Theorem: Let A be a $m \times n$ matrix, $A = (a_{ij})_{\substack{1\le i\le m \\ 1\le j\le n}}$. $\qquad (A : \underset{x}{\mathbb{C}^n} \to \mathbb{C}^m)$

Let a_j denote the j^{th} column of A and
a_i^* denote the ith row of A.

(i) $\|A\|_1 = \max\limits_{1\le j\le n} \|a_j\|_1 = \max\limits_j \sum\limits_{i=1}^{m} |a_{ij}|$ (maximum column sum.)

(ii) $\|A\|_\infty = \max\limits_{1\le i\le n} \|a_i^*\|_\infty = \max\limits_i \sum\limits_{j=1}^{n} |a_{ij}|$ (maximum row sum.)

Proof: (i) $\|Ax\|_1 = \left\| \begin{bmatrix} \sum_j a_{1j}x_j \\ \sum_j a_{2j}x_j \\ \vdots \\ \sum_j a_{mj}x_j \end{bmatrix} \right\|_1 = \sum\limits_{i=1}^{m} \left| \sum\limits_{j=1}^{n} a_{ij}x_j \right|$

$[\;\cdots\;][\vdots] = [\;\;]$

$$\le \sum \sum |\underbrace{a_{ij}x_j}| \quad (\Delta \text{ ineq.})$$

$$= \sum_{i=1}^{m}\sum_{j=1}^{n} |a_{ij}|\cdot|x_j|$$

$$= \sum_{j=1}^{n} |x_j| \underbrace{\sum_{i=1}^{m}|a_{ij}|}_{j^{th}\text{ column sum } =: c_j}$$

$$= \sum_{j=1}^{n} c_j|x_j| = c_1|x_1| + c_2|x_2| + \cdots + c_n|x_n|$$
$$\le \max c_j|x_1| + \max c_j|x_2| + \cdots + \max c_j|x_n|.$$

$$\le \left(\max_j c_j\right)\sum_{j=1}^{n}|x_j|$$

$$= \left(\max_j c_j\right)\cdot \|x\|_1 .$$

$\|A\|_1 \le$ max. column sum

$$\therefore \|Ax\|_1 \le \|x\|_1 \cdot \left(\max_j c_j\right)$$

$$\therefore \|A\|_1 = \sup_{x\ne 0}\frac{\|Ax\|_1}{\|x\|_1} \le \max_j \left\{\sum_{i=1}^{m}|a_{ij}|\right\}$$

[Suppose $\max\limits_j \left\{\sum\limits_{i=1}^{m}|a_{ij}|\right\}$ is attained at $j = j_0$, so $\sum\limits_{i=1}^{m}|a_{ij_0}|$ is the max.]

To show that the bound is attained, we need to demonstrate a vector u such that $\|Au\|_1$ equals the max. element $\sum\limits_{i=1}^{m}|a_{ij_0}|$. supposing that max. column sum occurs at column j_0.

Let $u = (0, 0 \dots 0, \underset{j_0}{1}, 0 \dots 0)$; then $\|Au\|_1 = \left\| \begin{bmatrix} \sum a_{1j} u_j = a_{1j_0} \\ a_{2j_0} \\ \vdots \\ a_{mj_0} \end{bmatrix} \right\|_1$

$$= \sum_{i=1}^{m} |a_{ij_0}| = \max_j \left\{ \sum_{i=1}^{m} |a_{ij}| \right\}$$

as reqd.

(ii) $\|Ax\|_\infty = \left\| \begin{bmatrix} \sum_i^n a_{1j} x_j \\ \vdots \\ \sum a_{mj} x_j \end{bmatrix} \right\|_\infty = \max_i \left\{ \left| \sum_{j=1}^{n} a_{ij} x_j \right| \right\}$

$\max\{|a_1 x_1 + a_2 x_2|, |b_1 x_1 + b_2 x_2|\}$.

$\leq \max\{|a_1 + a_2|, |b_1 + b_2|\}$

$\cdot \max\{|x_1|, |x_2|\}$

can be checked based on Δ ineq.

$$\leq \max_i \left\{ \sum_{j=1}^{n} |a_{ij}| \right\} \cdot \max_j \{|x_j|\}$$

$$= \max_i \left\{ \sum_{j=1}^{n} |a_{ij}| \right\} \cdot \|x\|_\infty .$$

$$\therefore \|Ax\|_\infty \leq \|x\|_\infty \cdot \underbrace{\max_i \left\{ \sum_{j=1}^{n} |a_{ij}| \right\}}_{i^{th} \text{ row sum.}}$$

$$\therefore \|A\|_\infty \leq \max_i \left\{ \sum_{j=1}^{n} |a_{ij}| \right\}$$

Suppose $i = i_0$ is the row at which max. is attained i.e.

$$\max_i \left\{ \sum_{j=1}^{n} |a_{ij}| \right\} = \sum_{j=1}^{n} |a_{i_0 j}|$$

Then we need to find a vector u such that $\|Au\|_\infty = \sum_{j=1}^{n} |a_{i_0 j}|$.

Let u be the vector $u_j = \begin{cases} \overline{a_{i_0 j}} / |a_{i_0 j}| & \text{if } a_{i_0 j} \neq 0 \\ 1 & \text{o.w.} \end{cases}$ (max row sum is achieved at row i_0).

$$\left[a_{i_0 j} = \begin{pmatrix} a \\ b \\ 0 \end{pmatrix}, \quad u = \begin{pmatrix} \bar{a}/|a| \\ \bar{b}/|b| \\ 1 \end{pmatrix} \right.$$

$$\|Au\|_\infty = \left\| \begin{bmatrix} \sum_{j=1}^{n} a_{1j} \cdot \overline{a_{i_0 j}} / |a_{i_0 j}| \\ \vdots \\ \sum a_{mj} \, \overline{a_{i_0 j}} / |a_{i_0 j}| \end{bmatrix} \right\|_\infty$$

$$\left(a_{11} \frac{\overline{a_{i_0 j}}}{|a_{i_0 j}|} + a_{12} \cdots + a_{13} \cdots + \cdots \right)$$

2 cases: ① $i \neq i_0$: then

$$\left| \sum a_{ij} \frac{\overline{a_{i_0 j}}}{|a_{i_0 j}|} \right| \leq \sum |a_{ij}| \cdot \frac{|\overline{a_{i_0 j}}|}{|a_{i_0 j}|}$$

$= \sum_j |a_{ij}| =$ ith row sum.

② if $i = i_0$: $\left| \sum a_{i_0 j} \cdot \frac{\overline{a_{i_0 j}}}{|a_{i_0 j}|} \right| = \left| \sum |a_{i_0 j}| \right|$

$= \sum |a_{i_0 j}|$

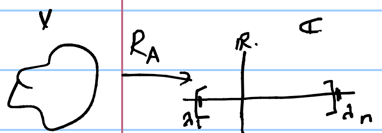
$= i_0$ row sum.

$= \max \left\{ \sum_{j=1}^{n} |a_{ij}| , \sum_{j=1}^{n} |a_{i_0 j}| \right\}$

$= \sum_{j=1}^{n} |a_{i_0 j}|$, which is as reqd. $\square$.

To find a formula for $\|A\|_2$, and for later purposes, lets look at the Rayleigh quotient.

Defn: Let $A: V \to V$ be a $\mathbb{C}$-linear map. The Rayleigh quotient of a matrix A is defined by $R_A : V \setminus \{0\} \to \mathbb{C}$



$$v \longmapsto \frac{\langle Av, v\rangle}{\langle v, v\rangle} = \frac{v^* A v}{v^* v} \quad (v \neq 0).$$

Note that $R_A(v) = R_A(\alpha v)$, $\forall \alpha \in \mathbb{C} \setminus \{0\}$.

$\therefore$ It suffices to consider vectors in the unit sphere (i.e. $v \in V$ such that $v^* v = 1$)

Properties of the Rayleigh quotient: $\qquad R_A(v)^* = R_A(v)$

(i) If A is Hermitian, then $R_A(v)$ is real $\forall v \in V$.

$\left\{ \text{Proof: } R_A(v) = \dfrac{v^* A v}{v^* v}, \quad (v^* A v)^* = v^* A^* v = v^* A v. \right\}$.

(ii) Theorem: If A is a $n \times n$ Hermitian matrix with eigenvalues $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ with associated eigenvectors $p_1, p_2, \ldots, p_n$ satisfying $p_i^* p_j = \delta_{ij}$ (Kronecker delta)

$\left(\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & \text{o.w.} \end{cases}\right)$

Then: (1) $R_A(p_k) = \lambda_k$, $\forall 1 \leq k \leq n$. (this is true by the spectral theorem.)

(2) $\lambda_k = \max\limits_{v \in V_k} R_A(v)$ where $V_k = \text{span}\{p_1, \ldots, p_k\}$.

(3) $\lambda_k = \min\limits_{v \perp V_{k-1}} R_A(v)$

$$\begin{array}{l} \overbrace{p_k \quad p_{k+1} \cdots p_n}^{V_{k-1}^{\perp}} \\ \lambda_1 \leq \cdots \leq \lambda_k \leq \lambda_{k+1} \cdots \leq \lambda_n \\ \underbrace{p_1, \cdots \quad p_k}_{v \in V_k} \end{array}$$

(4) $\{R_A(v) / v \in V\} \subset [\lambda_1, \lambda_n] \subset \mathbb{R}$.

Proof: Let $\mathcal{U} = [p_1 | \cdots | p_n]$ then $\mathcal{U}^* A \mathcal{U} = \text{diag}(\lambda_i) = D$.
($\mathcal{U}$ is unitary)

For any $v \in V$, $v \neq 0$, let $w = \mathcal{U}^{-1} v$, so that $v = \mathcal{U} w$.

Then $R_A(v) = \dfrac{v^* A v}{v^* v} = \dfrac{w^* \mathcal{U}^* A \mathcal{U} w}{w^* \mathcal{U}^* \mathcal{U} w} = \dfrac{w^* D w}{w^* w} = R_D(w)$.

Now for any $v \in V_k$, $v = \sum_{i=1}^{k} \alpha_i p_i$, $\therefore w = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_k \\ 0 \\ \vdots \\ 0 \end{pmatrix}$

$$\left[\begin{aligned} v &= \mathcal{U} w \\ &= [p_1 | \cdots | p_n] \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_k \\ 0 \\ \vdots \\ 0 \end{pmatrix} \\ &= \sum_{i=1}^{k} \alpha_i p_i \end{aligned}\right]$$

$$\therefore R_A(v) = R_A\left(\sum_{i=1}^{k} \alpha_i p_i\right) = R_D(w)$$

$$= \frac{(\bar{\alpha}_1 \cdots \bar{\alpha}_k\, 0 \cdots 0) \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_k \\ 0 \\ \vdots \\ 0 \end{pmatrix}}{w^* w}$$

★ $$R_A(v) = \frac{\sum_{i=1}^{k} \lambda_i |\alpha_i|^2}{\sum_{i=1}^{k} |\alpha_i|^2} \quad \left(\begin{array}{l}\text{weighted average of} \\ \text{eigenvalues } \lambda_1, \ldots, \lambda_k\end{array}\right)$$

(1) $R_A(p_k) = R_D(w)$, $w = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \cdots k$ $\qquad p_k \in V_k$, $w = \mathcal{U}^{-1} p_k = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \end{pmatrix}$

$= \lambda_k$.

(2) $$\max_{v \in V_k} R_A(v) = \max_{v \in V_k} \left\{ \frac{\sum \lambda_i |\alpha_i|^2}{\sum |\alpha_i|^2} \right\} = \lambda_k.$$

[Note that $\lambda_1 \le \cdots \le \lambda_k \le \cdots$

$$\Rightarrow \lambda_1 |\alpha_1|^2 + \cdots + \lambda_k |\alpha_k|^2$$
$$\le \lambda_k |\alpha_1|^2 + \cdots + \lambda_k |\alpha_k|^2$$
$$= \lambda_k \sum_{i=1}^{k} |\alpha_i|^2$$

$$\therefore \frac{\sum \lambda_i |\alpha_i|^2}{\sum |\alpha_i|^2} \le \lambda_k \frac{\sum |\alpha_i|^2}{\sum |\alpha_i|^2} = \lambda_k \Bigg]$$

(3) If $v \perp V_{k-1}$, then v is of the form $\sum_{i=k} \alpha_i p_i$

$\therefore w$ is of the form $\begin{pmatrix} 0 \\ \vdots \\ 0 \\ \alpha_k \\ \vdots \\ \alpha_n \end{pmatrix}$

$$R_A(v) = R_D(w) = \frac{\sum_{i=k}^{n} \lambda_i |\alpha_i|^2}{\sum_{i=k}^{n} |\alpha_i|^2}$$

But $\lambda_k \le \lambda_{k+1} \le \cdots \le \lambda_n \Rightarrow \dfrac{\sum_{i=k}^{n} \lambda_i |\alpha_i|^2}{\sum |\alpha_i|^2} \ge \lambda_k \dfrac{\sum_{i=k}^{n} |\alpha_i|^2}{\sum |\alpha_i|^2}$

so $\lambda_k = \min_{v \perp V_{k-1}} R_A(v)$.

$\operatorname{im} R_A = [\lambda_1, \lambda_n]$

(4) It follows from (2) & (3) that $\lambda_1 \le R_A(v) \le \lambda_n$ $\forall\, v \in V \setminus \{0\}$.

Since R_A is a continuous function on the "unit sphere" in V, $\operatorname{im} R_A = [\lambda_1, \lambda_n]$ follows from the intermediate value theorem.

Thus we have for a Hermitian matrix A:

- $\lambda_1 = \min \{ R_A(v) \mid v \in V \}$
- $\lambda_n = \max \{ R_A(v) \mid v \in V \}$
- The Rayleigh quotient gives a powerful method of calculating / estimating e-values / e-vectors. We will come back to this topic later.

Calculating $\|A\|_2$: Let A be a $m \times n$ matrix.

$$\|A\|_2 = \sup_{\substack{x \in \mathbb{C}^n \\ x \neq 0}} \frac{\|Ax\|_2}{\|x\|_2} = \sup_{\substack{x \neq 0 \\ x \in \mathbb{C}^n}} \frac{\sqrt{\langle Ax, Ax \rangle}}{\sqrt{\langle x, x \rangle}}$$

$$\therefore \|A\|_2^2 = \sup_{\substack{x \neq 0 \\ x \in \mathbb{C}^n}} \frac{\langle Ax, Ax \rangle}{\langle x, x \rangle}$$ (since these are positive quantities)

$$= \sup_{\substack{x \neq 0 \\ x \in \mathbb{C}^n}} \left(\frac{x^* A^* A x}{x^* x} \right) = \sup_{\substack{x \neq 0 \\ x \in \mathbb{C}^n}} R_{A^*A}(x)$$

A^*A is Hermitian (always), so $R_{A^*A}(v) \le$ largest e-value of A^*A $(\forall\, v \in \mathbb{C}^n)$

$\therefore \sup R_{A^*A}(x) =$ largest e-value of A^*A
$= \rho(A^*A)$ (notation)

A^*A is positive semidefinite $\left(x^* A^* A x = \langle Ax, Ax \rangle = \|Ax\|^2 \ge 0 \right)$.

$\therefore$ all e-values of A^*A are ≥ 0.

$\therefore \|A\|_2 = \sqrt{\rho(A^*A)}$ = largest singular value of A. $= \sigma_1(A)$ (notation).