Self-adjoint operators.

Note Title

Let V be a finite dim'l i.p.s. over field F. ($F = \mathbb{R}$ or $\mathbb{C}$).

Defn: A linear operator $T \in L(V)$ is said to be self-adjoint if
$$T = T^*.$$

- Consider $A \in F^{n \times n}$ as a linear operator on F^n, with the standard inner product. Then,

  A is a self-adjoint operator $\iff A^* = A$ i.e. A is a Hermitian matrix.

- If $T \in L(V)$ is self-adjoint, then the matrix of T w.r.t. an ordered orthonormal basis $\mathcal{B}$ of V is Hermitian, i.e. $[T]_{\mathcal{B}}$ is Hermitian.

  (since $[T]_{\mathcal{B}} = [T^*]_{\mathcal{B}} \underset{\substack{\uparrow \\ \text{needs orthonormality of } \mathcal{B}}}{=} [T]_{\mathcal{B}}^*$)

  (See Prop. 3.3

- If $F = \mathbb{R}$, then $T \in L(V)$ is self-adjoint $\iff$ its matrix w.r.t. an orthonormal basis is symmetric.

Proposition: Let $T \in L(V)$ be self-adjoint. Then

(i) for all $v \in V$, $\langle Tv, v \rangle$ is a real number.

(ii) If $\langle Tv, v \rangle = 0$ for all $v \in V$, then $T \equiv 0$

(iii) If $T^k v = 0$ for some $k > 1$, then $Tv = 0$.

(In general, if $Tv = 0$ then $T^k v = 0$ for any k i.e. $\ker T \subset \ker T^k$

If T is self-adjoint, we also have $\ker T^k \subset \ker T$ for some $k > 1$.)

(iv) All eigenvalues of T are real.
(v) E-vectors corr. to distinct e-values are orthogonal. } V. Imp.

Proof: (i) $\langle Tv, v\rangle = \overline{\langle v, Tv\rangle} = \overline{\langle v, T^* v\rangle} = \overline{\langle Tv, v\rangle}$,

hence $\langle Tv, v\rangle$ is a real number.

(ii) Suppose $\langle Tv, v\rangle = 0$ for all $v \in V$.

$\therefore$ $0 = \langle T(x+y), x+y\rangle = \underbrace{\langle Tx, x\rangle}_{=0} + \underbrace{\langle Tx, y\rangle + \langle Ty, x\rangle} + \underbrace{\langle Ty, y\rangle}_{=0}$
for any $x, y \in V$.

$= \langle Tx, y\rangle + \langle Ty, x\rangle$

$\langle Ty, x\rangle = \langle y, T^* x\rangle = \langle y, Tx\rangle$

$= 2\langle Tx, y\rangle$

$\therefore$ $\langle Tx, y\rangle = 0$ for any $x, y \in V$.
$\therefore$ $Tx = 0$ for any $x \in V$
$\Rightarrow T \equiv 0$.

(iii) If $k=1$, nothing to prove.
If $k>1$, choose $m \geq 1$ s.t. $2^{m-1} < k < 2^m$.
$T^k v = 0 \Rightarrow T^{2^m} v = 0$

Consider $\langle T^{2^{m-1}} v, T^{2^{m-1}} v\rangle = \langle T^{2^m} v, v\rangle = 0$

By induction hypothesis, $Tv = 0$.

(iv) If $F = \mathbb{C}$, let λ be an e-value with corr. e-vector v, then
$\lambda\langle v, v\rangle = \langle \lambda v, v\rangle = \langle Tv, v\rangle = \langle v, T^* v\rangle$
$= \langle v, Tv\rangle$
$= \langle v, \lambda v\rangle$
$= \bar{\lambda}\langle v, v\rangle$

Since $\langle v, v\rangle \neq 0$, we must have $\lambda = \bar{\lambda}$ i.e. λ is real.

If $F = \mathbb{R}$, then $C_T(x) \in \mathbb{R}[x]$ can still have complex roots.

To prove that all e-values are real, we need to prove that the roots of $C_T(x)$ are real.

T is self-adjoint $\Rightarrow$ matrix of T w.r.t. an ordered orthonormal basis $\mathcal{B}$ is symmetric.

i.e. $A = [T]_{\mathcal{B}}$ is symmetric.

Consider A as a linear operator on $\mathbb{R}^n$, then A is self-adj. & $C_A(x) = C_T(x)$; so all roots of $C_A(x)$ are reals $\Rightarrow$ all roots of $C_T(x)$ are real.

(v) If λ & μ are distinct e-values of T & u, v are corr. e-vectors, then

$$\lambda \langle u, v \rangle = \langle \lambda u, v \rangle = \langle Tu, v \rangle$$
$$= \langle u, Tv \rangle$$
$$= \langle u, \mu v \rangle$$
$$= \overline{\mu} \langle u, v \rangle$$
$$= \mu \langle u, v \rangle$$

Since $\lambda \neq \mu$, $\langle u, v \rangle = 0 \Rightarrow u$ is orthogonal to v. $\square$

Defn: A self-adj. operator T on V is said to be
- positive def. if $\langle Tv, v \rangle > 0$ for all $v \in V \setminus \{0\}$.
- positive semidef. if $\langle Tv, v \rangle \geq 0$ for all $v \in V \setminus \{0\}$.

Theorem: If $T \in L(V)$ is self-adjoint, then TFAE:

(i) T is pos. def.

(ii) eigenvalues of T are positive.

(iii) There exists a unique pos. def. operator A on V such that $A^2 = T$; A is also denoted by $\sqrt{T}$.

(iv) There exists an invertible linear operator $S \in L(V)$ such that $T = S^* S$.

Proof: (i) $\Rightarrow$ (ii) If λ is an e-value of T, then $\lambda \in \mathbb{R}$.

If v is the corr. e-vector, then $\langle Tv, v\rangle > 0$

i.e. $\langle \lambda v, v\rangle > 0$

i.e. $\lambda \cdot \underbrace{\langle v, v\rangle}_{>0} > 0$

$\Rightarrow \lambda > 0$.

(ii) $\Rightarrow$ (i) Let $\mathcal{B} = \{v_1, \ldots, v_n\}$ be an orthonormal basis of V (why?) consisting of e-vectors of T, so $Tv_i = \mu_i v_i$, $\mu_i > 0$ (by assumption)

For $v \in V$, suppose $v = \sum_{i=1}^{n} \alpha_i v_i$

$$\langle Tv, v\rangle = \langle T(\sum \alpha_i v_i), \sum \alpha_j v_j\rangle$$

$$= \langle \sum \alpha_i Tv_i, \sum \alpha_j v_j\rangle$$

$$= \langle \sum \alpha_i \mu_i v_i, \sum \alpha_j v_j\rangle$$

$$= \sum \sum \mu_i \alpha_i \bar{\alpha}_j \langle v_i, v_j\rangle$$

$$= \sum_{i=1}^{n} \mu_i |\alpha_i|^2 > 0 \quad \text{whenever } v \neq 0.$$

$\therefore T$ is pos. def.

(Rest of the proof - after spectral thm.)

Proposition: Let $T \in L(V)$ be self-adjoint. TFAE:

(i) T is pos. semidef.

(ii) The e-values of T are non-negative.

(iii) There exists a unique semi-def. operator A on V s.t. $A^2 = T$ (denote A by $\sqrt{T}$).

(iv) There exists $S \in L(V)$ such that $T = S^*S$.

Normal operators.

Defn: An operator $T \in L(V)$ is said to be normal if $T^*T = TT^*$

(Both unitary & self-adjoint operators are normal)

Proposition: If $T \in L(V)$ is normal, then:

(i) $\|Tv\| = \|T^*v\|$ for all $v \in V$.

(ii) If $Tv = \lambda v$ then $T^*v = \bar{\lambda} v$; i.e. if v is an e-vector of T with e-value λ, then it is also an e-vector of T^* with e-value $\bar{\lambda}$.

(iii) E-vectors corr. to distinct e-values of T are orthogonal.

(iv) If $T^k v = 0$ for some $k > 1$, then $Tv = 0$.

Proof: (i) Consider
$$\|Tv\|^2 = \langle Tv, Tv\rangle = \langle v, T^*Tv\rangle$$
$$= \langle v, TT^*v\rangle$$
$$= \langle T^*v, T^*v\rangle$$
$$= \|T^*v\|^2 \quad \text{for all } v \in V.$$
$\therefore \|Tv\| = \|T^*v\|$ for all $v \in V$.

(ii) Note that if T is normal then for any e-value λ of T, the operator $(T - \lambda I)$ is also normal

[i.e. $(T-\lambda I)(T-\lambda I)^* = (T-\lambda I)^*(T-\lambda I)$ (Exercise)

$\therefore \|(T-\lambda I)v\| = \|(T-\lambda I)^* v\|$

$\therefore Tv = \lambda v \iff T^*v = \bar{\lambda} v$.

(iii) Let u & v be e-vectors of T corr. to distinct e-values λ & μ.

$$\lambda\langle u, v\rangle = \langle \lambda u, v\rangle = \langle Tu, v\rangle = \langle u, T^*v\rangle = \langle u, \bar{\mu} v\rangle$$
$$= \mu\langle u, v\rangle$$

$\lambda \neq \mu$, so $\langle u, v\rangle = 0$ i.e. $u \perp v$.

(iv) If T is normal, then $S = T^*T$ is self-adjoint.

$\left(S^* = (T^*T)^* = T^*T \quad \therefore S = S^*\right)$

$T^k v = 0 \Rightarrow S^k v = 0$ $\left\{ S^k = (T^* T^k) \neq \underbrace{T^*T \cdot T^*T \cdots T^*T}_{k \text{ times.}} \right.$

$\Rightarrow Sv = 0.$ $\quad S^k v = 0 \}.$

$(\because S$ is self-adjoint$)$.

$$\|Tv\|^2 = \langle Tv, Tv\rangle = \langle v, T^*Tv\rangle = \langle v, \underbrace{Sv}_{=0}\rangle = 0$$

$$\Rightarrow Tv = 0. \qquad \square$$

Proposition: $T \in L(V)$ is normal $\Leftrightarrow \|Tv\| = \|T^*v\|$ for all $v \in V$.

Pf: Assume $\|Tv\| = \|T^*v\|$ for all $v \in V$

$$\Rightarrow \langle Tv, Tv\rangle = \langle T^*v, T^*v\rangle$$

$$\Rightarrow \langle T^*Tv, v\rangle = \langle TT^*v, v\rangle$$

$$\Rightarrow \langle T^*Tv, v\rangle - \langle TT^*v, v\rangle = 0$$

$$\Rightarrow \langle (T^*T - TT^*)v, v\rangle = 0 \rightarrow (\star)$$

Notice that $S = T^*T - TT^*$ is self-adjoint.

(Exercise)

So $(\star) \Rightarrow T^*T - TT^* = 0$

i.e. $T^*T = TT^*$. $\qquad \square$