G-S orthonormalization ; Adjoints.

Note Title 27-01-2022

Recall: If $\{x_1, \dots, x_n\}$ is an orthonormal basis of V, then for any $v \in V$, $v = \sum \langle v, x_i \rangle x_i$.

[firstly, $v = \sum \alpha_i x_i$, since $\{x_1, \dots, x_n\}$ is a basis.

Recall from last lecture:
if $w = v - \sum \langle v, x_i \rangle x_i$,
then $\langle w, x_i \rangle = 0 \ \forall i$, $\longrightarrow \because \{x_1, \dots, x_n\}$ is orthonormal basis.
$\therefore w = 0$
$\therefore v = \sum \underbrace{\langle v, x_i \rangle}_{\text{component of } v \text{ in direction } x_i} x_i$.

Thus, we have written v as a sum of its components in the n orthonormal directions.]

Gram-Schmidt orthonormalization:

Theorem: Let $\{x_1, \dots, x_n\}$ be a set of lin. indep vectors in an inner product space V. Then there exists a sequence of orthonormal vectors $\{y_1, \dots, y_n\}$ such that for every k -
$$\text{span}\{x_1, \dots, x_k\} = \text{span}\{y_1, \dots, y_k\}.$$

{This means: (i) $y_i \perp y_j$ whenever $i \neq j$, $1 \leq i, j \leq k$.
(ii) $\|y_i\| = 1 \ \forall i$.
(iii) $\langle x_1, \dots, x_k \rangle = \langle y_1, \dots, y_k \rangle \ \forall 1 \leq k \leq n$,

means:
$$\left\{\begin{aligned} &\langle x_1 \rangle = \langle y_1 \rangle \\ &\langle x_1, x_2 \rangle = \langle y_1, y_2 \rangle \\ &\vdots \\ &\langle x_1, x_2, \dots, x_k \rangle = \langle y_1, y_2, \dots, y_k \rangle \\ &\vdots \\ &\langle x_1, \dots, x_n \rangle = \langle y_1, \dots, y_n \rangle \end{aligned}\right\}.$$

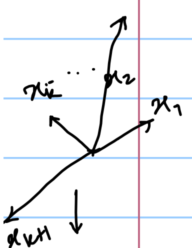
Proof: By induction on n -
$n = 1$: Given $\{x_1\}$, define $y_1 = \dfrac{x_1}{\|x_1\|}$, so that $\|y_1\| = 1$

Moreover, $\langle x_1 \rangle = \text{span}\left\{ \frac{x_1}{\|x_1\|} \right\} = \text{span}\{y_1\}$
$= \langle y_1 \rangle$.

Induction hypothesis: for k, $\exists$ $\{y_1, \dots, y_k\}$ s.t. the reqd. conditions are satisfied.

To prove the result for $k+1$:

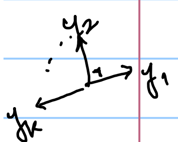
W.



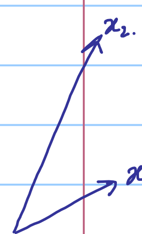
Have: $\{y_1, \dots, y_k\}$ s.t. $y_i \perp y_j$ $\forall\ 1 \le i, j \le k$,
$\|y_i\| = 1$ &

$\langle x_1, \dots, x_k \rangle = \langle y_1, \dots, y_k \rangle$

Define $y'_{k+1} = x_{k+1} - \left[\langle x_{k+1}, y_1 \rangle y_1 + \langle x_{k+1}, y_2 \rangle y_2 + \dots + \langle x_{k+1}, y_k \rangle y_k \right]$



& $y_{k+1} = \dfrac{y'_{k+1}}{\|y'_{k+1}\|}$. $\left(\Rightarrow \|y_{k+1}\| = 1 \right)$.

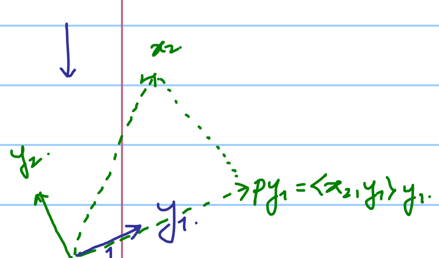


Condition (i) is satisfied because

$y_{k+1} \in \text{span}\{\underbrace{y_1, \dots, y_k}, x_{k+1}\}$

$= \text{span}\{x_1, \dots, x_k, x_{k+1}\}$.

$\therefore \langle y_1, \dots, y_{k+1} \rangle = \langle x_1, \dots, x_{k+1} \rangle$



$\underbrace{\langle x_2, y_1 \rangle}_{p} y_1$

$y_2 = \dfrac{x_2 - py_1}{\|x_2 - py_1\|} \in \text{span}\{x_2, y_1\} = \text{span}\{x_2, x_1\}$.

$\therefore \text{span}\{y_1, y_2\} = \text{span}\{x_1, x_2\}$.

Ex. $V = \mathbb{R}^3$, $x_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $x_2 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$, $x_3 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$.

(check that these are lin-indep.).

① $y_1 = \dfrac{x_1}{\|x_1\|}$, $\|x_1\| = 1$, $y_1 = x_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$.

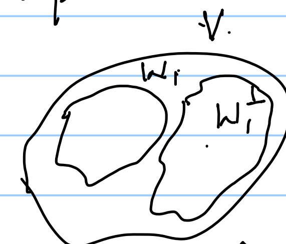
② $y_2 = \dfrac{x_2 - \langle x_2, y_1 \rangle y_1}{\|x_2 - \langle x_2, y_1 \rangle y_1\|} = \dfrac{\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} - 1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}{\| \text{ num } \|}$

$$= \frac{\begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}}{1} = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}.$$

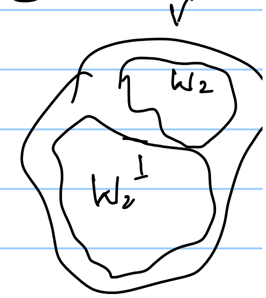
③ $y_3 = \dfrac{x_3 - \langle x_3, y_1 \rangle y_1 - \langle x_3, y_2 \rangle y_2}{\| \text{ num. } \|} =$

(exercise).

Proposition: Let W be a subspace of a f.d'l i.p.s. V then $V = W \oplus W^\perp$.



Proof: To show: ① $V = W + W^\perp$
② $W \cap W^\perp = \{0\}$.



Let $\{x_1, \dots, x_m\}$ be an orthonormal basis of W & $v \in V$.

Consider $w' = v - \underbrace{\sum_{i=1}^{m} \langle v, x_i \rangle x_i}_{= w \in W}$

Claim: $w' \in W^\perp$, we show that $w' \perp x_i \ \forall\, 1 \le i \le m$.

$\left\{ \langle w', x_i \rangle = \quad \cdots \quad \cdots = 0. \right\}$

$w' = v - w$, so $v = w + w' \in W + W^\perp$.

To show that $W \cap W^{\perp} = \{0\}$:

$$\text{let } v \in W \cap W^{\perp}$$

from above, $v = w + w'$, $w \in W$, $w' \in W^{\perp}$.

- $v \in W$:

- $v \in W^{\perp}$:

$\implies v = 0$. $\therefore W \cap W^{\perp} = \{0\}$. $\square$.

Corollary: Let W be a subspace of a f.d. i.p.s. V.

Then $(W^{\perp})^{\perp} = W$.

Direct sum

(Section 6, chapter 2, page 52 of Sahai-Bist).

① External : V_1 & V_2 are 2 v.s. over same field F. Then $V_1 \oplus V_2 = \{(v_1, v_2) / v_1 \in V_1, v_2 \in V_2\}$.

Can be shown that $V_1 \oplus V_2$ is also a v.s. over F w.r.t.

- addition : $(v_1, v_2) + (w_1, w_2) = (v_1 + w_1, v_2 + w_2)$
- scalar multi : $\alpha(v_1, v_2) = (\alpha v_1, \alpha v_2)$

examples : (i) $\mathbb{R}^2 = \{(x_1, x_2) / x_1, x_2 \in \mathbb{R}\}$.
$= \mathbb{R} \oplus \mathbb{R}$. (as v.s. over $\mathbb{R}$)

(ii) $\mathbb{R}^n = \{(x_1, \dots, x_n) / x_i \in \mathbb{R}\}$
$= \underbrace{\mathbb{R} \oplus \dots \oplus \mathbb{R}}_{n \text{ times}}$

(iii) $\mathbb{R} \oplus \mathbb{C}$ ($\mathbb{R}$ & $\mathbb{C}$ as v.s. over $\mathbb{R}$)
$= \{(x, z) / x \in \mathbb{R}, z \in \mathbb{C}\}$

(What is the diff. bet^n: ① $\mathbb{C}$ as a v.s. over $\mathbb{R}$ ② $\mathbb{C}$ as a v.s. over $\mathbb{C}$?)

In general, if $V_1, \dots, V_n$ are all v.s. over field F, then $V_1 \oplus V_2 \oplus \dots \oplus V_n$
$= \{(v_1, v_2, \dots, v_n) / v_i \in V_i\}$
is a direct sum.

② Internal : If $W_1, \dots, W_k$ are subspaces of V & if it turns out that $W_1 \oplus \dots \oplus W_k = V$, then we say that V is a direct sum of $W_1, \dots, W_k$.

Theorem: V is a direct sum of its subspaces $W_1, \dots, W_k$ iff

(i) $V = W_1 + \dots + W_2$ $\left\{ \begin{array}{l} \forall v \in V, \exists w_i \in W_i \\ \text{s.t. } v = w_1 + w_2 + \dots + w_k \end{array} \right.$

(ii) for each i, $W_i \cap \sum_{j \neq i} W_j = \{0\}$.

(It can be shown that $V \cong W_1 \oplus \dots \oplus W_k$)