Inner product spaces.

Note Title 12-02-2021

Let V be a vector space over F, $(F = \mathbb{R}$ or $\mathbb{C})$.

Defn: An inner product on V is a function $(\_,\_) : V \times V \to F$ satisfying :

(i) $(\alpha u + \beta v, w) = \alpha(u,w) + \beta(v,w)$ (linearity in first component)

(ii) $(v, u) = \overline{(u,v)}$ (conjugate symmetry)

(iii) $(u,u) > 0 \quad \forall\, u, v \in V \setminus \{0\}$.

Remarks: ① If $F = \mathbb{R}$, then $(u, \alpha v + \beta w) = (\alpha v + \beta w, u)$

$= \alpha(v,u) + \beta(w,u)$.

(linearity in the second component)

② If $F = \mathbb{C}$, then $(u, \alpha v + \beta w) = \overline{(\alpha v + \beta w, u)}$

$= (\bar{\alpha} v + \bar{\beta} w, u)$

$= \bar{\alpha}(v,u) + \bar{\beta}(w,u)$.

(Over $\mathbb{C}$, inner product is conjugate linear in the second component).

③ $(0, v) = (0+0, v) = (0,v) + (0,v) \implies (0,v) = 0 \;\forall v \in V$.

$\therefore (0,v) = (v,0) = 0$.

④ $(u,u) = 0 \iff u = 0$.

Defn: The norm (or length) of a vector v is defined as

$$\|v\| = \sqrt{(v,v)}.$$

Note : ① $\|v\| = 0 \iff v = 0$.

② If $\|v\| = 1$, v is called a unit vector.

Theorem : Let V be an inner product space over F. Then :

① $\|u \pm v\|^2 = \|u\|^2 \pm 2\,\mathrm{Re}\,(u,v) + \|v\|^2$.

② $\|u+v\|^2 + \|u-v\|^2 = 2\|u\|^2 + 2\|v\|^2$ (Parallelogram law).

③ $\|\lambda u\| = |\lambda| \cdot \|u\| \quad \forall\, \lambda \in F.$

(HW) ④ Polarization identities:

$$4(u,v) = \begin{cases} \|u+v\|^2 - \|u-v\|^2 & \text{if } F = \mathbb{R} \\ \|u+v\|^2 - \|u-v\|^2 + i\|u+iv\| - i\|u-iv\| & \text{if } F = \mathbb{C}. \end{cases}$$

⑤ $|(u,v)| \leq \|u\| \cdot \|v\|.$ (Cauchy Schwarz inequality).

⑥ $\|u \pm v\| \leq \|u\| + \|v\|.$ (Triangle inequality).

⑦ $\|u\| - \|v\| \leq \|u - v\|.$

Proof: ① $\|u \pm v\|^2 = (u \pm v,\ u \pm v) = (u,u) \pm (u,v) \pm (v,u) + (v,v)$

$$= \|u\|^2 \pm \left((u,v) + \overline{(u,v)}\right) + \|v\|^2.$$

$$= \|u\|^2 \pm 2\,\mathrm{Re}\,(u,v) + \|v\|^2.$$

② follows from ①.

③ $\|\lambda u\|^2 = (\lambda u, \lambda u) = \lambda \bar{\lambda}(u,u) = |\lambda|^2 (u,u)$

$$\therefore\ \|\lambda u\| = |\lambda| \cdot \|u\|.$$

④ follows from ①.

$$\lambda\bar{\mu}(u,v) = -\|v\|^2 \cdot \overline{(v,u)}\,(u,v) = -\|v\|^2 \cdot |(u,v)|^2$$

⑤ If $v = 0$, there is nothing to prove.

If $v \neq 0$, for any $\lambda, \mu \in F$,

$$0 \leq \|\lambda u + \mu v\|^2 = |\lambda|^2 \cdot \|u\|^2 + 2\,\mathrm{Re}\,\lambda\bar{\mu}(u,v) + |\mu|^2 \cdot \|v\|^2.$$

In particular if λ & μ are scalars $\lambda = \|v\|^2$, $\mu = -(u,v)$, then $\cancel{0 \leq \|u\|^2 \cdot \|v\|^4 - |(u,v)|^2 \cdot \|v\|^2}$

$$0 \leq \|u\|^2 \cdot \|v\|^4 - 2\|v\|^2 \cdot |(u,v)|^2 + |(u,v)|^2 \cdot \|v\|^2$$

$$= \|u\|^2 \cdot \|v\|^4 - |(u,v)|^2 \cdot \|v\|^2$$

$$\therefore\ |(u,v)|^2 \leq \|u\|^2 \cdot \|v\|^2 \quad \text{i.e.}\quad |(u,v)| \leq \|u\| \cdot \|v\|.$$

⑥, ⑦: HW.

Examples: (1) Standard inner product: $V = \mathbb{R}^n$

define $(x,y) = \sum_{i=1}^{n} x_i y_i$

$\left[n=3, \quad x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}, \quad x \cdot y = x_1 y_1 + x_2 y_2 + x_3 y_3 \right]$ (dot product)

$$(x,y) = \left(\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \right) = x_1 y_1 + \cdots + x_n y_n = (y_1 \cdots y_n) \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = y^t x$$

(2) Standard inner product: $V = \mathbb{C}^n$, $(x,y) = \sum x_i \bar{y}_i$

$$\left(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \right) = x_1 \bar{y}_1 + x_2 \bar{y}_2 + x_3 \bar{y}_3 = (\bar{y}_1 \ \bar{y}_2 \ \bar{y}_3) \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$= y^* x$. ('$*$' denotes conjugate transpose)

(3) $V = F^{m \times n}$, $(A,B) := tr(AB^*)$

Check: this is an inner product.

<u>Orthogonality</u>

Let V be an inner product space.

<u>Defn</u>: Two vectors u & v in V are said to be orthogonal if $(u,v) = 0$.

(denote $u \perp v$).

<u>Defn</u>: If S and T are subsets of V, then S is said to be orthogonal to T if for each $s \in S$ & $t \in T$, $(s,t) = 0$.

<u>Defn</u>: For any subset S of V, $S^\perp$ (S perp) will denote $\{ v \in V / (s,v) = 0 \ \forall s \in S \}$.