Note Title 07-03-2022

Stability of an algorithm -

- The meaning of $O(\varepsilon_{mach})$: The notation $\varphi(t) = O(\psi(t))$ asserts that $\exists$ some positive constant C such that for all t close to some understood limit (such as $t \to 0$ or $t \to \infty$)

$$|\varphi(t)| \le C \cdot |\psi(t)|.$$

(It is important to note that C is a uniform constant which works for all t.)

Thus, when we write

$$\| \text{ term } \| = O(\varepsilon_{mach})$$

we mean that $\exists$ some +ve constant C s.t.

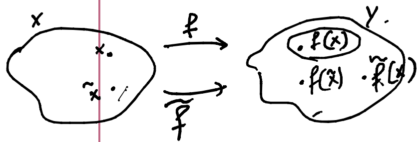
$$\| \text{ Term } \| \le C \cdot \varepsilon_{mach}.$$

- An algorithm can be viewed as a function $\tilde{f} : X \to Y$ where X & Y are normed v.s. of data & solutions respectively.

($f : X \to Y$ is the problem under consideration)

Defn: An algorithm $\tilde{f}$ is said to be accurate if

rel. error in computation using $\tilde{f}$ $\rightarrow$ $$\frac{\| f(x) - \tilde{f}(x) \|}{\| f(x) \|} = O(\varepsilon_{mach}).$$



<u>Defn</u>: An algorithm $\tilde{f}$ is said to be stable (i.e. forward stable) if for every $x \in X$,

$$\frac{\|\tilde{f}(x) - f(\tilde{x})\|}{\|f(\tilde{x})\|} = O(\varepsilon_{mach}) \quad \text{for some } \tilde{x}$$

with $\frac{\|x - \tilde{x}\|}{\|x\|} = O(\varepsilon_{mach})$.

★ i.e. $\tilde{f}$ gives nearly the right answer to nearly the right question.

↑ $\tilde{x}$ is a slight perturbation in x.

<u>Defn</u>: An algorithm is said to be backward stable if

for every $x \in X$, $\tilde{f}(x) = f(\tilde{x})$ for some $\tilde{x}$ with $\frac{\|x - \tilde{x}\|}{\|x\|} = O(\varepsilon_{mach})$.

i.e. a backward stable algorithm gives the right answer to
★ nearly the right question.

<u>Note</u>: (1) Backward stability $\Rightarrow$ forward stability

(2) Backward error analysis is easier to carry out.

<u>Some examples</u> -

(1) Subtraction : $f(x_1, x_2) = x_1 - x_2$. ← by hand.

$\tilde{f}(x_1, x_2) = fl(x_1) \ominus fl(x_2)$ ← by computer

(i) $fl(x_1) = x_1(1+\varepsilon_1)$ & $fl(x_2) = x_2(1+\varepsilon_2)$

(from fund. ppty. of fl. pt. repr.) where $|\varepsilon_1|$ & $|\varepsilon_2| \le \varepsilon_{mach}$.

(ii) $fl(x_1) \ominus fl(x_2) = \left(fl(x_1) - fl(x_2)\right)(1+\varepsilon_3)$,

(fund. ppty. of fl. pt. arithmetic) where $|\varepsilon_3| \le \varepsilon_{mach}$.

Putting these together:

$$\tilde{f}(x_1,x_2) = fl(x_1) \ominus fl(x_2) = \big(x_1(1+\varepsilon_1) - x_2(1+\varepsilon_2)\big)(1+\varepsilon_3)$$

$$= x_1\underbrace{(1+\varepsilon_1)(1+\varepsilon_3)}_{1+\underbrace{\varepsilon_1+\varepsilon_3+\varepsilon_1\varepsilon_3}_{\le 2\varepsilon_{mach}+\varepsilon_{mach}^2}} - x_2\underbrace{(1+\varepsilon_2)(1+\varepsilon_3)}_{1+\varepsilon_2+\varepsilon_3+\varepsilon_2\varepsilon_3}$$

$$= \underbrace{x_1(1+\varepsilon_4)}_{\tilde{x}_1} - \underbrace{x_2(1+\varepsilon_5)}_{\tilde{x}_2} = \tilde{x}_1 - \tilde{x}_2 = f(\tilde{x}_1,\tilde{x}_2)$$

where $|\varepsilon_4|, |\varepsilon_5| \le 2\varepsilon_{mach} + \varepsilon_{mach}^2$.

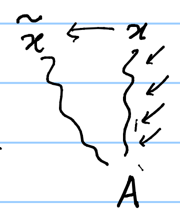
The computed result $= \tilde{x}_1 - \tilde{x}_2$, where $\dfrac{|\tilde{x}_1 - x_1|}{|x_1|} = O(\varepsilon_{mach})$

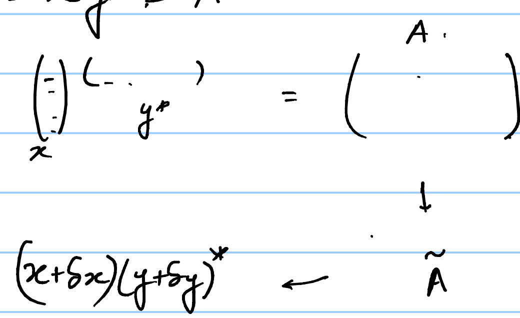
$\therefore$ Subtraction is backward stable. $\dfrac{|\tilde{x}_2 - x_2|}{|x_2|} = O(\varepsilon_{mach})$.

(2) Inner product:- can show that for vectors $x, y \in \mathbb{C}^m$, $\alpha = x^* y$ is backward stable.

(3) Outer product - is (forward) stable but not backward stable.

$x, y \in \mathbb{C}^m$, $f(x,y) = xy^* = A$.





(4) Example 15.4, page 110 T-B. ☆

☆ (5) An unstable algorithm - page 110, T-B.

Accuracy of a backward stable algorithm. (floating pt. error analysis.)

What can be said about the accuracy of a backward stable algorithm?

Theorem: Suppose a backward stable algorithm is applied to solve a problem $f: X \to Y$ with condition number κ on a computer that satisfies the fund. ppties of fl. pt. repr. & fl. pt. arithmetic. Then the relative errors satisfy -

$$\frac{\| \tilde{f}(x) - f(x) \|}{\| f(x) \|} = O\left(\kappa(x) \cdot \varepsilon_{mach} \right)$$

Proof: By defn. of backward stability, $\tilde{f}(x) = f(\tilde{x})$ for some $\tilde{x} \in X$ s.t. $\frac{\| x - \tilde{x} \|}{\|x\|} = O(\varepsilon_{mach})$.

Recall, $\kappa(x) = \sup_{\delta x \to 0} \left(\frac{\| \delta f \| / \| f(x) \|}{\| \delta x \| / \|x\|} \right)$ Take $\delta x = \tilde{x} - x$.

$$\kappa(x) \geq \frac{\| f(\tilde{x}) - f(x) \|}{\| f(x) \|} \Big/ \frac{\| \tilde{x} - x \|}{\|x\|}$$

$f(\tilde{x}) = \tilde{f}(x)$.

$$\therefore \frac{\| \tilde{f}(x) - f(x) \|}{\| f(x) \|} \leq O\left(\kappa(x) \cdot \varepsilon_{mach} \right).$$

Thus, if an alg. is backward stable, then the accuracy of the answer depends on the condition number at x.

Work involved in an algorithm.

- flops.
- cost of moving the data
- cost of storing the data.

G-L: 1.2.1 - 1.2.7
1.1.10 - 1.1.17
1.4 & 1.5

- communication-avoidance
- updation.

BLAS - basic lin. alg. subroutines LAPACK library.

"levels"

level 1 BLAS · saxpy : scalar $\underline{a}$ times $\underline{x}$ plus y, (x, y vectors)

$$a\begin{pmatrix}x_1\\ \vdots \\ x_n\end{pmatrix} \pm \begin{pmatrix}y_1\\ \vdots \\ y_n\end{pmatrix}$$

$y = ax + y$. (update form). $2n$.

level 2 BLAS · gaxpy : generalized saxpy - A is a matrix.

$y = Ax + y$. $\quad O(mn)$.

$\underline{mn} + m + n$.

level 3 BLAS · matrix-matrix multi.

	data	work
level - 1	linear $O(n)$	$O(n)$ linear
level - 2	$O(mn)$ quadratic	$O(mn)$ quadratic
level - 3	$O(mn)$ quadratic	$O(mnr)$ cubic. ✓

$\underline{m \times r}$ $\quad \underline{r \times n}$

How many flops reqd. for multi. 2 matrices.