

VIP Cheatsheet: Applications

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Physics Laws

□ **Gravitational force** – A mass m is subject to the gravitational force \vec{F}_g , which is expressed with respect to \vec{g} of magnitude $9.81 \text{ m} \cdot \text{s}^{-2}$ and directed towards the center of the Earth, as follows:

$$\vec{F}_g = m\vec{g}$$

□ **Spring force** – A spring of constant k and of relaxed position \vec{x}_0 attached a mass m of position \vec{x} has a force \vec{F}_s expressed as follows:

$$\vec{F}_s = -k(\vec{x} - \vec{x}_0)$$

□ **Friction force** – The friction force F_f of constant coefficient β applied on a mass of velocity \vec{v} is written as:

$$\vec{F}_f = -\beta\vec{v}$$

□ **Mass moment of inertia** – The mass moment of inertia of a system of mass m_i located at distance r_i from point O , expressed in point O is written as:

$$J_0 = \sum_i m_i r_i^2$$

□ **Torque** – The torque \vec{T} of a force \vec{F} located at \vec{r} from the reference point O is written as:

$$\vec{T} = \vec{r} \times \vec{F}$$

□ **Newton's second law** – A mass m of acceleration \vec{a} to which forces \vec{F}_i are applied verifies the following equation:

$$m\vec{a} = \sum_i \vec{F}_i$$

– In the 1-D case along the x axis, we can write it as $mx'' = \sum_i F_i$.

– In the rotational case, around point O , we can write it as $J_0\theta'' = \sum_i T_i$.

Spring-mass system

□ **Free undamped motion** – A free undamped spring-mass system of mass m and spring coefficient k follows the ODE $x'' + \frac{k}{m}x = 0$, which can be written as a function of the natural frequency ω as:

$$x'' + \omega^2 x = 0 \quad \text{with} \quad \omega = \sqrt{\frac{k}{m}}$$

□ **Free damped motion** – A free damped spring-mass system of mass m , of spring coefficient k and subject to a friction force of coefficient β follows the ODE $x'' + \frac{\beta}{m}x' + \frac{k}{m}x = 0$, which can be written as a function of the damping parameter λ and the natural frequency ω as:

$$x'' + 2\lambda x' + \omega^2 x = 0 \quad \text{with} \quad \lambda = \frac{\beta}{2m} \quad \text{and} \quad \omega = \sqrt{\frac{k}{m}}$$

which has the following cases summed up in the table below:

Condition	Type of motion
$\lambda > \omega$	Over damped
$\lambda = \omega$	Critically damped
$\lambda < \omega$	Under damped

□ **Forcing frequency** – A forcing function $F(t)$ is often modeled with a periodic function of the form $F(t) = F_0 \sin(\gamma t)$, where γ is called the forcing frequency.

□ **Forced undamped motion** – A forced undamped spring-mass system of mass m and spring coefficient k follows the ODE $x'' + \frac{k}{m}x = F_0 \sin(\gamma t)$, which can be written as a function of the natural frequency ω as:

$$x'' + \omega^2 x = F_0 \sin(\gamma t) \quad \text{with} \quad \omega = \sqrt{\frac{k}{m}}$$

which has the following cases summed up in the table below:

Condition	Type of motion
$\gamma \neq \omega$	General response
$\gamma \approx \omega$	Beats
$\gamma = \omega$	Resonance

□ **Forced damped motion** – A forced damped spring-mass system of mass m , of spring coefficient k and subject to a friction force of coefficient β follows the ODE $x'' + \frac{\beta}{m}x' + \frac{k}{m}x = F_0 \sin(\gamma t)$, which can be written as a function of the damping parameter λ and the natural frequency ω as:

$$x'' + 2\lambda x' + \omega^2 x = F_0 \sin(\gamma t) \quad \text{with} \quad \lambda = \frac{\beta}{2m} \quad \text{and} \quad \omega = \sqrt{\frac{k}{m}}$$

Boundary Value Problems

□ **Types of boundary conditions** – Given a numerical problem between 0 and L , we distinguish the following types of boundary conditions:

Name	Boundary values
Dirichlet	$y(0)$ and $y(L)$
Neumann	$y(0)$ and $y'(L)$
Robin	$y(0)$ and $\alpha y(L) + \beta y'(L)$

□ **Numerical differentiation** – The table below sums up the approximation of the derivatives of y at point x_j , knowing the values of y at each point of a uniformly spaced set of grid points.

Order of derivative	Name	Formula	Order of error
First derivative	Forward difference	$y'_j = \frac{y_{j+1} - y_j}{h}$	$O(h)$
	Backward difference	$y'_j = \frac{y_j - y_{j-1}}{h}$	$O(h)$
	Central difference	$y'_j = \frac{y_{j+1} - y_{j-1}}{2h}$	$O(h^2)$
Second derivative	Central difference	$y''_j = \frac{y_{j+1} - 2y_j + y_{j-1}}{h^2}$	$O(h^2)$

□ **Direct method** – The direct method can solve linear ODEs by reducing the problem to the resolution of a linear system $Ay = f$, where A is a tridiagonal matrix.

□ **Shooting method** – The shooting method is an algorithm that can solve ODEs through an iterative process. It uses a numerical scheme, such as Runge-Kutta, and converges to the right solution by iteratively searching for the missing initial condition $y'(0)$.

Remark: in the linear case, the shooting method converges after the first two initial guesses.