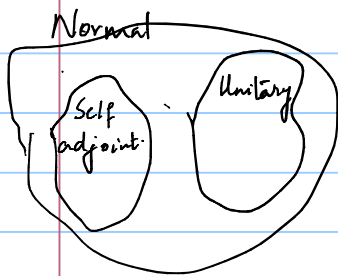
Note Title

Spectral theorem

Recap: V be a f.d. i.p.s. on field F. $T \in L(V)$.

Normal operators: defn. $TT^* = T^*T$.

Properties: ① T is normal $\iff \|Tv\| = \|T^*v\|$ for all $v \in V$.



② $Tv = \lambda v \Rightarrow T^*v = \bar{\lambda} v$

③ E-vectors corr. to distinct e-values are orthogonal

④ If $T^k v = 0$ for some $k \geq 1$, then $Tv = 0$

(i.e. if $v \in \ker T^k$ then $v \in \ker T$; thus $\ker T^k = \ker T$ for any $k \geq 1$)

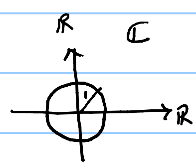
Unitary operators: defn. $TT^* = T^*T = I$. $(AA^* = A^*A = I_{n\times n})$

Properties: ① TFAE: (i) T is unitary

(ii) T preserves inner products

(iii) T preserves norm

(iv) T maps orthonormal bases to orthonormal bases.



② E-values of a unitary operator have absolute value 1

③ for $A \in F^{n\times n}$,

TFAE: ① A is unitary

② the columns of A form an orthonormal basis of F^n.

③ the rows of A form an orthonormal basis of F^n.

④ The change of basis matrix ${}_{\mathcal{B}_2}[T]_{\mathcal{B}_1}$ for orthonormal bases $\mathcal{B}_1$ & $\mathcal{B}_2$ is a unitary matrix.

Self-adjoint operators: Defn: $T = T^*$ (thus, $T^*T = TT^*$, so T is normal)

Properties: ① For every $v \in V$, $\langle Tv, v\rangle$ is real.

② $\langle Tv, v\rangle = 0 \ \forall v \in V \Rightarrow T = 0$.

③ If $T^k v = 0$ for some $k \geq 1$, then $Tv = 0$
(i.e. $\ker T^k = \ker T$)

④ All e-values of T are real.

⑤ e-vectors corr. to distinct e-values are orthogonal.

2 subclasses of self-adjoint operators -

	Positive semidefinite	Positive definite.
Defn:	$\langle Tv, v\rangle \geq 0 \quad \forall v \in V$	$\langle Tv, v\rangle > 0 \quad \forall v \in V$.
Equivalent conditions	• all e-values are non-neg. • pos. semidef. $\sqrt{T}$ exists. • $\exists\ S \in L(V)$ s.t. $T = S^*S$	• all e-values are positive • pos. def. $\sqrt{T}$ exists • $\exists$ invertible $S \in L(V)$ s.t. $T = S^*S$.

(Spectrum of T := set of e-values of T.)

The spectral theorem

Statement: Let T be a triangulable linear operator on a f.d. i.p.s. V. Then T is normal $\Longleftrightarrow$ V has an orthonormal basis consisting of eigenvectors of T.

Triangulable $\Rightarrow \exists$ inv. matrix P s.t. $T = P^{-1} \underbrace{T'}_{\text{upper } \Delta^r} P$.

Proof: Given assumption $\Rightarrow$ $[T]_{\mathcal{B}}$ is diagonal.

Let $[T^*]_{\mathcal{B}}$ denote matrix of T^* w.r.t this basis, then we know that $[T^*] = [T]^*$, which is also diagonal.

$T = P^{-1} D P$, $P \to$ unitary. "unitarily diagonalizable".

Since diagonal matrices commute,

$$[TT^*] = [T]\cdot[T^*] = [T^*][T] = [T^*T],$$

so the operators TT^* & T^*T must be equal. Hence T is normal.

Conversely, if T is triangulable, then its minimal polynomial splits over F, so

$$m_T(x) = (x-\lambda_1)^{m_1}(x_2-\lambda_2)^{m_2}\cdots(x-\lambda_k)^{m_k}$$

(λ_i's are distinct).

Primary decomposition thm. implies

$$V = \left[\ker(T-\lambda_1 I)^{m_1}\right] \oplus \cdots \oplus \ker\left[(T-\lambda_k I)^{m_k}\right]$$

T normal $\Rightarrow$ $(T-\lambda_i I)$ is normal

$\Rightarrow \ker(T-\lambda_i I)^{m_i} = \ker(T-\lambda_i I)$

$$\therefore V = \underbrace{\ker(T-\lambda_1 I)}_{\mathcal{B}_1} \oplus \cdots \oplus \underbrace{\ker(T-\lambda_k I)}_{\mathcal{B}_k}$$

Let $\mathcal{B}_i$ be orthonormal basis of $\ker(T-\lambda_i I)$,

& let $\mathcal{B} = \mathcal{B}_1 \cup \cdots \cup \mathcal{B}_k$,

Then $\mathcal{B}$ is orthonormal & it consists of e-vectors of T.

$T = P^{-1} \underbrace{T'}_{\substack{\text{upper} \\ \Delta^r}} P$ unitary. "unitarily triangulable".

Thm: Schur's theorem: Let T be a triangulable linear operator on a f.d.i.p.s. V. Then $\exists$ an orthonormal basis $\mathcal{B}$ of V such that the matrix of T w.r.t. $\mathcal{B}$ is upper Δ'r.

Proof: Since T is triangulable, by defn. $\exists$ a basis $\mathcal{B}' = \{v_1, \dots, v_n\}$ such $[T]_{\mathcal{B}'}$ is upper Δ'r.

If $\mathcal{B}'$ is not orthonormal, apply the G-S orthonormalisation and find orthonormal basis $\mathcal{B} = \{u_1, \dots, u_n\}$ such that -

$$u_1 = \frac{v_1}{\|v_1\|}, \qquad u_2 = \frac{v_2 - \overbrace{\langle v_2, u_1 \rangle}^{\alpha_{12}} u_1}{\underbrace{\| \cdots \|}_{\alpha_{22}}}, \dots$$

let α_{11} denote $\|v_1\|$,

$$v_1 = \alpha_{11} u_1, \qquad v_2 = \alpha_{22} u_2 + \alpha_{12} u_1, \dots$$

(continue this way & get $v_k = \alpha_{1k} u_1 + \alpha_{2k} u_2 + \dots + \alpha_{kk} u_k$)

$${}_{\mathcal{B}}[I]_{\mathcal{B}'} = \big[I(v_1) \big| \cdots \big| I(v_n) \big]_{\mathcal{B}}$$

$$= \begin{bmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1k} & \cdots & \alpha_{1n} \\ 0 & \alpha_{22} & & \vdots & & \vdots \\ \vdots & 0 & & \alpha_{kk} & & \vdots \\ 0 & \vdots & & 0 & & \alpha_{nn} \end{bmatrix}$$

$$[T]_{\mathcal{B}} = {}_{\mathcal{B}}[I]_{\mathcal{B}'} [T]_{\mathcal{B}'} {}_{\mathcal{B}'}[I]_{\mathcal{B}}$$

$\therefore [T]_{\mathcal{B}}$ is upper Δ'r.

Check: $T(n)$ = set of all $n \times n$ upper Δ'r matrices

1. $T(n)$ is closed under matrix multiplication
2. The inverse of an element in $T(n)$ is also in $T(n)$.

Corollary: If A is a unitary upper Δ'r matrix, then $A^{-1} = A^*$ (is also upper Δ'r)

hence A will be a diagonal matrix with diagonal entries having abs. value 1.

λ is an eigenvalue of $T \Leftrightarrow \bar{\lambda}$ is an eigenvalue of T^*

λ is an e-value of $T \Leftrightarrow (T-\lambda I)$ is invertible

i.e. $\exists S$ s.t. $S(T-\lambda I) = (T-\lambda I)S = I$

Taking adjoints, $S^*(T^* - \bar{\lambda} I) = (T^* - \bar{\lambda} I)S^* = I^* = I$.

Thus, $T^* - \bar{\lambda} I$ is invertible, which means $\bar{\lambda}$ is an e-value of T^*.

Converse holds by similar argument.

However, the corr. e-vectors need not be the same.

Eg: Let $T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, then $\lambda = 1$ is an e-value of T with e-vector $v = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$T^* = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ also has 1 as e-value but the corr. e-vector is $w = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, not v.!

(Notice that $T^* v = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ so $T^* v \neq \lambda v$.)

- Notice that every matrix is triangulable over $\mathbb{C}$, since every polynomial splits over $\mathbb{C}$.
- In matrix form, the spectral theorem says that if the matrix A is normal, then $\exists$ a unitary matrix $\mathcal{U}$ such that $\mathcal{U}^* A \mathcal{U}$ is diagonal; with e-values of A on the diagonal.
  (i.e. $A = \mathcal{U} D \mathcal{U}^*$)
- For self-adjoint matrix A, the spectral theorem says that $\exists$ an orthogonal matrix P such that $P^{-1}AP$ is diagonal, with e-values of A on the diagonal.
- The above statement can also be proved for a symmetric matrix or for self-adjoint operators over real vector spaces, although the proof is harder.