Divide & conquer method.

. The idea of this method is to divide a symmetric tridiagonal matrix into smaller & smaller matrices. We may continue this division till we get a diagonal matrix (with e-values on the diagonal) or use other algorithms to find e-values for the smaller blocks.

Let $T \in \mathbb{R}^{m \times m}$ be a symmetric, tridiagonal matrix with non-zero off-diagonal entries.

Let $n \approx m/2$. Split the matrix T into submatrices

$$[T_1]_{n\times n} \quad \text{and} \quad [T_2]_{(m-n)\times(m-n)}.$$

$$T = \left[\begin{array}{c|c} [T_1] & \beta \\ \hline \beta & [T_2] \end{array}\right]_{m\times m} = \left[\begin{array}{c|c} \hat{T}_1 & \\ \hline & \hat{T}_2 \end{array}\right] + \underbrace{\left[\begin{array}{c|c} \beta & \beta \\ \hline \beta & \beta \end{array}\right]}_{\text{rank-1 correction}}$$

$\hat{T}_1$, $\hat{T}_2$ are tridiagonal symm. + rank-1 correction

Suppose we have diagonalizations

$$\hat{T}_1 = Q_1 D_1 Q_1^T, \quad \hat{T}_2 = Q_2 D_2 Q_2^T,$$

$$T = \left[\begin{array}{c|c} \hat{T}_1 & \\ \hline & \hat{T}_2 \end{array}\right] + \left[\begin{array}{c|c} \beta & \beta \\ \hline \beta & \beta \end{array}\right] = \left[\begin{array}{c|c} Q_1 & \\ \hline & Q_2 \end{array}\right] \left(\left[\begin{array}{c|c} D_1 & \\ \hline & D_2 \end{array}\right] + \beta z z^T \right) \left[\begin{array}{c|c} Q_1^T & \\ \hline & Q_2^T \end{array}\right]$$

where $z^T = (q_1^T, q_2^T)$,

where q_1 is the last row of Q_1

& q_2 is the first row of Q_2.

(diagonal + rank-1 correction)

What are the e-values of $D + ww^T$, where D is diagonal is $w \neq 0$.

If λ is an e-value of $D + ww^T$, with e-vector $q\,(\neq 0)$

then $(D + ww^T)q = \lambda q$.

i.e. $(D - \lambda I)q + ww^Tq = 0$

i.e. $q + (D - \lambda I)^{-1} ww^Tq = 0$

i.e. $w^Tq + w^T(D - \lambda I)^{-1} w w^Tq = 0$.

i.e. $w^Tq\left(1 + w^T(D - \lambda I)^{-1} w\right) = 0$.

which is true if
$$\underbrace{1 + \sum_{i=1}^{m} \frac{w_i^2}{d_i - \lambda}}_{f(\lambda)\ \text{"secular function"}} = 0.$$

Roots of $f(\lambda)$ are the e-values of $D + ww^T$.

These can be found using Newton-like methods which are fast & inexpensive.