Projections

Note Title

Defn: The decomposition $V = W \oplus W^{\perp}$, for any subspace W of V, gives a unique decomposition for every $v \in V$ as -

$$v = w + w' \quad , \quad w \in W, \ w' \in W^{\perp}$$

In this case, the map $v \mapsto w$ is called "the orthogonal projection of v onto W". (Sometimes, w itself is called the orthogonal proj. of v onto W.).

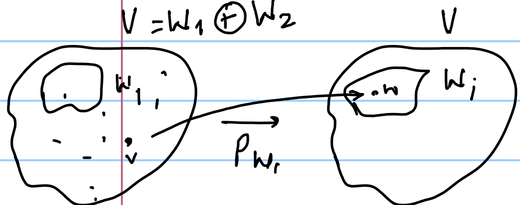
$W \subseteq V$, $V = W \oplus W^{\perp}$ (direct sum decomposition)

- $(W^{\perp})^{\perp} = W$
- $W_1 \subseteq W_2$ then $W_2^{\perp} \subseteq W_1^{\perp}$.

- In general you can define $P_{W_1} : V \to V$, for $V = W_1 \oplus W_2$
  $v \mapsto w$, $w \in W_1$

  & check that:
  1. P_{W_1} is a linear transf.
  2. $P_{W_1}(V) = W$.
  3. $P_{W_1}^2 = P_{W_1}$ (i.e. P_{W_1} is an idempotent.)

$v = w + w'$
$V = W_1 \oplus W_2$



$\mathrm{im}(P_{W_1}) = W_1$

$$P_{W_1}^2(v) = P_{W_1}(P_{W_1}(v)) = P_{W_1}(w) = w = P_{W_1}(v).$$

P_{W_1} is called the "projection of V onto W_1 along W_2".

- In particular, if $V = W \oplus W^{\perp}$ & we define

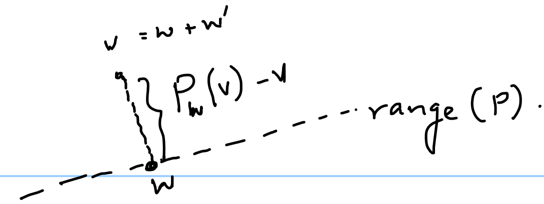
$$P_W : V \to V$$
$$v \mapsto w, \ w \in W$$

then P_W is called the "orthogonal proj. of V onto W".

Notice that for $v \in V$, $P_W(v) \in W$ & $v - P_W(v) \in W^{\perp}$

i.e. $\left(P_W(v) - v \in (\text{range } P_W)^{\perp} = W^{\perp}\right)$

Schematic :



If $V = W_1 \oplus \cdots \oplus W_k$.

P_{W_1} = proj. of V onto W_1 along $W_2 \oplus \cdots \oplus W_k$

Orthogonal projections occur often in practice.

How to compute the matrix of P_W?

Theorem: Let W be a subspace of a f.d. i.p.s. V & $v \in V$:

(i) $\|v - P_W(v)\| \leq \|v - w\|$ for any $w \in W$.

(ii) If $\{x_1, \ldots, x_m\}$ is an orthonormal basis of W, then $P_W(v) = \sum_{i=1}^{m} \langle v, x_i \rangle x_i$.

Proof: (i) $v - P_W(v) \in W^\perp$, so for any $w \in W$,

$$\underbrace{v - P_W(v)}_{\in W^\perp} \perp \underbrace{P_W(v) - w}_{\in W}.$$

Consider
$$\|v - w\|^2 = \|v - P_W(v) + P_W(v) - w\|^2$$
$$= \|v - P_W(v)\|^2 + \underbrace{\|P_W(v) - w\|^2}_{\geq 0}.$$

$$\Rightarrow \|v - w\|^2 \geq \|v - P_W(v)\|^2.$$

(ii) Recall that for $V = W \oplus W^\perp$
$$v = \underbrace{\sum_{i=1}^{m} \langle v, x_i \rangle x_i}_{\in W} + \underbrace{w'}_{\in W^\perp}$$

so $P_W(v) = \sum_{i=1}^{m} \langle v, x_i \rangle x_i$

- If $W = \langle a \rangle$, then $P_W(v) = \langle v, \frac{a}{\|a\|} \rangle \cdot \frac{a}{\|a\|}$

$$= \frac{1}{a^* a} \langle v, a \rangle a$$

$a = \begin{pmatrix} a_1 \\ \vdots \\ a_p \end{pmatrix} \in W \leq V = \dim p.$

$$= \frac{1}{a^* a} \underbrace{(a^* v) a}$$

$a a^* = \begin{pmatrix} a_1 \\ \vdots \\ a_p \end{pmatrix} (a_1 \cdots a_p)$

$$= \frac{1}{a^* a} a (a^* v)$$

$= \begin{pmatrix} & \end{pmatrix}_{p \times p}$

$$= \left(\frac{a a^*}{a^* a} \right) v.$$

What is rank P_W?

$(P_W : V \to W.$

$\therefore P_W = \frac{a a^*}{a^* a}.$

$\{$ • For a proj. P, $I - P$ is also a proj.

(check). $I - P_W = I - \frac{a a^*}{a^* a}$ is the proj. onto $W^\perp$.

- If $\dim W > 1$ & $\{a_1, \dots, a_k\}$ is a basis for W, let $A = [a_1 | a_2 | \cdots | a_k]$, then $P_W = A (A^T A)^{-1} A^T$.

If $\{q_1, \dots, q_k\}$ is an orthonormal basis for W, let $Q = [q_1 | \cdots | q_k]$, then $P_W = Q Q^T$.

Exercise: Let $W = \text{span} \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}, \begin{pmatrix} 5 \\ 6 \\ 7 \\ 8 \end{pmatrix} \right\}$ in $\mathbb{R}^4$.

(these are lin. indep.).

Let $P_W : \mathbb{R}^4 \to W$.

$$P_W = A\left(A^T A\right)^{-1} A^T, \quad \text{where } A = \begin{bmatrix} 1 & 5 \\ 2 & 6 \\ 3 & 7 \\ 4 & 8 \end{bmatrix}.$$

- calculate P_W.
- check that P_W does indeed map any vector of $\mathbb{R}^4$ onto W.

$$P_W \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} \\ \\ \end{pmatrix} = \underline{\underline{\alpha}} \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} + \underline{\underline{\beta}} \begin{pmatrix} 5 \\ 6 \\ 7 \\ 8 \end{pmatrix}.$$

<u>Note:</u> $\operatorname{rank} P_W = \dim W < \dim V$ $\qquad$ $\operatorname{rank} P_W = \dim W$.

$\therefore P_W$ is NOT an orthogonal matrix.

<u>Eigenvalues</u> & <u>eigenvectors</u>:

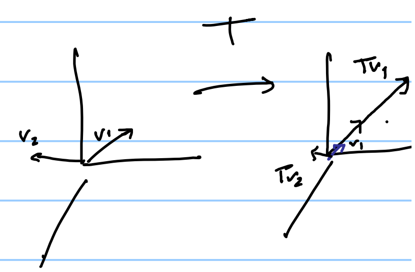
V is a v.s. over field K.

<u>Defn</u>: For a linear map $T : V \to V$, and a scalar $\lambda \in K$, if $\exists$ a non-zero vector $v \in V$ such that $\underline{Tv = \lambda v}$, then v is said to be an e-vector of T corresponding to the e-value λ.

$T : V \to V$.

$\{v_1, \dots, v_k\}$

$T v_i = \lambda_i v_i$



T is diagonalizable

$\therefore (P^{-1} T P$ is diagonal$)$

T is not diagonalizable.

$\downarrow$

Jordan canonical form.

(Tutorial : How to compute e-values & e-vectors?)

Remarks: ① Notice that $\underset{\sim}{T} v = \lambda v = \underbrace{\lambda \cdot I}\, v$

$$\therefore \quad (T - \lambda I) v = 0$$ $\longrightarrow$ you will use this in computation.

$$\Rightarrow \quad v \in \ker (T - \lambda I).$$

② If $Tv = \lambda v$, then $\ker (T - \lambda I)$ is non-empty & non-zero.

③ From the 1st isomorphism thm, if $T : V \to V$ is linear then

$$\dim V = \dim (\ker T) + \dim (\operatorname{im} T).$$

Here, $\dim V = \underset{>0}{\dim (\ker (T - \lambda I))} + \underset{< \dim V.}{\dim (\operatorname{im} (T - \lambda I))}.$

$$\text{so } \det (T - \lambda I) = 0.$$

④ To compute all the e-values of T, we solve the above equation.

Ex. $A = \begin{bmatrix} 6 & -1 & 2 \\ 4 & 1 & 2 \\ -10 & 0 & 3 \end{bmatrix}$ $\rightsquigarrow$ lin. trans. from $\mathbb{R}^3 \to \mathbb{R}^3$.

$$A - x I = \begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix} - \begin{bmatrix} x & & 0 \\ & x & \\ 0 & & x \end{bmatrix}$$

$$= \begin{bmatrix} 6 - x & -1 & 2 \\ 4 & 1 - x & 2 \\ -10 & 0 & 3 - x \end{bmatrix}$$

$$\det (A - x I) = (\text{poly. of deg. } 3 \text{ in } x).$$

solve to get all e-values of A.

(exercise).

$$\left(\overbrace{E_k \cdots E_1}^{P} A \overbrace{E_1' \cdots E_k'}^{Q}\right) \qquad A$$

Do A & PAQ have the same e-values?.

This happens $\Leftrightarrow$ PAQ is a similarity transf. of A.

Similarity: $A \sim B \Leftrightarrow B = P^{-1} A P$