Matrix norms.

Note Title

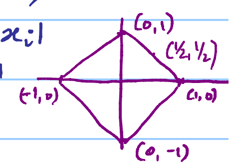
- Vector norm: a function $\|\cdot\| : V \to \mathbb{R}$ satisfying
  - $\|x\| \geq 0$ & $\|x\| = 0 \Leftrightarrow x = 0$
  - $\|x+y\| \leq \|x\| + \|y\|$ ✓
  - $\|\alpha x\| = |\alpha| \cdot \|x\|$. ✓

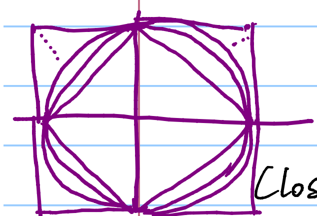
- An important class of vector norms is the class of p-norms defined for $p \geq 1$: ($x = (x_1, \dots, x_n)$)

$$\|x\|_p = \left(\sum |x_i|^p \right)^{1/p}.$$

$\mathbb{R}^2$. $n=2$, $x = (x_1, x_2)$

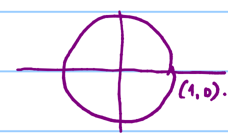
$p=1$: $\|x\|_1 = \sum |x_i| = |x_1| + |x_2|$





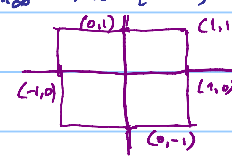
Closed unit discs in some p-norms.

$p=2$: $\|x\|_2 = \left(\sum |x_i|^2\right)^{1/2} = \left(x_1^2 + x_2^2\right)^{1/2}$



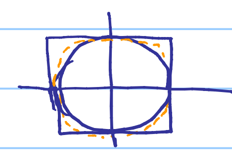
Defn: $\|x\|_\infty = \max_i \{ |x_i| \}$
(∞-norm)

$p=\infty$: $\|x\|_\infty = \max\{|x_1|, |x_2|\}$.



Another class: weighted p-norms
$\|x\|_W := \|Wx\|_p$ for any norm $\|\cdot\|$ & any non-singular matrix W.

$2 \leq p < \infty$



Matrix norms -

Defn: A matrix norm is a function $\|\cdot\| : \mathbb{C}^{m \times n} \to \mathbb{R}$ satisfying -

- $\|A\| \geq 0 \quad \forall A$ & $\|A\| = 0 \Leftrightarrow A = 0$
- $\|A+B\| \leq \|A\| + \|B\|$ ✓
- $\|\alpha A\| = |\alpha| \cdot \|A\|$ ✓
- $\|AB\| \leq \|A\| \cdot \|B\|$ (desirable ppty, whenever multiplication exists i.e. m=n.)

Important examples -

① Induced matrix norms: Suppose $A \in \mathbb{C}^{m \times n}$, consider $A : \mathbb{C}^n \to \mathbb{C}^m$
$\|\cdot\|_n$ ✓ $\|\cdot\|_m$ ✓

The induced matrix norm $\|A\|_{(m,n)}$ is the smallest scalar C such that -

$$\|Ax\|_m \leq C \|x\|_n$$

$\mathbb{C}^n \xrightarrow{A} \mathbb{C}^m$

i.e. $\dfrac{\|Ax\|_m}{\|x\|_n} \leq C.$ $\left\{ \begin{array}{l} C \text{ is the maximum} \\ \text{factor by which } A \text{ can} \\ \text{'stretch' } x. \end{array} \right.$

$\|\alpha x\| = |\alpha| \cdot \|x\|$, so $\dfrac{\|A\alpha x\|}{\|\alpha x\|} = \dfrac{|\alpha|\,\|Ax\|}{|\alpha|\,\|x\|} = \dfrac{\|Ax\|}{\|x\|}$

So it is sufficient to consider vectors x of norm 1.

$$\|A\|_{(m,n)} = \sup_{\substack{x \neq 0 \\ x \in \mathbb{C}^n}} \frac{\|Ax\|_m}{\|x\|_n}$$

$$= \sup_{\substack{x \in \mathbb{C}^n \\ \|x\|_n = 1}} \|Ax\|_m$$

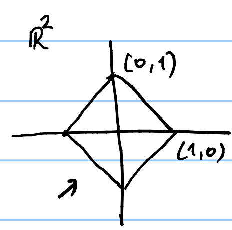
We will denote the induced matrix norm $\|A\|_{(p,p)}$ simply by $\|A\|_p$.

<u>Example</u> : $A = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$. Estimate the induced norms $\|A\|_1$, $\|A\|_2$ and $\|A\|_\infty$.
$(A : \mathbb{R}^2 \to \mathbb{R}^2)$.

Consider the action of A on unit discs in $\mathbb{R}^2$:

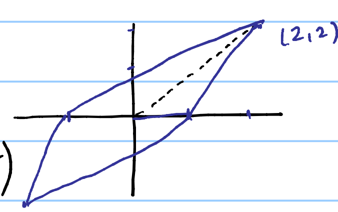
<u>1-norm:</u>

$\sup\limits_{\|x\|_1 = 1} \|Ax\|_1$



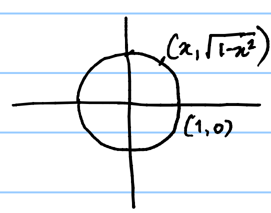
$\xrightarrow{A}$

$\begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + 2y \\ 2y \end{pmatrix}$

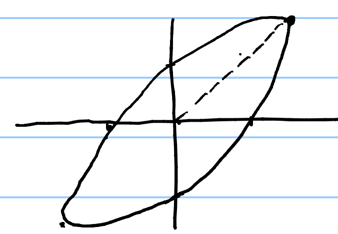


$\|A\|_1 = \sup\limits_{\|x\|_1 = 1} \|Ax\|_1$
$= 4.$

<u>2-norm:</u>



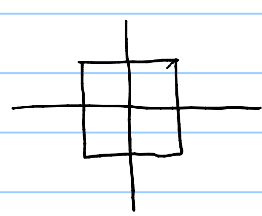
$\xrightarrow{A}$



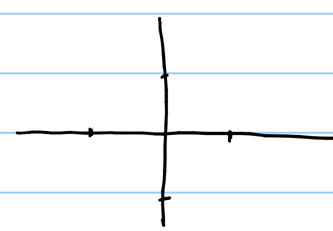
$$\begin{array}{ccc} A & x & Ax \\ \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix} & \begin{bmatrix} x \\ \sqrt{1-x^2} \end{bmatrix} = & \begin{bmatrix} x + 2\sqrt{1-x^2} \\ 2\sqrt{1-x^2} \end{bmatrix} \end{array}$$

$\|A\|_2 \approx 2.92 \left(\begin{array}{l} \geq \sqrt{2^2 + 2^2} \\ = \sqrt{8}. \end{array} \right)$

<u>∞-norm</u>



$\xrightarrow{A}$



$\|A\|_\infty =$

(Exercise).

Theorem: Let A be a $m \times n$ matrix, $A = (a_{ij})_{\substack{1 \le i \le m \\ 1 \le j \le n}}$.

Let a_j denote the j^{th} column of A and a_i^* denote the ith row of A.

(i) $\|A\|_1 = \max\limits_{1 \le j \le n} \|a_j\|_1 = \max\limits_j \sum\limits_{i=1}^m |a_{ij}|$ (maximum column sum.)

(ii) $\|A\|_\infty = \max\limits_{1 \le i \le n} \|a_i^*\|_\infty = \max\limits_i \sum\limits_{j=1}^n |a_{ij}|$ (maximum row sum.)

Proof: (i)

$$\|Ax\|_1 = \left\| \begin{bmatrix} \sum_j a_{1j} x_j \\ \sum_j a_{2j} x_j \\ \vdots \\ \sum_j a_{mj} x_j \end{bmatrix} \right\|_1 = \sum_{i=1}^m \left| \sum_{j=1}^n a_{ij} x_j \right|$$

$$\underset{(\Delta \text{ ineq.})}{\le} \sum \sum |a_{ij} x_j|$$

$$= \sum_{i=1}^m \sum_{j=1}^n |a_{ij}| \cdot |x_j|$$

$$= \sum_{j=1}^n |x_j| \underbrace{\sum_{i=1}^m |a_{ij}|}_{j^{th} \text{ column sum } =: c_j}$$

$$= \sum_{j=1}^n c_j |x_j|$$

$$\le \left(\max_j c_j\right) \sum_{j=1}^n |x_j|$$

$$= \left(\max_j c_j\right) \cdot \|x\|_1 .$$

$$\therefore \|Ax\|_1 \le \|x\|_1 \cdot \left(\max_j c_j\right)$$

$$\therefore \|A\|_1 = \sup_{x \ne 0} \frac{\|Ax\|_1}{\|x\|_1} \le \max_j \left\{ \sum_{i=1}^m |a_{ij}| \right\}$$

[Suppose $\max\limits_j \left\{ \sum\limits_{i=1}^m |a_{ij}| \right\}$ is attained at $j = j_0$, so $\sum\limits_{i=1}^m |a_{ij_0}|$ is the max.]

To show that the bound is attained, we need to demonstrate a vector $\underline{u}$ such that $\|Au\|_1$ equals the max. element $\sum\limits_{i=1}^m |a_{ij_0}|$.