Householder's matrices & Givens' matrices (for QR factorization).

Idea of Householder's method:

$$(m\geq n)\quad \underset{A}{\begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix}_{m\times n}} \xrightarrow{Q_1} \underset{A_1 = Q_1 A}{\begin{bmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & \vdots & & \vdots \\ \vdots & & & \\ 0 & * & * & * \end{bmatrix}} \xrightarrow{Q_2} \underset{A_2 = Q_2 A}{\begin{bmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & \vdots & \vdots \\ \vdots & \vdots & \vdots & \\ 0 & 0 & * & * \end{bmatrix}} \longmapsto \cdots \to \underset{R}{\begin{bmatrix} * & * & & \\ 0 & * & & \\ \vdots & \vdots & \ddots & * \\ 0 & 0 & & 0 \end{bmatrix}} \quad \cdots$$

$$\underbrace{Q_n \cdots Q_1}_{Q^{-1}} A = R \quad (\text{upper } \Delta^r\ R)$$

$$A = QR.$$

Each Q_k is a unitary $m\times m$ matrix of the form-

$$Q_k = \left[\begin{array}{c|c} I_{k-1} & 0 \\ \hline 0 & F \end{array}\right]$$

where F is $(m-k+1)\times(m-k+1)$ unitary matrix.

F must be such that premultiplication by F induces zeroes below diagonal in the k^{th} column of $A_{k-1} = Q_{k-1}\cdots Q_1 A$.

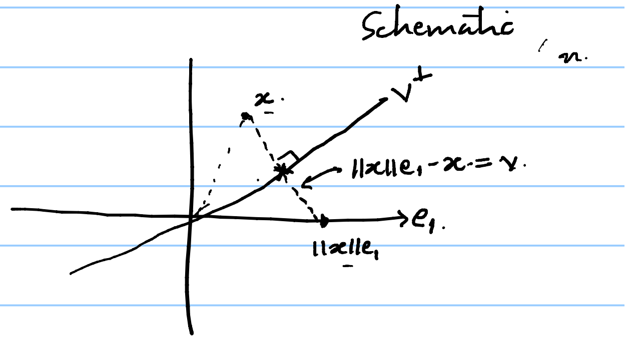
$$A_{k-1} = \begin{array}{c} \begin{array}{cccccc} & & k-1 & k & & \end{array} \\ \begin{bmatrix} * & & & * & \cdots & * \\ 0 & * & & & & * \\ \vdots & & * & \vdots & & \vdots \\ \vdots & & 0 & \to * & & \vdots \\ 0 & & 0 & \boxed{\begin{bmatrix} * \\ \vdots \\ * \end{bmatrix}} & & * \end{bmatrix}_{m\times n} \end{array}$$

For $x \in \mathbb{C}^{m-k+1}$, F is such that

$$x \xmapsto{F} \|x\|e_1 \in \mathbb{C}^{m-k+1}.$$

$$x = \begin{bmatrix} * \\ * \\ \vdots \\ * \end{bmatrix} \longmapsto \begin{bmatrix} \|x\| \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Householder's idea was to choose a vector v such that the reflection across $H = v^\perp$ maps x to $\|x\|e_1$.



The projection $P_{\perp v} = I - \dfrac{vv^*}{v^*v}$ maps any vector onto H; to reflect across H we must go twice that length in the same direction.

You can check that $\overset{\text{reflection across } H.}{\overset{\downarrow}{F}} = I - \frac{2vv^*}{v^*v}$ achieves this reflection.

F is called a "Householder matrix".

Note that F is unitary & hence full rank. $\left[\begin{array}{l} F^* = \left(I - 2\frac{vv^*}{v^*v}\right)^* = I - \frac{2\,vv^*}{v^*v} = F \\ \text{So } FF^* = F^*F = F^2 = I \end{array}\right.$

$$\downarrow$$

$$F^2 = \left(I - 2\frac{vv^*}{v^*v}\right)^2 = I^2 - 2\left(\frac{2vv^*}{v^*v}\right) + \frac{4(vv^*)^2}{(v^*v)^2}$$

We could choose v to be $z\|x\|e_1 - x$ where z is any scalar with $|z| = 1$.

For numerical purposes, we choose z such that $z \cdot \|x\|e_1$ is not too close to x (for if it is, then it is possible to get cancellation errors when calculating v.)

We choose $z = -\text{sign}(x_1)$

(we could've chosen $-\text{sign}(x_i)$ for any component x_i)

$\therefore -v = -\text{sign}(x_1) \cdot \|x\| e_1 - x$

or $v = \text{sign}(x_1)\|x\|e_1 + x$ (since $v^\perp$ & $-v^\perp$ are the same hyperplane.)

Example: $A = \begin{bmatrix} 1 & 2 \\ 4 & 5 \\ 4 & 8 \\ 4 & 2 \end{bmatrix}$

Find Q_1 such that $Q_1 A = \begin{bmatrix} * & * \\ 0 & * \\ 0 & * \\ 0 & * \end{bmatrix}$

Here, $x = \begin{pmatrix} 1 \\ 4 \\ 4 \\ 4 \end{pmatrix}$, want Q_1 s.t. $Q_1(x) = \|x\|_2 e_1 = \begin{pmatrix} \sqrt{1+4^2+4^2+4^2} \\ 0 \\ 0 \\ 0 \end{pmatrix}$

$$= \begin{pmatrix} 7 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

$$v = \text{sign}(x_1) \cdot \|x\| e_1 + x = \begin{pmatrix} 7 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 4 \\ 4 \\ 4 \end{pmatrix} = \begin{pmatrix} 8 \\ 4 \\ 4 \\ 4 \end{pmatrix}$$

$$Q_1 = I - \frac{2 vv^*}{v^* v} = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} - \frac{2}{v^* v} \begin{bmatrix} & \\ & \end{bmatrix}$$

(left as exercise). $= \; . \; . \; . \; = \begin{bmatrix} & \\ & \end{bmatrix}$

$$Q_1 A = \begin{bmatrix} * & * \\ 0 & * \\ 0 & * \\ 0 & * \end{bmatrix}$$

Operation counts -

Using G-S: The operation count is dominated by the inner loop which requires (m multi + $(m-1)$ additions for r_{ij}) & (m multi + m subtractions for v_j)

$$\sim \sum_{i=1}^{n} \sum_{j=i+1}^{n} 4m \sim \sum_{i=1}^{n} 4mi \sim \underline{2mn^2} \text{ flops.}$$

For Householders matrix:

The operation count is dominated by the innermost loop.

Let l denote $(m-k+1)$

- for dot product & scalar product:

l multi $+ (l-1)$ additions $+$
l multi

$= 3l - 1$ flops.

Algorithm:

```
for $k = 1$ to $n$
    $x = A_{k:m,k}$
    $v_k = \text{sign}(x_1) \|x\|_2 e_1 + x$
    $v_k = v_k / \|v_k\|_2$
    for $j = k$ to $n$
        $A_{k:m,j} = A_{k:m,j} - 2 v_k (v_k^* A_{k:m,j})$
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$$x = \begin{pmatrix} a_{kk} \\ \vdots \\ a_{mk} \end{pmatrix}$$

$$A_{k:m,k:n} = \begin{bmatrix} a_{kk} & \cdots & a_{kn} \\ \vdots & & \\ a_{mk} & \cdots & a_{mn} \end{bmatrix}$$

- for subtraction : l flops.

Total ~ $4l$ flops. → first step : $4mn$

Second step : $4(m-1)(n-1)$

Summing over 2 for loops :

$$\sum_{k=1}^{n}\sum_{j=k}^{n} 4l$$

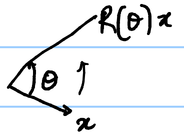
$$= \sum_{k=1}^{n-1} 4(m-k)(n-k) + 4mn$$

$$\vdots$$

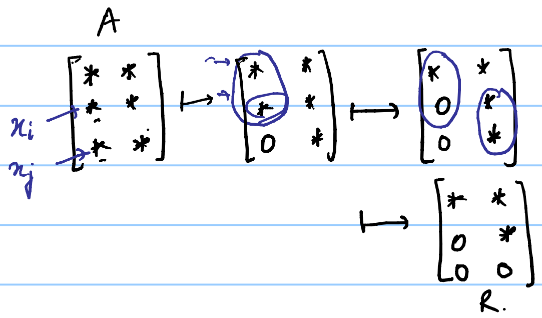
$$= 2mn^2 - \frac{2n^3}{3}.$$

Givens' matrices

A Givens' rotation $R(\theta) = \begin{bmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{bmatrix}$ rotates any vector $x \in \mathbb{R}^2$ counterclockwise by θ.



$$R(i,j,\theta) = \begin{array}{c|cccccc|} & & & i & & j & \\ \hline & 1 & & & & & \\ & & \ddots & & & & \\ i & & & \cos\theta & & -\sin\theta & \\ & & & & \ddots & & \\ j & & & \sin\theta & & \cos\theta & \\ & & & & & & \ddots \\ & & & & & & 1 \\ \hline \end{array}$$



$$\begin{pmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}x_i \\ x_j\end{pmatrix} = \begin{pmatrix}\sqrt{x_i^2+x_j^2} \\ 0\end{pmatrix}$$

i.e. $\cos\theta = \dfrac{x_i}{\sqrt{x_i^2+x_j^2}}$, $\sin\theta = \dfrac{-x_j}{\sqrt{x_i^2+x_j^2}}.$

Illustration :

$$\begin{pmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{pmatrix} \xmapsto{\begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & c & -s \\ & & & s & c \end{pmatrix}} \begin{pmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \end{pmatrix} \xmapsto{\begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & c' & -s' & \\ & & s' & c' & \\ & & & & 1 \end{pmatrix}} \begin{pmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \\ 0 & 0 & 0 & * \end{pmatrix}$$