Note Title

Jordan canonical form.

Defn: Suppose $T \in L(V)$, and $\lambda_1, \dots, \lambda_n$ are e-values of T.

Let $E_r(\lambda_i)$ denote $\ker (T - \lambda_i I)^r$

$$(\{0\} = E_0(\lambda_i) \subseteq E_1(\lambda_i) \subseteq \dots \subseteq E_p(\lambda_i) \subseteq \dots)$$

$E_r(\lambda_i)$ is called the generalized eigenspace of order r corr. to λ_i.

For an e-value λ of T, consider the sequence of non-zero vectors $x_1, \dots, x_k$ such that

$x_1 \in \ker(T - \lambda I)$

$$T x_1 = \lambda x_1 \qquad (x_1 \text{ is an e-vector corr. to e-value } \lambda)$$

$$T x_2 = \lambda x_2 + x_1$$

$$T x_3 = \lambda x_3 + x_2$$

$$\vdots$$

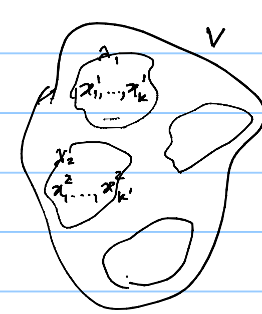
$$T x_k = \lambda x_k + x_{k-1}.$$

then the set of vectors $\{x_1, \dots, x_k\}$ is called a Jordan chain corr. to e-value λ.

Result: A Jordan chain consists of linearly indep. vectors.

If we consider the basis $\mathcal{B}' = \{x_1, \dots, x_k\}$ & T' denotes $T|_{\mathcal{B}'}$, then

$$[T']_{\mathcal{B}'} = \begin{bmatrix} \lambda & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda \end{bmatrix}$$



This matrix is called the Jordan block of order k with e-value λ.

Suppose we find Jordan chains corr. to each e-value; then the union of these Jordan chains forms a basis $\mathcal{B}$ of V, and the matrix of T w.r.t. $\mathcal{B}$ is of the form:

$$[T]_{\mathcal{B}} = \begin{bmatrix} [J_{\lambda_1}] & & & \\ & [J_{\lambda_2}] & & \\ & & \ddots & \\ & & & [J_{\lambda_n}] \end{bmatrix} \longrightarrow \textcircled{\star}$$

<u>Theorem</u>: (Existence of the Jordan canonical form)

T be a linear operator on n-dim'l v.s. V over field F, and suppose the characteristic poly. of T splits over F. Then there exists a basis $\mathcal{B}$ of V such that $[T]_{\mathcal{B}}$ has the form given in $\textcircled{\star}$.

<u>Triangulable matrices</u>:

<u>Defn</u>: A linear operator $T \in L(V)$ is said to be triangulable if there exists an ordered basis $\mathcal{B}$ of V such that $[T]_{\mathcal{B}}$ is upper Δ'r.

(Matrix analogue: for any square matrix A, A is said to be triangulable if there exists an invertible matrix P s.t. $P^{-1}AP$ is upper Δ'r.)

<u>Theorem</u>: Let V be f.d. over F, & $T \in L(V)$. TFAE:

(1) the char. poly. of T splits over F

(2) T is triangulable.

In particular, any matrix is triangulable over $\mathbb{C}$. (but not necessarily over $\mathbb{R}$).

- unitary operators
- Hermitian operators
- SVD.

Unitary operators

Defn: A linear operator T on V is said to be unitary if $TT^* = T^*T = I$. ($\Leftrightarrow T^*$ is unitary)

Properties: If S & T are unitary, then
- $S \circ T$ is unitary
- S^{-1} is unitary.

Proposition: Let V be f.dim'l i.p.s. over F & $T \in L(V)$.
TFAE:
1. T is unitary
2. $\langle Tu, Tv \rangle = \langle u, v \rangle$ for all $u, v \in V$. [T preserves inner products]
3. $\|Tu\| = \|u\|$ for all $u \in V$ [T preserves norms]
4. T maps orthonormal bases to orthonormal bases.

Proof: 1 $\Leftrightarrow$ 2 $\quad \langle Tu, Tv \rangle = \langle u, T^*Tv \rangle = \langle u, v \rangle$ if & only if $T^*T = I$.

2 $\Leftrightarrow$ 3 $\quad$ 2 $\Rightarrow$ 3 clear from defn. of norm.
3 $\Rightarrow$ 2 exercise using polarization identity.

2 $\Leftrightarrow$ 4 $\quad$ 2 $\Rightarrow$ 4 follows from defn. of orthonormality.
4 $\Rightarrow$ 2: Let $\{u_1, \dots, u_n\}$ be an orthonormal basis of V, let $u = \sum_{i=1}^{n} \alpha_i u_i$, $v = \sum_{i=1}^{n} \beta_i u_i$

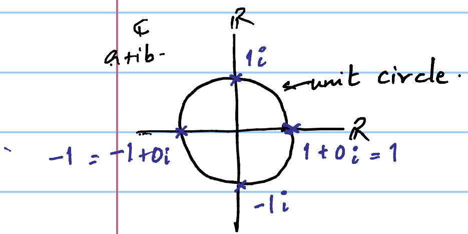
$$\langle Tu, Tv \rangle = \left\langle \sum_{i=1}^{n} \alpha_i Tu_i, \sum_{i=1}^{n} \beta_i Tu_i \right\rangle$$
$$= \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \overline{\beta_j} \langle Tu_i, Tu_j \rangle$$
$$= \sum_i \sum_j \alpha_i \overline{\beta_j} \delta_{ij}, \quad \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{o.w.} \end{cases}$$
(Kronecker delta)
$$= \sum \alpha_i \overline{\beta_i} = \langle u, v \rangle$$
$\square$

Corollary: Eigenvalues of a unitary operator have absolute value 1.

Proof: Suppose λ is an e-value of a unitary operator T and suppose v is the corr. e-vector,

$$\text{then} \quad \|v\| = \|Tv\| = \|\lambda v\| = |\lambda| \cdot \|v\|$$

$$\text{So} \quad |\lambda| = 1. \qquad (v \neq 0). \qquad \square$$



if $z = a+ib \in \mathbb{C}$, $|z| = \sqrt{a^2+b^2}$.

1."

$a^2+b^2=1$.

eqn. of unit circle: $x^2+y^2=1$.

Note that the only real e-values of a unitary operator can be ± 1.

(Terminology: If $F=\mathbb{R}$, then the unitary operator is called 'orthogonal'.)

Proposition: Suppose $T \in L(V)$ & $\mathcal{B}$ is an ordered orthonormal basis of V. Then $[T]_{\mathcal{B}}$ is unitary matrix $\Leftrightarrow$ T is unitary (over $\mathbb{C}$)

[$[T]_{\mathcal{B}}$ is orthogonal matrix $\Leftrightarrow$ T is orthogonal (over $\mathbb{R}$)].

Proposition: Suppose A is a $n \times n$ matrix. TFA.E

1. A is unitary
2. Columns of A form an orthonormal basis of F^n w.r.t. standard inner product
3. columns of A^t form an orthonormal basis of F^n w.r.t. the standard inner product.

Pf: ① $\Rightarrow$ ② apply A to the standard basis $\{e_1, \dots, e_n\}$ $\qquad e_i = (0, \dots, 0, \underset{i}{1}, 0 \dots 0)^t$

Ae_i = i^{th} column of A

A unitary $\Rightarrow$ A sends orthornormal basis $\{e_1, \ldots, e_n\}$ to an orthonormal basis $\underbrace{\{Ae_1, \ldots, Ae_n\}}_{\text{columns of } A}$.

② $\Rightarrow$ ① The $(i,j)^{th}$ entry of a matrix B $e_i^t B e_j$.

Suppose the columns of A are orthormal, then

$$\underset{\text{of } A^*A}{(i,j)^{th} \text{entry}} = e_i^t (A^* A) e_j = (Ae_i)^* (Ae_j) = \delta_{ij}$$

$$\therefore A^* A = I_n \Rightarrow A \text{ is unitary.} \qquad \square.$$

Corollary: If $T \in L(V)$ is unitary, and if $\mathcal{B}_1$ & $\mathcal{B}_2$ are ordered orthonormal bases of V then ${}_{\mathcal{B}_2}[T]_{\mathcal{B}_1}$ is a unitary matrix.