S is maximal orthonormal $\Rightarrow$ $w' = 0$.

i.e. $w - \sum (w, v_i) v_i = 0$.

$$\Rightarrow w = \sum \underbrace{(w, v_i)}_{\alpha_i} v_i = \sum_{i=1}^{n} \alpha_i v_i$$

$\therefore$ S spans V, so S is a basis. $\square$

Gram-Schmidt orthonormalization.

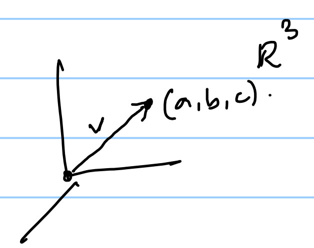
Given a lin. indep. set of non-zero vectors $\{v_1, \dots, v_n\}$, can we form a set of orthonormal vectors $\{q_1, \dots, q_n\}$ such that

1. $\text{span}\{v_1, \dots, v_k\} = \text{span}\{q_1, \dots, q_k\}$, $\forall k = 1, \dots, n$.
2. each $q_{j+1} \perp \{q_1, \dots, q_j\}$.

$$\begin{array}{cc} v & w \\ \begin{pmatrix} a \\ b \\ c \end{pmatrix}, & \begin{pmatrix} d \\ e \\ f \end{pmatrix} \end{array} \in \mathbb{R}^3$$

1. distance between 2 points: $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$.
2. length of v. $= \sqrt{v \cdot v}$ $\quad |v| = \sqrt{a^2 + b^2 + c^2}$
3. $v \cdot w = |v| \cdot |w| \cdot \cos\theta$

$$\cos^{-1}\left(\frac{v \cdot w}{|v| \cdot |w|}\right).$$



$\mathbb{R}^3$: • Standard inner product :

$$\left\langle \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \right\rangle = x_1 y_1 + x_2 y_2 + x_3 y_3 = \sum_{i=1}^{3} x_i y_i$$

$$\underset{x}{\overset{\shortparallel}{}} \qquad \underset{y}{\overset{\shortparallel}{}}$$

This is an inner product & in fact it is the dot product on $\mathbb{R}^3$!

$\mathbb{C}^3$ - Standard inner product

$$\left\langle \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \right\rangle = x_1 \bar{y}_1 + x_2 \bar{y}_2 + x_3 \bar{y}_3 = \sum_{i=1}^{3} x_i \bar{y}_i$$

$\mathbb{C}^n$: standard inner product :

$\underline{\underline{\langle x, y \rangle = y^* x}}$.

$$\left\langle \underset{x}{\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}}, \underset{y}{\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}} \right\rangle = \sum_{i=1}^{n} x_i \bar{y}_i = \underline{\underline{y^* x}}.$$

$$y^* x = (\bar{y}_1 \ \bar{y}_2 \cdots \bar{y}_n) \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = x_1 \bar{y}_1 + x_2 \bar{y}_2 + \cdots + x_n \bar{y}_n$$

lin indep $\{v_1, v_2, v_3\}$. $\longrightarrow$ $\{q_1, q_2, q_3\}$ orthonormal set.

① $q_1 = \dfrac{v_1}{\|v_1\|} \Rightarrow \|q_1\| = 1$.

② $(q_2, q_1) = 0$ & $\|q_2\| = 1$.

$$q_2 = \frac{v_2 - \langle v_2, q_1 \rangle q_1}{\| v_2 - \langle v_2, q_1 \rangle q_1 \|}.$$

③ $(q_3, q_1) = 0$, $(q_3, q_2) = 0$ & $\|q_3\| = 1$.

$$q_3 = \frac{w}{\|w\|},$$

$$w = v_3 - \langle v_3, q_1 \rangle q_1 - \langle v_3, q_2 \rangle q_2$$

lin. indep. $\{v_1, \dots, v_n\} \longrightarrow \{q_1, q_k, q_{k+1}, \dots q_n\}$.