Gaussian elimination

Note Title

Let A be a square $m \times m$ matrix.

Idea - to transform A into an upper triangular matrix by introducing zeroes below the diagonal.

Suppose $A = \begin{array}{c} \begin{array}{cccc} a_1 & a_2 & \cdots & a_m \end{array} \\ \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & & & \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mm} \end{bmatrix} \end{array}$ such that A is invertible & $a_{ii} \neq 0 \ \forall i$.

Step 1: To annihilate $a_{21}, \dots, a_{m1}$; assume $a_{11} \neq 0$.

- $R_2 - \frac{a_{21}}{a_{11}} R_1 \longleftrightarrow$ multiplying A on the left by

$$L_{21} = \begin{bmatrix} 1 & & & 0 \\ -\frac{a_{21}}{a_{11}} & 1 & & \\ \vdots & & \ddots & \\ 0 & & & 1 \end{bmatrix}$$

- $R_3 - \frac{a_{31}}{a_{11}} R_1 \longleftrightarrow$ left multi. by $\begin{bmatrix} 1 & & & 0 \\ 0 & 1 & & \\ -\frac{a_{31}}{a_{11}} & & \ddots & \\ \vdots & & & 1 \end{bmatrix} = L_{31}$

$\vdots$

- $R_m - \frac{a_{m1}}{a_{11}} R_1 \longleftrightarrow$ L.M. by $\begin{bmatrix} 1 & & & 0 \\ 0 & 1 & & \\ \vdots & & \ddots & \\ 0 & & & \\ -\frac{a_{m1}}{a_{11}} & & & 1 \end{bmatrix} = L_{m1}$

Let $l_{21} = \frac{a_{21}}{a_{11}}$, $l_{31} = \frac{a_{31}}{a_{11}}$, $\dots$, $l_{m1} = \frac{a_{m1}}{a_{11}}$.

$$L_1 = L_{m1} \cdots L_{31} L_{21} = \begin{bmatrix} 1 & & & 0 \\ -l_{21} & \ddots & & \\ -l_{31} & & \ddots & \\ \vdots & & & \\ -l_{m1} & & & 1 \end{bmatrix}$$

$$A_1 = L_1 A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ 0 & \boxed{a^1_{22}} & \cdots & a^1_{2m} \\ 0 & a^1_{32} & \cdots & a^1_{3m} \\ \vdots & \vdots & & \vdots \\ 0 & & & a^1_{mm} \end{bmatrix}$$

$A_2 = L_2 (L_1 A)$

Assuming $a^1_{22} \neq 0$,

$l_{32} = \frac{a^1_{32}}{a^1_{22}}$

$L_{32} = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & -l_{32} & \ddots & \\ & 0 & & \ddots \\ & \vdots & & \end{bmatrix}$ & so on.

Finally get L_2.

$$L_2 = \begin{bmatrix} 1 & & & 0 \\ & 1 & & \\ & -l_{32} & \ddots & \\ & -l_{42} & & \ddots \\ & \vdots & & \\ & -l_{m2} & & 1 \end{bmatrix}$$

Step k :

$$A_{k-1} = \begin{array}{c|c} & \begin{array}{ccccccc} & & & & k & & \end{array} \\ & \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1k-1} & a_{1k} & \cdots & a_{1m} \\ 0 & a_{22}^1 & a_{23}^1 & & & \vdots & & a_{2m}^1 \\ 0 & 0 & a_{33}^2 & & & \vdots & & \vdots \\ \vdots & \vdots & & \ddots & a_{k-1,k-1}^{k-1} & \vdots & & \\ k & & & & 0 & a_{kk}^{k-1} & \cdots & \\ & & & & \vdots & \vdots & & \\ & & & & 0 & \vdots & \cdots & \end{bmatrix} \end{array}$$

$$L_k = \begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & 1 & & \\ & & -l_{k+1,k} & \ddots & \\ & & \vdots & & \\ & & -l_{m,k} & & 1 \end{bmatrix}, \quad \text{where } l_{k+1,k} = \frac{a_{k+1,k}^{k-1}}{a_{kk}^{k-1}}$$

$$l_{jk} = \frac{a_{jk}}{a_{kk}}$$

$$\& \quad A_k = L_k A_{k-1}$$

Continuing in this way, $(L_{m-1} L_{m-2} \cdots L_2 L_1) A = \mathcal{U}$ (say) (upper Δ^r)

Let $L = L_1^{-1} L_2^{-1} \cdots L_{m-1}^{-1}$, then $A = L\,\mathcal{U}$.

(the $L\mathcal{U}$-factorization of A).

Note that $L = \begin{bmatrix} 1 & & & \\ l_{21} & 1 & & 0 \\ l_{31} & l_{32} & \ddots & \\ \vdots & \vdots & & \ddots \\ l_{m1} & & l_{m,m-1} & 1 \end{bmatrix}$; $\det L = 1$.

(check)

If A is any square matrix with a_{ii} not necessarily $\neq 0$, then we have to introduce "pivoting".

This is done as follows -

Step 1 : check whether $a_{11} \neq 0$. If yes, find L_1 & compute $L_1 A$.

$j \begin{bmatrix} 0 \\ 0 \\ 0 \\ a_{j1} \end{bmatrix}$ Else, suppose a_{j1} is the first nonzero element in the first column.

Then exchange rows j & 1.

i.e. left multiply by $E_{1j} = \begin{array}{c|c} & \begin{array}{ccc} 1 & & j \end{array} \\ \begin{array}{c} 1 \\ j \end{array} & \begin{bmatrix} 0 & \cdots & 1 & \\ \vdots & & \vdots & \\ 1 & 0 \cdots & 0 & \\ & & & \ddots & 1 \end{bmatrix} \end{array}$