Note Title

SVD – Singular value decomposition.

Defn: Given $A \in \mathbb{C}^{m\times n}$, a singular value decomposition (SVD) of A is a factorization $A = U \Sigma V^*$ where U & V are unitary & Σ is diagonal.

The diagonal entries of Σ are called the singular values of A & are usually arranged in decreasing order.

Dimension considerations: let $m \geq n$ (WLOG); for simplicity, assumed rank $= n$.

- Reduced SVD –

$$\underset{m\times n}{A} = \underset{m\times n}{U}\ \underset{n\times n}{\Sigma}\ \underset{n\times n}{V^*} \qquad \Sigma = \begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_n \end{bmatrix}$$

What happens if $m \leq n$?

- Full SVD –

$$\underset{m\times n}{A} = \underset{m\times m}{U}\ \underset{m\times n}{\Sigma}\ \underset{n\times n}{V^*} \qquad \Sigma = \begin{bmatrix} \sigma_1 \cdots \sigma_n \\ \hline 0 \\ \vdots \\ 0 \end{bmatrix}_{m\times n}$$

($m-n$ extra columns of U)

Columns of U are called "left singular vectors" of A

V - - "right singular vectors" of A.

- If $A = U\Sigma V^*$, then $AV = U\Sigma$ $\quad \left(\Sigma = \begin{bmatrix} \sigma_1 & & 0 \\ & \sigma_2 & \\ 0 & & \ddots\ \sigma_n \end{bmatrix}\right)$

$$A\,[v_1 | \cdots | v_n] = [u_1 | \cdots | u_n] \begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_n \end{bmatrix}$$

$$[Av_1 | Av_2 | \cdots | Av_n] = [\sigma_1 u_1 | \cdots | \sigma_n u_n]$$

i.e. $\boxed{Av_i = \sigma_i u_i}$ $\quad \forall\ 1 \leq i \leq n.$

Consequences –

① The e-values of A^*A are σ_i^2. The right singular vectors v_i are the corr. orthonormal e-vectors i.e.

$$\underbrace{\underset{n\times m}{A^*}\ \underset{m\times n}{A}}_{n\times n} \qquad (A^*A)\,v_j = \sigma_j^2\, v_j \qquad 1 \leq j \leq n.$$

Proof: $A^*A = (U\Sigma V^*)^* (U\Sigma V^*)$

$= V\Sigma^* \underbrace{U^* U} \Sigma V^*$

A is $m\times n$.
A^*A is $n\times n$.

$\Sigma^*\Sigma = \begin{bmatrix} \sigma_1^2 & & & \\ & \sigma_2^2 & & \\ & & \ddots & \\ & & & \sigma_n^2 \end{bmatrix} = \Sigma^2.$

$A^*A = \underline{V}\,\Sigma^2\,\underline{V}^*$ $\rightarrow$ this is the e-value decomposition of A^*A. $\square$

② The e-values of AA^* are σ_i^2 $(1\le i\le n)$ & $(m-n)$ zeroes.

The left singular vectors u_i are the corresponding orthonormal e-vectors i.e. $AA^* u_j = \sigma_j^2 u_j$ $(1\le j\le m)$

$\overbrace{\underset{m\times n}{A}\ \underset{n\times m}{A^*}}^{m\times m}$

Proof: If U is $m\times n$, choose $(m-n)$ orthogonal vectors $\{\tilde{u}_1, \dots, \tilde{u}_{m-n}\}$

& let $\tilde{U} = [\,U \,|\, \tilde{u}_1 \,|\, \cdots \,|\, \tilde{u}_{m-n}\,]$.

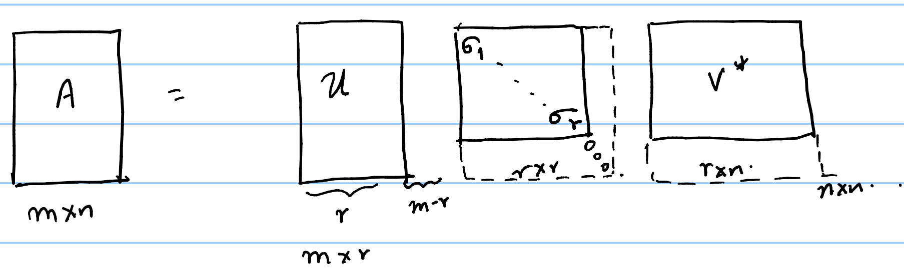
$AA^* = (U\Sigma \underbrace{V^*)(V} \Sigma^* U^*)$

$= U\Sigma^2 U^*$

$= \tilde{U} \left[\begin{array}{c|c} \Sigma^2 & 0 \\ \hline 0 & 0 \end{array}\right] \tilde{U}^*$ $\rightarrow$ this is the e-value decomp. of AA^*. $\square$

③ If rank $A = r \le \min\{m,n\}$, the reduced SVD of A takes the form:

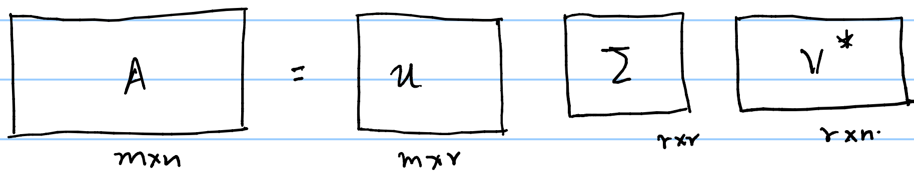
(Reduced)



(if $A = U\Sigma V^*$, then rank(A) = rank$(U\Sigma V^*)$ = rank Σ)

④ If $m \le n$ to begin with, then SVD of A is of the form (rank $A = r$)

(Check).



(5) SVD of A^*: If $A = \mathcal{U}\Sigma V^*$, $A^* = \underline{V\Sigma^*\mathcal{U}^*}$, $\underbrace{\Sigma = \Sigma^*}$

(Singular values of A = singular values of A^*)

$\therefore A^* = V\Sigma^*\mathcal{U}^*$ is the SVD of A^*.

Columns of V are left singular vectors of A^*
Columns of $\mathcal{U}$ are right singular vectors of A^*.

$$A^* = V\Sigma\mathcal{U}^*$$
$$A^*\mathcal{U} = V\Sigma$$
$$\boxed{A^* u_j = \sigma_j v_j}$$

$\bar{\sigma}_i = \sigma_i$ (why?)

Is $\Sigma = \Sigma^*$? $\Sigma = \begin{bmatrix} \sigma_1 & & & \\ & \ddots & & \\ & & \sigma_r & \\ & & & 0 \\ & & & & \ddots \\ & & & & & 0 \end{bmatrix}$, $\Sigma^* = \begin{bmatrix} \bar{\sigma}_1 & & & \\ & \ddots & & \\ & & \bar{\sigma}_r & \\ & & & 0 \\ & & & & \ddots \\ & & & & & 0 \end{bmatrix} = \begin{bmatrix} \sigma_1 & & & \\ & \ddots & & \\ & & \sigma_r & \\ & & & 0 \\ & & & & \ddots \\ & & & & & 0 \end{bmatrix}$

(1) the matrices A^*A & AA^* are symmetric (real e-value) positive semidef. (non-neg e-values)

Consider $\langle A^*Av, v\rangle$
$= \langle Av, Av\rangle = \|Av\| \geq 0$ $\forall v \in V$.

[T is pos. semidef if $\langle Tv, v\rangle \geq 0$ $\forall v \neq 0$.]

Similarly, $\langle AA^*v, v\rangle = \langle A^*v, A^*v\rangle = \|A^*v\| \geq 0$ $\forall v \in V$.

$\therefore$ The e-values of A^*A & AA^* are real & non-neg.
Their +ve sq. roots are the singular values of A (& A^*).

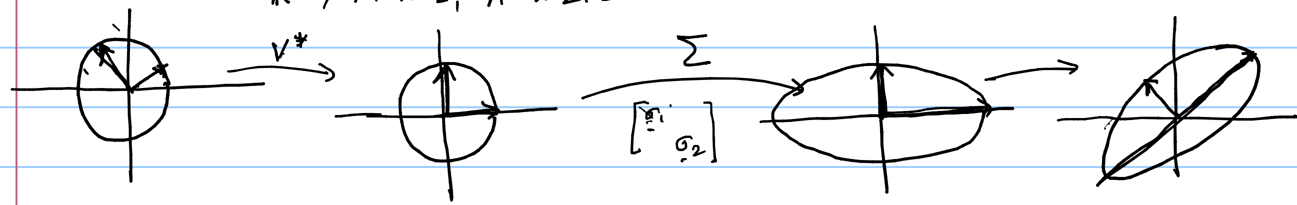
- Existence of SVD – after matrix norms is done.

- Geometry of the SVD: $A \in \mathbb{C}^{m\times n}$, $A = \mathcal{U}\Sigma V^*$

$A: \mathbb{R}^n \to \mathbb{R}^m$.

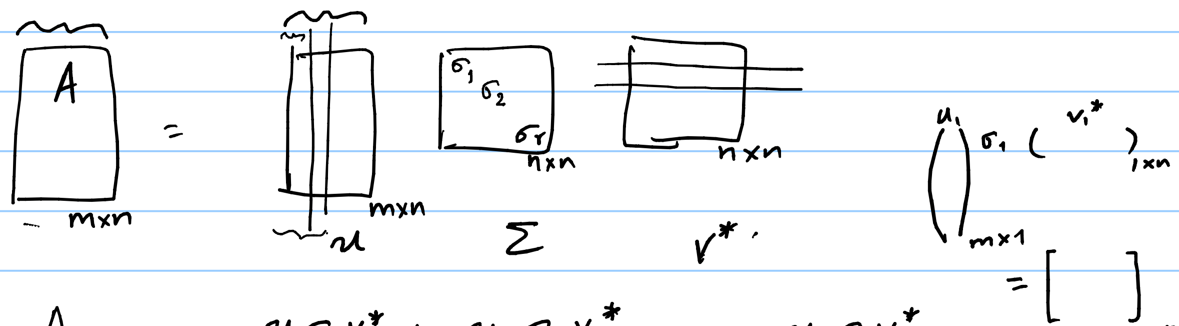
$$Ax = (\mathcal{U}\Sigma V^*)x = \mathcal{U}\Sigma(V^*x)$$
$$= \mathcal{U}\left(\Sigma(V^*x)\right)$$

$\mathbb{R}^2$, $m=n=2$, A is 2×2 matrix.



$$A = \mathcal{U}\,\Sigma\, V^*$$

Remark: Eckhart-Young theorem:



$$A \approx \underbrace{\underbrace{\underbrace{u_1\sigma_1 v_1^*}_{A_1 = \text{rank } 1} + u_2\sigma_2 v_2^*}_{\text{rank } 2 \quad A_2} + \cdots + u_r\sigma_r v_r^*}_{\text{rank } r \quad A_r}$$

$$\|A - A_k\| \leq \|A - B\| \quad \text{for rank } k \text{ matrix } B$$

SVD gives a way to approximate A (in the best possible way) by lower rank matrices.

<u>A is diagon...</u>

$$A = P^{-1} D P.$$

$Ax = b$	$Dx = b$
A	D
$\|Ax - b\|$	$\|Dx - b\|$

A is not diagonalizable?