Adjoints ; unitary operators.

Theorem: (Riesz representation theorem) Let V be a f.dim'l i.p.s. over field F. & f be a linear map $f : V \to F$ (such a map is called a "linear functional")

Then there exists a unique vector x in V such that

$$f(v) = \langle v, x \rangle \text{ for all } v \in V.$$

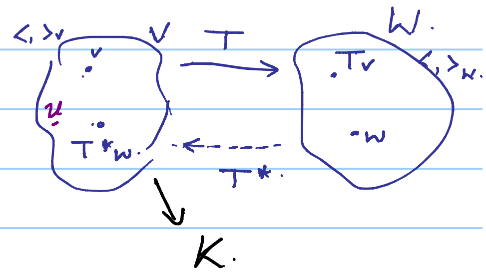
(So any linear functional on V can be expressed as an inner product with a fixed vector in V.
i.e. any linear $f : V \to F$ is of the form $\langle -, x \rangle$; so we may write f as $f_x = \langle -, x \rangle$)

$$\boxed{\begin{aligned}&\text{Check that for a fixed vector } y \in V, \text{ the map}\\ &\langle -, y \rangle : V \to F\\ &\quad v \mapsto \langle v, y \rangle\\ &\text{is a linear map on } V.\end{aligned}}$$

Theorem: Let V and W be f.d. i.p.s. over field K. and let $T \in L(V, W)$.
Then there exists a unique linear map $T^* : W \to V$ such that

$$\langle Tv, w \rangle_W = \langle v, T^* w \rangle_V \text{ for all } v \in V \text{ \& } w \in W.$$

Defn: T^* is called the adjoint of T.



Proof: For a fixed $w \in W$, consider the map

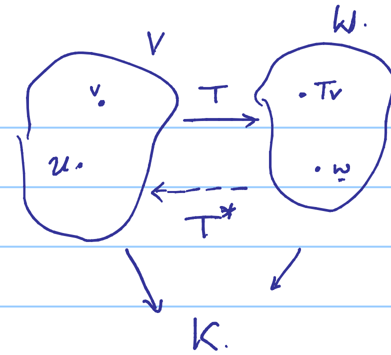
$$\begin{aligned} f_w : V &\to K \\ v &\mapsto \langle Tv, w \rangle_W. \end{aligned}$$

f_w is a linear map (check)

By the Riesz repr. theorem, $\exists$ a unique vector $u \in V$ such that $f_w(v) = \langle v, u \rangle_V$ for all $v \in V$.

Define $T^* w = u$.

① $\langle \underline{T}v, w\rangle = f_w(v) = \langle v, u\rangle = \langle v, \underline{T}^* w\rangle$



② T^* is linear -

Want to show: $T^*(\alpha x + \beta y) = \alpha T^* x + \beta T^* y$ for any $x, y \in W$.

We will show that for any $v \in V$,

$$\langle v, \underbrace{T^*(\alpha x+\beta y)}\rangle = \langle v, \alpha \underbrace{T^* x + \beta T^* y}\rangle$$

$$\begin{aligned} LHS = \langle Tv, \alpha x + \beta y\rangle &= \bar{\alpha}\langle Tv, x\rangle + \bar{\beta}\langle Tv, y\rangle \\ &= \bar{\alpha}\langle v, T^* x\rangle + \bar{\beta}\langle v, T^* y\rangle \\ &= \langle v, \alpha T^* x + \beta T^* y\rangle = RHS. \end{aligned}$$

$\square$

<u>Some properties of the adjoint</u> -

① If $S, T \in L(V,W)$, then $(S+T)^* = S^* + T^*$ & $(\lambda S)^* = \bar{\lambda} S^*$.

② If $S \in L(V,W)$, then $(S^*)^* = S$.

③ For $S, T \in L(V)$, $(S \circ T)^* = T^* \circ S^*$

④ If $T \in L(V)$ is invertible, then $(T^*)^{-1} = (T^{-1})^*$.

⑤ <u>Matrices</u>: (i) For $A \in K^{m\times n}$, the conjugate transpose $(\bar{A})^t$ is the adjoint of A, and is denoted by A^*.

For $v \in K^n$, $w \in K^m$,

$$\langle Av, w\rangle = \underbrace{w^* Av} = (A^* w)^* v = \langle v, A^* w\rangle.$$

(ii) If V and W are f.d. i.p.s. over K with ordered orthonormal bases $\mathcal{B}$ & $\mathcal{B}'$ resp., then for $T \in L(V,W)$,

$${}_{\mathcal{B}'}[T]_{\mathcal{B}}^* = {}_{\mathcal{B}}[T^*]_{\mathcal{B}'}.$$

(iii) For $T \in L(V)$, $\det T^* = \overline{\det T}$

and $\operatorname{tr} T^* = \overline{\operatorname{tr} T}$.

Unitary operators

Defn: A linear operator T on V is said to be unitary if $TT^* = T^*T = I$. ($\Leftrightarrow T^*$ is unitary)

Properties: If S & T are unitary, then
- $S \circ T$ is unitary
- S^{-1} is unitary.

Proposition: Let V be f.dim'l i.p.s. over F & $T \in L(V)$.
TFAE:
1. T is unitary
2. $\langle Tu, Tv \rangle = \langle u, v \rangle$ for all $u, v \in V$. [T preserves inner products]
3. $\|Tu\| = \|u\|$ for all $u \in V$ [T preserves norms]
4. T maps orthonormal bases to orthonormal bases.

Proof: 1 $\Leftrightarrow$ 2 $\quad \langle Tu, Tv \rangle = \langle u, T^*Tv \rangle = \langle u, v \rangle$ if & only if $T^*T = I$.

2 $\Leftrightarrow$ 3 $\quad$ 2 $\Rightarrow$ 3 clear from defn. of norm.
3 $\Rightarrow$ 2 exercise using polarization identity.

2 $\Leftrightarrow$ 4 $\quad$ 2 $\Rightarrow$ 4 follows from defn. of orthonormality.
4 $\Rightarrow$ 2: Let $\{u_1, \dots, u_n\}$ be an orthonormal basis of V, let $u = \sum_{i=1}^{n} \alpha_i u_i$, $v = \sum_{i=1}^{n} \beta_i u_i$

$$\langle Tu, Tv \rangle = \left\langle \sum_{i=1}^{n} \alpha_i Tu_i, \sum_{i=1}^{n} \beta_i Tu_i \right\rangle$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \overline{\beta_j} \langle Tu_i, Tu_j \rangle$$

$$= \sum_i \sum_j \alpha_i \overline{\beta_j} \delta_{ij}, \quad \delta_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{o.w.} \end{cases}$$

(Kronecker delta)

$$= \sum \alpha_i \overline{\beta_i} = \langle u, v \rangle$$

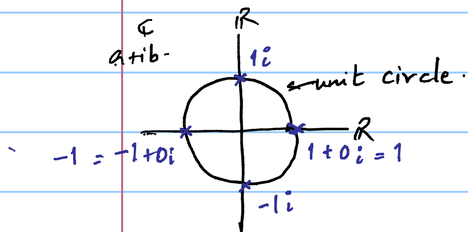
$\square$

Corollary: Eigenvalues of a unitary operator have absolute value 1.

Proof: Suppose λ is an e-value of a unitary operator T and suppose v is the corr. e-vector,

then $\|v\| = \|Tv\| = \|\lambda v\| = |\lambda| \cdot \|v\|$

So $|\lambda| = 1$. $(v \neq 0)$. $\square$



if $z = a+ib \in \mathbb{C}$, $\underset{1.}{|z|} = \sqrt{a^2+b^2}$. $a^2+b^2 = 1$.

eqn. of unit circle: $x^2+y^2=1$.

Note that the only real e-values of a unitary operator can be ± 1.

(Terminology: If $F = \mathbb{R}$, then the unitary operator is called 'orthogonal'.)

Proposition: Suppose $T \in L(V)$ & $\mathcal{B}$ is an ordered orthonormal basis of V. Then $[T]_{\mathcal{B}}$ is unitary matrix $\Leftrightarrow$ T is unitary (over $\mathbb{C}$)

[$[T]_{\mathcal{B}}$ is orthogonal matrix $\Leftrightarrow$ T is orthogonal (over $\mathbb{R}$)].

Proposition: Suppose A is a $n \times n$ matrix. TFAE

1. A is unitary
2. Columns of A form an orthonormal basis of F^n w.r.t. standard inner product
3. columns of A^t form an orthonormal basis of F^n w.r.t. the standard inner product.

Pf: (1) $\Rightarrow$ (2) apply A to the standard basis $\{e_1, \ldots, e_n\}$ $\qquad e_i = (0, \cdots, 0, \underset{i}{1}, 0 \cdots 0)^t$

$Ae_i = i^{th}$ column of A

A unitary $\Rightarrow$ A sends orthornormal basis $\{e_1, \dots, e_n\}$ to an orthonormal basis $\{\underbrace{Ae_1, \dots, Ae_n}_{\text{columns of } A}\}$.

② $\Rightarrow$ ① The $(i,j)^{th}$ entry of a matrix B $e_i^t B e_j$.

Suppose the columns of A are orthormal, then

$$\underset{\text{of } A^*A}{(i,j)^{th}\text{entry}} = e_i^t \left(A^* A\right) e_j = \left(A e_i\right)^* \left(A e_j\right) = \delta_{ij}$$

$$\therefore A^* A = I_n \Rightarrow A \text{ is unitary.} \qquad \square.$$

<u>Corollary</u>: If $T \in L(V)$ is unitary, and if $\mathcal{B}_1$ & $\mathcal{B}_2$ are ordered orthonormal bases of V then $[T]_{\mathcal{B}_2 \mathcal{B}_1}$ is a unitary matrix.