Defn: S is said to be an orthogonal set if $\forall\, v, w \in S$, $(v, w) = 0$.

Defn. S is said to be orthonormal if ① S is orthogonal
② $\|v\| = 1 \quad \forall\, v \in S$.

Proposition: An orthogonal set of nonzero vectors is linearly independent.

Proof Let $S = \{v_1, \ldots, v_n\}$ be an orthogonal set.

Let $\sum \alpha_i v_i = 0$.

Consider $\underset{0 =}{\left(\sum \alpha_i v_i, v_i\right)} = (\alpha_1 v_1 + \cdots + \alpha_n v_n, v_i)$

$$= \underbrace{\alpha_1(v_1, v_i)}_{=0} + \cdots + \underbrace{\alpha_i (v_i, v_i)}_{\text{only } \neq 0 \text{ Term}} + \cdots + \underbrace{\alpha_n(v_n, v_i)}_{=0}$$

$$= \alpha_i (v_i, v_i) \qquad \Rightarrow \alpha_i = 0$$

$(v_i, v_i) \neq 0$.

In this way, we can show $\alpha_i = 0$, $\forall i$. $\square$

Prop: A maximal orthonormal set is a basis.

($S = \{v_1, \ldots, v_n\}$ is said to be maximal orthonormal, if adding another element to S makes it no longer orthogonal.)

So for any other vector $v \in V$, (v, v_i) ..

Proof: Let V be f.d.v.s. & S be a maximal orthonormal set.

Let $S = \{v_1, \ldots, v_n\}$, let $w \in V$. consider

$$w' = w - \sum (w, v_i) v_i \qquad \text{component of } w \text{ in the direction } v_i$$

$$= w - \left[(w, v_1) v_1 + (w, v_2) v_2 + \cdots + (w, v_n) v_n\right]$$

~~Co~~ Consider $(w', v_k) = (w, v_k) - \left[(w, v_1)(v_1, v_k) + (w, v_2)(v_2, v_k) + \cdots + (w, v_k)\underbrace{(v_k, v_k)}_{=1} + \cdots + (w, v_n)(v_n, v_k)\right]$

$$= (w, v_k) - \left[(w, v_k)\right] = 0.$$

In this way, $(w', v_i) = 0 \ \forall i$.

S is maximal orthonormal $\Rightarrow$ $w' = 0$.

$$\text{i.e.} \quad w - \sum (w, v_i) v_i = 0.$$

$$\Rightarrow \quad w = \sum \underbrace{(w, v_i)}_{\alpha_i} v_i = \sum_{i=1}^{n} \alpha_i v_i$$

$\therefore$ S spans V, so S is a basis. $\square$