Proof of Feedback Control Convergence

Let $\boldsymbol{\rho}$ be the desired population density vector, and let $\boldsymbol{\rho}'_t$ be the AUV population density at time t. If the diagonal gain (i.e., the gain for transitions from nodes to themselves) is k and the off-diagonal gain (transitions from nodes to different nodes) is k', then for the matrix

$$[k] = \begin{bmatrix} 1-k & -k' & \dots & -k' \\ -k' & 1-k & \dots & -k' \\ \vdots & \vdots & \ddots & \vdots \\ -k' & -k' & \dots & 1-k \end{bmatrix}$$

and error $\boldsymbol{e}_t = \boldsymbol{\rho} - \boldsymbol{\rho}'_t$, we will show $\boldsymbol{e}_{t+1} = [k]\boldsymbol{e}_t$.

The feedback control specifies the transition matrix at time t as

$$[T]_t = \begin{bmatrix} 1+ke_0/\rho'_0 & k'e_1/\rho'_1 & \dots & k'e_n/\rho'_n \\ k'e_0/\rho'_0 & 1+ke_1/\rho'_1 & \dots & k'e_n/\rho'_n \\ \vdots & \vdots & \ddots & \vdots \\ k'e_0/\rho'_0 & k'e_1/\rho'_1 & \dots & 1+ke_n/\rho'_n \end{bmatrix}_t$$

so that after a single transition at time t, we have $\boldsymbol{\rho}'_{t+1} = [T]_t\boldsymbol{\rho}'_t$. Therefore

$$\begin{aligned} \boldsymbol{e}_{t+1} &= \boldsymbol{\rho} - \boldsymbol{\rho}'_{t+1} \\ &= \boldsymbol{\rho} - \begin{bmatrix} 1+ke_0/\rho'_0 & k'e_1/\rho'_1 & \dots & k'e_n/\rho'_n \\ k'e_0/\rho'_0 & 1+ke_1/\rho'_1 & \dots & k'e_n/\rho'_n \\ \vdots & \vdots & \ddots & \vdots \\ k'e_0/\rho'_0 & k'e_1/\rho'_1 & \dots & 1+ke_n/\rho'_n \end{bmatrix}_t \begin{bmatrix} \rho'_0 \\ \rho'_1 \\ \vdots \\ \rho'_n \end{bmatrix}_t \\ &= \boldsymbol{\rho} - \begin{bmatrix} p_0 + ke_0 + k'e_1 + \dots + k'e_n \\ k'e_0 + p_1 + ke_1 + \dots + k'e_n \\ \vdots \\ k'e_0 + k'e_1 + \dots + p_n + ke_n \end{bmatrix}_t \\ &= \boldsymbol{\rho} - \boldsymbol{\rho}'_t - \begin{bmatrix} k & k' & \dots & k' \\ k' & k & \dots & k' \\ \vdots & \vdots & \ddots & \vdots \\ k' & k' & \dots & k \end{bmatrix} \begin{bmatrix} e_0 \\ e_1 \\ \vdots \\ e_n \end{bmatrix}_t \\ &= [k]\boldsymbol{e}_t. \end{aligned}$$