Note Title 24-01-2022

Prerequisites -

1) Linear dep. & indep : ① to check a set of vectors is lin. dep. or indep.
   ② some examples.

2) Span of a set : ① to check if a set of vectors spans a subspace.
   ② what is subspace generated by a set of vectors.

3) Basis & dimension - ① check if a set of vectors is a basis
   ② calculate dimension.

4) Linear transformations : ① how to check if a map is a lin. transf.
   → ② {linear transf} $\leftrightarrow$ {matrices}
   → ③ Change of basis.
   ④ kernel & image.

5) Eigenvalues & e-vectors - ① calculate these.
   ② diagonalizability -
   - what does it mean?
   - how to check?
   - how to get to the diagonal form if the matrix is diagonalizable.
   ③ Jordan canonical form.

6) Inner product spaces, G-S orthonormalisation, adjoint, Normal operators & spectral theorem.

$\alpha \in K$, V_1 & V_2 are v.s. over K.

Linear transformations :

$$V_1 \xrightarrow{f} V_2$$

$$v \longrightarrow f(v)$$
$$w \longrightarrow f(w).$$

$$\begin{tikzcd}
v, w \quad V_1 \arrow[r, "f"] \arrow[d, "\text{add}^n"] \arrow[dr, bend left] & V_2 \quad f(v), f(w) \arrow[d, "\text{add}^n"] \\
v+w \quad V_1 \arrow[r, "f"] & V_2 \quad f(v+w)
\end{tikzcd}$$

① . $v+w \longrightarrow f(v+w)$, $\quad f(v)+f(w) = f(v+w)$

② $\alpha \cdot v \longrightarrow f(\alpha \cdot v)$, $\quad \alpha \cdot f(v) = f(\alpha \cdot v)$.

$K = \mathbb{R}$ or $\mathbb{C}$.

Let V_1, V_2 be f. dim'l. v.s. over K., $\dim V_1 = m$, $\dim V_2 = n$

$$\left\{ \begin{matrix} \text{lin transf} \\ V_1 \to V_2 \end{matrix} \right\} \longleftrightarrow \left\{ K^{n \times m} \right\}.$$

$$V_1 \xrightarrow{T} V_2.$$
$\dim V_1 = m$. $\qquad \dim V_2 = n$

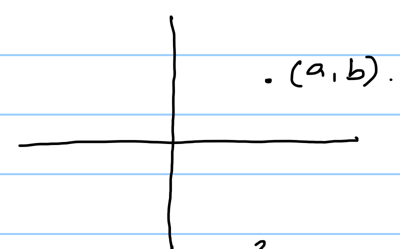
Suppose $T: V_1 \to V_2$ is a lin. Transf.

$\exists$ a basis for V_1 $\{x_1, \ldots, x_m\}$ $\qquad$ $\exists$ basis for V_2 $\{y_1, \ldots, y_n\}$.

$\{x_1, \ldots, x_m\} \longrightarrow \{Tx_1, Tx_2, \ldots, Tx_m\}$

$\mathbb{R}^2$ over $\mathbb{R}$.

$\mathbb{R}^2$.



$\forall \; v \in V_1$,

$$v = \alpha_1 x_1 + \ldots + \alpha_m x_m$$
$$= \sum_{i=1}^{m} \alpha_i x_i$$

$$T(v) = T\left(\sum \alpha_i x_i\right)$$
$$= \sum \alpha_i \, Tx_i.$$
$$\in V_2.$$

$e_1, e_2 \in \mathbb{R}^2$ s.t.

$$\begin{pmatrix} a \\ b \end{pmatrix} = a \begin{pmatrix} 1 \\ 0 \end{pmatrix} + b \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$\exists$ other pairs of basis vectors.

$$\begin{pmatrix} a \\ b \end{pmatrix} = \alpha_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \alpha_2 \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$Tx_1, Tx_2, \ldots, Tx_m \in V_2$.

$$\therefore \; Tx_1 = \sum_{i=1}^{n} t_{i1} \, y_i \quad , \quad Tx_2 = \sum_{i=1}^{n} t_{i2} \, y_i$$

$$\vdots$$

$$T x_m = \sum_{i=1}^{n} t_{im}\, y_i$$

$$\left[\, T x_1 \,\middle|\, T x_2 \,\middle|\, \cdots \,\middle|\, T x_m \,\right] = \begin{bmatrix} t_{11} & t_{12} & \cdots & t_{1m} \\ t_{21} & t_{22} & & t_{2m} \\ \vdots & \vdots & & \vdots \\ t_{n1} & t_{n2} & & t_{nm} \end{bmatrix}_{n \times m}.$$

Conversely, any $n \times m$ matrix with entries in K gives a linear transf. from V_1 to V_2

Exercise: $K = \mathbb{R}$. $A = \begin{bmatrix} 2 & 4 & 3 \\ 1 & 5 & 1 \end{bmatrix}_{2\times 3}$

Check: $T_A : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ given by

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \longmapsto A \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{bmatrix} 2 & 4 & 3 \\ 1 & 5 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$= \begin{pmatrix} 2x + 4y + 3z \\ x + 5y + z \end{pmatrix} \in \mathbb{R}^2.$$

Show that T_A is a linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^2$.

(check that $T_A(\alpha v_1 + \beta v_2) = \alpha T_A(v_1) + \beta T_A(v_2)$ for any $v_1, v_2 \in \mathbb{R}^3$ & $\alpha, \beta \in \mathbb{R}$.)