Note Title

Eigenvalues & eigenvectors.

Ex. $A = \begin{bmatrix} 6 & -1 & 2 \\ 4 & 1 & 2 \\ -10 & 0 & 3 \end{bmatrix}_{3\times 3}$. (over $\mathbb{C}$).

(Consider A as a linear transf. from $\mathbb{C}^3 \to \mathbb{C}^3$)

• To find eigenvalues: solve $\det(A - xI) = 0$.

$\det(A - xI)$ → characteristic polynomial of A.

$$A - xI = \begin{bmatrix} 6-x & -1 & 2 \\ 4 & 1-x & 2 \\ -10 & 0 & 3-x \end{bmatrix}$$

$$\det(A - xI) = \dots = (x-2)(-x^2 + 8x - 35)$$

Alg. multi. (λ) := # of times λ appears as a root of $\det(A - xI)$

Roots are: 2, $4 \pm i\sqrt{19}$.

$\lambda_1 = 2$, $\lambda_2 = 4 + i\sqrt{19}$, $\lambda_3 = 4 - i\sqrt{19}$.

• To find eigenvectors:

(i) $\lambda_1 = 2$ - Solve $(A - \lambda_1 I)v = 0$

$$A - 2I = \begin{pmatrix} 4 & -1 & 2 \\ 4 & -1 & 2 \\ -10 & 0 & 1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

$\vdots$

Geometric multiplicity (λ) := dimension of the eigenspace of λ.

$$v = t\begin{pmatrix} 1/10 \\ 12/5 \\ 1 \end{pmatrix}.$$

The e-space corr. to λ_1 is $\left\{ t\begin{pmatrix} 1/10 \\ 12/5 \\ 1 \end{pmatrix} \,/\, t \in \mathbb{C} \right\}$.

A particular e-vector can be chosen by making a choice of $t\,(\neq 0)$.

(ii) $\lambda_2 = 4 + i\sqrt{19}$

(iii) $\lambda_3 = 4 - i\sqrt{19}$

E-value	alg. multi.	geom. multi.
$\lambda_1 = 2$	1	1
$\lambda_2 = 4 + i\sqrt{19}$	1	~
$\lambda_3 = 4 - i\sqrt{19}$	1	~

Diagonalizability

Question: Given any matrix A, is there a "simpler form" of A, say B, such that A is 'similar to' B.

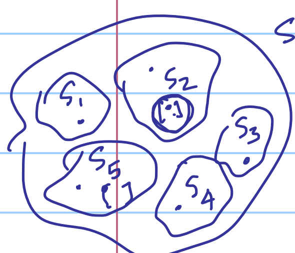
Similarity: A & B are square matrices of the same dimension ($n \times n$).

We define 'A similar to B' & denote $A \sim B$ if $\exists$ an invertible matrix P s.t.

$$A = P^{-1} B P.$$

- Similarity is an equivalence relation on the set S of all $n \times n$ matrices -
  - i) reflexive : $A \sim A \quad \forall A \in S$
  - ii) symmetric: if $A \sim B$ then $B \sim A$
  - iii) transitive: if $A \sim B$ & $B \sim C$ then $A \sim C$.

$\rightarrow S = S_1 \cup \dots \cup S_k$ s.t. $S_i \cap S_j = \phi$.



$\mathbb{Z}$: fix $m \in \mathbb{Z}$.
$a \sim b$ if $m \mid (a-b)$.

$m=5$: $\mathbb{Z} = \underbrace{5k}_{0} \cup \underbrace{5k+1}_{1} \cup \underbrace{5k+2}_{2} \cup \underbrace{5k+3}_{3} \cup \underbrace{5k+4}_{4}$.

Back to matrices: S breaks up into similarity classes.

- char. poly.
- e-values
- det.
- trace
- e-vectors & e-spaces.

① If $A \sim B$, then do A & B have the same det, tr, char. poly. etc.

② Is there a "nice" element that can be chosen as a repr. of the entire similarity class?

"nice" $\equiv$ diagonal.

Defn: A $n \times n$ matrix A is said to be diagonalizable if $\exists$ a diagonal matrix D & an invertible matrix P s.t.

$$A = P^{-1} D P \quad \left(\text{i.e. } A \sim \text{a diagonal matrix } D.\right).$$

Q: When is a given $n \times n$ matrix A diagonalizable?

Note: ① Given any polynomial $p(x) \in K[x]$, one can make sense of $p(T)$ for any linear transf. $T: V \to V$ (V is a v.s. over K).

eg. $p(x) = x^2 + 2x + 5$.

$$p(T) = T^2 + 2T + 5I. \qquad T^k = \underbrace{T \circ T \circ \cdots \circ T}_{k \text{ times}}.$$

($p(T): V \to V$ is also a linear transf).

If $\underset{(n\times n)}{A}$ is a square matrix: $p(A) = A^2 + 2A + 5I_{n \times n}$

② Any linear transf. T satisfies its char. poly.

$$\text{char}(T) = \det(T - \underset{\sim}{x} I). = p(x)$$

then $p(T) = \det(T - \underset{\sim}{T} I) = 0$

Defn: Minimal poly. of T:

$$\text{Let } S = \{ p(x) \in K[x] \mid p(T) = 0 \}$$

S is non-empty since $\text{char}(T) \in S$.

(partial order on S: $a(x) < b(x)$ if $a(x)$ divides $b(x)$ $(a(x) \mid b(x))$.

eg. ① $(x-1) < (x^2-1)$.

$(x-1) \mid (x^2-1)$.

② x & $x+1$ are not related)

W.r.t. this ordering, the minimal element of S is called the minimal poly. of T $(m_T(x))$.

Important properties:

1. $m_T(x)$ divides all other elements of S. In particular $m_T(x) \mid char(T)$.