Principal Component Risk Budgeting Portfolio) is challenging to interpret by a common man as its components are purely statistical. To address this, Gram Schimdt Orthonormalization is used to transform asset returns into uncorrelated risk source, that we can understand as unique fluctuation of each asset, which helps the investor to know how different asset independently contribute to portfolio risk.

4.1 Centralization of Returns:

The first step is to centralize the asset returns to ensure that the covariance of the transformed returns has zero value of mean. Let the centralized asset returns be denoted by ai for i=1,…,ni.

4.2 Gram-Schmidt Orthonormalization Process:

Now applying the Gram-Schmidt process to derive uncorrelated vectors bi and unit vectors ui as follows:

$$
\begin{aligned}
b_1 &= a_1 \qquad u_1=\frac{b_1}{|b_1|}=\frac{a_1}{|a_1|} \\
b_2 &= a_2-(a_2\cdot u_1)u_1 \qquad u_2=\frac{b_2}{|b_2|} \\
b_3 &= a_3-(a_3\cdot u_1)u_1-(a_3\cdot u_2)u_2 \qquad u_3=\frac{b_3}{|b_3|} \\
&\cdots \\
b_n &= a_n-(a_n\cdot u_1)u_1-(a_n\cdot u_2)u_2-\cdots-(a_n\cdot u_{n-1})u_{n-1} \qquad u_n=\frac{b_n}{|b_n|}
\end{aligned}
$$

we denote ai·uj=βi,j (for j=1,…,n−1), which leads to the following transformed returns:

$$
\begin{aligned}
b_1 &= a_1 \\
b_2 &= a_2-\beta_{2,1}u_1 \\
b_3 &= a_3-\beta_{3,1}u_1-\beta_{3,2}u_2
\end{aligned}
$$

and so on.

Then, the portfolio return Rp is represented as:

$$\begin{aligned} R_p &= w_1 a_1 + w_2 a_2 + \cdots + w_n a_n \\ &= w_1 b_1 \\ &+ w_2(\beta_{2,1} u_1 + b_2) \\ &+ w_3(\beta_{3,1} u_1 + \beta_{3,2} u_2 + b_3) \\ &\cdots \\ &+ w_n(\beta_{n,1} u_1 + \beta_{n,2} u_2 + \cdots + b_n) \end{aligned}$$

Because of equation bi=| bi |.ui

$$\begin{aligned} R_p &= (w_1|b_1| + w_2\beta_{2,1} + \cdots + w_n\beta_{n,1})u_1 \\ &+ (w_2|b_2| + w_3\beta_{3,2} + \cdots + w_n\beta_{n,2})u_2 \\ &+ (w_3|b_3| + w_4\beta_{4,3} + \cdots + w_n\beta_{n,3})u_3 \\ &\cdots \\ &+ w_n|b_n|u_n \end{aligned}$$

4.3 Variance of Portfolio Rp

Using the properties Var(ui)=1 and Cov(ui,uj)=0 (for i≠ji), the variance of Rp becomes:

$$\begin{aligned} Var(R_p) &= (|b_1|w_1 + \beta_{2,1}w_2 + \cdots + \beta_{n,1}w_n)^2 \\ &+ (|b_2|w_2 + \beta_{3,2}w_3 + \cdots + \beta_{n,2}w_n)^2 \\ &+ (|b_3|w_3 + \beta_{4,3}w_4 + \cdots + \beta_{n,3}w_n)^2 \\ &\cdots \\ &+ (|b_n|w_n)^2 \end{aligned}$$

4.4 Objective Function

The objective function for balancing the risk contributions according to specified risk budgets ci is defined as:

$$f(w\,;c) = \sum_{i=1}^{n} ((|b_i|w_i + \beta_{i+1,i}w_{i+1} + \cdots + \beta_{n,i}w_n)^2 - c_i \cdot Var(R_p))^2$$

Analysis

1. Risk Parity Portfolio (RPP)

Optimal Weights:

Variables	Sensex	Bonds	Nifty 50	Gold	USD_INR
Values	0.1522038	0.41265417	0.15133537	0.28380666	0.0

To achieve equal risk contribution the weights are adjusted. The value of one assets value of weight comes out to be zero because it is not required to balance the risk.

As, weight of USD_INR is 0 , i.e., it doesn't contribute to the portfolio. This can happen if the asset is either too low-risk to impact the total risk significantly or if adding it would disrupt the desired equal risk contribution.

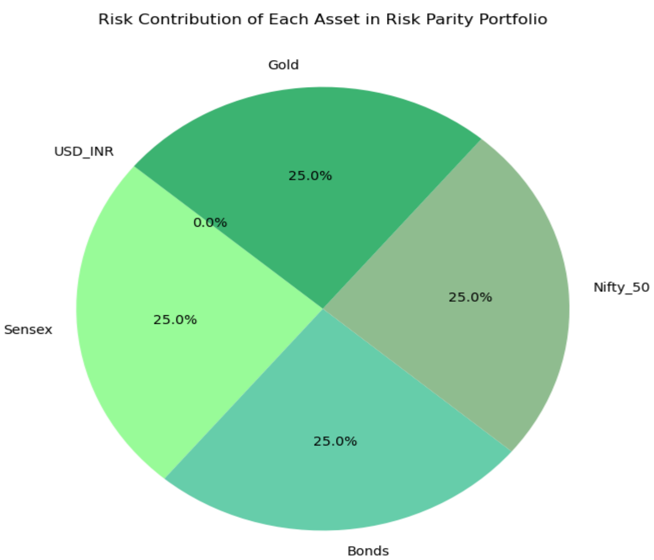
Risk Contributions:

Variables	Sensex	Bonds	Nifty 50	Gold	USD_INR
Values	0.25000996	0.2498667	0.25001706	0.25010628	0.0

Each asset contribute around 25% to the total portfolio risk , which is what RPP approach aims to achieve.

The negative risk contribution for the last asset aligns with the zero weight it has no presence in the portfolio and henceforth no contribution to total risk.

The total risk of the portfolio is very close to 1, ensuring the calculations are accurate.



2. Risk Budgeting Portfolio (RBP)

Optimal Weights:

Variables	Sensex	Bonds	Nifty 50	Gold	USD_INR
Values	0.18127401	0.39635772	0.18032156	0.24204671	0.0

The weights 0.181 for Sensex, 0.396 for Bonds, etc. were determined to achieved the risk of 0.25,0.15,0.25,0.1,0.25 respectively.

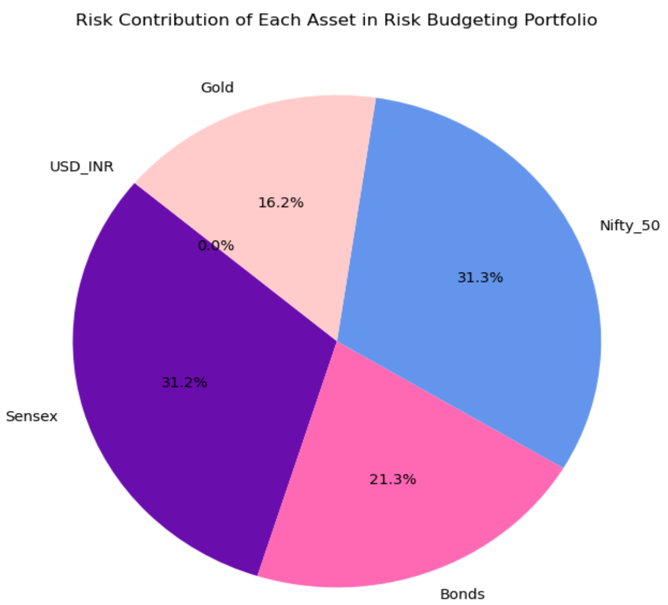
As, weight of USD_INR is 0 , i.e., it doesn't contribute to the portfolio as including may not align with the target risk contribution.

Risk Contributions:

Variables	Sensex	Bonds	Nifty 50	Gold	USD_INR
Values	0.31239135	0.21257473	0.31254491	0.16248901	0.0

These ae the percentage of risk that each asset contributes to the portfolio's overall risk. Sensex and Nifty 50 each contributes approximately 31% of the total risk, Bonds and Gold contribute approximately 21% and 16% respectively.

The total portfolio risk is exactly 1.



The Risk Budgeting Portfolio (RBP) method has few limitations:

1. Assigning the risk budgets for each asset can be a challenge. If the target are not well aligned with actual market risk , the portfolio may not perform well or as expected.
2. RBP usually relies on past volatility and correlations, that may not always predict the future performance, especially in fluctuating markets.
3. RBP may not completely tell about the diversification across independent factors, i.e. it may miss some hidden risks if different assets are correlated during market dynamics.

3. Principal Component Risk Budgeting Portfolio (PCRBP)

Optimal Weights:

Variables	Sensex	Bonds	Nifty 50	Gold	USD_INR
Values	0.19370513	0.19421129	0.19377883	0.14461104	0.27369371

The weights are balanced (e.g., 0.1937, 0.1942, etc.) but reflect small adjustments to manage risk according to each principal component's contribution.

Risk Contributions:

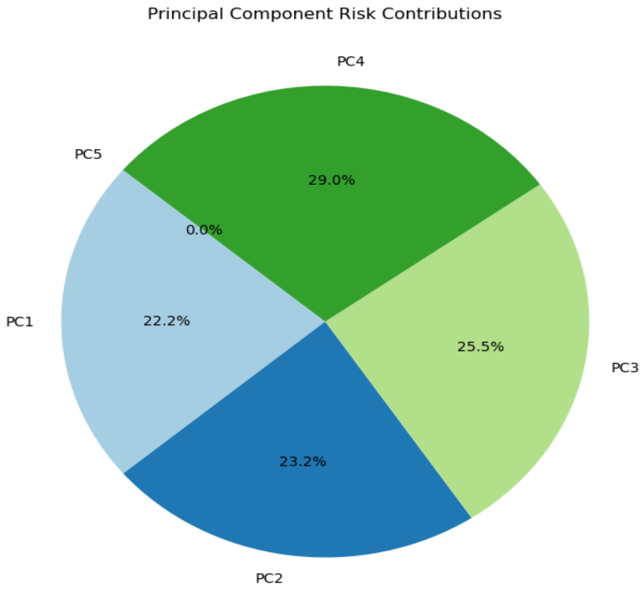
Variables	Sensex	Bonds	Nifty 50	Gold	USD_INR
Values	8.22e-03	8.58e-03	9.43e-03	1.07e-02	3.35e-09

This distribution reflects the PCRBP's goal of allocating risk according to principal components rather than individual assets.

These values show the actual risk contributed by each principal component to the portfolio's total risk, with the largest contributions from the first few components.

The small values (e.g., 0.0082, 0.0085, etc.) confirm that risk is spread across components, but with emphasis on the first few, as expected in PCRBP.

The total portfolio risk of 0.1923 is the combined risk after balancing risk across principal components.

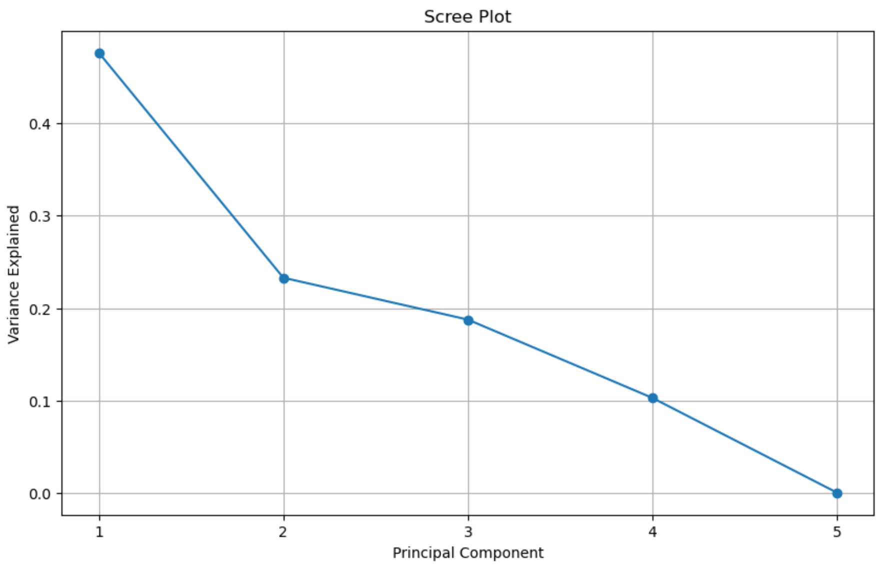


Explained Variance Ratios:

Variables	Sensex	Bonds	Nifty 50	Gold	USD_INR
Values	0.47592446	0.23282411	0.18758691	0.1030039	0.00066062

These values (e.g., 0.4759, 0.2328, etc.) represent the proportion of total portfolio variance explained by each principal component.

Higher values mean the first few components capture most of the portfolio's variance, indicating that the portfolio's risk is concentrated in those components. This is typical in PCRBP, where only a few components usually drive most of the risk.

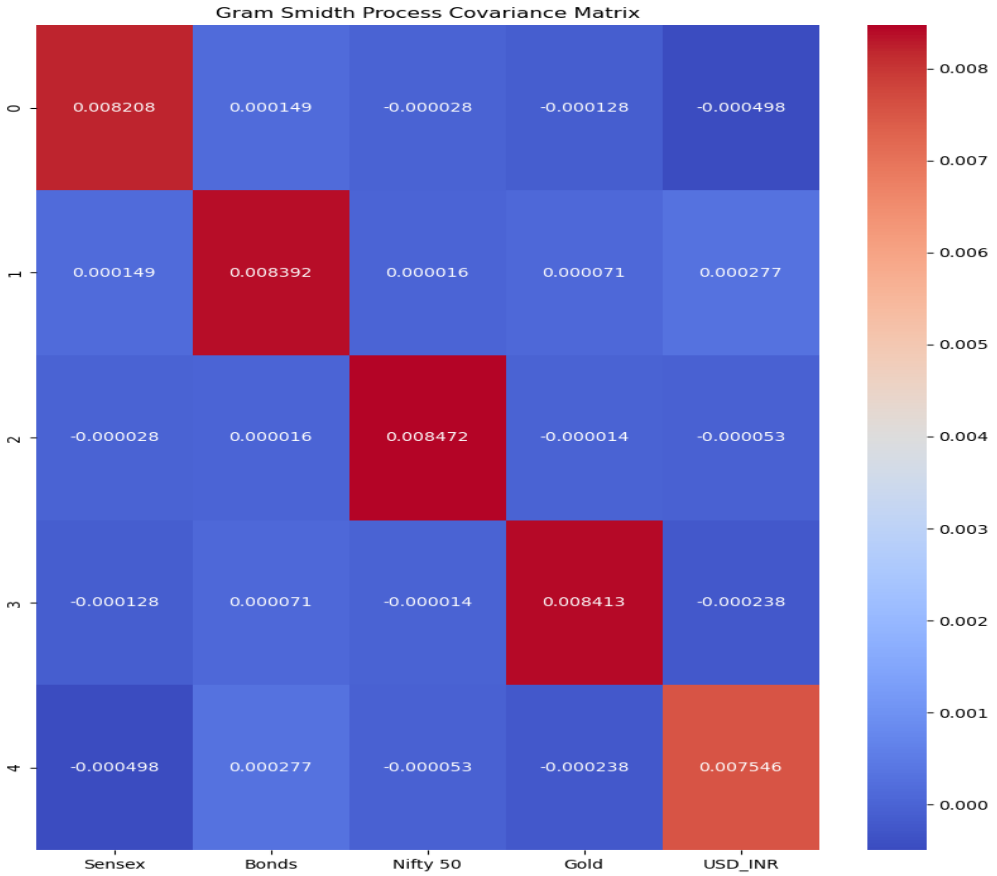


Effective Number of Bets (ENB)

The ENB of 3.0626910002597683 gives us an idea that while there are five assets, the portfolio has the equivalent of about three independent risk sources. This reflects the diversification achieved across principal components rather than individual assets.

The ENB close to 3 indicates moderate diversification, as PCRBP aims to balance risk across statistically independent sources rather than assets directly.

4. Gram-Schmidt Orthonormalization Portfolio (GSOP)



Covariance Matrix After Orthogonalization

After Orthogonalization, the variance of assets is almost equal.

Gram-Schmidt Orthonormalization Portfolio Risk: 0.04006946737344428

The total portfolio risk represents the variability or the overall level of risk in the portfolio's returns. The total portfolio risk is nothing but the standard deviation of portfolio's expected return, which is used commonly to measure risk.

Less magnitude would mean the stability or less volatility of the portfolio's risk. A higher number would mean more risk i.e., the returns could vary more widely from day to day.

So, a total portfolio risk is 0.0401 that indicates that, on average the returns from the portfolio won't fluctuate much. It is considered to be relatively more moderate level of risk for investment portfolio i.e., it is not very much risky or conservative.

Gram-Schmidt Orthonormalization Portfolio Weight:

Variables	Sensex	Bonds	Nifty 50	Gold	USD_INR
Values	0.2	0.2	0.2	0.2	0. 2

The equal weight allocation tells that each asset has the same proportion in portfolio. It indicates that GSOP method was applied with equal risk or budget objective, in which the asset is meant to contribute equal to portfolio's risk. In a fully diversified GSOP portfolio, weights are typically not exactly equal unless the covariance matrix and target risk contributions align in a way that leads to such a distribution.

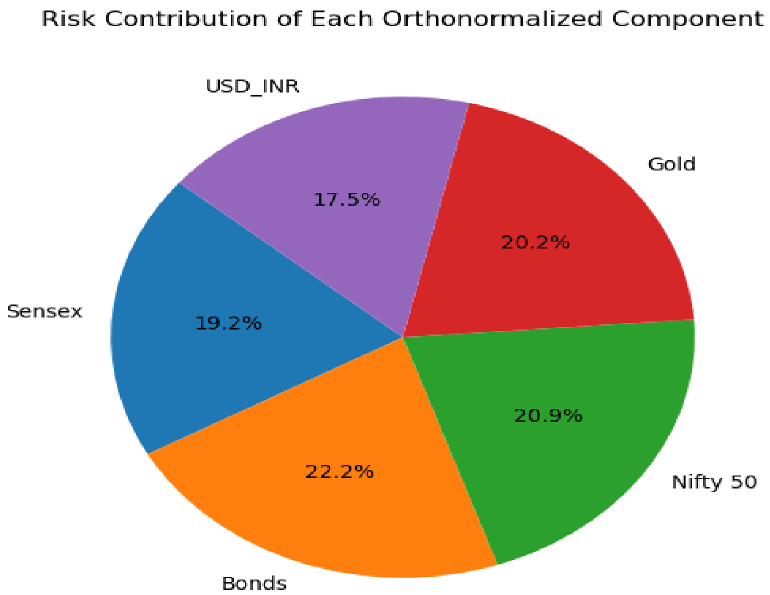
Normally, GSOP weights may vary depending on each asset's risk level. Equal weights might indicate that either:

1. The assets' individual risks (variances) were very similar.
2. The Gram-Schmidt orthonormalization process and optimization constraints pushed toward an equal-weight solution.

Risk Contributions:

Variables	Sensex	Bonds	Nifty 50	Gold	USD_INR
Values	0.192	0.222	0.209	0.202	0. 175

After applying the Gram-Schmidt orthogonalization, the portfolio shows balanced risk contributions across all assets: Bonds have the highest weight (22.2%), followed closely by Nifty 50 (20.9%), Gold (20.2%), Sensex (19.2%), and USD/INR (17.5%). This balance creates a diversified portfolio where each asset independently contributes to risk without overlap, enhancing stability and flexibility for risk management. By minimizing correlations, the portfolio achieves robust diversification, helping to mitigate potential market swings from any single asset class.



Order of Orthogonalization in Risk Parity/Budgeting Portfolio Using Gram-Schmidt Orthonormalization

In the Gram-Schmidt orthogonalization method, the order in which assets are orthogonalized impacts the resulting portfolio. Orthogonalization is the process of transforming assets so that each one contributes unique information that isn't linearly dependent on the others. This method is particularly useful in Risk Parity/Budgeting strategies, where the goal is to balance risk across portfolio components.

Why Order Matters in Orthogonalization?

- **Risk Distribution**: The order of asset orthogonalization directly influences the distribution of risk in the portfolio. If high-volatility or highly correlated assets are orthogonalized first, they are "de-risked" by isolating the unique risks associated with these assets before moving on to less risky or correlated assets.
- **Portfolio Performance**: Different sequences of orthogonalization yield different portfolio characteristics, such as overall volatility and risk contribution by each asset, affecting the portfolio's risk-adjusted returns.

Criteria for Optimal Orthogonalization Order

When seeking an optimal orthogonalization order, two main criteria are often considered:

1. **Volatility**:
    - Assets with higher individual volatility contribute more to the portfolio's total risk.
    - **Strategy:** Orthogonalizing high-volatility assets earlier can help to stabilize the portfolio by addressing major sources of risk upfront.
2. **Correlation**:
    - Assets with high correlations should be orthogonalized earlier. This separate overlapping risk factors, leading to a more balanced risk distribution.

- **Strategy**: Orthogonalizing highly correlated assets first can reduce redundancy in risk exposure, as these assets share similar behaviours in response to market changes.

Dynamic Ordering through Optimization

The optimal order of orthogonalization can change as market conditions evolve. A dynamic approach allows for re-evaluating and adjusting the asset order to adapt to changes in volatility, correlations, and overall market risk.

Steps in Dynamic Ordering with Optimization

1. **Define the Objective**:

- **Portfolio Objectives**: Start by setting clear objectives for the portfolio. Common objectives include:
  - **Minimizing Portfolio Volatility**: Focuses on reducing overall risk in the portfolio.
  - **Equal Risk Contribution**: Ensures each asset contributes equally to portfolio risk after orthogonalization.

2. **Generate Permutations**:

- **Permutations**: For a small number of assets, generate all possible permutations of the asset order.

  Example: For 4 assets, there are 4! =24 possible orders. Testing all 24 orders is feasible in small portfolios.

- **Optimization for Large Portfolios**: With a larger number of assets, testing all permutations is computationally impractical. Instead, use optimization algorithms such as:
  - **Genetic Algorithms**: Mimics natural selection to iteratively find the best asset order.
  - **Simulated Annealing**: A probabilistic technique to explore various combinations and converge on an optimal solution without testing every possibility.

3. **Evaluate Each Permutation**:

- **Application of Gram-Schmidt Orthogonalization**: For each permutation, apply the Gram-Schmidt process to orthogonalize the assets in the specified order.
- **Performance Metrics**: Calculate key metrics for the resulting portfolio, such as:
  - **Volatility**: Total portfolio volatility.
  - **Risk Contribution**: Risk allocated to each asset, checking if it aligns with equal risk objectives.
- **Objective Alignment**: Assess how well each permutation meets the portfolio objective (e.g., minimizing volatility or equalizing risk).

4. **Choose the Best Permutation**:

- **Optimal Order Selection**: After evaluating all or a sufficient subset of permutations, select the one that best aligns with the defined objective.

  Example: If minimizing volatility is the goal, choose the order that results in the lowest portfolio volatility.

5. **Dynamic Re-optimization**:

- **Periodic Adjustment**: Market conditions can change, with correlations and volatilities shifting over time.
- **Re-optimization**: Periodically re-run the optimization process to adjust the orthogonalization order based on updated market data.
- **Goal**: By adapting the asset order dynamically, the portfolio remains optimized for changing market environments, enhancing performance resilience.

Interpretation and Result

1. **Objective**: **Minimizing Portfolio Volatility**

```
Best Permutation: ('Sensex', 'Bonds', 'Nifty50', 'Gold', 'Currency')
Minimum Portfolio Volatility: 0.20066397716610532
```

This output indicates:

- The optimal permutation (Sensex, Bonds, Nifty50, Gold, Currency) that minimizes portfolio volatility.
- The **minimum portfolio volatility** achieved, which in this example is 0.2006.

2. **Objective**: **Minimizing Portfolio Risk**

```
Minimum Portfolio Risk: 0.03866292290795632
Optimal Asset Order for Minimum Risk: ('Bonds', 'Sensex', 'currency', 'Nifty 50', 'Gold')
```

This output indicates:

- The optimal permutation (Sensex, Bonds, Gold, Nifty50, Currency) that minimizes portfolio Risk.
- The **minimum portfolio Risk** achieved, which in this example is 0.03866

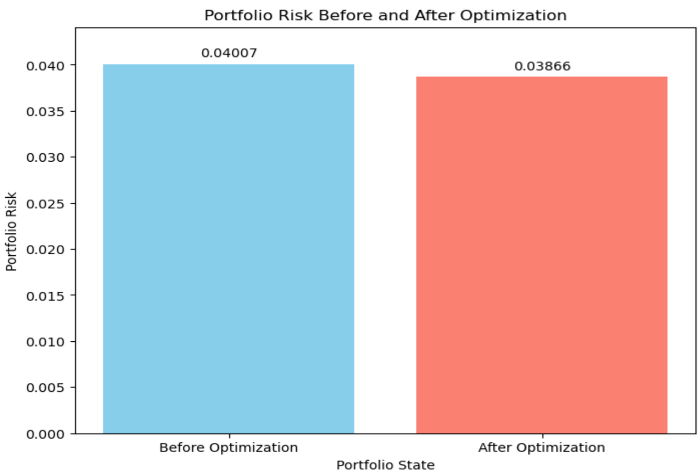


Fig: Comparison of Portfolio Risk before and after optimizing the order

Result and Conclusion

- **Risk Reduction Through GSOP:**
  Gram-Schmidt Orthonormalization reduces the risk of portfolio by converting the returns of assets into uncorrelated components. The portfolio risk of 0.0401 obtained using GSOP has lower variability of returns when compared to other methods.

- **Balanced Risk Contributions:**
  From GSOP we get a relatively balanced risk distribution of assets. For example, after orthogonalization, Bonds contributed 22.2% to the risk, Nifty 20.9%, Gold 20.2%, Sensex 19.2%, and Currency 17.5%.
  The removal of the unnecessary risk factors gives us a better diversified portfolio which is stable as well.

- **Identifying the Ideal Orders:**
  The asset order during the orthogonalization process impacts the performance of the portfolio. Reordering resulted in identifying permutations of orders that minimize overall volatility and risk with regard to the portfolio.
  The ideal permutations include orders such as Sensex, Bonds, Nifty50, Gold, Currency to reduce volatility and Sensex, Bonds, Gold, Nifty50, Currency to reduce risk which comes out to be 0.03866 in this case.

- **Comparison between the methods:**
  GSOP performed better than the other portfolios, such as RPP and RBP in terms of lower risk; whereas the PCRBP gives a different insight than the basis of diversification on statistical factors.

Limitations

- **Dependency:**

The analysis depends on historical data over the period of 2014–2023, which assumes that the past volatility and correlations will continue into the future, not at all times in scenarios with changing market conditions.

- **Computational Complexity:**

The number of combinations to be tested for optimal ordering are going to increase exponentially with the increase in portfolio size. Although it’s feasible for five assets, it is not feasible for a larger number of assets.

- **Static Covariance Assumption:**

The covariation matrices are assumed static for the entire length of the analysis period. This is, in fact, might not be a right estimation of dynamic behaviour of time in the market.

- **Sensitivity to Input:**

The approach is sensitive to initial assumptions such as weight constraints and choice of risk measures to be followed, which may affect the result.

Future Scope

- **Dynamic Re-optimization**

Apply ML or optimization algorithms to order portfolios dynamically with respect to changes in the market.

- **Wider Asset Class Coverage**

Expand the analysis to include other asset classes such as real estate, or cryptocurrencies to get a diversified of portfolio construction.

- **International Market Application**

Try Applying GSOP across other international markets which carry more volume than the Indian market to check whether it works equally well in different and complex market settings.

- **Real-Time Application**

Develop real-time portfolio management systems using GSOP with up-to-date market data for adaptive risk management.

- **Scenario Analysis**

Test the performance of portfolios under stress scenarios as well as the black swan events to analyse their vulnerability towards extreme market conditions.
Thus, this study underlines the possibility of realizing high-performance resilient, risk-optimized portfolios that can be specifically tailored toward market dynamics using advanced mathematical frameworks such as GSOP. The methodologies de veloped here may be refined and extended for further applications and impacts in future research work

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