$V \xrightarrow{T} W$

$V_{B_1} \xrightarrow{{}_{B_1'}[T]_{B_1}} W_{B_1'}$

$V_{B_2} \xrightarrow{{}_{B_2'}[T]_{B_2}} W_{B_2'}$

$P = {}_{B_2}[I]_{B_1}$ $\qquad$ ${}_{B_2}[I]_{B_1'} = Q.$

${}_{B_2}[T]_{B_2'} = Q \, {}_{B_1'}[T]_{B_1} \, P^{-1}$

$V = \mathbb{R}^3 \xrightarrow{T} \mathbb{R}^2$

$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x+y \\ x-z \end{pmatrix}$

$B_1 = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\} \quad B_1' = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$

${}_{B_2}[T]_{B_1} = \left[\, T\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}_{B_2} \,\middle|\, T\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}_{B_2} \,\middle|\, T\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}_{B_2} \, \right]$

$= \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & -1 \end{bmatrix}$

$B_2 = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\} \quad B_2' = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$

$$P^{-1} = \underset{B_1}{\underset{B_2}{[I]}} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \qquad Q = \underset{B_2'}{\underset{B_1'}{[I]}} = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix}$$

$$\boxed{\underset{B_2}{\underset{B_2'}{[T]}} = Q \; \underset{B_1'}{\underset{B_2}{[T]}} \; P^{-1}}$$

★ If $V = W$, then

$Q = P$

$$\boxed{\underset{B_2'}{\underset{B_2}{[T]}} = Q \; \underset{B_1'}{\underset{B_1}{[T]}} \; Q^{-1}}$$

★ $A \in M_{n \times m}(\mathbb{R})$

$A : \mathbb{R}^m \longrightarrow \mathbb{R}^n$

(a_{ij})

$$\begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix} \longmapsto \begin{pmatrix} \sum_{j=1}^{m} a_{1j} x_j \\ \sum a_{2j} x_j \\ \vdots \\ \sum a_{nj} x_j \end{pmatrix}$$

{ Linear transformation } $\longleftrightarrow$ { all matrices }

★ **Eigen Values and Eigen vectors**

→ Eigen vectors corresponding to distinct eigen values are linearly independent.

★ Let E_λ = subspace spanned by eigen vectors of λ

→ Algebraic multiplicity of eigen values = it's multiplicity as a root of characteristic polynomial.

geometric multiplicity of λ = $\dim E_\lambda$

In general geom. multiplicity (≤) ≤ alg. multiplicity of λ.

★ Diagonalizability: A is diagonalizable if $\exists$ invertible matrix P s.t. $A = P^{-1} \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_k \end{pmatrix} P$.

eigen values of A : $\lambda_1, \lambda_2 \cdots \lambda_k$

- alg. mult of $\lambda_i = 1 \ \forall i$ → A is diagonalizable.
- (alg. mult of $\lambda_j \geq 1$ for some j
  - alg. mult = geom. mult $\forall \lambda_j$ → A is diagonalizable
  - alg. mult ≠ geom. mult for some λ_j → A is not diagonalizable

When A is not diagonalizable.

"almost" diagonal JCF

diag + N.

★ Similar matrices?

★ **Inner products**

(1) $V \times V \longrightarrow \mathbb{R}$

$v, w \longmapsto \langle v, w \rangle$

① $\langle v, v \rangle \geq 0$. $\langle v, v \rangle = 0 \Leftrightarrow v = 0$.

②

★ **Norms**

$\|\cdot\| : V \to \mathbb{R}_{\geq 0}$.

$\|v\| = \sqrt{\langle v, v \rangle}$

★ **Standard inner product on V**

① over $\mathbb{R}^n$: dot product

$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = \sum x_i y_i$$

② over $\mathbb{C}^n$:

$\sum x_i \bar{y}_i$

★ Different norms on vectors

① $\|v\| = \langle v, v \rangle^{1/2} = (\Sigma v_i^2)^{1/2}$

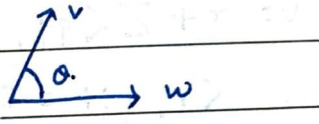
~~② $\|v\| = \sum v$~~

② $v = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$ $\quad \|v\| = (\Sigma v_i^p)^{1/p}$

$p \in \mathbb{Z}_{>0}$.

★ Angles between vectors

$$\cos\theta = \frac{\langle v, w \rangle}{\|v\| \, \|w\|}$$



If $\theta = 90°$, then we say v is orthogonal to w.

$$v \perp w \Leftrightarrow \langle v, w \rangle = 0$$

★ Orthogonal set

★ Orthonormal set.

Give any set $S = \{v_1, \dots v_n\}$ (linearly independent?), we can generate a set of orthonormal vectors $\{q_1, \dots q_n\}$ such that

$$\text{span}\{v_1 \dots v_k\} = \text{span}\{q_1, \dots q_k\}$$

General fact:

If $\{q_1 \dots q_n\}$ is a set of orthonormal vectors in V ($\dim V = m$), then for every $v \in V$ consider

$$r = v - \langle q_1 v \rangle q_1 - \langle q_2 v \rangle q_2 \dots - \langle q_n v \rangle q_n.$$

__/__/____

Then $\langle r, q_1 \rangle = 0$.

$\langle r, q_2 \rangle = 0$

$\vdots$

$\langle r, q_n \rangle = 0$.

r is orthogonal to all these vectors.

$$r = v - \sum \langle q_i, v \rangle q_i$$

Thus, for any $v \in V$, we may express v as

$$v = r + \sum \langle q_i, v \rangle q_i$$

where $\langle r, q_i \rangle = 0 \quad \forall q_i$

orthogonal decomposition of v

If the set $\{q_1 \cdots q_n\}$ is a basis for V then $r = 0$ &

$$v = \sum_{i=1}^{n} \langle q_i, v \rangle q_i$$

(orthogonal decomposition of v).

- Linear algebra Sahai Bist.
- Schaum's outline - (Linear algebra) (Lipschitz).

__/__/____

★ **Triangulable matrices**

→ Every matrix over $\mathbb{C}$ is triangulable.

★ **Self adjoint operators**

Symmetric (over $\mathbb{R}$) $A = A^t$

Hermitian (over $\mathbb{C}$) $A = A^*$

adjoint of a linear transformation.

$\langle Tx, y \rangle = \langle x, Ty \rangle$

★ **Symmetric / Hermitian matrix** (self adjoint)

① Eigen values of a self adjoint matrix are real, non-negative

★ Eigen vectors of a symmetric matrix are linearly independent and orthogonal.

→ **Spectral theorem for self-adjoint matrices**

★ If A is self-adjoint then $\exists$ invertible matrix P s.t. $P^{-1}AP$ is diagonal.

Moreover P is orthogonal (unitary).

★ Vector norms

Suppose V is a vector space over $\mathbb{C}$

Defⁿ: A norm is a function $\|\cdot\| : V \to \mathbb{R}$ s.t.

1. $\|v\| \geq 0$, $\|v\| = 0$ if $v = 0$.
2. $\|\alpha v\| = |\alpha| \|v\|$ $\forall \alpha \in \mathbb{C}$ & $v \in V$.
3. $\|v + w\| \leq \|v\| + \|w\|$ $\forall v, w \in V$.

Example: p-norm on V

Let $v = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$, $\|v\|_p = \left(\sum |v_i|^p \right)^{1/p}$

if $p = 1$

$$\|v\|_1 = \sum |v_i|$$

if $p = \infty$

$$\|v\|_\infty := \max_i \{ |v_i| \}$$

if $p = 2$

$$\|v\|_2 = \left(\sum |v_i|^2 \right)^{1/2}$$

(Euclidean norm).

$p_1 \geq 1$ & $p_2 \geq 1$

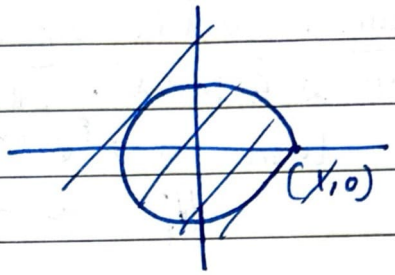
$$m \|v\|_{p_2} \leq \|v\|_{p_1} \leq n \|v\|_{p_2}$$

★ Closed unit balls in p-norms

$$B = \{ v \in V \mid \|v\| \leq 1 \}$$

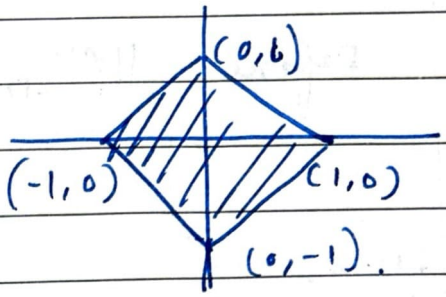
- $p = 1$

$v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ $\quad \|v\| = |v_1| + |v_2|$



$1 = |v_1| + |v_2|$

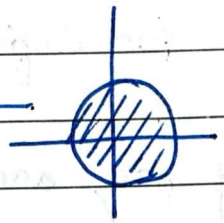
$1 = x + y$



- $p = 2$.

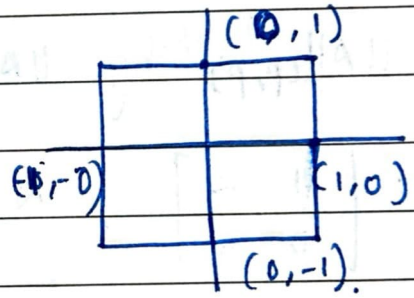
$v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$

$\|v\| = \sqrt{\|v_1\|^2 + \|v_2\|^2}$



- $p = \infty$

$\|v\|_\infty = \max\{\|v_1\|, \|v_2\|\}$



$\boxed{2 \le p \le \infty}$

★ **Matrix Norm**

Let $A \in M^{n \times m}(\mathbb{C})$

Then a matrix norm is a function $\|\cdot\| : M^{n \times m}(\mathbb{C}) \to \mathbb{R}$

Satisfying ① $\|A\| \ge 0$, $\|A\| = 0 \Leftrightarrow A = 0$.

② $\|\alpha A\| = |\alpha| \, \|A\| \quad \forall \alpha \in \mathbb{C}$

③ $\|A + B\| \le \|A\| + \|B\| \quad \forall A, B, C \in M^{n \times m}$.

Additionaley if we have desirable property

$\|AB\| \le \|A\| \, \|B\|$ whenever AB is defined.

Example: **Induced matrix norm**

Let $A \in M_{n\times m}(\mathbb{C})$

then $A : {}^{p}\mathbb{C}^{m} \to {}^{q}\mathbb{C}^{n}$

$\|x\|_p \quad x \longrightarrow Ax. \quad \|Ax\|_q$

Define $\|A\|_{(p,q)} = \sup_{\substack{x\in\mathbb{C}^m \\ x\neq 0}} \frac{\|Ax\|_q}{\|x\|_p}$

Since,

→ $\|\alpha A\| = |\alpha| . \|A\|$ we can define $\|A\|_{(p,q)} = \sup_{\|x\|_p=1} \|Ax\|_q$

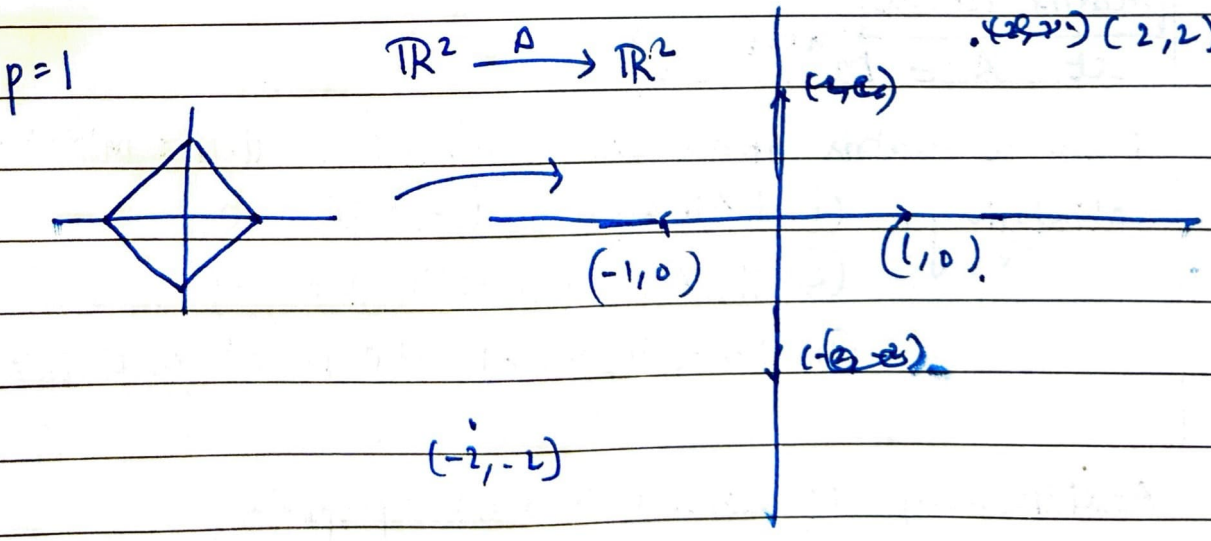
In practice, usually $p=q$ and we will denote

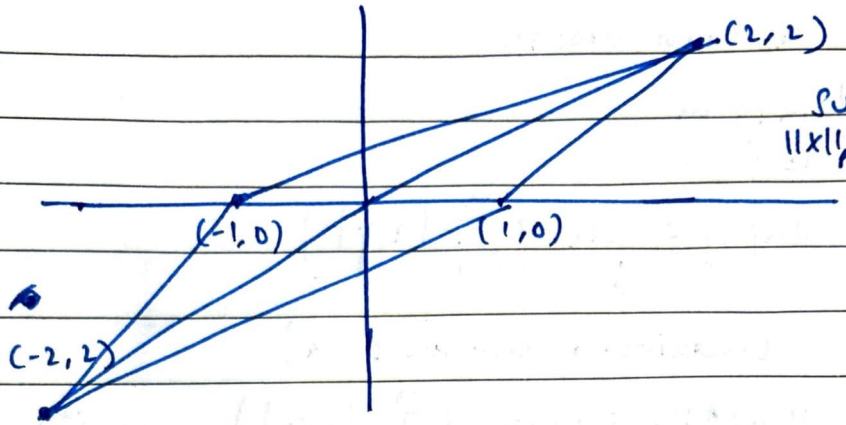
$\|A\|_{(p,p)}$ by $\|A\|_p$.

eg: $A = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$, $A\ \mathbb{R}^2 \to \mathbb{R}^2$

$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x+2y \\ 2y \end{pmatrix}$

$\begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+2y \\ y \end{bmatrix}$



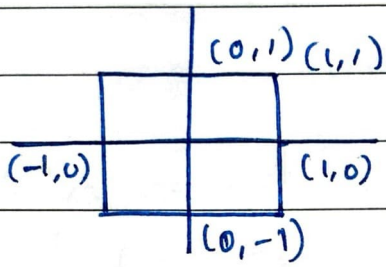


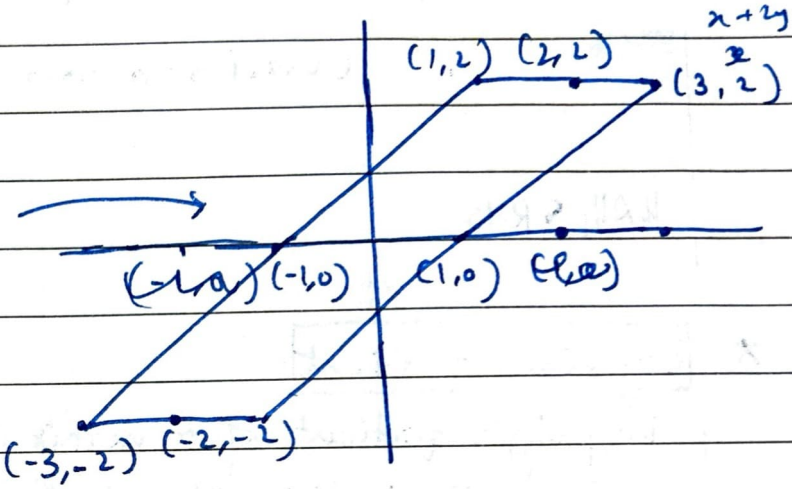
$$\sup_{\|x\|_p=1} \|Ax_p\|_1 = \|A\begin{pmatrix}0\\1\end{pmatrix}\|_1$$

$$= \|\begin{pmatrix}2\\2\end{pmatrix}\|_1$$

$$= 4$$

$p = \infty$.





$x+2y$

$-1,-1$

$x+2y$

$2y$

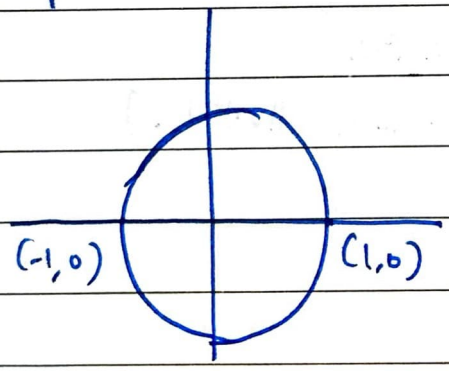
$$\sup_{\|x\|_\infty=1} \|Ax\|_\infty = 3.$$

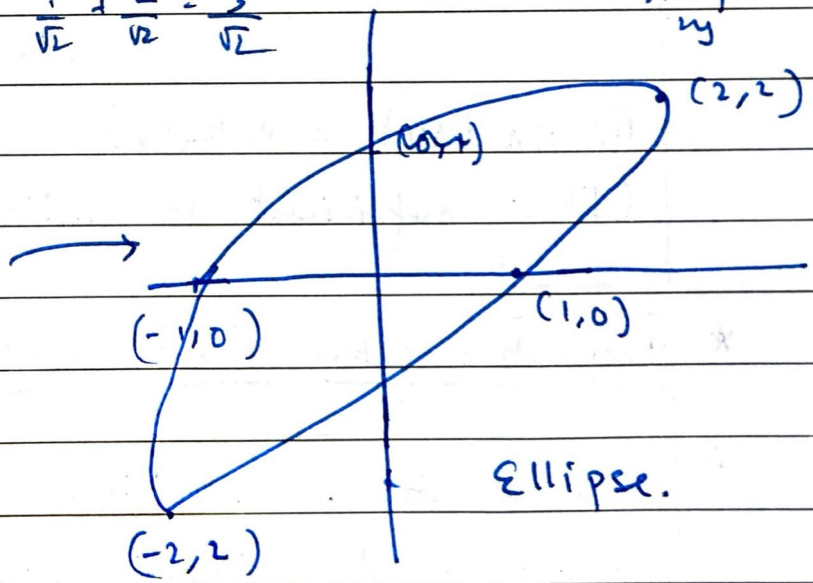
$x+2y$ $\quad \frac{1}{\sqrt{2}} + \frac{2}{\sqrt{2}} = \frac{3}{\sqrt{2}}$

$x+2y$

$2y$

$p=2$





Ellipse.

$\|A\|_2$

$$\sup_{\|x\|_2=1} \|Ax\|_p = 2\sqrt{2} \simeq 2.82?$$

Theorem: Let A be a $m \times n$ matrix

$$A = (a_{ij})_{\substack{1 \le i \le m \\ 1 \le j \le n}}$$

(i) $\|A\|_1 = \max_{1 \le j \le n} \|a_j\|_1 = \max_j \left(\sum_{i=1}^{m} |a_{ij}| \right)$

(maximum column sum)

(ii) $\|A\|_\infty = \max_i \|a_{i*}\|_1 = \max_i \left(\sum_{j=1}^{n} |a_{ij}| \right)$ a_i = i^{th} row of A.

(maximum row sum).

$\|A\|_1 \le$ RHS

* **Rayleigh quotient**

Rayleigh quotient of a matrix A is defined as

$$R_A : V \setminus \{0\} \to \mathbb{C}$$

$$v \longmapsto \frac{v^* A v}{v^* v} = \frac{\langle Av, v \rangle}{\langle v, v \rangle} \quad v \neq 0$$

① $R_A(\alpha v) = \alpha R_A(v) \quad \forall \alpha \neq 0.$

(It is sufficient to consider v s.t. $\|v\| = 1$.)

* **Properties of Rayleigh quotient**

i) If A is a Hermitian matrix, with e-values $\lambda_1 \le \lambda_2 \cdots \le \lambda_n$ and associated eigen vectors $p_1, \cdots p_n$ satisfying

$p_i^* p_j = \delta_{ij}$ (δ_{ij} (Kronecker delta))

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

↓ vectors are orthonormal

then

1) $\boxed{R_A(p_k) = \lambda_k}$

2) $\lambda_k = \max_{v \in V_k} R_A(v)$

$V_k = \text{span}\{p_1, \dots p_k\}$

3) $\lambda_k = \min_{v \in V_{k-1}^{\perp}} R_A(v)$, $V_{k-1}^{\perp} = \text{span}\{p_k, \dots p_n\}$

$V = V_{k-1} \oplus V_{k-1}^{\perp}$

↓ direct sum.

Theorem: A is Hermitian with e-values $\lambda_1 \le \lambda_2 \le \dots \le \lambda_n$ & e-vectors $\{p_1, \dots, p_n\}$ s.t. $\langle p_i, p_j \rangle \delta_{ij}$

then

(i) $R_A(p_k) = \lambda_k$

(ii) $\lambda_k = \max_{v \in V_k} R_A(v)$ $V_k = \text{span}\{p_1, \dots, p_k\}$

Proof: Let $U = [p_1 | p_2 | \dots p_n]$ then $U^* A U = \text{diag}(\lambda_i) = D$

for an $v \in V$, $v \neq 0$, let $U^{-1} v$ so that $v = Uw$

$$R_A(v) = \frac{v^* A v}{v^* v} = \frac{(Uw)^* A (Uw)}{(Uw)^* (Uw)} = \frac{w^* (U^* A U) w}{w^* U^* U w}$$

$$= \frac{w^* D w}{w^* w} = R_D(w).$$

for $v \in V_k$, $v = \begin{pmatrix} \\ \\ \end{pmatrix}$ $v = \sum_{i=1}^{k} \alpha_i p_i$, $w = U^{-1} v$ $\begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_k \\ 0 \\ \vdots \\ 0 \end{pmatrix}$

$$R_A(v) = R_D \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_k \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} \bar{\alpha_1} & \dots & \bar{\alpha_k} & 0 & \dots & 0 \end{pmatrix} \begin{pmatrix} \lambda_1 & & & & & \\ & \ddots & & & & \\ & & \lambda_n & & & \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_k \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

__/__/__

$$(\bar{\alpha_1} \cdots \bar{\alpha_k}\ 0 \cdots 0)\begin{pmatrix} \lambda_1 & & & \\ & \ddots & & \\ & & \lambda_k & \\ & & & \ddots \end{pmatrix}\begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_k \\ 0 \\ \vdots \end{pmatrix} = \frac{\sum_{i=1}^{k} \lambda_i |\alpha_i|^2}{\sum_{i=1}^{k} |\alpha_i|^2}$$

(1) $R_A(p_k) = \lambda_k$ since $p_k = \sum_{i=1}^{k} \alpha_i p_i$

where $\alpha_k = 1$

& $\alpha_i = 0 \ \forall\ i \neq k$.

★★★ for $v \in V_k$

$$R_A(v) = \frac{\sum_{i}^{k} \lambda_i |\alpha_i|^2}{\sum_{i}^{k} |\alpha_i|^2}$$

(2) $$\max_{v \in V_k} R_A(v) = \max_{v \in V_k} \left\{ \frac{\sum_{i}^{k} \lambda_i |\alpha_i|^2}{\sum |\alpha_i|^2} \right\}$$

Recall: $\lambda_0 \leq \lambda_1 \leq \cdots \leq \lambda_n$

each:

→ $\lambda |\alpha_i|^2 \leq \lambda_k |\alpha_i|^2 \ \forall\ i \leq k$

→ $\sum \lambda |\alpha_i|^2 \leq \lambda_k \left(\sum |\alpha_i|^2\right)$.

$\forall\ v \in V_k \quad R_A(v) \leq \lambda_k \Rightarrow \lambda_k = \max_{v \in V_k} \{R_A(v)\}$.

* Calculating $\|A\|_2$

Let A be a $n \times n$ matrix

$$\|A\|_2 = \sup_{\substack{x \in \mathbb{C}^n \\ x \neq 0}} \frac{\|Ax\|_2}{\|x\|_2} = \sup_{\substack{x \in \mathbb{C}^n \\ x \neq 0}} \frac{\sqrt{\langle Ax, Ax \rangle}}{\sqrt{\langle x, x \rangle}}$$

Squaring

$$\|A\|_2^2 = \sup_{\substack{x \in \mathbb{C}^n \\ x \neq 0}} \frac{\langle Ax, Ax \rangle}{\langle x, x \rangle} = \frac{(Ax)^* (Ax)}{x^* x}$$

$$= \sup_{\substack{x \neq 0 \\ x \in \mathbb{C}^n}} \frac{x^* A^* A x}{x^* x}$$

$$= \sup_{\substack{x \neq 0 \\ x \in \mathbb{C}^n}} R_{A^*A}(x).$$

$A^* A$ is hermitian. always. , so

$\forall v \in V \quad R_{A^*A}(v) \leq$ Largest e-value of A^*A

(attained at pts)

$\therefore$ Supremum $\sup R_{A^*A}(x) = \rho(A^*A) = \|A\|_2^2$

Note that: $A^* A$ is always positive semi definite

$$\boxed{\|A\|_2 = \sqrt{R_A(A^*A)}.} \rightarrow \text{largest singular value of } A.$$

1. Solve linear eqns.
2. Compute eigen vectors & eigen values } Aim of Numerical linear Algebra.

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★ Singular Value Decomposition

$A = U \Sigma V^*$

$m \times n$ — U: unitary; Σ: diagonal with singular value on the diagonal.

$$U^* U = I$$

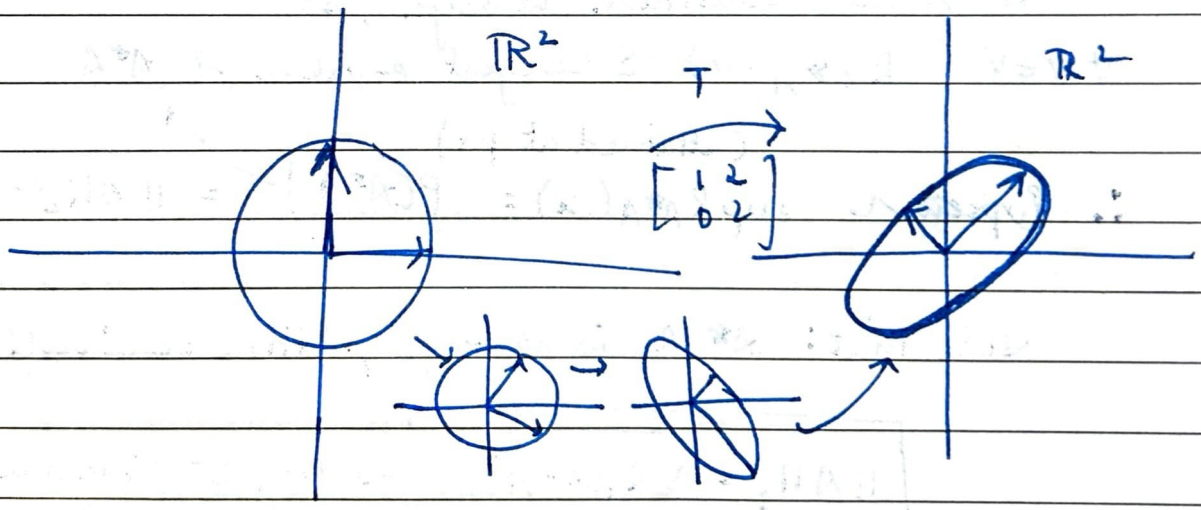
$$U^{-1} = U^*$$

$\mathbb{R}$ the cols of V are orthogonal for $\mathbb{R}$

Unitary if working over $\mathbb{C}$

→ Orthogonal (over $\mathbb{R}$) $\Rightarrow O^t O = O O^t = I$

→ Unitary (over $\mathbb{C}$) $= U^* U = U U^* = I$.



Any linear transformation from $\mathbb{R}^m \longrightarrow \mathbb{R}^n$

$\mathbb{R}^m \xrightarrow{T} \mathbb{R}^n$

unit ball $\longmapsto$ hyperellipse

(n- dimensional ⊕ ellipsoid).

$A(x) = U \Sigma V^* (x)$

- V^*: rotation
- Σ: scaling / stretching
- U: rotating

$A_{m \times n} = U \Sigma V^*$

- Σ: diagonal (singular values on diagonal)
- U, V^*: unitary

$A: \mathbb{R}^n \to \mathbb{R}^m$.

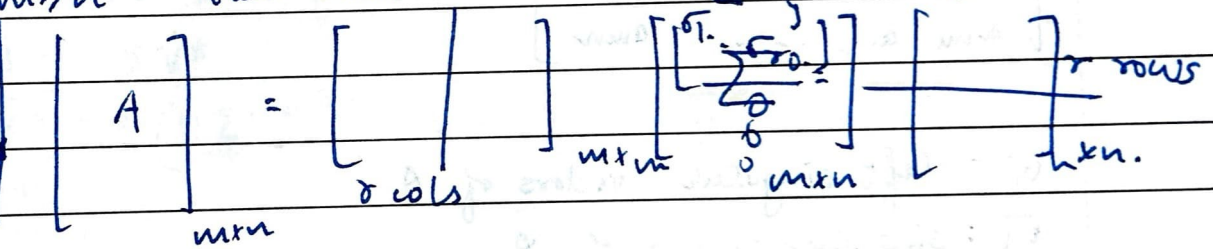
$A_{m \times n} = U_{m \times \_} \; \Sigma \; V^*_{n \times \_} (x)$

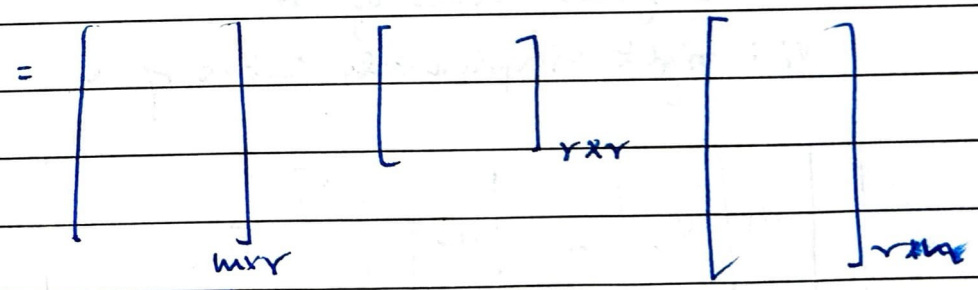
$r = \text{rank}(A)$ = # non-zero singular values of A. $\{\sigma_1 \dots \sigma_r\}$

$m \gg n$ tall matrix

Σ → singular values $\neq 0$.

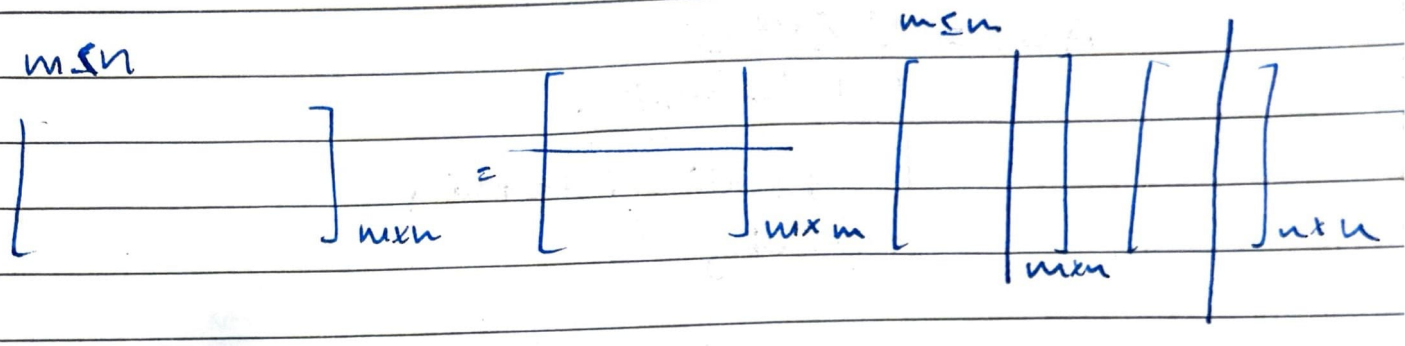
Full SVD





(Reduced SVD)

$m \leq n$



$m \times n$ matrix A $(m \geq n)$

$$A = U \Sigma V^*$$

diagonal

unitary

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

u_i = left singular vectors of A

σ_i : singular values of A

v_i : right singular vectors of A.

$$\underset{m \times n}{\begin{bmatrix} u_1 & u_2 & \cdots & u_m \end{bmatrix}} \underset{n \times n}{\begin{bmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_n \end{bmatrix}} \underset{n \times n}{\begin{bmatrix} v_1^* \\ v_2^* \\ \vdots \\ v_n^* \end{bmatrix}}$$

$A = U \Sigma V^*$

$AV = U\Sigma$

$$A \begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{bmatrix} = \begin{bmatrix} | & | & & | \\ u_1 & | & \cdots & u_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_n \end{bmatrix}$$

$$\begin{bmatrix} | & | & & | \\ Av_1 & Av_2 & \cdots & Av_n \\ | & | & & | \end{bmatrix} = \begin{bmatrix} | & | & & | \\ \sigma_1 u_1 & | & \cdots & \sigma_n u_n \\ | & | & & | \end{bmatrix}$$

$Av_i = \sigma_i u_i \quad \forall \; 1 \le i \le n$

Let $A = U\Sigma V^*$ be the SVD of A.

The eigen values of A^*A are σ_i^2. The right singular vectors are the corresponding orthonormal vectors of A^*A

i.e. $A^*Av_i = \sigma_i^2 v_i \quad \forall \; 1 \le i \le n$

eigen decomposition of A^*A

$A = U\Sigma V^*$

$A^*A = (U\Sigma V^*)^* (U\Sigma V^*) = V\Sigma^* U^* U\Sigma V^* = V \begin{bmatrix} \sigma_1^2 & & \\ & \ddots & \\ & & \sigma_n^2 \end{bmatrix} V^*$

A^*A → Hermitian.

The eigenvalues of AA^* are σ_i^2 & $(m-n)$ zeroes. The left singular vectors are the orthonormal eigenvalues of AA^*.

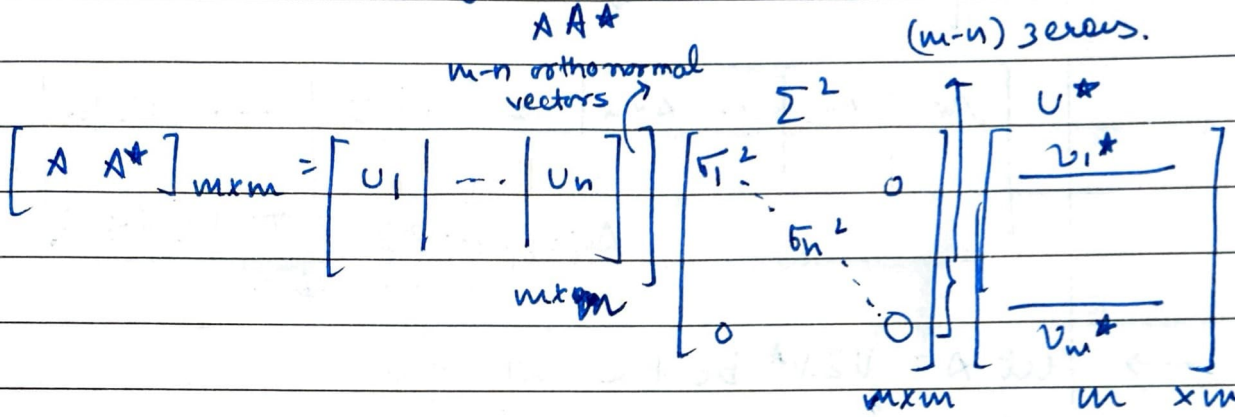
$A_{n \times m}$ $\qquad$ $(A^*A)_{n \times n}$ $\qquad$ $(AA^*)_{m \times m}$

$(AA^*)u_i = \sigma_i^2 u_i \quad \forall \; 1 \le i \le n.$

Proof:

$$AA^* = (U\Sigma V^*)(U\Sigma V^*)^*$$
$$= U\Sigma V^* (V\Sigma^* U^*)$$
$$= U\Sigma^2 U^*$$

← eigen decomposition of AA^*



★ **Solving a system of linear equations**

→ Gaussian Elimination.
→ Method
→ algorithm
→ conditioning
→ stability
→ work (no. of computation) → floating pt. arithmetic.

→ QR decomposition → projection matrices
→ Graham Schmidt orthonormalization.

→ SVD to solve system of linear eqns.
→ least squares methods.
→ Iterative methods.

* **Gaussian Elimination**

$AX = B$

$A_{m\times n}\ X_{n\times 1} = B_{m\times 1}$

linear system of equations →

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}_{m\times n} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}_{n\times 1} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}_{m\times 1}$$

Linear system of eqns →

$a_{11}x_1 + a_{12}x_2 - a_{1n}x_n = b_1$

$\vdots$

$a_{m1}x_1 + \cdots a_{mn}x_n = b_m$.

m eqns in n variables.

Gaussian elimination Let A be a $m\times n$ matrix.

Row reduction

$$3x + y + 5z = 12$$
$$x + y - z = 9$$
$$2x - 4y + z = 5.$$

$$\begin{bmatrix} 3 & 1 & 5 \\ 1 & 1 & -1 \\ 2 & -4 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 12 \\ 9 \\ 5 \end{bmatrix}$$

1. Augmented matrix

$$\left[\begin{array}{ccc|c} 3 & 1 & 5 & 12 \\ 1 & 1 & -1 & 9 \\ 2 & -4 & 1 & 5 \end{array}\right]$$

~~$R_3 = R_3 -$~~

$R_3 \leftrightarrow R_2$

$$\begin{bmatrix} 3 & & \\ 2 & & \\ 1 & 1 & -1 \end{bmatrix}$$

Row Reduced form	Row Reduced echelon form
① Upper ~~Δ~~gular matrix (atleast).	→ ① Upper ~~Δ~~gular matrix + first non-zero element of row should be 1!

$R_1 \leftrightarrow R_2$

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 9 \\ 3 & 1 & 5 & 12 \\ 2 & -4 & 1 & 5 \end{array}\right]$$

E_{12}

★ **Permutation matrix** $(n \times n)$

$P_{ij} =$ $\qquad\qquad$ P_{12}

$3 \times 3 \quad I_{3\times 3} \rightarrow P_{12}$

$$\left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{array}\right]$$

P_{12}: can be defined as exchange of rows 1 & 2 or exchange of columns 1 & 2.

P_{ij} = permutation matrix obtained by exchanging the i^{th} & j^{th} columns (rows) of $I_{n\times n}$.

Rows of 1st matrix × Cols of right matrix

eg.

$$\left[\begin{array}{ccc} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{array}\right] \left[\begin{array}{ccc} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{array}\right] = \left[\begin{array}{ccc} 4 & 5 & 6 \\ 1 & 2 & 3 \\ 7 & 8 & 9 \end{array}\right]$$

essentially exchanging the 1st & 2nd rows.

$P_{ij}\,A =$

↓ Left multiplication with P_{ij}

j^{th} row is going to be exchanged with i^{th} rows of A.

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 3 \\ 5 & 4 & 6 \\ 8 & 7 & 9 \end{bmatrix}$$

Right multiplication with P_{ij}

↳ j^{th} col is exchanged with i^{th} col of A.

$A P_{ij}$

$$\left[\begin{array}{ccc|c} 3 & 1 & 5 & 12 \\ 1 & 1 & -1 & 9 \\ 2 & -4 & 1 & 5 \end{array}\right] \xrightarrow[P_{12}[A;b]]{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|c} 1 & 1 & -1 & 9 \\ 3 & 1 & 5 & 12 \\ 2 & -4 & 1 & 5 \end{array}\right]$$

$R_2 = R_2 - 3R_1 \quad R_3 = R_3 - 2R_1$

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 9 \\ 0 & -2 & -8 & -15 \\ 0 & -6 & 3 & -13 \end{array}\right]$$

$$\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & -1 \\ 3 & 1 & 5 \\ 2 & -4 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & -1 \\ 0 & -2 & 8 \\ 0 & -3 & 3 \end{bmatrix}$$

$R_2 - 3R_1$ & $R_3 - 2R_1$

__/__/____

$A \xrightarrow{P_{12}} (P_{12}A) \xrightarrow{L_1} (L_1 P_{12} A) \xrightarrow{L_2} (L_2 L_1 P_{12} A)$

$$L_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix}$$ upper triangular matrix

$$\boxed{L_3 L_1 P_{12} A = U}$$

$\rightarrow a_{11}$ is non-zero. $\neq a_{11} \neq 0$.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$\Rightarrow$ L_i's are always lower Δgular

L_1

$$\begin{bmatrix} 1 & 0 & 0 \\ -\frac{a_{21}}{a_{11}} & 1 & 0 \\ -\frac{a_{31}}{11} & & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22}' & a_{23}' \\ 0 & a_{32}' & a_{33}' \end{bmatrix}$$

L_2 is applied to $L_1 A$.

assuming that a_{22}' is non-zero.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{a_{32}'}{a_{22}'} & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22}' & a_{23}' \\ 0 & a_{32}' & a_{33}' \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22}' & a_{23}' \\ 0 & 0 & a_{33}'' \end{bmatrix}$$

★ **Exercise**

4x4 matrix

See that $L_1 L_2 L_3$ obtained commute. with each other.

$$L = L_1 L_2 (A)$$

$$L_2 L_1 (A) = L$$

★ **Gaussian Elimination without pivoting**

Assumption the pivots encountered at every step of reduction are non-zero.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{1m} \\ \vdots & & \\ a_{n1} & a_{n2} & \cdots \; a_{nm} \end{bmatrix}$$

we can find lower Δ^r matrices $L_1, L_2, \cdots L_{n-1}$ such that

$$L_{n-1} \left(\cdots L_2 (L_1 A) \right) = U \quad (\text{upper } \Delta^r)$$

$$L_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ \frac{-a_{21}}{a_{11}} & 1 & & & & \\ \vdots & & & & & \\ \frac{-a_{n1}}{a_{11}} & 0 & \cdots & & 0 & 1 \end{bmatrix}_{n \times n}$$

$$L_2 = \begin{bmatrix} 1 & 0 & & & \\ 0 & 1 & & & \\ \vdots & -\frac{a_{32}'}{a_{22}'} & 1 & & \\ 0 & -\frac{a_{n2}'}{a_{22}'} & & & 1 \end{bmatrix}_{n \times n}$$

$$L_k = \begin{bmatrix} 1 & & & & \\ 0 & 1 & & & \\ \vdots & & 1 & & \\ 0 & & l_{k+1,k} & \ddots & \\ 0 & 0 & l_{n,k} & \cdots & 1 \end{bmatrix}$$

$$l_{k+1,k} = -a_{k+1,k}^{(k+1)}$$

$$l_{k+1,k} = -\frac{a_{k+1,k}^{(k-1)}}{a_{k,k}^{(k-1)}}$$

$$\vdots$$

$$l_{jk} = -\frac{a_{jk}^{(k-1)}}{a_{k,k}^{(k-1)}} \quad (j \geq k+1)$$

let

$L = L_n \cdot L_{n-1} \cdots L_2 \cdot L_1$ check $= \begin{bmatrix} 1 & & & \\ -\frac{a_{21}}{a_{11}} & 1 & & \\ & -\frac{a_{32}'}{a_{22}'} & 1 & \\ -\frac{a_{n1}}{a_{11}} & -\frac{a_{n2}'}{a_{22}'} & l_{nk} & 1 \end{bmatrix}$

$$\boxed{LA = U} \qquad \boxed{A = L^{-1}U}$$

now,

let L be L^{-1} & L^{-1} be L.

$$L^{-1}A = U \qquad \boxed{A = LU}$$

LU factorization of A).

* Solving a system of linear eqns.

$Ax = b$.

- factorize $A = LU$

$Ax = b$

$(LU)x = b$.

i.e. $L(\underbrace{Ux}_{y}) = b$.

- Solve $Ly = b$. for y. (by substitution)

- Solve $Ux = y$. (by back substitution).

* **Gaussian Elimination with pivoting**

If at the k^{th} step of Gaussian Elimination, the element

$a_{k,k}^{(k-1)} = 0$, then you need "pivoting"

(pivot at the k^{th} step).

$$A^{k-1} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22}^{(1)} & \cdots & a_{2n} \\ 0 & 0 & & \\ & & 0 & \\ & & * & \\ & & * & \\ 0 & 0 & * & \cdots 0 \end{bmatrix} \quad k$$

Strategy - I (Partial)

look at the elements below the ~~a_{kk}~~ k^{th} 0 & take the absolute value exchange with the one that has largest abs. value. (Row exchange).

* Gaussian Elimination is always done with pivoting because it's an unstable algorithm in itself. __/__/____
(theoretically). (even with pivot).

* **Partial Pivoting**

If during G.E., the pivot at the k^{th} step is zero, then choose $a_{k,k}^{(k-1)}$ to be the largest among ↓ (for stability)

$$|a_{jk}^{(k-1)}|, \quad k \le j \le n$$

$$L_n P_n (L_2 P_2 (L_1 (P_1 A))) = U$$

$$(L_n' L_{n-1}' \cdots L_1' \; P_n P_{n-1} \cdots P_1) A = U$$

$$L^{-1} P A = U$$

$$\boxed{PA = LU}$$

* **Complete pivoting**

Pivots are chosen among elements $\{a_{ij}\}$ $1 \le k \le n$, $j \le k \le n$

By looking at their absolute value.

$$L_n P_n \cdots L_2 P_2 \; L_1 P_1 A Q_1 Q_2 \cdots Q_n = U$$

$$L^{-1} PAQ = U$$

$$\boxed{PAQ = LU}$$

* Solving a system of eq"s using G.E. with pivoting

$Ax = b$, have $PAQ = LU$

$PAx = Pb$ → column

$PAQx = Pb$ — Q doesn't affect x.

$L(Ux) = Pb$ ← Solve for $Ly = Pb$

y.

$Ux = y$ ← Solve for.

I] G.E. without pivoting

$U = A$, $L = I$.

col → for $k = 1$ to $m-1$:

row → for $j = k+1$ to m :

$$l_{jk} = \frac{u_{jk}}{u_{kk}}$$

for $i = k$ to m :

$$u_{j,i} = u_{ji} - l_{jk} u_{ki}$$

II] G.E. with partial pivoting

We only calculating floating point operations. (no. of).

$U = A$, $L = I$, $P = I$.

for $k = 1$ to $m-1$:

- select $i \geq k$ to maximize $|u_{ik}|$
- interchange rows u_i & u_k

for $j = 1$ to $k-1$

$u_{kj} \leftrightarrow u_{ij}$

2nd
(inner loop)

for $j = k+1$ to m:

$$\ell_{jk} = \frac{U_{jk}}{U_{kk}}$$

for $i = k$ to m:

$$U_{ji} = U_{ji} - \ell_{jk} U_{ki}$$

★ Operation count

For each $k \le i \le m$,

the operation $U_{ji} = U_{ji} - \ell_{ji} U_{ii}$

takes 2 operations (1 multiplication
1 subtraction)

entries manipulated in

column $k = m-k = \ell$ (say).

division required in the k^{th} column.

column = ℓ.

1st column.

Total # of operations $= (m-1) + 2(m-1)$ for $(m-1)$ rows

2nd column

$= (m-2) + 2(m-2)$ operations

for $(m-2)$ rows.

$$\sum_{k=1}^{m-1} (m-k) + \sum_{k=1}^{m-1} 2(m-k)^2$$

$$\vdots$$

$$\frac{4m^3}{3} + \Theta(m^2)$$

If no of Gaussian elimination - $O(n^3)$
System of eq^n s ???

__/__/____

⇒ $PAQ = LU$ $\quad$ Q is orthogonal and invertible.

$PA = LUQ^T$

$PAx = Pb$

$LUQ^T x = Pb.$

- Solve $Lz = Pb$ for z
- Solve $Uy = z$ for y
- Set $y = Q^T$???

Notes

① $\det P = \pm 1 \ (\det Q)$

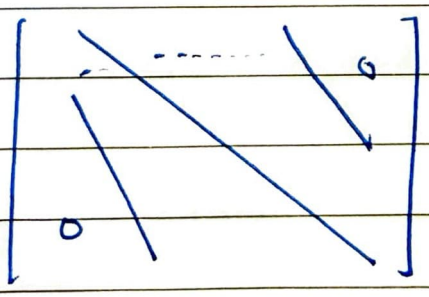
② $\det L = 1$

③ $\det U = \pm \det A.$
= product of pivots

④ LU factorization of A (or PA or PAQ) exists irrespective of whether A is invertible or not.

⑤ The LU factorization of A exists if and only if
$\iff \det \Delta_k \neq 0 \quad \forall \ 1 \leq k \leq n,$
where Δ_k = top left $k \times k$ submatrix of A.

★ **Band matrix**



LU factorization preserves band structure.
i.e. both L & U of band matrices are band matrices too.

__/__/____

★ Cholesky factorization

~~If A is~~

Let A be a symmetric 2×2 matrix

$$\begin{bmatrix} a & c \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ c/a & 1 \end{bmatrix} \begin{bmatrix} a & c \\ 0 & d-\left(\frac{c}{a}\right)c \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} 1 & 0 \\ c/a & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} a & 0 \\ 0 & d-\left(\frac{c}{a}\right)c \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} 1 & c/a \\ 0 & 1 \end{bmatrix}}_{L^T}$$

Theorem: If A is $\mathbb{R}^{n\times n}$ is symmetric and if Δ_k is non-singular $\forall\ 1\le k\le n$, then $\exists$ a unit lower Δ^r matrix L and a diagonal matrix D s.t. $A = LDL^T$.

This factorisation is unique.

Proof: $\det \Delta_k \neq 0 \quad \forall\ 1\le k\le n.$

$\therefore A$ has a LU factorization

Let $A = LU$.

$$L^{-1} A L^{-T} = U L^{-T} = \text{Diagonal matrix.}$$

($L^{-1}AL^{-T}$: symmetric; UL^{-T}: upper Δ^r matrix.)

$$(L^{-1} A L^{-1})^T = L^{-1} A^T L^{-T}$$

$$L^{-1} A L^{-T} = D.$$

$$\boxed{A = L D L^T}$$

* **Conditioning and Stability**

* **Conditioning of a problem**

X : normed vector space of data.

Y : normed vector space of soln's.

$$f : X \to Y$$

Defⁿ : **Absolute condition number**

Let δx denote a small perturbation in x & $\delta f = f(x + \delta x) - f(x)$

Then the abs. condition number $\hat{K}(x)$ of f at x is defined as

$$\hat{K}(x) = \lim_{\delta \to 0} \sup_{\|\delta x\| < \delta} \frac{\|\delta f\|}{\|\delta x\|}$$

← abs. change in soln

→ abs. change in data.

$$\approx \sup \frac{\|\delta f\|}{\|\delta x\|}$$

Defⁿ : **Relative condition number**

$$K(x) := \sup_{\delta x} \frac{\|\delta f\| / \|f\|}{\|\delta x\| / \|x\|}$$

← rel. change in soln

→ rel. change in data

Ex. f is the problem of computing $\sqrt{x}$ for $x > 0$.

$f: x \mapsto \sqrt{x}$.

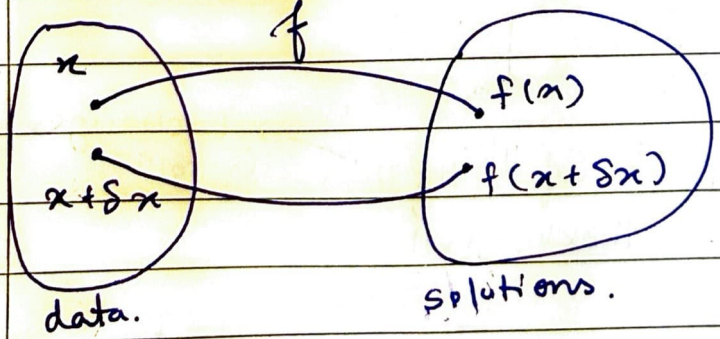
$$J(x) = \frac{1}{2\sqrt{x}}$$

$$K(x) = \frac{\|J(x)\|}{\|f(x)\| / \|x\|} = \frac{\|1/2\sqrt{x}\|}{\|\sqrt{x}\| / \|x\|}$$

$$= \frac{1}{2}$$ (small conditioning no. which says that this is a well conditioned problem).

* Recall : conditioning $\leftarrow$ abs. condition number $\hat{K}$
rel. condition number $k(x)$.

$f: x \longrightarrow \sqrt{x}$ $K(x) = 1/2$ (f is well-conditional)



Ex: f: finding sq. rt. of quadratic polynomial.

Input	o/p	o/p
(b, c)	$\sqrt{b^2-4c}$	$\frac{-b \pm \sqrt{b^2-4ac}}{2}$

$$K(x) \simeq \frac{\|\delta f\| / \|f(x)\|}{\|\delta x\| / \|x\|}$$

$$K(x) \simeq \sup_{\|x\| \to 0} \frac{\|\delta f\| / \|f(x)\|}{\|\delta(x)\| / \|x\|} = \frac{\|\delta f\| / \|\delta x\|}{\|\delta(x)\| \|f(x)\| / \|x\|}$$

$\therefore$ if f is continuous

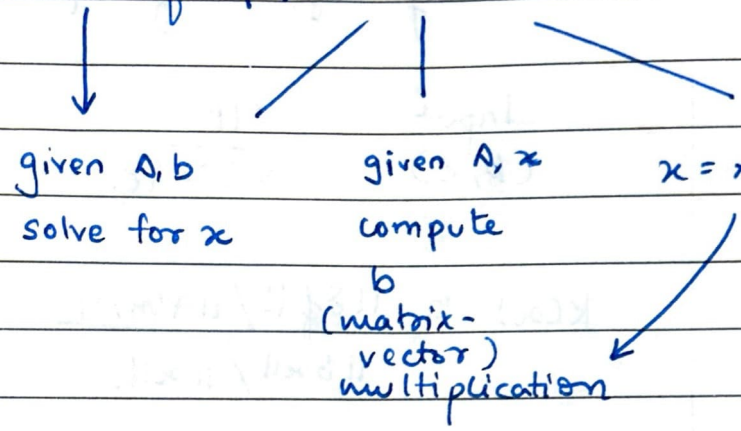
$$= \frac{J_f(x)}{\|f(x)\| / \|x\|}$$

$$J_f(x) = \begin{bmatrix} \frac{-1}{2} \pm \frac{b}{2\sqrt{b^2-4c}} & \mp \frac{1}{\sqrt{b^2-4c}} \end{bmatrix}$$

If we have a polynomial with repeated roots then there is small tolerance for perturbations in b and c.

★ If the polynomial has repeated roots $\|J(x)\|$ is ∞ in which case f is an ill-conditioned problem.

f: solving a system of eqn's — Consider → $Ax = b$.



* **Condition of matrix-vector multiplication**

(for A) 2 problems: $x \mapsto Ax \ (=b)$; $b \mapsto A^{-1}b \ (=x)$

$x \mapsto Ax$ $\quad \hat{K}(x) = \sup_{\delta x} \dfrac{\|A(x+\delta x) - A(x)\| / \|A(x)\|}{\|\delta(x)\| / \|x\|}$.

$= \sup_{\delta x} \dfrac{\|A(\delta(x))\| / \|\delta(x)\|}{\|A(x)\| / \|x\|}$. $\quad \because \sup_{\delta x} \dfrac{\|A\,\delta(x)\|}{\|\delta(x)\|} = \|A\|$

$= \|A\| \left(\dfrac{\|x\|}{\|Ax\|} \right)$

$\leq \|A\| \, \|A^{-1}\|$

$K(x) \leq \|A\| \, \|A^{-1}\| = K(A)$.

eg.

$\|A\|_p \, \|A^{-1}\|_p = K_p(A)$

↳ depends on the norm taken for A & A^{-1}.

★ **Condition of a system of eqns $Ax=b$ w.r.t. perturbations in A.**

Theorem: Let b be fixed and consider the problem of solving $Ax=b$. w.r.t. perturbations in A.
Then the condition number of the problem is $K = K(A)$.

Proof: If A is perturbed by an infinitesimal amount δA then x must change to $x+\delta x$. so,

δA is a matrix.

$$(A+\delta A)(x+\delta x) = b.$$

$$\therefore \underbrace{Ax}_{b.} + A\delta x + \delta A x + \underbrace{\delta A \delta x}_{\approx 0.} = b.$$

$$A\delta x + \delta A x = 0.$$

$$\delta x = -A^{-1}\delta A x.$$

$$\|\delta x\| \leq \|A^{-1}\delta A x\|$$

$$\leq \|A^{-1}\| \|\delta A\| \cdot \|x\|$$

$$\frac{\|\delta x\|}{\|x\|} \leq \|A^{-1}\| \|\delta A\|$$

$\frac{\|\delta x\|}{\|x\|}$ ↷ rel. change in soln $(A \longmapsto x)$. b is fixed.

$$K = \sup_{\delta A} \frac{\|\delta x\|/\|x\|}{\|\delta A\|/\|A\|} \leq \sup_{\delta A,} \|A^{-1}\| \|\delta A\| / \|A\|$$

$\|\delta A\|/\|A\|$ ↷ rel. change in data.

$$\leq \|A^{-1}\| \|A\|$$

$$\leq K(A).$$

There is some A for which this bound is reached.

★ It can be shown that δA and x can be chosen so that the bound is attained

$\therefore K = K(A)$

★ **Properties of $K(A)$**

① $K_p(A) = \|A\|_p \|A^{-1}\|_p$ is norm dependent.

② For every matrix A $K(A) \geq 1$ $(\|A\| \|A^{-1}\| \geq \|A A^{-1}\| = \|I\| = 1)$

③ $K(A) = K(A^{-1})$

④ $K(\alpha A) = K(A)$

⑤ If A is unitary or orthogonal
then $K_2(A) = 1$
(exercise) ??

⑤ implies that $Ax = b$ where A is unitary or orthogonal is a perfectly conditioned system.

→ most often 2 norm is used

⑥ $K_2(A)$ is invariant under unitary transformation.

⑦ By convention, $K(A) = \infty$ for a singular matrix A.

→ Does $\det A \approx 0$ ~~or a large~~ indicate a large $K(A)$?
This is not necessarily true, for example.

eg. (i) Let $B_n = \begin{bmatrix} 1 & -1 & -1 & \cdots & -1 \\ & 1 & & & \\ & & \ddots & & \\ 0 & & & & 1 \end{bmatrix} \in \mathbb{R}^{n \times n}$

$\det B_n = 1$.

$K_\infty(B_n) = n 2^{n-1}$ (large, badly conditioned).

eg. (11) Let $D_n = \text{diag}(10^{-1}, \ldots, 10^{-1}) \in \mathbb{R}^{n \times n}$.

$K_p(D_n) = 1$ but $\det D_n = 10^{-n}$ $(\simeq 0)$

Q. eg. **Effect of δA → perturbations**

$$\begin{pmatrix} 10 & 7 & 8 & 7 \\ 7 & 5 & 6 & 5 \\ 8 & 6 & 10 & 9 \\ 7 & 5 & 9 & 10 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 32 \\ 23 \\ 33 \\ 31 \end{pmatrix} \qquad \text{soln} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

Consider the perturbed data.

$$A(x + \delta x) = \begin{pmatrix} 32.1 \\ 22.9 \\ 33.1 \\ 30.9 \end{pmatrix} \qquad \text{soln} \begin{pmatrix} 9.2 \\ -12.6 \\ 4.5 \\ -1.1 \end{pmatrix} \qquad \delta b = \begin{pmatrix} 0.1 \\ -0.1 \\ 0.1 \\ -0.1 \end{pmatrix}$$

cond

Ex: **Consider perturbations in A**

$$\begin{pmatrix} 10 & 7 & 8.1 & 7.2 \\ 7.08 & 5.04 & 6 & 5 \\ 8 & 5.98 & 9.89 & 9 \\ 6.99 & 4.99 & 9 & 9.98 \end{pmatrix} \qquad x = \begin{pmatrix} -81 \\ 137 \\ -34 \\ 22 \end{pmatrix}$$

★ Compare

$$\frac{\|\delta b\| / \|b\|}{\|\delta x\| / \|x\|} \propto \qquad \frac{\|\delta A\| / \|A\|}{\|\delta x\| / \|x\|}$$

Example in Last class

Consider $A = \begin{pmatrix} 10 & 7 & 8 & 7 \\ 7 & 5 & 6 & 5 \\ 8 & 6 & 10 & 9 \\ 7 & 5 & 9 & 10 \end{pmatrix}$

Conditioning of $Ax = b$.

- w.r.t perturbations in b

$$K \approx \frac{\|\delta x\| / \|x\|}{\|\delta b\| / \|b\|} \approx \frac{8}{\frac{1}{200}} \approx 1600$$

- w.r.t. perturbations in A

$$K \approx \frac{\|\delta x\| / \|x\|}{\|\delta A\| / \|A\|} \approx \frac{160}{0.22 / 33} \approx \frac{160 \times 33}{0.22}$$

$$\delta A = \begin{pmatrix} 0 & 0 & 0.1 & 0.2 \\ 0.08 & 0.09 & 0 & 0 \\ 0 & -0.02 & -0.11 & 0 \\ -0.01 & -0.01 & 0 & -0.02 \end{pmatrix}$$

$$\boxed{K_2(A) \approx 2984}$$

__/__/____

★ **Floating point representation**

1985 - IEEE standard for floating point representation.
Kahan
& Stone ...

~~754-2008~~
↓
754-2019.

$\square\square \times 10^{C}$ (d1 d2) — **Base 10**

eg. $-99 \leq C \leq 99$.

$\square . \square\square \cdots \square \times \beta^{E}$ (d0 d1 d2 ... dp-1)

p - spaces

p - precision

→ $\beta = 2$, $p = 3$, $-1 \leq t \leq 2$.

each $0 \leq d_i \leq \beta - 1$

$d_i \in \{0, 1\}$ if $\beta = 2$.

$d_0 \neq 0$. | Normalized representation.

1.0 × 10^-1

1.00 × 2^0 ⟷ 1. } → ambiguous
.01 × 2^1 ⟷ 1

∴ $d_0 \neq 0$

* Construct a floating point system $\beta=2, p=3, -1 \le t \le 2$

$F \subseteq \mathbb{R}$

$\mathbb{F}$		$\mathbb{R}$	
1.00×2^{-1}	$\longleftrightarrow$	0.5	
1.01×2^{-1}	$\longleftrightarrow$	0.625	
1.10×2^{-1}	$\longleftrightarrow$	0.75	
1.11×2^{-1}	$\longleftrightarrow$	0.875	
		↓ x2	
1.00×2^{0}	$\longleftrightarrow$	1	
1.01×2^{0}	$\longleftrightarrow$	1.25	
1.10×2^{0}	$\longleftrightarrow$	1.5	
1.11×2^{0}	$\longleftrightarrow$	1.75	
		↓ x2	
1.00×2^{1}	$\longleftrightarrow$	2	
1.01×2^{1}	$\longleftrightarrow$	2.5	
1.10×2^{1}	$\longleftrightarrow$	3	
1.11×2^{1}	$\longleftrightarrow$	3.5	
		↓ x2.	
1.00×2^{2}	$\longleftrightarrow$	4	
1.01×2^{2}	$\longleftrightarrow$	5	
1.10×2^{2}	$\longleftrightarrow$	6	
1.11×2^{2}	$\longleftrightarrow$	7	

Non-uniform error bet^n interval and outside interval

0.1.5

(1)

$$TF = \{0\} \cup \pm \{ d_0 . d_1 d_2 \cdots d_{p-1} \times \beta^t \mid e_{min} \le t \le e_{max}, \; 0 \le d \le \beta - 1, \; d_0 \neq 0 \}.$$

density of representation is highest bet^n 0 & 1.
and as we move away to bigger no.s the representation becomes sparse.

x = actual real no. to be represented
x' = floating point representation of x.

$$\text{relative error representation} = \frac{|x - x'|}{|x|}.$$

- Note that the floating pt. no.s are not equally spaced
- Convention is 0 is represented by $\boxed{1.0 \times \beta^{e_{min}-1}}$

- Rel representation when $x \in \mathbb{R}$ is represented by $x' \in TF$ is defined as $\frac{|x - x'|}{|x|}$

Q. What is the max rel. repr. error that can occur in a given floating pt. system?

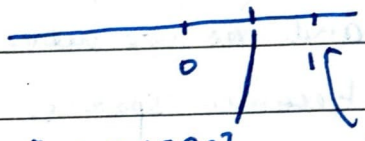
eg. ~~$x = 0.\overline{03}14159$ is represented by~~
$x = 3.14159$ is represented by 3.14×10^0
then rel. error is represented by $= \frac{|3.14159 - 3.14|}{3.14159}$
$\approx \underline{0.0005}$

eg. $\beta = 10$, $p = 5$. $e_{min} = -1$

Ans.

$$\frac{9}{10} + \frac{9}{100} + \frac{9}{1000} + \frac{9}{10000} + \frac{9}{100000} \times 10^{-1}$$

0.99999



$1 = 1.0000 \times 10^0$

1.0001×10^0

1.0001

$1 + \beta^{1-p}$ ↓ no. after 1.

max representation error $= \left| \frac{1.0000 - \{1.00005 - 1\}}{1.00005} \right|$

now, $\frac{1 + \beta^{1-p}}{2} \rightsquigarrow 1$

max. representation error $= \left| \frac{\frac{1 + \beta^{1-p}}{2} - 1}{\frac{1 + \beta^{1-p}}{2}} \right|$

Defⁿ: The floating point representation of a real no. is of the form

$d.ddd \times \beta^{e}$ — significant; e → exponent; β → base.

p-digits

p-precision

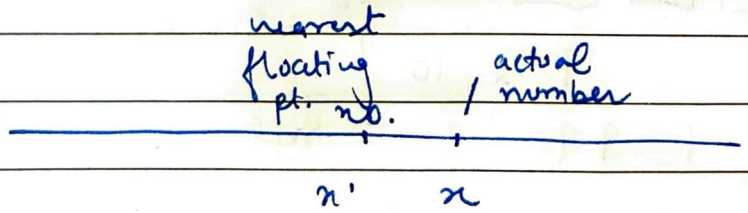
more precisely: the number represented by

$$\pm d.ddd \times \beta^{e} \text{ is } \left(d_0 + d_1 \times \beta^{-1} + \cdots + d_{p-1} \times \beta^{-(p-1)}\right) \times \beta^{e}$$

each d_i is betⁿ $0 \leq d_i \leq \beta - 1$

- Let $e_{min} \leq e \leq e_{max}$, then # possible exponents $= e_{max} - e_{min} + 1$.
- # possible significands β^p.
- In general the floating pt. repr. of a real no. may not be unique. To calculate this, we consider "normalized" representation which $d_0 \neq 0$.

★ **Ulps (Units in the last place)**



$\text{error} = |x - x'| \cdot \beta^{p-1}$ ulps (units in last place)

If the floating pt. number $d.d\ldots d \times \beta^{e}$ is used for rounding off a real no. z,

then the error is $\left|d.d\ldots - \frac{z}{\beta^{e}}\right| \times \beta^{p-1}$ ulps.

eg. $\beta = 10, \quad p = 3$

(i) Suppose $z = 0.0314$ & 3.12×10^{-2} is it's fl. pt. representation.

$$\text{error} = \left| 3.12 - \frac{0.0314}{10^{-2}} \right| \times 10^{2}$$

$$= 0.02 \times 10^{2}$$

$$= \underline{\alpha \text{ ulps.}}$$

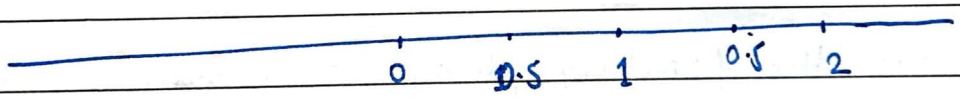
(ii) $z = 0.0314159$

$d.ddd - d \times \beta^{e} = 314 \times 10^{-2}$

$$\text{error} = \left| 3.14 - \frac{0.0314159}{10^{-2}} \right| \times 10^{2}$$

$$= 0.00159 \times 10^{2}$$

$$= 0.159 \text{ ulps.}$$



for base β & precision p what is floating pt. no. that comes after 1.

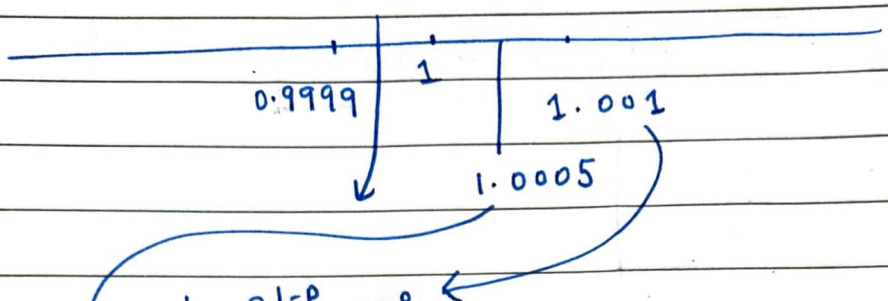
Base → 10 Precision → 4

$$9.999 \times 10^{-1}$$

$0.9999 \rightarrow$ last real no.

last floating point number before 1 is $\underline{9.999 \times 10^{-1}}$

first floating point number after 1 is $\underline{1.001 \times 10^{0}}$



$1 + \beta^{1-p} \times 10^{0}$

$1 + \frac{\beta^{1-p}}{2} \times 10^{0}$

$\left[1 \;,\; 1+\beta^{1-p}\right]$

↓

half of this interval is $\boxed{1 + \frac{\beta^{1-p}}{2}}$

→ For base β, the possible abs. error $|x - x'|$ can be as large as???

$0.0\ldots.0\,\beta' \times \beta^{e}$

(p zeros), where $\beta' = \beta/2$

this error is $\left(\frac{\beta}{2}\right)\beta^{-p} \times \beta^{e}$

Relative error $= \frac{|x - x'|}{|x|}$

$\boxed{\beta^{e} \le x \le \beta \times \beta^{e}.}$

$10^{e} \le 9.999 \times 10^{e} < 10 \times 10^{e}$

For a fixed e,

$$\beta^{e} \le |x| \le \beta \cdot \beta^{e}.$$

$$\frac{1}{\beta^{e}} \ge \frac{1}{|x|} \ge \frac{1}{\beta \cdot \beta^{e}}$$

$$\therefore \frac{|x - x'|}{|x|} \le \frac{(\beta/2)\,\beta^{-p} \times \beta^{e}}{\beta^{e}} \le \left(\frac{\beta}{2}\right)\beta^{-p}.$$

* Rounding to 0 gives 100% error

$$\varepsilon_{mach} := \frac{\beta^{1-p}}{2}$$

for a given machine with base β & precision p the relative error is always bounded by ε_{mach}.

The max^m rel. repr. error in a floating pt. system with base β & precision p is $\frac{\beta^{1-p}}{2} =: \varepsilon_{mach}$

$$a \oplus b = fl(a+b)$$

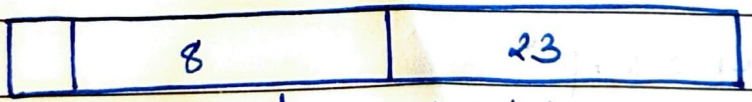
(a: rounded)

* Last time: max rel. representation error is $\frac{1}{2}\beta^{1-p} =: \varepsilon_{mach}$

• IEEE single precision

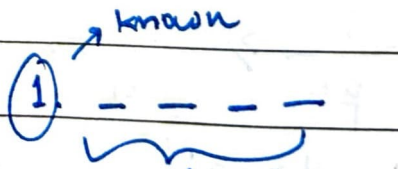
$-126 \le e \le 127$

$\beta = 2$, $p = 23$, available length = 32 bits



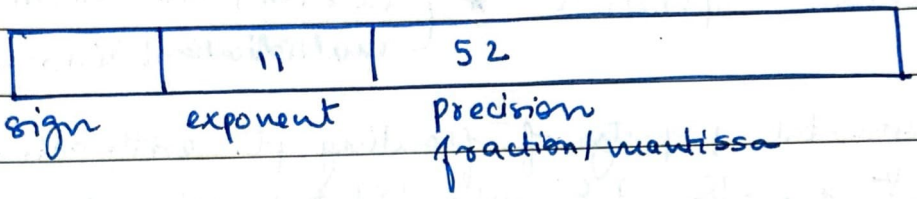
sign s | exponent e | precision p.

$$(-1)^s \, 2^{e-127} \, (1+f)$$



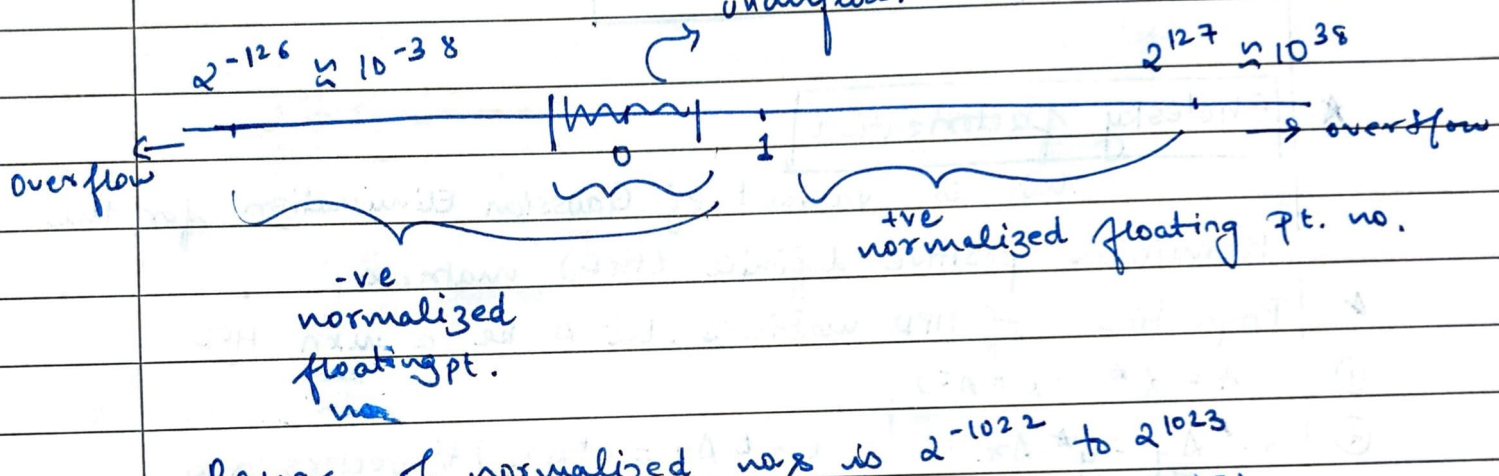
$$\varepsilon_{mach} = \frac{1}{2}\beta^{1-p} = \frac{1}{2}\beta^{-24}$$

- IEEE double precision $\beta = 2$, $p = 52$, 64 bits length.



$e_{min} = -1022$ $\quad e_{max} = 1023$.

$\varepsilon_{mach} = 2^{-53} \approx 10^{-16}$



Range of normalized nos is 2^{-1022} to 2^{1023}
$(\approx 10^{-308}$ to $10^{308})$.

① Fundamental property of floating pt. representation

$$\left| \frac{fl(x) - x}{x} \right| \leq \varepsilon_{mach}$$

for every $\forall\ x \in \mathbb{R}$, $\exists\ \varepsilon$ with $|\varepsilon| \leq \varepsilon_{mach}$
such that $fl(x) = x(1+\varepsilon)$

$$\left| \frac{fl(x) - x}{x} \right| = |\varepsilon| \leq \varepsilon_{mach}$$

② Let $x \circledast y$ denote the computed answer of the arithmetic operation $*$ (addition / subtraction / multiplication / division)

Fundamental property of floating pt. arithmetic —

$\forall\, x, y \in \mathbb{R}$, $\exists\, \varepsilon$ with $|\varepsilon| \leq \varepsilon_{mach}$ s.t.

$$x \circledast y = x * y\,(1+\varepsilon)$$

★ **Cholesky factorization**

This is variant of Gaussian Elimination for the Hermitian positive definite (HPD) matrices.

★ Properties of HPD matrices. Let A be a $m \times n$ HPD

① $A = A^*$ $(A = A^T)$

② $x^* A y = y^* A x$ $(x^t A y = y^t A x)$ $\forall$ vectors x, y.

③ $x^* A x$ is real $\forall\, x \in \mathbb{C}^m$ moreover

PD $\Leftrightarrow$ $x^* A x > 0$ $\forall\, x \in \mathbb{C}^m$

④ If X is a full rank $m \times n$ matrix then $X^* A X$ is also HPD

- $(X^* A X)^* = X^* A X^{**} = X^* A X$
- $x^* X^* A X x = (Xx)^* A (Xx) > 0$

(X - full rank $\Rightarrow Xx \neq 0$).

⑤ As a consequence of ④

① every principal submatrix of A is HPD

② letting $X_i = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \leftarrow i$, $X_i^* A X_i = a_{ii}$ is >0

$\therefore$ all diagonal elements of A must be non-zero.

$X = \begin{bmatrix} 1 & 0 & & \\ 0 & 1 & & \\ \cdots & & \cdots & 1 \\ 0 & 0 & & \end{bmatrix}_{n \times k}$ then X^*AX = top-left $k \times k$ submatrix.

* **Idea of Cholesky factorization**

Suppose $A = \left[\begin{array}{c|c} a_{11} & w^* \\ \hline w & K \end{array}\right]$ ← 1st row. , note $a_{11} > 0$.

$A = \begin{bmatrix} \sqrt{a_{11}} & 0 \\ 0 & \sqrt{\quad} \end{bmatrix}$ ~~$A = I$~~

R_1^* A_1 R_1

$A = \left[\begin{array}{c|c} \sqrt{a_{11}} & 0 \\ \hline w/\sqrt{a_{11}} & I \end{array}\right] \begin{bmatrix} 1 & \cdots & 0 \\ \vdots & & \frac{K - w^*}{a_{11}} \end{bmatrix} \begin{bmatrix} \sqrt{a_{11}} & w^*/\sqrt{a_{11}} \\ 0 & 1 \end{bmatrix}$

$A = R_1^* A_1 R_1$

__/__/__

$$A = R_1^* A_1 R_1$$
$$= R_1^* (R_2^* (A_2) R_2) R_1$$
$$= (R_1^* \dots R_m^*) A_m (R_m \dots R_1)$$
$$\downarrow$$
$$I$$

$$A = R^* R$$

Lower triangular. Upper triangular

★ **Algorithm** → HPD $\Rightarrow a_{ii} > 0 \ \forall i$

eg.
$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} \bar{r_{11}} & 0 & 0 \\ \bar{r_{12}} & \bar{r_{22}} & 0 \\ \bar{r_{13}} & \bar{r_{23}} & \bar{r_{33}} \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ 0 & r_{22} & r_{23} \\ 0 & 0 & r_{33} \end{bmatrix}$$

$$= \begin{bmatrix} r_{11}^2 & \bar{r_{11}} r_{12} & \bar{r_{11}} r_{13} \\ \bar{r_{12}} r_{11} & r_{12}^2 + r_{22}^2 & \bar{r_{12}} r_{13} + \bar{r_{22}} r_{23} \\ \bar{r_{13}} r_{11} & \bar{r_{13}} r_{12} + \bar{r_{23}} r_{22} & r_{13}^2 + r_{23}^2 + r_{33}^2 \end{bmatrix}$$

$r_{11} = \sqrt{a_{11}}$ & equality of matrices.

Algorithm

for $k = 1$ to m

$$r_{kk} = \left(a_{kk} - \sum_{j=1}^{k-1} r_{kj}^2 \right)^{1/2}$$

for $i = k+1$ to m

$$r_{ik} = \left(a_{ik} - \sum_{j=1}^{k-1} r_{kj}\, r_{ij} \right) / r_{kk}$$

$\frac{n(n-1)(2n-1)}{6}$

$\frac{(n^2-n)(2n-1)}{6}$

$\frac{n^3}{3} - \frac{2n^2}{6} - \frac{n^2}{6}$

$-3n^2$

Check operation count $\approx \frac{m^3}{3}$ flops.

* **Cholesky factorization is not same as LU factorization**

* **Stability of an algorithm**

The meaning of $\Theta(\varepsilon_{mach})$.

The notation $\underline{\varphi}(t) = \Theta(\psi(t))$ means that $\exists$ a positive constant C such that $\forall$ t close to some understood limit ($t \to 0$ or $t \to \infty$)

$$|\underline{\varphi}(t)| \leq C\,|\psi(t)|$$

(Note that C works uniformly for $\forall$ t).

Correct algorithm

Cholesky Algorithm

* for $k = 1$ to m:

$$r_{kk} = \left(a_{kk} - \sum_{j=1}^{k-1} r_{jk}^2\right)^{1/2}$$

for $i = k+1$ to m:

$$r_{ik} = \left(a_{ik} - \sum_{j=1}^{k-1} r_{jk} r_{ji}\right) / r_{kk}$$

$\sim O(m^3/3)$

* ### Stability of an algorithm

$\tilde{f}$ algorithm

$$X \xrightarrow{\tilde{f}} Y$$

X: normed vector space of i/p

Y: normed vector space of solution (output).

Defn: An algorithm $\tilde{f}$ is said to be accurate if $\dfrac{\|f(x) - \tilde{f}(x)\|}{\|f(x)\|} = 0$

$$\frac{\|f(x) - \tilde{f}(x)\|}{\|f(x)\|} = O(\varepsilon_{mach})$$

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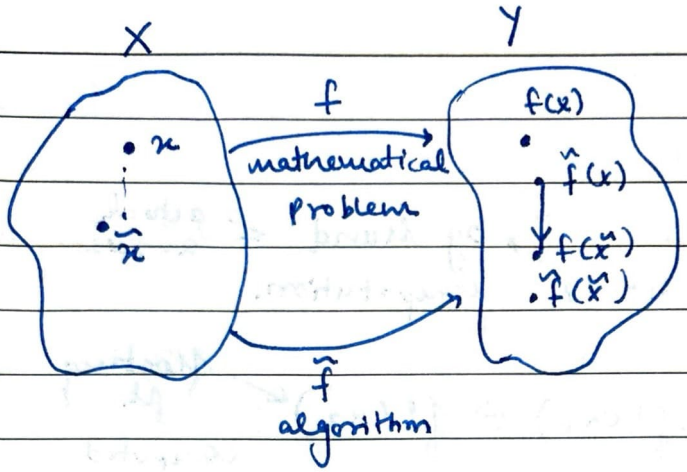

Defⁿ: An algorithm $\tilde{f}$ is said to be (forward) stable if $\forall x \in X$,

$$\frac{\|\tilde{f}(x) - f(\tilde{x})\|}{\|f(\tilde{x})\|} = O(\varepsilon_{mach}), \text{ whenever } \frac{\|x - \tilde{x}\|}{\|x\|} = O(\varepsilon_{mach})$$

Nearly correct answer to a nearly correct question.

Defⁿ: An algorithm $\tilde{f}$ is said to be backward stable if $\forall x \in X \quad \tilde{f}(x) = f(\tilde{x})$ for some $\tilde{x}$ with $\frac{\|x - \tilde{x}\|}{\|x\|} = O(\varepsilon_{mach})$.

(Right answer to the nearly right question).

Remarks * 1) Backward stability $\not\Rightarrow$ Forward stability

2) In general, backward stability analysis is easier to carry out.

Ex:

Subtraction

$f(x_1, x_2) = x_1 - x_2$ → by hand computation. ← actual answer

$\hat{f}(x_1, x_2) = fl(x_1) \ominus fl(x_2)$ ← floating pt. computed answer.

~~$fl(x_1 - x_2)$~~

① By the fundamental ppty of fl. pt. representation.

$fl(x_1) = x_1(1+\varepsilon_1)$ $\quad |\varepsilon_1| \le \varepsilon_{mach}$

$fl(x_2) = x_2(1+\varepsilon_2)$ $\quad |\varepsilon_2| \le \varepsilon_{mach}$.

② By fundamental ppty of fl. pt. arithmetic

$fl(x_1) \ominus fl(x_2) = (fl(x_1) - fl(x_2))(1+\varepsilon_3)$

where $|\varepsilon_3| \le \varepsilon_{mach}$.

$f(x_1, x_2)$ $\quad fl(x_1) \ominus fl(x_2) = x_1(1+\varepsilon_1) - x_2(1+\varepsilon_2)$

$= x_1 - x_2 + (x_1 - x_2)\varepsilon_1$

$= x_1 - x_2 + x_1\varepsilon_1 - x_2\varepsilon_2$

$\hat{f}(\hat{x}_1, \hat{x}_2)$ $\Rightarrow f(\hat{x}_1, \hat{x}_2) = \hat{x}_1 - \hat{x}_2$

$= x_1(1+\varepsilon_1) - x_2(1+\varepsilon_2)$

$= x_1 + x_1\varepsilon_1 - x_2 - x_2\varepsilon_2$

$= x_1 - x_2 + x_1\varepsilon_1 - x_2\varepsilon_2$

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$fl(x_1) \ominus fl(x_2) = (x_1(1+\varepsilon_1) - x_2(1+\varepsilon_2))(1+\varepsilon_3)$

$= x_1(1+\varepsilon_1)(1+\varepsilon_3) - x_2(1+\varepsilon_2)(1+\varepsilon_3)$

$\hat{f}(x_1, x_2)$ $\quad 1 + \varepsilon_1\varepsilon_3 + \varepsilon_2\varepsilon_3 + \varepsilon_3 + \ldots$

$\leq 2\varepsilon_{mach} + \varepsilon_{mach}^2$

$\hat{x}_1 = x_1(1+\varepsilon_4)$ $\quad = x_1(1+\varepsilon_4) - x_2(1+\varepsilon_5)$ where

$\hat{x}_2 = x_2(1+\varepsilon_5)$ $\quad |\varepsilon_4|, |\varepsilon_5| \leq 2\varepsilon_{mach} + \varepsilon_{mach}^2$

$= \hat{x}_1 - \hat{x}_2 = f(\hat{x}_1, \hat{x}_2)$

∴ Subtraction is backward stable.

$$\frac{\|x - \hat{x}\|}{\|x\|} = O(\varepsilon_{mach})$$

② Outer product : stable but not backward stable.

$f(x, y) = xy^* = A$

$x+\delta x \quad y+\delta y \quad (x+\delta x)(y+\delta y)^* = A + \delta A$

Rank 1? ??

★ **Accuracy of a backward stable algorithm**

Theorem Suppose a backward stable algorithm is applied on a computer (satisfying fundamental properties of floating pt. representation & floating pt. arithmetic) to solve a problem $f : X \to Y$ with condition number K

Then $$\frac{\|\hat{f}(x) - f(x)\|}{\|f(x)\|} = O(K\varepsilon_{mach}).$$

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functional stable algorithm.

$$\frac{\|\tilde{f}(x) - f(x)\|}{\|f(x)\|} = \Theta(\kappa\, \varepsilon_{mach})$$

Proof: def^n of backward stability : $\tilde{f}(x) = f(\tilde{x})$ for some $\tilde{x}$ s.t $\frac{\|x - \tilde{x}\|}{\|x\|} = \Theta(\varepsilon_{mach})$

$$\kappa(x) = \sup_{\delta x \to 0} \left(\frac{\|\delta f\| / \|f\|}{\|\delta x\| / \|x\|} \right)$$

$$\Rightarrow \kappa(x) \geq \frac{\|f(\tilde{x}) - f(x)\| / \|f(x)\|}{\|x - \tilde{x}\| / \|x\|}$$

choose $\delta x = x - \tilde{x}$

$$\kappa(x) \geq \frac{\|\tilde{f}(x) - f(x)\| / \|f(x)\|}{\|x - \tilde{x}\| / \|x\|}$$

$$\geq \Theta(\kappa)$$

$$\kappa(x) \frac{\|x - \tilde{x}\|}{\|x\|} \geq \frac{\|\tilde{f}(x) - f(x)\|}{\|f(x)\|}$$

$$\Theta(\kappa\, \varepsilon_{mach}) = \frac{\|\tilde{f}(x) - f(x)\|}{\|f(x)\|}$$

Thus, if an algorithm is backward stable, then the accuracy depends on the conditioning of the problem.

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★ **Error Analysis** for G.E. (Demme pages 47-48).
algorithm for (GEPP).

$$|E_{jk}| = \left| \sum_{i=1}^{j} l_{ji}\, u_{ik} \cdot \delta_i \right| \le \sum_{i=1}^{j} |l_{ji}|\,|u_{ik}| \cdot n\varepsilon = n\varepsilon$$

$$= n\varepsilon \left(|L|\cdot|U|\right)_{jk}$$

$$\boxed{\delta_j \le n\varepsilon}$$

$$|E_{jk}| = n\varepsilon \left(|L|\cdot|U|\right)_{jk}$$

$$A = LU + E \qquad \boxed{\|E\| \le n\varepsilon \, \| |L| \cdot |U| \|} \quad (\text{abs.})$$

for norms that do not depend on "signs of the matrix entries (eg. 1-norm, ∞-norm, Frobenious norm, but not the 2-norm)

we may simplify this to-

$$\boxed{\|E\| \le n\varepsilon \, \|L\| \, \|U\|} \; ★★$$

Remark: The above bound doesn't help much for larger n.
So in practice, error analysis is mostly computed on computed residual $r = Ax - b$. $(\vec{x} = A^{-1}r + A^{-1}b)$

$$\therefore \|\delta x\| = \|A^{-1}\| \cdot \|r\|$$

If GEPP is used $\boxed{\|L\| \le 1}$

* Projection matrices

A projection matrix is a matrix P which satisfies $P^2 = P$ (also called idempotent)
(In general, a linear map $T: V \to V$ is a projection if $T^2 = T$)

Note: ① If $v \in$ range P, then $P(v) = v$

If $x = P(v)$,

$P(x) = PP(v) = P^2(v) = P(v) = x$

$\therefore v = x$

② If $v \notin$ range P, then $P(v) - v \in$ kernel (P) / null (P)

$P(P(v) - v) = P^2(v) - P(v)$

$= P(v) - P(v) = 0$

③ If P is a projection then, $I - P$ is also a projection

Want $(I-P)^2 = (I-P)$

$(I-P)^2 v = (I-P)(I-P)v$

$= (I-P)(v - Pv)$

$= v - Pv - Pv + P^2 v.$

$= (I-P)v.$

④ range $(I-P)$ = null (P)

check.

5) null $(I-P)$ = range (P)
(let $P = I - (I-P)$)

④ range $(I-P)$ = null (P)
let $v \in (I-P)$
$\therefore (I-P)v = v.$
$v - Pv = v.$
$Pv = 0.$ $\therefore$ $v \in$ null space of P.
$v \notin (I-P)$
$\therefore (I-P)v = 0.$
$v - Pv = 0$
$v = Pv.$ $\therefore$ $v \in P$ v belongs to null space of $(I-P)$.

⑥ range $(P) \cap$ null space $(P) = \{0\}$

The projection P separates the underlying vector space into two (disjoint) complementary subspaces

$$V = \text{range } P \oplus \text{null } P$$

Note that : range (P) may not be orthogonal to the null space of P i.e. null (P).

Def^n : An orthogonal projection is a matrix P for which range $P \perp$ null P.
~~R is an orthogonal projection if $\Leftrightarrow$ PAPA~~