Bisection method.

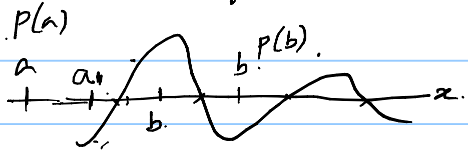
Idea: locate e-values of a symmetric matrix A by locating the roots of the polynomial $p(x) = \det(A - xI)$.

Let c be a reqd. root, suppose $c \in [a,b]$

Calculate $p(a)$, $p(b)$: if one of them is zero, we found a root.

if both are nonzero,

check signs.



if they have opposite signs,

we know that $c \in [a,b]$.

– now let $t = \frac{a+b}{2}$, find $p(t)$.

if $p(t)=0$ then $c=t$.

else, continue with $[a,t]$ & $[t,b]$.

The bisection method is most powerful in the case of symmetric matrices – because the e-values of symm. matrices have some nice properties.

Let $A \in \mathbb{R}^{m \times m}$ be symmetric. Then A can be reduced to tridiagonal form

$$\begin{pmatrix} a_1 & b_1 & 0 & \cdots & 0 \\ b_1 & a_2 & b_2 & \ddots & \vdots \\ \vdots & b_2 & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & b_{m-1} \\ 0 & \cdots & 0 & b_{m-1} & a_m \end{pmatrix}$$

Exercises ① Let $A^{(1)}, A^{(2)}, \ldots, A^{(m)}$ denote the top-left principal submatrices of A.

Suppose the off-diagonal entries are non-zero i.e. $b_i \neq 0$.

Define polynomial $p_k(x) = \det\left(A^{(k)} - x I_{k\times k}\right)$, for $2 \le k \le m$.

$= \text{char}\left(A^{(k)}\right)$.

These polynomials satisfy a recurrence relation –

(Exercise) $$p_k(x) = (a_k - x)\, p_{k-1}(x) - b_{k-1}^2\, p_{k-2}(x). \quad ☆$$

② If A is a hermitian, tri-diagonal matrix with off diagonal entries being non-zero, then the e-values of A will be distinct.

Let $\lambda_i^{(k)}$ be the i^{th} e-value of $A^{(k)}$.

Now suppose that $\lambda_1^{(k)} < \lambda_2^{(k)} < \cdots < \lambda_k^{(k)}$ are the distinct e-values of $A^{(k)}$.

The crucial idea that makes the bisection method work is that these eigenvalues "strictly interlace" –

<u>Theorem</u> (Sturm sequence property) – If the tridiagonal matrix A has no zero subdiagonal entries, then the e-values of $A^{(k)}$ strictly interlace with e-values of $A^{(k+1)}$ –

$$\lambda_j^{(k+1)} < \lambda_j^{(k)} < \lambda_{j+1}^{(k+1)} \quad \forall\ 1 \leq k \leq m-1 \ \ \& \ \ 1 \leq j \leq k-1.$$

Moreover, if $a(\lambda)$ denotes the number of sign changes in the Sturm sequence – (for some $\lambda \in \mathbb{R}$)

$$p_0(\lambda), p_1(\lambda), \ldots, p_m(\lambda)$$

then $a(\lambda)$ equals the number of eigenvalues of A that are less than λ.

(By convention, $p_0(\lambda) = 1$ & if $p_k(\lambda) = 0$ then $p_k(\lambda)$ is said to have opposite sign from $p_{k-1}(\lambda)$)

$$A^{(k-1)} \qquad \lambda_1^{(k-1)} < \lambda_2^{(k-1)} \cdots < \lambda_{j-1}^{(k-1)} < \lambda_j^{(k-1)} < \lambda_{j+1}^{(k-1)} < \cdots < \lambda_{k-1}^{(k-1)}$$

$$A^{(k)} \qquad \lambda_1^{(k)} < \lambda_2^{(k)} \cdots < \lambda_{j-1}^{(k)} < \lambda_j^{(k)} < \lambda_{j+1}^{(k)} < \cdots$$

$$A^{(k+1)} \qquad \lambda_1^{(k+1)} < \lambda_2^{(k+1)} \cdots < \lambda_{j-1}^{(k+1)} < \lambda_j^{(k+1)} < \lambda_{j+1}^{(k+1)} < \cdots$$

This property makes it possible to find the exact number of e-values of a matrix in a specified interval.

In particular, the above thm. says that -

① the # of e-values < 0, i.e. # of negative e-values of A (lie in $(-\infty, 0)$)

i.e. $a(0)$ equals the # of sign changes in the sequence

$$p_0(0),\ p_1(0),\ \dots,\ p_m(0)$$

i.e. $1,\ \det A^{(1)},\ \dots,\ \det A^{(m)} = \det A$.

② the # of e-values of A in $[a, b)$ equals -

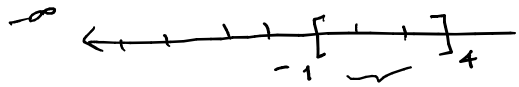
$$\left(\text{\# of e-values in } (-\infty, b)\right) - \left(\text{\# of evalues in } (-\infty, a)\right).$$

Note that the e-values of A in $(-\infty, b)$ can be found by looking for all -ve e-values of $A + bI$

(since $(A+bI)x = \lambda x \iff Ax = (\lambda - b)x$).

Thus, by applying the recurrence relation for $p_k(x)$ & counting the number of sign changes along the way, the bisection method can locate e-values in an arbitrarily small interval.

Example: Find number of e-values of $A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 1 & -4 \end{pmatrix}$ in the interval $[-1, 4]$.



Soln: # e-values of A in $[-1, 4]$

$$= \left(\#\text{ e-values of } A \text{ in } (-\infty, 4)\right) - \left(\#\text{ e-values of } A \text{ in } (-\infty, -1)\right)$$

$$= \left(\#\text{ neg. e-values of } A + 4I\right) - \left(\#\text{ neg. e-values of } A - I\right)$$

- Compute $A + 4I$, compute its Sturm seq.

  Note # sign changes (call it $a(4)$)

- Compute $A - I$ & its Sturm seq.

  Note # sign changes, call it $a(-1)$.

What to do if some off-diagonal entry b_i equals zero?

Apply deflation.

$$\left(\begin{array}{cccc|cccc} a_1 & b_1 & & & & & & \\ & a_2 & & & & 0 & & \\ & & \ddots & & & & & \\ & & a_{i-1} & b_{i-1} & b_i & & & \\ & & b_{i-1} & a_i & & & & \\ \hline & & & b_i & a_{i+1} & \ddots & & \\ & 0 & & & & \ddots & b_{m-1} & \\ & & & & & b_{m-1} & a_m & \end{array}\right)$$